Introduction to Fuzzy Logic

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It is the mark of an instructed mind to rest satisfied with that degree of precision which the nature of the subjects admits, and not to seek exactness where only an approximation of the truth is possible

Aristotle, 384-322 BC

All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.

Bertrand Russell, 1923

We must exploit our tolerance for imprecision

Lotfi Zadeh, 1973

- More the uncertainty in a problem, the less precise understanding of that problem
- Aristotelian logic does not admit imprecision in truth
- We should balance the precision we seek with the uncertainty that exists
- This module is dedicated to the characterization and quantification of uncertainty within engineering problems such that an appropriate level of precision can be expressed
- Why to engage in such pursuit?
  - High level of precision cost in terms of time and money
  - Are we solving problems that require precision?
- When considering use of fuzzy logic, one should ponder the need for exploiting the tolerance for imprecision
- Can human live with a little less precision?

- Lotfi Zadeh from UCB introduced fuzzy set theory in 1965
- Uncertainty can be manifested in many forms:
  - it can be fuzzy (not sharp, unclear, imprecise, approximate)
  - it can be vague (not specific, amorphous)
  - it can be ambiguous (too many choices, contradictory)
  - it can be in the form of ignorance (not knowing sth)
- Statement, ‘I shall return soon’ is vague
- Statement, ‘I shall return in a few minutes’ is fuzzy
- Semantically, vague and fuzzy are not synonyms
- Usually a vague proposition is fuzzy, but the converse is not generally true
- Consider the proposition that a person is young

- Primary benefit of fuzzy systems theory is to approximate system behavior where analytic function or numerical relation do not exists
- Fuzzy systems are useful in two general contexts:
    1. situation involving highly complex system whose behaviors are not well understood and
    2. situation where an approximate, but fast, solution is warranted
- Fuzzy systems are inherently robust

Fuzzy Sets and Membership

- A person is tall if s(he) is above 6 feet-can be assessed
- Is the person nearly 6 feet tall?
- People of Tutsi tribe in Rwanda and Burundi, a height for a male of 6 feet is considered short

Fuzzy Sets and Membership

- Classical sets contain objects that satisfy precise properties of membership
- Fuzzy set contain objects that satisfy imprecise properties of membership, i.e, membership of an object in fuzzy set can be approximate
- For crisp set, an element x in the universe X is either a member of some crisp set A or not, mathematically

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- Zadeh extended the notion of binary membership to accommodate various “degree of membership” on the real continuous interval $[0, 1]$, where end point 0 and 1 conform to no membership and full membership
- Sets on the universe X that can accommodate “degrees of membership” are called fuzzy sets

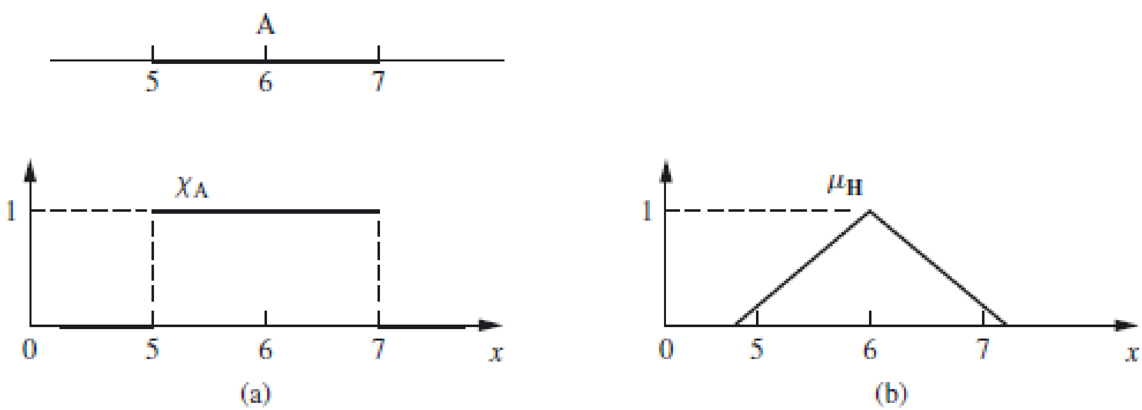


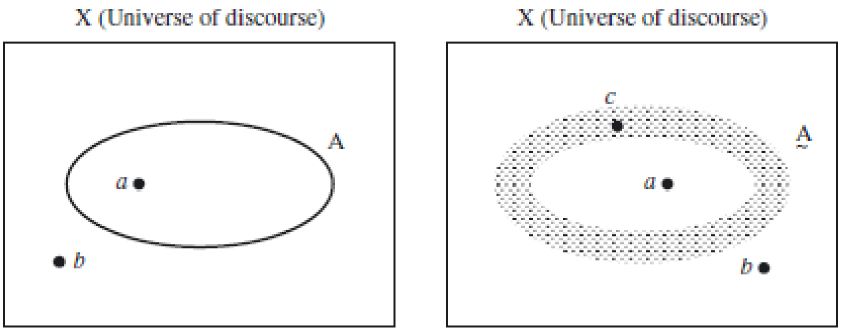
Figure 1: Height membership functions for (a) a crisp set A and (b) a fuzzy set H

- A crisp set has unique membership function
- A fuzzy set can have infinite number of membership function

Properties of fuzzy membership function:

1. Normality ($\mu_H(6) = 1$)
2. Monotonicity (the closer the H is to 6, the closer μ_H is to 1), and
3. Symmetric (numbers equidistant from 6 should have the same value of μ_H)

Classical Sets



(a) (b)

- **Universe of discourse** is the universe of all available information on a given problem
- Abstraction of the universe we denote by X, and a set on this universe by A
- A classical set is defined by crisp boundaries, i.e there is no uncertainty in the prescription or location of the boundaries
- A fuzzy set is described by vague or ambiguous properties; hence its boundaries are ambiguous

- $A \subset B$: A is fully contained in B (if $x \in A$, then $x \in B$)
- $A \subseteq B$: A is contained in or is equivalent to B
- $A \leftrightarrow B$: $A \subseteq B$ and $B \subseteq A$ (A is equivalent to B)
- All possible set of X constitute a power set, $P(X)$

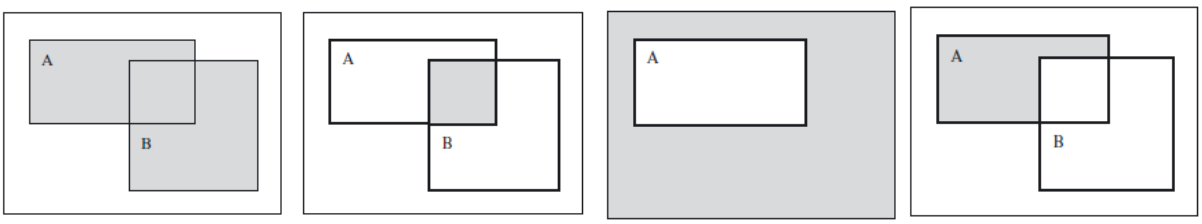
Example

Let $X = \{a, b, c\}$, so the power set is

$$P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

Operations on Classical Set

- **Union**: $A \cup B = \{x | x \in A \vee x \in B\}$
- **Intersection**: $A \cap B = \{x | x \in A \wedge x \in B\}$
- **Complement**: $\overline{A} = \{x | x \notin A, x \in X\}$
- **Difference**: $A | B = \{x | x \in A \wedge x \notin B\}$



Properties of Classical (Crisp) Set

Properties of classical sets and showing their similarity to fuzzy sets are as follows:

- **Commutativity**: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- **Associativity**: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributivity**: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **Idempotency**: $A \cup A = A$, $A \cap A = A$
- **Identity**: $A \cup \emptyset = A$, $A \cap X = A$, $A \cap \emptyset = \emptyset$, and $A \cup X = X$
- **Transitivity**: If $A \subseteq B$, and $B \subseteq C$ then $A \subseteq C$
- [1]**Involution**: $\overline{\overline{A}} = A$

[1] a function, transformation, or operator that is equal to its inverse, i.e. which gives the identity when applied to itself.

Following classical set properties are not valid for fuzzy set:

- **Axiom of excluded middle**: $A \cup \overline{A} = X$
- **Axiom of contradiction**: $A \cap \overline{A} = \emptyset$

Mapping Classical Sets to Functions

Mapping relates set-theoretic forms to function-theoretic representation of information

- Suppose X and Y are two different universe of discourse (information)
- If $x \in X$ corresponds to an element $y \in Y$, it is termed as mapping from X to Y or $f : X \mapsto Y$
- As a mapping, the characteristic (indicator) function χ_A is defined as

$$\chi_A = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- For any set A defined on the universe X, there exists a function-theoretic set, called a value set, $V(A)$, under the mapping of characteristic function, χ
- By convention, the null set $\emptyset$ is assigned the membership value 0 and the whole set X is assigned the membership value 1

Example

Consider a universe of three elements $X = \{a, b, c\}$, it is required to map the power set $P(X)$ to a universe Y, consisting of only two elements (the characteristic function).

- $P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
- The element in the value set $V(A)$ as determined from the mapping are

$$V\{P(X)\} = \{\{0, 0, 0\}, \{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}, \{1, 1, 0\}, \{0, 1, 1\}, \{1, 0,$$

Consider two sets A and B defined on the universe X. Lets defined set operation on these sets in terms of function-theoretic terms as

- **Union**:

$$A \cup B \rightarrow \chi_{A \cup B}(x) = \chi_A(x) \vee \chi_B(x) = max(\chi_A(x), \chi_B(x))$$

- **Intersection**:

$$A \cap B \rightarrow \chi_{A \cap B}(x) = \chi_A(x) \wedge \chi_B(x) = min(\chi_A(x), \chi_B(x))$$

- **Complement**: $\overline{A} \rightarrow \chi_{\overline{A}}(x) = 1 - \chi_A(x)$
- **Containment**: $A \subseteq B \rightarrow \chi_A(x) \leq \chi_B(x)$

Fuzzy Sets

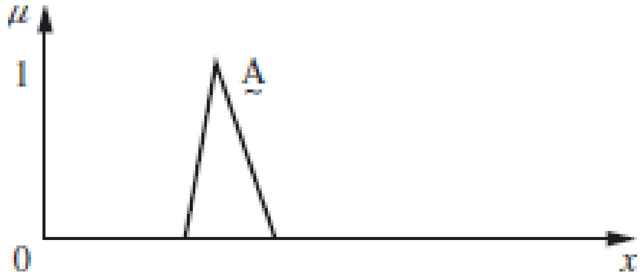
- In classical/crisp sets, the transition for $x \in X$ between membership and non-membership in a given set is abrupt and well defined
- In fuzzy sets, this transition can be gradual
- In fuzzy sets, transition among various degrees of membership can be thought of as conforming to the fact that the boundaries of fuzzy sets are vague and ambiguous
- Here, membership of $x \in X$ is measured by a function that attempts to describe vagueness and ambiguity
- A fuzzy set, is a set containing elements that have varying degrees of membership in the set
- Elements in a fuzzy set, because their membership need not be complete, can also be members of other fuzzy sets on the same universe

- Characteristic function of fuzzy set is denoted by $\mu_A(x)$
- Notation for fuzzy sets defined on a universe of discourse X:

$$\tilde{A} = \left\{ \frac{\mu_{\tilde{A}}(x_1)}{x_1} + \frac{\mu_{\tilde{A}}(x_2)}{x_2} + \ldots \right\} = \left\{ \sum_i \frac{\mu_{\tilde{A}}(x_i)}{x_i} \right\}$$

$$\tilde{A} = \left\{ \int \frac{\mu_{\tilde{A}}(x)}{x} \right\}$$

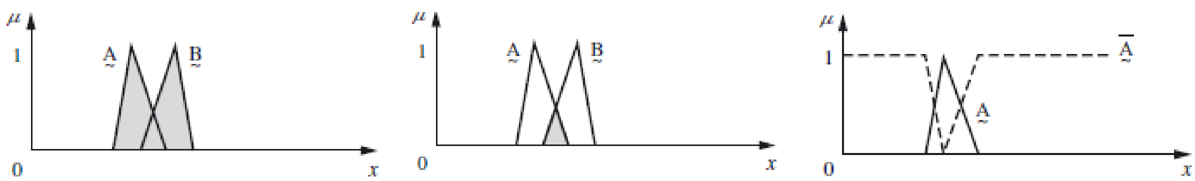
Here horizontal bar is not a quotient but rather a delimiter



Fuzzy Set Operations

Lets consider three fuzzy sets $\tilde{A}, \tilde{B}$ and $\tilde{C}$ on the universe X

- **Union**: $\mu_{\tilde{A}\cup\tilde{B}}(x) = \mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x) = max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$
- **Intersection**: $\mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x) = min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$
- **Complement**: $\mu_{\bar{\tilde{A}}}(x) = 1 - \mu_{\tilde{A}}$
- $\tilde{A} \subseteq X \rightarrow \mu_{\tilde{A}}(x) \leq \mu_X(x)$
- $\forall x \in X, \mu_{\emptyset}(x) = 0$
- $\forall x \in X, \mu_X(x) = 1$



- Collection of all fuzzy sets and fuzzy subsets on X is denoted as the fuzzy power set $P(X)$
- All fuzzy sets can overlap, so $n_{P(X)} = \infty$
- Fuzzy sets follow all the properties of crisp set except excluded middle axioms
- Excluded middle axioms do not hold for fuzzy sets:
  - $A \cup \overline{A} \neq X$
  - $A \cap \overline{A} \neq \emptyset$

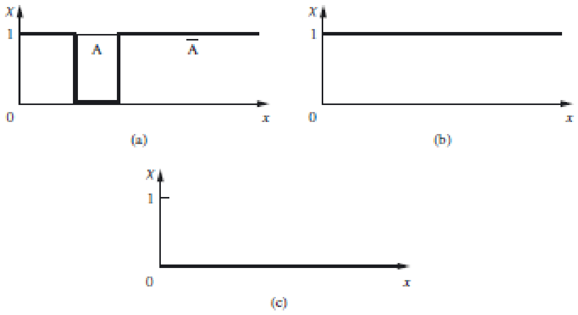


Figure 2: (a) Crisp set A and its complement (b) $A \cup \overline{A} = X$ and (c) $A \cap \overline{A} = \emptyset$

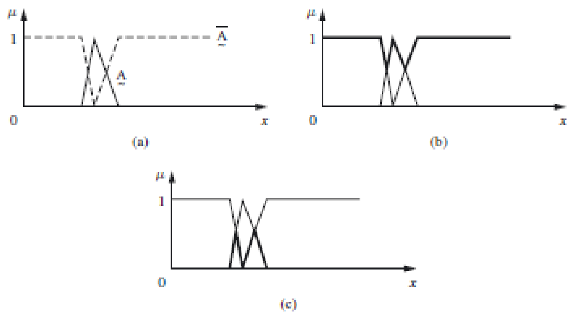


Figure 3: (a) Fuzzy set A and its complement (b) $A \cup \overline{A} \neq X$ and (c) $A \cap \overline{A} \neq \emptyset$

Example

Consider two fuzzy sets, namely

$$A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}, \; B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

Compute the following

- $A \cup B$
- $A \cap B$
- $A|B$
- $\overline{A \cup B}$

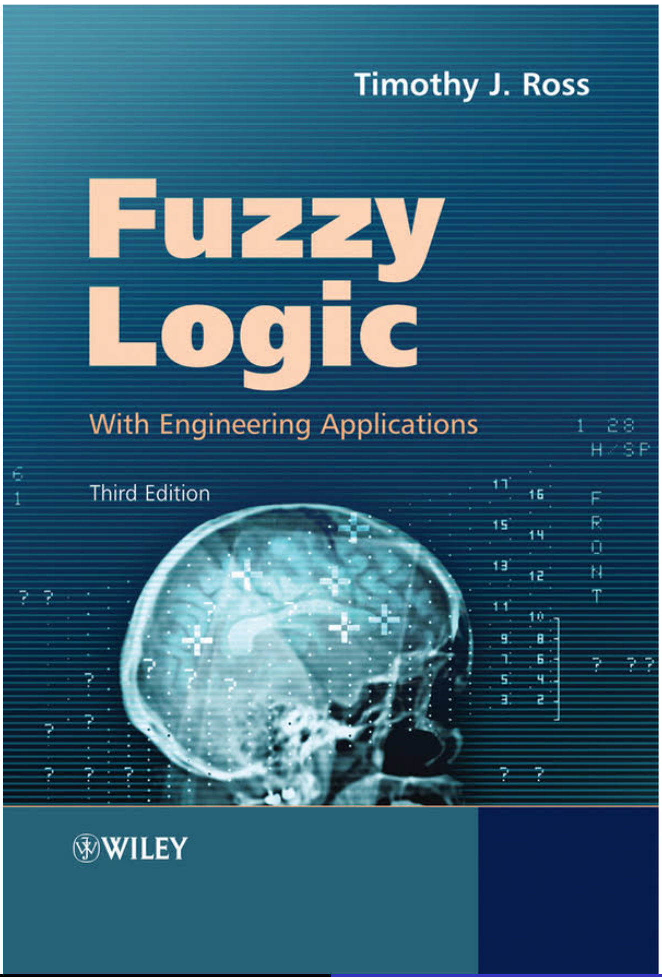

Timothy J. Ross
Fuzzy Logic
With Engineering Applications
Third Edition
WILEY

Figure 4