3 Expectation and Moment-Generating Function

LEARNING GOALS

After completing the chapter, you will be able to:

- Define expectation, mean and variance.
- Obtain mean and variance and other moments for the given data.
- Obtain expectations for discrete and continuous random variables.
- Understand moment-generating function and its use.
- Find out moment-generating function (discrete or continuous function).
- Determine mean and variance using moment-generating function.
- Learn about characteristic function and its use.

3.1 Introduction

Let us consider a situation where company A pays Rs. 10,000 to Mr. X and Rs. 15,000 to Mr. Y as salary. Now, if Mr. Z joins the company, he/she may expect a salary around (10,000 + 15,000)/2 = Rs. 12,500, an average of the salaries paid to Mr. X and Mr. Y.

Let us consider a more general situation in which we have drawn a sample $(x_1, x_2, \ldots, x_n)$ of size n, and we would like to get an average (mean or expected) value for this set of n observations. Normal practice is to add the n observations and divide the sum by n to get the required average value, say $\bar{x}$. That is,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Without loss of generality, let us assume that each of the n observations is equally likely to happen. Then there exists a probability $1/n$ for every x_i, i.e. $P(x_i) = 1/n$, $i = 1, 2, \ldots, n$. Now we can rewrite $\bar{x}$ as follows:

$$\begin{aligned}
\bar{x} &= \frac{x_1}{n} + \frac{x_2}{n} + \cdots + \frac{x_n}{n} \\
&= x_1\left(\frac{1}{n}\right) + x_2\left(\frac{1}{n}\right) + \cdots + x_n\left(\frac{1}{n}\right) \\
&= x_1 P(x_1) + x_2 P(x_2) + \cdots + x_n P(x_n) \\
&= \sum_{i=1}^{n} x_i P(x_i)
\end{aligned}$$

Therefore, if the events are not equally likely, we assume the respective $P(x_i)$ value for the event x_i and use the same formula for computing the expected value. In statistics, the average value computed in this way is also known as the *expected value.*

3.2 Definition and Properties of Expectation

If X is a random variable then the expectation of X, denoted as $E(X)$, is given as

$$E(X)=\sum_{x=-\infty}^{\infty} x\,P(X=x), \qquad \text{if } X \text{ is a discrete random variable}$$

$$E(X)=\int_{-\infty}^{\infty} x\,f(x)\,dx, \qquad \text{if } X \text{ is a continuous random variable}$$

$E(X)$ is often represented by the notation μ_X or simply μ

Properties of Expectation

(i) If X is a random variable (discrete or continuous), and $Y = aX + b$, where a and b are real numbers, then:

$$E(Y) = E(aX + b) = aE(X) + E(b) = aE(X) + b$$

It may be noted that $E(b) = b$ implies that the expected value of a constant is the same constant only.

(ii) If X is a random variable and $g(X)$ is a function of X, then

$$E[g(X)]=\sum_{x=-\infty}^{\infty} g(x)\,P(X=x), \qquad \text{if } X \text{ is a discrete random variable}$$

$$E[g(X)]=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx, \qquad \text{if } X \text{ is a continuous random variable}$$

(iii) *Variance*: If X is a random variable (discrete or continuous), then the variance of X, denoted by $V(X)$ or σ_X^2, or simply σ^2, is given as

$$V(X)=\sigma_X^2 = E[X - E(X)]^2$$
$$= E\{X^2 - 2XE(X) + [E(X)]^2\}$$
$$= E(X^2) - 2E[XE(X)] + [E(X)]^2$$

Since $E(X)$ is constant, we have

$$E[2XE(X)] = 2E(X)E(X) = 2[E(X)]^2$$

Therefore,

$$V(X)=\sigma_X^2 = E(X^2)-[E(X)]^2$$

Theoretically speaking, $V(X)$ is nothing but the average (mean or expectation) of the squared differences of each observation from its own mean value. If X is a random variable, and if we have drawn a sample of n observations that are equally likely and whose mean is $E(X)$, then we can define the variance of X as

$$V(X)=\frac{1}{n}\sum_{i=1}^{n}[x_i-E(X)]^2$$

(iv) *Standard deviation*: It may be noted that if the random variable X stands for length that is measured in metre, then the unit of variance becomes (meter)^2, because variance gives a squared average value. Now square root is taken over the variance to get a meaningful deviation (i.e. in same unit as meter) of each observation from its own mean. Hence, the standard deviation of the random variable X, denoted by σ_X, is the square root of the variance and is given by

$$SD(X)=\sigma_X=\sqrt{V(X)}$$

(v) If X is a random variable (discrete or continuous), and $Y = aX + b$, where a and b are real numbers, then:

$$V(Y)=a^2V(X)$$

We know that

$$\begin{aligned}V(Y)&=E[Y-E(Y)]^2\\&=E[aX+b-aE(X)-b]^2\\&=E\{a[X-E(X)]\}^2\\&=a^2E[X-E(X)]^2\\&=a^2V(X)\end{aligned}$$

3.3 Moments and Moment-Generating Function

It is well known that a distribution is said to be well defined or well characterized, if its mean (for central tendency), variance and standard deviation (for spread), skewness (for the symmetry), and kurtosis (for peakness) are known. The ***skewness*** is used to measure the degree of symmetry of the distribution that may be normal or left-skewed or right-skewed, depending upon the data under study. The details of skewness are shown in Figure 3.1. Similarly, the ***kurtosis*** measures the degree of peakness that may be normal (called as *mesokurtic*) or over-peaked (*leptokurtic*) or under-peaked (*platykurtic*) that are shown in Figure 3.2.

3.3.1 Raw and Central Moments

We have seen that mean is obtained as $E(X)$. For determining variance, we use the formula

$$V(X)=E[X-E(X)]^2=E(X^2)-[E(X)]^2$$

in which $E(X^2)$ is used. Similarly, skewness can be measured by

$$\frac{\{E[X-E(X)]^3\}^2}{\{E[X-E(X)]^2\}^3}$$

which requires $E(X^3)$. Finally, for kurtosis, we use the measure

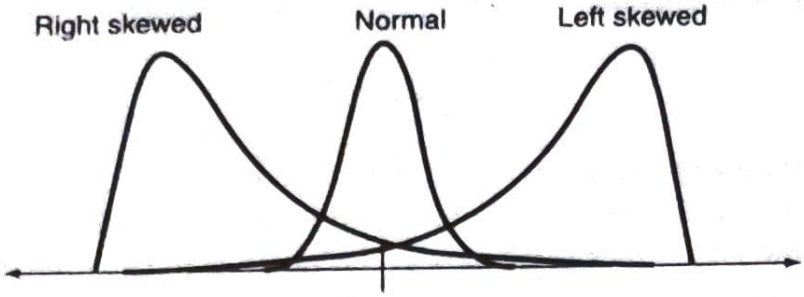

Figure 3.1 Skewness of a distribution.

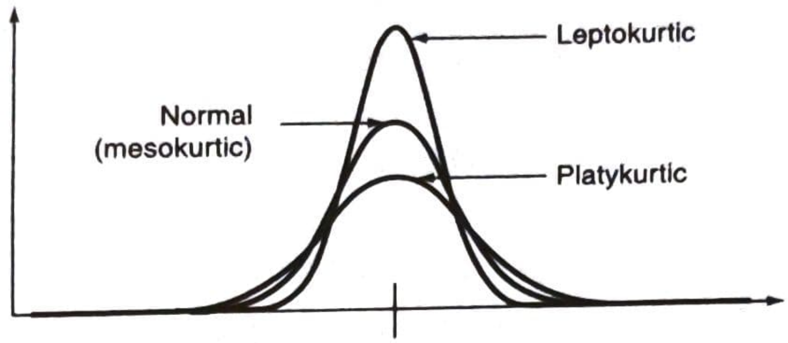

Figure 3.2 Peakness of a distribution.

$$\frac{E[X-E(X)]^4}{\{E[X-E(X)]^2\}^2}$$

that involves $E(X^4)$.

Now, $E(X)$, $E(X^2)$, …, $E(X^r)$ are called *moments* about origin or zero or simply *raw moments*. In general, we call $E(X^r)$ as the rth-order raw moment and for $r = 1, 2, \ldots$, we denote them as

$$E(X) = E(X-0) = \mu_1' \quad \text{(gives mean of } X\text{)}$$
$$E(X^2) = E(X-0)^2 = \mu_2'$$

and so on, and hence in general, we have

$$E(X^r) = E(X-0)^r = \mu_r', \text{ for } r = 1, 2, \ldots$$

If we consider $E[X - E(X)]$, $E[X - E(X)]^2$, …, $E[X - E(X)]^r$, these moments are called as *moments about mean* or simply *central moments*. In general, we call $E[X - E(X)]^r$ as the rth-order central moment and for $r = 1, 2, \ldots$, we denote them as

$$E[X-E(X)] = 0 = \mu_1$$
$$E[X-E(X)]^2 = \mu_2 \quad \text{(gives variance of } X\text{)}$$

and so on, and hence, in general, we have

$$E[X-E(X)]^r = \mu_r, \quad \text{for } r = 1, 2, \ldots$$

3.3.2 Relationship Between Central and Raw Moments

Now we can obtain mean, variance, skewness and kurtosis using the relationship between the central and raw moments:

(i) Mean $E(X) = \mu_1'$

(ii) Variance

$$
\begin{aligned}
V(X) &= E[X - E(X)]^2 = \mu_2 \\
&= E[X^2 - 2XE(X) + [E(X)]^2] \\
&= E(X^2) - 2E(X)E(X) + [E(X)]^2 \\
&= E(X^2) - [E(X)]^2 \\
&= \mu_2' - (\mu_1')^2
\end{aligned}
$$

Therefore, $\mu_2 = \mu_2' - (\mu_1')^2$.

(iii) Skewness $= \mu_3^2 / \mu_2^3$, where

$$
\begin{aligned}
\mu_3 &= E[X - E(X)]^3 \\
&= E(X^3) + 3[E(X)]^3 - 3E(X^2)\,E(X) - [E(X)]^3 \\
&= \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3
\end{aligned}
$$

(iv) Kurtosis $= \mu_4 / \mu_2^2$, where

$$
\begin{aligned}
\mu_4 &= E[X - E(X)]^4 \\
&= E(X^4) - 4E(X^3)E(X) + 6E(X^2)[E(X)]^2 - 3[E(X)]^4 \\
&= \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4
\end{aligned}
$$

3.3.3 Moments About an Arbitrary Value

Instead of considering $E[X - E(X)]^r$ to get the central moments, if we take an arbitrary constant, say a, then we have $\mu_r' = E(X - a)^r$, and are known as *moments about the arbitrary value a*. Accordingly, when $r = 1$, we have $\mu_1' = [E(X) - a]$, and it does not generate mean $= \mu_1'$ but we have $\mu_1 = \mu_1' + a$. All higher-order moments are not affected by the change of scale from mean $E(X)$ to the arbitrary value a.

3.3.4 Moment-Generating Function

It may be noted that if we know the raw moments $E(X)$, $E(X^2)$, $E(X^3)$ and $E(X^4)$, we can obtain all central moments. Apparently, we look for a function that can generate all such raw moments $E(X^r) = \mu_r'$ for all r. This function is known to be $E(e^{tX})$ and is called *moment-generating function* (abbreviated as MGF), and denoted as $M_X(t)$. If X is a random variable, then its MGF is given as

$$M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx} P(X = x), \quad \text{if } X \text{ is discrete}$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, \quad \text{if } X \text{ is continuous}$$

Now let us consider

$$M_X(t) = E(e^{tX}) = E\left(1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots + \frac{t^rX^r}{r!} + \cdots\right)$$

$$= E(1) + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

That is,

$$M_X(t) = 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

It can be verified that

$$\text{The coefficient of } \frac{t^r}{r!} = \mu_r' = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = E(X^r)$$

Hence, $M_X(t)$ generates all raw moments for $r = 1, 2, 3, \ldots$ and hence it is called MGF of the random variable X.

3.3.5 Properties of Moment-Generating Function

Property 1

If X is a random variable with MGF, $M_X(t)$, and if there exists a constant, say c, then the following is true:

$$M_{cX}(t) = M_X(ct)$$

where

$$\text{LHS} = M_{cX}(t) = E(e^{tcX}); \qquad \text{RHS} = M_X(ct) = E(e^{ctX}) = E(e^{tcX})$$

Property 2

If X_1 and X_2 are two independent random variables, then the MGF of the sum of these random variables is equal to the product of the MGF's of the random variables. Consider

$$M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}]$$

$$= E(e^{tX_1} e^{tX_2})$$

$$= E(e^{tX_1})E(e^{tX_2}) \quad (\text{since } X_1 \text{ and } X_2 \text{ are independent})$$

$$= M_{X_1}(t) M_{X_2}(t)$$

Note: Two random variables and independent random variables will be dealt in Chapter 7.

Property 3

The MGF of a random variable is unique. That is, If X and Y are two different random variables with MGFs $M_X(t)$ and $M_Y(t)$, then $M_X(t) = M_Y(t)$, if and only if X and Y have common distribution function.

3.3.6 Characteristic Function

Raw moments can also be generated by another function called *characteristic function*, denoted by $\Phi_X(t)$. If X is a random variable, then its characteristic function is defined as

$$\Phi_X(t)=\sum_{x=-\infty}^{\infty} e^{itX} P(X=x), \qquad \text{if } X \text{ is discrete}$$

$$\Phi_X(t)=\int_{-\infty}^{\infty} e^{itX} f(x)dx, \qquad \text{if } X \text{ is continuous}$$

where the imaginary number $i=\sqrt{-1}$.

Now the rth raw moments can be obtained as

$$\mu_r' = (-i)^r \left.\frac{d^r\Phi_X(t)}{dt^r}\right|_{t=0} = E(X^r)$$

Solved Examples

Example 3.1

The random variable X, representing the number of errors per 100 lines of software code, has the following probability distribution

x	2	3	4	5	6
$P(X=x)$	0.01	0.25	0.40	0.30	0.04

Find: (i) mean (expected value), (ii) variance and (iii) standard deviation of X.

Solution

(i) Since X is a discrete random variable, by definition, we know

$$\begin{aligned}\text{Mean} = E(X) &= \sum_{x=-\infty}^{\infty} x P(X=x)\\ &= \sum_{x=2}^{6} x P(X=x)\\ &= 2(0.01) + 3(0.25) + 4(0.40) + 5(0.30) + 6(0.04)\\ &= 0.02 + 0.75 + 1.60 + 1.50 + 0.24\\ &= 4.11\end{aligned}$$

(ii) For finding variance of X, by definition, we know

$$V(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

Consider $E(X^2)$. Now, let $g(X) = X^2$. Then, by definition we have

$$\begin{aligned} E(X^2) &= E[g(X)] \\ &= \sum_{x=-\infty}^{\infty} g(x)P(X=x) \\ &= \sum_{x=2}^{6} x^2 P(X=x) \\ &= 2^2(0.01) + 3^2(0.25) + 4^2(0.40) + 5^2(0.30) + 6^2(0.04) \\ &= 0.04 + 2.25 + 6.40 + 7.50 + 1.44 \\ &= 17.63 \end{aligned}$$

Therefore,

$$V(X) = 17.63 - (4.11)^2 = 17.63 - 16.8921 = 0.7379$$

(iii) For finding standard deviation, by definition, we know

$$SD(X) = \sqrt{V(X)} = \sqrt{0.7379} = 0.859$$

Example 3.2

The number of messages sent per hour over a computer network has the following probability distribution:

x	10	11	12	13	14	15
$P(X=x)$	0.08	0.15	0.30	0.20	0.20	0.07

Determine the mean and standard deviation of the number of messages sent per hour.

Solution

The mean of the number of messages sent per hour is obtained as follows:

$$\begin{aligned} \text{Mean} = E(X) &= \sum_{x=-\infty}^{\infty} x\,P(X=x) \\ &= \sum_{x=10}^{15} x\,P(X=x) \\ &= 10(0.08) + 11(0.15) + 12(0.30) + 13(0.20) + 14(0.20) + 15(0.07) \\ &= 12.5 \end{aligned}$$

Similarly, the variance of the number of messages sent per hour is obtained as

$$V(X) = E(X^2) - [E(X)]^2$$

Now $E(X^2)$ can be obtained as

$$\begin{aligned} E(X^2) &= E[g(X)] \\ &= \sum_{x=-\infty}^{\infty} g(x)P(X=x) \\ &= \sum_{x=10}^{15} x^2 P(X=x) \\ &= 10^2(0.08) + 11^2(0.15) + 12^2(0.30) + 13^2(0.20) + 14^2(0.20) + 15^2(0.07) \\ &= 158.1 \end{aligned}$$

Then

$$V(X) = 158.1 - (12.5)^2 = 1.85$$

Therefore, the standard deviation of the number of messages sent per hour is obtained as

$$\mathrm{SD}(X) = \sqrt{V(X)} = \sqrt{1.85} = 1.36$$

Example 3.3

Suppose that the random variable X stands for the number of cars that pass through a car wash between 4 P.M. and 5 P.M. on any day with the following probability distribution.

x	4	5	6	7	8	9
$P(X=x)$	1/12	1/12	1/4	1/4	1/6	1/6

Let $g(X) = 2X - 1$ be the amount of money in rupees paid to the attendant by the manager, then find the attendant's expected earnings for this particular time period. Also obtain the variance of the attendant's earnings.

Solution

We know that

$$\begin{aligned} E[g(X)] &= E(2X-1) \\ &= 2E(X) - 1 \quad \text{(by the property of expectation)} \end{aligned}$$

Now consider

$$\begin{aligned} E(X) &= \sum_{x=-\infty}^{\infty} x P(X=x) \\ &= \sum_{x=4}^{9} x P(X=x) \\ &= 4(1/12) + 5(1/12) + 6(1/4) + 7(1/4) + 8(1/6) + 9(1/6) \\ &= (4+5+18+21+16+18)/12 \\ &= 82/12 \end{aligned}$$

Therefore, the average earnings by the attendant is obtained as

$$E[g(X)] = 2E(X) - 1 = 2\left(\frac{82}{12}\right) - 1 = \frac{41}{3} - 1 = \text{Rs. } 12.67$$

Similarly, we know that

$$\begin{aligned} V[g(X)] &= V(2X-1) \\ &= 4V(X) \quad \text{(by the property of variance)} \end{aligned}$$

But $V(X) = E(X^2) - [E(X)]^2$. Consider,

$$
\begin{aligned}
E(X^2) &= E[g(X)] \\
&= \sum_{x=-\infty}^{\infty} g(x) P(X = x) \\
&= \sum_{x=4}^{9} x^2 P(X = x) \\
&= 4^2 (1/12) + 5^2 (1/12) + 6^2 (1/4) + 7^2 (1/4) + 8^2 (1/6) + 9^2 (1/6) \\
&= (16 + 25 + 108 + 147 + 128 + 162)/12 \\
&= 586/12
\end{aligned}
$$

Therefore,

$$V(X) = \frac{586}{12} - \left(\frac{82}{12}\right)^2 = \frac{308}{144}$$

Now, the variance of earnings by the attendant is

$$V[g(X)] = 4V(X) = 4\left(\frac{308}{144}\right) = \text{Rs. } 8.56$$

Example 3.4

Let X be a random variable with the following probability distribution

X	-3	6	9
$P(X = x)$	1/6	1/2	1/3

Then evaluate $E(2X + 1)^2$.

Solution

We know that

$$E(2X + 1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1$$

Now consider

$$E(X) = (-3)\frac{1}{6} + (6)\frac{1}{2} + (9)\frac{1}{3} = \frac{11}{12}; \quad E(X^2) = (-3)^2\frac{1}{6} + (6^2)\frac{1}{2} + (9^2)\frac{1}{3} = \frac{93}{2}$$

Therefore,

$$E(2X + 1)^2 = (4)\frac{93}{2} + (4)\frac{11}{2} + 1 = 209$$

Example 3.5

Let X be a random variable with probability density function

$$f(x)=\begin{cases}\dfrac{x^2}{3}, & -1<x<2\\ 0, & \text{elsewhere}\end{cases}$$

Find: (i) mean (ii) variance and (iii) standard deviation of X. Also obtain the expected value and variance of $g(X)=4X+3$.

Solution

(i) Since X is a continuous random variable, by definition, we know that

$$E(X)=\int_{-\infty}^{\infty} x f(x)dx=\int_{-1}^{2} x\frac{x^2}{3}dx=\left|\frac{x^4}{12}\right|_{-1}^{2}=\frac{16}{12}-\frac{1}{12}=\frac{15}{12}$$

(ii) For finding variance of X, by definition, we know that

$$V(X)=\sigma_x^2=E(X^2)-[E(X)]^2$$

Consider $E(X^2)$. Now, let $g(X)=X^2$. Then, by definition, we have

$$\begin{aligned}E(X^2)=E[g(X)]&=\int_{-\infty}^{\infty} g(x)f(x)dx\\&=\int_{-1}^{2}x^2\left(\frac{x^2}{3}\right)dx\\&=\left|\frac{x^5}{15}\right|_{-1}^{2}\\&=\frac{32}{15}-\frac{-1}{15}\\&=\frac{33}{15}\end{aligned}$$

Therefore,

$$V(X)=\frac{33}{15}-\left(\frac{15}{12}\right)^2=\frac{33}{15}-\frac{225}{144}=0.6375$$

(iii) For finding standard deviation, by definition, we know that

$$\text{SD}(X)=\sqrt{V(X)}=\sqrt{0.6375}=0.7984$$

Now we find the mean and variance of $g(X)=4X+3$ as

$$E[g(X)]=E(4X+3)=4E(X)+3=4\left(\frac{15}{12}\right)+3=\frac{15}{3}+3=8$$

and

$$
\begin{aligned}
V[g(X)] &= V(4X+3) \\
&= 16V(X) \qquad \text{(by the property of variance)} \\
&= 16\,(0.6375) \\
&= 10.2
\end{aligned}
$$

Example 3.6

With equal probability, the observations 5, 10, 8, 2 and 7 show the number of defective units found during five inspections in a laboratory. Find mean and variance.

Solution

Let the random variable X represents the number of defective units per inspection. Then

$$\text{Mean} = E(X) = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{5+10+8+2+7}{5} = \frac{32}{5} = 6.4$$

and

$$
\begin{aligned}
\text{Variance} &= E[X - E(X)]^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}[x_i - E(X)]^2 \\
&= \frac{1}{5}[(5-6.4)^2 + (10-6.4)^2 + (8-6.4)^2 + (2-6.4)^2 + (7-6.4)^2] \\
&= \frac{1}{5}(1.96 + 12.96 + 2.56 + 19.36 + 0.36) \\
&= \frac{1}{5}(37.2) \\
&= 7.44
\end{aligned}
$$

Example 3.7

With equal probability, the observations 5, 10, 8, 2 and 7 show the number of defective units found during five inspections in a laboratory. Find: (a) the first four central moments, (b) central moments from the moments about origin (raw moments) and (c) central moments from the moments about an arbitrary value 8.

Solution

(a) We know that the central moments can be obtained as:

$$\mu_r = E[(X - E(X)]^r, \qquad r = 1, 2, 3, 4$$

That is

$$\mu_1 = E[(X - E(X))]^1 = 0$$
$$\mu_2 = E[(X - E(X))]^2$$
$$\mu_3 = E[(X - E(X))]^3$$
$$\mu_4 = E[(X - E(X))]^4$$

In order to find the first four central moments, we first obtain the computations for $[X - E(X)]^r$ for $r = 1, 2, 3, 4$, as given in the following table.

	X	$X - 6.4$	$(X - 6.4)^2$	$(X - 6.4)^3$	$(X - 6.4)^4$
	5	−1.4	1.96	−2.744	3.8416
	10	3.6	12.96	46.656	167.9616
	8	1.6	2.56	4.096	6.5536
	2	−4.4	19.36	−85.184	374.8096
	7	0.6	0.36	0.216	0.1296
Total	32	0	37.2	−36.96	553.296

Now we have

$$\mu_1 = 0, \qquad \text{But mean} = E(X) = \frac{32}{5} = 6.4$$

$$\mu_2 = \frac{37.2}{5} = 7.44 \text{ (Variance)}$$

$$\mu_3 = \frac{-36.96}{5} = -7.392$$

$$\mu_4 = \frac{553.296}{5} = 110.66$$

(b) We know that given the raw moments, the first four central moments can be obtained as:

$$\mu_r = E[(X - E(X)]^r, \qquad r = 1, 2, 3, 4$$

That is,

$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

where

$$\mu_r' = \frac{d^r}{dr^r}[M_X(t)], \quad r = 1, 2, 3, 4$$

are the raw moments. Also we know that

$$\mu'_r = E(X^r), \quad r = 1, 2, 3, 4$$

Therefore, we obtain the raw moments, then substitute the same to get the central moments. Consider the following table.

	X	X^2	X^3	X^4
	5	25	125	625
	10	100	1000	10000
	8	64	512	4096
	2	4	8	16
	7	49	343	2401
Total	32	242	1988	17138

Then

$$\mu'_1 = E(X^1) = E(X) = \frac{\Sigma X}{n} = \frac{32}{5} = 6.4$$

$$\mu'_2 = E(X^2) = \frac{\Sigma X^2}{n} = \frac{242}{5} = 48.4$$

$$\mu'_3 = E(X^3) = \frac{\Sigma X^3}{n} = \frac{1988}{5} = 397.6$$

$$\mu'_4 = E(X^4) = \frac{\Sigma X^4}{n} = \frac{17138}{5} = 3427.6$$

Therefore, the first-order central moment is given by $\mu_1 = 0$, but mean $= \mu'_1 = 6.4$. The second-order central moment is given by

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = 48.4 - (6.4)^2 = 7.44 \text{ (variance)}$$

The third-order central moment is given by

$$\begin{aligned}\mu_3 &= \mu'_3 - 3\mu'_2\,\mu'_1 + 2(\mu'_1)^3 \\ &= 397.6 - 3(48.4)(6.4) + 2(6.4)^3 \\ &= 397.6 - 929.28 + 524.288 \\ &= -7.392\end{aligned}$$

The fourth-order central moment is given by

$$\begin{aligned}\mu_4 &= \mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu'_2\,(\mu'_1)^2 - 3(\mu'_1)^4 \\ &= 3427.6 - 4(397.6)(6.4) + 6(48.4)(6.4)^2 - 3(6.4)^4 \\ &= 3427.6 - 10178.56 + 11894.784 - 5033.1648 \\ &= 110.66\end{aligned}$$

(c) In order to find the first four central moments using an arbitrary value 8, we first obtain the computations for $(X - 8)^r$ for $r = 1, 2, 3, 4$, as given in the following table.

	X	$X-8$	$(X-8)^2$	$(X-8)^3$	$(X-8)^4$
	5	−3	9	−27	81
	10	2	4	8	16
	8	0	0	0	0
	2	−6	36	−216	1296
	7	−1	1	−1	1
Total	32	−8	50	−236	1394

Then

$$\mu_1' = E(X-8)^1 = \frac{\Sigma(X-8)}{n} = \frac{-8}{5} = -1.6$$

$$\mu_2' = E(X-8)^2 = \frac{\Sigma(X-8)^2}{n} = \frac{50}{5} = 10$$

$$\mu_3' = E(X-8)^3 = \frac{\Sigma(X-8)^3}{n} = \frac{-236}{5} = -47.2$$

$$\mu_4' = E(X-8)^4 = \frac{\Sigma(X-8)^4}{n} = \frac{11394}{5} = 278.8$$

The first-order central moment can be obtained as

$$\mu_1 = \mu_1' + 8 = -1.6 + 8 = 6.4 \text{ (Mean)}$$

By substituting the values of the moments about arbitrary value 8 in the expressions for central moments, we get $\mu_2 = 7.44$ (variance), $\mu_3 = -7.392$ and $\mu_4 = 110.66$. It may be noted that the values of central moments computed from raw moments and moments about arbitrary value are same and formulas are not changed except for μ_1.

Example 3.8

Find the moment-generating function of the random variable whose raw moments are given by $\mu_r' = (r+1)!2^r$.

Solution

We know that if X is a random variable, then its moment-generating function $M_X(t) = E(e^{tX})$ can be written as

$$\begin{aligned} M_X(t) &= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!}(r+1)!2^r \\ &= \sum_{r=0}^{\infty} (2t)^r (r+1) \\ &= 1 + 2(2t) + 3(2t)^2 + \cdots \\ &= (1-2t)^{-2} \end{aligned}$$

Example 3.9

A continuous random variable X has the probability density function

$$f(x)=\begin{cases} kx^2e^{-x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant. Find the rth moment about origin and hence find mean and variance of X.

Solution

Let us find the constant k using the property of probability density function. We know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{\infty} kx^2e^{-x}\,dx = 1$$

$$\Rightarrow k\int_0^{\infty} e^{-x}x^{3-1}\,dx = 1, \qquad \text{but } \int_0^{\infty} e^{-x}x^{n-1}\,dx = \Gamma(n)$$

$$\Rightarrow k\Gamma(3) = 1 \qquad [\text{as } \Gamma(n)=(n-1)!]$$

$$\Rightarrow k = \frac{1}{2}$$

Now the rth moment about origin can be obtained as

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f(x)\,dx$$

$$= \int_0^{\infty} x^r\left(\frac{1}{2}e^{-x}x^2\right)dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x}x^{(r+3)-1}\,dx$$

$$= \frac{1}{2}\Gamma(r+3)$$

Then

$$\frac{1}{2}\Gamma(r+3) = \frac{(r+2)!}{2}$$

Therefore, when $r=1$, we get mean of X as

$$E(X) = \mu_1' = \frac{(1+2)!}{2} = 3$$

When $r=2$, we have

$$\mu_2' = \frac{(2+2)!}{2} = 12$$

Therefore, variance of X is

$$V(X) = \mu_2 = \mu_2' - (\mu_1')^2 = 12 - 3^2 = 3$$

Example 3.10

If a random variable X has the MGF $M_x(t) = 3/(3 - t)$, obtain the standard deviation of X.

Solution

Given

$$M_x(t) = \frac{3}{3-t} = \frac{1}{1-t/3} = \left(1-\frac{t}{3}\right)^{-1}$$

But

$$\left(1-\frac{t}{3}\right)^{-1} = 1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\cdots$$

Now, we have

$$M_X(t) = 1+\frac{1}{3}\frac{t}{1!}+\frac{2}{9}\frac{t^2}{2!}+\frac{2}{9}\frac{t^3}{3!}+\cdots$$

Therefore, mean $= E(X) = \mu_1' = 1/3$, the coefficient of $t/1!$; $\mu_2' = 2/9$, the coefficient of $t^2/2!$; and hence

$$\text{Variance} = V(X) = \mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{9}-\left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

Therefore,

$$\text{Standard deviation} = \sqrt{V(X)} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Example 3.11

Find the characteristic function of the random variable X whose probability density function is given as

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and hence find the mean and variance of X.

Solution

We know that

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx = \int_0^{\infty} e^{itx} e^{-x}\,dx = \int_0^{\infty} e^{-(1-it)x}\,dx = \left.\frac{e^{-(1-it)x}}{-(1-it)}\right|_0^{\infty} = (1-it)^{-1}$$

Now the rth raw moments can be obtained as

$$E(X^r) = \mu_r' = (-i)^r \left.\frac{d^r \Phi_X(t)}{dt^r}\right|_{t=0}$$

Therefore, we have mean

$$E(X) = \mu_1' = (-i)^1 \left.\frac{d\Phi_X(t)}{dt}\right|_{t=0} = (-i)[-(1-it)^{-2}(-i)]_{t=0} = 1$$

Similarly,

$$E(X^2) = \mu_2' = (-i)^2 \left.\frac{d^2\Phi_X(t)}{dt^2}\right|_{t=0}$$

$$= (-i)^2 \left.\frac{d}{dt}[i(1-it)^{-2}]\right|_{t=0}$$

$$= (-i)^2[i(-2)(1-it)^{-3}(-i)]_{t=0}$$

$$= 2$$

Therefore,

$$\text{Variance} = V(X) = \mu_2 = \mu_2' - (\mu_1')^2 = 2 - 1^2 = 1$$

Summary

This chapter dealt with the concepts of expectation, mean, variance, stamdard deviation, higher-order moments, moment generating function, how to obtain mean and variance for a given data, how to obtain mean and variance for a given distribution using moment generating function and characteristic function. Formulas shown in Table 3.1 may be kept in mind for quick reference.

Table 3.1 Important formulas

Expectation	$E(X) = \sum_{x=-\infty}^{\infty} x\,P(X = x)$ if X is discrete random variable
	$E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$ if X is continuous random variable
	$E(X)$ is denoted as μ_X or simply μ to represent mean

(Continued)

Table 3.1 (Continued)

Properties of expectation	$E(aX+b) = aE(X) + E(b) = aE(X) + b$ $E[g(X)] = \sum_{x=-\infty}^{\infty} g(x)P(X = x)$ if X is discrete random variable $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$ if X is continuous random variable
Variance	$V(X) = \sigma_x^2 = E[X - E(X)]^2 = E(X^2) - [E(X)]^2$ $V(X) = \frac{1}{n}\sum_{i=1}^{n}[x_i - E(X)]^2$, for n observations $V(aX) = a^2 V(X)$
Standard deviation	$SD(X) = \sigma_X = \sqrt{V(X)}$
rth raw moment	$E(X^r) = E(X-0)^r = \mu_r'$
rth central moment	$E[X - E(X)]^r = \mu_r$
rth moment about arbitrary value a	$E(X-a)^r$
Mean	$E(X) = \mu_1'$
Variance	$\mu_2 = \mu_2' - (\mu_1')^2$
Skewness	$\frac{\mu_3^2}{\mu_2^3}$, $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$
Kurtosis	$\frac{\mu_4}{\mu_2^2}$, $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$
Moment-generating function (MGF)	$M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx}P(X = x)$, if X is discrete $M_X(t) = \int_{-\infty}^{\infty} e^{tx}f(x)dx$, if X is continuous
rth raw moment if MGF is known	$\mu_r' = \left.\frac{d^r M_X(t)}{dt^r}\right\|_{t=0} = E(X^r)$
Characteristic function	$\Phi_X(t) = \sum_{x=-\infty}^{\infty} e^{itx}P(X = x)$, if X is discrete $\Phi_X(t) = \int_{-\infty}^{\infty} e^{itx}f(x)dx$, if X is continuous
rth moment if characteristic function is given	$\mu_r' = (-i)^r \left.\frac{d^r \Phi_X(t)}{dt^r}\right\|_{t=0} = E(X^r)$

Problems

1. Let X be a random variable with probability distribution as

x	0	1	2	3
$P(X=x)$	1/3	1/2	0	1/6

Find mean and variance of X.

2. If the mean and variance of the random variable X are 3 and 2 respectively, then find the mean and variance of the random variable $Y = 4X + 9$.

3. A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs. 10 for each white ball and Rs. 5 for black ball. Find his expectation.

4. If X is RV of the outcome when a six-faced die is tossed, find the MGF and hence find mean and variance.

5. If the PDF of a random variable X is

$$f(x) = \begin{cases} \frac{1}{2}, & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find its MGF and hence mean and variance.

6. Let X be a uniform random variable with $f(x)$ as PDF in the interval $(0, 1)$. Determine the MGF of X, and hence find its mean and variance.

7. If a random variable X has the MGF $2/(3-t)$, obtain its standard deviation.

8. Find the MGF for the distribution whose PDF is given as

$$f(x) = \begin{cases} ke^{-x} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find also the standard deviation.

9. For the random variable X whose PDF is given as

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the MGF and hence find its mean and variance.

10. If a random variable has the probability function $P(x) = 1/2^x$; $x = 1, 2, \dots$ find the MGF, and hence mean and variance of X.

11. If X is a random variable with probability density function given as

$$f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

then obtain the MGF of X.

12. If a random variable has the MGF $M_x(t) = 1/(1-t)^3$, obtain the standard deviation of X.

13. Find the characteristic function of the random variable X whose PDF is given as

$$f(x) = \begin{cases} ke^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and hence find the mean and variance of X.

Multiple-Choice Questions

1. The average marks secured by 36 students were 52, but it was discovered that an item 64 was misread as 46. Then the correct mean of the marks is:
   (a) 52.5 (b) 55.2
   (c) 52.0 (d) 55.0

2. An average rainfall of a city from Monday to Saturday is 0.3 inch. Due to heavy rainfall on Sunday, the average rainfall for the week increased to 0.5 inch. The rainfall on Sunday is:
   (a) 1.8 (b) 3.5
   (c) 5.3 (d) 1.7

3. The mean of 5 observations 1, 2, 6, x_4 and x_5 is 4.4 and $x_4 - x_5 = 5$. The values of x_4 and x_5 are:
   (a) 9 and 4 (b) 8.5 and 3.5
   (c) 7.5 and 4.5 (d) None

4. If a random variable X assumes values -1, 0 and 1 with probabilities 0, 1/2 and 1/2, respectively, then the variance of $2X+3$ is:
   (a) 4 (b) 5
   (c) 1 (d) 2

5. If a random variable X assumes values -1, 0 and 1 with probabilities 1/3, 1/3 and 1/3, respectively, then the standard deviation of $5X+1$ is:
   (a) 1.08 (b) 2.08
   (c) 3.08 (d) 4.08

6. If the mean and variance of five observations are 6 and 9.2 respectively then the sum of squares of the numbers is:
   (a) 226 (b) 231.04
   (c) 256 (d) 236.04

7. If the mean of the numbers 2, 3, 7, 8 and 10 is 6, then the variance is:
   (a) 4 (b) 6.2
   (c) 9.2 (d) 8.0

8. The mean of 5 observations is 4.4, and their variance is 8.24. If three of the observations are 1, 2 and 6, then the other two observations are:
   (a) 4, 9 (b) 8, 5
   (c) 7, 6 (d) 3, 10

9. The average monthly wage of all workers in a factory is Rs. 444. If the average wages paid to male and female workers are Rs. 480 and Rs. 360 respectively, then the percentage of male and female workers employed by the factory are:
   (a) 30 and 70 (b) 70 and 30
   (c) 25 and 75 (d) 50 and 50

10. The mean of the values 0, 1, 2, ..., n with corresponding weights, ${}^nC_0, {}^nC_1, {}^nC_2, \ldots {}^nC_n$ respectively is:
   (a) $\dfrac{2^n}{n+1}$ (b) $\dfrac{2^{n+1}}{n(n+1)}$
   (c) $\dfrac{n+1}{2}$ (d) $\dfrac{n}{2}$

11. The standard deviation of first n natural numbers is:
   (a) $\sqrt{\dfrac{n^2-1}{12}}$ (b) $\sqrt{\dfrac{n^2-1}{6}}$
   (c) $\sqrt{\dfrac{n^2+1}{6}}$ (d) $\sqrt{\dfrac{n^2+1}{12}}$

12. The two numbers x and y, whose mean is 5 and variance is 4, are:
   (a) $x = 4$ and $y = 6$
   (b) $x = 3$ and $y = 7$

(c) $x = 1$ and $y = 9$
(d) $x = 3.5$ and $y = 6.5$

13. The standard deviation of 0, 1, 2, ..., 9 is σ, then the standard deviation of 100, 101, 102, ..., 109 is:

(a) σ (b) $\sigma + 100$
(c) $\sqrt{\sigma^2 + 9}$ (d) 100σ

14. If the random variable X has the probability density function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then the variance of X is:

(a) 1/2 (b) $\sqrt{2}$
(c) 2/9 (d) 1/18