Space-Time Clutter Covariance Matrix Computation and Interference Subspace Tracking

Steven T. Smith*
Lincoln Laboratory
Massachusetts Institute of Technology

Abstract

A fast method of computing ideal space-time clutter covariance matrices is described and the method's application to the study of clutter interference subspaces is considered. These areas are of principal importance in the study of space-time adaptive processing (STAP) for airborne pulse doppler radar arrays. Formulae for clutter covariance matrices assuming various stochastic models are first derived, then a fast algorithm is developed in the case of uncorrelated patch-to-patch clutter with arbitrary illumination and reflectance. Finally, the clutter interference subspace as a function of scan angle for the case of a rotating sensor array is studied.

1. Introduction

Ground clutter is problematic for airborne pulse doppler array radars because the ground spans a range of angles and doppler frequencies. Ground in front of the aircraft is approaching (positive doppler shift), ground on the side appears stationary (zero doppler shift), and ground in back is receding (negative doppler shift). This problem is further complicated if the antenna array is rotating in order to search the full 360° volume. It is apparent that mitigating clutter interference in this scenario requires a space-time filter that accounts for both angle and doppler dependent ground returns. This space-time filter can be constructed adaptively, hence the study of space-time adaptive processing (STAP).

This paper addresses aspects of the problem of adaptive clutter mitigation using STAP techniques. Several formulas are derived for the ideal space-time clutter covariance matrix of signals received by an airborne radar. A fast and efficient algorithm for computing covariance matrices for a wide range of scenarios, including arbitrary transmit antenna patterns and arbitrary, nonhomogeneous, clutter is developed. Finally, this algorithm is used to study the problem of tracking the clutter interference subspace of an rotating airborne antenna array.

*MIT Lincoln Laboratory, Lexington, MA 02173; stsmith@ll.mit.edu. This work was sponsored by ARPA under Air Force contract F19628-95-C-0002. Opinions, interpretations, conclusions, and recommendations are those of the author and are not necessarily endorsed by the United States Air Force.

2. Stochastic Model of the Clutter

The model for clutter interference is described in this section. The received signal due to ground reflections is modeled as a stochastic integral. This integral depends upon the parameters of an airborne radar [10, 12]. The basic idea of airborne radar is to transmit a train of coherent pulses and to receive the reflected energy on a phased array antenna. Detection of targets at specific angles and doppler frequencies is achieved by coherently integrating the measurements from all array elements and all pulses. This signal processing environment may be described by its salient parameters [12]

v_a = aircraft velocity
M = number of coherent pulses
N = number of antenna elements
T_r = pulse repetition interval
d = interelement spacing
λ_o = radar operating wavelength
ϕ_n = antenna normal azimuth w.r.t. aircraft broadside
ϕ = azimuth angle w.r.t. ϕ_n
θ = elevation angle w.r.t. horizon

From these parameters, there are two derived parameters which will appear frequently,

$$\beta = 2v_aT_r/d, \qquad \text{and} \qquad \xi = 2\pi d\lambda_o^{-1}\cos\theta.$$

The parameter β is the number of half-interelement spacings traversed by the aircraft per pulse, and the parameter ξ is the spatial frequency (in radians) across the array at endfire from a signal with elevation angle θ. For example, the case $\beta = 1$, $\xi = \pi$ (e.g., $d = \lambda_o/2$, $\theta = 0$), yields clutter returns that span the full range of unambiguous spatial and doppler frequencies. The idealized assumptions of a linear, calibrated array, and uniform, coherent pulses are held throughout the paper.

1058-6393/96 $5.00 © 1996 IEEE
Proceedings of ASILOMAR-29

Consider the clutter return from the ith pulse on the kth element at a fixed rangegate, $i = 0, 1, \ldots, M-1$, $k = 0, 1, \ldots, N-1$. Fix a polarization direction and an elevation angle θ of interest, so that the azimuth angle ϕ is the only free variable. The pulse propagates from the aircraft with a far field transmitted antenna pattern given by the complex valued function $A(\phi)$. The far field radiation intensity $I(\phi)$ is then given by $I(\phi) = |A(\phi)|^2$. The pulse strikes the ground at the rangegate of interest and propagates back to the aircraft, where the returns from all directions are integrated by the antenna elements. Due to fluctuations in the complex reflectivity of the ground, the complex return from an infinitesimal patch on the ground at azimuth ϕ is modeled by the random variable

$$\text{infinitesimal ground return} = A(\phi)e^{j\omega_{ik}(\phi)}\, d\gamma(\phi), \tag{1}$$

where $d\gamma(\phi)$ is the (unknown) infinitesimal angular reflectivity at ϕ, and

$$\omega_{ik}(\phi) = \xi\big(\beta \sin(\phi_n + \phi)i + \sin(\phi)k\big) \tag{2}$$

is the space-time frequency associated with the ith pulse and kth element. Note that $\gamma(\phi)$ is a second order stochastic process defined on the circle.

The only stochastic part of the model of Eqs. (1) and (2) is the azimuth dependent clutter reflectivity $\gamma(\phi)$. This is an appropriate model for systems with pulse widths shorter than the clutter's decorrelation time. However, this model excludes the possibility of internal clutter motion and terrain scattered jamming, which have time correlations as well as azimuth and elevation correlations. Furthermore, it is assumed that the doppler frequency associated with any patch on the ground is known precisely and does not change from pulse to pulse, excluding the possibilities of internal clutter motion or rapid scanning periods relative to the pulse repetition interval. The properties of $\gamma(\phi)$ may be determined experimentally; there is a wide body of literature on this subject [1, 6, 7, 11].

The output voltage of the kth antenna element from the ith pulse is given by the stochastic integral [3]

$$x_{ik} = \oint A(\phi)e^{j\omega_{ik}(\phi)}\, d\gamma(\phi), \tag{3}$$

which results from integrating Eq. (1) around the full circle. It is important to realize that x_{ik} is a random variable. For now, the only assumptions about the stochastic process $\gamma(\phi)$ are that

$$E[d\gamma(\phi)] = 0,$$
$$E[\gamma(\phi)\gamma^*(\phi')] = \Gamma(\phi, \phi') \text{ exists.}$$

The first assumption implies that $E[x_{ik}] = 0$ and the second assumption assures the existence of second moments of x_{ik}.

2.1. Space-Time Clutter Covariance Matrix

We are now in a position to compute the space-time clutter covariance matrix $\mathbf{R}_c$ using the stochastic model for the clutter returns given in Eq. (3). This matrix is obtained from the covariance of the return at the ith pulse and kth element with the return from the jth pulse and lth element. That is,

$$(\mathbf{R}_c)_{ik;jl} = E[x_{ik}x_{jl}^*]. \tag{4}$$

As the symbols i and j are used to denote specific pulses, the notation $i^2 = j^2 = -1$ will henceforth be dropped and $\sqrt{-1}$ will be used explicitly. Applying Eq. (4) to Eq. (3),

$$(\mathbf{R}_c)_{ik;jl} = \oint\!\!\oint A(\phi)A^*(\phi')e^{\sqrt{-1}\left(\omega_{ik}(\phi)-\omega_{jl}(\phi')\right)}\, dd'\Gamma(\phi, \phi'). \tag{5}$$

Eq. (5) provides a model for a variety of clutter interference effects, such as angle-dependent clutter correlations; however, this model is too complicated for several special cases. If some simplifying assumptions about the underlying stochastic model $\gamma(\phi)$ are made, more tractable formulae for the space-time clutter covariance matrix results. If a more complicated clutter model is desired—for example, one that considers internal clutter motion—then an azimuth dependent random phase must be added to $\omega_{ik}(\phi)$ in Eq. (2), and expectations of the second order terms taken, yielding a more complicated integral formula in Eq. (5).

2.2. Uncorrelated Patch-to-Patch Clutter

If the stochastic process $\gamma(\phi)$ has orthogonal increments, then

$$E\big[\big(\gamma(\phi_1) - \gamma(\phi_2)\big)\big(\gamma(\phi_1') - \gamma(\phi_2')\big)^*\big] = 0$$

whenever the circular arcs defined by the inequalities $\phi_1 \le \phi \le \phi_2$ and $\phi_1' \le \phi \le \phi_2'$ do not intersect (except at one or two points). Note that under the assumption $E[d\gamma(\phi)] = 0$, the weaker assumption of uncorrelated increments or the stronger assumption of independent increments would also suffice to describe this case. This amounts to an assumption that nonoverlapping clutter patches are uncorrelated, which is an appropriate model if the beamwidth is larger than the decorrelation angle of the clutter. Under this assumption, Eq. (5) becomes

$$(\mathbf{R}_c)_{ik;jl} = \oint |A(\phi)|^2 e^{\sqrt{-1}(z\sin\phi + w\cos\phi)}\, d\Gamma(\phi), \tag{6}$$

where, by overuse of notation,

$$d\Gamma(\phi) = E[|d\gamma(\phi)|^2] \tag{7}$$

denotes the infinitesimal ground reflectivity at azimuth ϕ, and

$$\begin{aligned} z &= \xi\big(\beta \cos\phi_n(i-j) + (k-l)\big), \\ w &= \xi\beta \sin\phi_n(i-j). \end{aligned}$$

It is useful to assume the existence of a function $\sigma^0(\phi) = d\Gamma/d\phi$ so that

$$d\Gamma(\phi) = \sigma^0(\phi)\, d\phi, \tag{8}$$

which may be inserted directly into the integral of Eq. (6). Note that $\sigma^0(\phi)$ is simply the angular reflectivity of the infinitesimal clutter patch between ϕ and $\phi + d\phi$. If the clutter is assumed to be wide sense stationary periodic (WSSP), a summation formula for $(\mathbf{R}_c)_{ij;kl}$ similar to Eq. (6) can be developed using the spectral decomposition theorem [3].

3. Fast Space-Time Clutter Covariance Matrix Computation

Now that formulas for space-time clutter covariance matrices have been developed for several stochastic models of the clutter, we turn to the problem of developing an efficient algorithm for computing the clutter covariance matrix. The most basic clutter model is the sandpaper earth model, $\sigma^0 \equiv$ constant. However, it is clear by examining Eqs. (6) and (8) that there is no additional computational burden in allowing the differential reflectivity σ^0 to vary with the azimuth, as it does not matter (computationally) if the function $|A(\phi)|^2$ or $\sigma^0(\phi)|A(\phi)|^2$ appears within the integral. Therefore, the uncorrelated clutter patches model will be used, and we shall consider space-time clutter covariance matrices of the form

$$(\mathbf{R}_c)_{ik;jl} = \oint c(\phi) e^{\sqrt{-1}(z\sin\phi + w\cos\phi)}\, d\phi, \tag{9}$$

where $c(\phi)\, d\phi$ is the reflected radiation intensity from the infinitesimal clutter patch between ϕ and $\phi + d\phi$:

$$c(\phi) = \sigma^0(\phi)|A(\phi)|^2. \tag{10}$$

Although this model does not address the general second order case (Eq. (5)), the methods used to analyze Eq. (9) may also be used for more general cases. Before evaluating the integral of Eq. (9), some notation to be used throughout this section is tabulated:

$$\begin{aligned} z &= \xi\big(\beta\cos\phi_n(i-j) + (k-l)\big), & a &= (z^2 + w^2)^{1/2}, \\ w &= \xi\beta\sin\phi_n(i-j), & \psi &= \arg(z, w). \end{aligned}$$

Note that the last two definitions imply that $z = a\cos\psi$ and $w = a\sin\psi$. It is important to keep in mind that the variables z, w, a, and ψ all depend upon the pulse and element indices i, j, k, and l. Furthermore, frequent use will be made of the well known Fourier series [13]

$$\cos(a\sin\phi) = J_0(a) + 2\sum_{n=1}^{\infty} J_{2n}(a)\cos 2n\phi, \tag{11}$$

$$\sin(a\sin\phi) = 2\sum_{n=0}^{\infty} J_{2n+1}(a)\sin(2n+1)\phi, \tag{12}$$

where $J_\nu(a)$ is a Bessel function of the first kind of order ν.

The definition of a and ψ allows Eq. (9) to be written as

$$(\mathbf{R}_c)_{ik;jl} = \oint c(\phi) e^{\sqrt{-1}\, a\sin(\psi+\phi)}\, d\phi \tag{13}$$

$$\begin{aligned} &= \oint c(\phi)\cos(a\sin(\psi+\phi))\, d\phi \\ &\quad + \sqrt{-1}\oint c(\phi)\sin(a\sin(\psi+\phi))\, d\phi. \end{aligned} \tag{14}$$

Employing the Fourier series of Eqs. (11) and (12), Eq. (14) becomes

$$\begin{aligned} (\mathbf{R}_c)_{ik;jl} &= J_0(a)\oint c(\phi)\, d\phi \\ &+ 2\sum_{n=1}^{\infty} J_{2n}(a)\oint c(\phi)\cos 2n(\psi+\phi)\, d\phi \\ &+ \sqrt{-1}\, 2\sum_{n=0}^{\infty} J_{2n+1}(a)\oint c(\phi)\sin(2n+1)(\psi+\phi)\, d\phi. \end{aligned} \tag{15}$$

The integrals that appear in Eq. (15) are seen to be the sine and cosine Fourier coefficients of the reflected clutter radiation intensity $c(\phi)$. Defining A_n and B_n by

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix} = \int_{-\pi}^{\pi} c(\phi) \begin{Bmatrix} \cos n\phi \\ \sin n\phi \end{Bmatrix} d\phi \quad n = 0, 1, \ldots,$$

$\mathbf{R}_c$ is given by the equations, expressed in polar form,

$$\begin{aligned} (\mathbf{R}_c)_{ik;jl} &= A_0 J_0(a) + 2\sum_{n=1}^{\infty} J_{2n}(a) C_{2n}\cos(2n\psi + \varphi_{2n}) \\ &+ \sqrt{-1}\, 2\sum_{n=0}^{\infty} J_{2n+1}(a) C_{2n+1}\sin\big((2n+1)\psi + \varphi_{2n+1}\big), \end{aligned} \tag{16}$$

where $A_n = C_n\cos\varphi_n$ and $B_n = C_n\sin\varphi_n$. Thus it is seen that the space-time clutter covariance matrix can be written compactly in a Fourier-Bessel series expansion. By Horn's large order expansion for Bessel functions of the first kind,

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ez}{2\nu}\right)^{\nu} \quad \text{as } \nu \to \infty \ (z \text{ fixed}), \tag{17}$$

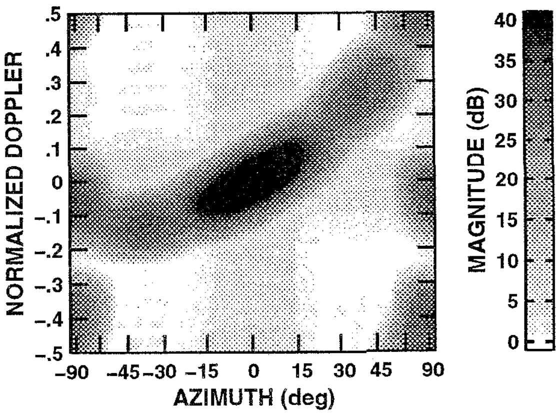

Figure 1. Clutter power spectral density of uniform transmit array and sandpaper earth model. The PSD is a function of azimuth and doppler frequency. Note that the transmit array's sidelobes are visible in the elliptical clutter ridge. The transmit array has 5 elements, its normal is set to $\phi_n = -45.7°$, the receive array has 14 elements, there are 16 coherent pulses, the clutter-to-noise ratio (per element per pulse) is 30 dB, the backlobe level is 60 dB below the frontlobe, $\beta = 0.93$, and $\xi = \pi \cdot 0.97$. Chebyshev windowing with 40 dB sidelobes is used for both spatial and doppler frequencies. The mainlobe clutter doppler ($\phi = 0°$), which is normally $\frac{1}{2}\beta \sin\phi_n$, has been shifted to zero doppler, emulating TACCAR compensation.

the infinite series in Eq. (16) are seen to converge quickly. Furthermore, the Fourier and Bessel coefficients may be computed quickly by using the fast Fourier transform (FFT) and order recursion properties of Bessel functions, respectively. Furthermore, the Hermitian Toeplitz-block Toeplitz structure of $\mathbf{R}_c$ can be exploited to greatly reduce the required number of computations. Applied to a scenario whose dimensionality MN is about 300, the resulting algorithm requires a few seconds on a computer workstation to compute $\mathbf{R}_c$, which is about 500–1000 times faster than the straightforward technique of using a Riemann sum with 1 degree spacings. An example scenario illustrating the clutter power spectral density (computed from $\mathbf{R}_c$) is shown in Figure 1.

4. Clutter Subspace Tracking

4.1. Clutter Rank

Brennan's rule asserts that for the case of antenna normal in the direction of broadside, i.e., $\phi_n = 0°$ or $180°$, the rank of the space-time clutter covariance matrix is given by the formula

$$\operatorname{rank} \mathbf{R}_c \doteq N + \beta(M-1). \tag{18}$$

This rule holds exactly for the case where β is an integer [12], and approximately for the noninteger case.

In the case of an airborne surveillance aircraft with a rotating antenna, the assumption that $\phi_n = 0°$ or $180°$ only holds twice per revolution. The clutter interference subspace changes as a function of the rotating array's antenna normal, and it is desirable to quantify these changes. To do this, the effective rank of $\mathbf{R}_c$ is computed as a function of scan angle.

The effective rank is defined to be the number of eigenvalues above a certain threshold, which in this case is chosen to be -10 dB relative to the noise floor. A uniformly weighted transmit array, with a backlobe level of -30 dB is used for the antenna pattern. The clutter-to-noise power per element per pulse is 30 dB. Note that the backlobe clutter eigenvalues exceed the rank threshold. Given the parameters $\beta = 1$, $\xi = \pi$, and $M = N = 18$, the effective rank of $\mathbf{R}_c$ is plotted as a function of scan angle in Figure 2. Brennan's rule, $\operatorname{rank} \mathbf{R}_c = 35$, is seen to hold at $\phi_n = 0°$ and $180°$, and the effective rank due to frontlobe and backlobe clutter is seen to double at other scan angles. It is concluded that if -10 dB is accepted as a reasonable cutoff for small eigenvalues, an adaptive algorithm with 70 degrees of freedom is sufficient to null the frontlobe and backlobe clutter for all scan angles in this scenario.

4.2. Clutter Subspace Distance

Not only does the effective rank of the covariance matrix change as a function of scan angle, but the interference subspace also changes. (In fact, the subspace varies continuously [8].) To quantify this change, the distance of the interference subspace from the interference at a fixed angle—aircraft broadside—is computed, as well as the differential or angle-to-angle subspace distance. If the clutter interference subspace changes smoothly enough as a function of time (scan angle), then a subspace tracking algorithm [2, 4, 9] may be used to compute the interference subspace in a computationally efficient manner.

Using the effective rank definition given above for the clutter subspace, the natural subspace distance

$$d_2(X,Y) = \left(\textstyle\sum_{i=1}^{p} \theta_i^2\right)^{1/2} \quad (\theta_i = \text{principal angles}),$$

between the clutter interference and the clutter interference at broadside is plotted in Figure 3. (N.b. this is not the subspace distance $d_\infty(X,Y) = \sin\theta_{\max}$ of Golub and Van Loan [5].) The parameters of the

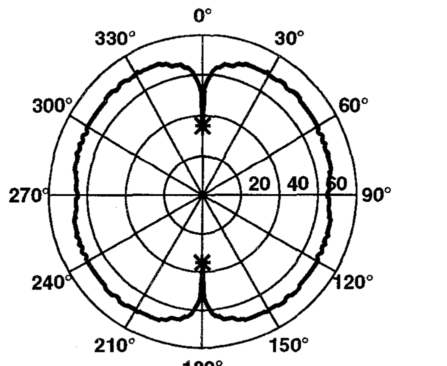

Figure 2. Clutter covariance matrix rank vs. scan angle. Brennan's rule, achieved at $\phi = 0°$ and $180°$, is denoted by two asterisks at which rank $\mathbf{R}_c = 35$. The maximum rank achieved equals 70.

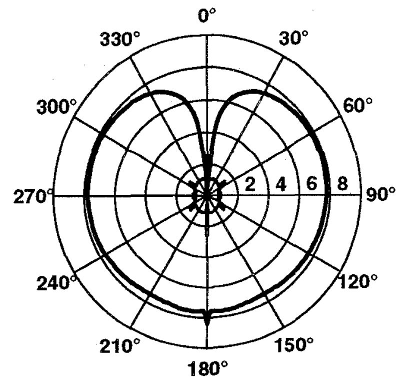

Figure 3. Natural subspace distance (rad) vs. scan angle. The subspace distance from broadside ($\phi = 0°$) is denoted by the thick curve (——); the differential subspace distance is denoted by the thin curve (——) below 2 rad.

scenario are identical to those of the rank computation above. Note that the subspace distance at broadside varies smoothly as a function of scan angle, and the differential subspace distance is small (usually about 1 rad, except for impulsive behavior near broadside), indicating that the interference subspace rotates smoothly as a function of time. Also, the natural subspace distance is relatively constant with $d_2 \doteq 8\,\mathrm{rad}$ over most scan directions. Thus the adaptive subspace tracking approach for clutter nulling may be appropriate computationally.

5. Conclusions

In this paper several formulas and a fast algorithm are derived for the space-time clutter covariance matrix of airborne radar scenario. The clutter returns are modeled as an azimuth dependent second order stochastic process. Several expressions for the clutter covariance matrix are derived for the case of a general second order process and the simplifying assumptions of uncorrelated patch-to-patch clutter, wide sense stationary periodic clutter, and uncorrelated WSSP (sandpaper earth) clutter. The computation of the clutter covariance is considered in the uncorrelated patch-to-patch case. It is seen that the elements of the covariance matrix are given by a Fourier-Bessel series. This results in a computationally efficient algorithm for computing the space-time covariance matrix for arbitrary far field transmit patterns and clutter reflectivity. Finally, this algorithm is used to study the problem of tracking the clutter interference subspace of a rotating airborne antenna array. It is seen that the rank of the clutter covariance matrix satisfies Brennan's rule at broadside, but doubles at other scan angles, and that the natural subspace distance between clutter subspaces changes smoothly, indicating the applicability of the adaptive subspace tracking approach for clutter nulling.

References

[1] Chan, H. C. Radar sea-clutter at low grazing angles, *Proc. IEE* **137** (2) : 102–112.

[2] Comon, P. and Golub, G. H. Tracking a few extreme singular values and vectors in signal processing. *Proc. IEEE* **78** (8) : 1327–1343, 1990.

[3] Doob, J. L. *Stochastic Processes.* New York: John Wiley & Sons, 1953.

[4] Edelman, A., Arias, T., and Smith, S. T. Conjugate gradient on the Stiefel and Grassmann manifolds, submitted to *SIAM J. Matrix Anal. Appl.*, 1995.

[5] Golub, G. H. and Van Loan, C. *Matrix Computations.* Baltimore, MD: Johns Hopkins University Press, 1983.

[6] Long, M. W. *Radar Reflectivity of Land and Sea.* Norwood, MA: Artech House, Inc., 1983.

[7] Nathanson, F. E. *Radar Design Principles*, 2d ed. New York: McGraw-Hill, Inc., 1991.

[8] Rellich, F. *Perturbation Theory of Eigenvalue Problems.* New York: Gordon and Breach, 1969.

[9] Smith, S. T. *Geometric Optimization Methods for Adaptive Filtering.* Cambridge University Press, to appear. See also Ph.D. diss., Harvard University, 1993.

[10] Staudaher, F. M. "Airborne MTI." In *Radar Handbook*, 2d ed., edited by M. Skolnik. New York: McGraw-Hill, 1990.

[11] Ulaby, F. T. and Dobson, M. C. *Handbook of Radar Scattering Statistics for Terrain.* Norwood, MA: Artech House, Inc., 1989.

[12] Ward, J. Space-Time adaptive processing for airborne radar, MIT Lincoln Laboratory Tech. Report 1015, 1994.

[13] Watson, G. N. *Theory of Bessel Functions*, 2d ed. Cambridge University Press, 1958.