1 Identidades

1.1 Series

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$$

$$g_k(n) = \sum_{i=1}^{n} i^k = \frac{1}{k+1}\left(n^{k+1} + \sum_{j=1}^{k}\binom{k+1}{j+1}(-1)^{j+1} g_{k-j}(n)\right)$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

1.2 Binomial Identities

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \qquad \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{k}\binom{n}{k}$$

$$\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h} \qquad \binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$$

$$\textstyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad \sum_{k=0}^{n} k^2\binom{n}{k} = (n+n^2)2^{n-2}$$

$$\textstyle\sum_{j=0}^{k}\binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k} \qquad \sum_{j=0}^{m}\binom{m}{j}^2 = \binom{2m}{m}$$

$$\textstyle\sum_{m=0}^{n}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1} \qquad \sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}$$

$$\textstyle\sum_{r=0}^{m}\binom{n+r}{r} = \binom{n+m+1}{m} \qquad \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k} = \mathrm{Fib}(n+1)$$

$$\textstyle(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \qquad (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$\textstyle 2\sum_{i=L}^{R}\binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^{R}\binom{n+1}{i}$$

2 Number Theory

2.1 Identities

$$\sum_{d|n}\varphi(d) = n$$

$$\sum_{\substack{i<n \\ \gcd(i,n)=1}} i = n\cdot\frac{\varphi(n)}{2}$$

$$|\{(x,y) : 1 \le x, y \le n,\, \gcd(x,y)=1\}| = \sum_{d=1}^{n}\mu(d)\left\lfloor\frac{n}{d}\right\rfloor^2$$

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\gcd(x,y) = \sum_{k=1}^{n} k\sum_{k|l}\left\lfloor\frac{n}{l}\right\rfloor^2\mu\left(\frac{l}{k}\right)$$

$$\sum_{x=1}^{n}\sum_{y=x}^{n}\gcd(x,y) = \sum_{g=1}^{n}\sum_{g|d}^{n} g\cdot\varphi\left(\frac{d}{g}\right)$$

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\mathrm{lcm}(x,y) = \sum_{d=1}^{n} d\,\mu(d)\sum_{d|l}^{n} l\binom{\lfloor\frac{n}{l}\rfloor+1}{2}^2$$

$$\sum_{x=1}^{n}\sum_{y=x+1}^{n}\mathrm{lcm}(x,y) = \sum_{g=1}^{n}\sum_{g|d}^{n} d\cdot\varphi\left(\frac{d}{g}\right)\cdot\frac{d}{g}\cdot\frac{1}{2}$$

$$\sum_{x\in A}\sum_{y\in A}\gcd(x,y) = \sum_{t=1}^{n}\left(\sum_{l|t}\frac{t}{l}\mu(l)\right)\left(\sum_{t|a}\mathrm{freq}[a]\right)^2$$

$$\sum_{x\in A}\sum_{y\in A}\mathrm{lcm}(x,y) = \sum_{t=1}^{n}\left(\sum_{l|t}\frac{l}{t}\mu(l)\right)\left(\sum_{a\in A,\, t|a} a\right)^2$$

2.2 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

2.3 Prime counting function - $\pi(x)$

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

2.4 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033
$10^{18}-11$ $10^{18}+3$ $2305843009213693951 = 2^{61}-1$

2.5 Number of Divisors

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

n	367567200	6983776800	13967553600	321253732800
$d(n)$	1152	2304	2688	5376

$18401055938125660800 \approx 2e18$ is highly composite with 184320 divisors.
For numbers up to 10^{88}, $d(n) < 3.6\sqrt[3]{n}$.

2.6 Lucas's Theorem

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod p$$

For p prime. n_i and m_i are the coefficients of the representations of n and m in base p. In particular, $\binom{n}{m}$ is odd if and only if n is a submask of m.

2.7 Fermat's Theorems

Let p be a prime number and a an integer, then:

$$a^p \equiv a \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a+b)^p \equiv a^p + b^p \pmod p$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod p$$

2.8 Taking modulo at the exponent

If a and m are coprime, then

$$a^n \equiv a^{n \bmod \varphi(m)} \pmod m$$

Generally, if $n \geq \log_2 m$, then

$$a^n \equiv a^{\varphi(m)+[n \bmod \varphi(m)]} \pmod m$$

2.9 Mobius invertion

If $g(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} g(d)\mu(\frac{n}{d})$.
A more useful definition is: $\sum_{d|n} \mu(d) = [n = 1]$
Example, sum of LCM:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\text{lcm}(i,j) &= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}[\gcd(i,j)=k]\frac{ij}{k} \\
&= \sum_{k=1}^{n}\sum_{a=1}^{\lfloor\frac{n}{k}\rfloor}\sum_{b=1}^{\lfloor\frac{n}{k}\rfloor}[\gcd(a,b)=1]abk \\
&= \sum_{k=1}^{n} k \sum_{a=1}^{\lfloor\frac{n}{k}\rfloor} a \sum_{b=1}^{\lfloor\frac{n}{k}\rfloor} b \sum_{d=1}^{\lfloor\frac{n}{k}\rfloor}[d|a]\,[d|b]\,\mu(d) \\
&= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor\frac{n}{k}\rfloor}\mu(d)\left(\sum_{a=1}^{\lfloor\frac{n}{k}\rfloor}[d|a]\,a\right)\left(\sum_{b=1}^{\lfloor\frac{n}{k}\rfloor}[d|b]\,b\right) \\
&= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor\frac{n}{k}\rfloor}\mu(d)\left(\sum_{p=1}^{\lfloor\frac{n}{kd}\rfloor} p\right)\left(\sum_{q=1}^{\lfloor\frac{n}{kd}\rfloor} q\right)
\end{aligned}$$

2.10 Chicken McNugget Theorem

Given two **coprime** numbers n, m, the largest number that cannot be written as a linear combination of them is $nm - n - m$.

- There are $\frac{(n-1)(m-1)}{2}$ non-negative integers that cannot be written as a linear combination of n and m;

- For each pair $(k, nm-n-m-k)$, for $k \geq 0$, exactly one can be written.

3 Geometry

3.1 Pythagorean Triples

For all natural a, b, c satisfying $a^2 + b^2 = c^2$ there exist $m, n \in \mathbb{N}$ and $m > n$ such that (reverse is also true):

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

3.2 Heron's Formula

The area of a triangle can be written as $A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$, where a, b, c are the lengths of its sides and $s = \frac{a+b+c}{2}$.
This can be generalized to compute the area A of a quadrilateral with sides a, b, c, d, with $s = \frac{a+b+c+d}{2}$ and α, γ any two opposite angles:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\left(\cos^2\left(\frac{\alpha+\gamma}{2}\right)\right)}$$

3.3 Pick's Theorem

The area of a simple polygon whose vertices have integer coordinates is:

$$A = I + \frac{B}{2} - 1$$

where I is the number of interior integer points, and B is the number of integer points in the border of the polygon.

3.4 Colinear Points

Three points are colinear on $\mathbb{R}^2$ iff:

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 0$$

The absolute value of this determinant is twice the area of the triangle ABC.

3.5 Coplanar Points

Four points are coplanar in $\mathbb{R}^3$ iff:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0$$

3.6 Trigonometry

3.6.1 Angle Sum

$$\sin(a \pm b) = \sin a \cos b \pm \cos\ \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

3.6.2 Sum-to-Product Transformation

$$\sin a \pm \sin b = 2 \sin \frac{a \pm b}{2} \cos \frac{a \mp b}{2}$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a \cos b}$$

3.7 Centroid of a polygon

The coordites of the centroid of a non-self-intersecting closed polygon is:

$$\frac{1}{3A}\left(\sum_{i=0}^{n-1}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \sum_{i=0}^{n-1}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)\right),$$

where A is twice the signed area of the polygon.

3.8 Two Ears Theorem

Every simple polygon with more than 3 vertices has at least two non-overlapping ears (a ear is a vertex whose diagonal induced by its neighbors which lies strictly inside the polygon). Equivalently, every simple polygon can be triangulated. Example of a simple polygon with exactly two ears:

img/two_ears.png

4 Other

4.0.1 String Matching with Wildcards

Consider a text T and a pattern P. P and T may have wildcards that will match any character. The problem is to get the positions where P occur in T. If we define the value of the characters such that the wildcard is zero and the other characters are positive, there is a matching at position i iff $\sum_{j=0}^{|P|-1} P[j]T[i+j](P[j]-T[i+j])^2 = 0$. Then, one can evaluate each term of

$$\sum_{j=0}^{|P|-1} (P[j]^3T[i+j] - 2P[j]^2T[i+j]^2 + P[j]T[i+j]^3)$$

using three convolutions.