1 Identidades

1.1 Series

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$$

$$g_k(n) = \sum_{i=1}^{n} i^k = \frac{1}{k+1}\left(n^{k+1} + \sum_{j=1}^{k}\binom{k+1}{j+1}(-1)^{j+1}g_{k-j}(n)\right)$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2}-(n+1)c^{n+1}+c}{(c-1)^2}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

1.2 Binomial Identities

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \qquad \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{k}\binom{n}{k}$$

$$\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h} \qquad \binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$$

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad \sum_{k=0}^{n} k^2\binom{n}{k} = (n+n^2)2^{n-2}$$

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k} \qquad \sum_{j=0}^{m}\binom{m}{j}^2 = \binom{2m}{m}$$

$$\sum_{m=0}^{n}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1} \qquad \sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}$$

$$\sum_{r=0}^{m}\binom{n+r}{r} = \binom{n+m+1}{m} \qquad \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k} = \mathrm{Fib}(n+1)$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \qquad (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$2\sum_{i=L}^{R}\binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^{R}\binom{n+1}{i}$$

2 Number Theory

2.1 Identities

$$\sum_{d|n}\varphi(d) = n$$

$$\sum_{\substack{i<n \\ \gcd(i,n)=1}} i = n \cdot \frac{\varphi(n)}{2}$$

$$|\{(x,y) : 1 \leq x, y \leq n,\ \gcd(x,y)=1\}| = \sum_{d=1}^{n}\mu(d)\left\lfloor\frac{n}{d}\right\rfloor^2$$

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\gcd(x,y) = \sum_{k=1}^{n} k \sum_{k|l}\left\lfloor\frac{n}{l}\right\rfloor^2 \mu\left(\frac{l}{k}\right)$$

$$\sum_{x=1}^{n}\sum_{y=x}^{n}\gcd(x,y) = \sum_{g=1}^{n}\sum_{g|d}^{n} g\cdot\varphi\left(\frac{d}{g}\right)$$

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\mathrm{lcm}(x,y) = \sum_{d=1}^{n} d\,\mu(d)\sum_{d|l}^{n} l\left(\frac{\lfloor\frac{n}{l}\rfloor+1}{2}\right)^2$$

$$\sum_{x=1}^{n}\sum_{y=x+1}^{n}\mathrm{lcm}(x,y) = \sum_{g=1}^{n}\sum_{g|d}^{n} d\cdot\varphi\left(\frac{d}{g}\right)\cdot\frac{d}{g}\cdot\frac{1}{2}$$

$$\sum_{x\in A}\sum_{y\in A}\gcd(x,y) = \sum_{t=1}^{n}\left(\sum_{l|t}\frac{t}{l}\mu(l)\right)\left(\sum_{t|a}\mathrm{freq}[a]\right)^2$$

$$\sum_{x\in A}\sum_{y\in A}\mathrm{lcm}(x,y) = \sum_{t=1}^{n}\left(\sum_{l|t}\frac{l}{t}\mu(l)\right)\left(\sum_{a\in A,\, t|a} a\right)^2$$

2.2 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

2.3 Prime counting function - $\pi(x)$

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

2.4 Some Primes

999999937	1000000007	1000000009	1000000021	1000000033
$10^{18} - 11$	$10^{18} + 3$	$2305843009213693951 = 2^{61} - 1$		

2.5 Number of Divisors

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

n	367567200	6983776800	13967553600	321253732800
$d(n)$	1152	2304	2688	5376

$18401055938125660800 \approx 2e18$ is highly composite with 184320 divisors.
For numbers up to 10^{88}, $d(n) < 3.6\sqrt[3]{n}$.

2.6 Lucas's Theorem

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod p$$

For p prime. n_i and m_i are the coefficients of the representations of n and m in base p. In particular, $\binom{n}{m}$ is odd if and only if n is a submask of m.

2.7 Fermat's Theorems

Let p be a prime number and a an integer, then:

$$a^p \equiv a \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a+b)^p \equiv a^p + b^p \pmod p$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod p$$

2.8 Taking modulo at the exponent

If a and m are coprime, then

$$a^n \equiv a^{n \bmod \varphi(m)} \pmod m$$

Generally, if $n \geq \log_2 m$, then

$$a^n \equiv a^{\varphi(m) + [n \bmod \varphi(m)]} \pmod m$$

2.9 Mobius invertion

If $g(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} g(d)\mu(\frac{n}{d})$.
A more useful definition is: $\sum_{d|n} \mu(d) = [n = 1]$
Example, sum of LCM:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{lcm}(i,j) &= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}[\gcd(i,j) = k]\frac{ij}{k} \\
&= \sum_{k=1}^{n}\sum_{a=1}^{\lfloor \frac{n}{k} \rfloor}\sum_{b=1}^{\lfloor \frac{n}{k} \rfloor}[\gcd(a,b) = 1]abk \\
&= \sum_{k=1}^{n} k \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} a \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} b \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} [d|a]\,[d|b]\,\mu(d) \\
&= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \left(\sum_{a=1}^{\lfloor \frac{n}{k} \rfloor}[d|a]\,a\right)\left(\sum_{b=1}^{\lfloor \frac{n}{k} \rfloor}[d|b]\,b\right) \\
&= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \left(\sum_{p=1}^{\lfloor \frac{n}{kd} \rfloor} p\right)\left(\sum_{q=1}^{\lfloor \frac{n}{kd} \rfloor} q\right)
\end{aligned}$$

2.10 Chicken McNugget Theorem

Given two **coprime** numbers n, m, the largest number that cannot be written as a linear combination of them is $nm - n - m$.

- There are $\frac{(n-1)(m-1)}{2}$ non-negative integers that cannot be written as a linear combination of n and m;

- For each pair $(k, nm - n - m - k)$, for $k \geq 0$, exactly one can be written.

2.11 Harmonic Lemma

This technique computes sums of the form $\sum_{i=1}^{n} f\left(\lfloor \frac{n}{i} \rfloor\right)$ in $O(\sqrt{n})$.
The value of $\lfloor \frac{n}{i} \rfloor$ is constant over blocks. If we are at index l, the value $v = \lfloor \frac{n}{l} \rfloor$ remains constant up to index $r = \lfloor \frac{n}{v} \rfloor$. We can iterate through these blocks instead of individual indices.

```
long long sum = 0;
for (int l = 1, r; l <= n; l = r + 1) {
    int val = n / l;
    r = n / val;
    sum += (long long)(r - l + 1) * f(val);
}
```

Generalization: For sums like $\sum_{i=1}^{n} g(i) \cdot f(\lfloor \frac{n}{i} \rfloor)$, the contribution of a block $[l, r]$ is:

$$(G(r) - G(l-1)) \cdot f\left(\left\lfloor \frac{n}{l} \right\rfloor\right)$$

where $G(k)$ is the prefix sum $\sum_{i=1}^{k} g(i)$, which must be efficiently computable.

3 Geometry

3.1 Pythagorean Triples

For all natural a, b, c satisfying $a^2 + b^2 = c^2$ there exist $m, n \in \mathbb{N}$ and $m > n$ such that (reverse is also true):

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

3.2 Heron's Formula

The area of a triangle can be written as $A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$, where a, b, c are the lengths of its sides and $s = \frac{a+b+c}{2}$.
This can be generalized to compute the area A of a quadrilateral with sides a, b, c, d, with $s = \frac{a+b+c+d}{2}$ and α, γ any two opposite angles:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\left(\cos^2\left(\frac{\alpha+\gamma}{2}\right)\right)}$$

3.3 Pick's Theorem

The area of a simple polygon whose vertices have integer coordinates is:

$$A = I + \frac{B}{2} - 1$$

where I is the number of interior integer points, and B is the number of integer points in the border of the polygon.

3.4 Colinear Points

Three points are colinear on $\mathbb{R}^2$ iff:

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 0$$

The absolute value of this determinant is twice the area of the triangle ABC.

3.5 Coplanar Points

Four points are coplanar in $\mathbb{R}^3$ iff:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0$$

3.6 Trigonometry

3.6.1 Angle Sum

$$\sin(a \pm b) = \sin a \cos b \pm \cos\ \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

3.6.2 Sum-to-Product Transformation

$$\sin a \pm \sin b = 2 \sin \frac{a \pm b}{2} \cos \frac{a \mp b}{2}$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a \cos b}$$

3.7 Centroid of a polygon

The coordites of the centroid of a non-self-intersecting closed polygon is:

$$\frac{1}{3A}\left(\sum_{i=0}^{n-1}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \sum_{i=0}^{n-1}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)\right),$$

where A is twice the signed area of the polygon.

4 Other

4.0.1 String Matching with Wildcards

Consider a text T and a pattern P. P and T may have wildcards that will match any character. The problem is to get the positions where P occur in T.
If we define the value of the characters such that the wildcard is zero and the other characters are positive, there is a matching at position i iff $\sum_{j=0}^{|P|-1} P[j]T[i+j](P[j] - T[i+j])^2 = 0$. Then, one can evaluate each term of

$$\sum_{j=0}^{|P|-1} (P[j]^3 T[i+j] - 2P[j]^2 T[i+j]^2 + P[j]T[i+j]^3)$$

using three convolutions.

"Eu vim fazer um anúncio, Shadow o ouriço é um filha da puta do caralho,
ele mijou na porra da minha esposa.

Isso mesmo, ele pegou a porra do pinto espinhoso dele
e mijou na porra da minha esposa
e disse que o pau dele era dessa tamanho
e eu disse "Credo, que nojo"
então estou fazendo um exposed no meu twitter.com.

Shadow o ouriço, você tem um pau pequeno,
que é do tamanho desta noz só que muito menor,
e adivinha, olha o tamanho do meu pirocão.

Isso mesmo bebê, pontas altas, sem pelos, sem espinhos,
olha só, parecem 2 bolas e 1 torpedo.

Ele fodeu a minha esposa, então adivinhem,
eu vou foder a terra.
Isso mesmo, isso que você ganha, meu super laser de mijo.

Exceto que eu não vou mijar na terra,
eu vou mijar na lua.

Você gostou disto Lula, eu mijei na lua, faz o L agora.
Vocês tem 23 horas antes que os meus perdigotos de mijo
atinjam a Terra.

Agora saiam da porra da minha frente,
antes que eu mije em vocês também."

~Robotnic