

1 Identidades

1.1 Series

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^n i \right)^2$$

$$g_k(n) = \sum_{i=1}^n i^k = \frac{1}{k+1} \left(n^{k+1} + \sum_{j=1}^k \binom{k+1}{j+1} (-1)^{j+1} g_{k-j}(n) \right)$$

$$\sum_{i=0}^n ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

1.2 Binomial Identities

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{k} \binom{n}{k}$$

$$\binom{n}{h} \binom{n-h}{k} = \binom{n}{k} \binom{n-k}{h}$$

$$\binom{n}{k} = \frac{n+1-k}{k} \binom{n}{k-1}$$

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$$

$$\sum_{k=0}^n k^2 \binom{n}{k} = (n+n^2)2^{n-2}$$

$$\sum_{j=0}^k \binom{m}{j} \binom{n-m}{k-j} = \binom{n}{k}$$

$$\sum_{j=0}^m \binom{m}{j}^2 = \binom{2m}{m}$$

$$\sum_{m=0}^n \binom{m}{j} \binom{n-m}{k-j} = \binom{n+1}{k+1}$$

$$\sum_{m=k}^n \binom{m}{k} = \binom{n+1}{k+1}$$

$$\sum_{r=0}^m \binom{n+r}{r} = \binom{n+m+1}{m}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = \text{Fib}(n+1)$$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

$$2 \sum_{i=L}^R \binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^R \binom{n+1}{i}$$

2 Number Theory

2.1 Identities

$$\sum_{d|n} \varphi(d) = n$$

$$\sum_{\substack{i < n \\ \gcd(i,n)=1}} i = n \cdot \frac{\varphi(n)}{2}$$

$$|\{(x,y) : 1 \leq x,y \leq n, \gcd(x,y) = 1\}| = \sum_{d=1}^n \mu(d) \left\lfloor \frac{n}{d} \right\rfloor^2$$

$$\sum_{x=1}^n \sum_{y=1}^n \gcd(x,y) = \sum_{k=1}^n k \sum_{k|l} \left\lfloor \frac{n}{l} \right\rfloor^2 \mu\left(\frac{l}{k}\right)$$

$$\sum_{x=1}^n \sum_{y=x}^n \gcd(x,y) = \sum_{g=1}^n \sum_{g|d} g \cdot \varphi\left(\frac{d}{g}\right)$$

$$\sum_{x=1}^n \sum_{y=1}^n \text{lcm}(x,y) = \sum_{d=1}^n d \mu(d) \sum_{d|l} l \left(\left\lfloor \frac{n}{l} \right\rfloor + 1 \right)^2$$

$$\sum_{x=1}^n \sum_{y=x+1}^n \text{lcm}(x,y) = \sum_{g=1}^n \sum_{g|d} d \cdot \varphi\left(\frac{d}{g}\right) \cdot \frac{d}{g} \cdot \frac{1}{2}$$

$$\sum_{x \in A} \sum_{y \in A} \gcd(x,y) = \sum_{t=1}^n \left(\sum_{l|t} t \mu(l) \right) \left(\sum_{t|a} \text{freq}[a] \right)^2$$

$$\sum_{x \in A} \sum_{y \in A} \text{lcm}(x,y) = \sum_{t=1}^n \left(\sum_{l|t} \frac{l}{t} \mu(l) \right) \left(\sum_{a \in A, t|a} a \right)^2$$

2.2 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.

For numbers until 10^{18} the largest gap is 1500.

2.3 Prime counting function - $\pi(x)$

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10 ²	10 ³	10 ⁴	10 ⁵	10 ⁶	10 ⁷	10 ⁸
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

2.4 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033
 $10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$

2.5 Number of Divisors

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

n	367567200	6983776800	13967553600	321253732800
$d(n)$	1152	2304	2688	5376

$18401055938125660800 \approx 2e18$ is highly composite with 184320 divisors.
 For numbers up to 10^{88} , $d(n) < 3.6\sqrt[3]{n}$.

2.6 Lucas's Theorem

$$\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod{p}$$

For p prime. n_i and m_i are the coefficients of the representations of n and m in base p . In particular, $\binom{n}{m}$ is odd if and only if n is a submask of m .

2.7 Fermat's Theorems

Let p be a prime number and a an integer, then:

$$a^p \equiv a \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a + b)^p \equiv a^p + b^p \pmod{p}$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2} :

$$a^{-1} \equiv a^{p-2} \pmod{p}$$

2.8 Taking modulo at the exponent

If a and m are coprime, then

$$a^n \equiv a^{n \bmod \varphi(m)} \pmod{m}$$

Generally, if $n \geq \log_2 m$, then

$$a^n \equiv a^{\varphi(m) + [n \bmod \varphi(m)]} \pmod{m}$$

2.9 Mobius inversion

If $g(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} g(d) \mu\left(\frac{n}{d}\right)$.

A more useful definition is: $\sum_{d|n} \mu(d) = [n = 1]$

Example, sum of LCM:

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \text{lcm}(i, j) &= \sum_{k=1}^n \sum_{i=1}^n \sum_{j=1}^n [\gcd(i, j) = k] \frac{ij}{k} \\ &= \sum_{k=1}^n \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} [\gcd(a, b) = 1] abk \\ &= \sum_{k=1}^n k \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} a \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} b \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} [d|a] [d|b] \mu(d) \\ &= \sum_{k=1}^n k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \left(\sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} [d|a] a \right) \left(\sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} [d|b] b \right) \\ &= \sum_{k=1}^n k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \left(\sum_{p=1}^{\lfloor \frac{n}{kd} \rfloor} p \right) \left(\sum_{q=1}^{\lfloor \frac{n}{kd} \rfloor} q \right) \end{aligned}$$

2.10 Chicken McNugget Theorem

Given two **coprime** numbers n, m , the largest number that cannot be written as a linear combination of them is $nm - n - m$.

- There are $\frac{(n-1)(m-1)}{2}$ non-negative integers that cannot be written as a linear combination of n and m ;

- For each pair $(k, nm - n - m - k)$, for $k \geq 0$, exactly one can be written.

2.11 Harmonic Lemma

This technique computes sums of the form $\sum_{i=1}^n f\left(\left\lfloor \frac{n}{i} \right\rfloor\right)$ in $O(\sqrt{n})$.

The value of $\left\lfloor \frac{n}{i} \right\rfloor$ is constant over blocks. If we are at index l , the value $v = \left\lfloor \frac{n}{l} \right\rfloor$ remains constant up to index $r = \left\lfloor \frac{n}{v} \right\rfloor$. We can iterate through these blocks instead of individual indices.

```
long long sum = 0;
for (int l = 1, r; l <= n; l = r + 1) {
    int val = n / l;
    r = n / val;
    sum += (long long)(r - l + 1) * f(val);
}
```

Generalization: For sums like $\sum_{i=1}^n g(i) \cdot f\left(\left\lfloor \frac{n}{i} \right\rfloor\right)$, the contribution of a block $[l, r]$ is:

$$(G(r) - G(l - 1)) \cdot f\left(\left\lfloor \frac{n}{l} \right\rfloor\right)$$

where $G(k)$ is the prefix sum $\sum_{i=1}^k g(i)$, which must be efficiently computable.

3 Combinatória

3.1 Stars and Bars Bounded

De quantas formas podemos definir uma sequência x tal que:

$$\sum_{i=0}^n x_i = M$$

$$l_i \leq x_i \leq r_i$$

3.1.1 Existência de solução

A existência de solução é garantida se e somente se:

$$\sum_{i=0}^n l_i \leq M \leq \sum_{i=0}^n r_i$$

3.2 Solução com DP em $O(NM^2)$

$$DP(N, M) = \sum_{i=l_n}^{r_n} DP(N - 1, M - i)$$

$$DP(0, M) = \begin{cases} 0, & M \neq 0 \\ 1, & M = 0 \end{cases}$$

3.3 Caso Particular $l_i = 0, r_i = M$

Stars and Bars com N-1 Barras

$$Resposta = \binom{M + N - 1}{M}$$

3.4 Redução do problema para $l_i = 0$

$$M \leftarrow M - \sum_{i=0}^n l_i$$

$$x_i \leftarrow x_i - l_i$$

$$r_i \leftarrow r_i - l_i$$

$$l_i \leftarrow 0$$

3.5 Caso Especial $r_i = R$

$$Resposta = \sum_{k=0}^N (-1)^k \binom{N}{k} \binom{M - (R + 1)k + N - 1}{N - 1}$$

4 Geometry

4.1 Pythagorean Triples

For all natural a, b, c satisfying $a^2 + b^2 = c^2$ there exist $m, n \in \mathbb{N}$ and $m > n$ such that (reverse is also true):

$$a = m^2 - n^2 \quad b = 2mn \quad c = m^2 + n^2$$

4.2 Heron's Formula

The area of a triangle can be written as $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the lengths of its sides and $s = \frac{a+b+c}{2}$.

This can be generalized to compute the area A of a quadrilateral with sides a, b, c, d , with $s = \frac{a+b+c+d}{2}$ and α, γ any two opposite angles:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \left(\cos^2 \left(\frac{\alpha + \gamma}{2} \right) \right)}$$

4.3 Pick's Theorem

The area of a simple polygon whose vertices have integer coordinates is:

$$A = I + \frac{B}{2} - 1$$

where I is the number of interior integer points, and B is the number of integer points in the border of the polygon.

4.4 Colinear Points

Three points are colinear on \mathbb{R}^2 iff:

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 0$$

The absolute value of this determinant is twice the area of the triangle ABC .

4.5 Coplanar Points

Four points are coplanar in \mathbb{R}^3 iff:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0$$

4.6 Trigonometry

4.6.1 Angle Sum

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

4.6.2 Sum-to-Product Transformation

$$\sin a \pm \sin b = 2 \sin \frac{a \pm b}{2} \cos \frac{a \mp b}{2}$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a \cos b}$$

4.7 Centroid of a polygon

The coordites of the centroid of a non-self-intersecting closed polygon is:

$$\frac{1}{3A} \left(\sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i) \right),$$

where A is twice the signed area of the polygon.

5 Other

5.0.1 String Matching with Wildcards

Consider a text T and a pattern P . P and T may have wildcards that will match any character. The problem is to get the positions where P occur in T .

If we define the value of the characters such that the wildcard is zero and the other characters are positive, there is a matching at position i iff

$\sum_{j=0}^{|P|-1} P[j]T[i+j](P[j] - T[i+j])^2 = 0$. Then, one can evaluate each term of

$$\sum_{j=0}^{|P|-1} (P[j]^3T[i+j] - 2P[j]^2T[i+j]^2 + P[j]T[i+j]^3)$$

using three convolutions.

*"Eu vim fazer um anúncio, Shadow o ouriço é um filha da puta do caralho,
ele mijou na porra da minha esposa.*

*Isso mesmo, ele pegou a porra do pinto espinhoso dele
e mijou na porra da minha esposa
e disse que o pau dele era dessa tamanho
e eu disse "Credo, que nojo"
então estou fazendo um exposed no meu twitter.com.*

*Shadow o ouriço, você tem um pau pequeno,
que é do tamanho desta noz só que muito menor,
e adivinha, olha o tamanho do meu pirocão.*

*Isso mesmo bebê, pontas altas, sem pelos, sem espinhos,
olha só, parecem 2 bolas e 1 torpedo.*

*Ele fodeu a minha esposa, então adivinhem,
eu vou foder a terra.
Isso mesmo, isso que você ganha, meu super laser de mijo.*

*Exceto que eu não vou mijar na terra,
eu vou mijar na lua.*

*Você gostou disto Lula? Eu mije na lua, faz o L agora.
Vocês tem 23 horas antes que os meus perdígotos de mijo
atinjam a Terra.*

*Agora saiam da porra da minha frente,
antes que eu mije em vocês também."*

~Robotnic