

## 1 Identidades

### 1.1 Series

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4} = \left( \sum_{i=1}^n i \right)^2$$

$$g_k(n) = \sum_{i=1}^n i^k = \frac{1}{k+1} \left( n^{k+1} + \sum_{j=1}^k \binom{k+1}{j+1} (-1)^{j+1} g_{k-j}(n) \right)$$

$$\sum_{i=0}^n ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

### 1.2 Binomial Identities

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{k} \binom{n}{k}$$

$$\binom{n}{h} \binom{n-h}{k} = \binom{n}{k} \binom{n-k}{h}$$

$$\binom{n}{k} = \frac{n+1-k}{k} \binom{n}{k-1}$$

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$$

$$\sum_{k=0}^n k^2 \binom{n}{k} = (n+n^2)2^{n-2}$$

$$\sum_{j=0}^k \binom{m}{j} \binom{n-m}{k-j} = \binom{n}{k}$$

$$\sum_{j=0}^m \binom{m}{j}^2 = \binom{2m}{m}$$

$$\sum_{m=0}^n \binom{m}{j} \binom{n-m}{k-j} = \binom{n+1}{k+1}$$

$$\sum_{m=k}^n \binom{m}{k} = \binom{n+1}{k+1}$$

$$\sum_{r=0}^m \binom{n+r}{r} = \binom{n+m+1}{m}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = \text{Fib}(n+1)$$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

$$2 \sum_{i=L}^R \binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^R \binom{n+1}{i}$$

### 1.3 Progressão Aritmética (PA)

**Termo Geral:**  $a_n = a_1 + (n-1)r$

**Soma dos Termos:**  $S_n = \frac{(a_1 + a_n)n}{2}$

**PA de 2ª Ordem:**  $a_n = An^2 + Bn + C$

### 1.4 Progressão Geométrica (PG)

**Termo Geral:**  $a_n = a_1 \cdot q^{n-1}$

**Soma Finita:**  $S_n = a_1 \frac{q^n - 1}{q - 1}$

**Soma Infinita ( $|q| < 1$ ):**  $S_{\infty} = \frac{a_1}{1-q}$

## 2 Number Theory

### 2.1 Identities

$$\sum_{d|n} \varphi(d) = n$$

$$\sum_{\substack{i \leq n \\ \gcd(i,n)=1}} i = n \cdot \frac{\varphi(n)}{2}$$

$$|\{(x, y) : 1 \leq x, y \leq n, \gcd(x, y) = 1\}| = \sum_{d=1}^n \mu(d) \left\lfloor \frac{n}{d} \right\rfloor^2$$

$$\sum_{x=1}^n \sum_{y=1}^n \gcd(x, y) = \sum_{k=1}^n k \sum_{k|l} \left\lfloor \frac{n}{l} \right\rfloor^2 \mu\left(\frac{l}{k}\right)$$

$$\sum_{x=1}^n \sum_{y=x}^n \gcd(x, y) = \sum_{g=1}^n \sum_{g|d} g \cdot \varphi\left(\frac{d}{g}\right)$$

$$\sum_{x=1}^n \sum_{y=1}^n \text{lcm}(x, y) = \sum_{d=1}^n d \mu(d) \sum_{d|l} l \left( \left\lfloor \frac{n}{l} \right\rfloor + 1 \right)^2$$

$$\sum_{x=1}^n \sum_{y=x+1}^n \text{lcm}(x, y) = \sum_{g=1}^n \sum_{g|d} d \cdot \varphi\left(\frac{d}{g}\right) \cdot \frac{d}{g} \cdot \frac{1}{2}$$

$$\sum_{x \in A} \sum_{y \in A} \gcd(x, y) = \sum_{t=1}^n \left( \sum_{l|t} \frac{t}{l} \mu(l) \right) \left( \sum_{t|a} \text{freq}[a] \right)^2$$

$$\sum_{x \in A} \sum_{y \in A} \text{lcm}(x, y) = \sum_{t=1}^n \left( \sum_{l|t} \frac{l}{t} \mu(l) \right) \left( \sum_{a \in A, t|a} a \right)^2$$

## 2.2 Large Prime Gaps

For numbers until  $10^9$  the largest gap is 400.  
For numbers until  $10^{18}$  the largest gap is 1500.

## 2.3 Prime counting function - $\pi(x)$

The prime counting function is asymptotic to  $\frac{x}{\log x}$ , by the prime number theorem.

x	10	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$	$10^7$	$10^8$
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

## 2.4 Some Primes

999999937    1000000007    1000000009    1000000021    1000000033  
 $10^{18} - 11$      $10^{18} + 3$      $2305843009213693951 = 2^{61} - 1$

## 2.5 Number of Divisors

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

n	367567200	6983776800	13967553600	321253732800
$d(n)$	1152	2304	2688	5376

$18401055938125660800 \approx 2e18$  is highly composite with 184320 divisors.  
For numbers up to  $10^{88}$ ,  $d(n) < 3.6\sqrt[3]{n}$ .

## 2.6 Lucas's Theorem

$$\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod{p}$$

For  $p$  prime.  $n_i$  and  $m_i$  are the coefficients of the representations of  $n$  and  $m$  in base  $p$ . In particular,  $\binom{n}{m}$  is odd if and only if  $n$  is a submask of  $m$ .

## 2.7 Fermat's Theorems

Let  $p$  be a prime number and  $a$  an integer, then:

$$a^p \equiv a \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

**Lemma:** Let  $p$  be a prime number and  $a$  and  $b$  integers, then:

$$(a + b)^p \equiv a^p + b^p \pmod{p}$$

**Lemma:** Let  $p$  be a prime number and  $a$  an integer. The inverse of  $a$  modulo  $p$  is  $a^{p-2}$ :

$$a^{-1} \equiv a^{p-2} \pmod{p}$$

## 2.8 Taking modulo at the exponent

If  $a$  and  $m$  are coprime, then

$$a^n \equiv a^{n \bmod \varphi(m)} \pmod{m}$$

Generally, if  $n \geq \log_2 m$ , then

$$a^n \equiv a^{\varphi(m) + [n \bmod \varphi(m)]} \pmod{m}$$

## 2.9 Mobius inversion

If  $g(n) = \sum_{d|n} f(d)$ , then  $f(n) = \sum_{d|n} g(d) \mu\left(\frac{n}{d}\right)$ .

A more useful definition is:  $\sum_{d|n} \mu(d) = [n = 1]$

Example, sum of LCM:

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n \text{lcm}(i, j) &= \sum_{k=1}^n \sum_{i=1}^n \sum_{j=1}^n [\text{gcd}(i, j) = k] \frac{ij}{k} \\
&= \sum_{k=1}^n \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} [\text{gcd}(a, b) = 1] abk \\
&= \sum_{k=1}^n k \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} a \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} b \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} [d|a] [d|b] \mu(d) \\
&= \sum_{k=1}^n k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \left( \sum_{a=1}^{\lfloor \frac{n}{k} \rfloor} [d|a] a \right) \left( \sum_{b=1}^{\lfloor \frac{n}{k} \rfloor} [d|b] b \right) \\
&= \sum_{k=1}^n k \sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \mu(d) \left( \sum_{p=1}^{\lfloor \frac{n}{kd} \rfloor} p \right) \left( \sum_{q=1}^{\lfloor \frac{n}{kd} \rfloor} q \right)
\end{aligned}$$

## 2.10 Chicken McNugget Theorem

Given two **coprime** numbers  $n, m$ , the largest number that cannot be written as a linear combination of them is  $nm - n - m$ .

- There are  $\frac{(n-1)(m-1)}{2}$  non-negative integers that cannot be written as a linear combination of  $n$  and  $m$ ;
- For each pair  $(k, nm - n - m - k)$ , for  $k \geq 0$ , exactly one can be written.

## 2.11 Harmonic Lemma

This technique computes sums of the form  $\sum_{i=1}^n f\left(\left\lfloor \frac{n}{i} \right\rfloor\right)$  in  $O(\sqrt{n})$ .

The value of  $\left\lfloor \frac{n}{i} \right\rfloor$  is constant over blocks. If we are at index  $l$ , the value  $v = \left\lfloor \frac{n}{l} \right\rfloor$  remains constant up to index  $r = \left\lfloor \frac{n}{v} \right\rfloor$ . We can iterate through these blocks instead of individual indices.

```

long long sum = 0;
for (int l = 1, r; l <= n; l = r + 1) {
    int val = n / l;
    r = n / val;
    sum += (long long)(r - l + 1) * f(val);
}

```

**Generalization:** For sums like  $\sum_{i=1}^n g(i) \cdot f\left(\left\lfloor \frac{n}{i} \right\rfloor\right)$ , the contribution of a block  $[l, r]$  is:

$$(G(r) - G(l-1)) \cdot f\left(\left\lfloor \frac{n}{l} \right\rfloor\right)$$

where  $G(k)$  is the prefix sum  $\sum_{i=1}^k g(i)$ , which must be efficiently computable.

## 2.12 Goldbach's Conjecture

Every even integer greater than 2 is the sum of two prime numbers. Verified for all even numbers up to  $4 \times 10^{18}$

# 3 Combinatória

## 3.1 Stars and Bars Bounded

De quantas formas podemos definir uma sequencia  $x$  tal que:

$$\sum_{i=0}^n x_i = M$$

$$l_i \leq x_i \leq r_i$$

### 3.1.1 Existência de solução

A existência de solução é garantida se e somente se:

$$\sum_{i=0}^n l_i \leq M \leq \sum_{i=0}^n r_i$$

## 3.2 Solução com DP em $O(NM^2)$

$$DP(N, M) = \sum_{i=l_n}^{r_n} DP(N-1, M-i)$$

$$DP(0, M) = \begin{cases} 0, & M \neq 0 \\ 1, & M = 0 \end{cases}$$

### 3.3 Caso Particular $l_i = 0, r_i = M$

Stars and Bars com N-1 Barras

$$Resposta = \binom{M+N-1}{M}$$

### 3.4 Redução do problema para $l_i = 0$

$$M \leftarrow M - \sum_{i=0}^n l_i$$

$$x_i \leftarrow x_i - l_i$$

$$r_i \leftarrow r_i - l_i$$

$$l_i \leftarrow 0$$

### 3.5 Caso Especial $r_i = R$

$$Resposta = \sum_{k=0}^N (-1)^k \binom{N}{k} \binom{M - (R+1)k + N - 1}{N - 1}$$

## 4 Geometry

### 4.1 Pythagorean Triples

For all natural  $a, b, c$  satisfying  $a^2 + b^2 = c^2$  there exist  $m, n \in \mathbb{N}$  and  $m > n$  such that (reverse is also true):

$$a = m^2 - n^2 \quad b = 2mn \quad c = m^2 + n^2$$

### 4.2 Heron's Formula

The area of a triangle can be written as  $A = \sqrt{s(s-a)(s-b)(s-c)}$ , where  $a, b, c$  are the lengths of its sides and  $s = \frac{a+b+c}{2}$ .

This can be generalized to compute the area  $A$  of a quadrilateral with sides  $a, b, c, d$ , with  $s = \frac{a+b+c+d}{2}$  and  $\alpha, \gamma$  any two opposite angles:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \left( \cos^2 \left( \frac{\alpha + \gamma}{2} \right) \right)}$$

### 4.3 Pick's Theorem

The area of a simple polygon whose vertices have integer coordinates is:

$$A = I + \frac{B}{2} - 1$$

where  $I$  is the number of interior integer points, and  $B$  is the number of integer points in the border of the polygon.

### 4.4 Colinear Points

Three points are colinear on  $\mathbb{R}^2$  iff:

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 0$$

The absolute value of this determinant is twice the area of the triangle  $ABC$ .

### 4.5 Coplanar Points

Four points are coplanar in  $\mathbb{R}^3$  iff:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0$$

### 4.6 Trigonometry

#### 4.6.1 Angle Sum

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

#### 4.6.2 Sum-to-Product Transformation

$$\sin a \pm \sin b = 2 \sin \frac{a \pm b}{2} \cos \frac{a \mp b}{2}$$

$$\cos a + \cos b = 2 \cos \frac{a + b}{2} \cos \frac{a - b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a + b}{2} \sin \frac{a - b}{2}$$

$$\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a \cos b}$$

### 4.7 Centroid of a polygon

The coordites of the centroid of a non-self-intersecting closed polygon is:

$$\frac{1}{3A} \left( \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i) \right),$$

where  $A$  is twice the signed area of the polygon.

### 4.8 2D Shapes

#### 4.8.1 Square

- **Perimeter:**  $P = 4s$
- **Area:**  $A = s^2$

Where  $s$  is the side length.

#### 4.8.2 Rectangle

- **Perimeter:**  $P = 2(l + w)$
- **Area:**  $A = l \cdot w$

Where  $l$  is the length and  $w$  is the width.

#### 4.8.3 Triangle

- **Perimeter:**  $P = a + b + c$
- **Area:**  $A = \frac{1}{2}b \cdot h$
- **Heron's Formula (Area):**  $A = \sqrt{s(s-a)(s-b)(s-c)}$ , where  $s = \frac{a+b+c}{2}$ .

Where  $a, b, c$  are the side lengths,  $b$  is the base, and  $h$  is the height.

#### 4.8.4 Circle

- **Circumference:**  $C = 2\pi r = \pi d$
- **Area:**  $A = \pi r^2$

Where  $r$  is the radius and  $d$  is the diameter.

#### 4.8.5 Parallelogram

- **Perimeter:**  $P = 2(a + b)$
- **Area:**  $A = b \cdot h$

Where  $a, b$  are adjacent side lengths,  $b$  is the base, and  $h$  is the height.

#### 4.8.6 Trapezoid

- **Area:**  $A = \frac{1}{2}(a + b)h$

Where  $a$  and  $b$  are the parallel side lengths and  $h$  is the height.

### 4.9 3D Shapes

#### 4.9.1 Cube

- **Surface Area:**  $SA = 6s^2$
- **Volume:**  $V = s^3$

Where  $s$  is the side length.

#### 4.9.2 Rectangular Prism (Cuboid)

- **Surface Area:**  $SA = 2(lw + lh + wh)$
- **Volume:**  $V = lwh$

Where  $l, w, h$  are the length, width, and height.

#### 4.9.3 Sphere

- **Surface Area:**  $SA = 4\pi r^2$
- **Volume:**  $V = \frac{4}{3}\pi r^3$

Where  $r$  is the radius.

#### 4.9.4 Cylinder

- **Lateral Surface Area:**  $A_L = 2\pi rh$
- **Total Surface Area:**  $SA = 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$
- **Volume:**  $V = \pi r^2 h$

Where  $r$  is the radius and  $h$  is the height.

#### 4.9.5 Cone

- **Lateral Surface Area:**  $A_L = \pi rl$
- **Total Surface Area:**  $SA = \pi rl + \pi r^2 = \pi r(l + r)$
- **Volume:**  $V = \frac{1}{3}\pi r^2 h$

Where  $r$  is the radius,  $h$  is the height, and  $l = \sqrt{r^2 + h^2}$  is the slant height.

#### 4.9.6 Pyramid

- **Volume:**  $V = \frac{1}{3}A_b \cdot h$

Where  $A_b$  is the area of the base and  $h$  is the height.

## 5 Grafos

### 5.1 Min Cut Max Flow Duality

We seek to construct a binary string  $S$  of length  $n$  that minimizes a total cost. The cost is defined as follows:

- If  $S_i = 0$ , a cost of  $A_i$  is incurred.
- If  $S_i = 1$ , a cost of  $B_i$  is incurred.
- If  $S_i = 1$  and  $S_j = 0$ , a penalty of  $C_{i,j}$  is incurred.

The total cost is the sum of all such costs and penalties for the chosen string  $S$ .

1. For each node  $i$ , add an edge  $s \rightarrow i$  with capacity  $B_i$ . This represents the cost of setting  $S_i = 1$ .
2. For each node  $i$ , add an edge  $i \rightarrow t$  with capacity  $A_i$ . This represents the cost of setting  $S_i = 0$ .
3. For each pair  $(i, j)$  with a penalty, add an edge  $i \rightarrow j$  with capacity  $C_{i,j}$ . This represents the penalty for setting  $S_i = 1$  and  $S_j = 0$ .

The capacity of a cut in this graph corresponds to the total cost of the binary string defined by the partition. For example, if node  $i$  is in the  $T$ -partition ( $S_i = 1$ ) and node  $j$  is in the  $S$ -partition ( $S_j = 0$ ), the edge  $i \rightarrow j$  must be cut, adding the penalty  $C_{i,j}$  to the total cost. By the max-flow min-cut theorem, the minimum cost is equal to the maximum flow from  $s$  to  $t$ .

## 6 Other

### 6.0.1 String Matching with Wildcards

Consider a text  $T$  and a pattern  $P$ .  $P$  and  $T$  may have wildcards that will match any character. The problem is to get the positions where  $P$  occur in  $T$ .

If we define the value of the characters such that the wildcard is zero and the other characters are positive, there is a matching at position  $i$  iff

$\sum_{j=0}^{|P|-1} P[j]T[i+j](P[j] - T[i+j])^2 = 0$ . Then, one can evaluate each term of

$$\sum_{j=0}^{|P|-1} (P[j]^3 T[i+j] - 2P[j]^2 T[i+j]^2 + P[j]T[i+j]^3)$$

using three convolutions.

*"Eu vim fazer um anúncio, Shadow o ouriço é um filha da puta do caralho, ele mijou na porra da minha esposa.*

*Isso mesmo, ele pegou a porra do pinto espinhoso dele  
e mijou na porra da minha esposa  
e disse que o pau dele era dessa tamanho  
e eu disse "Credo, que nojo"  
então estou fazendo um exposed no meu twitter.com.*

*Shadow o ouriço, você tem um pau pequeno,  
que é do tamanho desta noz só que muito menor,  
e adivinha, olha o tamanho do meu pirocão.*

*Isso mesmo bebê, pontas altas, sem pelos, sem espinhos,  
olha só, parecem 2 bolas e 1 torpedo.*

*Ele fodeu a minha esposa, então adivinhem,  
eu vou foder a terra.  
Isso mesmo, isso que você ganha, meu super laser de mijo.*

*Exceto que eu não vou mijar na terra,  
eu vou mijar na lua.*

*Você gostou disto Lula? Eu mije na lua, faz o L agora.  
Vocês tem 23 horas antes que os meus perdígotos de mijo  
atinjam a Terra.*

*Agora saiam da porra da minha frente,  
antes que eu mije em vocês também."*

*~Robotnic*