1 Identidades

1.1 Series

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$$

$$g_k(n) = \sum_{i=1}^{n} i^k = \frac{1}{k+1}\left(n^{k+1} + \sum_{j=1}^{k}\binom{k+1}{j+1}(-1)^{j+1}g_{k-j}(n)\right)$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2}-(n+1)c^{n+1}+c}{(c-1)^2}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

1.2 Binomial Identities

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \qquad \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{k}\binom{n}{k}$$

$$\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h} \qquad \binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$$

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad \sum_{k=0}^{n} k^2\binom{n}{k} = (n+n^2)2^{n-2}$$

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k} \qquad \sum_{j=0}^{m}\binom{m}{j}^2 = \binom{2m}{m}$$

$$\sum_{m=0}^{n}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1} \qquad \sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}$$

$$\sum_{r=0}^{m}\binom{n+r}{r} = \binom{n+m+1}{m} \qquad \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k} = \mathrm{Fib}(n+1)$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \qquad (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$2\sum_{i=L}^{R}\binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^{R}\binom{n+1}{i}$$

1.3 Progressão Aritmética (PA)

Termo Geral: $a_n = a_1 + (n-1)r$
Soma dos Termos: $S_n = \frac{(a_1+a_n)n}{2}$
PA de 2ª Ordem: $a_n = An^2 + Bn + C$

1.4 Progressão Geométrica (PG)

Termo Geral: $a_n = a_1 \cdot q^{n-1}$
Soma Finita: $S_n = a_1\frac{q^n-1}{q-1}$
Soma Infinita ($|q| < 1$): $S_\infty = \frac{a_1}{1-q}$

2 Number Theory

2.1 Identities

$$\sum_{d|n}\varphi(d) = n$$

$$\sum_{\substack{i<n \\ \gcd(i,n)=1}} i = n \cdot \frac{\varphi(n)}{2}$$

$$|\{(x,y) : 1 \le x, y \le n, \gcd(x,y) = 1\}| = \sum_{d=1}^{n}\mu(d)\left\lfloor\frac{n}{d}\right\rfloor^2$$

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\gcd(x,y) = \sum_{k=1}^{n}k\sum_{k|l}\left\lfloor\frac{n}{l}\right\rfloor^2\mu\left(\frac{l}{k}\right)$$

$$\sum_{x=1}^{n}\sum_{y=x}^{n}\gcd(x,y) = \sum_{g=1}^{n}\sum_{g|d}^{n} g\cdot\varphi\left(\frac{d}{g}\right)$$

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\mathrm{lcm}(x,y) = \sum_{d=1}^{n}d\,\mu(d)\sum_{d|l}^{n}l\binom{\lfloor\frac{n}{l}\rfloor+1}{2}^2$$

$$\sum_{x=1}^{n}\sum_{y=x+1}^{n}\mathrm{lcm}(x,y) = \sum_{g=1}^{n}\sum_{g|d}^{n}d\cdot\varphi\left(\frac{d}{g}\right)\cdot\frac{d}{g}\cdot\frac{1}{2}$$

$$\sum_{x\in A}\sum_{y\in A}\gcd(x,y) = \sum_{t=1}^{n}\left(\sum_{l|t}\frac{t}{l}\mu(l)\right)\left(\sum_{t|a}\mathrm{freq}[a]\right)^2$$

$$\sum_{x\in A}\sum_{y\in A}\mathrm{lcm}(x,y) = \sum_{t=1}^{n}\left(\sum_{l|t}\frac{l}{t}\mu(l)\right)\left(\sum_{a\in A,\, t|a}a\right)^2$$

2.2 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

2.3 Prime counting function - $\pi(x)$

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

2.4 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033
$10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$

2.5 Number of Divisors

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

n	367567200	6983776800	13967553600	321253732800
$d(n)$	1152	2304	2688	5376

18401055938125660800 $\approx$ 2e18 is highly composite with 184320 divisors.
For numbers up to 10^{88}, $d(n) < 3.6\sqrt[3]{n}$.

2.6 Lucas's Theorem

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod p$$

For p prime. n_i and m_i are the coefficients of the representations of n and m in base p. In particular, $\binom{n}{m}$ is odd if and only if n is a submask of m.

2.7 Fermat's Theorems

Let p be a prime number and a an integer, then:

$$a^p \equiv a \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a+b)^p \equiv a^p + b^p \pmod p$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod p$$

2.8 Taking modulo at the exponent

If a and m are coprime, then

$$a^n \equiv a^{n \bmod \varphi(m)} \pmod m$$

Generally, if $n \geq \log_2 m$, then

$$a^n \equiv a^{\varphi(m) + [n \bmod \varphi(m)]} \pmod m$$

2.9 Mobius invertion

If $g(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} g(d)\mu(\frac{n}{d})$.
A more useful definition is: $\sum_{d|n} \mu(d) = [n = 1]$
Example, sum of LCM:

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{lcm}(i,j) &= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}[\gcd(i,j)=k]\frac{ij}{k} \\
&= \sum_{k=1}^{n}\sum_{a=1}^{\lfloor \frac{n}{k}\rfloor}\sum_{b=1}^{\lfloor \frac{n}{k}\rfloor}[\gcd(a,b)=1]abk \\
&= \sum_{k=1}^{n} k \sum_{a=1}^{\lfloor \frac{n}{k}\rfloor} a \sum_{b=1}^{\lfloor \frac{n}{k}\rfloor} b \sum_{d=1}^{\lfloor \frac{n}{k}\rfloor}[d|a]\,[d|b]\,\mu(d) \\
&= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k}\rfloor}\mu(d)\left(\sum_{a=1}^{\lfloor \frac{n}{k}\rfloor}[d|a]\,a\right)\left(\sum_{b=1}^{\lfloor \frac{n}{k}\rfloor}[d|b]\,b\right) \\
&= \sum_{k=1}^{n} k \sum_{d=1}^{\lfloor \frac{n}{k}\rfloor}\mu(d)\left(\sum_{p=1}^{\lfloor \frac{n}{kd}\rfloor} p\right)\left(\sum_{q=1}^{\lfloor \frac{n}{kd}\rfloor} q\right)
\end{aligned}
$$

2.10 Chicken McNugget Theorem

Given two **coprime** numbers n, m, the largest number that cannot be written as a linear combination of them is $nm - n - m$.

- There are $\frac{(n-1)(m-1)}{2}$ non-negative integers that cannot be written as a linear combination of n and m;
- For each pair $(k, nm - n - m - k)$, for $k \geq 0$, exactly one can be written.

2.11 Harmonic Lemma

This technique computes sums of the form $\sum_{i=1}^{n} f\left(\lfloor \frac{n}{i} \rfloor\right)$ in $O(\sqrt{n})$.
The value of $\lfloor \frac{n}{i} \rfloor$ is constant over blocks. If we are at index l, the value $v = \lfloor \frac{n}{l} \rfloor$ remains constant up to index $r = \lfloor \frac{n}{v} \rfloor$. We can iterate through these blocks instead of individual indices.

```
long long sum = 0;
for (int l = 1, r; l <= n; l = r + 1) {
    int val = n / l;
    r = n / val;
    sum += (long long)(r - l + 1) * f(val);
}
```

Generalization: For sums like $\sum_{i=1}^{n} g(i) \cdot f(\lfloor \frac{n}{i} \rfloor)$, the contribution of a block $[l, r]$ is:

$$(G(r) - G(l-1)) \cdot f\left(\left\lfloor \frac{n}{l} \right\rfloor\right)$$

where $G(k)$ is the prefix sum $\sum_{i=1}^{k} g(i)$, which must be efficiently computable.

2.12 Goldbach's Conjecture

Every even integer greater than 2 is the sum of two prime numbers. Verified for all even numbers up to 4×10^{18}

3 Combinatória

3.1 Stars and Bars Bounded

De quantas formas podemos definir uma sequencia x tal que:

$$\sum_{i=0}^{n} x_i = M$$

$$l_i \leq x_i \leq r_i$$

3.1.1 Existência de solução

A existência de solução é garantida se e somente se:

$$\sum_{i=0}^{n} l_i \leq M \leq \sum_{i=0}^{n} r_i$$

3.1.2 Solução com DP em $O(NM^2)$

$$DP(N, M) = \sum_{i=l_n}^{r_n} DP(N-1, M-i)$$

$$DP(0, M) = \begin{cases} 0, M \neq 0 \\ 1, M = 0 \end{cases}$$

3.1.3 Caso Particular $l_i = 0, r_i = M$

Stars and Bars com N-1 Barras

$$Resposta = \binom{M+N-1}{M}$$

3.1.4 Redução do problema para $l_i = 0$

$$M \leftarrow M - \sum_{i=0}^{n} l_i$$

$$x_i \leftarrow x_i - l_i$$

$$r_i \leftarrow r_i - l_i$$

$$l_i \leftarrow 0$$

3.1.5 Caso Especial $r_i = R$

$$Resposta = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \binom{M-(R+1)k+N-1}{N-1}$$

3.2 Formulas Básicas

Permutação Circular

O número de maneiras de ordenar n objetos distintos em um círculo.

$$PC_n = (n-1)!$$

Permutação com Repetição

O número de maneiras de ordenar n objetos, onde existem n_1 objetos idênticos de um tipo, n_2 de outro, e assim por diante.

$$P_n^{n_1,n_2,\ldots,n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}$$

Combinação Caótica (Desarranjo)

O número de permutações de n objetos onde nenhum objeto permanece em sua posição original.

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Aproximação para n grande:

$$D_n \approx \frac{n!}{e}$$

4 Geometry

4.1 Pythagorean Triples

For all natural a, b, c satisfying $a^2 + b^2 = c^2$ there exist $m, n \in \mathbb{N}$ and $m > n$ such that (reverse is also true):

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

4.2 Heron's Formula

The area of a triangle can be written as $A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$, where a, b, c are the lengths of its sides and $s = \frac{a+b+c}{2}$.
This can be generalized to compute the area A of a quadrilateral with sides a, b, c, d, with $s = \frac{a+b+c+d}{2}$ and α, γ any two opposite angles:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\left(\cos^2\left(\frac{\alpha+\gamma}{2}\right)\right)}$$

4.3 Pick's Theorem

The area of a simple polygon whose vertices have integer coordinates is:

$$A = I + \frac{B}{2} - 1$$

where I is the number of interior integer points, and B is the number of integer points in the border of the polygon.

4.4 Colinear Points

Three points are colinear on $\mathbb{R}^2$ iff:

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 0$$

The absolute value of this determinant is twice the area of the triangle ABC.

4.5 Coplanar Points

Four points are coplanar in $\mathbb{R}^3$ iff:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0$$

4.6 Trigonometry

4.6.1 Angle Sum

$$\sin(a \pm b) = \sin a \cos b \pm \cos\ \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

4.6.2 Sum-to-Product Transformation

$$\sin a \pm \sin b = 2 \sin \frac{a \pm b}{2} \cos \frac{a \mp b}{2}$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a \cos b}$$

4.7 Centroid of a polygon

The coordites of the centroid of a non-self-intersecting closed polygon is:

$$\frac{1}{3A}\left(\sum_{i=0}^{n-1}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \sum_{i=0}^{n-1}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)\right),$$

where A is twice the signed area of the polygon.

4.8 2D Shapes

4.8.1 Square

- **Perimeter:** $P = 4s$
- **Area:** $A = s^2$

Where s is the side length.

4.8.2 Rectangle

- **Perimeter:** $P = 2(l + w)$
- **Area:** $A = l \cdot w$

Where l is the length and w is the width.

4.8.3 Triangle

- **Perimeter:** $P = a + b + c$
- **Area:** $A = \frac{1}{2} b \cdot h$
- **Heron's Formula (Area):** $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$.

Where a, b, c are the side lengths, b is the base, and h is the height.

4.8.4 Circle

- **Circumference:** $C = 2\pi r = \pi d$
- **Area:** $A = \pi r^2$

Where r is the radius and d is the diameter.

4.8.5 Parallelogram

- **Perimeter:** $P = 2(a + b)$
- **Area:** $A = b \cdot h$

Where a, b are adjacent side lengths, b is the base, and h is the height.

4.8.6 Trapezoid

- **Area:** $A = \frac{1}{2}(a + b)h$

Where a and b are the parallel side lengths and h is the height.

4.9 3D Shapes

4.9.1 Cube

- **Surface Area:** $SA = 6s^2$
- **Volume:** $V = s^3$

Where s is the side length.

4.9.2 Rectangular Prism (Cuboid)

- **Surface Area:** $SA = 2(lw + lh + wh)$
- **Volume:** $V = lwh$

Where l, w, h are the length, width, and height.

4.9.3 Sphere

- **Surface Area:** $SA = 4\pi r^2$
- **Volume:** $V = \frac{4}{3}\pi r^3$

Where r is the radius.

4.9.4 Cylinder

- **Lateral Surface Area:** $A_L = 2\pi rh$
- **Total Surface Area:** $SA = 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$
- **Volume:** $V = \pi r^2 h$

Where r is the radius and h is the height.

4.9.5 Cone

- **Lateral Surface Area:** $A_L = \pi rl$
- **Total Surface Area:** $SA = \pi rl + \pi r^2 = \pi r(l + r)$
- **Volume:** $V = \frac{1}{3}\pi r^2 h$

Where r is the radius, h is the height, and $l = \sqrt{r^2 + h^2}$ is the slant height.

4.9.6 Pyramid

- **Volume:** $V = \frac{1}{3}A_b \cdot h$

Where A_b is the area of the base and h is the height.

5 Grafos

5.1 Min Cut Max Flow Duality

We seek to construct a binary string S of length n that minimizes a total cost. The cost is defined as follows:

- If $S_i = 0$, a cost of A_i is incurred.
- If $S_i = 1$, a cost of B_i is incurred.
- If $S_i = 1$ and $S_j = 0$, a penalty of $C_{i,j}$ is incurred.

The total cost is the sum of all such costs and penalties for the chosen string S.

1. For each node i, add an edge $s \to i$ with capacity B_i. This represents the cost of setting $S_i = 1$.

2. For each node i, add an edge $i \to t$ with capacity A_i. This represents the cost of setting $S_i = 0$.
3. For each pair (i, j) with a penalty, add an edge $i \to j$ with capacity $C_{i,j}$. This represents the penalty for setting $S_i = 1$ and $S_j = 0$.

The capacity of a cut in this graph corresponds to the total cost of the binary string defined by the partition. For example, if node i is in the T-partition ($S_i = 1$) and node j is in the S-partition ($S_j = 0$), the edge $i \to j$ must be cut, adding the penalty $C_{i,j}$ to the total cost. By the max-flow min-cut theorem, the minimum cost is equal to the maximum flow from s to t.

5.2 Notable Applications and Equivalences on Flow

- **Bipartite Matching:**
  The size of the maximum matching in a bipartite graph is equal to the maximum flow in a network constructed from the graph.
- **Kőnig's Theorem:**
  In any bipartite graph, the number of edges in the maximum matching is equal to the number of vertices in the minimum vertex cover.

$$\text{Maximum Matching} = \text{Minimum Vertex Cover}$$

$$\text{Maximum Independent Set} = |V| - \text{Maximum Matching}$$

- **Menger's Theorem:**
  The maximum number of vertex-disjoint paths between two vertices u, v is equal to the minimum number of vertices to be removed to disconnect u and v.
- **Project Selection Problem (Min-Cut):**
  Binary decision problems with interdependent costs and profits can be modeled as a minimum cut problem, where the cut separates the "chosen" decisions from the "not chosen" ones.

6 Strings

6.0.1 The Concatenation Trick

An elegant way to solve the string matching problem is to construct a new string $S = P + \# + T$, where # is a delimiter character not present in P or T. We then compute the prefix function π_S for this combined string.

6.0.2 Finding the Smallest Period of a String

The smallest period of a string S of length n can be found using $\pi[n-1]$. The value $k = n - \pi[n-1]$ is a potential period. If n is divisible by k, then k is the length of the smallest period. Otherwise, the smallest period is n itself. The intuition is that $\pi[n-1]$ represents the largest overlap between the beginning and the end of the string, so the non-overlapping part, $n - \pi[n-1]$, must be the repeating unit.

6.0.3 String Compression

This is equivalent to finding the smallest period. The problem asks for the shortest string t such that S can be represented as k concatenations of t. The length of this base string t is $n - \pi[n-1]$, provided n is divisible by this length.

7 Game Theory

The Game of Nim

Nim is the canonical impartial game. It consists of several piles of stones. A move consists of choosing one pile and removing any positive number of stones from it.

Winning Condition: The winning strategy is determined by the **Nim-Sum** of the pile sizes, which is their bitwise XOR sum. Let the pile sizes be $p_1, p_2, \ldots, p_k$.

$$\text{Nim-Sum} = p_1 \oplus p_2 \oplus \cdots \oplus p_k$$

A position is a P-position (losing) if and only if its Nim-Sum is 0. Otherwise, it is an N-position (winning).

The Sprague-Grundy Theorem

This is the fundamental theorem of impartial games. It states that every impartial game under the normal play convention is equivalent to a Nim pile of a certain size. This "equivalent size" is called the **Grundy number** (or **nim-value**).

Grundy Numbers (g-numbers): The Grundy number of a game state S, denoted $g(S)$, is defined recursively as the smallest non-negative integer that is not among the Grundy numbers of the states reachable in one move from S. This is the **Minimum Excluded value (MEX)** of that set.

$$g(S) = \text{mex}\{g(S') \mid S' \text{ is reachable from } S \text{ in one move}\}$$

The MEX of a set of non-negative integers is the smallest non-negative integer not in the set. For example, $\text{mex}\{0,1,3,4\} = 2$.

Sum of Games: Many games can be decomposed into a sum of independent sub-games (e.g., a game played on multiple disconnected boards). The Sprague-Grundy theorem states that the g-number of a sum of games is the Nim-Sum of the g-numbers of the sub-games.

$$g(G_1 + G_2 + \cdots + G_k) = g(G_1) \oplus g(G_2) \oplus \cdots \oplus g(G_k)$$

Winning Condition (General Games): Combining these ideas provides a universal winning condition for any impartial game:

A game state is a P-position (losing) if and only if its Grundy number is 0.

Classic Games and their Grundy Numbers

- **A single pile of Nim:** For a pile of size n, the g-number is simply n. So, $g(n) = n$. This is why the XOR sum works for multiple piles.
- **Subtraction Games:** A game with a single pile where a player can remove any number of stones $s \in \{s_1, s_2, \ldots, s_k\}$. The g-number for a pile of size n is:
$$g(n) = \text{mex}\{g(n - s_i) \mid s_i \in S, n \geq s_i\}$$

8 Other

8.0.1 String Matching with Wildcards

Consider a text T and a pattern P. P and T may have wildcards that will match any character. The problem is to get the positions where P occur in T.

If we define the value of the characters such that the wildcard is zero and the other characters are positive, there is a matching at position i iff $\sum_{j=0}^{|P|-1} P[j]T[i+j](P[j] - T[i+j])^2 = 0$. Then, one can evaluate each term of

$$\sum_{j=0}^{|P|-1} (P[j]^3 T[i+j] - 2P[j]^2 T[i+j]^2 + P[j]T[i+j]^3)$$

using three convolutions.