Module 1: Integral calculus.

Multiple Integrals :-

A repeated process of integration of a function of two and three variables reffered to as double integrals: $\iint f(x,y)\,dx\,dy$ and triple integrals: $\iiint f(x,y,z)\,dx\,dy\,dz$.

* Evaluation of double and triple integrals :-

1> Evaluate $\int_0^2\int_0^2 (x^2+y^2)\,dx\,dy$

$$I = \int_0^2 \left[\int_0^2 (x^2+y^2)\,dx\right] dy$$

$$I = \int_0^2 \left[\frac{x^3}{3} + y^2 x\right]_0^2 dy$$

$$I = \int_0^2 \left(\frac{8}{3} + 2y^2\right) - \left(\frac{1}{3} + y^2\right) dy$$

$$I = \int_0^2 \left(\frac{7}{3} + y^2\right) dy = \left[\frac{7}{3}y + \frac{y^3}{3}\right]_0^2 = \frac{14}{3} + \frac{8}{3} = \underline{\frac{22}{3}.}$$

2> Evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$.

$$I = \int_{x=0}^1 \frac{1}{\sqrt{1-x^2}} \left\{\int_{y=0}^1 \frac{dy}{\sqrt{1-y^2}}\right\} dx$$

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \{\sin^{-1} y\}_0^1\, dx$$

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \{ \sin^{-1} 1 - \cancel{\sin^{-1} 0} \} \, dx$$

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left\{ \frac{\pi}{2} - 0 \right\} dx = \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx$$

$$I = \frac{\pi}{2} \{ \sin^{-1} x \}_0^1 = \frac{\pi}{2} \{ \sin^{-1} 1 - \cancel{\sin^{-1} 0} \} = \frac{\pi}{2} \left(\frac{\pi}{2} \right).$$

$$\underline{I = \frac{\pi^2}{4}}.$$

3) Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx\, dy$

$$I = \int_{y=0}^1 y \cdot \left. \frac{x^4}{4} \right]_{x=0}^{\sqrt{1-y^2}} dy = \frac{1}{4} \int_{y=0}^1 y \cdot (1-y^2)^2 \, dy.$$

$$I = \frac{1}{4} \int_{y=0}^1 y (1 + y^4 - 2y^2) \, dy = \frac{1}{4} \int_{y=0}^1 (y + y^5 - 2y^3) \, dy.$$

$$I = \frac{1}{4} \left[\frac{y^2}{2} + \frac{y^6}{6} - \frac{2y^4}{4} \right]_0^1 = \frac{1}{4} \left[\frac{1}{2} + \frac{1}{6} - \frac{1}{2} \right] = \frac{1}{4} \left(\frac{1}{6} \right)$$

$$\underline{I = \frac{1}{24}}.$$

4) Evaluate $\displaystyle\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dy \, dx = \underline{\frac{3}{35}}$.

5) Evaluate $\displaystyle\int_1^4 \int_0^{\sqrt{4-x}} xy \, dy \, dx = \underline{\frac{9}{2}}$.

6) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

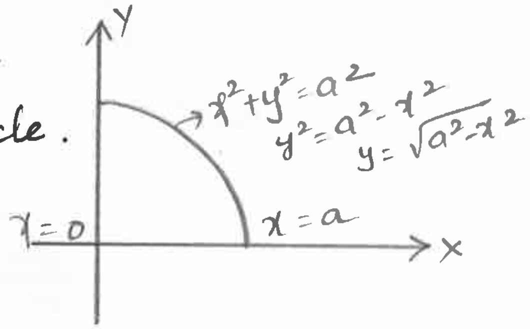
$$I = \int_{x=0}^{1} dx \int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2 + y^2} = \int_{x=0}^{1} dx \left\{ \frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right\}_0^{\sqrt{1+x^2}}$$

$$I = \int_{x=0}^{1} dx. \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} = \frac{\pi}{4} \int_{x=0}^{1} \frac{dx}{\sqrt{1+x^2}}$$

$$I = \frac{\pi}{4} \left[\log(x + \sqrt{x^2+1}) \right]_0^1 = \underline{\frac{\pi}{4} \log(\sqrt{2}+1)}$$

7) Evaluate $\iint_R xy\,dx\,dy$ where R is the Quadrant of the circle $x^2+y^2=a^2$ where $x \geq 0$, $y \geq 0$.

Sol^n: The region of integration be the first quadrant of the Circle.



$$\iint_R xy\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx.$$

$$= \int_{x=0}^{a} x \frac{y^2}{2} \Big]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2} \int_{x=0}^{a} x(a^2-x^2)\,dx.$$

$$= \frac{1}{2} \int_{x=0}^{a} xa^2 - x^3\,dx = \frac{1}{2} \left[\frac{x^2 a^2}{2} - \frac{x^4}{4} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right] = \underline{\frac{a^4}{8}}.$$

8) Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$.

$$I = \int_0^1 \left(x e^{y/x}\right)_0^{x^2} dx = \int_0^1 (xe^x - x)\, dx$$

$$I = xe^x - \int e^x dx - \frac{x^2}{2}\Big]_0^1 = xe^x - e^x - \frac{x^2}{2}\Big]_0^1$$

$$I = \left(e - e - \frac{1}{2}\right) - (0 - 1 - 0) = \underline{\frac{1}{2}}.$$

9) Evaluate $\int_0^1 \int_0^2 \int_1^2 xyz^2\, dx\, dy\, dz$

$$I = \int_0^1 \int_0^2 \frac{x^2 y z^2}{2}\Big]_1^2 dy\, dz = \int_0^1 \int_0^2 \left(2yz^2 - \frac{yz^2}{2}\right) dy\, dz$$

$$I = \int_0^1 \left(\frac{2y^2z^2}{2} - \frac{y^2z^2}{4}\right)_0^2 dz = \int_0^1 \left[y^2z^2 - \frac{y^2z^2}{4}\right]_0^2 dz$$

$$I = \int_0^1 \frac{3y^2z^2}{4}\Big]_0^2 dz = \int_0^1 3z^2\, dz = \frac{3z^3}{3}\Big]_0^1 = \underline{1}$$

10) Evaluate $\int_0^a \int_0^a \int_0^a (x^2+y^2+z^2)\, dx\, dy\, dz$

Sol^n: $I = \int_0^a \int_0^a \frac{x^3}{3} + y^2x + z^2x\Big]_0^a dy\, dz$

$$I = \int_0^a \int_0^a \left(\frac{a^3}{3} + ay^2 + az^2\right) dy\, dz$$

$$I = \int_0^a \left(\frac{a^3}{3}y + \frac{ay^3}{3} + az^2y\right)\Big]_0^a dz$$

$$I = \int_0^a \left(\frac{a^4}{3} + \frac{a^4}{3} + a^2z^2\right)dz = \frac{a^4}{3}z + \frac{a^4}{3}z + \frac{a^2z^3}{3}\Big]_0^a$$

$$I = \frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3} = \underline{a^5}.$$

11) Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$.

$$I = \int_{-c}^{c}\int_{-b}^{b}\left(\frac{x^3}{3} + xy^2 + xz^2\right)_{-a}^{a} dy\,dz.$$

$$I = \int_{-c}^{c}\int_{-b}^{b}\left[\left(\frac{a^3}{3} + ay^2 + az^2\right) - \left(-\frac{a^3}{3} - ay^2 - az^2\right)\right]dy\,dz.$$

$$I = \int_{-c}^{c}\int_{-b}^{b} \frac{2a^3}{3} + 2ay^2 + 2az^2\,dy\,dz$$

$$I = \int_{-c}^{c} \frac{2a^3}{3}y + \frac{2ay^3}{3} + 2az^2y\Big]_{-b}^{b} dz.$$

$$I = \int_{-c}^{c}\left[\left(\frac{2a^3b}{3} + \frac{2ab^3}{3} + 2az^2b\right) - \left(-\frac{2a^3b}{3} - \frac{2ab^3}{3} - 2az^2b\right)\right]dz$$

$$I = \int_{-c}^{c}\left(\frac{4a^3b}{3} + \frac{4ab^3}{3} + 4az^2b\right)dz.$$

$$I = \left[\frac{4a^3b}{3}z + \frac{4ab^3}{3}z + \frac{4az^3b}{3}\right]_{-c}^{c}$$

$$I = \left(\frac{4a^3bc}{3} + \frac{4ab^3c}{3} + \frac{4abc^3}{3}\right) - \left(-\frac{4a^3bc}{3} - \frac{4ab^3c}{3} - \frac{4abc^3}{3}\right)$$

$$I = \frac{8a^3bc}{3} + \frac{8ab^3c}{3} + \frac{8abc^3}{3} = \underline{\frac{8abc}{3}(a^2+b^2+c^2)}.$$

12> Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

Sol^n:-

$$I = \int_0^1\int_0^{\sqrt{1-x^2}} xy\,\frac{z^2}{2}\Big]_0^{\sqrt{1-x^2-y^2}}\,dy\,dx.$$

$$I = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy\,(1-x^2-y^2)\,dy\,dx = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy - x^3y - xy^3\,dy\,dx$$

$$I = \frac{1}{2}\int_0^1 \frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\Big]_0^{\sqrt{1-x^2}}\,dx.$$

$$I = \frac{1}{2}\int_0^1 \frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\,dx.$$

$$I = \frac{1}{8}\int_0^1 2x - 2x^3 - 2x^3 + 2x^5 - x[1+x^4 - 2x^2]\,dx.$$

$$I = \frac{1}{8}\int_0^1 2x - 4x^3 + 2x^5 - x - x^5 + 2x^3\,dx.$$

$$I = \frac{1}{8}\int_0^1 (x - 2x^3 + x^5)\,dx. = \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1$$

$$I = \frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{1}{8}\left(\frac{1}{6}\right) = \underline{\frac{1}{48}}.$$

13> Evaluate $\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dx\,dy = \underline{\frac{4}{35}}.$

14> Evaluate $\int_{-1}^1\int_0^z\int_{x-z}^{x+z} (x+y+z)\,dy\,dx\,dz = \underline{0}.$

15) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{(\sqrt{1-x^2-y^2})^2 - z^2}}$$

Let $K = \sqrt{1-x^2-y^2}$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{K} \frac{dz}{\sqrt{K^2-z^2}}\,dy\,dx .$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}\frac{z}{K}\Big]_{z=0}^{K} dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} [\sin^{-1}(1) - \sin^{-1}(0)]\,dy\,dx$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0\right) dy\,dx = \frac{\pi}{2}\int_0^1 y\Big|_0^{\sqrt{1-x^2}} dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\,dx$$

$$\left(\because \int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right).$$

$$I = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$I = \frac{\pi}{2}\left[0 + \frac{1}{2}(\sin^{-1}1 - \sin^{-1}0)\right] = \frac{\pi}{2}\left(\frac{1}{2}\left(\frac{\pi}{2}\right)\right)$$

$$I = \frac{\pi^2}{8}.$$

16) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}} = \frac{\pi^2 a^2}{8}.$

17> Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$.

$$I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y} . e^{z}\, dz\, dy\, dx.$$

$$I = \int_0^a \int_0^x e^{x+y}.\; e^{z}\Big|_0^{x+y}\, dy\, dx$$

$$I = \int_0^a \int_0^x e^{x+y}\,[e^{x+y} - 1]\, dy\, dx.$$

$$I = \int_0^a \int_0^x (e^{2x}.e^{2y} - e^{x}.e^{y})\, dy\, dx$$

$$I = \int_0^a \left(e^{2x}.\ \frac{e^{2y}}{2}\Big]_{y=0}^{x} - e^{x} e^{y}\Big]_{y=0}^{x} \right) dx.$$

$$I = \int_0^a \frac{e^{2x}}{2}(e^{2x}-1) - e^{x}(e^{x}-1)\; dx$$

$$I = \int_0^a \left(\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^{x} \right) dx.$$

$$I = \frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x}\Big]_0^a$$

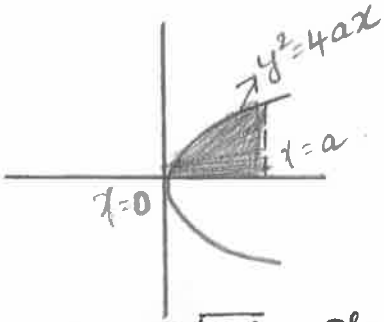
$$I = \left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} \right) - \left(\frac{1}{8} - \frac{3}{4} + 1 \right)$$

$$I = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8}$$

* Evaluation of double integrals by change of order of integration:-

1) Change the order of integration and evaluate

$$\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dy \, dx.$$



Sol^n: Let $I = \int_{x=0}^{a} \int_{y=0}^{2\sqrt{ax}} x^2 \, dy \, dx.$

$y = 2\sqrt{ax}$, $x = a$

$y^2 = 4ax$

$y^2 = 4a^2$

$\Rightarrow y = 2a.$

$$I = \int_{y=0}^{2a} \int_{x=\frac{y^2}{4a}}^{a} x^2 \, dx \, dy$$

$$I = \int_{y=0}^{2a} \left.\frac{x^3}{3}\right]_{y^2/4a}^{a} dy = \frac{1}{3}\int_{y=0} \left(a^3 - \frac{y^6}{64a^3}\right) dy$$

$$I = \frac{1}{3}\left\{a^3 y - \frac{y^7}{64a^3 \cdot 7}\right\}_0^{2a}$$

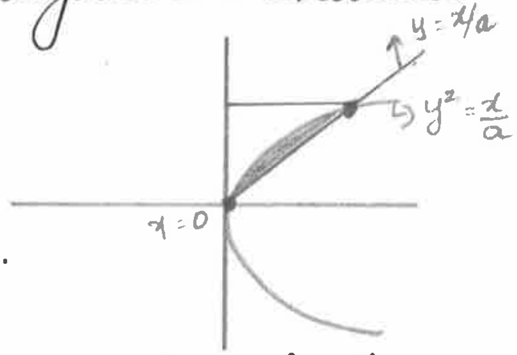
$$I = \frac{1}{3}\left\{2a^4 - \frac{1}{64a^3} \cdot \frac{1}{7} \cdot 128a^7\right\}$$

$$I = \frac{1}{3}\left\{2a^4 - \frac{2}{7}a^4\right\}$$

$$I = \frac{4a^4}{7}$$

2) By changing the order of integration evaluate

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx .$$



Sol^n:- Let $I = \int_{x=0}^{a} \int_{y=x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx .$

$y = \frac{x}{a}, \quad y^2 = \frac{x}{a}$

$\Rightarrow y = y^2$

$\Rightarrow y = 1.$

$$\therefore I = \int_{y=0}^{1} \int_{x=ay^2}^{ay} (x^2+y^2)\,dx\,dy$$

$$I = \int_{y=0}^{1} \left.\frac{x^3}{3} + xy^2\right]_{ay^2}^{ay} dy = \int_{y=0}^{1} \left[\frac{a^3y^3}{3} + ay^3\right] - \left[\frac{a^3y^6}{3} + ay^4\right] dy$$

$$I = \frac{a^3}{3}\left(\frac{y^4}{4}\right)_0^1 + a\left(\frac{y^4}{4}\right)_0^1 - \left(\frac{a^3y^7}{21}\right)_0^1 - \left(\frac{ay^5}{5}\right)_0^1$$

$$I = \frac{a^3}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + a\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$I = \frac{a^3}{\cancel{3}}\left(\frac{\cancel{3}}{28}\right) + a\left(\frac{1}{20}\right)$$

$$I = \frac{a^3}{28} + \frac{a}{20}.$$

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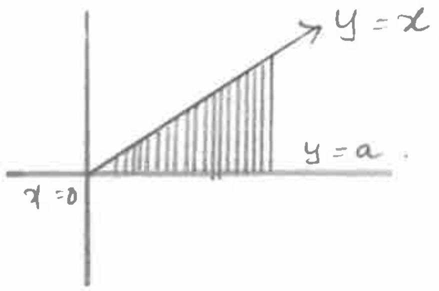
3) By changing the order of integration, show that

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \frac{\pi a}{4}.$$

Sol^n: Let $I=\int_{y=0}^{a}\int_{x=y}^{a}\frac{x}{x^2+y^2}\,dx\,dy$

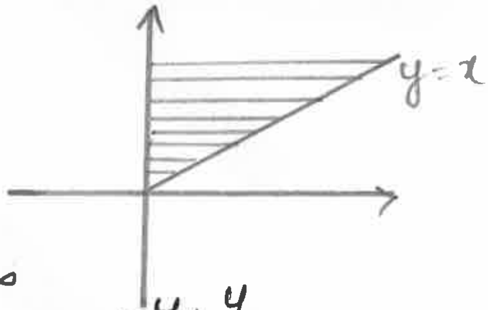


$$I=\int_{x=0}^{a}\int_{y=0}^{x}\frac{x}{x^2+y^2}\,dy\,dx$$

$$I=\int_{x=0}^{a}\cancel{x}\cdot\frac{1}{\cancel{x}}\tan^{-1}\frac{y}{x}\Big|_0^x\,dx=\int_0^a\frac{\pi}{4}\,dx=\frac{\pi}{4}(x)_0^a$$

$$I=\frac{\pi a}{4}.$$

4) Change the order of integration and evaluate integral $\int_0^\infty\int_x^\infty\frac{e^{-y}}{y}\,dy\cdot dx$.



Sol^n: Let $I=\int_{x=0}^{\infty}\int_{y=x}^{\infty}\frac{e^{-y}}{y}\,dy\cdot dx$

$$I=\int_{y=0}^{\infty}\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy=\int_{y=0}^{\infty}\frac{x\cdot e^{-y}}{y}\Big|_0^y\,dy$$

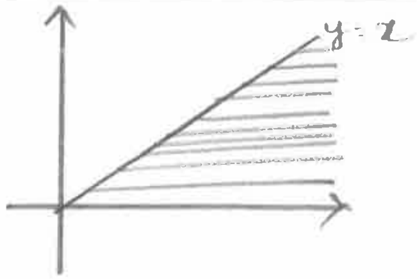
$$I=\int_{y=0}^{\infty}\cancel{y}\cdot\frac{e^{-y}}{\cancel{y}}\,dy=-e^{-y}\Big|_0^\infty=-(e^{-\infty}-e^{-0})$$

$$I=-(0-1)=\underline{1}.$$

5) By changing the order of integration. show that

$$\int_0^\infty\int_0^x xe^{-x^2/y}\,dy\,dx=\frac{1}{2}$$

Sol^n:- $I=\displaystyle\int_{x=0}^{\infty}\int_{y=0}^{x} xe^{-x^2/y}\,dy\,.dx$



$$I=\int_{y=0}^{\infty}\int_{x=y}^{\infty} xe^{-x^2/y}\,dx\,dy$$

Put $-x^2/y=t \Rightarrow -\frac{2x}{y}dx=dt$.

When $x=y$, $t=-y$

when $x=\infty$, $t=-\infty$

$$I=\int_{0}^{\infty}\int_{-y}^{-\infty} e^{t}\left(-\frac{y\,dt}{2}\right)dy$$

$$I=-\frac{1}{2}\int_{0}^{\infty}\{e^{t}\}_{-y}^{-\infty}\,y\,dy=-\frac{1}{2}\int_{0}^{\infty}\{0-e^{-y}\}\,y\,dy$$

$$I=\frac{1}{2}\int_{0}^{\infty}ye^{-y}\,dy=\frac{1}{2}\left[\underbrace{y(-e^{-y})\Big|_{0}^{\infty}}_{0}-\int_{0}^{\infty}-e^{-y}.1\,dy\right]$$

$$I=\frac{1}{2}\{-e^{-y}\}_{0}^{\infty}=-\frac{1}{2}(0-1)=\frac{1}{2}.$$

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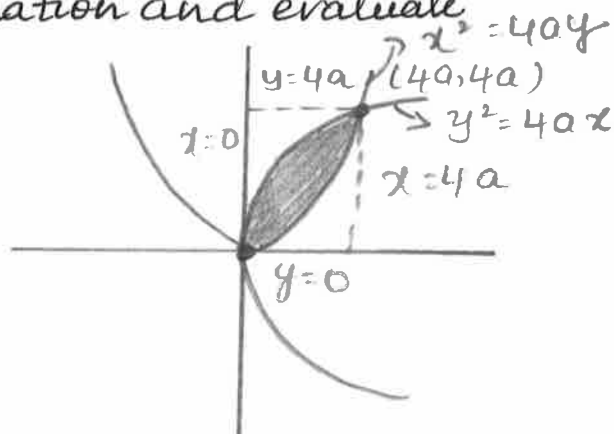
6) Change the order of integration and evaluate $\displaystyle\int_{0}^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$.



Sol^n:- Let $I=\displaystyle\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx.$

$$I=\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$y^2=4ax \Rightarrow x=\dfrac{y^2}{4a}$

$x^2=4ay \Rightarrow x=2\sqrt{ay}$

$$I=\int_{y=0}^{4a}[x]_{y^2/4a}^{2\sqrt{ay}}\,dy$$

$$I = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy = \left. 2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

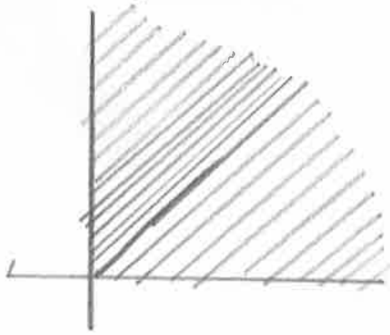
$$I = \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a} = \frac{32}{3}a^2 - \frac{16a^2}{3} = \underline{\frac{16}{3}a^2}$$

* Evaluation of double integrals by chainging into polar coordinates:—

1) Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates.

Soln:- We have $x = r\cos\theta$, $y = r\sin\theta$; $x^2+y^2 = r^2$

$$dx\,dy = r\,dr\,d\theta$$



Since x, y varies from 0 to ∞

r also varies from 0 to ∞.

In the first quadrant θ Varies from 0 to $\pi/2$.

Let $I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}.r\,dr\,d\theta$.

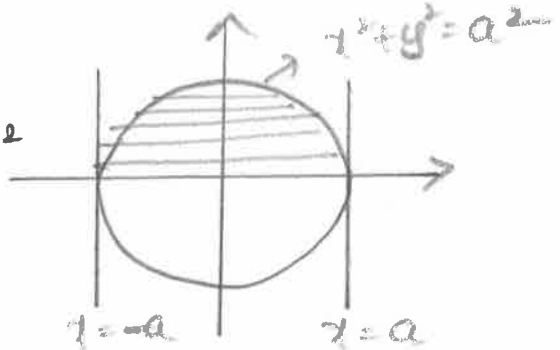
Put $r^2 = t$ $\therefore$ $2r\,dr = dt \Rightarrow r\,dr = \frac{dt}{2}$

$r \to 0$ to ∞ $\therefore$ $t \to 0$ to ∞.

$$I = \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}.\frac{dt}{2}.d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} \left. -e^{-t}\right]_0^{\infty} d\theta = +\frac{1}{2}\int_0^{\pi/2} d\theta.$$

$$I = +\frac{1}{2}\left.\theta\right]_0^{\pi/2} = \frac{1}{2}.\frac{\pi}{2} = \underline{\frac{\pi}{4}}.$$

2) Change the integral $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ into polars and hence evaluate.



Soln:- $\theta \to 0$ to π.

Let $x = r\cos\theta,\ y = r\sin\theta,\ x^2+y^2 = r^2$

$\Rightarrow\ a^2 = r^2 \Rightarrow a = r$.

$\therefore\ r \to 0$ to a.

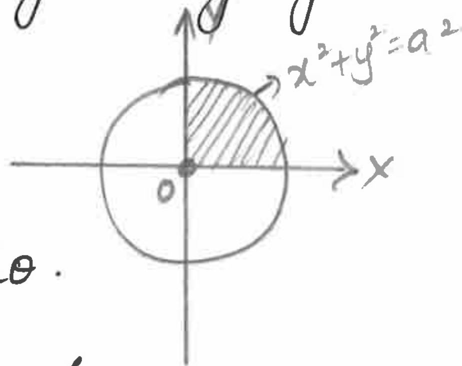
Also, $dx\,dy = r\,dr\,d\theta$.

$$I = \int_{\theta=0}^{\pi}\int_{r=0}^{a} r.r\,dr\,d\theta = \int_{\theta=0}^{\pi} \frac{r^3}{3}\,d\theta = \frac{a^3}{3}\int_0^{\pi} d\theta.$$

$$I = \frac{a^3}{3}\ \theta\Big]_0^{\pi} = \underline{\frac{\pi a^3}{3}}.$$

---

3) Evaluate $\iint xy\,dx\,dy$ over the positive quadrant bounded by the circle $x^2+y^2=a^2$ by changing into polar coordinates.



Soln:- Consider $\iint xy\,dx\,dy$.

$x = r\cos\theta$; $y = r\sin\theta$ $\quad \theta \to 0$ to $\pi/2$; $r \to 0$ to a.

$$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\underline{\cos\theta}\,.\,r\underline{\sin\theta}\ r.\,dr\,d\theta.$$

$$I = \int_{\theta=0}^{\pi/2} \frac{r^4}{4}\Big|_0^a \frac{\sin 2\theta}{2}\,d\theta = \frac{a^4}{8}\int_0^{\pi/2} \sin 2\theta\,d\theta.$$

Put $2\theta = t$

$2\,d\theta = dt$.

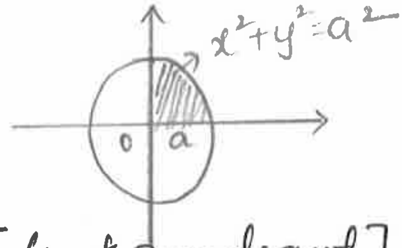
$d\theta = \frac{dt}{2}$. $\quad$ if $\theta = 0,\ t = 0$
$\theta = \pi/2,\ t = \pi$

$$I = \frac{a^4}{8}\int_0^{\pi} \sin t\,\frac{dt}{2} = \frac{a^4}{8}\int_0^{\pi/2} \sin t\,dt = \frac{a^4}{8}(-\cos t)_0^{\pi/2}$$

$$I = \frac{a^4}{8}(-0+1) = \frac{a^4}{8}.$$

4) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dy\, dx$ by transforming to polar coordinates.

Soln: We have $x = r\cos\theta$, $y = r\sin\theta$



$$x^2+y^2 = r^2.$$

$$r^2 = a^2 \Rightarrow r = a.$$

$\therefore$ $r \to 0$ to a ; $\theta = 0$ to $\pi/2$ [first Quadrant]

$$dx\,dy = r\,dr\,d\theta.$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2\sin^2\theta \cdot r \cdot r\,dr\,d\theta.$$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^4 \sin^2\theta\, dr\, d\theta = \int_{\theta=0}^{\pi/2} \frac{r^5}{5}\Big|_0^a \sin^2\theta\, d\theta$$

$$I = \frac{a^5}{5} \int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{a^5}{10}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2}$$

$$I = \frac{a^5}{10}\left[\frac{\pi}{2} - \frac{1}{2}(\cancel{\sin\pi - \sin 0})\right]$$

$$I = \frac{\pi a^5}{20}.$$

5) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{y^2+x^2}\, dx\, dy$ by changing into polars.

Soln:- $I = \dfrac{a^4}{4}.$

* Applications to find Area and volume by double integral.

* $\iint_R dx\,dy$ = Area of the region R in the cartesian form.

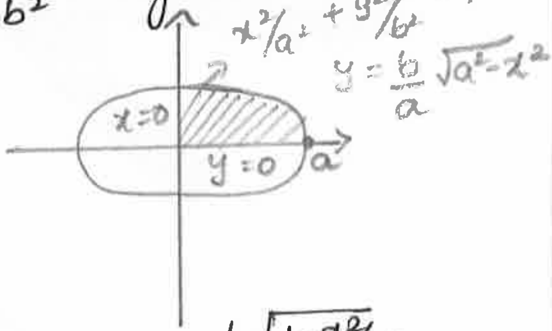
* $\iint_R r\,dr\,d\theta$ = Area of the region R in the polar form.

* $\iint_R z\,dy\,dx$ = Volume $\rightarrow$ Cartesian form.

* $\iint_R 2\pi r^2 \sin\theta\,dr\,d\theta \rightarrow$ Polar form.

Problems :—

1) Find the area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by double integration.



Sol^n :- Area (A) $=\iint_R dx\,dy$

The Area in the first Quadrant.

x varies from 0 to a

y varies from 0 to $b\sqrt{1-x^2/a^2}$

Required area $= \int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-x^2/a^2}} dx\,dy = \int_{x=0}^{a} y\Big|_0^{b\sqrt{1-x^2/a^2}} dx$

$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$ $\quad \left(\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)$

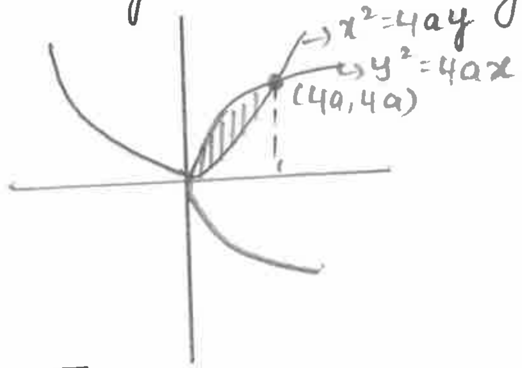
$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= \frac{b}{a}\left[\left(0+\frac{a^2}{2}\sin^{-1}(1)\right)-(0+0)\right]$

$= \frac{\pi ab}{4}.$

$\therefore$ Total Area of the ellipse $= 4\left(\frac{\pi ab}{4}\right) = \pi ab$ Sq units.

2) Find the area between the parabolas $y^2=4ax$ & $x^2=4ay$.



Sol^n :- Solving equations $y^2=4ax$ & $x^2=4ay$ for x, we get the points of intersection as $(0,0)$ & $(4a,4a)$.

$\therefore$ x varies from 0 to $4a$

y varies from $y=x^2/4a$ to $2\sqrt{ax}$

Required Area $= \int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dx\,dy$

$$= \int_{x=0}^{4a} y\Big]_{x^2/4a}^{2\sqrt{ax}} dx = \int_0^{4a}\left(2a^{1/2}x^{1/2} - \frac{x^2}{4a}\right)dx$$

$$= \frac{2a^{1/2}x^{3/2}}{3/2} - \frac{x^3}{4a\times 3}\Big|_0^{4a} = \frac{32a^2}{3} - \frac{16}{3}a^2$$

Required Area $= \frac{16}{3}a^2$ Sq.units.

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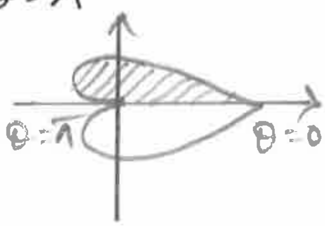
3) Find the area enclosed by the curve cardioid $r=a(1+\cos\theta)$ between $\theta=0$ and $\theta=\pi$



Sol^n:- θ varies from 0 to π

r varies from 0 to $a(1+\cos\theta)$

Required area $= \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,d\theta\,dr$.

$$= \int_{\theta=0}^{\pi} \frac{r^2}{2}\Big]_0^{a(1+\cos\theta)} d\theta = \int_0^{\pi}\frac{(a(1+\cos\theta))^2}{2}d\theta$$

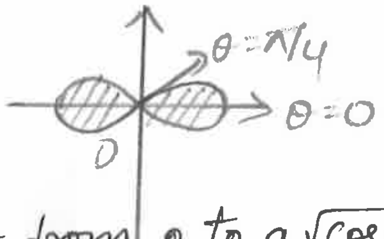
$$= \frac{a^2}{2}\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta = \frac{a^2}{2}\int_0^{\pi}\left(1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right)d\theta$$

$$= \frac{a^2}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

$$= \frac{3a^2\pi}{4} \text{ Sq units.}$$

4) Find the area of the lemniscate $r^2 = a^2\cos 2\theta$.

Sol^n:- Here the required area is four times the area determined by the curve.



θ varies from 0 to $\pi/4$; r varies from 0 to $a\sqrt{\cos 2\theta}$.

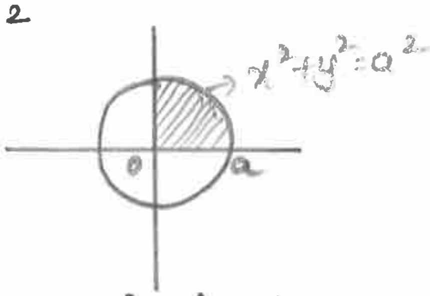
$$\therefore \text{Area} = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\, d\theta\, dr = 4\int_{\theta=0}^{\pi/4} \frac{r^2}{2}\Big|_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$\text{Area} = 4\cdot\frac{a^2}{2}\int_0^{\pi/4} \cos 2\theta\, d\theta = 2a^2 \frac{\sin 2\theta}{2}\Big]_0^{\pi/4} = 2a^2\left(\frac{1}{2}\right)$$

$$\text{Area} = a^2 \text{ Squnits.}$$

5) Find the area of a circle $x^2+y^2=a^2$

Sol^n:- Area $= \iint dx\,dy$



$$\text{Area} = 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} dy\, dx$$

$$\text{Area} = 4\int_{x=0}^{a} y\Big|_0^{\sqrt{a^2-x^2}} dx$$

$x^2+y^2=a^2$
$y^2 = a^2-x^2$
$y = \sqrt{a^2-x^2}$

$$= 4\int_{x=0}^{a}\sqrt{a^2-x^2}\, dx = 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right] = 4\,\frac{a^2\pi}{4}$$

$$\text{Area} = \pi a^2 \text{ Squnits.}$$

* Volumes as double integrals :-

1) Using double integrals find the volume of the sphere $x^2+y^2+z^2=a^2$.

Soln:- $V=\iint z\,dx\,dy$

$$V=8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dy\,dx$$

Put $k^2=a^2-x^2 \Rightarrow k=\sqrt{a^2-x^2}$

$$V=8\int_0^a\int_0^k\sqrt{k^2-y^2}\,dy\,dx$$

$$V=8\int_0^a\left\{\frac{y}{2}\sqrt{k^2-y^2}+\frac{k^2}{2}\sin^{-1}(y/k)\right\}_0^k dx$$

$$V=8\int_0^a\frac{k^2}{2}\cdot\frac{\pi}{2}\,dx=\frac{8\pi}{4}\int_0^a k^2\,dx.$$

$$V=\frac{8\pi}{4}\int_0^a(a^2-x^2)\,dx=\frac{8\pi}{4}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$V=\frac{8\pi}{4}\left\{a^3-\frac{a^3}{3}\right\}=\frac{8\pi}{4}\cdot\frac{2a^3}{3}$$

$$V=\frac{4}{3}\pi a^3 \text{ cubic units.}$$

2) Using double integrals find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Soln:- $V=\iint z\,dx\,dy$.

But $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2} \Rightarrow z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$V = 8\int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx.$$

put

$$k = \frac{b}{a}\sqrt{a^2-x^2} \;;\; k^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$\frac{k^2}{b^2} = 1-\frac{x^2}{a^2}.$$

$$V = 8\int_0^a\int_0^k c\sqrt{\frac{k^2}{b^2}-\frac{y^2}{b^2}}\,dy\,dx.$$

$$V = \frac{8c}{b}\int_0^a\int_0^k\sqrt{k^2-y^2}\,dy\,dx.$$

$$V = \frac{8c}{b}\int_0^a\left\{\frac{y}{2}\sqrt{k^2-y^2}+\frac{k^2}{2}\sin^{-1}(y/k)\right\}_0^k dx.$$

$$V = \frac{8c}{b}\int_0^a\frac{k^2}{2}\cdot\frac{\pi}{2}\,dx = \frac{2c\pi}{b}\int_0^a k^2\,dx = \frac{2c\pi}{b}\int_0^a\frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$V = \frac{2c\pi b}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a = \frac{2bc\pi}{a^2}\left[a^3-\frac{a^3}{3}\right]$$

$$V = \frac{2bc\pi}{a^2}\left[\frac{2a^3}{3}\right] = \frac{4}{3}abc\pi$$

$$\therefore\; V = \frac{4}{3}\pi abc \text{ cubic units.}$$

3) Find the volume of the tetrahedron $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Soln: $V = \iint z\,dx\,dy$

But $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$$V=\int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx.$$

$$V=c\int_{x=0}^{a}\left(y-\frac{x}{a}y-\frac{1}{b}\frac{y^2}{2}\right)_0^{b(1-x/a)}dx.$$

$$V=c\int_{x=0}^{a} b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{1}{2b}b^2(1-x/a)^2\,dx.$$

$$V=c\int_{x=0}^{a}\left\{b-\frac{bx}{a}-\frac{x}{a}b+\frac{bx^2}{a^2}-\frac{b}{2}\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)\right\}dx$$

$$V=c\int_{x=0}^{a}\left(\frac{b}{2}-\frac{2bx}{a}+\frac{bx^2}{2a^2}+\frac{bx}{a}\right)dx$$

$$V=c\int_0^a\left(\frac{b}{2}-\frac{bx}{a}+\frac{bx^2}{2a^2}\right)dx$$

$$V=c\left\{\frac{b}{2}.x-\frac{bx^2}{a2}+\frac{b}{2a^2}\cdot\frac{x^3}{3}\right\}_0^a$$

$$V=c\left\{\frac{ab}{2}-\frac{ab}{2}+\frac{ab}{6}\right\}$$

$$V=\frac{abc}{6}\text{ cubic units.}$$

4) A pyramid is bounded by three co-ordinate planes and the plane $x+2y+3z=6$. Compute the volume by double integration.

Soln:- Let $x+2y+3z=6$ or $\frac{x}{6}+\frac{y}{3}+\frac{z}{2}=1$

$$\Rightarrow z=2\left(1-\frac{x}{6}-\frac{y}{3}\right).$$

If $z=0$, $\frac{x}{6}+\frac{y}{3}=1 \Rightarrow y=3\left(1-\frac{x}{6}\right)$

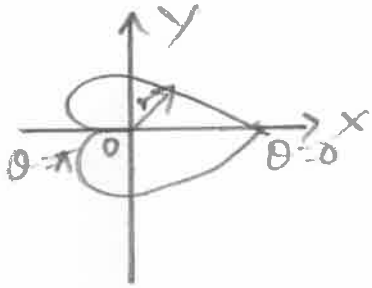
If $z=0$, $y=0$ then $x=6$

$$V=\iint z\,dx\,dy$$

$$V=\int_{x=0}^{6}\int_{y=0}^{3(1-x/6)} 2\left(1-\frac{x}{6}-\frac{y}{3}\right)dy\,dx = \underline{6 \text{ cubic units}}.$$

5) Find the volume generated by the revolution of the cardiod $r=a(1+\cos\theta)$ about the initial line.

Solⁿ:- Required volume =



$$\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} 2\pi r^2\sin\theta\,d\theta\,dr$$

$$= 2\pi\int_0^{\pi}\left.\frac{r^3}{3}\right|_0^{a(1+\cos\theta)}\sin\theta\,d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta.$$

Put $1+\cos\theta=t$

$-\sin\theta\,d\theta=dt$

$\theta=0,\ t=2;\ \theta=\pi,\ t=0$.

$$V=\frac{2}{3}\pi a^3\int_2^0 -t^3.\,dt = \frac{2\pi a^3}{3}\int_0^2 t^3\,dt = \frac{2\pi a^3}{3}\cdot\left.\frac{t^4}{4}\right|_0^2$$

$$\underline{V=\frac{8\pi a^3}{3}\text{ cubic units.}}$$

* Beta and Gamma functions :-

$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx}$ $(m, n > 0)$ is called Beta function.

$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx}$ $(n > 0)$ is called Gamma function.

* Properties of Beta and Gamma functions :

1. $\beta(m,n) = \beta(n,m)$.

2. $\Gamma(n+1) = n\,\Gamma n$

3. $\Gamma(n+1) = n!$ for a positive integer n.

4. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

5. $\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt = \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt$.

** 6. $\Gamma\frac{1}{2} = \sqrt{\pi}$ (S.T $\Gamma\frac{1}{2} = \sqrt{\pi}$)

Proof: By def^n, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$.

Put $x = t^2 \Rightarrow dx = 2t\,dt$.

$$\Gamma n = \int_0^\infty e^{-t^2} (t^2)^{n-1} 2t\,dt$$

$$\Gamma n = 2\int_0^\infty e^{-t^2} t^{2n-1} dt$$

Put $n = \frac{1}{2}$, $\Gamma\frac{1}{2} = 2\int_0^\infty e^{-t^2} dt$

Consider $\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \Gamma\frac{1}{2}\,\Gamma\frac{1}{2}$.

$$= \left\{2\int_0^\infty e^{-x^2}dx\right\}\left\{2\int_0^\infty e^{-y^2}dy\right\}$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta.$$

Put $r^2 = t$

$2r\,dr = dt$

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-t}\,\frac{dt}{2}\,d\theta$$

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 2\int_0^{\pi/2} e^{-t}\Big|_0^\infty d\theta = 2\int_0^{\pi/2} 1.\,d\theta$$

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 2.\,\theta\Big|_0^{\pi/2} = \cancel{2}.\frac{\pi}{\cancel{2}} = \pi$$

$$\therefore \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}.}$$

---

* Relation between Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

Proof: By definition,

$$\Gamma n = \int_0^\infty e^{-x}\,x^{n-1}dx$$

put $x = t^2$

$dx = 2t\,dt$

$$\Gamma n = \int_0^\infty e^{-t^2}(t^2)^{n-1}\,2t\,dt$$

$$\Gamma n = 2\int_0^\infty e^{-t^2}\,t^{2n-1}\,dt$$

Consider,

$$\Gamma m \,\Gamma n = \left\{ 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \right\} \left\{ 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy \right\}$$

$$\Gamma m \,\Gamma n = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy$$

Converting into polar coordinates

$$\Gamma m \,\Gamma n = 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} \cdot r\,dr\,d\theta$$

$$\Gamma m \,\Gamma n = \left\{ 2\int_0^\infty e^{-r^2} (r^2)^{m+n-1}\, r\,dr \right\} \left\{ 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta \right\}$$

$$\Gamma m \,\Gamma n = \left\{ 2\int_0^\infty e^{-r^2} (r^2)^{m+n-1}\, r\,dr \right\} \beta(m,n)$$

put $r^2 = u$

$2r\,dr = du$

$r\,dr = \frac{du}{2}$

$$\Gamma m \,\Gamma n = \left\{ 2\int_0^\infty e^{-u} u^{(m+n)-1} \frac{du}{2} \right\} \beta(m,n)$$

$$\Gamma m \,\Gamma n = \Gamma(m+n)\, \beta(m,n)$$

$$\therefore \boxed{\beta(m,n) = \frac{\Gamma m \,\Gamma n}{\Gamma m+n}}$$

* Prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$. Hence deduce that $\int_0^{\pi/2} \sin^p\theta \, d\theta = \int_0^{\pi/2} \cos^p\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$

Proof :- By definition,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta \, d\theta$

when $x = 0, \ \theta = 0$

$x = 1, \ \theta = \pi/2$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} \, 2\sin\theta\cos\theta \, d\theta.$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

put $2m-1 = p$, $2n-1 = q$

$m = \frac{p+1}{2}$ $\quad n = \frac{q+1}{2}$

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = 2\int_0^{\pi/2} \sin^{2\left(\frac{p+1}{2}\right)-1}\theta \cos^{2\left(\frac{q+1}{2}\right)-1}\theta \, d\theta$$

$$\boxed{\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta.} \rightarrow (1)$$

Put $q = 0$ in (1)

$$\int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \cdot \Gamma\frac{1}{2}}{2\Gamma\frac{p+2}{2}} = \frac{\Gamma\frac{p+1}{2} \cdot \sqrt{\pi}}{2\Gamma\frac{p+2}{2}} = \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+2}{2}} \cdot \frac{\sqrt{\pi}}{2}$$

Replace p by 0 and q by p in (1).

$$\therefore \int_0^{\pi/2} \cos^p\theta \, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+2}{2}}$$

---

* **NOTE:**

*1) $\Gamma n \, \Gamma 1-n = \dfrac{\pi}{\sin n\pi}$

2) $\Gamma\frac{3}{2} = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$ $\quad (\because \Gamma n+1 = n\Gamma n)$

3) $\Gamma\frac{11}{2} = \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \dfrac{945\sqrt{\pi}}{32}$.

4) $\Gamma -\frac{1}{2}$ $\quad \left(\because \Gamma n+1 = n\Gamma n \right.$
$\left. \Gamma n = \dfrac{\Gamma n+1}{n}\right)$

$$\Gamma -\frac{1}{2} = \frac{\Gamma\frac{1}{2}}{-\frac{1}{2}} = \frac{\sqrt{\pi}}{-\frac{1}{2}} = -2\sqrt{\pi}.$$

5) $\Gamma -\frac{5}{2} = \dfrac{1}{-\frac{5}{2} \cdot -\frac{3}{2} \cdot -\frac{1}{2}} \cdot \sqrt{\pi} = \dfrac{\sqrt{\pi}}{-\frac{15}{8}} = -\frac{8}{15}\sqrt{\pi}$.

6) $\dfrac{6\,\Gamma 8/3}{5\,\Gamma 2/3} = \dfrac{6 \cdot \frac{5}{3} \cdot \frac{2}{3} \cancel{\Gamma\frac{2}{3}}}{5\cancel{\Gamma\frac{2}{3}}} = \dfrac{20}{15} = \dfrac{4}{3}$.

7) $\beta(5,6) = \dfrac{\Gamma 5 \, \Gamma 6}{\Gamma 11} = \dfrac{\Gamma(4+1)\,\Gamma(5+1)}{\Gamma(10+1)} = \dfrac{4! \cdot 5!}{10!}$

8) $\beta\left(\frac{9}{2}, \frac{7}{2}\right) = \dfrac{\Gamma\frac{9}{2}\,\Gamma\frac{7}{2}}{\Gamma\frac{16}{2}} = \dfrac{1575\pi}{128}$.

Problems:—

1) Evaluate $\int_0^\infty x^3 e^{-4x^2} dx$

Sol^n:- Put $t = 4x^2 \Rightarrow x^2 = \frac{t}{4}$; $x = \frac{\sqrt{t}}{2}$

$dx = \frac{1}{2} \cdot \frac{1}{2\sqrt{t}} dt = \frac{1}{4\sqrt{t}} dt$

$$I = \int_0^\infty \left(\frac{\sqrt{t}}{2}\right)^3 e^{-t} \cdot \frac{1}{4\sqrt{t}} dt = \frac{1}{32} \int_0^\infty e^{-t} \cdot t^{3/2 - 1/2} dt$$

$$I = \frac{1}{32} \int_0^\infty e^{-t} t \, dt = \frac{1}{32} \int_0^\infty e^{-t} \cdot t^{2-1} dt$$

$$I = \frac{1}{32} \Gamma 2 = \frac{1}{32} \Gamma 1+1 = \frac{1}{32} \Gamma 1$$

$$\underline{I = \frac{1}{32}.}$$

2) S.T $\int_0^\infty \sqrt{x} e^{-x^3} dx = \frac{\sqrt{\pi}}{3}$.

Sol^n:- Let $I = \int_0^\infty \sqrt{x} e^{-x^3} dx$.

Put $t = x^3 \Rightarrow x = \sqrt[3]{t}$

$dx = t^{-2/3} \times \frac{1}{3} dt$

$$I = \int_0^\infty \sqrt{t^{1/3}} \cdot e^{-t} \cdot \frac{1}{3} t^{-2/3} dt$$

$$I = \frac{1}{3} \int_0^\infty t^{1/6 - 2/3} \cdot e^{-t} dt = \frac{1}{3} \int_0^\infty e^{-t} t^{-1/2} dt.$$

$$I = \frac{1}{3} \int_0^\infty e^{-t} t^{1/2 - 1} dt \qquad \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$I = \frac{1}{3} \Gamma \frac{1}{2} = \underline{\frac{\sqrt{\pi}}{3}}.$$

3) Evaluate $\int_0^\infty e^{-4x} x^{3/2}\, dx$.

Soln:—

Put $t = 4x$, $x = \frac{t}{4}$

$dx = \frac{1}{4} dt$.

$$I = \int_0^\infty e^{-t}\left(\frac{t}{4}\right)^{3/2} \frac{1}{4}\, dt = \frac{1}{4}\int_0^\infty e^{-t} \frac{\sqrt{t^3}}{8}\, dt.$$

$$I = \frac{1}{32}\int_0^\infty e^{-t} t^{3/2}\, dt = \frac{1}{32}\int_0^\infty e^{-t} t^{5/2-1}\, dt.$$

$$I = \frac{1}{32}\Gamma\frac{5}{2} = \frac{1}{32}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{32}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$\underline{I = \frac{3\sqrt{\pi}}{128}}.$$

4) Evaluate $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx \int_0^\infty x^2 e^{-x^4}\, dx$.

Soln: $I_1 = \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx$.

Put $x^2 = t \Rightarrow x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}} dt$.

$$I_1 = \int_0^\infty \frac{e^{-t}}{\sqrt{t^{1/2}}}\cdot\frac{1}{2\sqrt{t}}\, dt.$$

$$I_1 = \frac{1}{2}\int_0^\infty e^{-t}\cdot t^{-3/4}\, dt$$

$$I_1 = \frac{1}{2}\int_0^\infty e^{-t} t^{1/4-1}\, dt$$

$$I_1 = \frac{1}{2}\Gamma\frac{1}{4}.$$

$I_2 = \int_0^\infty x^2 e^{-x^4}\, dx$.

Put $x^4 = t \Rightarrow x = t^{1/4}$

$dx = \frac{1}{4} t^{-3/4} dt$

$$I_2 = \int_0^\infty t^{1/2} e^{-t} \frac{1}{4} t^{-3/4}\, dt.$$

$$I_2 = \frac{1}{4}\int_0^\infty e^{-t} t^{-1/4}\, dt$$

$$I_2 = \frac{1}{4}\int_0^\infty e^{-t} t^{3/4-1}\, dt$$

$$I_2 = \frac{1}{4}\Gamma\frac{3}{4}$$

Consider $I_1\cdot I_2 = \frac{1}{2}\Gamma\frac{1}{4}\cdot\frac{1}{4}\Gamma\frac{3}{4} = \frac{1}{8}\Gamma\frac{1}{4}\Gamma\frac{3}{4} = \frac{1}{8}\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)$

$$= \frac{1}{8}\cdot\frac{\pi}{\sin\left(\frac{1}{4}\right)\pi} = \frac{1}{8}\cdot\frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$\left(\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right).$$

$$= \underline{\frac{\pi}{4\sqrt{2}}}.$$

5) Prove that $\int_0^\infty e^{-x^4}dx \int_0^\infty \sqrt{x}\, e^{-x^2}dx = \frac{\pi}{4\sqrt{2}}$.

Sol^n: $I_1 = \int_0^\infty e^{-x^4}dx$

Put $x^4 = t$

$x = t^{1/4}$

$dx = \frac{1}{4}t^{-3/4}dt$.

$I_1 = \int_0^\infty e^{-t}\cdot \frac{1}{4}t^{-3/4}dt$

$I_1 = \frac{1}{4}\int_0^\infty e^{-t}t^{1/4-1}\,dt$

$I_1 = \frac{1}{4}\Gamma\frac{1}{4}$.

$I_2 = \int_0^\infty \sqrt{x}\, e^{-x^2}dx$

Put $x^2 = t$

$x = t^{1/2}$

$dx = \frac{1}{2}t^{-1/2}dt$.

$I_2 = \int_0^\infty \sqrt{t^{1/2}}\, e^{-t}\,\frac{1}{2}t^{-1/2}dt$

$I_2 = \frac{1}{2}\int_0^\infty e^{-t}\cdot t^{1/4}\cdot t^{-1/2}dt$

$I_2 = \frac{1}{2}\int_0^\infty e^{-t}\cdot t^{-1/4}dt$

$I_2 = \frac{1}{2}\int_0^\infty e^{-t}\cdot t^{3/4-1}dt$

$I_2 = \frac{1}{2}\Gamma\frac{3}{4}$.

$I_1\cdot I_2 = \frac{1}{4}\Gamma\frac{1}{4}\cdot\frac{1}{2}\Gamma\frac{3}{4} = \frac{1}{8}\Gamma\frac{1}{4}\Gamma\frac{3}{4}$

$= \frac{1}{8}\Gamma\frac{1}{4}\cdot\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{8}\cdot\frac{\pi}{\sin\frac{\pi}{4}} =$

$= \underline{\pi/4\sqrt{2}}$.

6) Prove that $\int_0^\infty x.e^{-x^8}dx \int_0^\infty x^2.e^{-x^4}dx = \frac{\pi}{16\sqrt{2}}$.

Sol^n:- $I_1 = \int_0^\infty x.e^{-x^8}dx$

Put $x^8 = t$

$x = t^{1/8}$

$dx = \frac{1}{8}t^{-7/8}dt$.

$I_1 = \int_0^\infty t^{1/8}e^{-t}\cdot\frac{1}{8}t^{-7/8}dt$

$I_1 = \frac{1}{8}\int_0^\infty e^{-t}t^{-3/4}\,dt$

$I_2 = \int_0^\infty x^2e^{-x^4}dx$

Put $x^4 = t$

$x = t^{1/4}$

$dx = \frac{1}{4}t^{-3/4}dt$

$I_2 = \int_0^\infty t^{1/2}e^{-t}\cdot\frac{1}{4}t^{-3/4}dt$

$I_2 = \frac{1}{4}\int_0^\infty e^{-t}t^{-1/4}dt$

$$I_1 = \frac{1}{8}\int_0^{\infty} e^{-t}\, t^{1/4-1}\,dt \qquad I_2 = \frac{1}{4}\int_0^{\infty} e^{-t}\, t^{3/4-1}\,dt$$

$$I_1 = \frac{1}{8}\,\Gamma\frac{1}{4} \qquad I_2 = \frac{1}{4}\,\Gamma\frac{3}{4}.$$

$$I_1 \cdot I_2 = \frac{1}{8}\Gamma\frac{1}{4} \cdot \frac{1}{4}\Gamma\frac{3}{4} = \frac{1}{32}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$

$$= \frac{1}{32}\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{32}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{32}\cdot\frac{\pi}{1/\sqrt{2}}$$

$$= \underline{\frac{\pi}{16\sqrt{2}}}.$$

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7) Evaluate $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$.

Soln:- Let $I = \int_0^{\pi/2}\frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\,d\theta = \int_0^{\pi/2}\sin^{1/2}\theta\,\cos^{-1/2}\theta\,d\theta$.

We know that $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

$\therefore\ p = 1/2,\ q = -1/2.$

$$I = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2\Gamma 1} = \frac{\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)}{2} = \frac{1}{2}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{2}\cdot\frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$\underline{I = \frac{\pi}{\sqrt{2}}}$$

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8) Evaluate $\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta$.

Soln: Let $I = \int_0^{\pi/2}\frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\,d\theta = \int_0^{\pi/2}\cos^{1/2}\theta.\sin^{-1/2}\theta\,d\theta.$

$$p = -1/2\ ;\ q = 1/2$$

$$I = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\Gamma 1} = \frac{\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)}{2} = \frac{1}{2}\,\frac{\pi}{\sin\pi/4} = \frac{1}{2}\cdot\frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \underline{\pi/\sqrt{2}}$$

9) Prove that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$.

Soln :- $I = \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta$

We know that $\int_0^{\pi/2} \sin^p\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+2}{2}} \cdot \frac{\sqrt{\pi}}{2}$

$$I = \frac{\cancel{\Gamma\frac{3}{4}}}{\Gamma\frac{5}{4}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\frac{1}{4}}{\cancel{\Gamma\frac{3}{4}}} \frac{\sqrt{\pi}}{2}$$

$$I = \frac{\pi}{4} \frac{\Gamma\frac{1}{4}}{\Gamma 5/4} = \frac{\pi}{4} \cdot \frac{\Gamma\frac{1}{4}}{\Gamma\frac{1}{4}+1} = \frac{\pi}{4} \frac{\cancel{\Gamma\frac{1}{4}}}{\frac{1}{4}\cancel{\Gamma\frac{1}{4}}}$$

$$I = \pi$$

10) Evaluate $\int_0^a x^4 \sqrt{a^2 - x^2}\, dx$

Soln: put $x^2 = a^2\sin^2\theta \Rightarrow x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$.

When $x = 0$, $\theta = 0$; $x = a$; $\theta = \pi/2$

$$\therefore \int_0^{\pi/2} a^4 \sin^4\theta \underline{\sqrt{a^2 - a^2\sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} a^4 \sin^4\theta\, \underline{a\cos\theta}\, a\cos\theta\, d\theta$$

$$= a^6 \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta \quad \text{Here } p = 4, q = 2$$

$$= a^6 \cdot \frac{\Gamma\frac{5}{2}\Gamma\frac{3}{2}}{2\Gamma\frac{8}{2}} = \frac{a^6 \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\frac{1}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}{2\Gamma 4.}$$

$$= \frac{a^6 \frac{3}{4} \cdot \frac{1}{2} \pi}{2\Gamma 3+1} = \frac{\pi a^6 \cdot 3}{8 \cdot 2 \cdot 3!} = \frac{\pi a^6}{32.}$$

$(\Gamma n+1 = n!)$

11) Evaluate $\int_0^{\infty} \frac{x^4}{1+x^6}\, dx$

Sol^n:

Put $x^6 = \tan^2\theta$.

$x = \tan^{1/3}\theta \Rightarrow dx = \frac{1}{3}\tan^{-2/3}\theta \sec^2\theta \, d\theta$.

when $x=0$; $\theta=0$; $x=\infty$; $\theta=\pi/2$.

$$= \int_0^{\pi/2} \frac{\tan^{4/3}\theta}{\cancel{1+\tan^2\theta}} \cdot \frac{1}{3}\tan^{-2/3}\theta \, \cancel{\sec^2\theta}\, d\theta.$$

$$= \frac{1}{3}\int_0^{\pi/2} \tan^{2/3}\theta \, d\theta = \frac{1}{3}\int_0^{\pi/2} \frac{\sin^{2/3}\theta}{\cos^{2/3}\theta}\, d\theta = \frac{1}{3}\int_0^{\pi/2} \sin^{2/3}\theta \cos^{-2/3}\theta \, d\theta$$

$$= \frac{1}{3}\, \frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{6}}{2\Gamma 1} = \frac{1}{6}\cdot \Gamma\frac{1}{6}\,\Gamma\left(1-\frac{1}{6}\right) = \frac{1}{6}\cdot\frac{\pi}{\sin\frac{\pi}{6}}$$

$$I = \frac{1}{6}\cdot\frac{\pi}{1/2} = \frac{\pi}{3}.$$

12) S.T $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

Sol^n:- put $x^4 = \tan^2\theta \Rightarrow x = \tan^{1/2}\theta \Rightarrow dx = \frac{1}{2}\tan^{-1/2}\theta \sec^2\theta\, d\theta$

when $x=0$; $\theta=0$: $x=\infty$, $\theta=\pi/2$

$$= \int_0^{\pi/2} \frac{1}{2}\cdot \frac{\tan^{-1/2}\theta \cdot \cancel{\sec^2\theta}\cdot d\theta}{\cancel{1+\tan^2\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta}\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$$= \frac{1}{2}\, \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\Gamma 1} = \frac{1}{4}\cdot \Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{4}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4}\cdot\frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{4}\cdot\sqrt{2} = \frac{\pi}{2\sqrt{2}}$$

13) ST $\int_0^\infty \frac{x}{1+x^6}\,dx = \frac{\pi}{3\sqrt{3}}$.

14) Evaluate $\int_0^1 x^{3/2}(1-x)^{1/2}\,dx$

Sol^n:- $I = \int_0^1 x^{5/2-1}(1-x)^{3/2-1}\,dx$.

$$I = \beta\left(5/2, 3/2\right) = \frac{\Gamma 5/2\ \Gamma 3/2}{\Gamma \frac{5}{2}+\frac{3}{2}} = \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{3\times 2\times 1}$$

$$I = \frac{\pi}{16}.$$