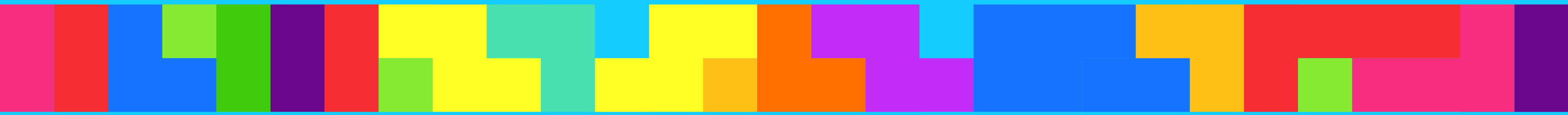

HERMAN TULLEKEN

POLYOMINOES 3.6

SHAPES AND TILINGS.

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Preface

On 8 September 2017 I set out to discover as much about polyominoes as I could. I thought it would be nice to collect what I found in a few essays, and those grew into this book.

I tried to keep the mathematics simple; you will not need to know much graph theory, combinatorics, or other machinery to understand the book. At times this means I cover only a very concrete special case of a more general (and maybe more beautiful) result, and some proofs in their naive disguise may be a bit clunky; this is the trade-off. I *do* try to give as many references to the bigger theory as possible.

Current Status. This book is a work in progress. Some chapters are incomplete or may have more errors than others. If you find any errors, please let me know[1].

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Acknowledgments. I want to thank Justin Southey for reading a draft of the second chapter. He made many suggestions that improved it greatly. I also want to thank several people that brought additional references to my attention: Viorel Nitica, Jonathan Lenchner, and Marcus Garvie. I also want to thank several people that sent me corrections: Peter Kagey, John Mason, James Stein, and Nicolau Saldanha.

Problems. There are three types of problems:

- Exercises that are interesting ideas or problems I discovered or thought of as I worked through the topics that could be helpful in building intuition and understanding, but not important enough for the main text. These are marked with a dagger†.
- Questions that occurred to me that I have not solved. Sometimes these are ideas for further investigation on which I did not spend any time; in other cases these are problems I could not solve even after some effort. These are marked with an asterisk*.

- Open questions from the literature that usually has been open for some time; most of these can be considered very difficult. These are marked with a double asterisk**.

Unmarked problems are not classified.

NOTATION. Numbers in orange like **A000105** refer to integer sequences in the *Online Encyclopedia of Integer Sequences* (Sloane, 2007). Clicking on them takes you to the relevant entry. These numbers also appear in the index (under A).

Notation	Meaning	Page
$\|R\|$	The area of R, the number of cells of R	12
R^*	Compact subregion of R	68
0_R, 1_R	Minimum and maximum tilings of R	90
$\ominus$	Cylinder deletion	51
$\vee$ $\wedge$	Join and meet operations of domino tilings.	87
$\#_{\mathcal{T}}R, \#R$	The number of tilings of a region R by tiles from the $\mathcal{T}$	106
$\mathcal{T}^+$	The tileset with augmented with the monomino.	114
$A(n)$	Aztec diamond	93
All, **None**, **Rot**	Polyomino symmetry classes	16
$B(R), W(R)$	The set of white (black) cells of R	41
$B(a_1 \cdot a_2 \cdots a_n)$	Bar graph	20
$C(P)$, $C(m,n)$	Chromatic number of P, chromatic number of $R(m,n)$	161
$\mathrm{clum}_{\mathcal{T}}(R)$	Clumsiness of R tiled by $\mathcal{T}$	346
$\mathrm{CO}_P(P,Q)$	Compatibility order	340
$\mathcal{C}_n$	Set of polyominoes with n corners	27
$\Delta(R)$	Deficiency of R	41
$\phi(R)$	The flow of R	43
$F_{m,n}$	Flag coloring	161
$\mathcal{F}_n$	Fountain sets	351
$G_{\mathcal{T}}(R), G(R)$	The gap number of region R with respect to tileset $\mathcal{T}$	114
$g(k,m,n,\cdots)$	The Frobenius number of $k,m,n,\cdots$	190
$h(v_0,\cdots,v_k)$	Height difference of path $v_0,\cdots,v_k$	80
$h_T(v)$	Height function of tiling T at v	80
$H_c(P)$, $H_h(P)$	Heesch Number	325
$L(a,b,c)$	L-shaped polyomino	206
$\mu(T,U)$	The number of moves from tiling T to U	92
$\mathcal{P}$	The set of all polyominoes	13
$\mathcal{P}_n$	The set of polyominoes with n cells	13
P, **R**, **Q**	Classes of the tiling hierarchy	134
$\mathrm{RO}(P)$, $\mathrm{SO}(P)$, $\mathrm{OO}(P)$, $\mathrm{IO}(P)$, $\mathrm{HO}(P)$	The various orders of P	198
$\rho(T)$	The ranking of a tiling	91
$R(m,n)$	Rectangle	19
S_n	Square coloring	162
sp	Spin	80
V_n	$B((n-2)\cdot 1^2)$	231
W_n	W-polyomino with n cells	170

Table 1: Notation.

1
Foundations

Informally, a polyomino is a shape made from squares joined edge-to-edge. This simple definition already allows for a lot of informal reasoning, and is used by authors such as Golomb (1996), and Martin (1991).

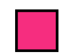


Figure 1: Monomino

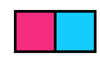


Figure 2: Domino

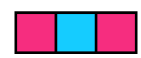


Figure 3: Bar tromino

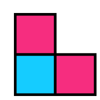
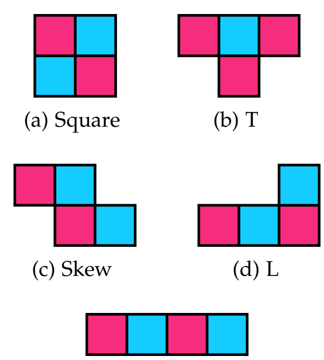


Figure 4: Right tromino



(a) Square (b) T (c) Skew (d) L (e) Bar

Figure 5: Tetrominoes

We need something a bit more formal. The next most convenient definition defines a polyomino as certain subsets of $\mathbb{Z}^2$, used by authors such as Beauquier and Nivat (1990) and Pak (2000). However, by itself this does not allow us to work easily with the borders of polyominoes. Authors that focus on the border use a slightly different formulation—see for example Conway and Lagarias (1990) and Qureshi (2012). Here we follow a hybrid approach. Our definition of a polyomino is as certain subsets of $\mathbb{Z}^2$, but we construct a border using the ideas in Conway and Lagarias (1990) and Qureshi (2012). Despite superficial differences, these descriptions are all similar; we are all trying to capture the same features of the same objects after-all.

A **point** is an element from $\mathbb{Z}^2$; a **shape** is a subset of $\mathbb{Z}^2$. (Very seldomly, we will work in $\mathbb{R}^2$ instead; in this case we use the term **lattice point** for an element of $\mathbb{Z}^2$.)

When talking about a point belonging to a shape, we use the word **cell**, and you should think as the point referring to the bottom left corner of a unit square. (Using a synonym makes it easier to distinguish between points of the polyomino and points of the border conceptually).

Shapes inherit terminology from set theory, and *subset*, *union*, *intersection* and *compliment* all have their usual meanings. The number of cells in a shape S—the **area** of S— is written $|S|$.

Two points are **neighbors** when the distance between them is 1.

A **path** is a sequence of points from $\mathbb{Z}^2$, $p_0, p_1, \cdots p_n$ such that consecutive points are neighbors. If $p_0 = p_n$, we say the path is **closed**, otherwise it is **open**. If no points, except possibly $p_0 = p_n$ repeat, we say the path is **simple**. The **length** of a path P is defined as n, and written $|P|$. The **reverse** of a path is the points in reverse order, and is written $\widehat{P}$.

We will work exclusively with simple paths, and for the remainder, "path" will mean "simple path".

A path lies in a shape if all its points lie in the shape. Two points p_0 and p_n are **connected** in a shape if there is a path $p_0, p_1, \cdots p_n$ that lies in the shape. If every pair of points in a shape are connected in that shape, the shape is called **connected**.

A **polyomino**[1] is a finite, connected shape. Note that points are always connected relative to some shape, but shapes are connected independently. We also use the term n-omino for polyominoes with n cells.

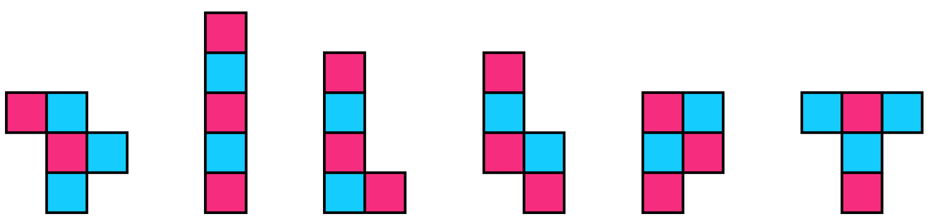
[1] Some authors use the term *domino* for polyomino; but for us a domino will always be a polyomino with two cells.

The class of polyominoes with n cells for various n are given special names. The names for $n \leq 10$ is shown in Table 2. Although we can construct names for larger n, it is customary to simply use a term like 111-omino.

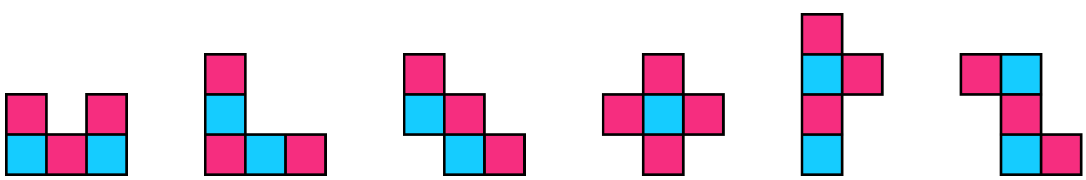
n	Name	$\lvert\mathcal{P}_n\rvert$
1	Monomino	1
2	Domino	1
3	Tromino	2
4	Tetromino	5
5	Pentomino	12
6	Hexomino	36
7	Septomino	108
8	Octomino	369
9	Nonomino	1285
10	Decomino	4655

Table 2: Names and counts for small polyominoes. More comprehensive tables of counts are given in Chapter 9.

How many polyominoes there are depends on which symmetries we distinguish. The most common case is not to distinguish any symmetries; this will be clarified in the next section. The set of all polyominoes with n cells (not distinguishing any symmetries) is denoted $\mathcal{P}_n$, and the set of all polyominoes is denoted $\mathcal{P}$. The counts for small free polyominoes are also given in Table 2, and small polyominoes are shown in Figures 1–6.



(a) F (b) I (c) L (d) N (e) P (f) T



(g) U (h) V (i) W (j) X (k) Y (l) Z

Figure 6: Pentominoes

A **partition** of a non-empty set S is a collection of sets S_i such that:

(1) $S \neq \emptyset$

(2) $S = \bigcup_i S_i$

(3) $S_i \cap S_j = \emptyset$ if $i \neq j$.

To partition a set means to give such a partition. If two shapes each have a point that are neighbors, we say the shapes **touch**. Each shape can be partitioned into a collection of sets such that each set is connected, and no two sets touch; these sets are called the **connected components** of the shape.

Problem† 1. *Show that each of a finite shape's connected components is a polyomino.*

Define a **rectangle** as the set

$$R(x_0, y_0, w, h) = \{(x, y) \mid x_0 \leq x < x_0 + w, y_0 \leq y < y_0 + h\}.$$

Let P be a polyomino, and let R be some rectangle that contains P as subset. If every point in R not in P is connected in P' to a point in R, we say the polyomino is **simply-connected**. The connected components of the points that fail this condition are called holes.[2] Holes are subsets of the rectangle, so must be finite in both size and number, and they are connected by definition. The holes are therefore polyominoes themselves; in fact we will see later that holes are in fact simply-connected.

To summarize, we have a polyomino as a connected subset of $\mathbb{Z}^2$, and it can be classified as being simply-connected or not; in the latter case it has a finite number of holes that are themselves polyominoes.

Figure 7 shows examples of polyominoes that are not simply-connected as we use the term here.

[2] Polyominoes with holes are also called *holey* or *multiply-connected* (Golomb, 1996, p. 74); polyominoes without holes are also called *simple* (Herzog et al., 2015) or *profane* (Toth et al., 2017, p. 367). Some authors use a slightly different notion of simply-connected, so that the second polyomino in Figure 7 is simply connected.

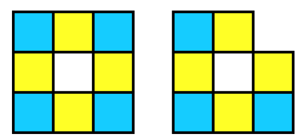
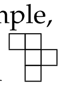
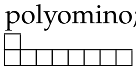
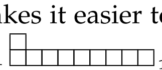


Figure 7: Two examples of polyominoes with holes.

1.1 *Symmetry*

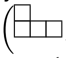
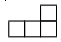
An **isometry** ϕ is a bijection $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ that *preserves distance between points*, that is: $|u - v| = |\phi(u) - \phi(v)|$. We will define some basic isometries below; all isometries can be expressed as compositions of these.

A **translation** t_v is an isometry $t_v(u) = u + v$ for some fixed point v. Translations move all points by the same amount in the same direction.

When the position of a polyomino is not important, we usually give a visual representation of a polyomino instead of the set of points; for example, the polyomino $\{(0, 2), (1, 2), (1, 1), (2, 1), (1, 0)\}$ is denoted with  $_5$. The subscript is the number of cells in the polyomino; it makes it easier to distinguish polyominoes such as  $_9$ and  $_{10}$.

A **rotation** r is $r(u) = u^{\perp}$. The composition of any number of translations and rotations gives us a **direct isometry**.

A **reflection** f is $r((x, y)) = (-x, y)$.

If S is a shape and ϕ an isometry, then $\phi(S)$ is the shape defined[3] by $\{\phi(v) \mid v \in S\}$. For example, $f\Big($$_4\Big) \equiv_t$ $_4$.

Two shapes S_1 and S_2 are **congruent** (Conway and Lagarias, 1990, p.184) if there is some isometry ϕ such that $S_1 = \phi(S)$, and we write $S_1 \equiv S_2$. Corresponding equivalence classes are called **free** (Redelmeier, 1981, Section 3).

[3] In other words, $\phi(S)$ is the *image* of S under ϕ.

S_1 and S_2 are **directly-congruent** if there is some a direct isometry d such that $S_1 = d(S)$, and we write $S_1 \equiv_{\text{d.}} S_2$. Corresponding equivalence classes are called **one-sided** (Golomb, 1996, p. 70). One-sided tetrominoes are shown in Figure 8. Compare this with the free tetrominoes shown in Figure 5.

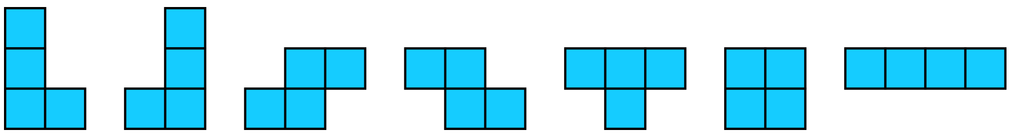


Figure 8: One-sided tetrominoes.

S_1 and S_2 are **equivalent** (Conway and Lagarias, 1990, p.184) if there is some translation t such that $S_1 = t(S)$, and we write $S_1 \equiv_t S_2$. Corresponding equivalence classes are called **fixed** (Redelmeier, 1981, Section 3). Fixed tetrominoes are shown in 9.

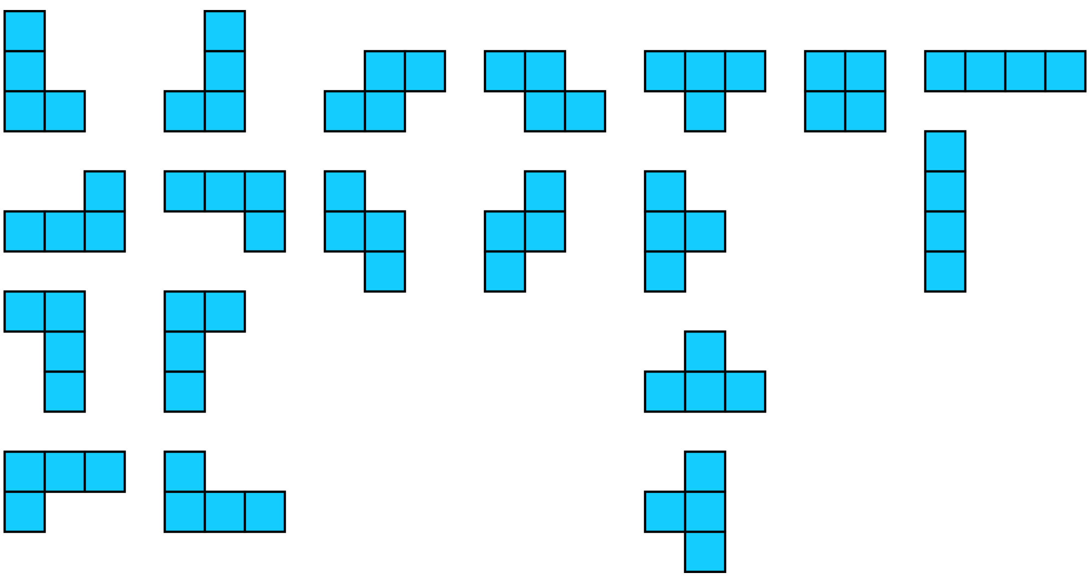


Figure 9: The set of fixed tetrominoes.

A polyomino P has 8 orientations, given by the following:

- $P_0 = P$
- $P_1 = r(P)$
- $P_2 = r^2(P)$
- $P_3 = r^3(P)$
- $P_4 = f(P)$
- $P_5 = r(f(P))$
- $P_6 = r^2(f(P))$
- $P_7 = r^3(f(P))$

A polyomino can be congruent to none, some, or all of its orientations; and we classify polyominoes by which orientations it is congruent to.

The **symmetry index** of a (free) polyomino is the number of fixed polyominoes congruent to it (Redelmeier, 1981, Section 3).

Table 3 shows details of the symmetry types, including an example of each.

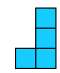
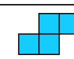
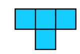
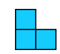
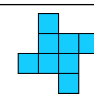
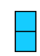
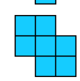
Type	Congruent orientations	Index	Symmetries	Smallest example
None	0	8		
Rot	0, 2	4	180° rotation	
Axis	0, 4 or 0, 6	4	horizontal or vertical reflection	
Diag	0, 5 or 0, 7	4	diagonal-reflection	
Rot2	0, 1, 2, 3	2	90° rotation	
Axis2	0, 2, 4, 6,	2	horizontal and vertical reflection or 180° rotation	
Diag2	0, 2, 5, 7	2	diagonal-reflection or 180° rotation	
All	0, 1, 2, 3, 4, 5, 6, 7, 8	1	all	

Table 3: The symmetry types of free polyominoes.

1.2 *Tilings*

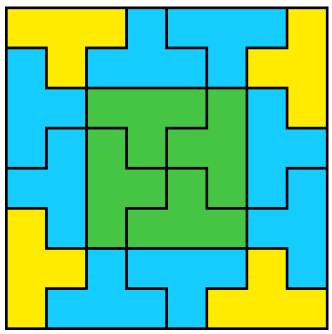
Given a shape, called a **region**[4] R, and a set of shapes, called a **tile set** $\mathcal{T}$, and a set of allowed isometries Φ, a **tiling** is a partition of R such that there is a map from the allowed symmetries from a tile to each set in the partition. In symbols, if the sets of the partition are S_i, then each $S_i = \phi(T)$ for some $\phi \in \Phi$ and some $T \in \mathcal{T}$.

[4] Also called *figure* or *picture*.

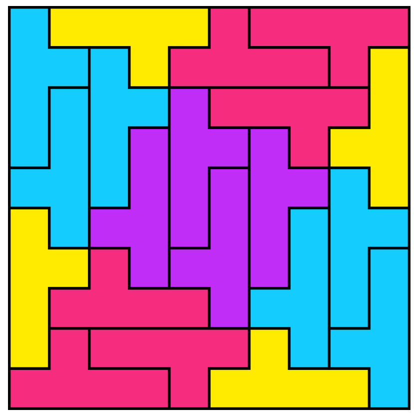
Such a partition need not exist. If it does, we say the region can be **tiled** by the tile set. We will mostly be interested in tile sets where the elements are polyominoes. If the tile set has a single tile, we simply say the region can be tiled by the tile.

In most cases the allowed set is all isometries; in this case we do not explicitly say so. Regions and tiles are both synonyms for shapes; we use them to make it easier to keep track of the various shapes' roles in a tiling problem. We use the term **subregion** for a subset of a region.

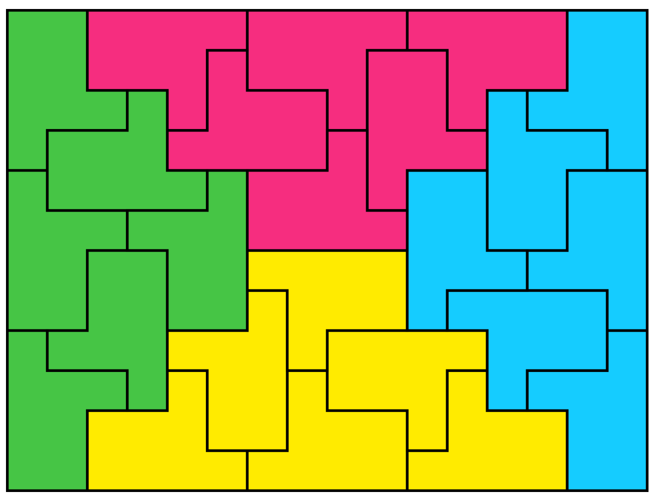
Figure 10 give shows some examples of tilings.



(a) A tiling of a rectangle by the T tetromino.



(b) A tiling of a rectangle by Y pentominoes.



(c) A tiling of a rectangle by an octomino.

Figure 10: Examples of tilings.

Theorem 1 (Area Criterion)**.** *If we have a tile set where all tiles have area n, then we can only tile regions whose area is divisible by n.*

[Referenced on pages 18, 31, 36, 39, 59, 100, 110, 115, 117, 128, 184, 199, 232, 235, 249, 352, 361, 365 and 367]

Example 1. *If a region is tileable by dominoes, then its area is even.*

Theorem 2 (Partitions)**.** *If all the subregions in a partition of a region are tileable, then so is the region.*

[Referenced on pages 18, 31, 58, 63, 65, 67, 68, 126, 127 and 185]

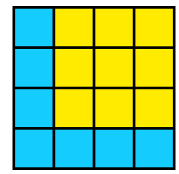


Figure 11: In this example, the 4×4 square is the region, and both the yellow and blue regions are subregions of the square.

Example 2. *There is a region with $2n$ cells not tileable by $\mathcal{P}_n$ for all $n > 1$.*

Consider a cross with area $2n$ and arms of length $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$. If we place a polyomino to cover one arm, it must cover the center of the cross, and another arm. This partitions the cross into three pieces—the covered arms and the two non-covered arms. Each of the non-covered arms is smaller than n and non-zero, so by Theorem 1 cannot be tiled. And hence, the whole figure cannot be tiled. See also Example 22.

Theorem 3 (Transitivity)**.** *If a set $\mathcal{T}_2$ can tile all tiles in a set $\mathcal{T}_2$, and $\mathcal{T}_2$ tiles R, then $\mathcal{T}_1$ tiles R.*

[Referenced on pages 31, 133, 134, 135, 186, 200 and 339]

Proof. The tiling of R by $\mathcal{T}_2$ gives as a partition in regions (the tiles of $\mathcal{T}_2$) that are each tileable by $\mathcal{T}_1$, and therefore by Theorem 2 R is tileable by $\mathcal{T}_1$ $\square$

We define an ordering on points which will make it convenient to sort points. We write $(x_1, y_1) < (x_2, y_2)$ if $y_1 < y_2$, or $y_1 = y_2$ and $x_1 < x_2$. The smallest point of a finite shape S is denoted $\min S$, and is called an anchor of the shape.

In a tiling of a finite region, the anchor of the region corresponds with the anchor of the tile that covers it.

Theorem 4 (Tilings by completely symmetric tiles)**.** *If a tiling of a region R by a polyomino $P \in$ **All** exists, it is unique.*

[Referenced on pages 31, 198 and 241]

Proof. (Adapted from nickgard (un.) (2017).) The left-most cell in the top row of the region can only be tiled by the left-most cell of the top row of the tile in any orientation (which are all equivalent because the tile has symmetry type **All**). We can form a new region from the untiled part, which by a similar argument can only be tiled one way. We can repeat this until eventually the complete figure is tiled. Since in each step only one placement is possible, the entire tiling is unique. □

Since squares have symmetry type **All**, it follows that if a region has a tiling by a single square, the tiling is unique.

The proof gives us an algorithm for finding a tiling if it exists, and a way to tell when it doesn't: we start the tiling by placing a tile so that its anchor matches the region's anchor if possible, and repeat with the remaining region. Eventually, we must either have the empty region (which means we have found a tiling of the region), or a region that is not tileable (that is, when we place the tile so that anchors match, a piece of the tile falls outside the region.) This tiling algorithm is *linear* in the number of cells of the region..

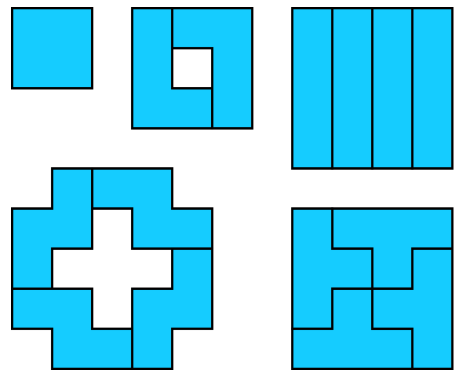


Figure 12: Minimal Baiocchi figures for the tetrominoes (Sicherman, 2015a).

[5] The idea was suggested by Claudio Baiocchi in January 2008, and appeared in that month in Friedman (2008, https://erich-friedman.github.io/mathmagic/0108.html).

Problem† 2. *Show that for a polyomino $P \in$* **All** *with perimeter p,*

(1) $|P| \equiv 0$ *or* $1 \pmod 4$

(2) $p(P) \equiv 0 \pmod 4$.

Problem† 3. *A **Baiocchi figure** of a polyomino is a region with symmetry type* **All** *that is tileable by that polyomino*[5]*. A Baiocchi figure of a polyomino is **minimal** if there is no smaller Baiocchi figure for that polyomino. Figure 12 shows minimal Baiocchi figures for the tetrominoes, and Figure 13 shows holeless variants.*

(1) *Find minimal Baiocchi figures for the pentominoes.*

**(2) *Is there a Baiocchi figure for the U-pentomino without holes? This seems to be an open problem (Sicherman, 2015a).*

(3) *Find the minimal holeless Baiocchi figures for other pentominoes.*

(4) *Is every polyomino in* **All** *the Baiocchi figure of* some *polyomino?*

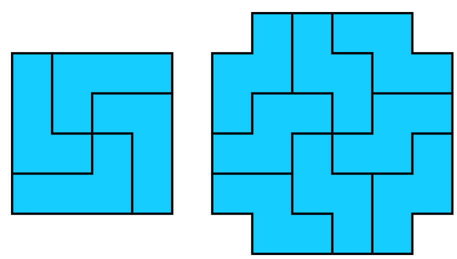


Figure 13: Holeless variants for minimal Baiocchi figures for the L-tetromino and skew tetromino (Sicherman, 2015a).

1.3 *Classes of Polyominoes*

A **rectangle** $R(m, n)$ is a polyomino

$$R(m, n) \equiv_t \{(x, y) \mid 0 \le x < m, 0 \le y < n\}.$$

Notice that the rectangle we defined earlier is equivalent to a rectangle with this new definition, that is

$$R(x_0, y_0, w, h) \equiv_t R(w, h).$$

The **width** is the value m, and the height is the value n. When $m = n$, we call the rectangle a **square**. It should be clear that $|R(m, n)| = mn$.

A shape S_1 **holds** another shape S_2 if there is a isometry ϕ such that $\phi(S_2) \subset S_1$, and we write $S_2 \sqsubset S_1$.

The smallest rectangle that holds a finite shape is called the shape's **hull**[6]. The hull of a rectangle is the rectangle itself. The area of the hull of a shape is always bigger than the area of the shape, unless the shape is a rectangle in which case the areas are the same. The **width** of a shape is the width of its hull, and the **height** of a shape is the height of its hull.

[6] Also called *bounding box.*

We will prove many tiling theorems involving rectangles in Chapter 5, but for now we state a simple theorem that we will use until then:

A **bar** is a rectangle with width or height equal to 1. For each bar, there is one bar with n cells, denoted by $I(n)$.

A **bar chart**[7] $B(c_1 \cdot c_2 \cdots c_n)$ is a polyomino (Bousquet-Mélou et al., 1999)

[7] Also called *Manhatten polyomino* (See for example Bodini and Lumbroso, 2009). The notation used here is from Martin (1986).

$$B(c_1 \cdot c_2 \cdots c_n) \equiv_t \bigcup_{i=1}^{n} \{(x,y) \mid x = i, 0 \le y < c_i\}.$$

(You can think of a bar chart as bars stacked together on the same base.)

We use exponentiation notation for repeated columns, for example $B(1 \cdot 1 \cdot 5 \cdot 5 \cdot 5 \cdot 5)$ can be written as $B(1^2 \cdot 5^4)$. Bar charts include a large number of polyominoes, and because of the handy notation make for excellent examples (when possible). The bottom row is called the **base**.

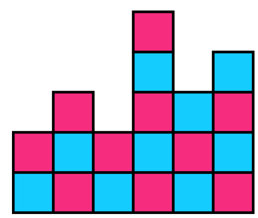


Figure 14: A bar graph with notation $B(2 \cdot 3 \cdot 2 \cdot 5 \cdot 3 \cdot 4)$.

Problem† 4.

(1) *Show that* $R(m,n) \equiv_t B(n^m)$.

(2) *Show that* $|B(c_1 \cdot c_2 \cdots c_n)| = \sum_i c_i$.

An **L-shape** is a polyomino with the form $B(a^b \cdot c^d)$. Note that rectangles are L-shapes with $b = 0$ or $c = 0$.

1.4 *Corners and Sides*

The four **vertices** of a cell (x,y) are the points (x,y), $(x+1,y)$, $(x+1,y+1)$, $(x,y+1)$. You can see that we use the bottom left vertex of a cell to represent it; shapes are sets of bottom-left vertices of the set of squares they are meant to represent. (This is a bit unusual; cells are normally represented by their center coordinates.)

An **edge** is a set of two neighboring points. Notice that a set of consecutive points in a path is an edge. All such edges are called the path's edges.

The four edges of a cell are the four pairs of the cell's vertices that are neighbors,

$$a^{\square} = \{\{(x,y),(x+1,y)\},\{(x+1,y),(x+1,y+1)\},\{(x+1,y+1),(x,y+1)\},\{(x,y+1),(x,y)\}\}.$$

If S is a shape, then the collection of edges is defined as $S^{\square} = \bigcup_{s \in S} s^{\square}$.

These edges can be sorted into two sets. The **interior edges** of S are those edges shared between two cells; the **exterior edges** are those shared by only cell. In a tiling, we say a tile **crosses** an edge if the edge is an internal edge of the tile. An edge **crosses** a path if one of the path's edges is an internal edge of tile tile.

If the shape is a simply-connected polyomino, we can construct a closed path from the set of exterior edges $p_0, p_1, \cdots p_n$ such that p_i, p_{i+1} is an edge, and that the cell that the edge belongs to is to the left of the edge. (You should check that this is possible when the shape is simply-connected.)

We could start with any node, but to make paths comparable we choose the minimum point from the path, defined below. This path is the border of the polyomino, and we denote it by $\partial_0 S$.

We write $(a_x, a_y) \leq (b_x, b_y)$ if $a_y < b_y$, or if $a_y = b_y$ and $a_x < b_x$. This allows us to sort points, and in a finite set of points S we have a minimum point denoted by $\min S = s_0$ where $s_0 \leq s$ for all $s \in S$.

If P is a polyomino with holes H_i, the filled polyomino $\bar{P}$ is defined as

$$\bar{P} = P \cup \bigcup_{i=0}^{H} H_i$$

(informally, this is P with the holes filled up). Note that $\bar{P}$ is a simply-connected polyomino.

We now define the border of P as the collection of paths

$$\partial_0 P = \partial_0 \bar{P} \tag{1.1}$$

$$\partial_i P = \widehat{\partial_0 \bar{H}_i}. \tag{1.2}$$

$\partial_0 P$ is called the outer border, and $\partial_i P$ for $i > 0$ is called the border around hole i. The fact that we reverse the path is significant; this is so that the polyomino is on the left as we travel around the path. (Otherwise the hole would be to the left).

A corner of a polyomino is a point in the path where the direction changes, that is if p_i is a corner if $p_i - p_{i-1} \neq p_{i+1} - p_i$. If p_i represents a left turn, then we call the corner convex. If p_i represents a right turn, we call the corner concave.

A **side** of the polyomino is a piece of path between two corners of a border where none of the points in between are corners. The side can be horizontal (when all the y-coordinates of the points have the same value) or vertical (when all the x-coordinates are the same).

The length of the side is the length of the piece of path, equal to the distance between the two endpoints of the path.[8]

[8] Many theorems in this section apply to a more general object called a *rectilinear polygons* (also *orthogonal polygon* or *axis-aligned polygon*)—polygons whose sides meet perpendicularly. The *Further Reading* section gives additional resources.

Theorem 5 (Wikipedia contributors (2017))**.** *The number of vertical and horizontal sides of a simply-connected polyomino are equal, and therefore the number of sides is even.*

[Referenced on pages 22, 24 and 46]

Proof. Horizontal and vertical sides alternate. In fact, the first side is horizontal (since it starts at the anchor), and the last side side is vertical (if it wasn't, the its first point would be to the left of the anchor, which is by definition left-most). Therefore each horizontal side is followed by a distinct vertical side, and there must be the same number of horizontal and vertical sides. □

The **perimeter** $\Pi(P)$ of a polyomino P is the sum of the lengths of all the paths in its border.

$$\Pi(P) = \sum_i |\partial_i P|$$

Theorem 6 (*)**.** *In a simply-connected polyomino the total horizontal edge length and total vertical edge length are even, and so is the perimeter.*

[Referenced on pages 22, 26 and 46]

Proof. If traveling along the border path, we go x units left, and x' units right, then $x + x' = 0$ (because the path is closed, and we end where we start). Then the total horizontal distance is $|x| + |x'| = 2|x|$, which is even since x is an integer. The same goes for the vertical length.

Since the perimeter is the sum of the vertical and horizontal length, which are both even, it must be even too. □

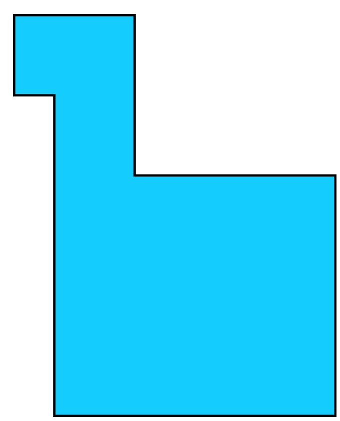


Figure 15: Golygon with 8 sides. This polyomino tiles the plane.

Theorem 7 (Csizmadia et al. (2004), Lemma 3.3)**.** *If the lengths of either all horizontal or all vertical sides are odd, then there is an even number of horizontal and vertical edges, and the total number of edges is divisible by four.*

[Referenced on page 46]

Proof. WLOG suppose the lengths of all horizontal sides are odd. Since the total length must be even (Theorem 6), there must be an even number of them, and because the number of horizontal and vertical sides are equal (Theorem 5), the total must be divisible by four. □

Problem 5** (Sallows et al. (1991))**.** *A golygon*[9] *is a closed path whose side lengths forms a sequence* $1, 2, 3, 4, \cdots$*; usually, self-intersections are allowed. An example of a polyomino whose border is a golygon with 8 sides is shown in Figure 15. Show the number of sides of a golygon must be divisible by 8.*

The golygon in Figure 15 is the only one with 8 sides. It is the only known golygon that can tile the plane. It seems to be an open problem whether any other golygons can tile the plane (Sallows, 1992).

[9] Also called a 90° *serial isogon*.

Theorem 8 (*Wikipedia contributors (2017))**.** *The border of a simply-connected polyomino has 4 more convex corners than convex corners.*

[Referenced on pages 23, 24, 28 and 125]

Proof 1. Suppose we travel along the path starting at the anchor, facing the second point of the path. At concave corners, we need to make a left turn, and at a convex corner, we need to make a right turn. When we return to our starting point, we make one more turn so we face the original direction. Thus we made a full turn of 360 degrees, so we must have made four more left turns than right turns. □

Proof 2. (Essentially O'Rourke (1987, Lemma 2.12)) The sum of interior angles of a polygon with k sides is $180°(k-2)$. Suppose we have m convex corners and n convex corners. We then have the equation $90° m + 270° n = 180°(k-2) = 180°(m+n-2)$. Solving this, we obtain $m = n + 4$. □

It follows that the border must have at least 4 convex corners.

Problem† 6. *Prove that (up to symmetry) there is a unique polyomino with 6 sides whose lengths are given.*

Example 3. *Suppose we can place* n 2×2*-squares anywhere on the lattice. They are allowed to overlap but not coincide. Then they must cover at least* $n + 3$ *cells.*

Here is why: Any cell can be covered by at most 4 squares. Convex corners can only be covered by one square. There must be at least 4 convex corners (Theorem 8). So all possible cells that can potentially be covered by 4 squares is given by $\frac{4n-4}{4} = n - 1$*. So there must be at least* $n - 1 + 4 = n + 3$ *cells in total.*

A side that lies between two convex corners is called a **knob**; a side that lies between two concave corners is called a **anti-knob**; a side between a convex corner and concave corner is called a **flat**.

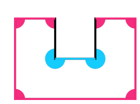


Figure 16: In this region, convex corners are red, and concave corners are blue. The five knobs that lie between pairs of convex corners are marked red, and the anti-knob between the pair of concave corners is blue. The two remaining (black) edges are flats.

Theorem 9 (ccorn (un.) (2017) via Wikipedia contributors (2017))**.** *Let K be the number of knobs and L the number of anti-knobs of a simply-connected polyomino. Then $K - L = 4$.*

[Referenced on pages 24 and 25]

Proof. Let n_x be the number of convex corners, and n_y the number of concave corners. Let n_{xx} be the number of convex corners followed by a convex corner, n_{xy} the number of convex corners followed by a concave corner, and so on.

Then $n_x = n_{xx} + n_{xy} = n_{xx} + n_{yx}$ and $n_y = n_{yx} + n_{yy} = n_{xy} + n_{yy}$, which means $n_{xx} = n_x - n_{xy} = n_x - (n_y - n_{yy}) = n_{yy} + (n_x - n_y)$. But by Theorem 8, $n_x - n_y = 4$, so $n_{xx} = n_{yy} + 4$. □

It follows that for a simply-connected polyomino, the number of knobs must always be at least 4. Furthermore:

Theorem 10. *If F is the number of flats simply-connected polyomino, then F is even.*

[Not referenced]

Proof. Since $K - L = 4$ (Theorem 9), it follows that K and L must either both be odd or both be even. So their sum is even. But $K + L + F = k$, and k is even (Theorem 5), therefore F must be even. □

Theorem 11 (Herzog et al. (2014), Lemma 1.1)**.** *Every hole is simply-connected.*

[Not referenced]

Theorem 12 (Herzog et al. (2014), Lemma 1.2)**.** *Let a and b be diagonally opposite points. They share two neighbors, call them b and d. In a simply-connected polygon, if $a, c \in P$ but $b \notin P$, then $d \in P$.*

[Referenced on page 26]

Proof. Suppose the opposite: suppose $d \notin P$. Because P is simply-connected, there is a path from a to b, and this path must either go around c or around d; if it goes around c, then c must be in a hole of P, and similarly if the path goes around d instead, d is in a hole of P. In either case, we have a hole, which contradicts the fact that P is simply-connected. Therefore, our assumption is incorrect and therefore $d \in P$. □

Theorem 13. *For a polyomino, the number of holes H is related to the number of knobs K and anti-knobs L by the following equation:*

$$H = \frac{L - K}{4} + 1.$$

[Referenced on page 124]

Proof. The border is defined as the paths $P_0 = \partial \bar{P}$ and $P_i = \widehat{\partial H_i}$ for $i = 1..H$. Suppose ∂H_i has K_i knobs and L_i anti-knobs, then $\widehat{\partial H_i}$ has L_i knobs and K_i anti-knobs (Theorem 9).

The total number of knobs and anti-knobs are thus given by

$$K = K_0 + \sum_1^H L_i \tag{1.3}$$

$$L = L_0 + \sum_1^H K_i \tag{1.4}$$

Subtracting, we get:

$$K - L = (K_0 - L_0) \sum_i^H (L_i - K_i) \tag{1.5}$$

And all the paths are anchored, we can use Theorem 9.

$$K - L = 4 \sum_1^H (-4) \tag{1.6}$$

$$= 4 - 4H. \tag{1.7}$$

After rearrangement, we get the required result:

$$H = \frac{L - K}{4} + 1.$$

□

A **lattice polygon** is a polygon whose vertices are in $\mathbb{Z}^2$.

Theorem 14 (Pick's Theorem, Niven and Zuckerman (1967), Theorem 4)**.** *The area A of a lattice polygon, is given by $A = b/2 + r - 1$, where b is the number of lattice points on the boundary of the polygon, and r is number of interior lattice points.*[10]

[10] This theorem was discovered by Pick (Pick (1899)); Bruckheimer and Arcavi (1995) provides some history and context; more references are given in the *Further Reading* section.

[Referenced on pages 26, 316 and 344]

Proof. The details of the following outline is easy to fill in:

(1) The theorem is true for rectangles.

(2) The theorem is true for right triangles.

(3) The theorem is true for triangles.

(4) The theorem is true for polygons.

For details, see the reference. □

Theorem 15 (Williams and Thompson (2008), Theorem 1)**.** *If the number of interior points of a simply-connected n-omino is given by r, the perimeter p is given by:*

$$p = 2n - 2r + 2.$$

[Referenced on pages 26, 27 and 43]

Proof. We use induction on n.

The theorem is true for the monomino and domino.

Suppose then it is true for a polyomino with n cells and r interior points, so that the perimeter $p = 2n - 2r + 2$.

We can attach another cell to the polyomino to form a new polyomino with $n+1$ cells. Let p' be the perimeter of this polyomino, r' be the number of interior points of the new polyomino. We want to show that $p' = 2(n+1) - 2r + 2$.

There are four possibilities (shown in Figure 17).

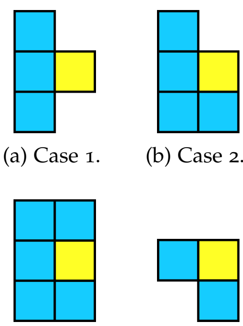
- Case 1: No new interior points are added. The new perimeter is 2 units longer; so $p' = p' + 2 = 2n - 2r + 4 = 2(n+1) - 2r' + 2$.

- Case 2: The perimeter stays the same, but one more interior point is added. So $p' = p = 2n - 2r + 2 = 2(n+1) - 2(r+1) + 2 = 2(n+1) - 2r' + 2$.

- Case 3: In this case two new interior points are added, and the perimeter is reduced by 2. Thus $p' = p - 2 = 2n - 2r + 2 = 2(n+1) + 2(r-2) = 2(n+1) - 2r'$.

- Case 4: This case is actually impossible if the original polyomino is simply-connected by Theorem 12.

□



(a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4.

Figure 17: The cases of Theorem 15. The blue cells are from the old polyomino, and the yellow cell is added to form the new polyomino.

The proof is also easily derived from Pick's Theorem (Theorem 14), by noting that for polyominoes, $n = A$ and $b = p$ (Williams and Thompson, 2008). This theorem gives us another proof that the perimeter is even as was proved in Theorem 6. The following is an extension of the theorem above to polyominoes with holes.

Theorem 16. *For a polyomino with H holes, the perimeter is given by:*

$$p = 2n - 2r - 2H + 2.$$

[Not referenced]

Proof. Let the holes be $H_1, H_2, \cdots$ Let $p_i = |\partial_i P|$. Let $n_0 = |\bar{P}|$, and $n_i = |H_i|$ the cells in hole i. Let r_0 be the number of interior points of $\bar{P}$, and r_i the interior points of H_i.

The number of points on the perimeter of a polyomino is the same as the length of the perimeter of the polyomino. Therefore $r = r_0 - \sum_{i=1}^{H}(r_i + p_i)$. We also have $n = n_0 - \sum_{i=1}^{H} n_i$ and $p = p_0 + \sum_{i=1}^{H} p_i$, and $p_i = 2n_i - 2r_2 + 2$ by Theorem 15.

Let us expand $p_0 - \sum_{i=1}^{H} p_i$:

$$2n - 2r - 2H + 2 = 2(n_0 - \sum n_i) - 2(r_0 - \sum(r_i + p_i)) - \sum 2 + 2 \tag{1.8}$$
$$= (2n_0 - 2r_0 + 2) + \sum [-2n_i + 2r_i - 2 + 2p_i] \tag{1.9}$$
$$= p_0 + \sum [-p_i + 2p_i] \tag{1.10}$$
$$= p_0 + \sum p_i \tag{1.11}$$
$$= p. \tag{1.12}$$

□

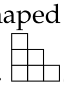
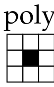
Problem† 7.

(1) *Do any of the polyominoes with n cells and minimum perimeter have holes?*

*(2) *Can you characterize the minimum perimeter polyominoes?*

See also Problem 15.

Let $\mathcal{C}_n$ be the class of polyominoes with n corners (and note that n is always even, and greater or equal to 4). The particular $\mathcal{C}_n$ a polyomino belongs to is called its **corner class**.

- $\mathcal{C}_4$ is the set of rectangles.
- $\mathcal{C}_6$ is the set of L-shaped polyominoes.
- $\mathcal{C}_8$ is the set of T-shaped (eg. T-tetromino), S-shaped (eg. skew-tetromino), U-shaped (eg. U-pentomino), and A-shaped (eg. A-heximonio, i.e. $_6$) polyominoes, and rectangles with one rectangular hole (eg. $_8$).

Problem† 8. *Let the number of polyomino-types with* $2n$ *corners and no holes, disregarding side length, be* $p(n)$*.*
$p(n) = 0, 1, 1, 4, 8, 29, 79, 280 \cdots$ (*A256456*)

(1) *How many types of polyominoes with 10 corners, and 0 or more holes, are there?*

(2) *Can you find a general formula for the number of types of polygons with* $2n$ *corners (any number of holes) in terms of* $p(n)$*?*

Theorem 17. *Every simple polyomino can be partitioned into* $\frac{n-2}{2}$ *or less rectangles.*

[Referenced on pages 29 and 31]

Proof. This is true for rectangles. Suppose then we have more than 4 vertices. We need at least one concave vertex (by Theorem 8). We can make a straight cut into the polyomino at this vertex, to meet the polyomino edge again in either another concave corner (Figure 18), or not a corner (Figure 19):

(1) In the first case, we have 3 more convex corners, and one less concave corner.

(2) In the second case, we have 3 more convex corners and two less concave corners.

Either way, we reduce the number of concave corners by at least one. This process must stop eventually, after we made at most C cuts, where C is the number of concave corners. When there are no more convex corners left, all the figures of the partition are rectangles, so we have $C + 1$ rectangles (or less). Let R be the number of rectangles. Then $R \leq C + 1$, and since $C = n/2 - 2$ (Theorem 8), we must have $R \leq \frac{n-2}{2}$. □

Sometimes $\frac{n-2}{2}$ is necessary (see for example Figure 20), but not always (see for example Figure 21).

Problem† 9. *What is the fewest number of rectangles a simply-connected polyomino with n corners can be partitioned into?*

Theorem 18. *Every polyomino can be partitioned into* $\lfloor \frac{n}{4} \rfloor$ *or less pieces where each piece is in* C_6 *or* C_4*.*

[Referenced on pages 29, 31 and 32]

The partitioning is the basis of the proof of the main theorem in O'Rourke (1983), and also covered in O'Rourke (1987, Theorem 2.5) and Győri (1986). The case for rectilinear polygons with holes is covered in Hoffmann et al. (1991).

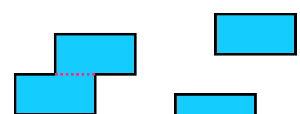


Figure 18: Cut meets in another concave corner.

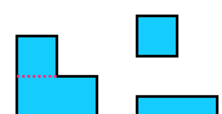


Figure 19: Cut meets in a non-corner.

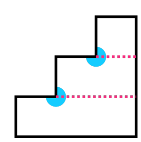


Figure 20: An example were no fewer than $\frac{n-2}{2}$ rectangles are possible.

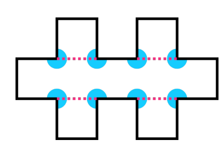


Figure 21: An example that has partitions with fewer than $\frac{n-2}{2}$ rectangles.

Theorem 19 (Győri and Mezei (2016), Thereom 4). *Every polyomino can be partitioned into*

$\left\lfloor \frac{3n+4}{16} \right\rfloor$ *or less pieces where each set in the partition is in $\mathcal{C}_8, \mathcal{C}_6$ or $\mathcal{C}_4$.*

[Referenced on pages 29, 31 and 32]

See the reference for proof.

Problem* 10. *Every polyomino can be partitioned into $P(m, n)$ partitions or less, where each set in the partition is in $\mathcal{C}_i$ with $i \leq m$, with $m \geq 4$ even and $n \geq 4$. Theorems 17, 18, and 19 give values for $P(4, n)$, $P(6, n)$ and $P(8, n)$. Can you give this function for other values of m?*

Theorem 20 (The knob criterion). *Suppose we have a region with knobs of length $e_1, e_2, \ldots$ and a set of tiles with knobs of length $e'_1, e'_2, \ldots$. A necessary condition for the region to be tileable is that each knob of length e_i, we have $e_i = \sum_j n_j e'_j$ for some $n_i \geq 0$.*

[Referenced on pages 30, 31, 184, 186, 187, 188 and 365]

Example 4. *The 9×9 square with the center removed cannot be tiled by 2×2 squares. The area of the region is 80, and so it satisfies the area criterion. However, we cannot express the side of length 9 as a multiple of 2, and so it fails the knob criterion. In fact, we cannot tile a 9×9 square with a cell removed from anywhere, since we always have at least two borders of odd length. Surprisingly, the minimum number of monominoes necessary to to tile a 9×9 square is actually 17! (See Theorem 183.)*

Problem* 11.

(1) *Determine what is the biggest rectangle that can tile the L-shape $B(a^b \cdot c^d)$.*

(2) *Can the rectangle found in the previous question always tile $R(a, b)$ and $R(c, d)$?*

Theorem 21 (The row criterion). *Suppose our region rows of length e_i and our tiles have rows of length e'_i. If no rotation is allowed, then a tiling can exist only if $e_i = \sum_j n_j e'_j$ for some $n_i \geq 0$.*

[Referenced on pages 31, 186, 187 and 189]

Example 5. *The tile in Figure 1.4 can only tile regions with all rows and columns containing an even number of cells. (In fact, if a row or column is not connected, each piece must have an even number of cells.)*

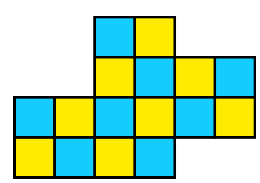


Figure 22: A tile that can only tile regions that have an even number of rows and columns.

Theorem 22 (Bower and Michael (2004), Observation, Section 1). *Let $\mathcal{T} = \{R(p, q)\}$. A tiling by translation exists if and only if $p \mid m$ and $q \mid n$.*

[Referenced on pages 31, 135, 183, 184, 185, 186, 187, 189, 192, 227, 230, 241, 281, 282, 283 and 339]

Proof.

If. Suppose $m = pk$ and $n = q\ell$. A tiling is given by placing ℓ rows with k tiles in each row.

Only if. All sides of the rectangle are knobs, so necessity follows directly from Theorem 20. □

The following fact is used to show that if we rotate a polyomino $90°$around a point and the rotated copy does not intersect the original, then a copy rotated $180°$ also does not intersect the original. (Later, we use this to put constraints on how certain polyominoes can fit into rectangles.)

This fact seems reasonable, but requires some ideas from topology to prove, and therefore I omit the proof.

Theorem 23 (*)**.** *Suppose P and Q are points such that P is Q rotated about $180°$ around O, and there is a continuous path p between P and Q. Suppose we rotate p $90°$ around O to form a new path p'. Then p and p' intersect.*

[Referenced on pages 30 and 32]

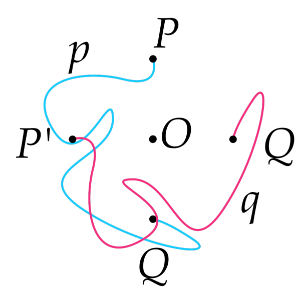


Figure 23: A curve between two points rotated intersects itself under certain conditions.

For a proof, see Hagen von Eitzen (un.) (2019). The Further Reading section has some references to related ideas.

Theorem 24. *Suppose that P_0 is a polyomino, and we rotate it about any point $90°$, $180°$ and $270°$ degrees to get P_1, P_2 and P_3. If $P_0 \cap P_1$ is empty, then $P_i \cap P_j$ is empty for all $i \neq j$.*

[Referenced on pages 201 and 202]

Proof. If P_0 does not overlap P_1, then by symmetry P_1 does not overlap P_2; P_2 does not overlap P_3; and P_3 does not overlap P_0. We will show P_0 does not overlap P_2; the remaining cases follow by symmetry.

Let's assign coordinates such that the center of rotation is at $(0,0)$, one cell is a unit square, with its edges aligned to the axes.

Note, the center of rotation can fall on either a cell center, or a vertex; it does not affect the proof.

Also note that if the center of rotation falls on a cell, that cell cannot be part of P_0, since otherwise it will also be part of P_1, which contradicts the hypothesis.

Now suppose that P_0 overlaps P_2. This means there is are two cells (x,y) and $(-x,-y)$ that are part of P_0. Since P_0 is connected, there is a path from (x,y) to $(-x,-y)$. By Theorem 23 this path must intersect with a rotated copy, which means P_0 must intersect P_1, which is impossible. Therefore, P_0 cannot overlap P_2. □

1.5 Summaries of Theorems

Name	Principle	Thm.
Area Criterion	If each tile in a tileset is divisible by n, they can only tile a region whose area is divisible by n.	1
Partitions	If a region can be partitioned into subregion that are tileable, then the region is tileable.	2
Rectangles	$R(m,n)$ can tile $R(am,an)$.	22
Transitivity	If a set $\mathcal{T}_2$ can tile all polyominoes in a set $\mathcal{T}_2$, and $\mathcal{T}_2$ tiles R, then $\mathcal{T}_1$ tiles R.	3
Perfect Symmetry	A tiling by a single polyomino that has perfect symmetry is unique.	4
Knob Criterion	If a set tiles a region, then every knob of the region must be expressible as an integer sum of knobs from the set.	20
Row Criterion	If a set tiles a region, then every row of the region must be expressible as an integer sum of rows (and columns if rotation is allowed) from the set.	21

Table 4: Fundamental principles.

Partitions	k or less	Thm.
$\mathcal{C}_4$	$\frac{n-2}{2}$	17
$\mathcal{C}_4, \mathcal{C}_6$	$\frac{n}{4}$	18
$\mathcal{C}_4, \mathcal{C}_6, \mathcal{C}_8$	$\frac{3n+4}{16}$	19

Table 5: Partition theorems.

1.6 Further Reading

Golomb (1996) and Martin (1991) are the classical references on polyominoes; both focus on their recreational aspects and do not require much mathematical background. Toth et al. (2017, Chapter 15) gives a broad overview of polyominoes, and Soifer (2010, Chapter 1) discusses several problem-solving techniques and contains various polyomino tiling exercises[11]. Guttmann (2009) is an advanced book, and deals with the types of problems that have applications in physics.

Winslow (2018) gives a list of interesting open problems concerning polyominoes.

There is a substantial number of web sites dealing with polyominoes. Below are the ones I think are most useful for reference purposes; they all contain lots of data.

- Grekov (2013) lists the number of known tilings of rectangles by various polyomino tile sets.
- Reid (2003a) contains lists of prime rectangles for polyominoes that tile rectangles.

[11] Chapter 5 is a note that also deals with a polyomino tiling problem.

- Dahlke (2001) contains some tilings of rectangles by various polyominoes, as well as analysis for polyominoes that don't tile rectangles.
- Myers (2019) contains lists of the number of polyominoes that can tile the plane in various ways.
- Oliveira e Silva (2015) contains lists of counts of polyominoes, broken down by the area of their holes and symmetries.

The magnificent Grünbaum and Shephard (1987) covers the general topic of tiling in depth. For a lighter overview, see Ardila and Stanley (2010). The lecture notes Bassino et al. (2015) covers many topics using ideas similar to the ones in this book. There are more references for general tiling in Section 7.9.

For more on rectilinear polygons, see O'Rourke (1987, Chapter 2) and Preparata and Shamos (2012). Theorems 18 and 19 can be used to prove certain *art gallery theorems*, which is the topic of O'Rourke (1987), Toth et al. (2017, Chapter 28) and Urrutia (2000).

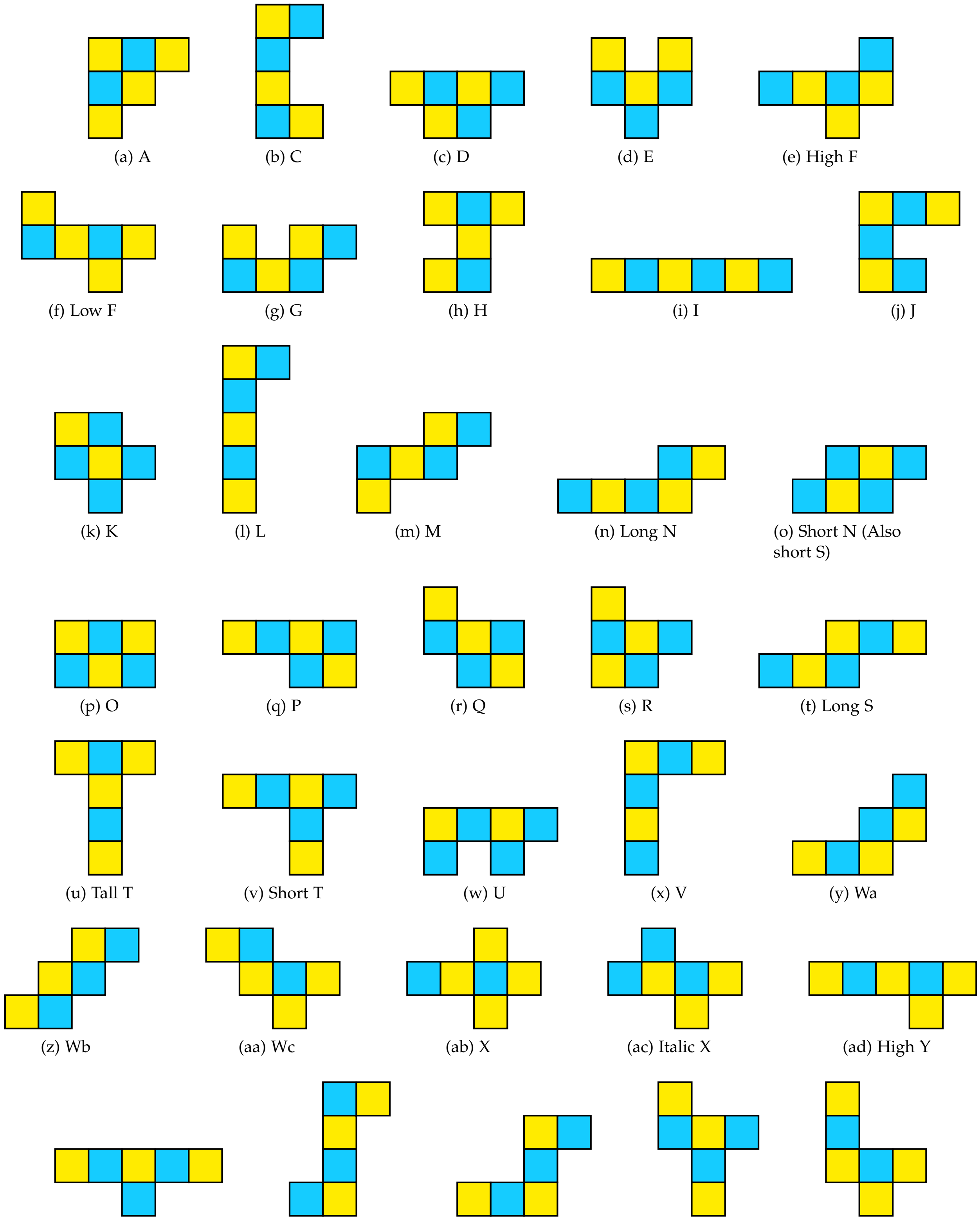
For more on Pick's Theorem, see Varberg (1985) and Grünbaum and Shephard (1993). For a more extensive list of references on Pick's Theorem, see Dubeau and Labbé (2007, Section 1). A very nice application of Pick's Theorem to solve a polyomino compatibility problem is given as Theorem 330. (Polyominoes are compatible if they can tile the same region, discussed in Section 8.1).

I asked for a proof of Theorem 23 on Mathematics Stack Exchange (see Herman Tulleken (un.) (2019)). This question prompted and uncovered several related questions. See YiFan (un.) (2019), Calum Gilhooley (un.) (2019a), Calum Gilhooley (un.) (2019b) and Hugo Tremblay (un.) (2019).

If you read Latvian, then Cibulis (2001a) and Cibulis (2001b) contains a large number of pentominoes puzzles with solutions and related information.

1.7 *Hexominoes and Heptominoes*

This section shows the hexominoes and heptominoes. The names given for hexominoes are from Kadon Enterprises, Inc. (2001), although these names are not standard (for example, Reid (2003a) use different names for some). It is in fact quite tricky to remember the names and distinguish similarly looking pieces; using the visual symbols directly avoids confusion and is what I have done (mostly) in this book.



(a) A (b) C (c) D (d) E (e) High F (f) Low F (g) G (h) H (i) I (j) J (k) K (l) L (m) M (n) Long N (o) Short N (Also short S) (p) O (q) P (r) Q (s) R (t) Long S (u) Tall T (v) Short T (w) U (x) V (y) Wa (z) Wb (aa) Wc (ab) X (ac) Italic X (ad) High Y (ae) Low Y (af) Tall Z (ag) Short Z (ah) High 4 (ai) Low 4

Figure 24: Hexominoes

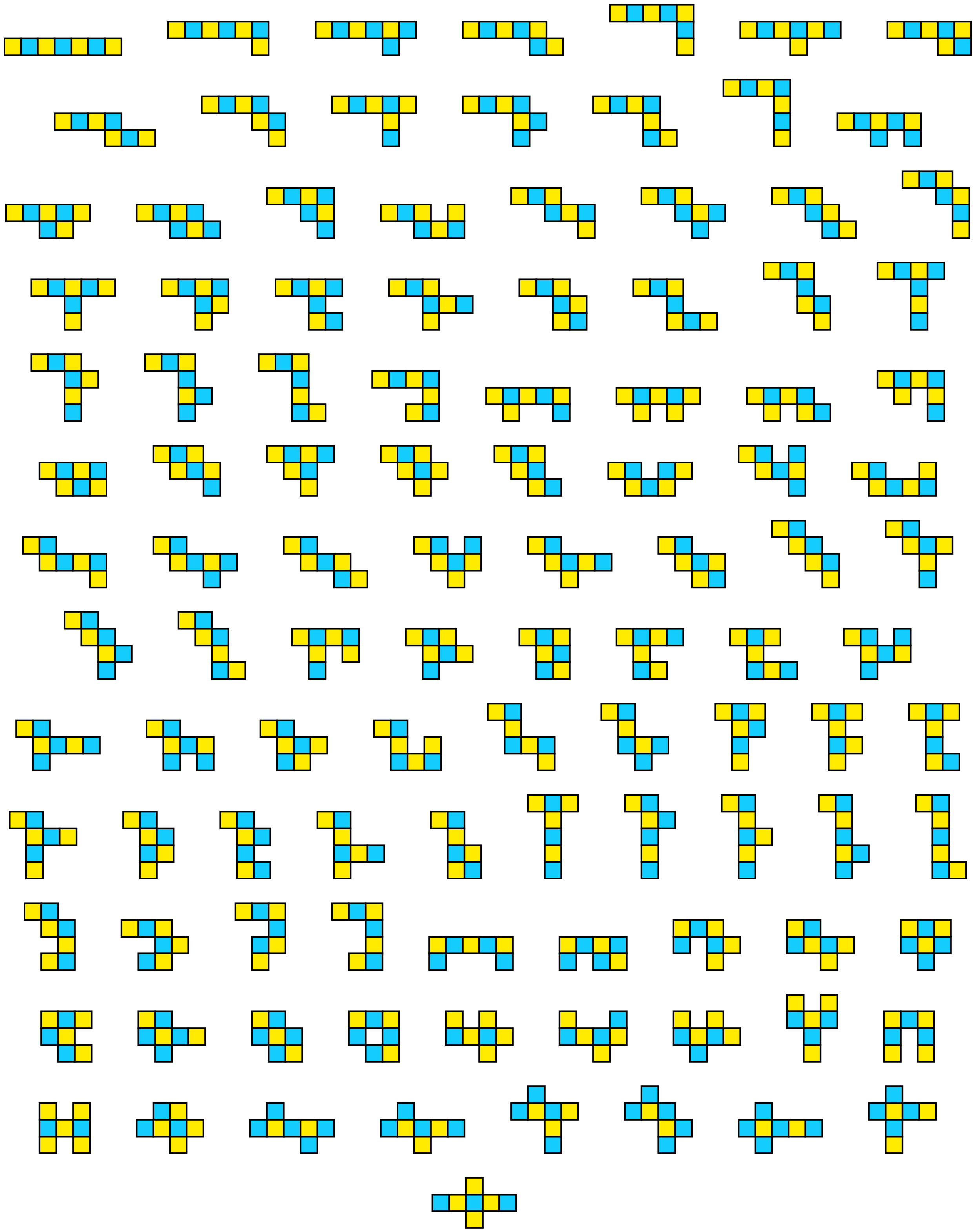


Figure 25: Heptominoes

2

Dominoes I

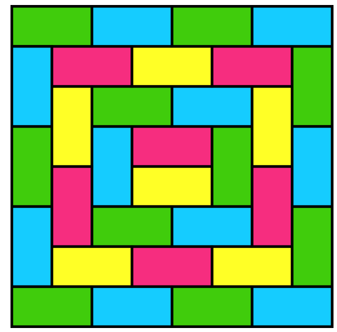


Figure 26: A tiling of a 8×8 square by dominoes.

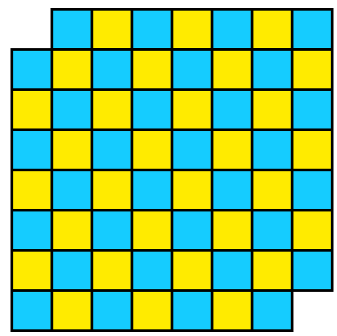


Figure 27: The mutilated chessboard.

FIGURE 30 SHOWS some examples of regions that cannot be tiled by dominoes. That some regions cannot be tiled by dominoes is intriguing. How can shapes as small as dominoes not be arranged to fit into a region as big as in Figure 30(e), when there are so many options?

This is the topic of the first section: understanding why some regions have a tiling by dominoes and others don't.

Figure 70 provides us with another mystery. In each region, there are dominoes that fit into the tiling in just one way (they are marked in yellow). In the first region it is easy to see why this must be so, but what is happening in the last region? How is it not possible to find a tiling so that at least some of those dominoes lie in a different position?

This is the topic of the second section: understanding how the same region can be tiled in different ways and how the region can force dominoes into certain positions.

A FAMOUS PROBLEM, the *mutilated chessboard problem*[1], illustrates some key ideas from each section.

Consider an 8×8 chessboard, with two opposite corners removed as in Figure 27. Can the board be tiled? If you know this problem, you know that it cannot. Because, every domino must cover exactly one black and one white square, and with opposite corners removed, there are more cells of one color than the other, and so a tiling is impossible. Of course, this principle can be applied to any region, and gives us a valuable tiling criterion. We will use this as the basis for constructing tiling criteria more powerful than the area criterion (Theorem 1) that I gave in the previous chapter.

If we remove two squares of opposite colors instead, is a tiling always possible? The answer is, *yes*, a tiling is always possible. Here they key is to notice that we can divide the board into two strips, as

[1] The mutilated chessboard problem, was first proposed by Max Black in Black (1947), and has been discussed in various places, including (Golomb, 1996, p. 4), (Martin, 1991, p. 1-4, 7-9), (Mendelsohn, 2004) and Engel (1998).

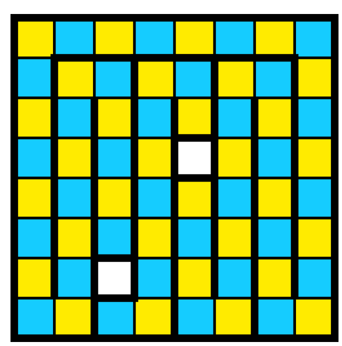


Figure 28: Dividing the chessboard into two strips

shown in Figure 28. And because the cells have different colors, no matter where we remove the two cells from, the end-points of each strip must have different colors, and so it has an even number of cells. Each of these strips has an even number of cells, and is tileable in the obvious way. (We will go over this logic in more detail once we made some proper definitions.) See Figure 29.

One way to show that a region is tileable, is to partition it into strips with an even number of cells. Strips correspond to paths, and hence can be open or closed. Closed strips have at least two tilings, and this is the basis of Section 2.2. We will see that cells that can only be tiled one way can never be part of a closed strip, and that all tilings of a region can be obtained from the two tilings of each closed strip in it.

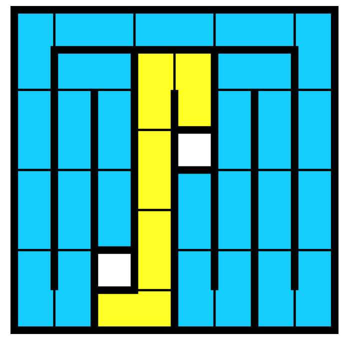


Figure 29: A tiling along the strips.

2.1 *Tiling Criteria*

In this section we develop some tools with which we can tell whether a region is tilable by dominoes or not.

We could, of course, use brute force to check all the regions in Figure 30 (or get a computer to do it). There are two reasons to look for something better:

- So that we can write faster programs. While a naive program might deal with all our examples in seconds, scientists using tilings to understand how molecules fit in solids need something better. Their "tiling problems" may be regions with millions of cells!

- So that we can understand how these tilings work, and understand other related phenomena. Section 2.2 gives a taste of this.

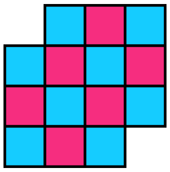
Our goal is to develop four techniques:

- A flow criterion: a technique of coloring cells in a region and analyzing the counts of dominoes that must cross the borders of subregions.

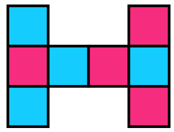
- Cylinder reduction: a geometric transformation that preserves the tileability of a region and can be used to simplify regions.

- The marriage theorem: another way to determine if a tiling exists.

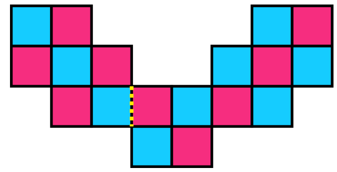
- A generalized way of coloring regions to exhibit their untiliablity.



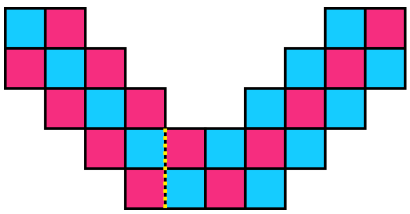
(a) An example of an unbalanced polyomino.



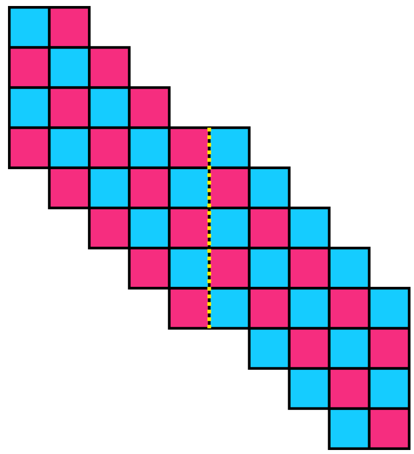
(b) An example of a balanced non-compact polyomino that cannot be tiled.



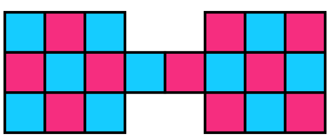
(c) An example of a balanced compact shape that cannot be tiled.



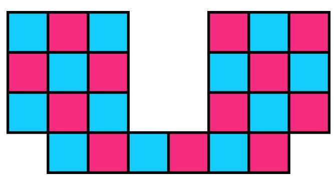
(d)



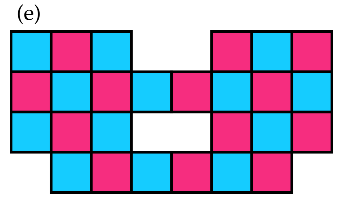
(e)



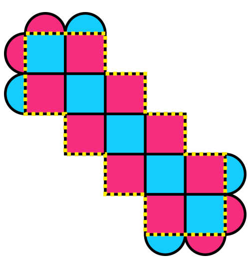
(f)



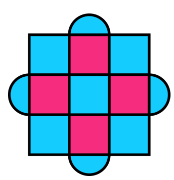
(g)



(h)



(i) No matter how this region and the next one are extended at the blobs, the resulting regions will not be tileable



(j)

Figure 30: Regions that cannot be tiled with dominoes.

We do not arrive at the state-of-the-art in solving domino tiling problems (just yet), however, the tools will help us deal with all the regions in our examples in Figure 30, and give us an intuitive understanding of how to tackle tiling problems.

A key point to understand from this section is: the border of a region is important, and tell us a lot about how a region can be tiled.

2.1.1 *Flow*

SINCE ALL TILINGS MUST SATISFY the area criterion (Theorem 1), we know a tiling by dominoes can exist only if the area of the region we wish to tile is even. Not all regions with even area are tileable (all the regions in Figure 30 have even area).

Now suppose we divide a region into two parts, each with an odd area. If the original region is tileable, then we know that there must be at least one domino that lies in both of the two parts. In fact, we know the number of dominoes that crosses the border between the two parts must be odd, otherwise the remaining regions will have odd area and not be tileable. Figure 31 illustrates this principle.

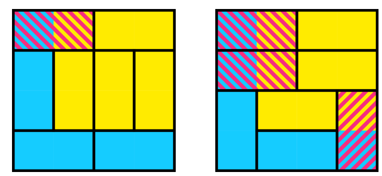


Figure 31: Two examples of the tiling of a square. The square is divided into two partitions (yellow and blue). Since both partitions have odd area, any tiling such as the two shown above must have an odd number of dominoes that cross the border.

Theorem 25 (Border Crossings Theorem)**.** *In a subregion S of a region with a tiling, the number of dominoes that cross the border of the subregion must have the same parity*[2] *as the area of the subregion.*

[2] Even or odd.

[Referenced on pages 40, 43, 62, 100, 122 and 128]

Proof. Let k be the number of dominoes that cross the border of S. Each of these dominoes has only one cell in S. If we remove these cells to form a new region S', we have a region that is tileable, with area $|S'| = |S| - k$. Since S' is tileable, $|S'|$ must be even (by Theorem 1), and so $|S|$ and k must either both be odd, or both be even. □

Below we give two applications of this theorem; as a theorem and an example.

A *bridge* is a sequence of cells in a region such that each has exactly two neighbors, and such that removing any cell in the bridge will leave the region disconnected. For example, Figure 30(f) and (g) each has a bridge, Figure 30(h) does not (since no cell splits the region when removed). Figure 30(c) also does not have a bridge; although there are single cells that will split the region if removed, they have more than two neighbors.

Theorem 26. *Each cell in a bridge can be tiled in only one way.* [3]

[3] Such cells are called *frozen*. We will define this concept in the next section.

[Not referenced]

Proof. Let R be a region with a bridge, and let v be a cell in that bridge. Because v is a cell in a bridge, it has exactly two neighbors– let's call them u and w. Further, $R - \{v\}$ is disconnected. Let S_u and S_w be the two disconnected subregions that contains u and w respectively. The border of each of these share exactly one edge with the cell v, in particular, S_u shares a border with v at the edge between u and v.

Now consider S_u. If $|S_u|$ is odd, then the number of dominoes that crosses the border of S_u is odd. The only place where a domino can cross is for a domino to cover both u and v, and therefore any tiling of R must have a domino in this position.

If on the other hand $|S_u|$ is even, then the number of dominoes that cross the border of S_u is even. Since there is only one place where a domino can cross, it means the number of dominoes that cross must be 0. Therefore, a single domino cannot cover both u and v, and so, a single domino must cover v and w. Therefore, every tiling of R must have a domino in this position.

Taken these together, v can only be tiled one way. The same argument applies to all other cells in the bridge, and therefore the bridge has a unique tiling in R. □

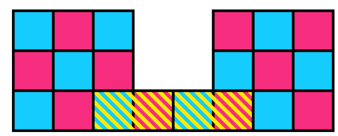


Figure 32: The cells marked yellow can be tiled in only one way.

The next example uses Theorem 25 in a more sophisticated way to prove that a tiling does not exist. [4]

[4] The author mentions this idea has been known before.

Example 6 ((Mendelsohn, 2004))**.** *Let's look at the mutilated chessboard. The top row has an odd area. Therefore, an odd number of dominoes must cross its border (Theorem 25), and only vertical dominoes that also lie in the second row can do that (otherwise, parts of dominoes will fall outside the mutilated chessboard).*

The second row has even area, and so an even number of dominoes must cross its border. We already have an odd number of dominoes that cross the border from the top, therefore we must have an odd number of vertical dominoes that cross the border to the bottom.

Following the same argument, we get that there must lie an odd number of vertical dominoes between each pair of adjacent rows. There are 7 such pairs, so we can conclude the total number of vertical dominoes must be odd.

Applying this idea to the columns, we find there must be an odd number of horizontal dominoes too. So the total number of dominoes must be even. But to cover the 62 squares we need 31 dominoes, which is odd. And therefore, a tiling is impossible.

This example shows that we can always determine the parity of the number of vertical and horizontal dominoes in a tiling. For a tiling to exist, this must be consistent with the parity of the total number of dominoes (which we can deduce from the area).

SATISFYING THE AREA CRITERION is not enough (all the regions in Figure 30(a)-(h) are untileable and have even area), and checking the parity as in the previous example is also not enough (for example, Figure 30(f)).

We will now look at the color argument we discussed at the beginning of the chapter, and see how far it gets us. The **checkerboard coloring** plays an important role in our discussions going forward, and to make it easier to talk about and prove details we introduce some additional notation and terminology.

We use $\mathbb{Z}_n$ for the set $\{0, 1, 2, \cdots, n-1\}$. A coloring is a function $K : \mathbb{Z}^2 \to \mathbb{Z}_n$; each cell is assigned an integer value, which we think of as a color. The checkerboard coloring is the coloring denoted F_2, defined as $F_2 = (x+y) \bmod 2$; it has two colors that we will call black (for 0) and white (for 1).

In a region R with checkerboard coloring applied, let $W(R)$ denote the white cells in R, and let $B(R)$ denote the black cells in R. The **deficiency**[5] of R is defined as

[5] Also called *bias*.

$$\Delta(R) = |B(R)| - |W(R)|. \tag{2.1}$$

If the deficiency of a region is 0, the region is **balanced**.[6]

[6] The term *balanced* is used informally in (Golomb, 1966, p. 17). The term , as well as the white and black functions, are defined explicitly in for example Thiant (2003). The term is also used in Herzog et al. (2015) for a completely different concept. The words *biased* and *unbiased* are sometimes used instead of unbalanced and balanced, for example Mason (2014).

The functions W, B and Δ depend on which of two ways the image has been colored. However, the absolute value of the deficiency and "being balanced" are inherent features of a region, and are independent of the coloring used.

Theorem 27 (Checkerboard Criterion. golomb1996polyominoes, p.4)**.** *For a region to be tileable by dominoes, it must be balanced .*

[Referenced on pages 50, 54, 55, 80, 114, 116 and 128]

Problem† 12.

(1) *Prove that* $\Delta(R) \equiv |R| \pmod 2$. *Note that this implies that the area criterion is redundant.*

(2) *Show that* $\Delta(R(m,n))$ *equals* 0 *or* 1 *depending on whether the area of the rectangle is odd or even.*

(3) *Prove* $|\Delta(R)| \leq |R|$.

(4) *Prove that if* R *is partitioned into two subregions* S_1 *and* S_2*, then* $|\Delta(R)| \leq |\Delta(S_1)| + |\Delta(S_2)|$.

Using this criterion, we can prove Figure 30(a) is not tileable. However, it *still* is not enough, since all the regions in Figure 30(b)-(h) are balanced and untileable.

Problem† 13. *Find examples of non-tileable balanced regions.*

What is the largest the deficiency can be? Figure 33 shows we can make the deficiency as large as we want by extending it as shown. However, it is bounded by the number of cells, as stated in the following theorem.

Theorem 28. *The deficiency of a connected region R is bounded by the number of cells as follows:*

$$|\Delta(R)| \leq \frac{|R|+1}{2}.$$

[Not referenced]

Proof. Let $B = B(R)$ and $W = W(R)$. WLOG, assume that $B > W$, so $\Delta(R) > 0$. We will shows that $B \leq 3W + 1$, from which the result follows. Because if $B \leq 3W + 1$, we have $2B - 2W \leq W + B + 1$, that is, $2\Delta(R) \leq |R| + 1$, or $\Delta(R) \leq \frac{|R|+1}{2}$.

We now prove $B \leq 3W + 1$. Suppose not; that is, suppose $B > 3W + 1$. Then there must be at least one white cell with 4 black neighbors, with none of these 4 black cells having any other white neighbors. If there are no other cells than these 5 cells, we have $4 = B = 3W + 1$, so there must be more. But then these 5 cells cannot be connected to any other cells, and we have a disconnected region which contradicts our hypothesis that the region is connected. □

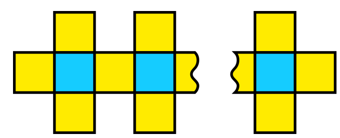


Figure 33: A family of regions that obtain maximum deficiency.

This upper bound is achievable by regions like the one shown in Figure 33. A polyomino that achieves the maximum amount of bias possible for its area is called **maximally biased** (Mason, 2014). The number of maximally biased polyominoes is shown in Table 6.

n	A234013	n	A234013
1	1	11	36
2	1	12	8
3	2	13	2
4	1	14	540
5	1	15	164
6	11	16	31
7	8	17	4
8	3	18	3174
9	1	19	749
10	79	20	103

Table 6: The number of maximally biased polyominoes.

$\|R\|$	$\|\Delta(R)\|$
$4k$	$2k$
$4k+1$	$2k+1$
$4k+2$	$2k$
$4k+3$	$2k+1$

Table 7: Maximum deficiency.

Problem† 14. *(Mason, 2014) Show the following facts about $4n+1$-sized biased polyominoes, with $B(R) > W(R)$:*

- *They are all of made from a monomino and T-tetrominoes like the one shown in Figure 33.*
- *The removal of any white square will result in an illegal, disconnected polyomino.*
- *The same for any twice-connected black square.*
- *The removal of any once-connected black square will result in a legal, maximally biased polyomino of size $4n$.*
- *The addition of a white square will result in a maximally biased $(4n+2)$–polyomino;*

Not all maximally biased $(4n+2)$-polyominoes can be generated from $(4n+1)$-maximally biased polyominoes, as shown in Figure 34 (Mason, 2014).

Problem† 15.

(1) *Show that the maximum perimeter for a polyomino with area n is* $2n+2$. *(See also Theorem 15).*

(2) *Show that the number of polyominoes with area n that have a perimeter of* $2n+2$ *(**A131482**) is the number of maximally biased polyominoes for* $n = 4k+1$.

See also Problem 7.

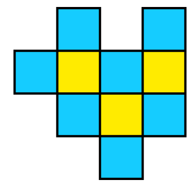


Figure 34: A maximally biased polyomino that cannot be formed by extending a maximally biased polyomino with one less cell.

In the same way that we turned the area criterion into something more useful by looking at what must happen at the border of subregions, we now turn the color criterion into something more powerful by looking at the border of subregions.

Let R be a region with the checkerboard coloring applied, and let S be a subregion of R. If w is the number of dominoes covering the border of S with a white cell inside S, and b is the number of dominoes covering the border with their black cells inside R, then we define the **flow**[7] of S as

$$\phi(S) = b - w. \tag{2.2}$$

[7] This definition is essentially given in Saldanha et al. (1995).

See Figure 35.

When $S = R$, the flow is 0, since there are no dominoes that cross the border, and so $w = b = 0$.

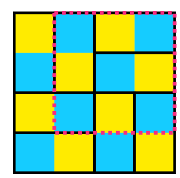


Figure 35: Let S be the region inside the red border. Then $b = 2$, $w = 1$, and so $\phi(S) = 2 - 1 = 1$. Also, $B(S) = 5$, $W(S) = 4$, and so $\Delta(S) = 5 - 4 = 1$.

Theorem 29 (The Flow Theorem)**.** *Let S be a subregion of R, and suppose R has a tiling by dominoes. Then the flow of S equals the deficiency of S, that is,*

$$\phi(S) = \Delta(S). \tag{2.3}$$

[Referenced on pages 44, 62, 63 and 128]

Proof. Let W (B) be the number of white (black) cells inside S, let w (b) be the number of dominoes that cross the border with their white (black) cells inside S, and let k be the number of dominoes completely in S. Then $B - W = (b+k) - (w+k)$, and so $B - W = b - w$, and thus $\Delta(S) = B - W = b - w = \phi(S)$.

□

Note the similarities between border crossings theorem (Theorem 25) and the flow theorem above. Both give us information about what happens at the border based on what is inside the region. But the flow theorem gives us much more information; Theorem 25 merely tells us the parity of the number of dominoes that cross the border. The flow theorem gives us the difference of the two different types of dominoes.

Also note that the area must have the same parity as the flow: if the area is even, then W and B are both odd or both even. In either case, their difference is even. This means with the flow theorem in place, the border crossing theorem is now redundant. [8]

[8] One reason to differentiate the two theorems is that the border crossings theorem is easier to generalize to other tile sets.

The following examples show how to use the flow theorem to prove a region is not tileable.

Example 7. *Consider Figure 30(c), and let's choose the subregion as the shape left of the dotted line. Now since $W = 3$ and $B = 4$, we have $|W - B| = 1$, so we know a domino must cross the dotted line. Removing this domino partitions the shape into two shapes, each of which is untileable because they don't satisfy the area criterion.*

Example 8. *Consider Figure 30(d), and choose the subregion as the region the right of the dotted line. We have $|B - W| = 2$, which implies dominoes must overlap the dotted line in two places. But this means $b = w = 1$, and so $|b - w| = 0$, which is impossible if the region is tileable. Therefore, it is not tileable.*

Example 9. *Consider Figure 30(e). Partition it in halves by a vertical cut through the middle. Then $|B - W| = 4$, but the maximum value that $|b - w|$ can have is 3. Therefore, the region is not tileable.*

Example 10. *Consider Figure 30(i), and consider the colored subregion. The deficiency $|B - W| = |8 - 5| = 3$. This means, at least three dominoes must overlap the border. However this is done, we are always left with a region with 4 black and 3 white squares, which is untileable. Therefore, the entire region is untileable.*

Problem† 16. *If we have a domino tiling of a region, show that we can determine the parity of vertical dominoes with their top squares black. (Follow the type of reasoning used in Example 6.)*

If you tried your hand at Problem 13 and followed the examples above, you may have noticed that you can create an untileable region by having an abundance of black on the one side of the region, and a abundance of white on the other side, with a choke point between the two parts. The following theorem gives a formulation of this idea.

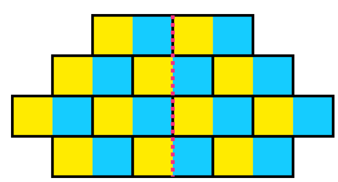


Figure 36: The deficiency of the left partition is -2, so the length of a cut through the center must be at least 3.

Theorem 30. [9] *Suppose we apply the checkerboard coloring to a tileable region, and partition it into two subregions with a straight cut. If one subregion has W white cells and B black cells, then the cut must have length at least $2|B - W| - 1$.*

[9] In Kenyon (2000c) the author mentions that a very similar theorem has been proven in Fournier (1996) (in French).

[Referenced on page 79]

Proof. Assume $B \geq W$. From the flow theorem (Theorem 29), we know that the number of tiles that cross the cut must be at least $B - W = b - w$. From this we get $b = B - W + w$, so $b \geq B - W$ (since w is non-negative). So the number of dominoes that cross the cut with a white cell inside the subregion must be at least $B - W$. Along a straight cut, there are at most $\left\lceil \frac{L}{2} \right\rceil$ places where this can happen, so it needs to have length at least $2(B - W) - 1$.

If we assume $W \geq B$, we can show the length of the cut must be at least $2(W - B) - 1$ following the same argument as above, reversing the roles of black and white.

Putting these together, we arrive at the result: the length of the cut is at least $2|W - B| - 1$. □

See Figure 36 for an example.

Problem† 17. *What if the cut is not straight?*

You may also have noticed that to create unbalanced regions or parts of regions, you have to manipulate the border of the region so you have a lot of corners of the same color. Also, we have already seen some theorems that relate the border of a region to what is going on inside. The following theorem shows we can determine the deficiency from what is going on at the border alone.

Theorem 31. *Let b be the number of black edges on the border, and w be the number of white edges on the border. Let B be the number of black squares and W be the number of white squares. Then $b - w = 4(B - W) = 4\Delta(R)$.*

[Referenced on pages 46, 79, 80 and 124]

Proof. Consider building a region cell by cell. At each stage, we can either add a white or a black square. If we add a white square, it is a neighbor of 0, 1, 2, 3, or 4 other cells, all of which must be black. Each exposed edge must be white, and each unexposed edge must reduce the total number of black edges. So the total amount that $b - w$ decreases is 4. A similar argument shows that if we add a black square, $b - w$ is increased by four. In other words: $b - w = 4(B - W)$. □

It follows that $B = W$ if and only if $b = w$.

Problem† 18. *A corner cell is a cell with two adjacent edges that are not shared with other cells. Let R be a region such that each of its corner cells have no neighbors that are on the border (that is, all neighbors of each corner cell are interior cells). Prove R is untilable.*

Theorem 32 (Csizmadia et al. (1999), Theorem 2.1, Theorem 3.1)**.** *If all n edges of a simply-connected region R have odd length,*

(1) *the deficiency of R is given by*

$$\Delta(R) = \frac{n}{4},$$

(2) *the region is unbalanced, and*

(3) *the region is untilable by dominoes.*

[Referenced on page 46]

Proof. [10] Suppose one corner edge is black. Then all corner edges are black. This means the ends of sides are all black. If B_i and W_i is the number of edges on side i that border black and white cells respectively (in the region), for each side i we have:

$$B_i - W_i = 1,$$

and summing over all sides:

$$\sum_{i=1}^{n}(B_i - W_i) = n.$$

But $\Delta(R) = \frac{\sum_{i=1}^{n}(B_i - W_i)}{4}$ (by Theorem 31), and thus $\Delta(R) = \frac{n}{4} > 0$, and so R is unbalanced, and hence untilable by dominoes. Note we already proved that n is divisble by 4 in Theorem 7. □

[10] The proof here is new. The proofs in Csizmadia et al. (1999) is quite complicated.

The theorem does not hold for regions with holes. For example, the 3×3 square with its center removed has all its sides odd, but it is balanced and even tileable (Figure 37).

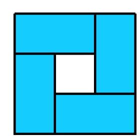


Figure 37: A region with a hole that has all its edges with odd length that is balanced and tileable by dominoes.

Theorem 33. *A balanced polyomino must have at least two sides of even length.*

[Referenced on page 128]

Proof. For the polyomino to be balanced, it must at least have one even side (Theorem 32). We also know the perimeter must be even (Theorem 6). Therefore, the number of odd sides must be even. But the total number of sides must be even (Theorem 5), and so the number of even sides must *also* be even. Thus, there must be two or more even sides. □

Theorem 34. *A polyomino with all sides even is tileable. (Mentioned in Kenyon, 2000b, Section 8).*

[Referenced on page 128]

Proof.

(1) For regions with no holes:

  (a) If the figure has exactly four knobs, it must be a rectangle. In this case we can tile the region with horizontal dominoes only.

  (b) If the figure has more than four knobs, we can remove one. Suppose a knob B lies between sides A and C, with $A \leq C$. We can remove a rectangle S_1 with sides of length B and A, and since both are even we can tile S_1 with horizontal dominoes. We can repeated this process until a single rectangle is left, which we can tile as we did above.

  Note that the long edge of a domino is shared with at most one domino.

(2) For regions with holes:

  (a) First tile the filled polyomino (i.e. the polyomino with all holes filled) with horizontal dominoes as in the procedure above.

  (b) Remove all dominoes that lie completely inside a hole.

  (c) The ones that lie partially in a hole come in pairs, since all sides are even. Each pair can be replaced with a single domino that covers only the cells that do not lie in the hole.

  This completes the tiling.

□

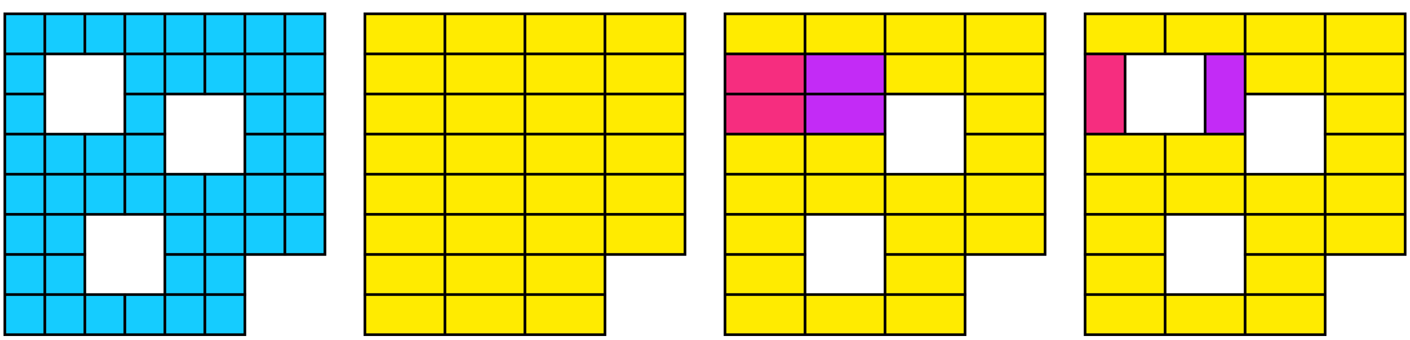


Figure 38: An example showing the procedure in the proof.

Problem† 19.

(1) *Can you modify the theorem's conditions to be weaker?*

(2) *Prove that if the sides of a region is divisible by n, it can be tiled by bars of length n if holes are at least $n-1$ units apart.*

(3) *If we use all n-ominoes for the tileset, can holes be anywhere?*

2.1.2 *The Marriage Theorem*

INFORMALLY, THE *marriage theorem* states the following[11]: Suppose we have a group of k white cells from a region R, and they have n black neighbors in R. If $n < k$, then no tiling by dominoes exists. Moreover, if every group of white cells in R has at least as many neighbors as white cells in the group, a tiling exists.

[11] This is a specific version of a much more general theorem given in Hall (1935) that is now called the *marriage theorem*, with applications that go beyond tiling. Covering the more general results fall outside the scope of this book, but we give some references in the *Further Reading* section. This version is given in Ardila and Stanley (2010).

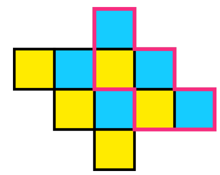


Figure 39: In this region, the cells surrounded by a pink line is a black bad patch. The rest of the region is a white bad patch.

Before we prove it, we need some terminology to make it easier to make our statements.

A subregion S of R is called a **white patch** of R if all the neighbors of its white cells are also in S, and there are no other black cells in S. In other words, all black cells in S have at least one neighbor also in S. A similar definition can be made for **black patch**.

A region itself is a patch. If a patch is not equal to the region we call it a **proper patch**. A white patch can be, but is not necessarily, a black patch. In fact, if R is connected, a white patch is also a black patch only if it is the entire region.

Theorem 35. *Suppose R is connected, and S is both a white patch and a black patch. Then $S = R$.*

[Not referenced]

Proof. Suppose R has some cells that are not in S. Since R is connected, we must have at least one of these cells be a neighbor to a cell in S. Let's call this cell u, and its neighbor in S, v. Suppose v is black, then because S is a black patch, all its neighbors must lie in S, and this contradicts that u lies outside S. And if v was white, because it is a white patch, all its neighbors must lie in S. Therefore, there can be no cells in R that are not also in S, and therefore $S = R$. □

Theorem 36. *Let S be a white patch of a region R. Then $R - S$ is a black patch of R. Similarly, if S is a black patch of R, then $R - S$ is a white patch.*

[Referenced on page 49]

Proof. We only prove the first part; the second part can be proven with the exact same argument with the roles of white and black reversed.

Suppose S is a white patch, but $R - S$ is not a black patch. Then there is a black cell u in $R - S$ with a neighbor v *not* in $R - S$. Then v must lie in S, and it is white. Since S is a white patch, all the neighbors of v, including u, must lie in S. We arrive at a contradiction, and so $R - S$ must be a black patch. □

We call a white patch **bad** if it has fewer black than white neighbors, and **good** otherwise. Similarly for black patches.

Theorem 37. *A balanced region has a bad black patch if and only if it has a white bad patch.*

[Referenced on pages 49, 50, 61 and 97]

We only prove the first part, the second part follows from the same argument reversing the roles of white and black.

Proof.
If. Suppose that S is a bad white patch. Then $R - S$ is a black patch (Theorem 36). Since S has fewer black than white cells and R is balanced, $R - S$ must have fewer white cells than black cells, and so it is bad. □

It follows directly from this that if all white patches in a region are good, then so are all black patches.

Theorem 38 (The Marriage Theorem)**.** *A region is tileable if and only if it has no bad patches.*

[Referenced on pages 51, 52, 61 and 128]

Proof.
If.[12] Suppose all the white patches of a region are good. It follows that all the black patches of the region must be good too (and vice versa) by Theorem 37. We can now partition the region into two subregions S and $R - S$ in one of two ways:

(1) If all the proper white patches of R are unbalanced, then each patch of R must have strictly more black cells than white cells. Let S be a white cell and its neighbor. Then S is tileable (with a single domino), and $R - S$ is a region with all its white patches good. This follows from the fact that we removed only one black cell, so the number of black cells in each patch can can drop by at most 1, and since these are strictly bigger than the number of white cells, after the drop there must be at least as many black cells as white cells in the patch.

(2) If at least one of the patches in R is balanced, we let S be such a patch. Then S must have all its white patches good, and $R - S$ must have all its black patches good (and so, it must also have all its white patches good). Both partitions must also be balanced.

Note that a patch in S or $R - S$ need not be a patch in R.

[12] The logic of this part of the proof follows the proof in (Kung et al., 2009, p. 56), due to Easterfield (Easterfield, 1946) and Halmos and Vaughan (Halmos and Vaughan, 2009).

We can continue this process, and it must eventually end, since the number of cells in R is finite. And so, we eventually arrive at a bunch of subsets of two cells each, all tileable by a single domino, and so the entire region must be tileable by dominoes.

Only if. If the region is not balanced, we know it is not tileable by Theorem 27.

Suppose then it is balanced, and suppose it has a bad white patch. (If it had a bad black patch, it must also have a bad white patch by Theorem 37, so there is no loss in generality.)

If there is a tiling, each white cell in the white patch has an associated black neighbor that lies in the same domino, and there must be at least as many of these black cells as white cells. However, the patch is bad so this is not the case, a contradiction. Therefore no tiling exist. □

We finally have a criterion that can work for all tilings. However, the problem is that it can be difficult to find a bad patch, or to show there aren't any. For example, in a region R, the flow theorem is easier to apply (although, of course, you *could* find a bad patch.)

That does not mean it is not useful: We will use this theorem a few times to prove some other things about tilings. And it also gives us a way the come up with untileable regions easily as the next theorems show.

An **extension** of a region R is a region P such that R is a subregion of P. Informally, an extension of R is some region formed by adding cells to R.

Theorem 39. *Suppose R is a region with an odd number of cells. If we apply the checkerboard coloring such that $W(R) < B(R)$, then any extension of R that adds no new neighbors to black cells in the polyomino is untileable.*

[Not referenced]

Proof. Let the original region be R, and its extension P. Since $W < B$, R is a bad patch with respect to itself. But since we do not add any neighbors to form P, R is also a bad patch with respect to P, and therefore untileable. □

Theorem 40. *If a region has a bad patch, it has a connected bad patch.*

[Not referenced]

Proof. Let S be the bad patch. WLOG assume that $B(S) < W(S)$. Now partition S into connected disjoint sets S_i such that the (black) neighbors of all the white cells in S_i is also in S_i. We want to show one of these sets is a bad patch. If each S_i is a good patch, then

$B(S_i) \geq W(S_i)$ for all i. But then $B(S) = \sum_i B(S_i) \geq \sum_i W(S_i) = W(S)$, a contradiction. □

This theorem shows the process of generating untileable regions described above can generate every untileable region.

2.1.3 *Cylinder Deletion*

WE CAN ALSO USE the Marriage Theorem (Theorem 38) to prove certain reductions do not affect the tileability of a region.

A **vertical** n**-cylinder** is a simply-connected region where each row has n cells (see Figure 40). A **horizontal** n**-cylinder** has n cells in each column (Hochberg, 2015).[13]

We use the following notation to denote horizontal cylinders: $C_n(a_1 \cdot a_2 \cdots a_m) = \bigcup_{i=1}^{n} \{(i,y) \mid a_1 \leq y < a_1 + n\}$. We use exponentiation to denote repeated columns.

[13] Cylinders also play an important role in tiling extensions (see for example Theorem 127) and tilings of the infinite strip.

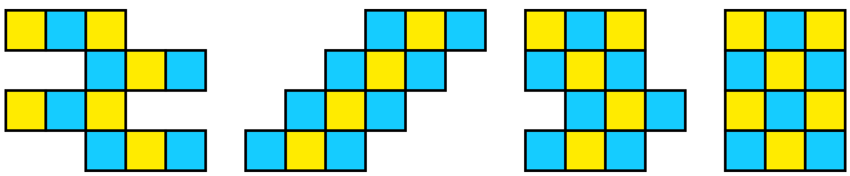
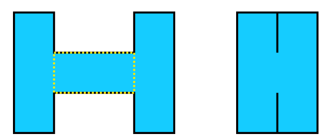


Figure 40: Examples of vertical 3-cylinders.

An n-cylinder is tileable by dominoes if n is even, since each row has an obvious tiling by horizontally-placed dominoes.



(a) (b)

Figure 41: Cylinder deletion. (a) The cylinder S is the shape contained in the yellow dotted line, and R is the entire region. (b) $R \ominus S$. Note the barriers that cannot be crossed by dominoes.

Suppose a vertical cylinder S is a subregion of R such that it shares is top and bottom borders with the border of R. If we remove S from R, and move the two pieces together, we get the new region $R \ominus S$. We call this operation a **deletion**. An analogous definition can be made for a horizontal cylinder S. After a deletion, we my be left with a region that has a barrier. For example, in Figure 41 we delete a cylinder from the double-T region. This yields a new shape with two internal barriers that cannot be crossed by dominoes. In particular, it means the two top cells are not neighbors of each other. Note that we cannot delete a horizontal cylinder from the reduced region because of the barriers.[14]

[14] We will later see how this geometrical operation is equivalent to simplifying the border word algebraically.

The operation is also very similar to deletions on rhombus tilings, also called *contractions*. In this context, the deleted shape is called a *de Bruijn section*. See for example Chavanon and Remila (2006) and Chavanon et al. (2003).

Theorem 41. *Let S be an n-cylinder with n even. Then R is tileable if $R \ominus S$ is tileable.*

[Referenced on pages 58, 96, 116 and 128]

Proof. WLOG, assume S is a vertical cylinder.

Suppose $R \ominus S$ is tileable. Then we can find a tiling for R as follows: In $R \ominus S$, the cut line is either covered by a domino or not.

Tile all the cells in R that are also in $R \ominus S$ the same way. If there are dominoes that cross the cut in $R \ominus S$, there will be two dominoes that cross the two cuts in R (in the same row). So any row in S will either have n untiled cells, or $n-2$ untiled cells. Since this number is even, we can fill the row with horizontal dominoes. this gives us a tiling for R.

□

Unfortunately, the converse of this theorem is not true (see for example Figure 42). However, a somewhat weaker version of it is true. A deletion is called **safe** when all the cells in $R \ominus S$ have neighbors in the same directions as they had in R.

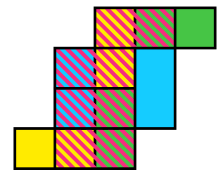


Figure 42: An example of a region R that is tileable, but $R \ominus S$ is not, for the 2-cylinder S marked pink.

Theorem 42. *Suppose S is a n-cylinder of R with n even, and that deleting it from R is safe. Then if R is tileable, then so is $S \ominus R$.*[15]

[15] I suspect something stronger is true, but have not been able to work out the details.

[Referenced on page 128]

Proof. The reduced region $R \ominus S$ has exactly the same neighbor setup as R. If R is tileable, then by the marriage theorem (Theorem 38), each set S of white cells has at least $|S|$ black neighbors. This must also hold for $R \ominus S$, since if it had a bad patch, so would R. Therefore, $R \ominus S$ is tileable. □

This method gives us a way to reduce some regions to a more manageable level.

But this method *also* gives us an algorithm to construct a tiling for R if we can find a tiling for $R \ominus S$.

Example 11. *In Figure 43 we show how a tiling can be constructed for a given region.*

First (shown in the left column of Figure 43), we successively delete 2-cylinders from the region, until the resulting region is simple enough to tile. We then tile it (if we couldn't, then its possible that no tiling exists for R. We do not know for sure, unless all the removals were safe).

Then, working backwards (shown on the right), we reinsert cylinders until we have rebuilt the original region. If the cutline goes between dominoes, we simply insert a domino perpendicular to the cutline; otherwise we insert a domino domino perpendicular to the cutline offset by a cell.

Problem† 20. *Establish the tileability of the regions in 30 by deleting cylinders until each region is manageable.*

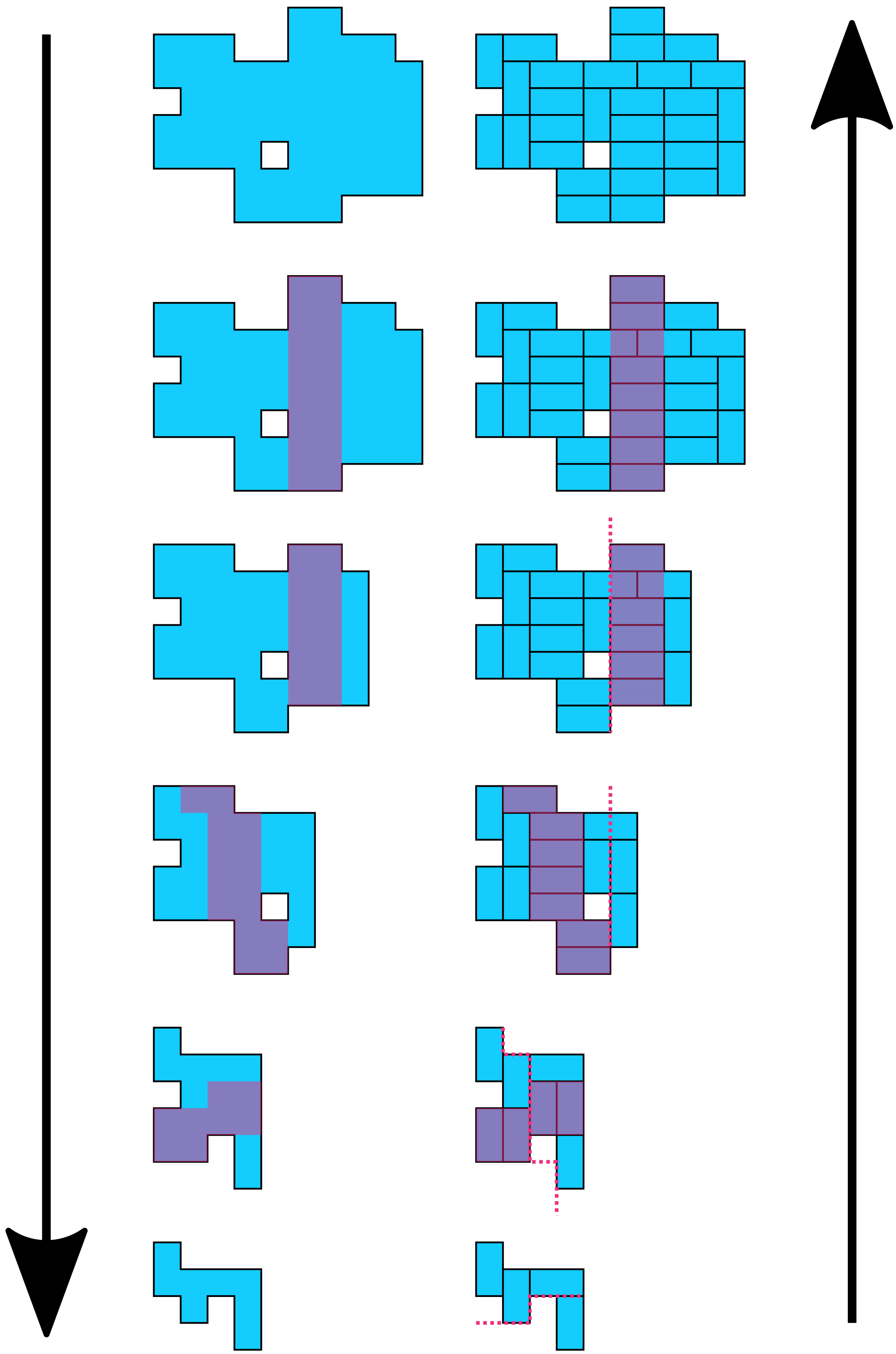


Figure 43: How to use cylinder deletion to find a tiling. On the left, from top to bottom, we delete 2-cylinders until we cannot. This region is easy to tile, shown on the bottom right. We then reinsert 2-cylinders in the reverse order (shown on the right-hand side from bottom to top). At each stage we complete the tiling in the obvious way.

Problem† 21. *Give some examples of regions that do not allow us to delete a cylinder from them. Are any of them tileable?*

Problem* 22. *Can you characterize the regions from which we can delete cylinders?*

A **Young diagram**[16] is a bar graph with columns in non-increasing order (eg. Pak, 2000). In a Young diagram, we have $a_i \geq a_j$ when $i < j$. A **triangle**[17] is a Young diagram $B(n \cdot n-1 \cdot n-2 \cdots 2 \cdot 1)$, denoted $T(n)$ (Friedman, 2006).

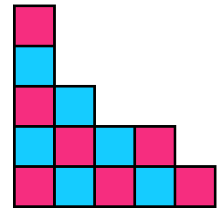


Figure 44: A Young diagram with notation $B(5 \cdot 3 \cdot 2^2 \cdot 1)$.

[16] Also called a *Ferrers diagram* (eg. Delest and Fedou, 1993) or *trapezoidal polyomino* (eg. Bodini and Lumbroso, 2009) or *partition polyomino* (eg. Leroux et al., 1998).

[17] Tilings of triangles are considered in section 8.6.

Theorem 43. *A triangle $T(n)$, with $n > 0$, is unbalanced (and therefore untileable).*[18]

[18] Compare this with Problem 15. Both are consequences of Lemma 4 in Thiant (2003). That lemma requires the notion of a *staircase* that we don't define here.

[Referenced on pages 54 and 362]

Proof. Apply the checkerboard coloring to the triangle such that the last column is black (Figure 45). Then

$$\begin{aligned} B - W &= \sum_{i=1}^{n} \left\lfloor \frac{i+1}{2} \right\rfloor - \sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor \\ &= \sum_{i=2}^{n+1} \left\lfloor \frac{i}{2} \right\rfloor - \sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor \\ &= \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{1}{2} \right\rfloor \\ &= \left\lfloor \frac{n+1}{2} \right\rfloor > 0. \end{aligned}$$

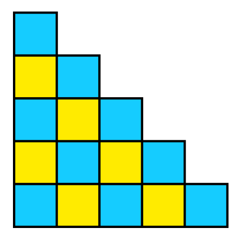


Figure 45: The triangle $T(5) = B(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$.

So the polyomino is unbalanced, and therefore untileable (Theorem 27). □

Theorem 44. *A balanced Young diagram must either have at least two adjacent columns equal, or at least two adjacent rows equal.*

[Referenced on pages 54, 116 and 356]

Proof. [19] If a Young diagram does not satisfy those conditions, it must be a triangle $T(n)$, which is not balanced by Theorem 43. Therefore, a balanced polyomino cannot have this form, and so a balanced polyomino must satisfy the conditions given. □

[19] See also Problem 117(3a).

Theorem 45. *A Young diagram is tileable if and only if it is balanced*[20].

[20] See also Problem 18.

[Referenced on pages 55 and 128]

Proof.

If. By Theorem 44 we know the polyomino has either two adjacent columns of equal length, or two adjacent rows of equal length. In either case, we can delete a 2-cylinder from the region. Since the resulting region must still be balanced, the conditions apply again, and we can repeat the process. The process must eventually end with the empty region, and this proves the Young diagram is tileable.

Only if. If a Young diagram is tileable it is balanced by Theorem 27. □

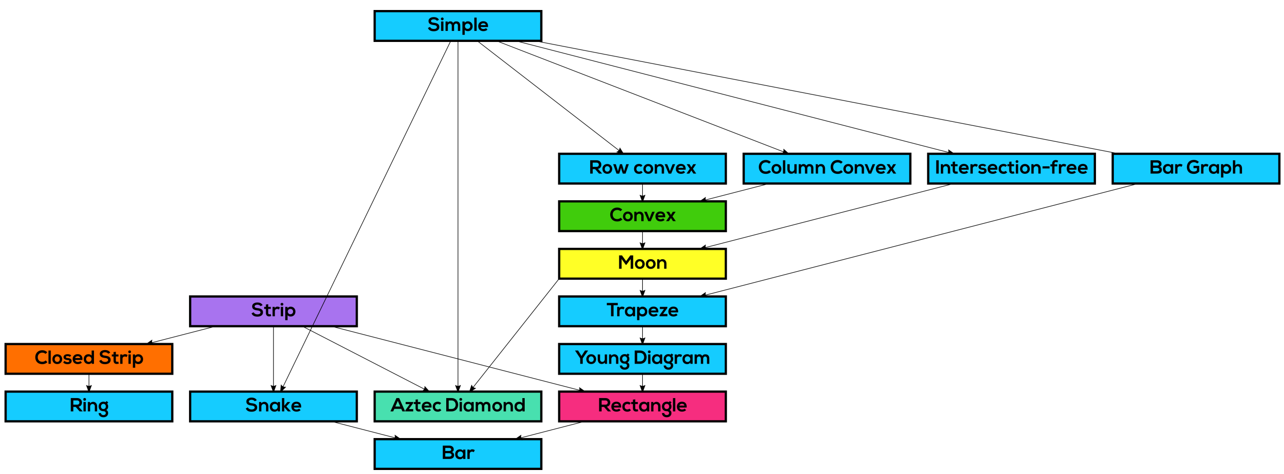


Figure 46: The relationship between some polyomino classes. Most of these are obvious.

See also Problem 38 and Theorem 57.

Row convex, column convex, and convex polyominoes are discussed in Section 8.4.

We already saw a class of polyominoes for which the Checkerboard Criterion (Theorem 27) is enough (namely Young diagrams, see Theorem 45), and we may wonder if there are other classes of polyominoes this is true.

There is, and this section we prove that a large class of polyominoes—that include Young diagrams—are tileable when they satisfy the checkerboard criterion. I will give three proofs of this fact, but before we get there we need a few definitions and helper theorems.

A domino in a tiling of some region is called **exposed** if at least one of its long sides lies completely on the border of the tiled region.

Theorem 46 (Ivan Neretin (un.) (2017)). *Any domino tiling with more than one tile has at least two exposed dominoes.*

[Referenced on pages 56, 57, 58 and 75]

Proof. We give an algorithm to produce two exposed long edges.

(1) Take any tile. If it is horizontal, look at its top (long) side. Otherwise look at the right (long) side.

(2) Introduce the coordinates: $(x, y) = (0, 0)$ at the middle of that side.

(3) If that side is free, we're done. If it is not, there must be another tile blocking it (maybe partially). Switch to that tile (or the rightmost/topmost of the two, if there are two).

(4) If it is horizontal, look at its top side. Otherwise look at the right side.

(5) Check the value of $x + y$ at the middle of that side. Make sure it increases when we step from a horizontal tile to vertical or vice versa, or (in the worst case) stays constant when we step to a tile of the same orientation.

(6) Go to step 3 and continue. It must end somewhere, for there are only so many tiles and they never repeat. (There can't be a cycle of tiles in different orientation, because $x + y$ increases when we change orientation, and never decreases. Neither can there be a cycle made of horizontal tiles only, for in that case y steadily increases in every step.)

To locate the second exposed long side, return to the initial tile and repeat everything in the opposite direction.

In fact, there must be at least two *opposite* long edges exposed.

The procedure to find the first edge always produces a right or top edge, and the procedure to find the second edge always produces a left or bottom edge. If the edges are opposite, we are done. Suppose they are not different, and WLOG let the first edge be a right edge and the second edge a bottom edge. Use the procedure to find a third edge, this time going top left (rotate the coordinate system 90 degrees anticlockwise). The procedure must either produce a left or top edge; in either case, it is opposite with one of the other two exposed edges. □

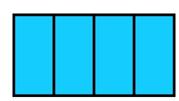


Figure 47: An example of a tiling that achieves the minimum number of exposed edges.

There are tilings that achieve the minimum number of exposed edges. An example is shown in Figure 47.

It is easy to modify this proof to apply to all rectangles (in the case of a square, all edges are "long").

Problem† 23. *Extend Theorem 46 to all rectangles. Can it be extended to an even bigger class of regions?*

Problem† 24. *Show that every region that can be tiled by X-pentominoes has at least four peaks.*

Rows are **comparable** if the column coordinates of one is a subset of the others.

A polyomino in a fixed orientation is **row convex** when in each row, there are no spaces between any two cells.

A **stack polyomino** is a bar graph that is row-convex and each pair of rows is comparable.[21]

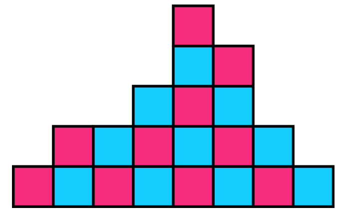


Figure 48: A stack polyomino corresponding to the vector $B(1 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 4 \cdot 2 \cdot 1)$.

[21] Also called a *trapeze* Beauquier et al. (1995).

Theorem 47. *A balanced stack polyomino contains a cylinder that can be deleted.*

[Referenced on page 58]

Proof. If the top row has two or more cells, we can remove a vertical cylinder and we are done.

Suppose then the top row has only one cell, and that it is in column k. Now we can remove a cylinder if, for some $i < k$, column i differs from the next by 2 or more, and for some $j \geq k$, column j differs from the next more than 2 cells.

So suppose one of these conditions fail, WLOG suppose for all $i < k$ the column differs from the next by just one cell (if they are equal, we can remove a vertical cylinder).

Now divide the region into two regions, with all k columns $i \leq k$ in S_1 and the other n columns in S_2. The deficiency in S_1 is given by $\pm \left\lceil \frac{m+1}{2} \right\rceil$. If $n < m$, it must have an absolute deficiency that is less than that of S_1. If $n > m$, there must be two columns that are equal. And if $n = m$, we have two cells in the top row. Therefore, none of the configurations are possible, and therefore both conditions must hold, and therefore, we can remove a cylinder. □

Note that if we know the region is tileable, we can use Theorem 46 to prove a cylinder can be removed. However, knowing that we can remove a cylinder from a tileable region is not as useful, since we often want to remove cylinders to establish whether a region is tileable or not in the first place.

Theorem 48. *Suppose R is a stack polyomino, and S is a cylinder that can be deleted from R. Then $R \ominus S$ is a stack polyomino.*

[Referenced on page 58]

Proof. Suppose the polyomino is $B(a_1 \cdots a_m \cdots a_n)$, and $a_{m-1} < a_m \geq a_{m+1}$.

Suppose the cylinder we delete is vertical. This is equivalent to removing two adjacent columns; the resulting polyomino is a bar graph, whose columns still satisfy thee inequalities that make the polyomino a stack polyomino.

Suppose the cylinder we delete is horizontal, and it lies in columns k to k'. Then, $a_{k-1} \leq a_k + 2$, and $a_{k'} + 2 \geq a_{k'+1}$. The new polyomino is a bar graph with $B(a_1 \cdots (a_k - 2) \cdots (a_m - 2) \ldots a_{k'2} \cdots a_n)$.

Combining all the inequalities, we see this new bar graph is a stack polyomino. □

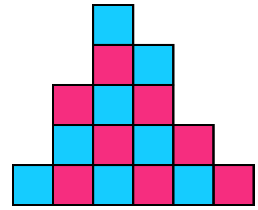


Figure 49: A balanced stack polyomino.

A **jig-saw** region is a region formed by removing all cells of one color from the top of a balanced stack polyomino with the checkerboard coloring applied (Bougé and Cosnard, 1992). A jig-saw region is sometimes, but not always, a stack polyomino. It is always a bar graph.

Theorem 49 (Bougé and Cosnard (1992)). *A balanced jig-saw region is tileable by dominoes.*

[Referenced on pages 58 and 128]

Proof. Suppose R is a balanced jig-saw region. Now make a partial tiling by placing a domino on each cell in the top row and its neighbor, and place horizontal dominoes on the second row starting from the outermost cells. If we remove all cells that are covered by dominoes, the resulting region R' is balanced, and a jig-saw region with one less row. Eventually, we must arrive at the empty figure, and so by the partition theorem (Theorem 2), R must be tileable. □

Theorem 50 (Bougé and Cosnard (1992), Beauquier et al. (1995)). *A balanced stack polyomino is tileable.*[22]

[Referenced on page 128]

Proof 1. We can delete a 2-cylinder from the stack polyomino (Theorem 47) and the result a new stack polyomino (by Theorem 48) or the empty region. We can continue this process until the last region is empty. By Theorem 41 this means the region is tileable. □

Proof 2. (Bougé and Cosnard (1992), Korn (2004, Part of Theorem 11.1, p. 156)) There must be at least two sides with length greater than two (Theorem 46), and so at least one side that is not the base. If this side is vertical, remove the two cells connected to this side and the top corner; if this side is horizontal, remove two cells connected to this side and the left or right corner. In all cases, the remaining figure is also a stack polyomino. Eventually, we must arrive at a single domino. Re-inserting dominoes in the reversed order in the same positions now yields a tiling of the stack polyomino. □

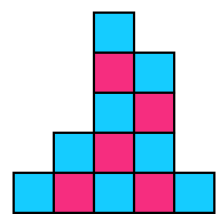


Figure 50: The jig-saw polyomino that corresponds to the stack polyomino in Figure 49.

Proof 3. (Bougé and Cosnard, 1992) Suppose the top row has an odd number of cells. Then put horizontal dominoes on the top row to leave one cell not tiled. If we remove all cells tiled, we get a new region R' which is a balanced jig-saw region, which is tileable (Theorem 49). Therefore, R must be tileable. □

[22] A fourth proof can be derrived from a more general theorem in Beauquier et al. (1995). Their result applies to tiling a stack polyomino with two bars, one horizontal and the other vertical, with no rotations allowed. If both bars have length two, we have the case of dominoes where any orientation is allowed. They introduce a lot of machinery to prove the general theorem. However, if it is made specific for dominoes, the ideas used resemble those of the second proof given here.

2.1.4 Coloring

We may wonder if other colorings exist that could give us information when the checkerboard coloring does not.

In a sense, the checkerboard coloring gives us the best information we can hope for in the general case. However, other colorings give us information in specific cases.

We first see what happens if we add a color.

Let us color a region with three colors, amber, blue, and cherry, such that colors in each row cycle, and diagonals have the same color. We will call this type of coloring a **flag coloring**.[23]

[23] Golomb (1996, p. 4) calls this (jokingly) a *patriotic coloring*, presumably because he used the three colors of the American flag. We define generalized flag colorings in Section 4.2.

When we place a domino, it always covers two different colors, and there are three such arrangements. Let's denote the number of times a domino of each type is used by k_{AB}, k_{AC}, and k_{BC}, and the number of cells of each color in a region by A, B, and C. We have the following equations relating these values:

$$A = k_{AB} + k_{AC} \tag{2.4}$$

$$B = k_{AB} + k_{BC} \tag{2.5}$$

$$C = k_{AC} + k_{BC} \tag{2.6}$$

With a bit of algebra, we get the following solutions to the equations above:

$$k_{AB} = \frac{A + B - C}{2} \tag{2.7}$$

$$k_{AC} = \frac{A - B + C}{2} \tag{2.8}$$

$$k_{BC} = \frac{-A + B + C}{2} \tag{2.9}$$

These must all be whole numbers and non-negative (because, remember, they correspond to the numbers of dominoes), and this becomes a criterion: for a tiling to exist, it is necessary that k_{AB}, k_{AC} and k_{BC} are all non-negative integers.

Figure 51 shows an example of a region that is balanced, but does not satisfy this new criterion.

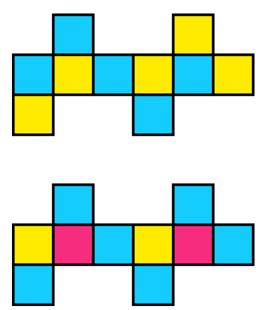


Figure 51: A region that is balanced, but does not satisfy another color criterion.

If you experiment a bit with this criterion, you will find that the regions are quite pathological and in general it is usually easy to find out they cannot be tiled *without* using the criterion. What is going on?

From the equations, we can make the following observations: k_{xy} will be an integer unless the number of cells is odd. So this part is of little help (we already know the number of cells must be even from Theorem 1). For k_{xy} to be negative, one of the colors must exceed the sum of the other two. And regions that does this tend to be spiky and uninteresting.

What if we changed the pattern? We have not used any detail of the pattern, except that neighboring cells cannot have the same color. If we color with only this restriction, it's rather easy to come up with a coloring where one color dominates. Figure 52 shows this coloring applied to Figure 30(h).

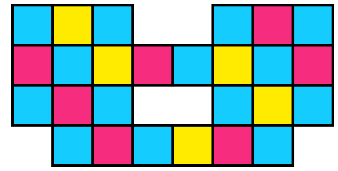


Figure 52: A coloring showing the region is not tileable.

But do we really need the two "loser" colors to be different? The answer is we don't. Let's set this up as we did before.

Color a region with amber and blue such that blue cells do not have any blue neighbors. We call such a coloring a **discriminating coloring**, with blue cells the **isolated** color. The tiles can now be of two types, blue-and-amber and amber only, and let's use k_{AB} and k_{AA} to denote how many of each tile we have.

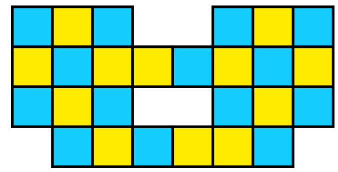


Figure 53: A coloring showing the region is not tileable. There are 14 blue cells, and 12 yellow cells. Since blue cells have no neighbors, if the region was tileable we would have at least at most as many blue cells as yellow cells.

Figure 53 shows this coloring applied to Figure 30(h).

We now have the following equations:

$$A = k_{AB} + 2k_{AA} \tag{2.10}$$

$$B = k_{AB} \tag{2.11}$$

Solving, we get

$$k_{AB} = B \tag{2.12}$$

$$k_{AA} = \frac{A - B}{2} \tag{2.13}$$

$$\tag{2.14}$$

For a tiling, k_{AB} and k_{AA} must be non-negative integers. k_{AB} will always be a non-negative integer; however, k_{AA} will be negative when $B > A$. The number k_{AA} will always be an integer as long as the number of cells is even.

If a region has a coloring for which k_{AA} is negative, we call the region **unfair** by that coloring.

Example 12. *Color the double-T as shown in Figure 12. With this coloring, $B > W$, and since no black cells have black neighbors, we conclude the tiling is impossible.*

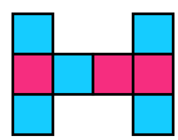


Figure 54: Double-T with alternative coloring

Problem† 25. *For each remaining region in Figure 30, find a discriminating coloring by which it is unfair.*

We state this as a theorem that we will call the **two-color criterion**. Note that it actually generalizes (and contains) the checkerboard criterion. However, the concept of balanced polyominoes is still useful, and when it works it's a very efficient way to describe the particular coloring.

Theorem 51 (Two-color criterion)**.** *If there is a discriminating coloring by which a region is unfair, then the region not tileable by dominoes.*

[Referenced on pages 61 and 128]

Proof. WLOG let black be the isolated color. If a tiling exists, there must be b dominoes that cover a black and white cell, and w dominoes that cover only white cells. The number of black cells in the region is given by $B = b$, and then number of white cells $W = 2w + b$. Now if $B > W$, then $b > 2w + b$, and hence $2w < 0$, which is impossible (the number of white only dominoes must be positive or zero). Therefore, no such tiling exists. □

Problem† 26. *Show that we cannot get more information by adding more colors using this type of criterion.*

In a sense, the new color theorem is really just the marriage theorem in disguise. The following theorem shows the connection between them.

Theorem 52. *A region has an unfair discriminating coloring if and only if it has a bad patch.*

[Referenced on page 61]

Proof.

If. Color the region with the checkerboard coloring. If the region has a bad patch, it has a black bad patch (Theorem 37). Let S be the biggest set that contains the black bad patch and is also a bad black patch.

Then $|W(S)| < |B(S)|$, and so $|W(R - S)| \geq |B(R - S)|$ (if it was not, we could make S bigger). Now swap all the colors in $R - S$. We now have a coloring in which there are more white cells than black cells, and all white cells only have black neighbors. Therefore we have an unfair discriminating coloring.

Only if. Suppose we have an unfair discriminating coloring of a region. Then the region is untilable (Theorem 51), and hence, in a checkerboard tiling, there is a bad patch (Theorem 38). □

From this theorem, we get the following:

Theorem 53. *A region is tileable if and only if all its discriminating colorings are fair.*

[Referenced on page 128]

Proof. If. Suppose all the discriminating colorings of a region are fair, but it is untilable. Then, by Theorem 38 there is a bad patch, and by Theorem 52 there exists an unfair discriminating coloring, which contradicts our initial assumption, therefore, the region must be tileable. *Only if.* Suppose a region is tileable. If it had a unfair discriminating coloring, it would not be tileable (by Theorem 51), and so all discriminating colorings must be fair. □

At the beginning of this section, we said that the checkerboard coloring gives us the best information in the general case. The reason for this is that of all discriminating colorings, the checkerboard has the highest density of isolated color, and can therefore discriminate the most figures (of a certain area). I am not going to make this idea more precise here. Furthermore, it makes ideas such as flow, and (as we will see in the next chapter) height functions possible, which other discriminating colors do not.

2.2 *The Structure of Domino Tilings*

SOME TILEABLE REGIONS HAVE cells that are tiled the same way in any tiling of the region. We call such cells **frozen**.

As we will see, there are three ways in which cells can become frozen:

- When it is part of a region that can geometrically only be tiled one way (when it is part of a peak, a notion we shall define shortly).
- When other placements violate the border crossings theorem 25.
- When other placements violate the flow theorem (Theorem 29).

We will see that cells that are not frozen are always part of some closed strip, and that by rotating the closed strip we can find a new tiling of the region.

A key point from this section is the following: any two tilings are connected through a series of simple operations that involve rotations on closed strips of dominoes.

2.2.1 *Closed Strips*

A **strip polyomino** is a polyomino whose cells form a path $v_0 = v_{n-1}$. If we also have $v_0 = v_{n-1}$, then we call the strip polyomino **closed**.[24]

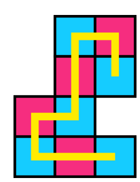


Figure 55: Strip polyomino

[24] In Beauquier et al. (1995) the word *ring* is used for a closed strip.

Theorem 54 (Beauquier et al. (1995))**.** *If we apply a checkerboard coloring to a strip polyomino, and its two ends have opposite colors, the path has an even number of cells, otherwise it has an odd number of cells.*

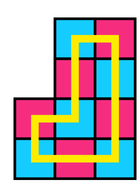


Figure 56: A closed strip

[Referenced on pages 63, 64, 72, 110 and 359]

Proof. We note that it is true for a strip polyomino with area 1. Now suppose it holds for strip polyominoes with area k. Consider now a strip polyomino with area $k+1$, and suppose its two ends have the same color. Now remove one end, to get a path of length k. Since the new end must have opposite color of the one removed, the two ends are now of opposite color, and hence n must be even, and so $k+1$ must be odd. Suppose then the two ends have different colors. If we remove one end from the strip polyomino, the new polyomino must have ends of the same color, and hence k must be odd, and so $k+1$ must be even. We proved that the theorem also holds for $k+1$, and therefore, it must hold for all k. □

Theorem 55 (Beauquier et al. (1995)). *Strip polyominoes with even area are tileable. Closed strip polyominoes have even area, and each has eat least two different tilings.*

[Referenced on pages 63, 64, 65, 66, 68, 69, 93, 115, 126, 128 and 359]

Proof. The cells of the strip polyomino forms a path $v_1, v_2, \ldots, v_n$, with n even. We can therefore partition the cells into sets $\{v_1, v_2\}$, $\{v_3, v_4\}, \ldots, \{v_{n-1}, v_n\}$, each containing two elements. Since in each set the two cells are neighbors (Theorem 2), the set can be tiled by a domino, and so the entire region can be tiled by dominoes .

If the strip is closed, then v_n and v_1 are neighbors, and therefore has opposite colors, and so n must be even (Theorem 54). One tiling is given above; another tiling is given by $\{v_n, v_1\}$, $\{v_2, v_3\}, \ldots,$ $\{v_{n-2}, v_{n-1}\}$. □

When trying to find a tiling of a region by hand, it is often easier to see if you can divide it into even strips (because you can quickly draw lines through a region, and just check that endpoints have different colors). This also helps you identify suitable choke points if no tilings exist so that you can apply the flow theorem (Theorem 29) and prove it. The following two theorems illustrate this idea.

Theorem 56. *All even-area rectangles are tileable.*

[Referenced on page 199]

Proof. If the area of a rectangle is even, then either its width or height must be even. WLOG say the width is even. Now partition the rectangle into rows. Each row is a strip polyomino with even area, and therefore tileable (Theorem 55), and so is the whole region (Theorem 2). □

In fact, rectangles are strip polyominoes, and what is more, rectangles with even area are closed strips.

Theorem 57. *Even-area rectangles are closed strips.*

[Referenced on page 55]

Proof. Let the width and height of the rectangle be m and n. WLOG suppose the width is even. Partition the rectangle into $m+1$ strip polyominoes as follows: form the first m snakes S_i from all cells in each column except the top one; form the last snake S_{m+1} from the cells in the top rows. To each strip assign a head and tail: Except for the last snake, odd numbered snakes get a tail at the top and head at the bottom. Even-numbered snakes get a head at the bottom and a tail at the top. The last snake gets a head at the left and a tail at the right.

Now if we join the snakes head to tail in order, and join the head of the last to the tail of the first, we have a path through all the cells in the rectangle, which proves it is a closed strip. □

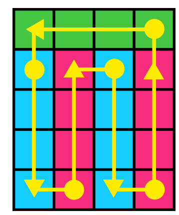


Figure 57: A rectangle is a strip polyomino.

Call a strip polyomino with odd area and with black endpoints **black**, and one with white endpoints **white**.

Theorem 58. *Let R be a white strip polyomino.*

(1) *If we remove a white cell from the strip, the remaining figure(s) are tileable.*

(2) *If we add a black cell (that is not already part of R) to the polyomino, then the resulting figure is tileable.*

The theorem holds of we interchange black and white.

[Not referenced]

Proof. If we remove a white endpoint, the resulting figure is a strip with even area and thus tileable (Theorem 55). If, on the other hand, we remove another white strip v_i, then $v_0, \cdots, v_{i-1}$ and $v_{i+1}, \cdots, v_{n-1}$ are two strip polyominoes, both with even area. (This is because v_0 is white, and v_{i-1}, a neighbor of v_i, is black, and so Theorem 54 applies; the same for the other strip.)

If we add the black cell to one of the strip polyominoes endpoints, we have a strip polyomino with even area, and so it is tileable. If, on the other hand, we add a black cell u to a strip polyomino of odd area next to a cell that is not an endpoint, v_i (necessarily white), then we can partition the figure into three figures: $\{u, v_i\}$, $\{v_0, \cdots, v_{i-1}\}$ and $\{v_{i+1}, \cdots, v_{n-1}\}$. The first is a domino, and hence tileable. The last two are strips with even area (as argued above) and hence are tileable. □

It is easy to extend this theorem to remove a white strip or add a black strip (that don't overlap with the original).

Let R be a region, and let R_1 be all the cells on the border of R. Let $S_i = R - R_i$, and R_i are all the cells on the border of S_i. We call R **saturnian** if for some k,

(1) $R_i = \varnothing$ for all $i > k$,

(2) R_k is tileable by dominoes

(3) R_i is a set of closed strips for all $i < k$.

S_i need not be connected. If a cell c is in R_i, we define its **level** by $\operatorname{lev}(c) = i$ (Parlier and Zappa, 2017).

The number k is called the **number of levels** of R; and the subregion R_k is called the **core**.

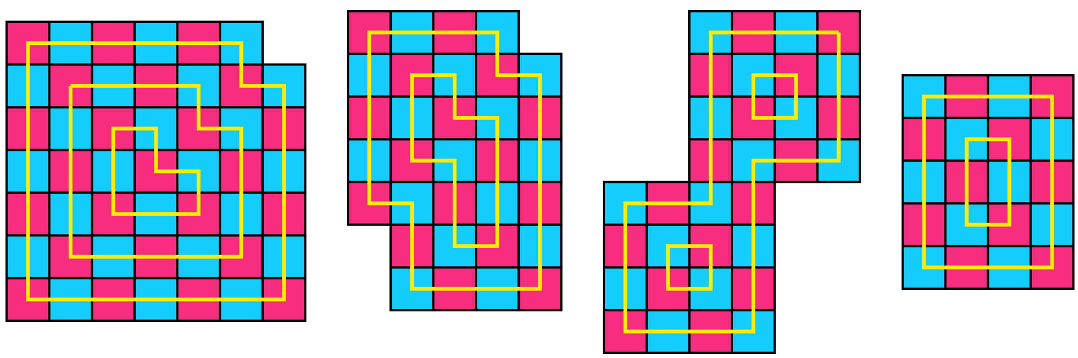


Figure 58: Examples of Saturnian regions. The third figure is an example of a Saturnian figure for which R_2 is disconnected.

Theorem 59. *Saturnian polyominoes are tilable by dominoes.*

[Referenced on page 128]

Proof. We can partition R as follows $R = R_1 + R_2 + \cdots R_k$. Each of $R_{i<k}$ is a closed strip and tileable (Theorem 55), and R_k is tileable by definition; hence, so is R (Theorem 2). $\square$

Problem† 27.

(1) *Prove that rectangles with even area are Saturnian.*

(2) *When is a rectangle with some corners removed Saturnian?*

(3) *Prove that the core of a Saturnian region cannot contain a* 3×3 *square subregion.*

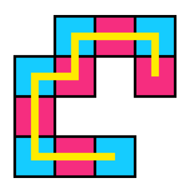


Figure 59: Snake

A **peak** is a cell of a region with only one neighbor in the region (Beauquier et al., 1995).

A **snake** is a polyomino that has two peaks, and every other cell has exactly two neighbors (Goupil et al., 2013). Snakes are strip polyominoes, and therefore snakes with even area are tileable. Table 8 shows the number of snakes with a given number of cells up to 10 cells.

A **ring** is a polyomino where each cell has exactly two neighbors. Rings are closed strip polyominoes.

Rings are closed strips, and therefore they have even area, and exactly two tilings.

Although we prove theorems for strip polynomials, it is not in general easy to determine whether a given polyomino is a strip polyomino (for example, it is not immediately obvious that Figure 30(e) is not a strip polyomino). On the other hand, it is easy to recognize snakes and rings.

k	$P(k)$ A000105	$S(k)$ A002013
1	1	1
2	1	1
3	2	1
4	5	2
5	12	3
6	35	7
7	108	13
8	369	31
9	1285	65
10	4655	154
α	3.6	2.3

Table 8: Number of free snakes $S(k)$ compared to the number if free polyominoes $P(k)$. The last row gives the ratio between the last two terms of the sequence in the table.

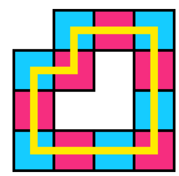


Figure 60: Ring

Problem† 28. *Show that polyominoes that maximize the hull are convex snakes.*

Problem† 29. *Define the dilation of a region as the region with all cells of the original region and their neighbors. Let R' be the dilation of R. When is $R' - R$ a ring?*

Problem† 30. *Define the erosion of a region as the region with all cells of the original region that has four neighbors. Let R' be the erosion of R. When is $R' - R$ a ring?*

Problem† 31. *What if in the definitions in the above we use 8-neighbors instead of 4-neighbors?*

Theorem 60 (Gomery's Theorem)**.** *If we remove a white and black cell from a checkerboard colored strip polyomino, the remaining region is tileable by dominoes.*[25]

[25] The theorem, by Ralph Gomery, was originally given in only for the 8×8 square (Honsberger, 1973, p. 65–67), and the proof was slightly different. However, the main idea of the proof is the same as given here.

[Referenced on page 128]

Proof. Let the cells along the path of the strip polyomino be v_1, v_2, v_3, $\dots$, v_k, and suppose the removed cells are v_m and v_n with $m < n$.

If the removed cells are consecutive in the path (Figure 61), then the region has a path $v_{n+1}, v_{n+2}, \dots v_k, v_1, v_2, \dots, v_{n-2}$. This is an even strip, and so is tileable (Theorem 55).

If the removed cells are not consecutive in the path (Figure 62), they partition the strip into two strips. The ends of each strip neighbor cells of opposite color, and therefore the ends of each strip is of opposite color. Therefore, the strips are tileable, and so is the whole region. □

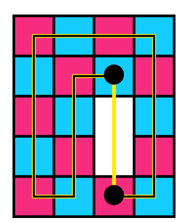


Figure 61: Removing two cells consecutive in the path.

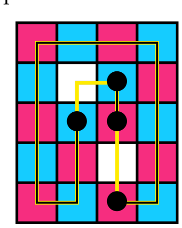


Figure 62: Removing two cells not consecutive in the path.

2.2.2 Frozen Cells

IN FIGURE 64, we can see a domino can fit only one way in the two yellow cells. We can then remove these cells, and consider whether the rest of the region can be tiled. Since this is a 2×2 square, it can be, and so we conclude the whole region is tileable. The same technique works for more complicated regions, such as in Figure 63.

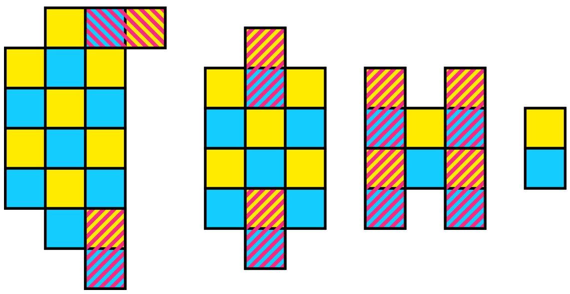


Figure 63: Illustrating peak removal.

When you apply this technique to Figure 30(b), you get a disconnected monomino after the first step, which is not tileable, and so the whole region is not tileable.

We will now formalize this process in a series of definitions and theorems.

A subregion of R is **frozen** if it can be covered by dominoes in only one way in any tiling of R.

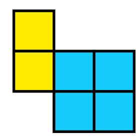


Figure 64: The yellow cells can only be covered in one way by a domino.

Theorem 61. *Suppose a region R is partitioned into two partitions, S_1 and S_2, and S_1 is tileable, and would be frozen in any tiling of R if it exists. Then the region is tileable if and only if S_2 is tileable.*

[Referenced on page 68]

Proof.

If. If S_2 is tileable, then R is tileable by Theorem 2.

Only if. Suppose R is tileable. Because dominoes can cover the frozen partition in only one way, in a tiling of R the dominoes that are part of S_1 must also tile S_1, since it is indeed tileable. Therefore, the dominoes *not* part of S_1, must tile S_2. □

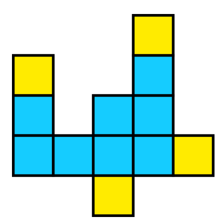


Figure 65: A region with 4 peaks, shown in yellow.

Theorem 62. *In a tileable region, a peak and its neighbor are frozen.*

[Not referenced]

Proof. There is only one way for the domino to cover the peak, and in that one way it also covers the neighbor. So if the region has a tiling, the neighbor cell can also only be covered one way. □

From this theorem, it follows that a region with a peak is tileable if and only if the region(s) that remain after we remove a peak and its connected cell. We can repeatedly remove peaks and their connected cells until no peaks remain. The resulting region (which may not necessarily be connected) is called the **compact subregion**. The compact subregion of R is denoted R^*. Note that R^* can be empty (Figure 66), or disconnected (Figure 67). It is possible that $R^* = R$, in which case we call R **compact**.

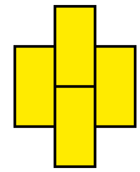


Figure 66: A region whose compact subregion is empty.

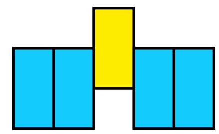


Figure 67: A region whose compact subregion (blue) is disconnected.

Problem† 32. *Suppose R has no holes. Prove R^* does not have any holes.*

Problem† 33. *What is the bound on $\Delta(R)$ as a function of the number of cells in R if R is compact?*

Theorem 63. *A region R is tileable if and only if its* ***compact subregion*** *R^* is tileable.*

[Not referenced]

Proof. Let $R' = R - R^*$, so that R' and R^* are partitions of R. Since all the cells in R' are frozen, by Theorem 61 R is tileable if and only if R^* is tileable. □

If a region consists out of disconnected parts, it is tileable if and only if each part is tileable (Theorem 2.) The tileability of all regions thus boils down to whether connected compact shapes are tileable.

Theorem 64 (Beauquier et al. (1995), Theorem 3.5)**.** *If a region has a unique tiling, it must have at least two peaks.*

[Referenced on pages 72 and 77]

Proof. Suppose a region has no peaks, and that it has a tiling. Now construct a sequence of cells as follows: v_1 is any cell, and v_2 is the cell in the same domino. Choose v_3, a neighbor of v_2 that is not v_1 (we can do this, since v_2 is not a peak).

We can continue in this fashion, always choosing new cells that haven't been chosen before, until eventually, we are "trapped" or run out of cells. Now the last cell, v_n, must have a neighbor that is not v_{n-1}, and is already part of the sequence, say v_i. (If it did not, then v_n would be a peak). So we have a sub-sequence $v_i, v_{i+1}, \ldots v_n$ that is a closed strip. Closed strips are tileable in more than one way (Theorem 55), and therefore, the entire region must be tileable in more than one way.

Suppose the region does have one peak. Construct the same sequence as above, but choose v_1 as the peak. □

This implies that compact shapes have at least two tilings. Figure 68 shows a big figure with a unique tiling, mentioned in Pachter and Kim (1998).

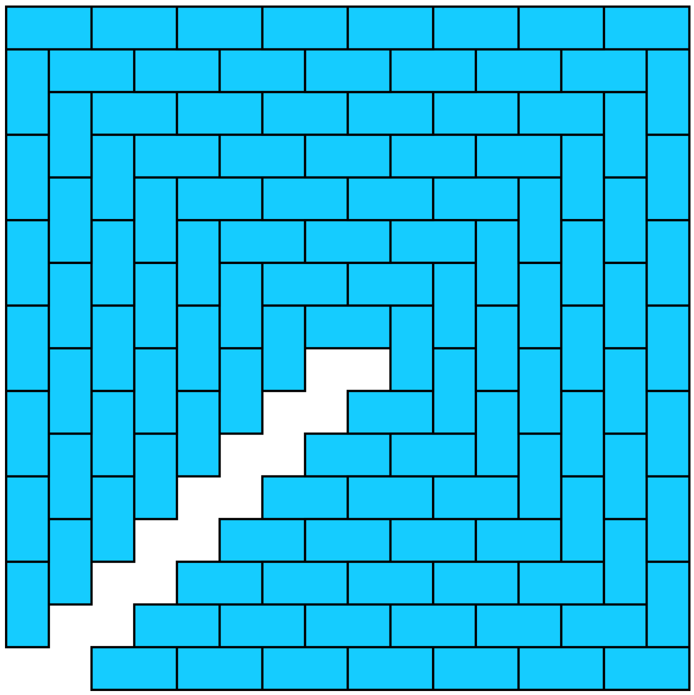


Figure 68: A figure with a unique tiling

If we recursively remove peaks and their connected cells, we are eventually left with either a compact region, or the empty region. If the former is the case, the region is uniquely tileable. Otherwise, it has more than one tiling.

2.2.3 *Tiling Transformations*

In this section we look at how one tiling can be transformed into another with a series of "primitive" transformations.

Theorem 65. *A cell is not frozen if and only if it is part of a tiled closed strip.*

[Referenced on pages 71, 77 and 117]

Proof. If. Suppose a cell v_1 is part of a tiled closed strip, and its neighbor in the the same domino is v_0, and its other neighbor along the strip is v_1. Then by Theorem 55 there is another tiling of the strip, and in that tiling v_1 and v_2 are in the same domino. Therefore, there is more than one way for v_1 to be covered by dominoes, and therefore it is not frozen.

Only if. Suppose a cell v_1 can be tiled in two ways. In one tiling v_0 is the same domino, and in the other tiling v_2 is in the same domino. Then let v_3 be v_2's neighbor in the first tiling, v_4 is v_3's neighbor in the second tiling, and so on. We can always extend the sequence, but since there are only a finite number of cells in the region, eventually the next cell must be one that is already in the sequence. Moreover, this repeated must be v_0, since a cell can be in the same domino in two different tilings at most two times, and any other cells v_i is already in the same domino as either v_{i-1} and v_{i+1}.

Therefore we have a sequence of cells $v_0, v_1, \ldots, v_k$, where v_{i+1} is a neighbor of v_i, and v_n is a neighbor of v_0. Therefore, the cells $v_0, v_1, \ldots, v_n$ form a closed strip. □

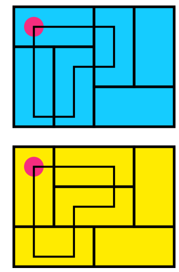


Figure 69: An example of a closed strip that contains a point and is tiled in both tilings.

The regions in Figure 70 all have frozen cells that cannot be part of any tiled closed strip.

Problem† 34. *Show that if we insert a cylinder into a figure where all the cells that surround the insertion section is frozen, then frozen cells stay frozen, and the entire cylinder is also frozen.*

From the result of the problem above, we know that we can get the ratio of frozen cells to total cells as close to 1 as we want, even in figures without peaks, bridges, or holes.

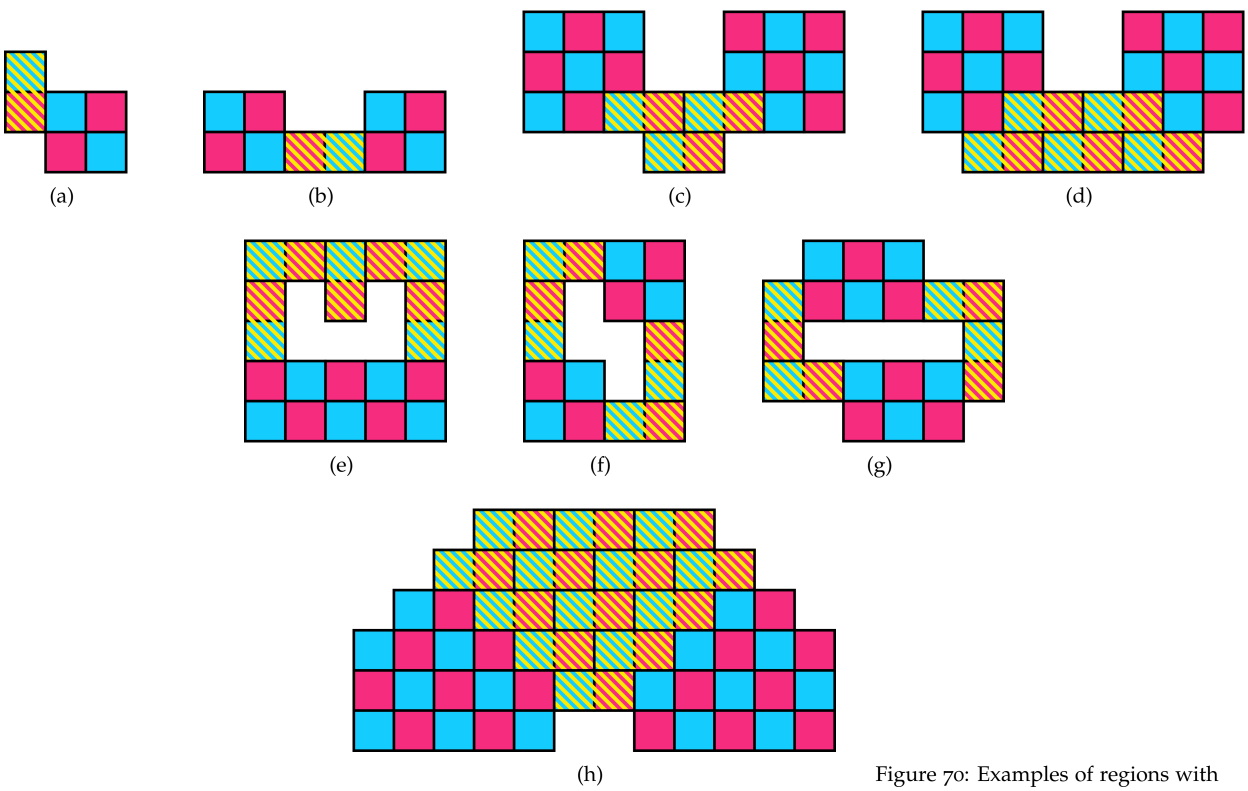


Figure 70: Examples of regions with frozen cells, marked with yellow stripes.

Problem† 35 ("Matematická Olympiáda" Contributors (2019), Problem 2[26])**.** *Find all the frozen cells in Figure 71, and conclude that the number of tilings must be a square number.*

[26] I learned about this problem through ACheca (un.) and Jaap Scherphuis (un.) (2019), where an analysis is also given.

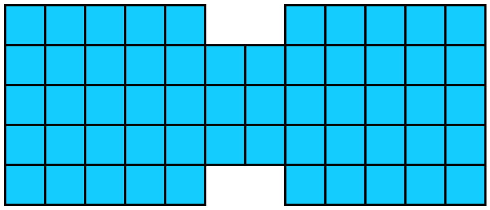


Figure 71: Which cells in the following figure are frozen?

Theorem 66. *If two closed strips are put side-to-side, then together they form a strip polyomino.*

[Not referenced]

Proof. Let the two closed strips be $u_1, u_2, \ldots, u_m$, and $b_1, b_2, \ldots, b_n$, and suppose u_i is neighbors with b_j. Then a strip is given by

$$u_{i+1}, u_{i+2}, \ldots u_m, u_0, u_1, \ldots, u_i, v_j, v_{j+1}, \ldots, v_n, v_0, v_1, \ldots v_{j-1}.$$

□

Figure 72 shows an example of three connected closed strips (2×2 squares) that are connected, but do not form a strip.

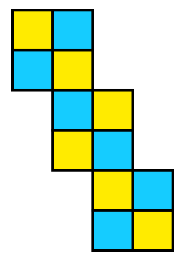


Figure 72: An example of 3 connected closed strips that do not form a strip.

An analogous argument shows any non-frozen cell tiled by a vertical domino, if it has a non-frozen horizontal neighbor, it can also be tiled horizontally.

In a region, if we replace a closed strip with its dual tiling, we call this operation a **strip rotation**[27] (Figure 73). When the closed strip is a 2×2 square, we call the strip rotation a **flip**, and the two dominoes that form the strip a **flippable pair**. In Theorem 65 we saw how to construct a closed strip that contains a cell from two different tilings. In the theorem below, we exploit this idea to prove that one tiling can be transform into any other tiling through a sequence of strip rotations.

[27] In Propp (2002)the other calls this operation (in the context of oriented graphs), a *face twist*.

Theorem 67 (The tiling connection theorem)**.** *We can obtain one tiling from another by a sequence of non-overlapping strip rotations.*

[Referenced on pages 76 and 119]

Proof. Pick any cell with a different tiling in the two tilings. Construct the closed strip as in Theorem 65, and perform a strip rotation on that strip in the first tiling. All the dominoes in that strip now has the same tiling as in the second tiling. Repeat the process. Notice,

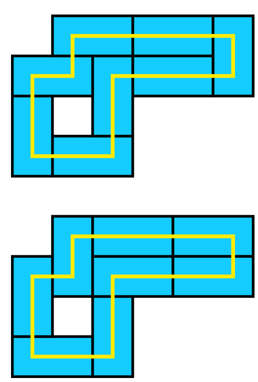


Figure 73: An example of a strip rotation.

that subsequent rounds do not change any dominoes that are already in the same position as in the second tiling, so non of the strips overlap. Since the region is finite, and we reduce the number of dominoes that differ from the second tiling in each step, the process must end when all the dominoes match. □

Theorem 68. *Let R be a tiled region, and S a subregion of R. Performing a strip rotation on R does not affect the flow on S.*[28]

[28] These ideas are more or less presented in Saldanha et al. (1995). They consider flow through cuts that do not disconnect the region.

[Not referenced]

Proof. This is clearly true when the closed strip is entirely within S or entirely out of S.

Suppose then the closed strip goes over the border of S.

If a strip outside S meets S at cells u and v, then either the strip has an even number of cells (and so u and v must have opposite colors, by Theorem 54), or an odd number of cells (and u and v must have the same colors, by Theorem 54).

In the first case, dominoes must either cross S at both points, or neither, and the net contribution of dominoes at those two points are zero. Doing a strip rotation will cause the opposite; either dominoes don't cross at either point, or they do at both. Again, the net contribution to the flow is 0.

In the second case, a domino must cross at the one point, and not the other, and therefore the contribution to the flow is $+1$ or -1, depending on the color of the cell where the domino crosses inside S. Say the crossing happens at u, and that u is black. Then the domino contributes +1 to the flow. v must be white too. When we do a rotation, there is no crossing at u, and a crossing at v. Since v is white, the contribution to the flow is 1. A similar argument shows when u is white the contribution before and after the strip rotation is -1.

So in all cases, for every strip, part of the closed strip, that meats S, we have that their contributions to the flow stays unchanged by strip rotations, and so the overall flow is unchanged. □

Theorem 69. *Every tileable region contains at least one of the following:*

- *Two peaks.*
- *A 2×2 square subregion.*
- *A hole.*

[Referenced on pages 73, 100, 101 and 274]

Proof. Suppose the region has a unique tiling. Then it contains two peaks by Theorem 64. Suppose then the region has more than one tiling. Pick a cell v that is not frozen. This cell must be part of a closed strip. Inside this closed strip, choose the left-most bottom-most cell. Because it is part of a closed strip, it must have at two neighbors inside the strip, and since this is a left-most bottom-most cell, it must have a top neighbor v_T and right neighbor v_R. Now consider the position u to the top of v_R. Either there is a cell, or there is not. In the former case, we have a 2×2 square. Suppose then there is not a cell in that spot. Since all the cells in the closed strip cannot lie to the left of v, they must surround u, and therefore there is a hole. $\square$

The following theorem allows us to do induction on closed strips: it gives us a way of breaking closed strips into smaller closed strips.

Theorem 70. *Every closed strip without holes and more than 4 cells contains a smaller tiled closed strip as subregion.*

[Referenced on page 74]

Proof. By Theorem 69 a closed script must have a either a hole or a 2×2 square as subregion, and since this closed script does not have a hole, it must contain a 2×2 subregion.

Now consider how that square is tiled. Since the flux is 0, we must have that 0, 2, or 4 dominoes overlap the border.

- If 0 dominoes overlap the border, we have a flippable pair, which is a tiled closed strip, and a subregion smaller than the original.

- If 2 dominoes overlap, the cells where dominoes overlap must be neighbors. There are four ways in which we can form a closed strip with this configuration (Figure 74). In two of them the branches of the strip must overlap (Figure 74(a) and (b)), and in one of the remaining the two branches are untileable (Figure 74(c)). That leaves only one configuration (Figure 74(d)).

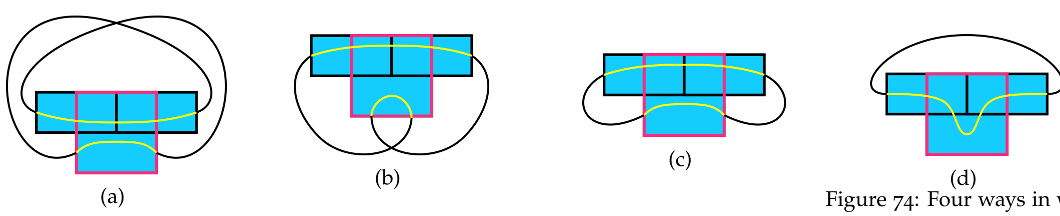


Figure 74: Four ways in which closed strips can be formed when 2 dominoes overlap.

In this configuration we can form a smaller closed strip by removing the domino that lies completely within the square (Figure 75).

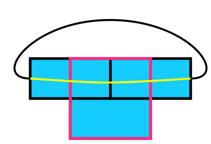


Figure 75: How a smaller closed strip can be formed this configuration.

- If 4 dominoes overlap, there are several possibilities of forming a closed strip, some are shown in Figure 76. Most of these have branches that cannot be tiled, (Figure 76(a)), or leave out one of the dominoes (Figure 76(b)); many also have branches that intersect. Two cases work (Figure 76(c) and (d)).

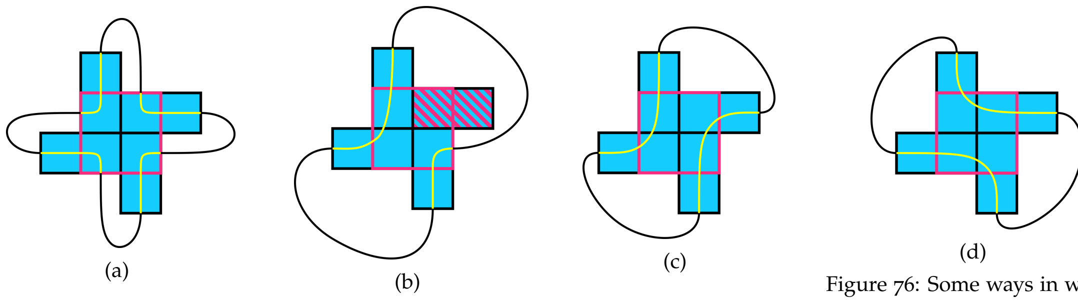


Figure 76: Some ways in which closed strips can be formed when 4 dominoes overlap.

In these cases we can split the path into two as shown in Figure 77. Either path is shorter than the original.

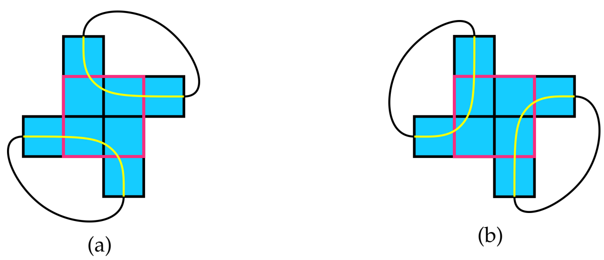


Figure 77: Splitting paths.

□

Theorem 71. *A tiling of a closed strip without holes must have at least one flippable pair.*

[Referenced on page 74]

Proof. If the closed strip is a 2×2 square, the closed strip is a flippable pair and we are done.

Otherwise, by Theorem 70, the region contains a smaller closed strip as subregion. Continue to find a smaller and smaller subregion. Since the tiling is finite, eventually we must find a 2×2 square, which is a flippable pair. □

Theorem 72. *A strip rotation (of a strip without holes) is equivalent to a sequence of flips with the same orientation.*

[Referenced on page 76]

Proof. It is true for a closed strip of 4 cells.

Find a flippable pair (it must exist by Theorem 71). If it is a 2×2 square, a flip rotates the entire strip.

Otherwise, the closed strip has more than 4 cells, and it can have either one branch (Figure 78(a)) or two branches (Figure 79(a)) to complete the strip.

(1) If it has one branch. Do the flip (Figure 78(a)). One domino is now in the right position, the other, with the branch, makes a closed strip with less than n cells, so we can do a rotation by induction. After this rotation, the complete strip is rotated.

(a) (b)

Figure 78: A strip with one branch before and after the flip.

(2) If it has two branches. Do the flip. We can now form two closed strips with one domino in each (Figure 79(a)). Each of these can be rotated by induction. After the rotation of each, the full strip is rotated too.

(a) (b)

Figure 79: A strip with two branches before and after the flip.

□

Theorem 73. *If a closed strip has a filled interior, a rotation of the strip is equivalent to a sequence of flips.*

[Referenced on page 76]

Proof. Find a domino with its long edge on the border of the interior (such a domino exist by Theorem 46). The long edge on the border neighbors the outer closed strip in two cells; either they lie in the same domino, or they don't.

(1) If they don't, they lie in two dominoes, who may or may not be neighbors in the strip. If they are, we can form a new closed strip by simply including the interior domino between them. Otherwise, we form two strips, a closed one with the interior domino between them, and the other what used to be between the two outer dominoes. These are closed if there is more than one domino.

(2) If they do, we flip the pair, and loop them in.

We continue this process (in each case using either of the available closed strips constructed so far) until the entire interior is enclosed within one of the closed strips $R_1, R_2, \ldots$. We may have shedded several strips S_1, $S_2, \ldots$ in the process.

Now each of R_i is a closed strip without holes, so a rotation on these are equivalent to flips. Perform the rotation on each.

Each of S_i may either be a closed strip without holes, or a single domino.

(1) In the case of the former, a rotation is equivalent to a sequence of flips, so we can do a rotation.

(2) In the case of the latter, after all the rotations, either one of the dominoes that used to be a neighbor inside the strip is now a flippable pair with it, or it is not.

  (a) If it is, do the flip.

  (b) Otherwise, perform a sequence of flips as shown in Figure 80.

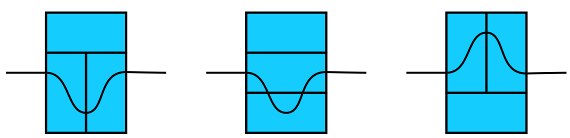


Figure 80: A sequence of flips to rotate a shedded single domino with now neighbors after the rotation.

All the dominoes in in the entire closed-strip are now rotated. Some dominoes in the interior might be in wrong positions, so they need to be restored. Pick a cell, and find a closed strip from the current and desired tiling. If this strip has an empty interior, we can perform a rotation equivalent to flips. Otherwise, we re-apply this theorem (it must end, because with each re-application we have fewer tiles and there is only a finite amount.) We repeat until the interior is completely restored.

We have performed a rotation of the outer strip, using only flips and flip-equivalent rotations. Therefore, the entire operation is flip-equivalent. □

Theorem 74 (Saldanha et al. (1995) [29]). *We can obtain one tiling of a region without holes from another by a sequence of flips.*

[Referenced on page 77]

Proof. Any rotation of a strip is equivalent to flips, whether it's interior is empty (Theorem 72) or filled (Theorem 73), and any tiling can be transformed into another by a series of rotations (Theorem 67). Therefore, any tiling can be transformed into another with a series of flips.[30] □

[29] This theorem was first discussed in Thurston (1990).

[30] See also Rémila (2004) which gives an easier treatment of flips and domino tilings.

Theorem 75. *A tileable stack polyomino with the bottom two rows equal has at least one flippable pair.*

[Not referenced]

Proof. A stack polyomino can have at most 3 peaks (the first and last cell in the bottom row, and the top row). So if the bottom rows have the same number of cells, there can be a most one peak. But for tilings to be unique, we need at least two peaks (Theorem 64). So the tiling is not unique, and there must be another tiling. We can reach the other tiling by a sequence if flips (Theorem 74), which means the original tiling must have at least one flippable pair. □

2.3 *Further Reading*

In Mendelsohn (2004) the author gives a gentle introduction to the relationship between domino tilings and graph theory.

We use a variety of color arguments elsewhere in in this book (for example Sections 5.3 and 5.1.2). See Engel (1998) for other applications and ways to structure coloring arguments.

As mentioned in a side note, the marriage theorem we presented here is a specific example of the much more general theorem in the area of *matching theory*. For a survey on the development of matching history, see Plummer (1992), and for a detailed treatment, see the book Lovász and Plummer (2009). Many books on combinatorics and graph theory contain chapters on matching, see for example Harris et al. (2008, Section 1.7), Diestel (2000), Bondy and Murty (2008, Chaper 16) and Bondy and Murty (1976).

We showed that multiple tilings in a region is the result of certain closed strips (Theorems 64, 65). If we remove dominoes to remove all these closed strips, the tiling is unique. The number of dominoes to remove is called the *forcing number* (for example, if we remove n dominoes along the diagonal of a $2n \times 2n$ square as in Figure 68, the resulting figure has a unique tiling. These ideas are discussed in for example Pachter and Kim (1998) and Lam and Pachter (2003).

Related to the idea of frozen cells is the following: There are certain edges in a region that will never be covered by a domino in any tiling. If we consider the dual graph, the edge is called a *fixed single edge*. (In this setup, there is an edge between two cells that are neighbors. A tiling is a matching of the graph, and edges can be double or single, depending on whether there is a matching edge or not. Frozen tiles correspond to fixed double edges.) See for example

Zhang (1996). The edges (in the original region) that are not crossed by dominoes in any tiling are called *fracture edges* by Fournier (1997).

Domino tilings and their statistics are of interest to physicists because they model the behavior of certain types of molecules. In this context, a domino is usually called a *dimer*, a monomino is called a *monomer*, and the tilings may be considered on more general graphs than square grids. The statistical model that deal with these tilings (or *coverings*) is called the *dimer model*. For a survey on the dimer model, see Kenyon (2000c). In the paper Cohn et al. (2001) the authors give a detailed analysis of the statistics of arbitrary regions.

For a less naive view on the topics discuss here, I recommend the following sequence of papers to get familiar with the algebra of polyomino tilings:

(1) Propp (1997) is a gentle introduction to Conway and Lagarias's work,

(2) Hitchman (2017) expands on these ideas, and includes a discussion on tiling invariants ,

(3) Conway and Lagarias (1990) is the paper where Conway and Lagarias introduces some group theoretic tools to study tilings, and

(4) Thurston (1990) is where height functions are introduced for the first time, and a polynomial time algorithm is described.

The thesis Donaldson (1996) discusses these ideas in a more leisurely fashion, and among other things, works through a proof of Thurston's algorithm.

For domino tilings, specifically, the following papers are a useful start[31]:

- *Tiling figures of the plane with two bars* (Beauquier et al., 1995).
- *Spaces of domino tilings* (Saldanha et al., 1995).
- *The lattice structure of the set of domino tilings of a polygon* (Rémila, 2004).
- *Optimal partial tiling of Manhattan polyominoes* (Bodini and Lumbroso, 2009).

[31] I will give a more expansive bibliography after we covered more topics. The papers listed here deal more or less with the same topics as this chapter

3
Dominoes II

In this chapter, we look further into the structure of domino tilings, how to count domino tilings, and what happens if we add or remove constraints from the tiling problem.

3.1 The Structure of Domino Tilings

We have seen that the domino tilings of a region are connected through strip rotations, and indeed that if the region is simply-connected, that it is connected through flips .

In this section we examine this structure further. We are working towards ways to answer questions like the following:

- What is the minimum number of flips we must perform on a tiling T to produce a tiling U?
- What can we say about how many (and where) flippable pairs occur in a tiling of a region?

We will do this by showing that the set of tilings of a region have a certain structure–they form a *distributive lattice*.

3.1.1 Height Functions

In this section we develop the *height function*, which will allow us to define an important tiling criterion and algorithm. We already encountered the two main ideas on which the height function is based:

(1) We can know what is the deficiency of a region by only looking at the edges on the border. (This is Theorem 31).

(2) If a region is partitioned into two subregions by a cut, the length of the cut must just about be double the absolute value of the deficiency of the one region so that enough dominoes can overlap it to compensate for the deficiency (Theorem 30).

The general idea is to look at a specific set of partitions, and see if the cut length is always long enough among these. (Instead of straight cuts, we will use cuts with a different shape.) If it is not for one of these partitions, then we know a tiling is impossible. If it is long enough for *all* these partitions, then we can conclude a tiling is possible.

Suppose u and v are neighboring vertices. Then we define the **spin** $\text{sp}(v, u)$ as 1 when we go from u to v and the cell to the left is black, and -1 if it is white (Rémila, 2004). Suppose $v_0, v_1, \cdots v_k$ is a path of vertices. Let us define the **height difference** of this path $h(v_0, v_1, \cdots v_k) = \sum_{i=1}^{k} \text{sp}(v_k, v_{k-1})$ (Rémila, 2004)[1].

Theorem 76. [2] *If $c_0, \cdots, c_n$ are the vertices of the border of a closed region (with $c_0 = c_n$), then if the region is balanced, $h(v_0, v_1, \cdots v_k) = 0$.*

[Referenced on page 80]

Proof. Along the border of a balanced region, deficiency is 0 (Theorem 27), and the difference in white and black edges are equal to 4 times the deficiency (Theorem 31), and therefore also 0, and so $h(c_0, \cdots, c_n) = \sum_{i=1}^{k} \text{sp}(c_k, c_{k-1}) = 0$. □

A path that does not cross any dominoes is called a **domino edge path** (Ito, 1996).

Theorem 77 (Rémila (2004), Corollary 1)**.** *Suppose we have a tiling of a region, and P and P' are two different domino edge paths from vertex u to vertex v. Then $h(P) = h(P')$.*

[Not referenced]

Proof. WLOG assume the paths do not cross. (If the paths cross at a vertex w, we can consider the two paths from u to w and from w to v.)

Let $P = p_0 \cdots p_m$ and $P' = p'_0 \cdots p'_n$ (with $p_0 = p_0 = u$ and $p'_m = p'_n$). The path $Q = p_0 \cdots p_m p'_{k-1} \cdots p'_0$ does not cross any dominoes, its interior is therefore tileable, and therefore balanced (Theorem 27). So $h(Q) = 0$ (Theorem 76). But $h(Q) = h(P) - h(P')$, and so $h(P) = h(P')$. □

This shows h only depends on the start and end nodes. We now pick any vertex, v^*, and define $h(v^*) = 0$, and $h(v) = h(v^*, v_1, v_2, \cdots v)$ for any domino edge path. The function h is called a **height function** (Rémila, 2004, Definition 1).

The height function on the border of a region is the same for all tilings of a region.

[1] Ito (1996) calls this the *value* of the path, and use the symbol $\langle v_0, v_1, \cdots, v_k \rangle$ to denote it.

[2] This is a stronger version of Rémila (2004, Proposition 1).

Theorem 78 (Rémila (2004), Proposition 2)**.** *Given the height function of a tiling, we can reconstruct the tiling. Therefore, if for two tilings we have $h_T(v) = h_U(v)$ for all v, then $T = U$.*

[Referenced on pages 83 and 91]

Proof. The height difference between neighboring vertices can only be:

(1) ± 1, if there is not a domino that crosses the edge between them, or

(2) ± 3, if there is a domino that crosses the edge between them.

So for any two neighboring vertices we can determine from the height function whether a domino crosses the edge between them or not, and doing this for all neighboring pairs will show us where all the dominoes in the tiling lie. □

Theorem 79 (Rémila (2004), Proposition 3)**.** *An integer function f defined on the vertices of a region that satisfy the following properties is the height function of a tiling:*

(1) *There exists a vertex v^* such $f(v^*) = 0$.*

(2) *For neighboring vertices u and v such that $\mathrm{sp}(u, v) = 1$, we either have $f(v) = f(u) + 1$, or $f(v) = f(u) - 3$.*

(3) *If these vertices are on the boundary, then $f(v) = f(u) + 1$.*

[Referenced on page 89]

Proof. Let $w_0, w_1, w_2, w_3, w_4 = w_0$ be a path around a black cell (going anticlockwise). Then by (2) the only possibility is if there is one vertex w_j such that $f(w_{j+1}) = f(w_j) - 3$, and for other the other three vertices we have $f(w_{i+1}) = f(w_i) + 1$ (otherwise, we cannot have $f(w_0) = f(w_4)$).

A symmetric argument can be made for white cells. We can now place a domino over the edges $w_i - w_{j+1}$. Note that dominoes cannot overlap, since if they overlapped in a black cell, for example, it means there must be *two* edges $w_j - w_{j+1}$, and not just one, such that $f(w_{i+1}) = f(w_i) + 1$.

This gives us a tiling of the region. We can now verify that $f(v) = h(v)$ by induction on the distance of v from v^*, which completes the proof. □

Theorem 80. *Let R be a region and S be a strip tiled in a tiling T of R. Let U be T with S rotated clockwise. Then the height of all vertices on and outside the outside border of S stays the same. The heights of all vertices on and inside the inside border increases by 4.*

[Referenced on pages 83 and 84]

Proof. All vertices outside and on the outside border of the closed strip can be reached by paths that are unaffected by the strip rotation; so these vertices' height stays the same.

Now consider a vertex u outside the closed strip or on the outer border, and one v on the inside (or on the inside border). Before the strip rotation, there is a path from u to v from which we can calculate the height of v given that of u. This path must intersect the strip at least once, and WLOG lets say it does cross it once. Since it cannot pass through a domino, it must pass between two dominoes on the closed strip. Let the vertices where this happens be w_o (on the outside border) and w_i (on the inside border).

After the strip rotation, we cannot go from w_o directly to w_i as before; instead, we need to pass around the domino. Suppose we go clockwise. Since the strip rotation was also clockwise, it means we are passing a black node on the left. We are also skipping the original section with a white domino on the left. Therefore, the net difference in height is $3 - (-1)$, which means the new height of v is now $h' = h + 4$. □

We will call a strip rotation **up** if it takes a anti-clockwise tiling to a clockwise tiling, and **down** otherwise.

A special case of this is when S is a flippable pair. An up flip increases a single vertex's height by 4. The height of a vertex between two tilings can only differ by multiples of 4, because we can get one tiling from any tiling by strip rotations[3]. (Note that while these strip rotations don't overlap, one can completely surround another, and so it is possible for the height of a vertex to differ by more than 4.)

[3] This is Rémila (2004, Lemma 1).

Problem† 36.

(1) *Suppose u and v are the two vertices of a cell edge of a tileable region R. Show that for any two tilings T and U of R, that the value of $(h_T(v) - h_U(v)) - (h_T(u) - h_U(u))$ is 4, 0 or -4.*

(2) *When is it true that if we have a tiling T of a region R, and a function g defined on the vertices of R such that for any two vertices u and v that share a cell edge, we have $(h_T(v) - g(v)) - (h_T(u) - g(u))$ is 4, 0 or -4, then there is a tiling U such that $h_U = g$.*

Suppose T and U are two tilings of a region, with height functions h_T and h_U. Then we write $T \leq U$ iff $h_T(v_i) \leq h_U(v_i)$ for all i.

A relation $\leq$ defined on a set is called a partial order if it satisfies these three properties for any elements A, B, and C of the set:

$$\begin{array}{lll} A \leq A & \text{Reflexive} & (1) \\ A \leq B \text{ and } B \leq A \text{ implies } A = B & \text{Commutative} & (2) \\ A \leq B \text{ and } B \leq C \text{ implies } A \leq C & \text{Anti-symmetric} & (3) \end{array}$$

Theorem 81. *The operation $\leq$ is a partial order.*

[Referenced on pages 87, 88, 89 and 90]

Proof. We need to prove the above three properties hold.

(1) $h_A(v) \leq h_A(v)$, and so $A \leq A$.

(2) $A \leq B$ implies $h_A(v) \leq h_B(v)$, and $B \leq A$ implies $h_B(v) \leq h_A(v)$, so $h_A(v) = h_B(v)$. And therefore, $A = B$ (Theorem 78).

(3) $A \leq B$ implies $h_A(v) \leq h_B(v)$ and $B \leq C$ implies $h_B(v) \leq h_C(v)$, so we have $h_A(v) \leq h_B(v)$, and so $A \leq C$.

□

Theorem 82. *Suppose that T and U are tilings that differ by a flip, and $T \leq U$. Then if X is a tiling such that $T \leq X \leq U$, then either $X = T$ or $X = U$.*

[Referenced on page 84]

Proof. From Theorem 80 it follows that for any two tilings T and U, we have $h_T(v) - h_U(v) = 4k$ for some integer k.

Now if T and U are two tilings that differ by a flip, then by Theorem 80 there is a single vertex u such that $h_T(u) + 4 = h_U(u)$, and for all other vertices v we have $h_T(v) = h_U(v)$. Suppose then there is a tiling X such that $T \leq X \leq U$. Then for all v, we must have $h_T(v) \leq h_X(v) \leq h_U(v)$. Then for $v \neq u$, we have $h_T(v) = h_X(v) = h_U(v)$, and for u we have $h_T(v) \leq h_X(v) \leq h_T(v) + 4$, or $0 \leq h_X(u) - h_T(v) \leq 4$. Since $h_X(u) - h_T(v) = 4k$ for some integer k, we must have $h_X(u) - h_T(u) = 0$ or 4, or $h_X(u) = h_T(u) \text{or} h_U(u)$. Therefore, for all v, we have $h_X(v) = h_T(v)$ or $h_U(v)$, and so by Theorem 78, $X = T$ or $X = U$. □

Theorem 83 (Rémila (2004), First part of Proposition 7)**.** *Suppose T and U are tilings of a region, and that $T < U$. Then we can perform an up flip on T to obtain a new tiling X such that $T < X \leq U$.*

[Referenced on pages 84 and 85]

Proof. Take the smallest vertex v such that $h_T(v) < h_U(v)$. (This implies $h_T(v) \leq h_U(v) - 4$.)

Now let v' be a neighbor of v such that $\operatorname{sp} v' = -1$. If a domino does not cross the edge $v - v'$, then by Theorem 78, $h_T(v') = h_T(v) - 1 < h_U(v) - 1 \leq h_U(v') + 1 - 1 = h_U(v')$. So $h_T(v') < h_U(v)$. But then v cannot be the smallest vertex, and so a domino *must* cross $v - v'$.

This is true for both neighbors of v with $\operatorname{sp} v' = -1$, and so we have a flippable pair. Moreover, it is anticlockwise, and so we can perform an upflip to obtain tiling X. By theorem 80 we have $h_X(v) = h_T(v) + 4 \leq h_T(v)$, and for all other vertices u we have $h_T(v) = h_X(u) \leq h_U(u)$. Thus, we have $T < X \leq U$. □

Let us write $T < U$ when $T \leq U$ and $T \neq U$. A **covering relation** $\prec$ is a such that if $T \prec U$, then $T < U$, and there is no element X such that $T < X < U$ (Davey and Priestley, 2002, 1.14, p.11).

Theorem L1. *(Davey and Priestley, 2002, 1.14, p.11) If $T \leq U$ are tilings of a simply connected region, then one of the following hold:*

(1) $T = U$.

(2) $T \prec U$.

(3) *There are tilings V_i such that $T \prec V_1 \prec V_2 \prec \cdots \prec U$.*

[Referenced on page 85]

Proof. Suppose that $T < U$. Then either there is a tiling X such that $T < X < U$, or there is not. In the latter case, $T \prec U$. In the former case, repeat the argument with T and X, and with X and U, until we recover the sequence $T \prec V_1 \prec V_2 \prec \cdots \prec U$. The process must end because no tiling is repeated (at each stage, we can write $T < X_1 < \cdots < U$ for the tilings found so far) and there are only a finite number or tilings. □

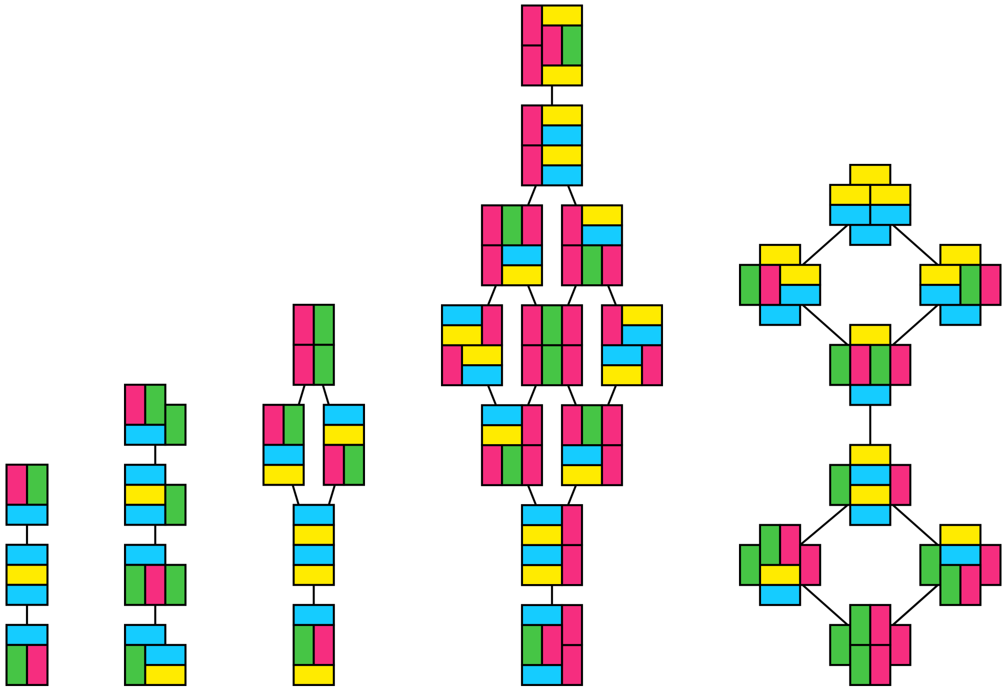
Theorem 84. *If R is a simply-connected region with tilings T and U, the following are equivalent:*

(1) $T \prec U$

(2) *T and U differ by a flippable pair, and the pair is anticlockwise in T.*

[Referenced on page 85]

Proof. Suppose $T \prec U$, which means $T < U$. Then by Theorem 83 there is a tiling X that differs by a flip from T such that $T < X \leq U$. If $X \neq U$, then we have $T < X < U$, which contradicts $T \prec U$, and so $X = U$, which means U differs from T by a single flip.



(a) (b) (c) (d) (e)

Figure 81: Examples of Hasse diagrams for the tilings of various regions without holes.

Suppose T and U differ by a flippable pair, anticlockwise in T. Now suppose X is a tiling such that $T \leq X \leq U$. Then by Theorem 82 either $X = T$ or $X = U$, and so $T \prec U$. $\square$

Alternative proof of second part. Suppose that for some V, we have $T \leq V \leq U$. This means, that $T \prec V_1 \prec \cdots \prec V_k \prec U$ (by Theorem L1), where one of V_i equals V, or, if we let $V_0 = T$ and $V_{k+1} = U$, we have $V_0 \prec V_1 \prec \cdots \prec V_k \prec V_{k+1}$. Let f_i be the flippable pair that takes V_{i-1} to V_i, and g the flippable pair that takes T to U.

Now consider the top-most (and left-most if there is more than one) flippable. The top left corner is not moved by any other flippable pair. This corner must then be different in both T and U, and so must be moved by g. And in fact, this flippable pair must equal g. Similarly, we can show bottom-most, left-most and right-most flips from f_i must all equal g. But then all f_i must equal g, but $f_i \neq f_{i+1}$ (the second one is already clockwise). So this situation is only possible if there are no V_i. Thus, no V can lie between T and U. $\square$

Theorem 85 (Rémila (2004), Proposition 7)**.** *Suppose $T < U$. Then there is a sequence of up flips that takes T to U.*

[Not referenced]

Proof. This follows from combining Theorem L1 with Theorem 84, or by induction from Theorem 83. $\square$

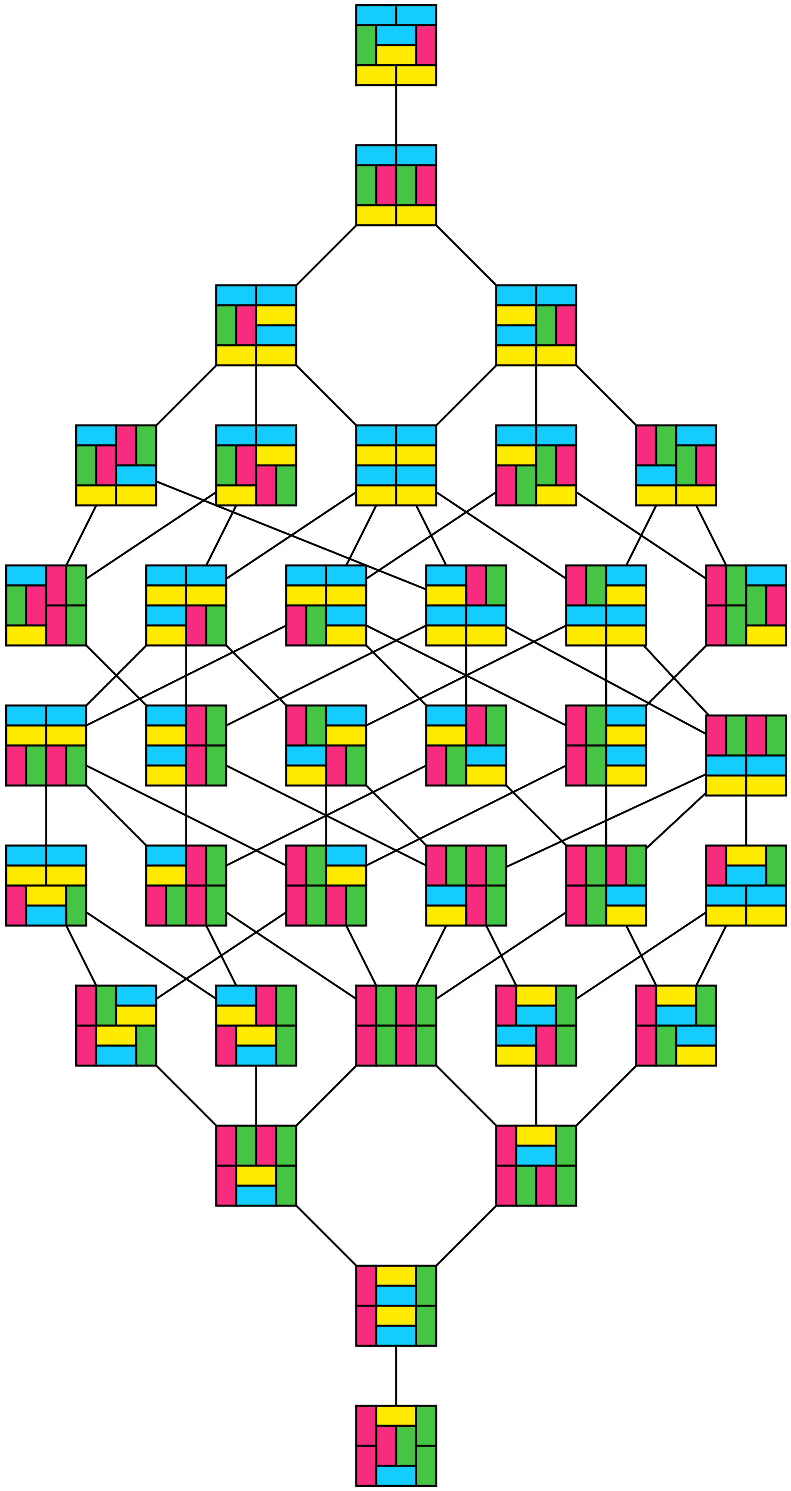


Figure 82: The flip graph of tilings of the 4×4 square.

Theorem 86. *If $A \prec T$, $B \prec T$, $A \prec U$ and $B \prec U$, then either $A = B$ or $T = U$.*

[Not referenced]

Proof. Let $A \xrightarrow{f_1} T \xrightarrow{f_2} B$ and $A \xrightarrow{g_1} U \xrightarrow{g_2} B$. If f_1 and f_2 are the same, then $A = B$. If they are different, then f_1 moves at least 2 cells that is not moved by f_2. Those same cells must be moved by g_1 (they cannot be moved by g_2 because of the orientation). But then $f_1 = g_1$, and $f_2 = g_2$, and so $T = U$. □

This theorem is generalized in Theorems L2 and L4.

Theorem 87. *Suppose that $A \leq T$, $A \leq U$, $B \leq T$, $B \leq U$, and $A \neq B$. Then there exist a tiling V such that $A \leq V$, $B \leq V$, $V \leq T$ and $V \leq U$.*

[Referenced on page 87]

Proof. If $T \leq U$, we can choose $V = T$. If $U \leq V$ we can choose $V = U$. Suppose then T and U cannot be compared.

Now A and B differ by closed strips, and say the outer ones are S_i and they surround vertices v_j. Now consider the tiling X which has all these strips in clockwise orientation. Clearly, $A \leq X$ and $B \leq X$. But clearly $X \leq T$ and $X \leq U$. □

Theorem L2. *Suppose Q is the set of all tilings bigger than both A and B. Then there exist a unique tiling V in Q smaller or equal to any element in Q.*

[Referenced on pages 87 and 88]

Proof. Let $Q = \{X_1, X_2, \cdots, X_n\}$. Now let V_1 be a tiling bigger than A and B but smaller than X_1 and X_2 (such a tiling must exist by Theorem 87). Let V_i be a tiling bigger than A and B but smaller than V_{i-1} and X_{i+1} (again, such a tiling exists by Theorem 87). Then V_{n-1} is in Q, and $V_{n-1} \leq X_i$ for all i.

To prove uniqueness, suppose there is another element V' such that it is bigger than A and B and smaller than all elements in Q. In particular, $V' \leq V$. But V is smaller than all elements of Q, and $V' \in Q$, so $V \leq V'$. Therefore, $V' = V$ (by Theorem 81.2). □

We call V the **join** of A and B and denote it by $A \vee B$. The following is basically a restatement of the definition in a more convenient form.

Theorem L3.

(1) $T \leq T \vee U$

(2) $U \leq T \vee U$

(3) *If $T \leq A$ and $U \leq A$, then $T \vee U \leq A$.*

[Referenced on pages 88 and 89]

Theorem L4. *Suppose Q is the set of all tilings smaller than both A and B. Then there exist a unique tiling V in Q bigger or equal to any element in Q.*

[Referenced on page 87]

Proof. The proof is almost exactly like the one in Theorem L2. □

We call V the **meet** of A and B and denote it by $A \wedge B$. As before, the following is basically a restatement of the definition in more convenient form.

Theorem L5.

(1) $T \wedge U \leq T$

(2) $T \wedge U \leq U$

(3) *If $A \leq T$ and $A \leq U$, then $A \leq T \wedge U$.*

[Referenced on page 88]

A set with a partial order for which the join and meet always exists is called a **lattice**.

Theorem L6. *Let T, U, and V be tilings of a region. The join and meet operations satisfy the following laws (Davey and Priestley, 2002, Thereom 2.9, p. 39):*

$$T \vee U = U \vee T \qquad T \wedge U = U \wedge T \qquad \textit{Commutative} \quad (1)$$

$$(T \vee U) \vee V = T \vee (U \vee V) \qquad (T \wedge U) \wedge V = T \wedge (U \wedge V) \qquad \textit{Associative} \quad (2)$$

$$T \vee (T \wedge U) = T \qquad T \wedge (T \vee U) = T \qquad \textit{Absorption} \quad (3)$$

$$T \vee T = T \qquad T \wedge T = T \qquad \textit{Idempotent} \quad (4)$$

[Not referenced]

Proof. Here we prove only the first of each of the four pairs of equations. The remainder is proven with dual arguments using (Theorem L5).

(1) This follows directly from the symmetry in the definition of $\vee$.

(2) Let $A = (T \vee U) \vee V$. Then, $(T \vee U) \leq A$ (Theorem L3.1), and $V \leq A$ (Theorem L3.2). From the first of these, we have $T \leq A$ and $U \leq A$. But then $U \vee V \leq A$ (Theorem L3.3), and so $T \vee (U \vee V) \leq A$ (Theorem L3.3), or $T \vee (U \vee V) \leq (T \vee U) \vee V$. Similarly, (by putting $B = T \vee (U \vee V)$) we can show $(T \vee U) \vee V \leq T \vee (U \vee V)$. Taken together, we have $T \vee (U \vee V) = (T \vee U) \vee V$ (Theorem 81.2).

(3) $T \wedge U \leq T$ and $T \leq T$, so $T \vee (T \wedge U) \leq T$. But $T \leq T \vee (T \wedge U)$, so $T \vee (T \wedge U) = T$ (Theorem 81.2).

(4) $T \leq T \vee T$ (Theorem L3.1), and (since $T \leq T$) $T \vee T \leq T$ (Theorem L3.3). Thus, $T = T \vee T$ (Theorem 81.2).

□

Theorem 88. [4] *Let T and U be tilings of a region. Then*

(1) $\min(h_T(v), h_U(v)) = h_{T \wedge U}(v)$

(2) $\max(h_T(v), h_U(v)) = h_{T \vee U}(v)$

[Referenced on page 91]

[4] This is essentially Rémila (2004, Proposition 5)

Proof. We will only prove (1), (2) can be proven symmetrically.

First, let us show that $f(v) = \min(h_T(v), h_U(v))$ corresponds to a tiling. We prove the three conditions of Theorem 79.

(1) $f(v^*) = \min(h_T(v^*), h_U(v^*)) = \min(0, 0) = 0$.

(2) Consider neighbors u and v such that $\mathrm{sp}(u, v) = 1$, and assume $h_T(u) < h_U(u)$, or $h_T(u) \leq h_U(u) - 4$. We also have $h_T(v) \leq h_T(u) + 1$, and $h_U(v) \geq h_U(u) - 3$, so $h_T(v) \leq h_T(u) + 1 \leq h_U(u) - 4 + 1 \leq h_U(v) + 3 - 4 + 1 = h_U(v)$.

   This proves if $h_T(u) < h_U(u)$, then $f(v) = h_T(v)$. But, since $f(u) = h_T(u)$, we have $f(v) - f(u) = h_T(v) - h_T(u)$, which means the second condition is satisfied. Similar arguments hold when $h_T(u) > h_U(u)$ or $h_T(u) = h_U(u)$.

(3) The above also shows the third condition is met when u and v lie on the border.

Therefore, f corresponds to a tiling.

Now let this tiling be X. Since $f = h_X = \min(h_T, h_U)$, we have $X \leq T, U$. Suppose then there is a tiling Y such that $X \leq Y \leq T, U$. Then we have $\min(h_T, h_U) = h_x \leq h_y \leq h_T, h_U$, which is possible only if $h_Y = h_X$, and therefore $Y = X$. Therefore X is the maximum tiling smaller than T and U, and therefore $X = T \wedge U$. Thus, $\min(h_T(v), h_U(v)) = h_{T \wedge U}(v)$. □

Theorem L7. *Let R be a region with a tiling. Then*

(1) *There exist a unique tiling E such that $T \leq E$ for all tilings T of R.*

(2) *There exist a unique tiling Z such that $Z \leq T$ for all tilings T of R.*

(3) *$E = Z$ if only if R has a unique tiling.*

[Referenced on pages 91 and 101]

Proof. Define $E = \bigwedge T_i$. Clearly, $T_i \leq Z$ for all i. Suppose another tiling E' has this property. We have both $E \leq E'$ and $E' \leq E$, so $E = E$ (Theorem 81.2).

Similarly, define $Z = \bigvee T_i$. $Z \leq T_i$ for all i. And if another tiling Z' has this property, we have both $Z \leq Z'$ and $Z' \leq Z$, and so $Z = Z'$ (Theorem 81.2).

Suppose R has a unique tiling T. Then clearly $T = Z = E$. Suppose on the other hand $E = Z$, and consider any tiling T of R. Then we have $Z \leq T \leq E$, or $Z \leq T \leq Z$, which implies $T = Z$ (Theorem 81.2), so all tilings of R are equal, and so R has a unique tiling. □

The tiling E in the theorem above is called the **maximum tiling**, and is denoted by 1_R (this assumes the tileset $\mathcal{T}$ is understood), or if the region is understood, simply by 1. Similarly, the tiling Z is called the **minimum tiling**, and it is denoted by 0_R, or simply 0, if R is understood. The minimum and maximum tilings for some rectangles are shown in Figure 83.

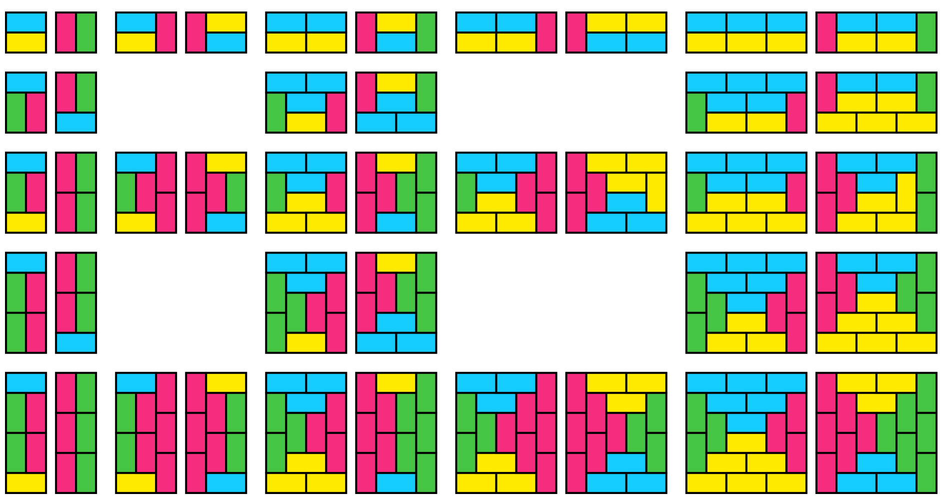


Figure 83: The minimum and maximum tilings for small rectangles.

A **distributive lattice** is a lattice that satisfies the following two distribution properties (Davey and Priestley, 2002, Def. 4.4, p. 86):

$$T \vee (U \wedge V) = (T \vee U) \wedge (T \vee V) \tag{3.1}$$

$$T \wedge (U \vee V) = (T \wedge U) \vee (T \wedge V) \tag{3.2}$$

Theorem 89 (Thurston (1990))**.** *The set of tilings of a simply-connected region forms a distributive lattice.*

[Not referenced]

Proof. We need to prove that any three tilings T, U, and V satisfy the distributive properties in Equations 3.1 and 3.2.

For the first equation, we use Theorem 88 to find an expression for the tilings on the left hand and right hand of the expression.

Let $X = T \vee (U \wedge V)$. This gives us $h_x = max(h_T, \min(h_U, h_V)) = \min(\max(h_T, h_U), \min(h_T, h_V))$.

Let $Y = (T \vee U) \wedge (T \vee V)$. Then $h_Y = \min(\max(h_T, h_U), \min(h_T, h_V))$, so $h_X = h_Y$, and so $X = Y$ by Theorem 78.

A similar argument can be used to show the second equation is also satisfied. $\square$

Problem† 37. *Find an algorithm to find the minimum and maximum tilings of a region given an arbitrary tiling of the region.*

Theorem 90. *Suppose that $T \prec X_1 \prec \cdots \prec X_m \prec U$ and $T \prec Y_1 \prec \cdots \prec Y_n \prec U$. Then $m = n$.*

[Referenced on page 91]

Proof. Let u_i be all the vertices of R. Let v_i be the vertex that changes height when we go from X_{i-1} to X_i (and let $X_0 = T$ and $X_{m+1} = U$). Similarly, let w_i be the vertices as we move along Y_i. Then we have $\sum_i h_U(u_i) - h_T(u_i) = \sum_{i=1} m + 1 v_i = \sum_{i=1} n + 1 w_i$, or $\sum_{i=1} m + 14 = \sum_{i=1} n + 14$, or $4(m+1) = 4(n+1)$, which means $m = n$. $\square$

The **rank** $\rho(T)$ of a tiling T is defined as follows[5]:

(1) $\rho(0) = 0$.

(2) If $T \prec U$, then $\rho(T) + 1 = \rho(U)$.

Theorem L7 ensures that 0 exists, and Theorem 90 ensures this definition is consistent: it is not possible to arrive at different rank depending on the chain chosen.

Note that if $T \leq U$, then $\rho(T) \leq \rho(U)$.

[5] Rank functions can be defined for general partially orders, see for example Stanley (1986, p. 99).

Theorem 91. *Suppose that $A, B \prec X$, and $A \neq B$. Then there is a tiling Y such that $Y \prec A, B$.*

[Referenced on page 92]

Proof. Because $A, B \prec X$, and $A \neq B$, A and B differ by two flips, and that these flips do not overlap (if they did, then $A \prec X \prec B$, or $B \prec X \prec A$, which is not the case). Suppose we perform flip f from A to X, and g from X to A. We can then perform g on A to obtain Y, and f on Y to obtain B, which shows that $Y \prec A, B$. □

Theorem 92. *Let $\mu(T, U)$ be the minimum number of flips necessary to turn T into U. Then $\mu(T, U) = 2\rho(T \wedge U) - \rho(T) - \rho(U) = \rho(T) + \rho(U) - 2\rho(T \vee U)$.*

[Referenced on page 92]

Proof. Suppose a sequence of tilings from T to V via flips is $U_1, U_2, \cdots U_k$.

A triangle on A and B is a sequence of tilings such that $A \prec X_1 \prec X_2 \prec \cdots \prec X_m \prec C$, and $B \prec Y_1 \prec Y_2 \prec \cdots \prec Y_m \prec C$.

Now there is a tiling U_i such that $U_i \prec U_{i-1}, U_{i+1}$ (unless this sequence is a triangle on T and U). By Theorem 91 there is a U_i' such that $U_{i-1}, U_{i+1} \prec U_i$. We can repeat this procedure until we have a triangle. Note that the operation does not change the number of flips.

Now let C be the top of this triangle. Then $T, U \leq C$, thus $T \vee U \leq C$. And since $T, U \leq T \vee U$, by Theorem we have a flip path from T to U via $T \vee U$, and this path of length $2\rho(T \wedge U) - \rho(T) - \rho(U)$. And since $T \vee U \leq C$, this path must be smaller than the path through C, and hence, the shortest path via flips between two tilings is through the join, given by $2\rho(T \wedge U) - \rho(T) - \rho(U)$.

A symmetric argument shows another shortest path is through the meet, which gives us the length $\rho(T) + \rho(U) - 2\rho(T \vee U)$.

So $\mu(T, U) = 2\rho(T \wedge U) - \rho(T) - \rho(U) = \rho(T) + \rho(U) - 2\rho(T \vee U)$. □

Theorem L8 (Parlier and Zappa (2017))**.** *Let T and U be two tilings of a region. Then $\mu(T, U) \leq \rho(1)$.*

[Not referenced]

Proof. Suppose that $\rho(T) + R(U) > \rho(1)$. By Theorem 92 $\mu(T, U) = 2\rho(1) - (\rho(T) + \rho(U))$, so $\mu(T, U) < \rho(1)$. Suppose on the other hand that $\rho(T) + \rho(U) \leq \rho(1)$. By Theorem 92, $\mu(T, U) = \rho(T) + \rho(U) - 2\rho(T \vee U) \leq \rho(T) + \rho(U) \leq \rho(1)$. □

We next give the maximum ranking for rectangles and Aztec diamonds. Using the above theorem, we know what is the most flips we need to turn one tiling of one of these figures into another tiling.

Theorem 93 (Parlier and Zappa (2017)). *For rectangles,*

$$\rho(1_{R(m,n)}) = \begin{cases} \frac{mn^2}{4} - \frac{n^3}{12} - \frac{n}{6} & \text{for } n \text{ even} \\ \frac{mn^2}{4} - \frac{n^3}{12} + \frac{n}{12} - \frac{m}{4} & \text{otherwise} \end{cases}. \tag{3.3}$$

For squares,

$$\rho(1_{R(n,n)}) = \frac{n^3 - n}{6}. \tag{3.4}$$

[Not referenced]

Table 9 shows some values for $\rho(1_{R(m,n)})$.

	2	4	6	8	10	12
1	0	0	0	0	0	0
2	1	3	5	7	9	11
3	2	6	10	14	18	22
4	3	10	18	26	34	42
5	4	14	26	38	50	62
6	5	18	35	53	71	89
7	6	22	44	68	92	116
8	7	26	53	84	116	148
9	8	30	62	100	140	180
10	9	34	71	116	165	215
11	10	38	80	132	190	250
12	11	42	89	148	215	286

Table 9: $\rho(1)$ for $R(m,n)$.

An **Aztec diamond** $A(n)$ is a region defined as follows Elkies et al. (1992a, p. 111)[6]:

$$A(n) = \left\{ (x,y) \mid |x - \frac{1}{2}| + |y - \frac{1}{2}| \leq n \right\}.$$

[6] The form of the definition in Elkies et al. (1992a) is slightly different because they defined it in terms of the cell vertices.

The following equivalent definition avoids fractions:

$$A(n) = \{ (x,y) \mid |2x-1| + |2y-1| \leq 2n \}.$$

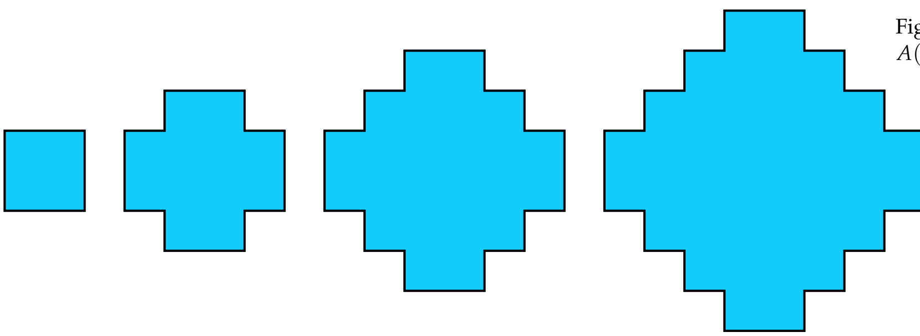


Figure 84: Aztec diamonds $A(1)$, $A(2)$, $A(3)$ and $A(4)$.

It is easy to see that the area of an Aztec diamond is given by $2n(n+1)$ (if you divide the Aztec diamond into quarters, each quarter is a triangle $T(n)$ with $1+2+3+n\cdots = n(n+1)/2$ cells).

Each row of an Aztec diamond is a strip polyomino with an even number of (connected) cells, and is therefore tileable by dominoes (Theorem 55). In fact, Aztec diamonds are strip polyominoes themselves.

Problem† 38. *Prove that Aztec diamonds are*

(1) *strip polyominoes*

(2) *Saturnian*

Theorem 94 (Parlier and Zappa (2017)). *For Aztec diamonds,*

$$\rho(1_{A(n)}) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

[Not referenced]

Table 10 shows some values for $\rho(1_{A(n)})$.

The following relates the height function of a tiling with its ranking.

Theorem 95. *The ranking of a tiling is given by*

$$\rho(T) = \frac{1}{4}\left(\sum_i h_T(v_i) - \sum_i h_0(v_i)\right).$$

[Not referenced]

n	A000330
1	1
2	5
3	14
4	30
5	55
6	91
7	140
8	204
9	285
10	385
11	506
12	650

Table 10: $\rho(1)$ for $A(n)$.

Proof. Clearly, $\rho(0) = 0$. Suppose now the above is true for all tilings T with $\rho(T) < k$. Suppose now T is any tiling with rank $\rho(U) = \rho(T) + 1$. This means, that U and T differ by a flip, and so there is a vertex v_m such such $h_U(v_m) = h_T(v_m) + 4$, and $h_U(v) = h_T(v)$ for all other vertices. Thus

$$\begin{aligned}\rho(U) &= \rho(T) + 1 \\ &= \frac{1}{4}\left(\sum_i h_T(v_i) - \sum_i h_0(v_i)\right) + 1 \\ &= \frac{1}{4}\left(-4 + \sum_i h_U(v_i) - \sum_i h_0(v_i)\right) + 1 \\ &= \frac{1}{4}\left(\sum_i h_U(v_i) - \sum_i h_0(v_i)\right).\end{aligned}$$

□

A **local minimum** is a vertex v, such that $h(v) < h(v')$ for all its neighbors v'. Similarly, a **local maximum** is a vertex v such that $h(v) > h(v')$ for all its neighbors v'.

Theorem 96 (Thurston (1990))**.** *Algorithm for finding a tiling of a region if it exists.*

[Referenced on pages 128 and 183]

Proof.

(1) An interior vertex is a local minimum or maximum if and only if it lies at the center of a flippable pair.

(2) The minimal tiling of a figure has no local maximums in the interior.

(3) If it did, we could get a tiling smaller than the minimum by performing a flip.

(4) Therefore, the vertices for which $h_0(v)$ is maximum lie on the border. Let v be such a vertex, and let u and w be its neighbors so that u, v and w is in clockwise order. Note that $h_0(u), h_0(w) < h(v)$, so the edge between u and v must be white, and the edge between v and w must be black.

(5) Therefore v cannot be a corner, which means u, v and w lie in a straight line.

(6) There cannot be a domino path from v to the interior neighboring vertex v', otherwise it will be higher than the maximum (since it has a black sell on the left). Therefore, a domino must cover the edge between v and v'. We found one domino of the tiling 0 (if it exists).

(7) We can now remove the two covered cells, update the height function on their edges, and repeat the process. If, when we update the height function we cannot assign consistent heights, a tiling is impossible.

□

A **direct path** from one vertex to another is a path from the one vertex to the other so that it always has a black cell on the left. Note that this path is not symmetric (See Figure 85). The **distance** between two vertices is the length of the shortest direct path between them (note that since the path has black cells on the left, the spin from one vertex on the path to the next is always 1).

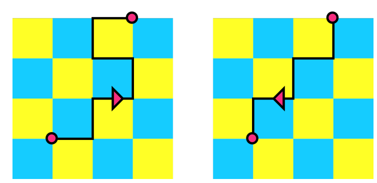


Figure 85: A direct path between two vertices. If we change the direction of the path, some edges are different.

Removing a cylinder from a region does not affect the heights of the vertices of the border that remain. To make this statement precise, we need a function that maps the cells of the reduced region to the cells of the original region. If R is a region, and S is a 2-cylinder, then $f_S : R \ominus S \to R$ maps the cells in $R \ominus S$ to their original counterparts in S.

Theorem 97. *If R is a region and S is a removable 2-cylinder, and h a height function. Then $h_{R\ominus S}(v_i) = h_R(f_S(v_i))$ for all vertices.*

[Not referenced]

Proof. Let v_k be a vertex on the border of $R - S$, with height function defined relative to the vertex v_0. Let $v_0, v_1, v_2, \cdots v_k$ be a path that does not cross any dominoes. The cylinder can cut through this path zero times, once, or twice.

(1) If it does not cut through the path, then the height of v_k is the same in both R and $R \ominus S$.

(2) If it cuts through the path once, there is one extra white and one extra black edge in R, which means the net effect is 0.

(3) If it cuts through the path twice, there is two extra white and two extra black edges, and the net effect on the height is 0.

In all cases, $h_{R\ominus S}(v_k) = h_R(f_S(v_k))$ □

Theorem 98. *Let S be a removable cylinder of R, and suppose u and v are vertices with not in the interior of the cylinder, or the borders of the cylinder shared with R. Then*

$$d_{R\ominus S}(u,v) < d_R(f(u), f(v)) < d_{R\ominus S}(u,v) + 4.$$

[Not referenced]

Proof. If u and v are on the same side of the cylinder, the shortest path between them is unaffected, and therefore the length is the same.

If they lie on opposite sides of the cylinder, we can find a shortest path between them that has at most 4 edges in the cylinder. This is because a piece of the path with more than 4 edges must have a section of 3 consecutive edges where the path goes around a cell, and can be changed to go around a different cell outside the cylinder. See Figure 86 for an example. And when we remove the cylinder, this path is at most 4 units shorter. □

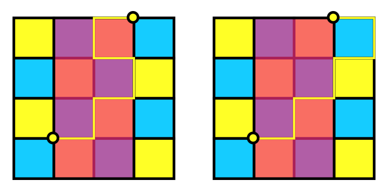


Figure 86: Rerouting a direct path.

Problem† 39.

(1) *The two theorems above show us that if $|h_{R\ominus S}(f(x), f(y))| \leq d_{R\ominus S}(f(x), f(y))$, then $|h_R(x,y)| \leq d_R(x,y)$. Use this to show that if $R \ominus S$ is tileable, then so is R. You also need to take the vertices of S into account. This is a different way to prove Theorem 41.*

(2) *Show that the converse does not hold by checking that Figure 42 provides a counter example.*

Theorem 99. *A simply-connected region R is tileable by dominoes if and only if $\Delta(R) = 0$ and $|h(x,y)| \leq d(x,y)$ for all vertices x and y on the border.*

[Referenced on page 128]

Proof. Suppose that there are vertices x and y such that $h > d$. Let S be a subregion of R between a shortest direct path between x and y and the border of R.

If R is tileable, then $\phi(S) = \Delta(S) = \frac{h+d}{4}$. But the only place where dominoes can cross is on the direct path between x and y. Note that because of the path's shape, we have

$$\frac{d}{3} \leq \phi(S) \leq \frac{d}{2}. \tag{3.5}$$

Thus $\frac{h+d}{4} \leq \frac{d}{2}$, or $h + d \leq 2d$, or $h \leq d$, which contradicts our assumption, and therefore R is not tileable.

On the other hand, suppose R is not tileable. Then there is a bad patch S. This bad patch cannot lie completely in the interior of the region (because all such cells have 4 neighbors in R, and therefore cannot be a bad patch). So suppose the bad patch shares a border with R between vertices x and y. WLOG assume the patch is black (if it is white, we can also find a black bad patch by Theorem 37). Now take a shortest direct path between x and y, and let S' be the subregion between this path and the border of R, with $\phi(S') > \Delta(S')$ (This is possible since the region is not tileable, and either the flux must be bigger or smaller; since there are two possible subregion, we can find one with the necessary constraint.)

By Equation 3.5 we have $\frac{h+d}{4} < \phi(S') \leq \frac{h}{2}$, so $h + d < 2h$, or $d < h$. □

The section *Further Reading* gives some references in dealing with regions with holes, which we don't cover here.

3.1.2 *Forced Flippable Pairs*

The following section is from Kranakis (1996). A **pyramid**[7] is stack polyomino with the following forms:

- For an even number of $2k$ columns: $B(3 \cdot 4 \cdot 5, \cdots (k+2)^2 \cdots 5 \cdot 4 \cdot 3)$. See Figure 87.
- For an odd number of $2k+1$ columns: $B(3 \cdot 4 \cdot 5, \cdots (k+2)^3 \cdots 5 \cdot 4 \cdot 3)$. See Figure 88.

[7] This is a slightly different definition than given in Kranakis (1996), which does not account correctly for all the cases in the pyramid lemma.

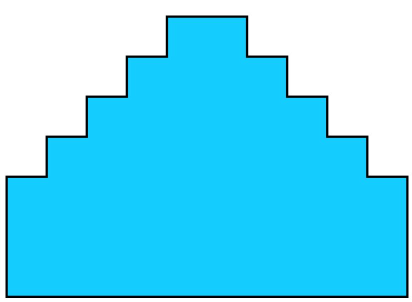


Figure 87: An even pyramid with vector $B(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7^2 \cdot 6 \cdot 5 \cdot 4 \cdot 3)$.

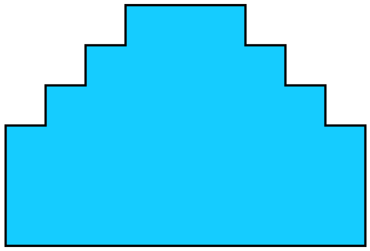


Figure 88: An odd pyramid with vector $B(3 \cdot 4 \cdot 5 \cdot 6^3 \cdot 5 \cdot 4 \cdot 3)$.

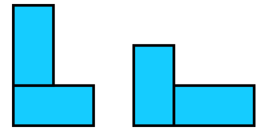


Figure 89: An L-configuration.

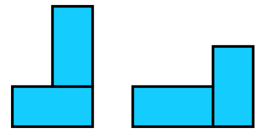


Figure 90: An R-configuration.

An **L-configuation** is two dominoes at 90 degrees that form a "left corner", (Figure 89); an **R-configuration** is two dominoes at 90 degrees that form a "right corner" (Figure 90).

If each of these occur on the same height within a tiling with the L-configuration on the left, we call the combination an **LR-configuration**.

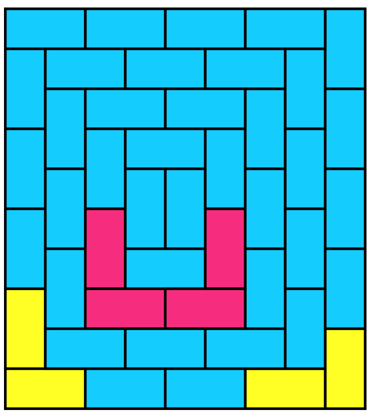


Figure 91: This tiling has several LR-configurations. One is shown in yellow, another one in red.

Theorem 100 (The Pyramid Lemma, Kranakis (1996), Lemma 4)**.** *If a domino tiling of a rectangle is given, and a LR-domino configuration is present in the tiling, then there is a flippable pair present in the smallest pyramid that contains the LR-configuration on its base.*

[Referenced on pages 99, 100, 102, 105, 131 and 274]

Proof. There are a few cases to consider, but the general pattern is the same. Take the LR configuration shown. The left horizontal domino can either be covered by another horizontal domino (in which case we have a flippable pair), or by a vertical domino. This vertical domino either has another vertical domino to the right (in which case we have a flippable pair) or a horizontal domino (forming an L-configuration). A similar argument on the right shows we either have a flippable pair, or another R-configuration. We continue this process, until eventually a flippable pair is forced.

Figures 92 and 93 shows all the different cases for different LR-configurations and whether the smallest pyramid is even or odd. □

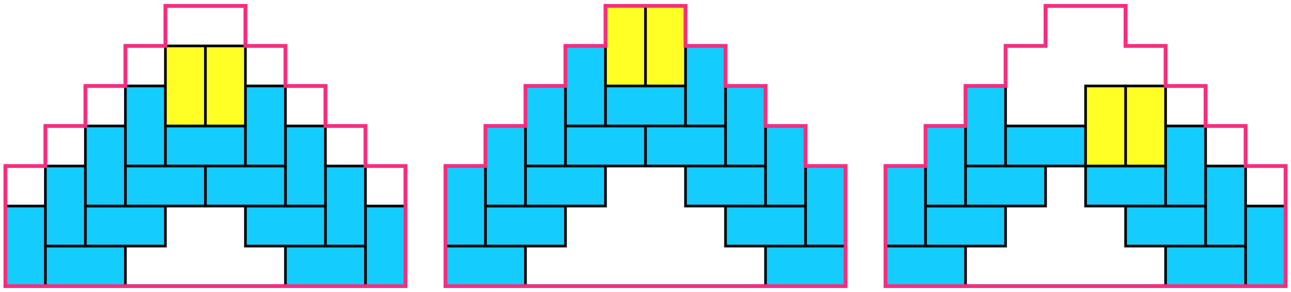


Figure 92: Forced flippable pair in even pyramids.

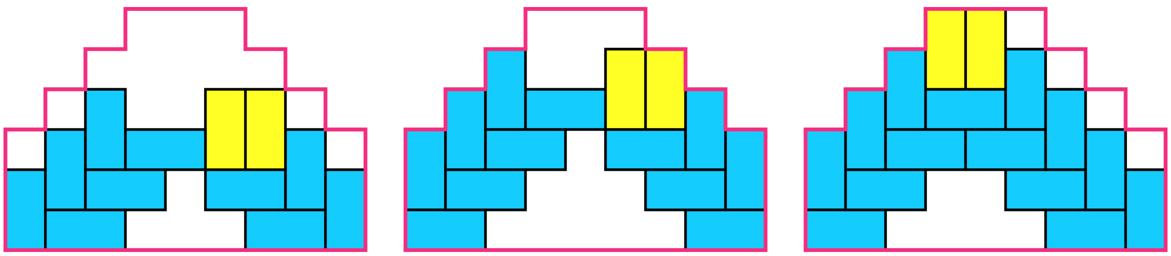


Figure 93: Forced flippable pair in odd pyramids.

Theorem 101. *Suppose there is a vertical domino on the bottom border or a rectangular tiling, with no other vertical dominoes between it and the left (or right) border of the rectangle. Then there is a flippable pair inside the pyramid with the vertical domino in one corner, and the other corner just outside the left (or right) rectangle border.*

[Referenced on pages 102, 105 and 131]

Proof. The logic is exactly the same as used in the proof of Theorem 100: If we try to fill the tiling without adding any flippable pairs, we are eventually forced to add a flippable pair. Figure 94 shows the setup.

□

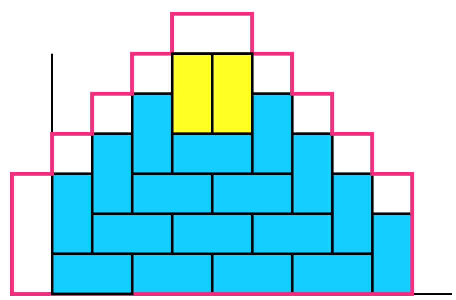


Figure 94: A forced flippable pair in the pyramid between a vertical domino and the rectangle border.

Theorem 102. *If there is an R-configuration in a tiling if a rectangle, and we draw a diagonal to the top left, there is a flippable pair that crosses this diagonal.*

[Referenced on page 131]

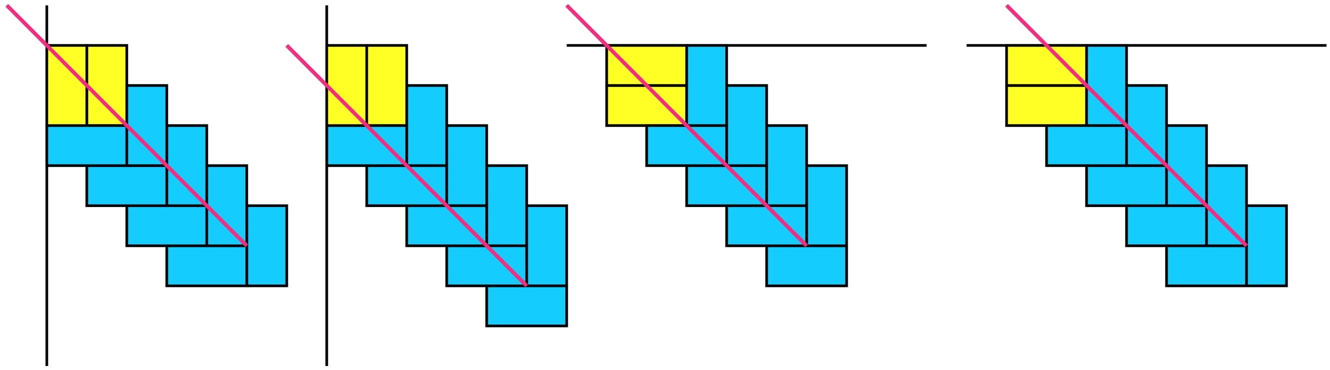


Figure 95: Forced flippable pair occurs on the diagonal that starts from the R-configuration.

Proof. Again, the logic is the same as in Theorem 100. If we fill in the tiling by avoiding flippable pairs, we are eventually forced to add a flippable pair. Figure 95 shows the setup.

□

Theorem 103 (Kranakis (1996), Theorem 5)**.** *Domino tilings of rectangles with both sides of length greater than 1 must always have a flippable pair.*

[Referenced on page 105]

Proof. This follows directly from Theorem 69, since if the rectangle has all sides greater than 1, it has no peaks, and also it has no holes.

We give here an alternative proof given in Kranakis (1996).

Suppose the region rectangle is $R(m, n)$. We prove it by induction on the area mn.

If $m = 2$, then either the first two cells are covered by two horizontal dominoes (a flippable pair), or by a vertical domino as in Figure 105. In the latter case, the next two cells are either covered by two horizontal dominoes (forming a flippable pair), or a vertical domino, forming a flippable pair with the first vertical domino. In all cases we have a flippable pair, and by symmetry the same is true if $n = 2$.

Suppose then both $m, n > 2$. We know (Theorem 1) that mn is even, so either m or n must be even. WLOG suppose m (the horizontal dimension) is even. If there are now vertical dominoes touching a horizontal border, we can strip off a row to get a new rectangle $R(m, n-1)$ with area $m(n-1) < mn$, and the theorem follows by induction.

Suppose there are vertical dominoes along the (say bottom) border. Since m is even, there must be at least two (Theorem 25). Choose two adjacent such dominoes. Then all the dominoes next to the border are horizontal, and by Theorem 100 there is a flippable pair in the pyramid induced by the corner configurations. □

Problem† 40. *Show that if a tiling of $R(m,n)$ with either $m > 3$ or $n > 3$ has two overlapping flippable pairs in a subregion S, then it has two flippable pairs that don't overlap.*

Theorem 104.

(1) *In a rectangle, there are at most two tilings with exactly only one flippable pair.*

(2) *If there are such tilings, the one is a rotation of the other.*

(3) *$R(m,n)$ have 0 such tilings if $m > n+2$ or $n > m+2$.*

(4) *$R(m,n)$ has 1 such tiling if $m = n+2$ or $n = m+2$.*

(5) *$R(m,n)$ has 2 such tilings otherwise, that is $m < n+2$ and $n < m+2$, or $n-2 < m < n+2$.*

[Referenced on page 102]

Proof. A tiling with only one flippable pair must be either minimal or maximal (since its rank can only be reduced or increased by flipping the flippable pair, it cannot be both increased and decreased since there is only one flippable pair.) Since there is exactly one minimal and maximal tiling (Theorem L7), there can be at most two tilings with flippable pairs.

Suppose there are two tilings with exactly one flippable pair.

(1) If $m = n$, we have a square. Now if we rotate one tiling by 90°, we get a different tiling with one flippable pair. But there can be at most two, so this tiling must equal the other tiling, and so the two tilings are rotations of each other.

(2) If $m \neq n$, suppose one tiling is not symmetric w.r.t. a 180° turn. Then by turning it, we can get another tiling. Again, there must be at least two, so it must equal the other tiling. If both are symmetric with respect to a 180° turn, then they must be equal (?), but then there cannot be two tilings with one flippable pair.

So in all cases, there if there are two tilings with one flippable pair, the two tilings are rotational copies of each other.

$R(2,m)$ has more than two flippable pairs for $m > 4$. Each vertical domino must be adjacent to another vertical domino, forming a flippable pair. But if there is only one such, all other dominoes must be horizontal, and so if $m > 4$, there must be at least 3, and they and any arrangement must result in at least one more flippable pair.

$R(3, m)$ must have a fault by Theorem 122. If the fault is vertical, we have $R(2, m)$ subtiling, which must have at least two flippable pairs (as we proved above). If the fault is horizontal, we have two rectangles with height at least two, and so neither has a peak and so each must have a flippable pair (Theorem 69), giving the total figure at least two flippable pairs. □

Theorem 105 (Kranakis (1996), Corollaries 7 and 8)**.** *For squares $R(n, n)$ with n even, there are*

(1) *A unique tiling (up to rotation) that has 1 flippable pair.*

(2) *A unique tiling (up to rotation) that has 2 flippable pairs.*

[Not referenced]

Proof. We showed that any rectangle has at most 2 tilings with 1 flippable pair, and they are rotations of each other (Theorem 104). (This first part is a different proof from the one given in Kranakis (1996)).

For the second part, we consider several cases.

(1) Suppose there are 4 vertical dominoes on the border. Some of these may be adjacent to form groups.

   (a) If there are 4 groups, there are 3 flippable pairs by Theorem 100.

   (b) If there are 3 groups, one group has a flippable pair, and there are 2 additional flippable pairs by 100.

   (c) If there are 2 groups:

      i. If one group is not in the corner, we have one flippable pair in one of the groups, one flippable pair between them (Theorem 100), and one flippable pair between the one group and rectangle border (101).

      ii. If both groups are in the corner, and both are flippable pairs, we have a third flippable pair between them (Theorem 100). If they are not both flippable pairs, then one is a flippable pair, and two more flippables are forced on the diagonals shown in Figure 96.

   In all cases, if we have 4 vertical dominoes, there are at least 3 flippable pairs, so this situation cannot occur.

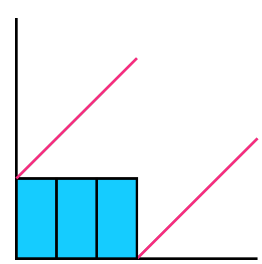


Figure 96: Two additional flippable pairs are forced.

(2) Suppose then there are 2 vertical dominoes.

(a) If there is one group, we have a flippabe pair. If it is not in the corner, we have two additional flippable pairs by Theorem 101, if it is, we have two additional flippable pairs on the diagonals as shown in Figure 97.

(b) If there are two groups

i. If neither is in the corner, we have a flippable pair between them, and between each and a rectangle border, giving three in total.

ii. If one domino is in the corner, we have have one flippable pair between them, and one between the other and the rectangle border. There is a third flippable pair forced by the top corners of the rectangle, and this flippable pair cannot coincide with either of the other two. See Figure 98.

This only leaves the case where there is a vertical domino in each corner.

The final case if there are no vertical dominoes.

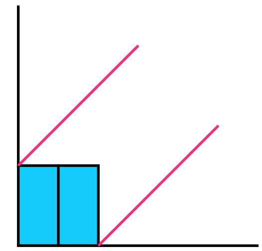


Figure 97: Two additional flippable pairs are forced.

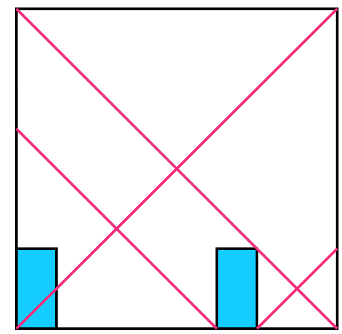


Figure 98: 3 forced flippable pairs.

We have shown that if there are exactly two flippable pairs, we either need no vertical dominoes, or there are exactly two, one in each corner. This can only be realized in one of two ways.

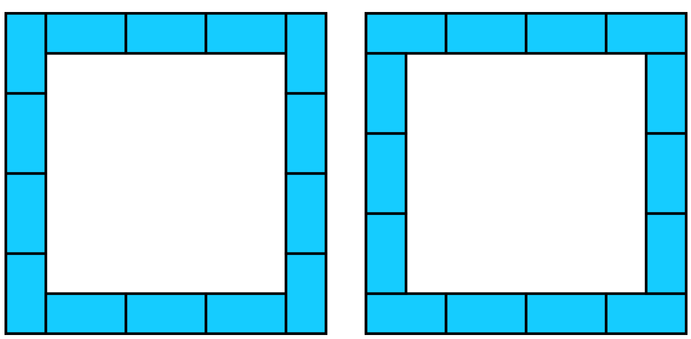
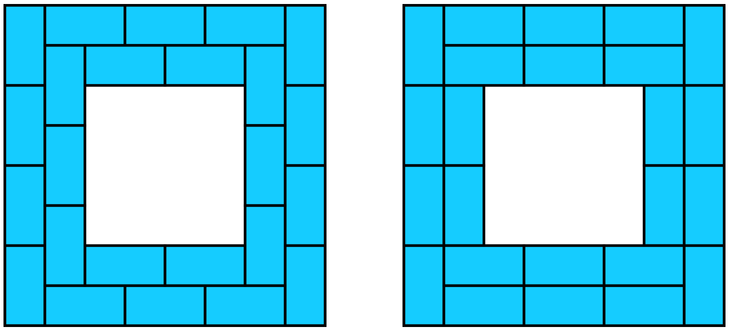


Figure 99: Two ways to complete the border.

In either case, the two flippable pairs are in a sub-square $R(n-2, n-2)$, and we can repeat the process above on this sub-square. Going forward, we only have one option to build the border each time.

Eventually, we arrive at one of the two tilings of the 4×4 square shown in Figure 101. This shows there are only two tilings with 2 flippable pairs, and they are 90 degree rotations of each other.



(a) A legal way to complete the border in the next step.

(b) Not a legal way to complete the border in the next step.

Figure 100: The next step.

□

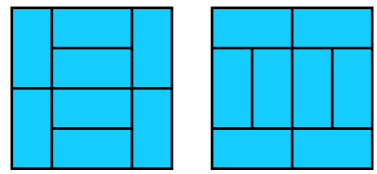


Figure 101: Two ways in which a 4×4 square can have exactly two flippable pairs.

In Kranakis (1996, Theorem 9) the following claim is made: *In a tiling of a* $n \times n$ *square with exactly 3 non-overlapping squares, there is a* 8×8 *square in the center that contains the three non-overlapping squares.* Although a proof is given the claim is in fact false, as the counterexample in Figure 102 shows. This pattern can be also constructed for larger squares, showing there is no bound on the subregion that contains the three squares.

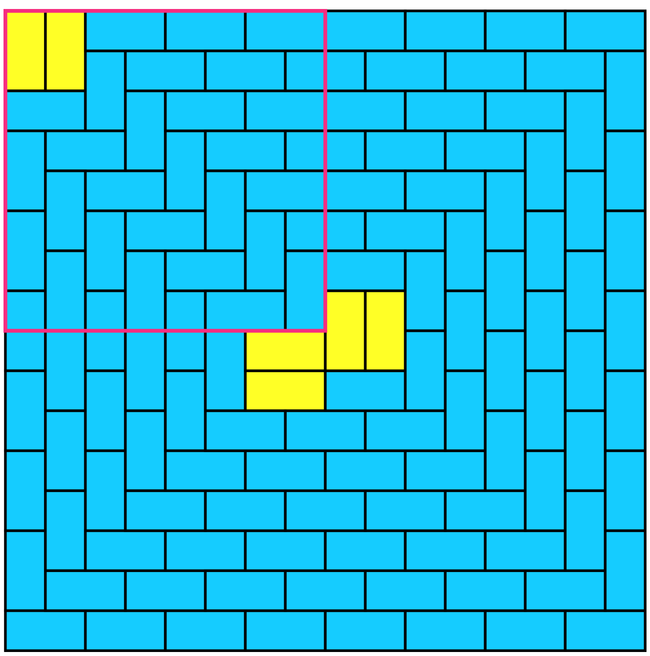


Figure 102: A counterexample to Kranakis's Theorem.

The author also conjectures that in tilings with exactly four squares, either all the tilings lie within a bounded square, or the tiling can be partitioned into four squares with has a flippable pair in the center of each. This is also false, as shown in Figure 103.

It *does* seem plausible that in a tiling with more than one flippable pair, there is not a square larger than $R(n/2, n/2)$ that contains only one flippable pair.

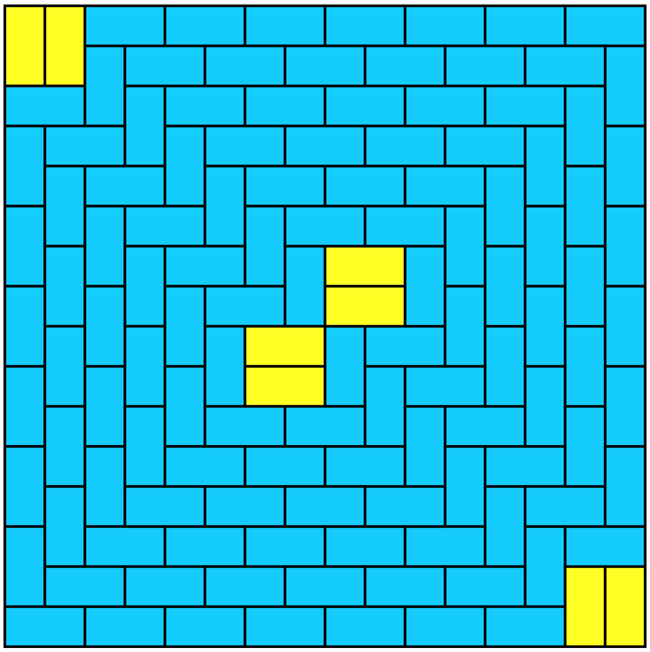


Figure 103: A counter example to Kranakis's Conjecture.

Theorem 106 (Kranakis (1996), Theorem 10)**.** *A domino tiling of a rectangle $R(m,n)$, with $1 < m \leq n$ must have at least $\lfloor \frac{n}{m+1} \rfloor$ non-overlapping flippable pairs.*

[Referenced on page 189]

Proof. The argument is similar to that used for Theorem 103, using Theorems 100 and 101. □

Figure 104 shows several tilings that achieves the minimum.

3.2 *Counting Tilings*

We have already seen a few results about the number of tilings of certain regions:

- Snakes have a unique tiling.
- Rings have two tilings.
- Any region with a 2×2 square as subregion has at least two tilings.

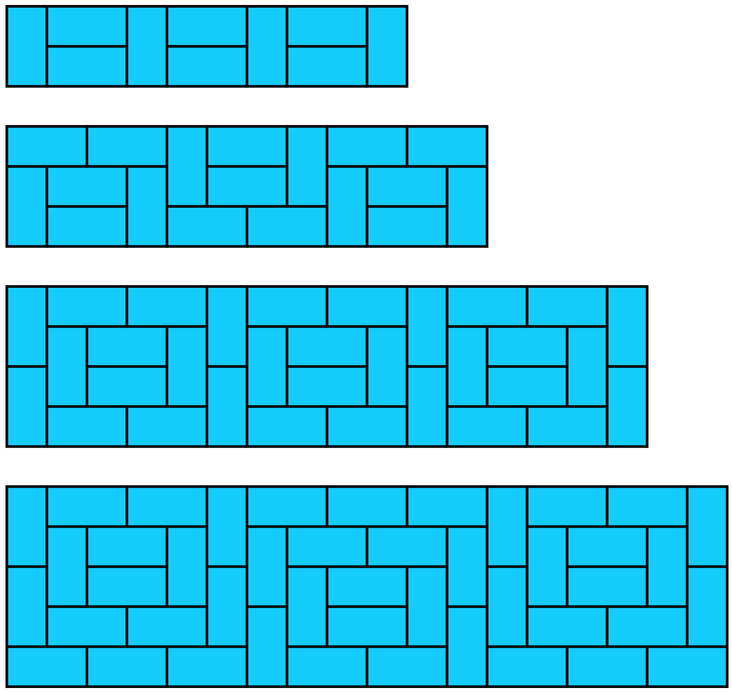


Figure 104: Tilings that achieves the minimum number of flips.

In this section we look at the number of tilings of rectangles and Aztec diamonds. We will denote the number of tilings of a region R by a tileset $\mathcal{T}$ as $\#_{\mathcal{T}} R$, or if $\mathcal{T}$ is understood, simply $\#R$.

3.2.1 Rectangles

Theorem 107. *Let F_n be the Fibbonacci numbers $1, 1, 2, 3, 5, 8, \cdots$ (**A000045**).*

$$\#R(2, n) = F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right). \tag{3.6}$$

[Referenced on page 108]

Proof. There are two ways in which to start a tiling: with a vertical domino, or two horizontal dominoes.

The first case leaves us with a $R(2, n-1)$ and the second case leaves us with a $R(2, n-2)$. From this we find the recursion: $\#R(2, n) = \#R(2, n-1) + \#R(2, n-2)$. And by checking that $\#R(2, 1) = 1$ and $\#R(2, 2) = 2$, we are lead to the familiar Fibonacci sequence. That is, $\#R(2, n) = F_{n+1}$. See Knuth et al. (1989, Section 6.6) for how to derive the expression for Fibonacci numbers. □

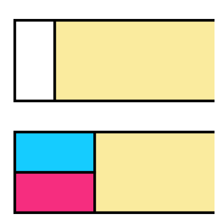
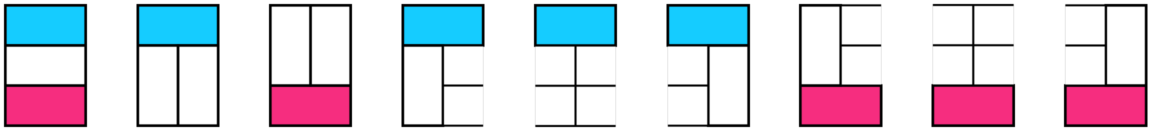


Figure 105: The two ways a domino tiling of $R(2, n)$ can start.

With a bit more work, we can also find a recursion for $R(3, n)$.

First, n must be even. That means we can divide the rectangle into $R(2, 3)$ blocks, and the only possibilities of how they are covered with dominoes are the ones shown in Figure 375.



(a) Universal (b) Blue Tower (c) Red Tower (d) Blue Start (e) Blue Middle (f) Blue End (g) Red Start (h) Red Middle (i) Red End

Figure 106: The only blocks that can make up $R(3, n)$. The terminology is from Butler et al. (2010)

Theorem 108.

$$\#R(3,n) = \frac{3+\sqrt{3}}{3}\left(2+\sqrt{3}\right)^{n/2} + \frac{3-\sqrt{3}}{3}\left(2-\sqrt{3}\right)^{n/2}$$

[Referenced on page 108]

Proof. Let S_n be the number of tilings possible for an $3 \times n$ rectangle starting with either the blue or red start piece. We can only start a rectangle with the universal piece, one of the towers, or one of the start pieces. This gives us:

$$\#R(3,n) = 3\#R(n-2) + 2S_n.$$

For a rectangle starting with a start piece, the only next piece is either the corresponding middle piece, or the corresponding end piece. This gives us:

$$S_n = \#R(3, n-4) + \#R(3, n-6) + \ldots + \#R(3, 2) + 1$$

Popping this into the equation above gives us:

$$\begin{aligned}
\#R(3,n) =& 3\#R(3,n-2) + 2(\#R(3,n-4) + \#R(3,n-6) + \ldots + \#R(3,2) + 1) \\
\#R(3,n) - \#R(3,n-2) =& 3\#R(3,n-2) + 2(\#R(3,n-4) + \#R(3,n-6) + \ldots + \#R(3,2) + 1) \\
& - 3\#R(3,n-4) - 2(\#R(3,n-6) + \#R(3,n-8) + \ldots + \#R(3,2) + 1) \\
=& 3\#R(3,n-2) - \#R(3,n-4)
\end{aligned}$$

Which gives us:

$$\#R(3,n) = 4\#R(3,n-4) - \#R(3,n-4) \tag{3.7}$$

Solving[8] this recurrence gives us:

[8] See Further Reading for references on how to do this.

$$\#R(3,n) = \frac{3+\sqrt{3}}{6}\left(2+\sqrt{3}\right)^{n/2} + \frac{3-\sqrt{3}}{6}\left(2-\sqrt{3}\right)^{n/2}$$

□

We get the sequence for even n: 3, 11, 41, 153, 571, ... This is **A001835** (skipping the first two terms).

The number of tilings of $R(3, n)$ *without* the universal tile is given by $2 \cdot 3^{n/2-1}$ for even n (Butler et al., 2010).

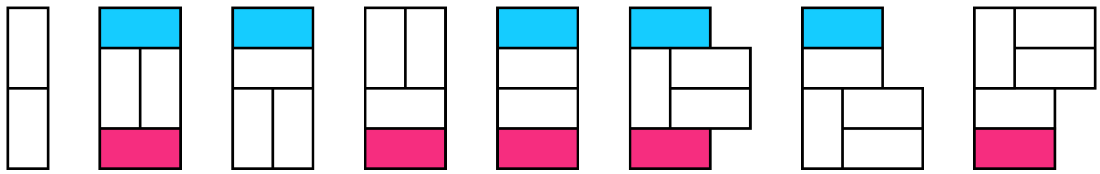


Figure 107: All the ways $R(4, n)$ can begin.

There are eight ways in which an $R(4, n)$ tiling can begin, shown and named in the picture below. We'll count the number with each kind of beginning configuration and add the results up to get $\#R(4, n)$. This gives us

$$\#R(4, n) = \#R(4, n-1) + 5\#R(4, n-2) + \#R(4, n-3) - \#R(4, n-4), \tag{3.8}$$

with $\#R(4, 0) = 1$, $\#R(4, 1) = 1$, $\#R(4, 2) = 5$, and $\#R(4, 3) = 11$.

How this occurrence is derived is shown by Steve Kass (un.) (2014). An explicit formula is given in the entry for this sequence: **A005178**.

From 5 onwards it becomes very tedious to do this type of recurrence derivation by hand. Figure 108 lists the 25 ways a $5 \times n$ rectangle can begin.

Theorem 109 (Kasteleyn (1961))**.** *The number of tilings of a $m \times n$ rectangle is given by*

$$\#R(m, n) = \prod_{k=1}^{m} \prod_{l=1}^{n} 2\sqrt{\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{l\pi}{n+1}} \tag{3.9}$$

[Not referenced]

The proof uses advanced ideas and is lengthy, so I omit it. For a readable exposition of this theorem, see Stucky (2015). Table 11 tabulates some values. Showing that this theorem is equivalent to Theorems 107 and 108 for $m = 2$ and $m = 3$ is a tricky exercise.

Problem† 41. *How many tilings does $B(m^2 \cdot 2^n)$ have? (See Figure 109.)*

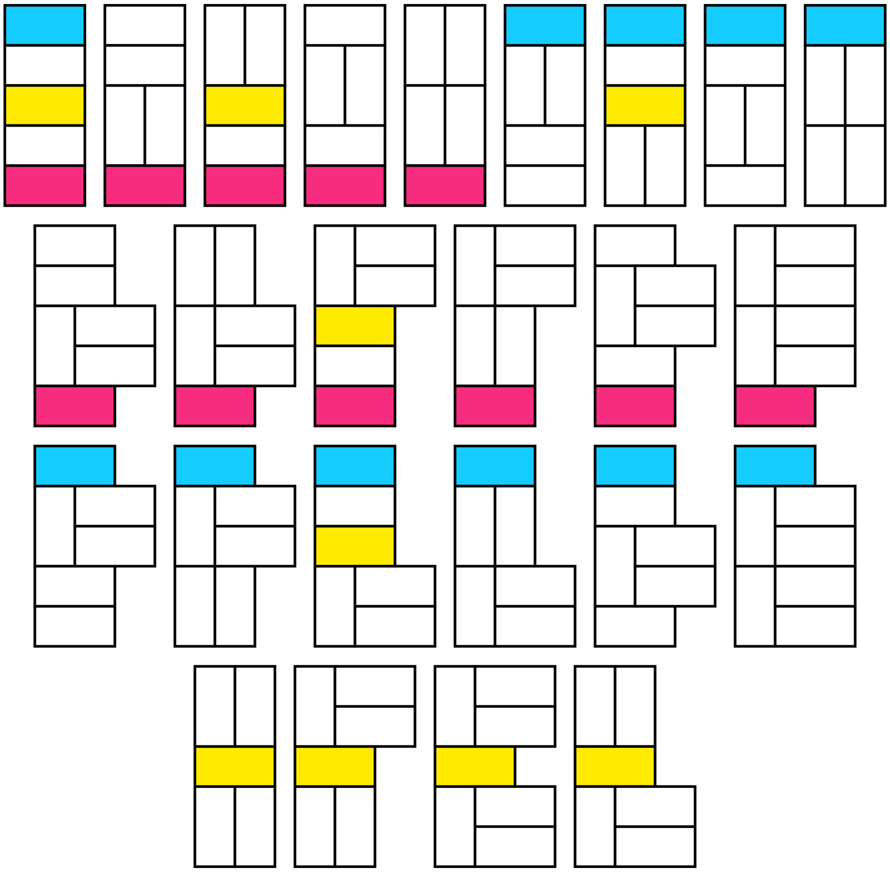


Figure 108: All the ways a $n \times 5$-rectangle can begin.

A099390	1	2	3	4	5	6	7	8	9	10	11
1	0	1	0	1	0	1	0	1	0	1	0
2	1	2	3	5	8	13	21	34	55	89	144
3	0	3	0	11	0	41	0	153	0	571	0
4	1	5	11	36	95	281	781	2245	6336	18061	51205
5	0	8	0	95	0	1183	0	14824	0	185921	0
6	1	13	41	281	1183	6728	31529	167089	817991	4213133	21001799
7	0	21	0	781	0	31529	0	1292697 0	53175517	0	
8	1	34	153	2245	14824	167089	1292697	12988816	108435745	1031151241	
9	0	55	0	6336							
10	1	89	571								
11	0	144									

Table 11: Number of domino tilings of a $m \times n$ rectangle.

3.2.2 *Aztec Diamond*

The number of tilings of $A(n)$ has been determined, but a proof of this is difficult, and I won't give it here. Instead, I will prove some other interesting properties.

Theorem 110. *Let R be an Aztec diamond, and S a subregion of R also an Aztec diamond with radius one less, sharing some border with R. Then in any tiling of R, either 0 or 2 dominoes cross the border of S.*

[Not referenced]

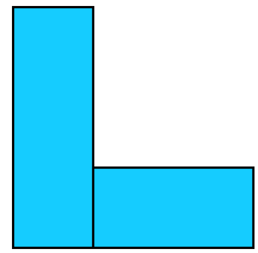


Figure 109: An example of a region with an L-shape composed from two rectangles of width 2.

Proof. Since S and $R - S$ is tileable, it is possible that a tiling of R has no dominoes that cross the border of S. Suppose one cross the border, and that the cell in S covered by that domino is white. Another domino must cross with the cell in S covered by it black (to keep the balance). So if one domino cross, then at least two must cross.

Any dominoes that crosses the border with the same color cells in S, must lie on the same side of S, and therefore two with no other border-crossing dominoes between them demarcates a snake with endpoints of the same color, which then has odd area (Theorem 54) and is untileable (Theorem 1). □

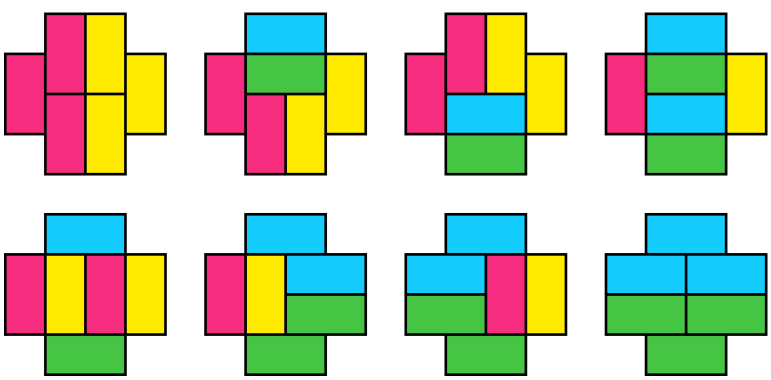


Figure 110: The 8 tilings of AD(2)

Figure 111 shows an example. Note also that most dominoes in the subregion is aligned with similar orientation. This partially explains the Arctic behavior we discuss in the next subsection.

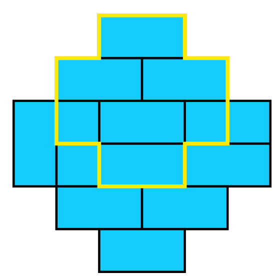


Figure 111: At most two dominoes can cross the border of the subregion.

Let $A(n,i,j)$ be an Aztec diamond with two cells on the border removed in columns i and j as in the theorem above. We can then write:

$$\#A(n+1) = \#A(n) + 2\sum_{i=1}^{n}\sum_{j=n+1}^{2n} \#A(n,i,j). \tag{3.10}$$

Theorem 111. *The number of domino tilings of an Aztec diamond $A(n)$ is given by*

$$\#A(n) = 2^{\binom{2}{n}} = 2^{\frac{n(n+1)}{2}}$$

[Not referenced]

The theorem was first proved in Elkies et al. (1992a) and Elkies et al. (1992b) (they give four different proofs). Simpler proofs are given in Eu and Fu (2005) and Fendler and Grieser (2016), with more details provided in Pardo Simón (2016). Table 12 tabulates some values.

n	$\lvert A(n)\rvert$ A046092	$\#A(n)$ A006125
0	0	1
1	4	2
2	12	8
3	24	64
4	40	1024
5	60	32768
6	84	2097152
7	112	268435456
8	144	68719476736
9	180	35184372088832
10	220	36028797018963968
11	264	73786976294838206464
12	312	302231454903657293676544
13	364	2475880078570760549798248448
14	420	40564819207303340847894502572032

Table 12: Number of tilings of the Aztec diamond.

3.2.3 *Arctic Behavior*

In Jockusch et al. (1998), the authors proved that randomly generated tilings of large Aztec diamonds had some peculiar patterns in the corners: the dominoes tend to be oriented in specific ways. This phenomenon does not occur for rectangles, where the orientation of dominoes are arbitrary.

Figures 112 and 113 show examples.[9]

Why is this happening? The formal statement and proof of this theorem (called the *Arctic Circle Theorem*) is beyond the scope of this book. But we can get of an intuitive understanding of what is going on. We have already seen that in certain subregions next to the border, dominoes tend up to line a certain way (Sections 3.1.2 and 2.2.2). Another way to look at this phenomenon is to look at the flow. Divide the region into four partitions as shown. In each, we calculate

[9] The software used to generate this software is available online; see Doeraene (2017).

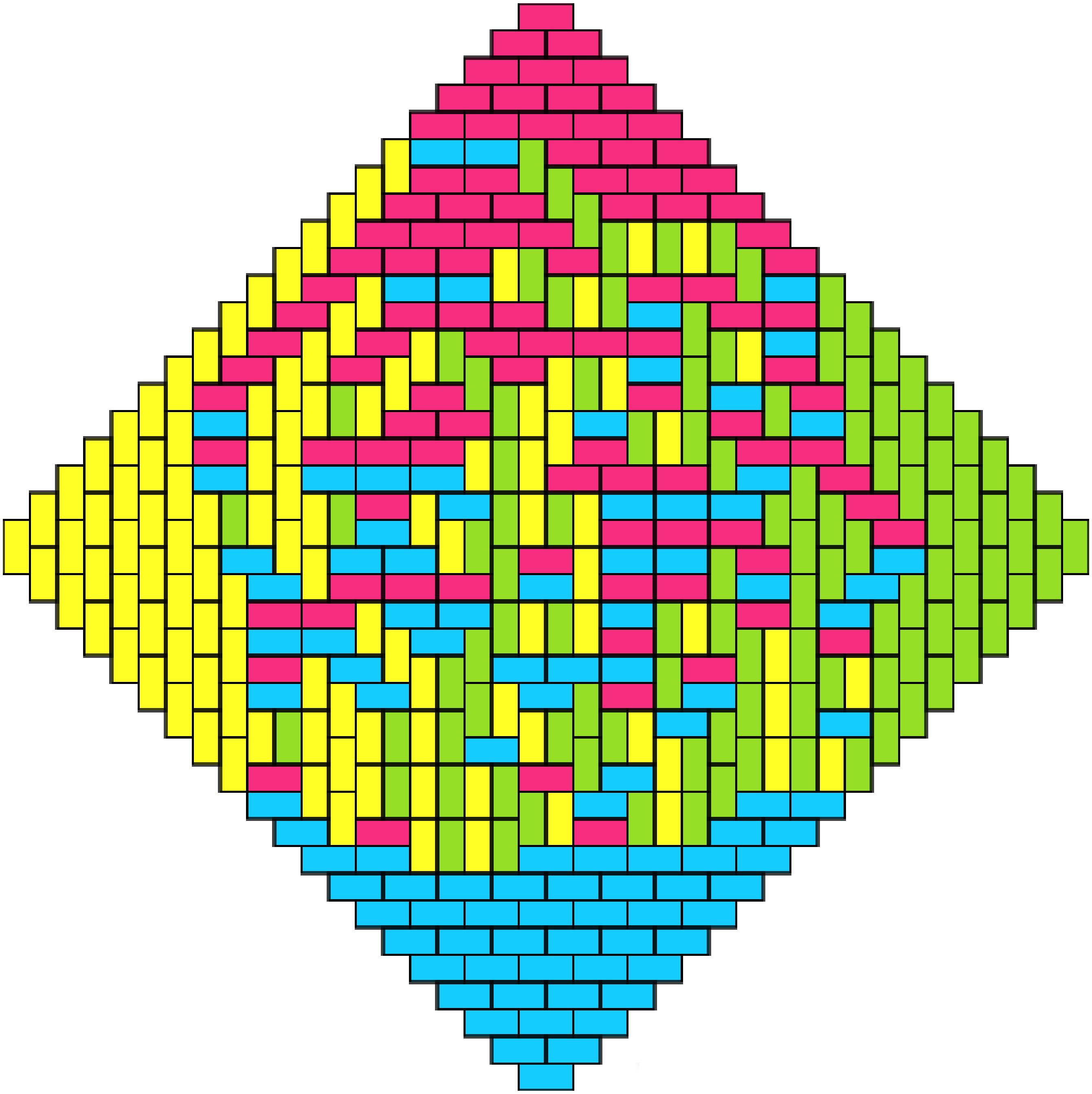


Figure 112: A tiling of an Aztec diamond.

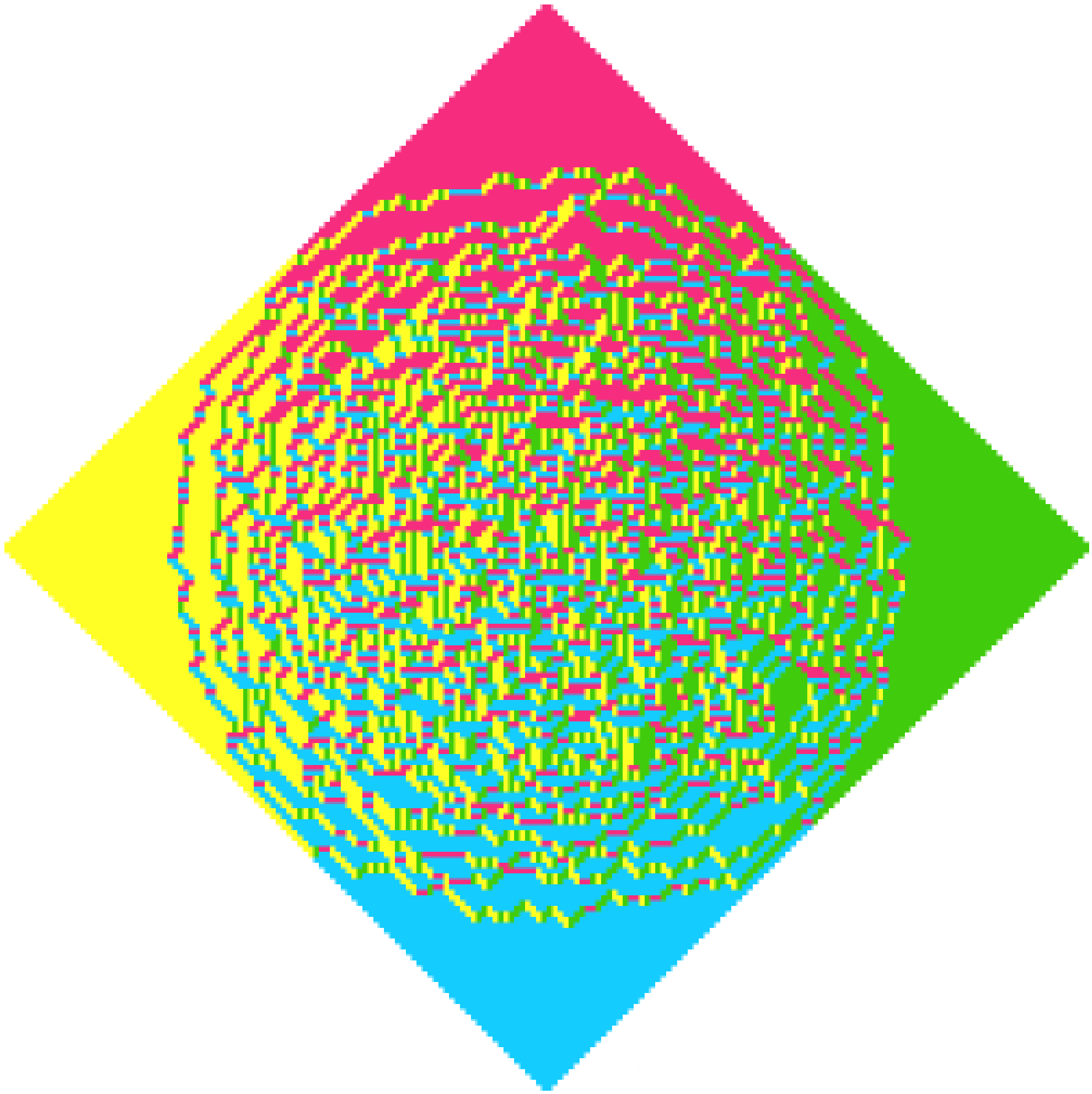


Figure 113: A tiling of an Aztec diamond.

the deficiency as $\frac{2(n+1)}{4}$. That means, we need as many dominoes to cross over the axes. The axis has $2\left\lfloor \frac{n}{2} \right\rfloor + 1$ crossing points. A quick calculation shows in this setup

$$c(n) = \begin{cases} 2d(n)+1 & \text{if } n \text{ is even} \\ 2d(n)-1 & \text{if } n \text{ is odd.} \end{cases}$$

This means about at least half the potential crossing points need to be actually crossed. Now when two such dominoes lie next to each other, the also determine the orientation of two dominoes between them. If there are three next to each other, then they determine the orientation of 6; in general the more crossing dominoes lie next to each other, the more other dominoes' orientation are forced. Because there is a large amount of dominoes that need to cross relative to the number of crossing points, on average there will be plenty of groups of adjacent dominoes oriented the same way, and therefore, on average, dominoes in this region tend to line up.

With this type of reasoning, we can see that this type of behavior will be typical of regions with reasonably sized stretches of border of the same color.

3.3 *Constraints and Monominoes*

3.3.1 *Optimal Tilings*

If a region is not tileable, how close can we get to tiling it? Asked differently: Given a set of dominoes and monominoes, what is the least number of monominoes required to tile the region?

Let $\mathcal{T}$ be a tileset without any monominoes, and let $\mathcal{T}^+$ be the tile set with a monomino added. Given some region R to tile with $\mathcal{T}^+$, the least number of monominoes we require is called the **gap number** of the region, with respect to the tileset $\mathcal{T}$ (Hochberg, 2015), and we will denote it by $G_{\mathcal{T}}(R)$, or simply $G(R)$ if the tile set is clear. (In this section, $\mathcal{T}$ is always the set containing the domino.) If $G(R) = 0$, then R is tileable.

A tiling of R that uses exactly $G(R)$ monominoes is called **optimal** (following Bodini and Lumbroso (2009)).

Theorem 112. *For dominoes, the gap number must at least be the absolute deficiency:*

$$G(R) \geq |\Delta(R)|.$$

[Referenced on pages 116 and 127]

Proof. Let M be the set of cells that are covered by monominoes in a optimal tiling of R. Then $R - M$ is tileable by dominoes, so

$\Delta(R - M) = 0$ (Theorem 27), and $|\Delta(M)| \leq |M| = G(R)$ (See Problem 12.3). But $|\Delta(R)| \leq |\Delta(R - M)| + |\Delta(M)|$ (Problem 12.4), so $|\Delta(R)| \leq G(M)$. □

Theorem 113. *For a tileset $\mathcal{T}$ where $|P| = n$ for all $P \in \mathcal{T}$, we have*

$$G_{\mathcal{T}}(R) \equiv |R| \pmod{n}. \tag{3.11}$$

[Referenced on page 115]

Proof. If $G_{\mathcal{T}}(R) = g$, then there is a tiling with tiles from $\mathcal{T}$ and g monominoes. Remove the cells covered by monominoes to form R'. Now since R' is tileable by $\mathcal{T}$, we know that $|R'| \equiv 0 \pmod{n}$ (Theorem 1). Thus $g + |R'| \equiv g \pmod{n}$, or $|R| \equiv g \pmod{n}$. □

Note that this theorem is stated for any tileset, not just for dominoes. For dominoes, it follows that $G(R) \equiv |R| \pmod{2}$.

Theorem 114. *If R has a tiling with dominoes and a single domino, then $G(R) = 1$, and so the tiling is optimal.*

[Referenced on page 115]

Proof. $G(R) \leq 1$ (since we have a tiling realized with one monomino), and it cannot be 0 since $|R|$ must be odd (Theorem 113), and so $G(R) = 1$. □

Theorem 115. *Strip polyominoes with odd area has gap number 1.*

[Not referenced]

Proof. Put the monomino on the first cell of the strip polyomino. The remainder of the figure is a strip polyomino with an even number of cells, and therefore tileable by dominoes (Theorem 55). We have a tiling with single monomino, and so the gap number must be 1 (Theorem 114). □

Theorem 116. *Monominoes in optimal domino tilings cannot be neighbors.*

[Referenced on page 119]

Proof. Any two neighboring monominoes can be replaced with a domino, and so the original tiling cannot be optimal. □

Theorem 117. *All optimal tilings of a region have the same number of black, and the same number of white monominoes.*

[Not referenced]

Proof. Suppose we have two tilings, T with W white monominoes and B black monominoes, and T' with W' and B' white and black dominoes respectively, and let S be the subregion covered by dominoes in T, and S' be the subregion covered by dominoes in T'.

Now we have $\Delta(R) = \Delta(S) + B - W = \Delta(S') + B' - W'$. But S and S' are tileable by dominoes, so $\Delta(S) = \Delta(S') = 0$ (Theorem 27). So $B - W = B' - W'$. But also, since the tilings are minimal, we have $B + W = B' + W'$, which gives is $B = B'$ and $W = W'$. $\square$

Class	$G(R)$
Strip polyomino with odd area (includes snake, rectangle)	1
Column convex polyomino with k odd columns (includes cylinder)	$\leq k$
Bar graph	See Bodini and Lumbroso (2009)
Young diagram (includes L-shaped polyomino, rectangle)	$\|\Delta(R)\|$

Table 13: Gap numbers for some classes of polyominoes.

Theorem 118. *The gap number of a Young diagram R is $|\Delta(R)|$.*

[Not referenced]

Proof. Suppose we cannot remove a cylinder from the region. Then, the Young diagram must have top cells in every column the same color (Theorem 44). If we put a monomino on each of the k top cells of odd columns, the remainder can be tiled with dominos. Clearly $|\Delta(R)| = k$, so the gap number must be at least k (Theorem 112) and since we can find a tiling with k monominoes and dominoes, the gap number is exactly k.

Suppose we *can* remove a cylinder. We can continue this until we cannot. We now have the shape described above. Find a tiling for it, and note that the absolute value of the deficiency is equal to the number of monominoes. It remains to show the original figure can be tiled with the required monominoes.

Now re-insert cylinders that were remove (using the process of Theorem 41) to find a tiling of the original region k monominoes. This means $G(R) \leq k$. But $G(R) \geq |\Delta(R)| = k$, therefore $G(R) = k = |\Delta(R)|$. $\square$

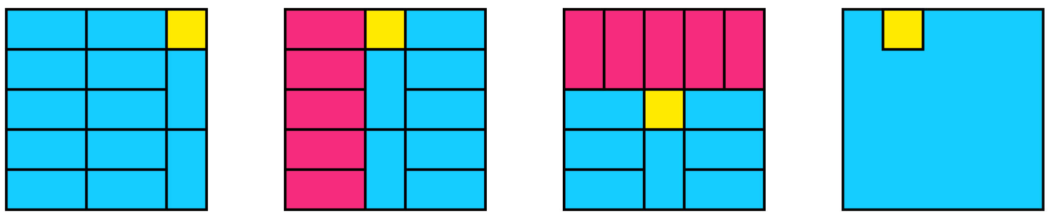
Once we have determined the gap number, we can also ask: where are legal positions for these monominoes to occur?

Example 13. *In a rectangle with odd area, the deficiency is 1, and say WLOG that $B > W$. Then at the very least the monomino must be placed on a black square. It is easy to see that a monomino can occur on the corner of such a rectangle. Using this, we can show it is possible for this monomino to occur on* any *black cell.*

First, suppose we want to place a monomino somewhere on border of the rectangle. We can now divide the rectangle into two rectangles such that one is even, and the other is odd with the monomino in the corner. Note that we can only do this when the monomino is placed on a black square, as is required. Both these are tileable, and so is the whole.

Suppose then the monomino is somewhere in the interior of the rectangle. Again, we can divide the rectangle into two rectangles, one which is even, and one which is odd with the monomino on the border. And again we can only do this because the monomino has to be placed on a black square. Both these are tileable, and therefore so is the whole.

Table 14 tabulates the number of tilings for some rectangles with odd area by dominoes and a single monomino.



(a) A rectangle with odd area and a monomino placed in the corner is tileable.

(b) A rectangle with odd area and a monomino placed on a black cell the edge is tileable.

(c) A rectangle with odd area and a monomino placed on a black cell in the interior is tileable.

(d) A rectangle with a monomino on a white cell is not tileable, since the remainder of the region is unbalanced.

It is not always the case that monominoes can occur on any cell of the correct color. In Figure 114 for example, there is no optimal tiling of the region with the monomino on the red cell. In this case removing the red cell partitions the remaining region into two regions with an odd area each, so neither is tileable by dominoes (Theorem 1). The same applies if the cell would partition the region into two regions and one is unbalanced, as shown in Figure 115.

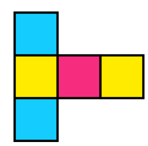


Figure 114: There is no optimal tiling of this region with the monomino covering the red cell.

Problem† 42. *Show that a rectangle tiled with dominoes and X-pentominoes need to have an even number of pentominoes.*

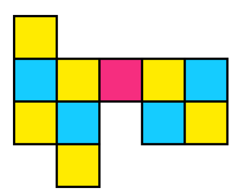


Figure 115: The gap number of this region is 1, but there is no optimal tiling with the monomino covering the red cell.

	1	3	5	7	9
1	1	2	3	4	5
3	2	18	106	540	2554
5	3	106	2180	37924	608143
7	4	540	37924	2200776	116821828
9	5	2554	608143	116821828	

Table 14: Number of tilings of odd-area rectangles with one monomino and dominoes.

The following Theorem generalizes Theorem 65.

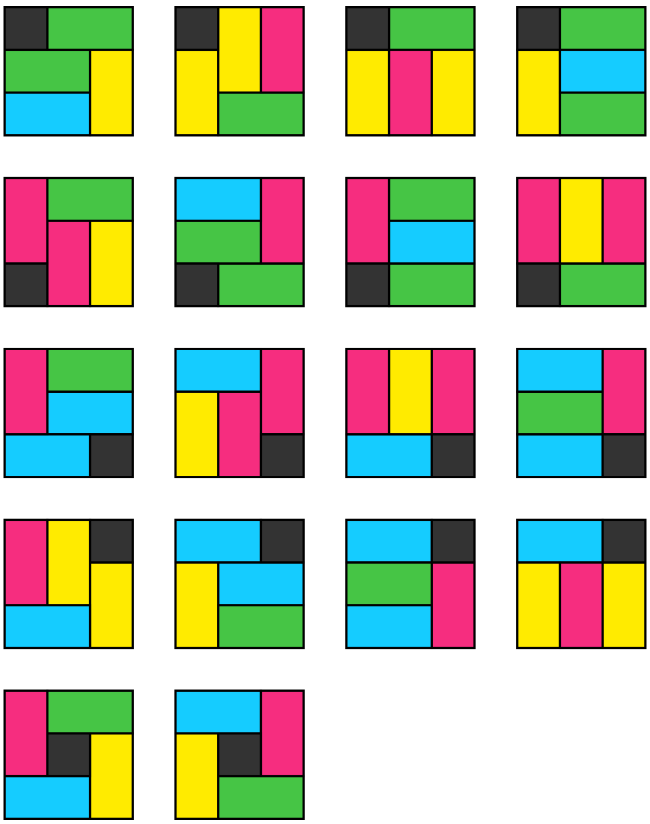


Figure 116: The optimal tilings of a 3×3 square.

Theorem 119. *Suppose R has two tilings T and U by dominoes and monominos. Then for each monomino in T that is not in U, there is a strip S that is tiled by a set of dominoes and a monomino in each tiling, and the monomino is at one of the strips ends in each tiling.*

[Referenced on page 119]

Proof. Let C_1 be a cell tiled by the monomino in T. That cell is covered by a domino in U, and has neighbor in that domino C_2. C_2 is covered by a domino in T, it's neighbor is C_3. We can continue this process until we reach C_k, which covers a monomino in U.

Note the following:

- k must be odd (this can be seen from the process itself, or from the fact that the two monominoes must have the same color, and hence any strip of which they are the endpoints must have an odd number of cells).
- $C_i \neq C_j$ if $i \neq j$. For suppose $C_i = C_j$ and $i < j$. It follows that $C_{i-1} = C_{j-1}$, and so on until we have $C_1 = C_{j-i+1}$. But unless $i = j$, it means that C_{j-i+1} is not covered by a domino in T (it is covered by a monomino), and therefore cannot have been part of the sequence in the first place, because all cells in the strip $C_{i\neq 1}$ are covered by dominoes in T.

□

Suppose that T is a tiling of R, and that $C_1, \cdots C_k \in R$ is a strip, and that C_1 is covered by a domino in T, and C_{2i}, C_{2i+1} is covered by a single domino in T. We can now form a new tiling T' by covering each C_{2i-1}, C_{2i} by a domino, and C_k by a monomino. This transformation is called a **strip shift**. See Figure 117 for an example.

The following Theorem generalizes Theorem 67.

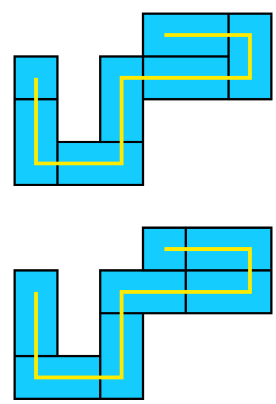


Figure 117: An example of a strip shift.

Theorem 120. *Any tiling of a region R by dominoes and k monominoes can be transformed into any other tiling with dominoes and k monominoes by performing strip rotations and strip shifts.*

[Referenced on page 119]

Proof. Let T and U be two tilings of a R. For each monomino in T not in U, there is a strip S_i such that if we transform a strip shift in T, we get the strip's tiling in U (Theorem 119). Let M_i be the cells covered by monominoes in both T and U. Then $R - \bigcup_i S_i - \bigcup_i M_i$ is a region with subtilings in T and U, that is covered by only dominoes in both tilings. These tilings are connected by strip rotations (Theorem 67).

□

Here are some consequences of Theorems 119 and 120:

(1) In an optimal tiling, no two monominoes can lie in the same tiled strip. This is a generalization of Theorem 116.

(2) A region R cannot be partitioned into strips so that fewer than $G(R)$ of them have odd length.

3.3.2 *Tatami Tilings*

[10] A **tatami tiling** is a tiling such that no four corners of tiles meet (Knuth (2009, Problem 7.1.4.215); *see also* Ruskey and Woodcock (2009) and Hickerson (2002)). Notice that this definition does not reference any tileset. There are two types of tatami tilings that are usually considered: tatami tilings with dominoes, and tatami tilings with dominoes and monominoes.

[10] This is a placeholder section which I hope to expand in a future version.

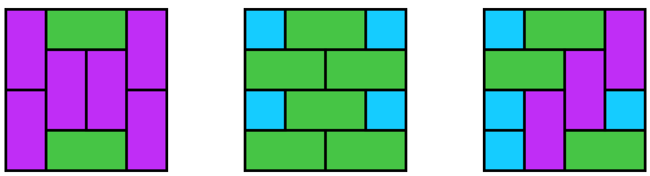
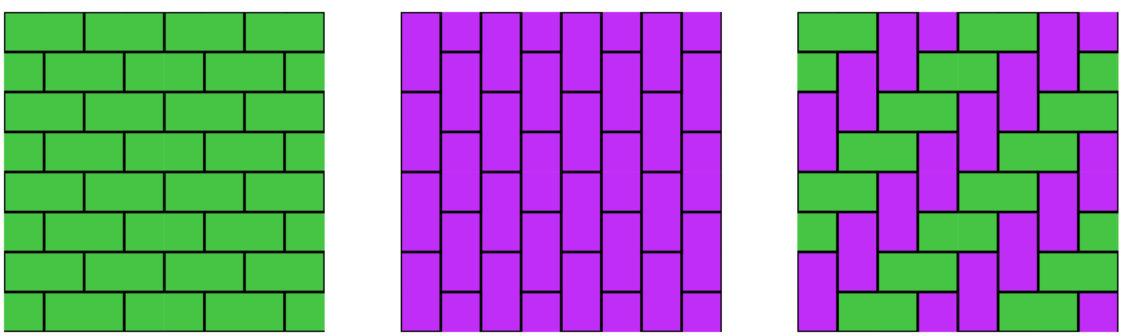


Figure 118: Tatami tilings of the 4×4 square with dominoes and monominoes.

The plane has many tatami tilings with dominoes; several are shown in Figure 119.



(a) Horizontal Bond Pattern (b) Vertical Bond Pattern (c) Herringbone Pattern

Figure 119: Tatami tilings of the plane.

3.3.3 *Fault-free Tilings*

A **fault** in a tiling is a line on the grid (horizontal and vertical) that goes through the region and is not crossed by any tile (Golomb, 1996, p. 18).[11] See Figure 120 for an example. A tiling with no faults are called **fault-free**. We are mostly interested in fault-free tilings of *rectangles*. See Figures 121 and 122 for examples.

[11] Also called *line of cleavage*, Reid (e.g. 2005). The problem of finding fault-free tilings was first proposed by Robert I. Jewett according to Golomb.

Theorem 121 (Golomb (1996), p.18)**.** *Domino tilings of* $2 \times n$ *rectangles cannot be fault-free.*

[Referenced on page 123]

Proof. There are only two configurations in which a rectangle can start (Figure 105). Both of these have faults. □

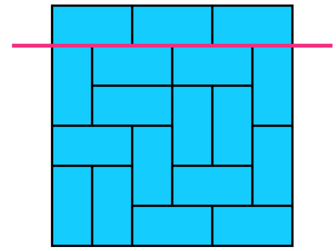


Figure 120: A tiling with a horizontal fault.

Theorem 122 (Golomb (1996), p.18 (no proof))**.** *Domino tilings of* $3 \times n$ *rectangles cannot be fault-free.*

[Referenced on pages 101 and 123]

Proof. A rectangle can only start with a universal block, a tower, or one of the start blocks (see Figure 375). The universal block and towers all have faults. A start block can only be followed by an end block, or a mid-block; and a mid-block can only be followed by another mid-block or an end block. So, since the rectangle is finite, if it starts with a start block, it must eventually have an end block. If that is the final block, the rectangle has a horizontal fault. Otherwise, it has a vertical fault. □

Theorem 123 (Golomb (1996), p.18)**.** *Domino tilings of* $4 \times n$ *rectangles cannot be fault-free.*

[Referenced on page 123]

Proof. (Martin, 1991, p. 18) All the ways a $4 \times n$ rectangle can begin is shown in Figure 107. The first 5 can also end the rectangle, or they can be extended. In the former case, each of those 5 has a fault. In the later case, there is a vertical fault at the end of the block.

The third last can be extended only by adding horizontal dominoes to the top and last row to the right. This shape can either be completed with a vertical domino, in which case there are two horizontal faults, or with two horizontal dominoes, which gives us the same situation as we started with (so eventually, a fault must be introduced).

The final two shapes can be extended by either a vertical domino, or two horizontal dominoes. In the former case, we have a horizontal fault, or if the rectangle continues, a vertical fault. The the latter case, we have the same situation we started with, so eventually a fault must be introduced. □

Theorem 124 (Golomb (1996), p.18)**.** *The* 5×6 *rectangle has a fault-free tiling by dominoes.*

[Referenced on page 123]

Proof. See Figure 121. ☐

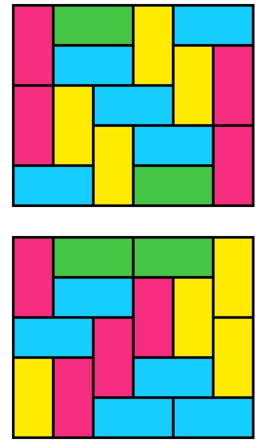


Figure 121: The two fault-free tilings of a 6×5 rectangle.

Theorem 125 (Golomb (1996), p.18)**.** *The* 6×6 *has no fault-free tiling by dominoes.*

[Referenced on pages 123 and 184]

Proof. Consider any potential fault line. This line must be covered by a domino. If no other domino covers it, then the line divides the region into two regions of odd area each (the area is odd since the cells covered by the fault-covering domino are not included). Then there must be at least one other domino on the line (Theorem 25). Since there are 10 lines, we need 20 dominoes. But the area is only 36, which means it will be covered by 18 dominoes. It is therefore impossible to cover all faults and still tile the figure. Thus, the figure has no fault-free tiling. ☐

Theorem 126 (Golomb (1996), p.19)**.** *A* 6×8 *rectangle has a fault-free tiling.*

[Referenced on page 123]

Proof. See Figure 122. ☐

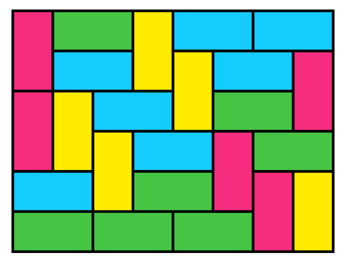


Figure 122: A fault-free tiling of a 6×8 rectangle.

A tiling **extension** is a a systematic way to find a tiling from a smaller tiling. Extensions are used to show tilings are possible for a sequence of of regions; they are implicitly induction proofs. Rectangular extensions are the most common, and we will see many types in Section 5.4. But there are also more complicated extensions that we will see in Section 6.1.

Theorem 127 (Golomb (1996), p.19)**.** *If we have a fault-free tiling of a* $R(m, n)$*, we can find fault-free tilings of* $R(m+2, n)$ *and* $R(m, n+2)$*.*

[Referenced on pages 51 and 123]

Proof. The same argument works for both types of extensions, so let's prove it for horizontal extensions. Pick a vertical domino on the border. (Such a domino must exist, for if it does not, then there would be a fault.) Remove it from the tiling. Now add a vertical 2-cylinder of dominoes along the crooked edge. Add the removed domino in the opening. The new rectangle is 2 cells wider, and is still fault-free. To see this, note that all original fault lines are still covered. We introduced two new potential faults. The inner most is covered by the dominoes that fit into the opening of the crooked edge; the other is covered by the remaining dominoes. ☐

Theorem 128 (Golomb (1996), p.18). *All even-area rectangles $R(m, n)$ (except $R(6, 6)$) has fault-free tilings by dominoes if $m, n \geq 5$.*

[Referenced on page 240]

Proof. By Theorem 127, we can use suitable extensions to find fault-free rectangles for $R(5+2p, 6+2q)$ and $R(6+2p, 8+2q)$, since both 5×6 and 6×8 rectangles as has fault-free tilings (Theorems 124 and 126). Clearly a $1 \times m$ rectangle must have faults. Theorems 121, 122, and 123 shows we cannot have fault-free tilings when $1 < m \leq 4$ or $1 < n \leq 4$. And finally, Theorem 125 shows there are no fault-free tilings of the 6×6 rectangle. □

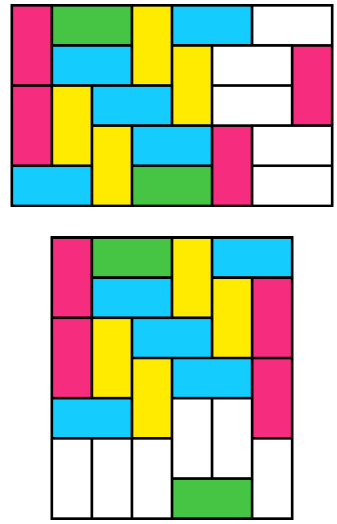


Figure 123: Extending a fault-free tiling of a 6×5 rectangle horizontally and vertically.

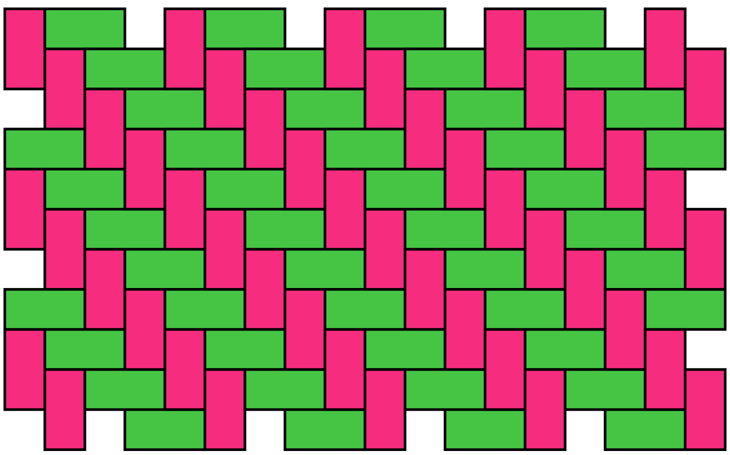


Figure 124: Fault-free tiling of the plane.

Problem† 43.

(1) *What rectangles have fault-free tatami tilings with*

  (a) *dominoes only,*

  (b) *dominoes and monominoes?*

(2) *Prove any tiling of an Aztec diamonds has at least three faults.*

3.3.4 Even Polyominoes

An **even polyomino**[12] is a polyomino where all corners (of inside and outside borders) neighbor black cells if we color the polyomino and its surroundings with the checkerboard coloring (Kenyon, 2000b). Note that the holes of an even polyomino are even polyominoes themselves.

[12] The terminology is unfortunate, as the term even polyomino is used for polyominoes that can only tile rectangles with an even number of tiles (p. 197). Since the notion defined here is not used outside this section else, I kept the term.

Theorem 129. *Let R be a even polyomino with H holes. Then*

$$\Delta(R) = 1 - H. \tag{3.12}$$

[Referenced on page 127]

Proof. For each edge E_i, notice that:

(1) if E_i is a knob, it lies between two black cells, and hence $b(E_i) - w(E_i) = 1$,

(2) if E_i is a anti-knob, it lies between two white cells, and hence $b(E_i) - w(E_i) = -1$, and

(3) if E_i is a flat, it lies between a black and white cell, and hence $b(E_i) - w(E_i) = 0$.

Therefore, we have for the entire polyomino, $b(R) - w(R) = P - V$. By Theorem 31 we have $\Delta(R) = \frac{b(R)-w(R)}{4}$, so $\Delta(R) = \frac{P-V}{4}$. And by Theorem 13 we have $\frac{P-V}{4} = 1 - H$, so $\Delta(R) = 1 - H$. □

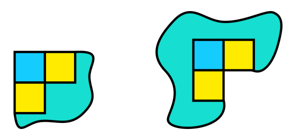


Figure 125: Valid corners in even polyominoes.

It follows immediately that even polyominoes without holes have an odd area. Also note that in even polyominoes, knobs and anti-knobs have odd length, and flats have even length.

Problem† 44.

(1) *Show that if we insert a 2-cylinder into an even polyomino, the result is an even polyomino.*

(2) *If we remove a 2-cylinder from an even polyomino, is the result an even polyomino?*

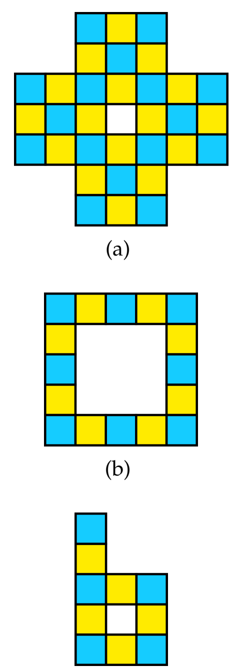


Figure 126: Examples of even polyominoes.

Theorem 130. *Even polyominoes with one hole are tileable.*

[Referenced on pages 126, 127 and 128]

Proof. For this proof to work, we will manipulate the border (either the outside or inside border) to give a new polyomino with a hole, and we will consider regions where borders overlap (but not cross) also valid polyominoes, and they are even if the corners are black as with the regular definition. Here are some examples of such polyominoes (the outside border has been slightly displaced where it coincides with the inside border).

(1) In an even polyomino, all peaks must be black (otherwise there are white corners). Also, all black peaks must be attached to a white cell with exactly two neighbors. Therefore, in an even polyomino, if we move the border to exclude the

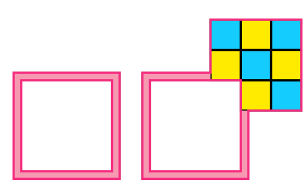


Figure 127: Border polyominoes.

peak and its neighboring cell, we are still left with an even polyomino.

An example of this manipulation is shown in Figure 128.

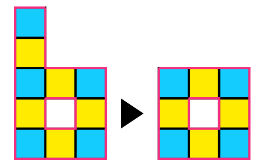


Figure 128: Peak removal

(2) A 2×2 square can be of two types. Type I has a black square in the top right, a Type II square has a black square in the top left. If a Type I square occurs in a corner of the polyomino, we can move the border to exclude it, and keep the polyomino even.

An example of this manipulation:

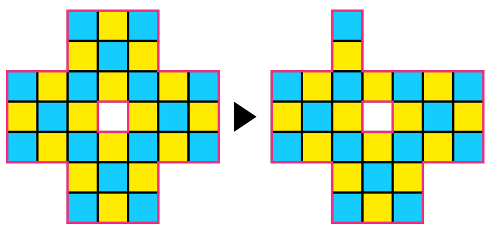
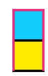
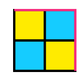


Figure 129: Corner removal

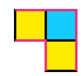
(3) All convex corners of a even polyomino are one of the four types shown in Figure 130.



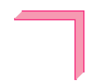
(a) Type 1



(b) Type 2



(c) Type 3



(d) Type 4

Figure 130: Corner Types

(4) If we perform the manipulations in 1 and 2 until we cannot, we are only left with Type 1 and Type 2 corners.

(5) A region with only type 3 and 4 convex corners and one hole cannot contain a 2×2 subregion.

Suppose there is such a square. Find the largest connected set of cells that contains this square such that each cell is part of a 2×2 subregion. This set of cells must have at least four convex corners (Theorem 8). For these corners to *not* be corners of the entire region, each needs to be "covered" by a "strand" (the picture below shows a possibility.) These strands must either result in peaks, or they must connect in pairs. If they connect in pairs, there are two holes, and this is impossible (Figure 131). Therefore, there cannot be any 2×2 subregion.

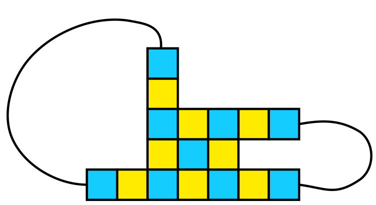


Figure 131: An example where strands that cover potential corners meet up in pairs, leading to two holes.

In this image, there is a 2×2 square, contained in a 3×3 square where each corner is covered by a white cell that stars the strand. In this example, the strands connect in pairs, leading to two holes.

(6) We cannot have any X- or T-junctions in a region with no 2×2 subregions, no peaks, and one hole.

Suppose we have an X-junction. Each of the four connectors must eventually lead to a peak, or pairs of them must join. In the latter case, we have two holes, which is impossible. Therefore there are no X junctions.

Suppose we have a T-junction. Each of the three connectors must either lead to a peak, or they must join other connectors. We can only join up if we have at least another T junction, in which case we will have at least have two holes, which is impossible. Therefore there are no T-junctions.

(7) This means we either have a ring, or a polyomino with no area (thus the inner and outer border coincide completely).

(8) This procedure shows we can partition any even polyomino with one hole into a set with dominoes, 2×2 squares, and either a ring (tileable by Theorem 55) or empty polyomino. All the elements of this partition are tileable, and therefore, so is the whole region (Theorem 2).

□

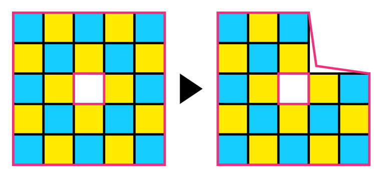


Figure 132: An example of a polyomino for which our proof would fail if we did not use the double border definition of a polyomino.

The double border definition is necessary to deal with polyominoes such shown in Figure 3.3.4. If we did not have that, removing a 2×2 corner would not leave an even polyomino. (It would be nice to find a proof that does not require this trick.)

One consequence of using this formulation is that the hole need not be an actual hole using the normal definition of a polyomino, and we exploit this in Theorem 132.

An algorithm follows immediately from the border manipulations.

Theorem 131. *A rectangle with odd area with an odd hole is tileable if it is balanced.*

[Not referenced]

Proof. Let $R = R_1 - R_2$ be our region, and R_1 the filled rectangle, and R_2 the hole. The deficiency of an odd-area rectangle is 1 or -1. (You can pair rows to cancel out, leaving the last row with one more cell of one color than the other.) Suppose the corners of the rectangle is black. Then, for the region to be balanced, the corners of the hole must also be black, because we have $\Delta(R) = \Delta(R_1) - \Delta(R_2)$, or $0 = 1 - (B - W)$, So $B - W = 1$, and so we have $B > W$, which means the rectangle must have black corners. Therefore R is an even polyomino, and so is tileable by dominoes (Theorem 130). $\square$

We can "fix" the deficiency of even polyominoes in a specific way to make a class of polyominoes that are tileable by dominoes.

A **Temperleyan polyomino** is a polyomino formed from an even polyomino by removing one black cell from the outer border, and appending a black cell to each inner border Kenyon (2000b).

Theorem 132. *Temperleyan polyominoes are tileable by dominoes.*

[Referenced on pages 126 and 127]

Proof. Case $h = 0$. Suppose R is the polyomino, and H is the black cell removed from the border. Then let $R' = R \cup H$. We can now view this polyomino as a double border polyomino, with outer border the border of R', and the inner border the border of H. Since R' is even and H is an even hole, R must be tileable by Theorem 130.

General case. See Kenyon (2000b, Section 7) for a sketch of a proof. $\square$

Example 14. *Suppose R is a region partitioned into two simply-connected subregions S_1 and S_2, with S_1 a black even polyomino and S_2 a white even polyomino. Then R is tileable. Let $v_1 \in S_1$ and $v_2 \in S_2$ be neighbors with v_1 black (and v_2 white), and let $S_1' = S_1 - \{v_1\}$ and $S_2' = S_2 - \{v_2\}$. Then S_1' and S_2' are both Temperleyan polyominoes, and tileable. And $\{v_1, v_2\}$ is tileable by a single domino since v_1 and v_2 are neighbors. The three regions form a partition of R, and are all tileable, and therefore R is tileable (Theorem 2).*

Theorem 133. *The gap number of an even polyomino with H holes is $|H - 1|$.*

[Not referenced]

Proof. Case $H = 0$. If we remove a black corner cell, the resulting polyomino is Temperleyan and thus tileable by Theorem 132. Therefore, the gap number is 1.

Case $H = 1$. The region is tileable (Theorem 130), and so the gap number is 0.

Case $H > 1$. Form a Temperleyan polyomino R' by appending black cells v_i to all but one interior border; there are $H - 1$ such cells. This polyomino is tileable (Theorem 132). Let u_i be the neighbor of v_i in the same domino of a tiling of R', and let $R'' = R - \{u_i\}$. Then R'' is tileable, and R therefore has a tiling by dominoes and $H - 1$ monominoes. Since $g \geq |\Delta(R)|$ (Theorem 112), and $\Delta(R) = 1 - H$ (Theorem 129), the tiling is optimal, and therefore $g = |1 - H|$. $\square$

3.4 *Miscellaneous*

3.4.1 *Polyominoes that Can be Tiled by Dominoes*

How many regions of a given area is tileable by dominoes? Table 15 gives us a sense. A polyomino tileable by dominoes is called a **polydomino**.

n	2n-ominoes A210996	Balanced A234012	Tileable A056785	Uniquely tileable A213377
1	1	1	1	1
2	5	4	4	3
3	35	24	23	20
4	369	230	211	170
5	4655	2601	2227	1728
6	63600	32810	25824	18878
7	901971	433855	310242	214278
8	13079255	5923677	3818983	2488176
9	192622052		47752136	29356463
α	14.7*	13.6*	12.5*	11.8*

Table 15: Balanced, tileable and uniquely tileable dominoes.

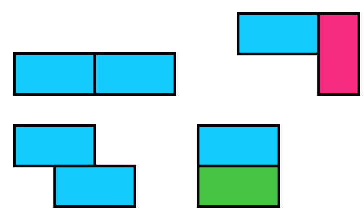


Figure 133: The four tileable tetrominoes.

3.5 *Summary of Tiling Criteria*

All tiling criteria we covered in this and the last chapter is summarized in Table 16. The table includes criteria from other chapters.

3.6 *Further Reading*

Berlov and Kokhas (2004) give some interesting problems (with solutions) on domino tilings.

For more on orders and lattices, see for example Davey and Priestley (2002) and Roman (2008). The following properties of lattices are well-known and it is worth considering their implication on domino tilings:

(1) M_3 is not isomorphic to any sublattice (Davey and Priestley, 2002, Thereom 4.10, p. 89).

(2) N_5 is not isomorphic to any sublattice (Davey and Priestley, 2002, Thereom 4.10, p. 89).

(3) The covering relation $\prec$ forms a median graph (Roman, 2008, Ex. 9, p. 125).

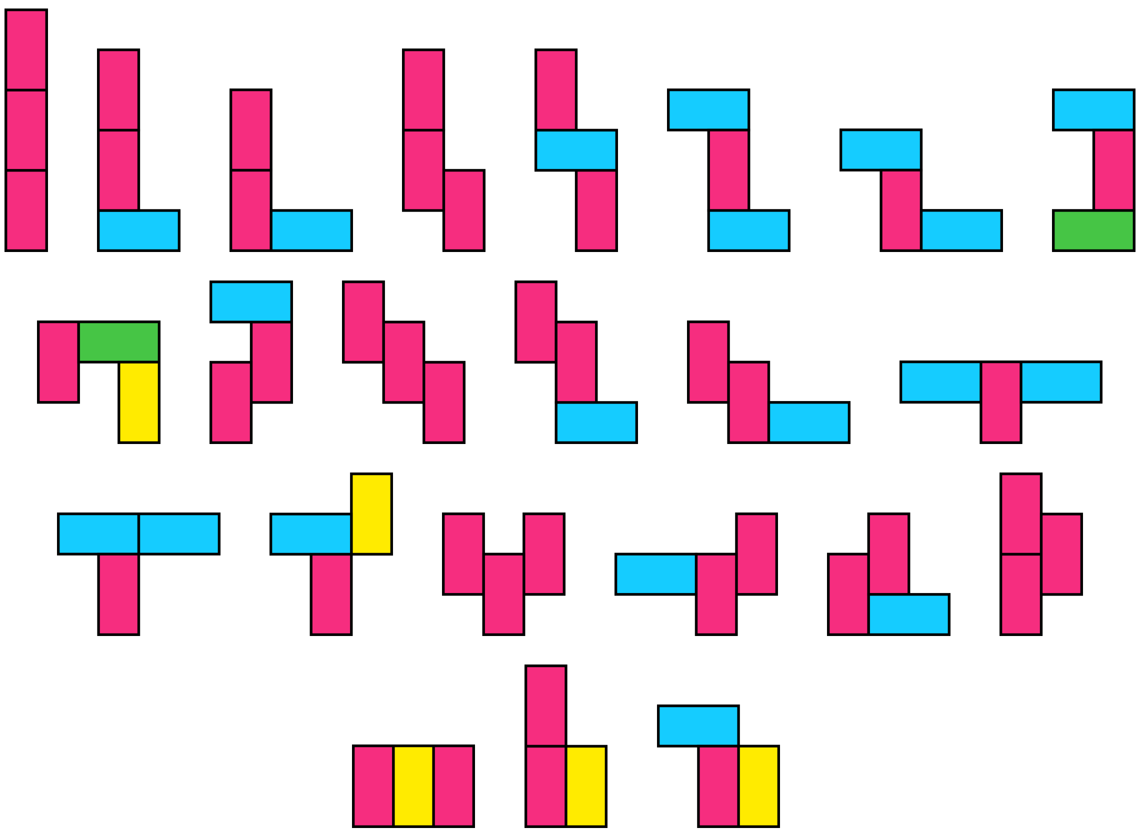


Figure 134: The 23 tileable hexominoes. All except the last three have unique tilings.

For more on the lattice structure of domino tilings specifically, see Rémila (2004), Caspard et al. (2003). For slightly more general results on height functions and dominoes tilings see Ito (1996), Beauquier and Fournier (2002) and Bodini and Latapy (2003).

Domino tilings (especially the use of height functions and the structure of tilings) have also been considered for regions more general than was covered here. For example:

- plane regions with barriers (Propp and Stanley, 1999)
- three-dimensional regions (Milet, 2015)
- regions on a torus (Impellizieri, 2016)
- quadriculated surfaces (Saldanha et al. (1995) and Saldanha and Tomei (2003))

Height functions and lattices can also be used to study other tiling problems[13], for example, tilings with:

[13] I use here the convenient list provided by Kenyon (1999).

- bars (Kenyon and Kenyon, 1992)
- rectangles (Kenyon and Kenyon, 1992)
- squares (Kenyon, 1993)
- tetrominoes (Muchnik and Pak, 1999)
- bibones (Kenyon and Rémila, 1996)
- tribones (Thurston, 1990)
- lozenges (Rémila (1996) and Thurston (1990))
- leaning dominoes (Rémila, 1996)
- triangles (Rémila, 1996)

In our treatment of height functions, we did not deal with regions with holes. The best explanation for dealing with holes is Bodini and Fernique (2006), but it is covered in the slightly more general setting of *planar dimer tilings*. Desreux et al. (2004) gives a similar technique specific for domino tilings, including algorithms for finding minimum tilings, to generate all tilings, and to do uniform sampling of tilings. Saldanha et al. (1995) gives an alternative approach, but requires some ideas from topology. Thiant (2003) gives an algorithm for tiling any region with dominoes.

In many of the proofs I omitted the details of solving recurrence relations. How to solve these is explained in many books on combinatorics or discrete mathematics; see for example (Brualdi, 2009, Chapter 7), (Knuth et al., 1989, Chapters 1 and 7) and (Wilf, 2005, Chapter 1).

Propp and Lowell (2015) discusses enumeration of tilings in general, and is a good overview of some of the techniques used. Mathematicians calculated the number of domino tilings of various classes, for example:

- quartered Aztec diamond (Lai, 2014)

- deficient Aztec rectangle (Krattenthaler, 2000)
- holy square (Tauraso, 2004)
- L-shaped domain (Colomo et al., 2018)
- expanded Aztec diamond Oh (2018)

Counting more general types of matchings is discussed in Propp et al. (1999). Special attention has been given to matchings whose counts are perfect powers (such as the Aztec diamond). See for example Pachter (1997), Ciucu (2003) and Lai (2013). One notable result is a factorization theorem for graphs with reflective symmetry given in Ciucu (1997).

Numerical results and generating functions for various tilings of rectangles by small rectangles (including dominoes) is given in Mathar (2013) and Mathar (2014). This includes counts of tatami tilings. In addition to generating functions, Gershon (2015) gives exact formulas for rectangles of fixed width up to 6.

For more details on tatami tilings with dominoes only, see Alhazov et al. (2010) and Ruskey and Woodcock (2009). For more on tatami tilings with dominoes and monominoes, see Erickson (2013), Erickson et al. (2011) and Erickson and Schurch (2012). The idea of *water striders* in Erickson et al. (2011) is closely related to Theorems 100, 101, and 102.

We consider optimal tilings of rectangles by rectangles and the associated gap number in Section 5.1.2, and for rectangles by other polyominoes in Section 5.5. Fault-free tilings of other polyominoes is covered in Section 5.4. Temperleyan polyominoes are closely related to spanning trees of certain graphs. This is discussed in Kenyon et al. (2001), with applications to the dimer model in Kenyon (2000b), Kenyon (2001) and Kenyon (2000a).

Conditions for tileability	Theorem
Necessary	
$\lvert R \rvert$ must be even.	1
R is balanced.	27
The parity of dominoes that cross the border of any subregion S of R must equal the parity of the deficiency of S.	25
The deficiency of any subregion must match the flow (and therefore, the number border of S must be able to accommodate the necessary flow.	29
R must be have least 2 sides of even length.	33
$R \ominus S$ is tileable if S is an n-cylinder with n even, and removing S from R is safe.	42
R is fair by some discriminating coloring.	51
R satisfies the diagonal rectangle criterion for dominoes	188
Sufficient	
All sides of R is even.	34
$R \ominus S$ is tileable, with S a n-cylinder with n even	41
R is a strip-polyomino with even area, which includes rectangles with even area.	55
R is saturnian	59
R is a checkerboard-colored strip polyomino with two cells of opposite colors removed	60
R is a balanced stack polyomino (which includes balanced Young diagrams).	50 (45)
R is a balanced jig-saw region.	49
R is an even polyomino with one hole.	130
Necessary and Sufficient	
R has no bad patches	38
R is fair by all discriminating colorings	53
We can consistently assign heights in each step of Thurston's Algorithm.	96
R is simply connected, balanced, and the difference in height is smaller than the distance between any two vertices on the border.	99

Table 16: Summary of tiling criteria for domino tilings.

4
Tiling Basics

In this chapter we develop some tools that we will find useful in the following chapters.

In the first section, we look at the relationships between polyominoes and sets that can tile certain fundamental regions. This gives us a way to organize the information about what regions a polyomino or set can tile.

In the second section, we look at colorings. For a long time coloring arguments (similar to the checkerboard logic we used for domino tilings) was the most important way to analyze tilings of polyominoes.

In the third section we briefly introduce border words; an algebraic way to describe the border of (simply-connected) polyominoes that is the first step towards the more powerful tools.

4.1 *The Tiling Hierarchy*

Theorem 3 says, paraphrased, if X tiles Y and Y tiles Z, then X tiles Z. This is the basic idea behind the tiling hierarchy. We take a bunch of region types (such as rectangles or the plane), and then we define a polyomino to belong to one or another class depending on which types of regions it can tile.

We then arrange these classes in a hierarchy so that if a polyomino belongs to a class, it also belongs to classes below in the hierarchy. If a polyomino tiles a rectangle, it also tiles a plane; and in general, the reverse need not hold. The tiling hierarchy is defined by the tiling implications of Theorem 137.

This setup simplifies the process of finding the basic tiling capabilities of a polyomino. If we proved a polyomino tiles a rectangle, we know it tiles the plane. And if instead we proved a polyomino does not tile the plane, we need not check whether it tiles a rectangle.

The basic classes of regions are rectangles, half-strips, bent strips, quadrants, reptiles, and the plane. As we will see (Section 4.1.3),

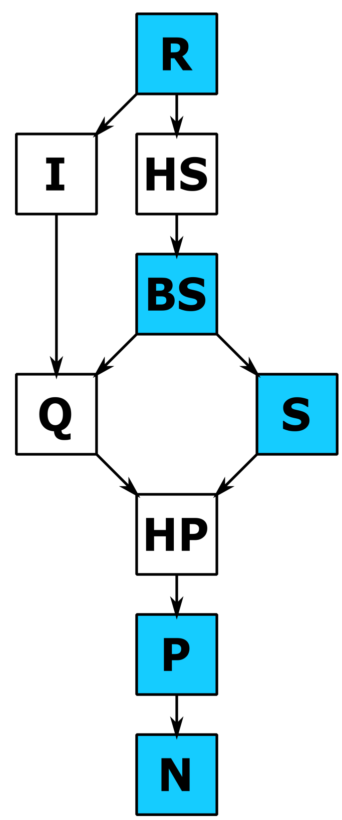


Figure 135: The relationship between classes. Classes in blue blocks have elements that are not in any higher class; white ones have no examples of polyominoes that are not also in a higher class.

there are other ways to form a hierarchy. But this basic one (as set out by Golomb (1966)) turns out to serve us well. Table 17 defines the classes and what they tile.

There are two interesting features about Golomb's hierarchy.

First, not all tiling relationships in the hierarchy derive from Theorem 3. As you will see in the proof of the main theorem, there are two unexpected implications; polyominoes that can tile scaled copies of themselves can also tile a quadrant; and polyominoes that can tile bent strips can also tiles strips. These implications follow from other important tiling principles which we have not encountered before, both have to do with tilings of infinite regions; we explore this in this section and in Chapter 5.

Second, it is not clear whether all the classes are in fact distinct. For example, all polyominoes that can tile scaled copies of themselves that we know of also tile rectangles. It is also not clear that the classes are the same for individual polyominoes and sets of polyominoes. These are difficult problems, and little progress has been made to bring us closer to understand them.

The **plane** is the set $\mathbb{Z}^2$.

A **half-plane** is a set congruent to

$$\{(x,y) \mid 0 \le x\}.$$

A **quadrant** is a set congruent to

$$\{(x,y) \mid 0 \le x, 0 \le y\}.$$

A **strip** is a set congruent to

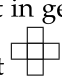
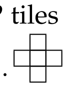
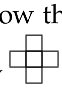
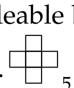
$$\{(x,y) \mid 0 \le x < w\}.$$

A **half-strip** is a set congruent to

$$\{(x,y) \mid 0 \le x < w, 0 \le y\}.$$

A polyomino P scaled by a factor k is written kP, and is the set defined as

$$kP = \{pk + r \mid p \in P, r \in R(k,k)\}.$$

Notice that $kR(m,n) = R(km,kn)$. While it is true that $R(m,n)$ tiles $kR(m,n)$, it is not in general true that P tiles kP. It is, for example, easy to show that  $_5$ does not tile $k \cdot$  $_5$, since any finite region tileable by  $_5$ must have four knobs of size 1, but all the knobs in $k \cdot$  $_5$ have size k.

A polyomino that *can* tile a scaled copy of itself is called a **reptile**. The smallest number of tiles used in such a construction is called

Class	Can Tile
N	Nothing
P	Plane
HP	Half plane
Q	Quadrant
S	Strip, Cylinder
Q $\cap$ **S**	Quadrant and Strip
HS	Half-strip
BS	Bent-strip
R	Rectangle
I	Scaled copy of itself

Table 17: Tiling classes. For example, if some polyomino $P \in$ **R** it means it can tile a rectangle. In all cases but the last the tiles are the same size.

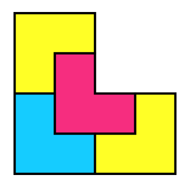


Figure 136: The right tromino is a reptile.

the **reptile order**. Reptile orders are squares. If a polyomino can tile a copy scaled by a factor of k, we say the polyomino is rep-k^2. A polyomino can be rep-k^2 for different values of k; the lowest k^2 (that is greater than 1) is the reptile order.

Monominoes tile everything, including bigger squares, therefore monominoes are reptiles. So is any rectangle. The smallest interesting reptile is the right tromino. The skew tetromino is the smallest polyomino that is not a reptile. The T-tetromino is a reptile — but in a boring way: it tiles a square, and so we can use the square to tile a bigger T-tetromino. Of course this is a general idea: if a polyomino tiles a rectangle, it can tile a square, and if it tiles a square, it can tile any scaled polyomino, including itself. Although the converse is not proven to be true, it is an open problem (Winslow, 2018, Open Problem 13); all reptiles we know also tile rectangles.

That reptiles tile the plane should be clear: take the polyomino, dissect it in smaller polyominoes of the same shape. Scale the figure so that the polyominoes in the dissection are the same size as the original. Repeat. This way, any size patch can be covered. (This type of tiling is called a **substitution tiling**. We look more at these tilings in Section 7.8.)

What is perhaps surprising is that reptiles always tile a quadrant. We prove this in the next few theorems.

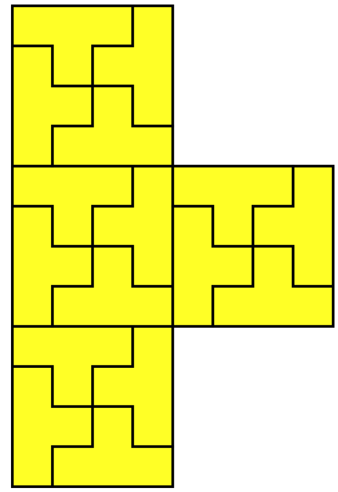


Figure 137: The T-tetromino is a reptile.

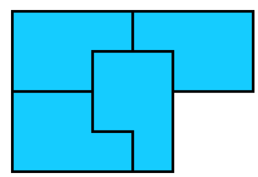


Figure 138: The P-pentomino is a reptile.

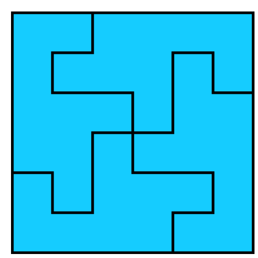


Figure 139: An example of a reptile that covers only one corner of its rectangular hull.

Theorem 134 (Golomb (1966), Theorem 4)**.** *A reptile covers at least one corner of its rectangular hull.*

[Referenced on pages 136 and 199]

Proof. Make a big version of the reptile by the replication process, (as big as you want). Clearly, if the reptile does not cover all four corners, there is no way it can cover the corner of the bigger copy; there will always be a gap. □

A reptile can fail to cover two corners; 139 shows an example.

Theorem 135. *A polyomino that can tile a rectangle, can also tile a square.*

[Referenced on page 135]

Proof. Suppose P tiles $R(m,n)$. $R(m,n)$ tiles $R(mn,mn)$ by Theorem 22, and so by Theorem 3 P tiles $R(mn,mn)$. □

Theorem 136. *A polyomino that tiles a rectangle can tile any polyomino scaled by a factor. In other words, all rectifiable polyominoes are also reptiles.*

[Referenced on pages 159, 199, 291, 296 and 297]

Proof. By Theorem 135 a polyomino P that tiles a rectangle can tile a square, say $R(k, k)$. These squares can tile the scaled polyomino kP, and so by Theorem 3 the polyomino can tile kP. □

The constructions in the proofs of the two theorems above does not necessarily construct the smallest possibility. For example, neither the tilings in Figures 136 and 138 can be tiled by smaller tileable rectangles.

The size of the smallest reptile-tiling is shown in Tables 39–43.

Theorem 137 (Golomb (1966)). *For a single polyomino:*

$$\mathbf{BS} \subset \mathbf{Q} \subset \mathbf{HP} \tag{4.1}$$

$$\mathbf{R} \subset \mathbf{I} \subset \mathbf{Q} \subset \mathbf{HP} \tag{4.2}$$

$$\mathbf{R} \subset \mathbf{HS} \subset \mathbf{BS} \subset \mathbf{S} \subset \mathbf{HP} \subset \mathbf{P} \subset \mathbf{N} \tag{4.3}$$

[Referenced on pages 133, 143, 149, 153, 198, 202, 231 and 310]

Proof. All the inclusions are obvious (see Figure 140), except for the following two[1]:

$$\mathbf{BS} \subset \mathbf{S} \tag{4.4}$$

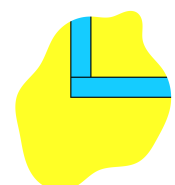
$$\mathbf{I} \subset \mathbf{Q} \tag{4.5}$$

[1] Golomb (1966, p. 287) mentions that easy but non-constructive proofs can be derived from *Kőnig's Infinity Lemma*, which can be found in for example Diestel (2000, Lemma 8.1.2), Bondy and Murty (2008, Exercise 1.21) and Harris et al. (2008, Theorem 3.2).

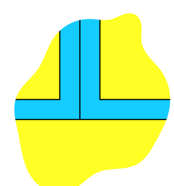
To prove 4.4, let P tile a bent strip. Let m be the maximum dimension of the polyomino's rectangular hull, and let w be the width of one arm of the bent strip. Now in any $w \times m$ rectangle of that arm, we can draw at least one path from the top of the arm to the bottom. There are only finitely many of such patterns, so in the arm with infinitely many such rectangles, at least one pattern must repeat. The segment of the arm between such repeated patterns is a cylinder, and it can used to tile a strip by placing them end to end.

To prove 4.5, let P be a reptile. It must cover a corner of its rectangular hull (by Theorem 134). Place the polyomino in the first quadrant with this corner at the origin. Now perform the replication process until the covered corner of the big polyomino is covered by a small polyomino in the same orientation. (This process can go on for at most 8 iterations, since there are only 8 distinct orientations.) Move this bigger polyomino to the origin, and call the new tiling $F(P)$. By repeatedly applying F, we can expand the tiling, which will cover the entire quadrant in the limit. □

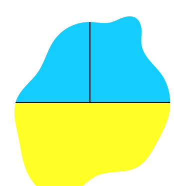
While we often use this theorem in the "forward" direction, it is worth considering reverse implications. For example, if a polyomino does not tile a quadrant, it can also not tile a rectangle. The following example shows how knowledge about how a tile tiles the quadrant informs us about how it tiles a rectangle.



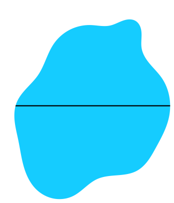
(a) Half-strips tile a bent strip.



(b) Bent-strips tile a branched strip.



(c) Quadrants tile a half plane.



(d) Half planes tile the plane.

Figure 140: Some inclusions of the tiling hierarchy demonstrated.

Example 15 (Reid (2003b), Example 6.7)**.** *The polyomino* $B(2 \cdot 3 \cdot 2^2)$ *tiles* $R(m,n)$ *iff* $6 \mid m$ *and* $6 \mid n$*.*

Proof. First, note that of all 8 possible placements along an edge, only 4 are possible. Second, by examining all cases (not hard to do) we observe that the edge must always be made of pairs as shown in Figure 141. Further examination shows the only way to tile the quadrant is to use 6×6 squares. And since no other tiling of the quadrant exists, it implies all rectangles must be tiles using these squares too, which is only possible if $6 \mid m$ and $6 \mid n$. □

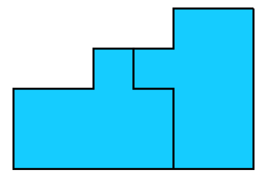


Figure 141: These polyominoes can only pack an edge in the configuration shown.

Problem† 45. *Flesh out the details in the example above.*

Problem 46** (Winslow (2018), Open Problems 12, 13; Golomb (1996), p. 107–108)**.** *No-one knows whether the following statements are true:*

(1) $\mathbf{I} = \mathbf{R}$

(2) $\mathbf{HS} = \mathbf{R}$

(3) $\mathbf{Q} = \mathbf{BS}$ *(and indeed, if* $\mathbf{I} \neq \mathbf{R}$*, then whether* $\mathbf{Q} = \mathbf{I} \cup \mathbf{BS}$*).*

(4) $\mathbf{HP} = \mathbf{S}$

(5) $\mathbf{P} = \mathbf{AP}$

Can you prove or disprove any of them?

There is no reason to believe any of the statements of Problem 46, other than that no counter examples have been found. There are suggestions that they may not be true:

- The statements (2)–(5) do not hold for tile sets that have more than one polyomino. See section 4.1.2 for examples.
- The Y-pentomino tile half-strips of width 6 and 8, bit not any rectangles with sides of length 6 or 8. This suggests $\mathbf{HS} \neq \mathbf{R}$.
- The P-pentomino and right-tromino are examples of reptiles that don't require construction of rectangles first, as shown in Figures 138 an 136.

Tables 18–19 list the smallest (most specific) classes that polyominoes belong to. All polyominoes with area 6 or less can (at least) tile the plane. Proof that polyominoes do or don't tile rectangles is given in Chapter 5. Proof that polyominoes do or don't tile the plane is given in Chapter 7. We give tilings for strips, bent strips in the next section.

Problem* 47. *Is the following true or false: If a polyomino tiles another polyomino with a scaling factor of $n > k$, where k is the largest side of the rectangular hull of the polyomino, it tiles a rectangle.*

Problem* 48. *Complete the Tables 18–19.*

4.1.1 *Strips, Bent Strips, and Half Strips*

We make the following set of definitions:

- A strip is **prime** if no smaller strip tiles it.
- A half-strip is prime if no smaller half-strip tiles it.
- A bent strip is prime if no smaller bent strip tiles it.
- A rectangle is prime if no smaller rectangle tiles it.

Similar definitions hold for tile sets.

From prime regions of a tile set, we can construct all regions of the same type that is tileable by the set. It also tells us what is possible for more restricted classes. For example, the L-pentomino does not tile any strips of width 3; therefore, it also does not tile a rectangle with width 3. The primes for rectangle tilings are interesting, and treated in detail in the chapter *Rectangles*.

Problem† 49. *Prove that the L-pentomino cannot tile a strip of width 3.*

For strips and half-strips, finding all strips or half-strips from primes is equivalent to the Frobenius problem, and for bent strips it is equivalent to the 2D Frobenius problem.

The prime strips and half-strips are given in Table 20. A rectangle $R(m, n)$ can be stacked to form a strip of widths m or n, but not all strips correspond to prime rectangles. Those that don't are marked in the table.

Problem† 50. *Determine all the bent strips that can be tiled by the*

(1) *skew-tetromino*

(2) *W-pentomino*

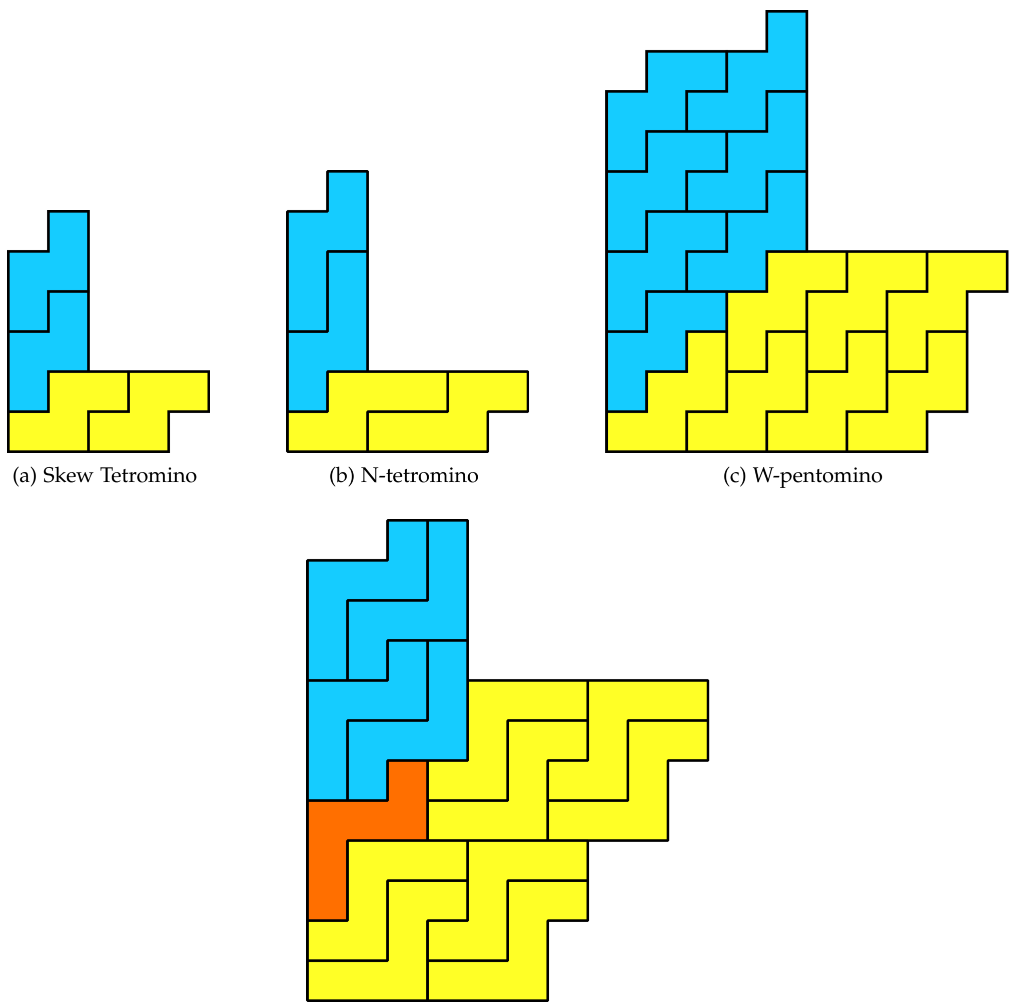
Problem* 51. *What snakes that are in C_8 tile bent strips?*

Problem* 52. *Complete Table 20.*

Reid (1997, Section 4)[2] gives three families of polyominoes that tile half-strips, but (for most of them) it is not known whether they tile rectangles. All of them are dissection of cylinders. I will call these **Reid polyominoes** of type I, II and III.

A path with 180°-rotational symmetry is called **centrosymmetric**.

[2] The description of the dissection is slightly different than given here. The starting cylinders of type II and III is given as half the size, which results in half-squares. To keep all dimensions integers, I doubled them. The shapes, however, are the same; just don't get confused when you compare with the original.



(a) Skew Tetromino

(b) N-tetromino

(c) W-pentomino

(d) S-hexomino

Figure 142: Bent strip tilings for various polyominoes.

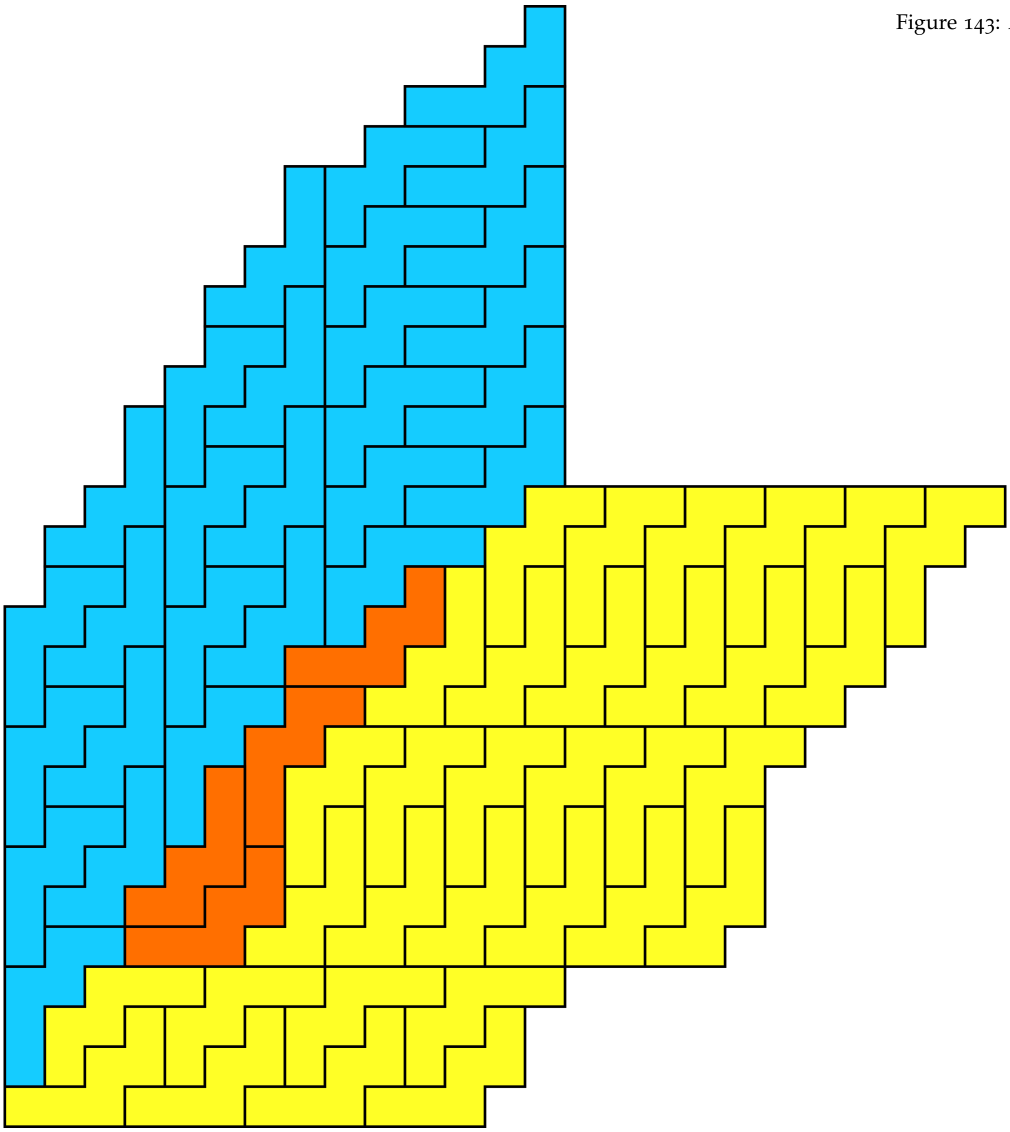
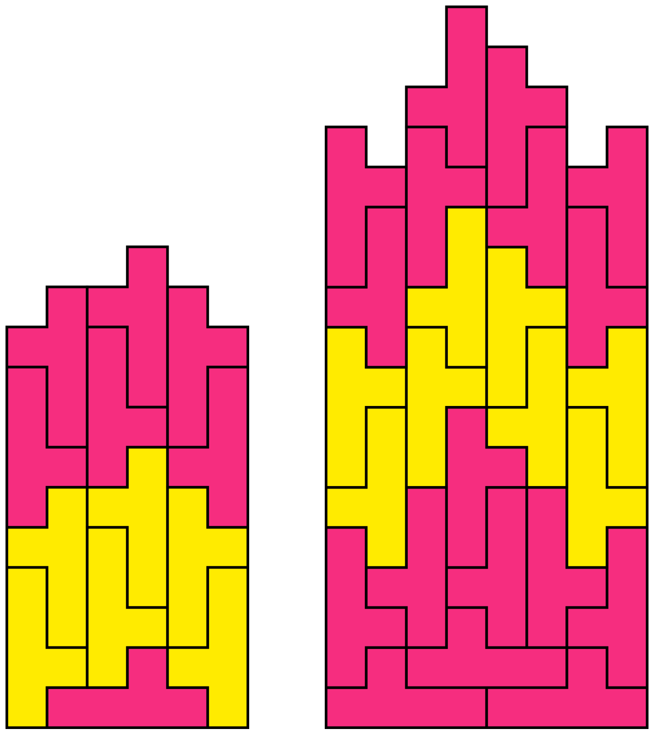


Figure 143: A bent strip.



(a) HS(6) (b) HS(8)

Figure 144: Half-strips of the Y-pentomino that does not correspond to widths of prime rectangles.

To construct a Type I Reid polyomino, take a cylinder $C_{8n-1}(0,0,1,1)$. Bisect the polyomino with a centrosymmetric path from A to B that alternates between 2 horizontal and one vertical segments, starting with the two horizontal segments. For a given n, we can get 2^{n-1} polyominoes from this construction. The resulting polyomino is a bar graph: $B(2 \cdot a_1{}^2 \cdot \cdot a_2{}^2 \cdot \cdots \cdot a_{4n-1}{}^2)$, with $a_i = 4 - a_{4n-i}$, and $a_n - 1 - a_n = \pm 1$.

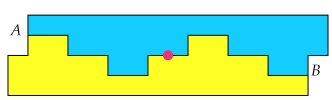


Figure 145: Construction of a Type I Reid polyomino.

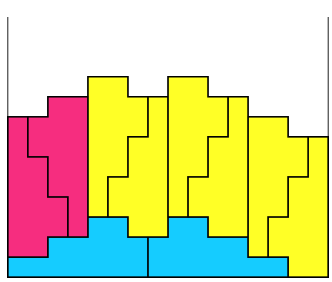


Figure 146: An example of type I Reid polyomino. In this case, $n = 2$.

To construct a Type II Reid polyomino, take a cylinder $C_{2nk-2}(0,0,2,2,4,4,\cdots,2(k-1),2(k-1))$, with $n,k > 1$. Bisect it with a centrosymmtric path from

A to B which alternates between $k-1$ steps (size 2) and vertical segments (none when $n=2$).

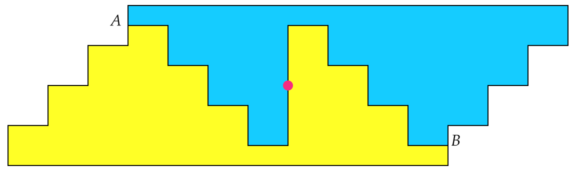


Figure 147: Construction of Type II Reid polyomino. In this case, $k = 4$ and $n = 3$.

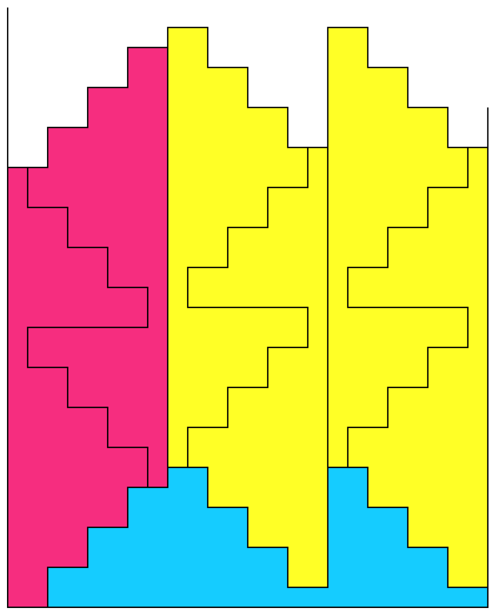


Figure 148: An example of type II Reid polyomino.

To construct a Type III Reid polyomino, start with a cylinder $C_{2nk+2}(0,0,2,2,4,4,\cdots,2(k-1),2(k-1))$, with $n, k > 1$. Dissect it with the same rules.

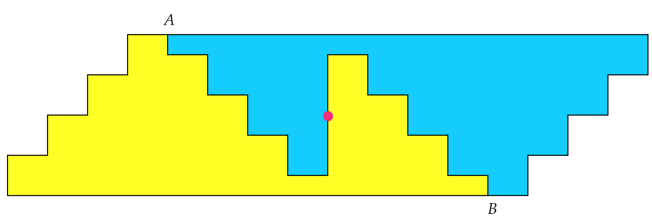


Figure 149: Construction of a type III Reid polyomino.

These families can probably be expanded. For example, we can follow the same procedure as constructing a type II Reid polyomino, but start from the cylinder $C_{3nk-2}(0,0,0,3,3,3,6,6,6,\cdots,3(k-1),3(k-1),3(k-1))$ instead. The construction is shown in Figure 151 and the strip tiling in Figure 152.

We will encounter similar constructions for families of polyominoes that tile rectangles (Section 5.2.4(4)).

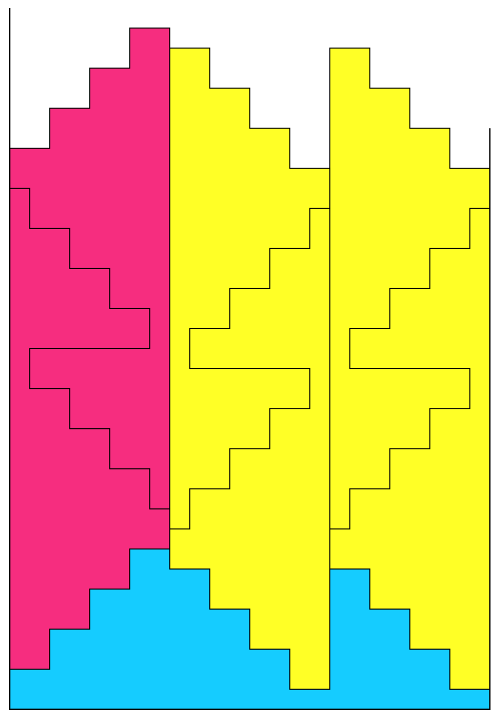


Figure 150: An example of type III Reid polyomino.

Problem* 53. *See how far you can extend the three constructions described here.*

Strip tilings are vital for studying rectangular tilings; the following theorem is at the heart of what makes investigations practical.

Theorem 138 (Reid (1997), Preposition 2.1 [3]). *Let $\mathcal{T}$ is a finite set of polyominoes, and w a fixed width. Then there is a finite, deterministic algorithm to decide:*

(1) *if $\mathcal{T}$ tiles an strip of width w,*

(2) *if $\mathcal{T}$ tiles an half-strip of width w,*

(3) *if $\mathcal{T}$ tiles any rectangle of width w,*

(4) *all lengths ℓ such that $\mathcal{T}$ tiles $R(w, \ell)$.*

[Not referenced]

The following theorems are more variations on the same theme:

Theorem 139. *If a tile set tiles a strip, there exists a periodic tiling by the same set.*

[Not referenced]

[3] Reid (1997) traces the idea, specifically of part (3) to David Klarner via Gardner (1989, p.184), Golomb (1966, Theorem 3)(which we stated as part of Theorem 137), and Bitner (1974). I have not been able to consult this last reference.

Proof. Let's suppose we have a tiling of a strip by a set of polyominoes.

Now make new tiles by taking each column as a tile. Tiles are the same if they cut though the original polyominoes in exactly the same way. This new tile set is finite. Now construct a directed graph with the new tiles as vertices, with an edge from T_i to T_j if T_j can be placed on the right of T_i and is legal by considering the original set (that is, all the polyominoes formed are from the original set).

The tiling if the strip corresponds to an infinite path along the directed edges. But since there is a finite number of vertices, there must be a cycle in the graph. This cycle corresponds to a periodic tiling with the tiles, and so also a periodic tiling with the original tiles. □

A cycle in the proof above corresponds to a cylinder. The theorem is then equivalent to the following:

Theorem 140. *If a tile tiles a strip of width m, it also tiles a cylinder of width m.*

[Not referenced]

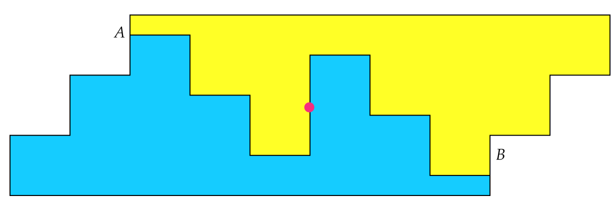


Figure 151: A construction similar to type II Reid polyomino, but using a different starting cylinder.

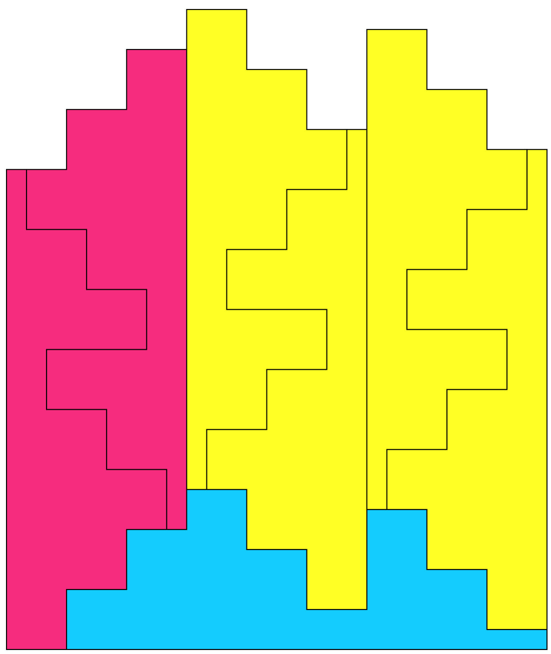


Figure 152: The tiling for the type II alternative construction.

Polyomino	Class
1	R
2	R
3	R
3	R
4	R
4	R
4	R
4	R
4	BS
5	R
5	R
5	R
5	R
5	BS
5	BS
5	S
5	S
5	P
5	P
5	P
5	P
6	R
6	R
6	R
6	R
6	R
6	R
6	R
6	R
6	R
6	R
6	BS
6	BS
6	BS
6	BS
6	BS
6	BS
6	S
6	S
6	S
6	S
6	S
6	S
6	S
6	S
6	S
6	P
6	P
6	P
6	P
6	P
6	P
6	P
6	P
6	P
6	P
7	R
7	R
7	R
7	R
7	R
7	R
7	N*
7	BS*, Q′
7	BS
7	S*, Q′
7	S*
7	S
7	S*
7	S*
7	P*
7	II*
7	P*
7	BS
7	P*
7	P*
7	P*
7	P*
7	P*
7	P*
7	P*
7	P*
7	P*
7	P*
7	P*
7	P*
7	P*
7	N
7	
7	
7	

Table 18: Tiling classes for polyominoes. The table gives the smallest class a polyomino is part of. Classes marked with an asterisk is not guaranteed to be the smallest. An accent shows the polyomino is *not* in that class.

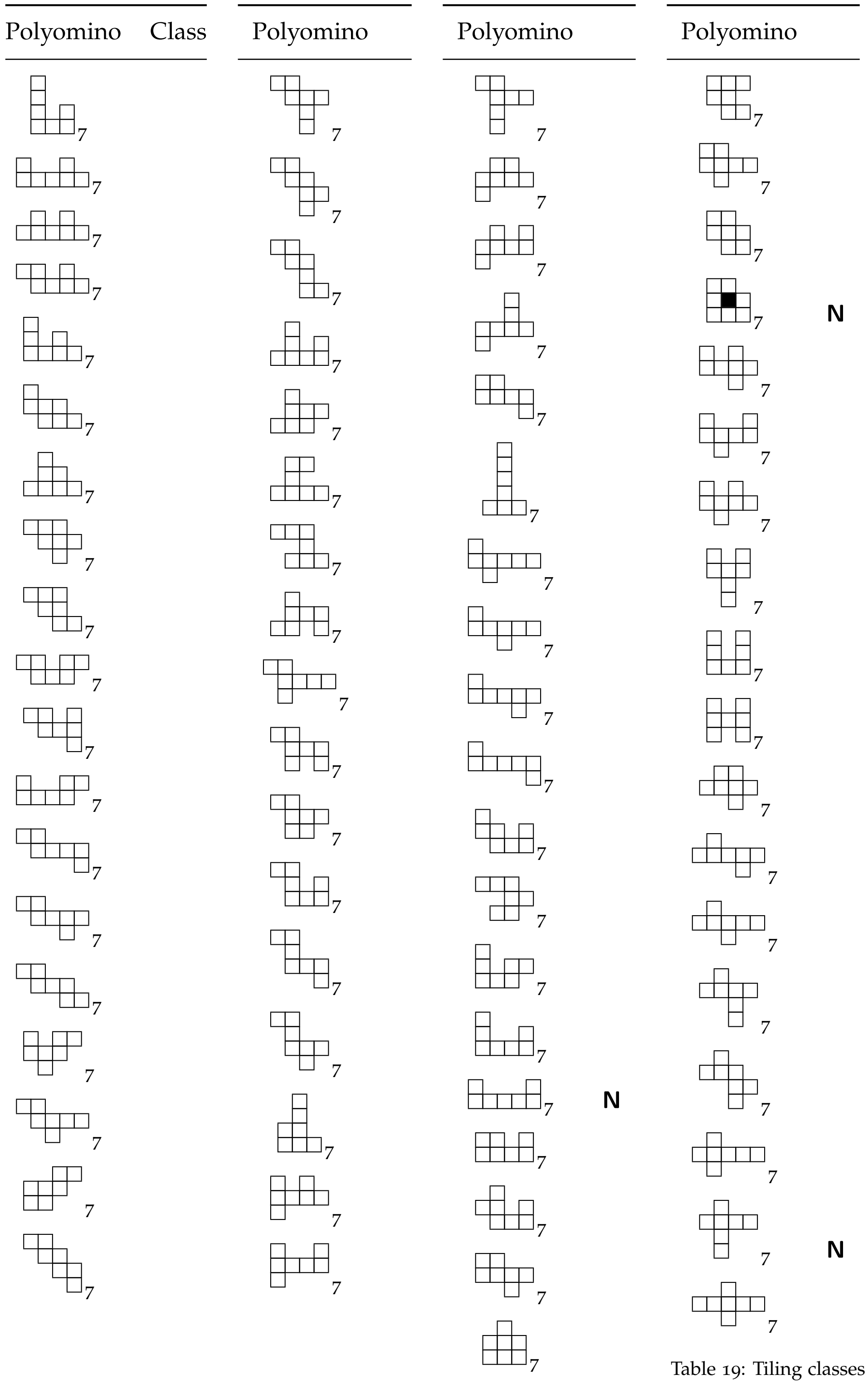


Table 19: Tiling classes for polyominoes.

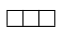
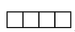
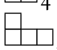
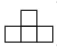
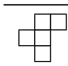
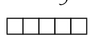
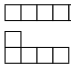
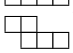
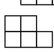
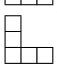
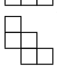
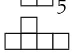
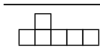
Polyomino	Prime Half-strips	Prime Strips
 1	1	1
 2	1	1
 3	1	1
 3	2, 3	2, 3
 4	1	1
 4	2	2
 4	2, 3	2, 3
 4	4,...?	2, 13[d]
 5	-	5, ...?
 5	1	1
 5	2, 5	2, 5
 5	-	2
 5	2, 3	2, 3
 5	-	5, ...?
 5	-	5, ...?
 5	5, 6[a], 8[a], 9	2[b], 5
 6 [c]	12[a], 23, 29, 30, 32	2, ...?

Table 20: Prime half-strips and strips.
[a]No rectangles with this width.
[b]No half-strips with this width.
[c]Prime ?
[d]Hochberg (2015); see also Figures 273–275.

4.1.2 *The Tiling Hierarchy for Sets*

We can extend the tiling hierarchy in a straightforward way to tile sets. We need to make one modification.

A tile set has the weak reptile property if the set can tile a scaled copy of each element from the set. A tile set has the strong reptile property if the set can tile a copy of each element of the set scaled by the same amount.

The tile set will tile a quadrant if it has the strong reptile property, but not necessarily if it has the week reptile property. For sets of single elements, there is no distinction between the two properties.

We use **I** for sets that have the strong reptile property, and **IW** for sets with the weak reptile property.

Theorem 141 (Golomb (1966))**.** *For sets of polyominoes:*

$$\mathbf{BS} \subset \mathbf{Q} \subset \mathbf{HP} \tag{4.6}$$

$$\mathbf{R} \subset \mathbf{I} \subset \mathbf{Q} \subset \mathbf{HP} \tag{4.7}$$

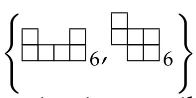
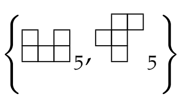
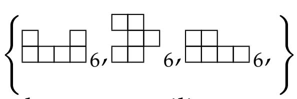
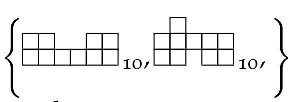
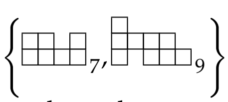
$$\mathbf{R} \subset \mathbf{HS} \subset \mathbf{BS} \subset \mathbf{S} \subset \mathbf{HP} \subset \mathbf{P} \subset \mathbf{N} \tag{4.8}$$

[Not referenced]

The proof of this Theorem is essentially the same as that of Theorem 137, so I omit it.

Problem 54** (Golomb (1970), p. 62)**.** *Are there sets with the weak reptile property that do not tile a rectangle?*

Here are some examples (all from Golomb (1970)).

-  has the strong reptile property. Neither polyomino is a reptile. Note these tile a right tromino double the side, and so also tile a rectangle.
-  tiles a square.
-  is reptilic, but no two taken by themselves are reptilic.
-  tiles a strip, but not a bend strip or quadrant.
-  tiles a quadrant, but not a strip and it does not have the strong reptile property.

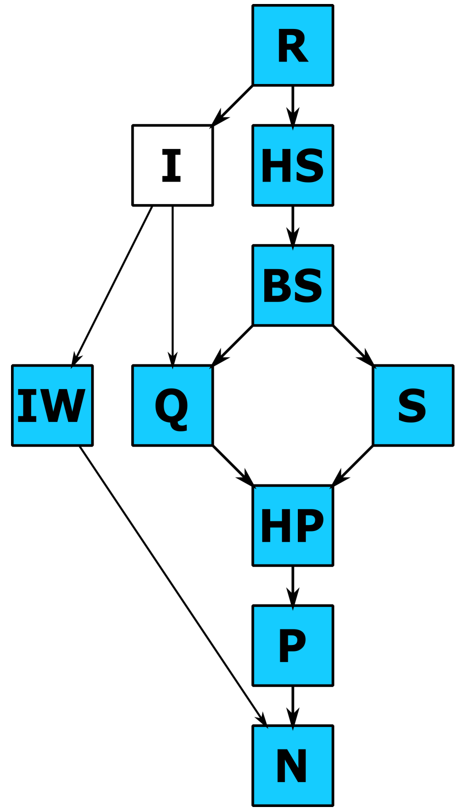


Figure 153: The tiling hierarchy for sets of polyominoes.

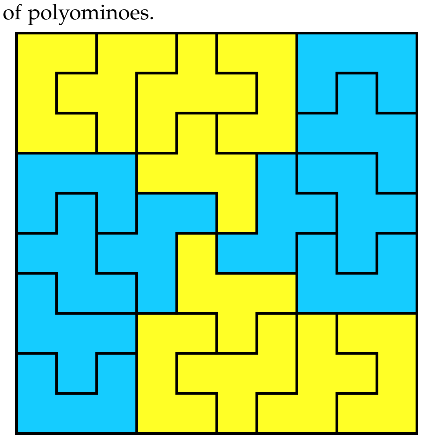


Figure 154: A set that tiles a rectangle (a square).

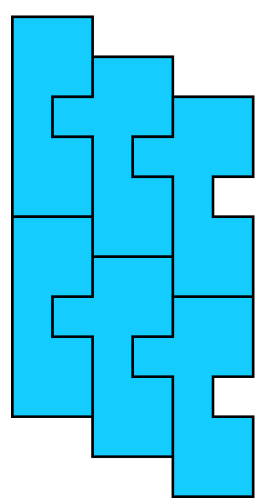
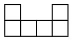
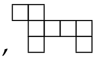
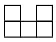
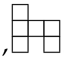
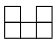
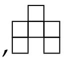


Figure 155: A set that tiles a half-plane

- $\{$ $_6$, $_8$ $\}$ tiles a quadrant and strip, but not a bent strip.
- $\{$ $_5$, $_6$ $\}$ tiles a quadrant and strip, but not a bent strip.
- $\{$ $_5$, $_6$ $\}$ tiles a half-strip, but not a rectangle.

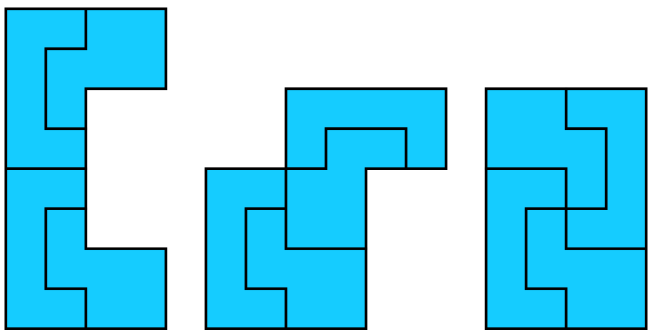


Figure 156: A reptilic set. This set also tiles a rectangle, also shown.

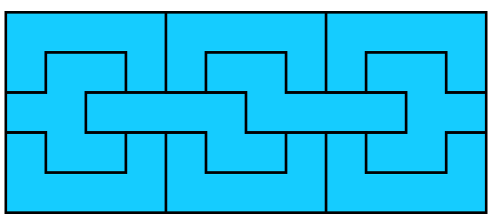


Figure 157: A set that tiles a rectangle.

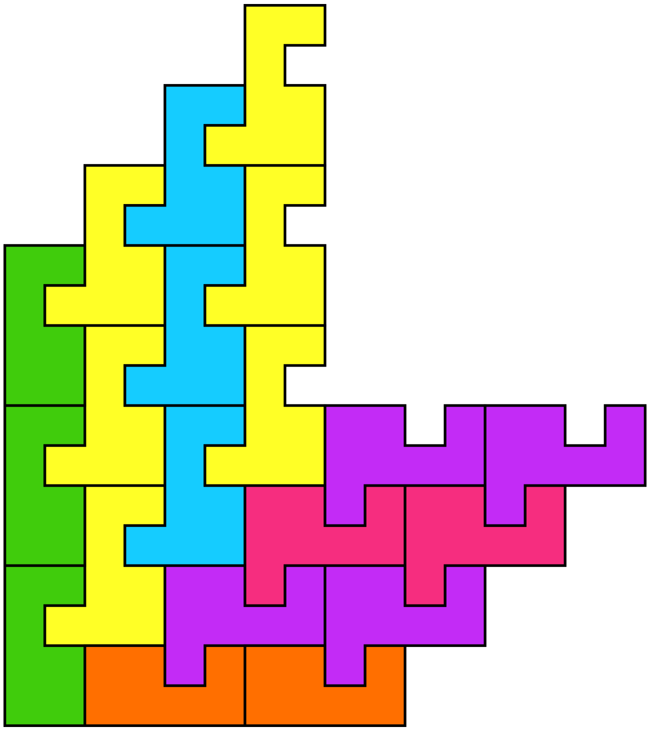


Figure 158: A set that tiles a quadrant, but not a strip.

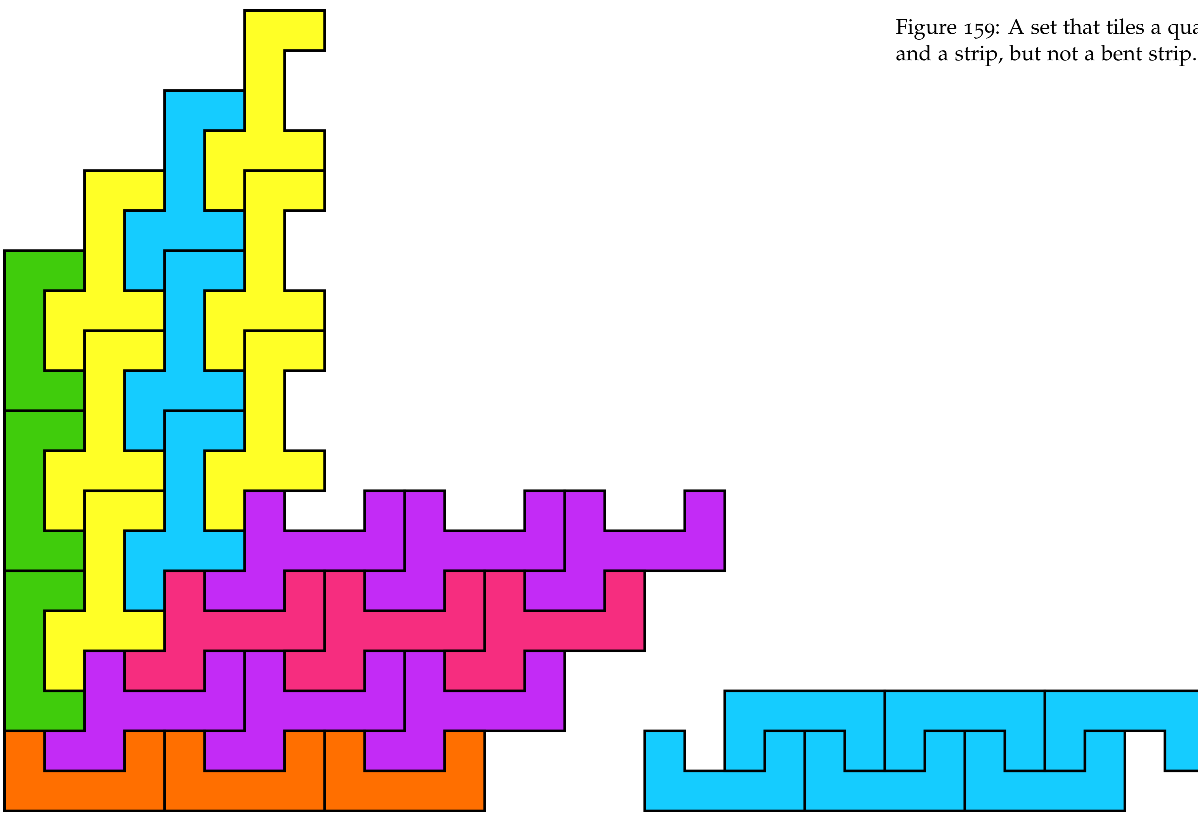


Figure 159: A set that tiles a quadrant and a strip, but not a bent strip.

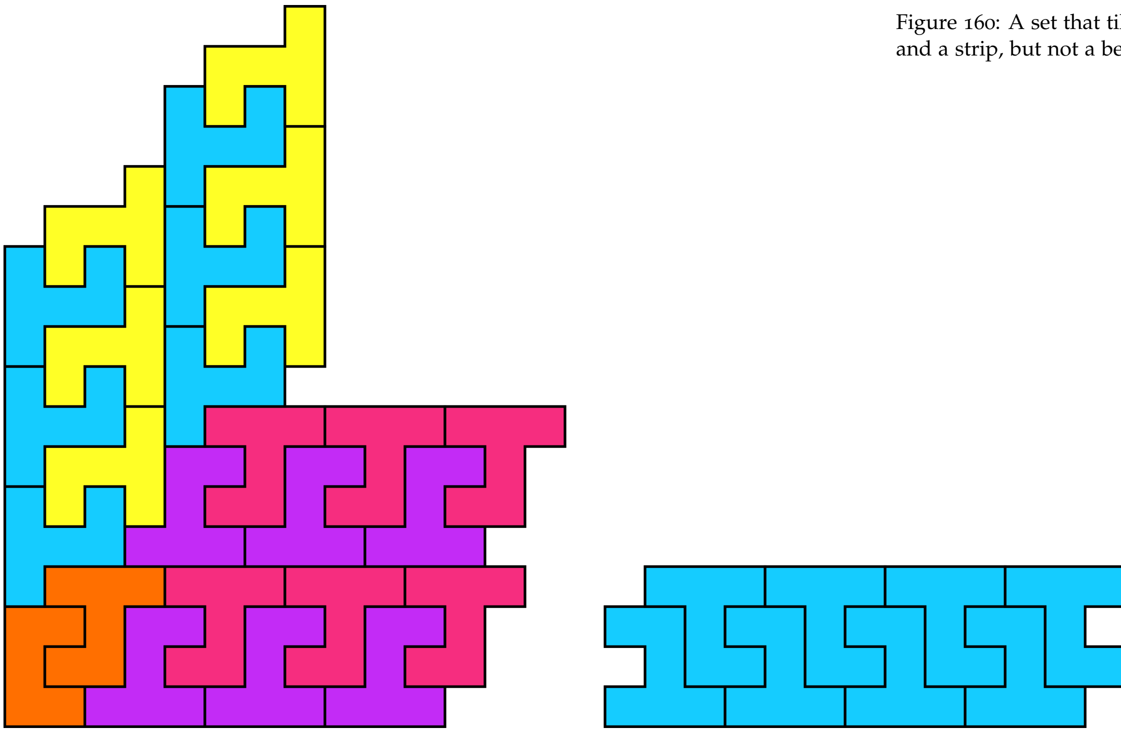


Figure 160: A set that tiles a quadrant and a strip, but not a bent strip.

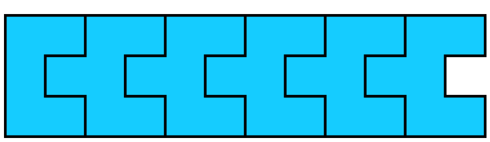
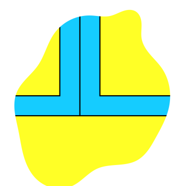


Figure 161: A set that tiles a strip but not a rectangle.

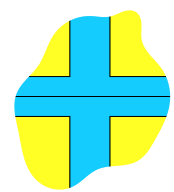
4.1.3 *Extending the Hierarchy*

It is not hard to come up with various other classes that fits into the existing hierarchy. It is much harder to tell whether such extensions would be useful. In this section, I give some ideas for classes that might be.

Rectangles and the Plane. Tilings of rectangles and the plane are important and have been extensively studied. We refine the classification of rectangular and plane tilings in Chapters 5 and 7.



(a) Bent strips tile a branched strip.



(b) Branched strips tile a crossed strip.

Figure 162: Some inclusions of the tiling hierarchy as extended by Liu demonstrated.

Liu's Extension. Liu (2018a) proposed two additional classes:

- **TS** T-strip, which means it can tile an infinite T-junction.
- **XS** X-strip, which means it can tile an infinite X-junction.

Theorem 142.

$$\mathbf{HS} \subset \mathbf{TS} \subset \mathbf{XS} \tag{4.9}$$

$$\mathbf{XS} \subset \mathbf{S} \tag{4.10}$$

[Not referenced]

Proof. 4.9 should be clear, see Figure 162. To show 4.10 we can use exactly the same argument as we did to prove 4.5 in Theorem 137. □

Table 21: Characteristic examples for Liu's classes.

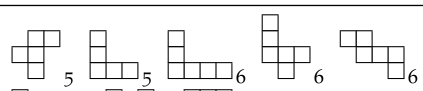
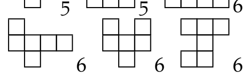
Class	Characteristic examples
TS	
XS	

Table 21 shows pentominoes and hexominoes that are characteristic examples of the new classes. All other polyominoes with 6 cells or less have the same classes as in the original Golomb hierarchy.

Nitica's Extension. Nitica (2018a) considers a tiling hierarchy for sets of fixed polyominoes; that is, only translations are allowed. If reflections and rotations are allowed, then a polyomino that can tile a quadrant can tile tile any quadrant; however, if only translations are allowed, a set can tile one quadrant but none of the other.

There is two unexpected implication in this hierarchy:

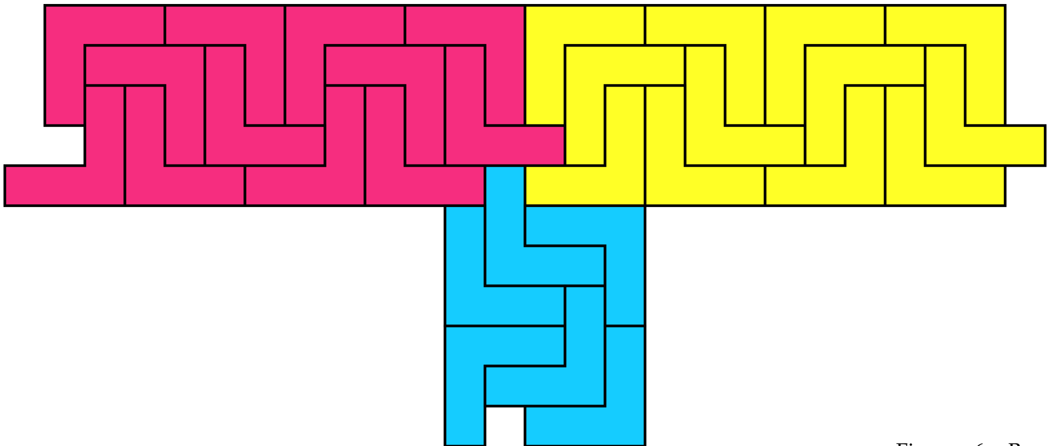


Figure 163: Branched strip for the V-pentomino.

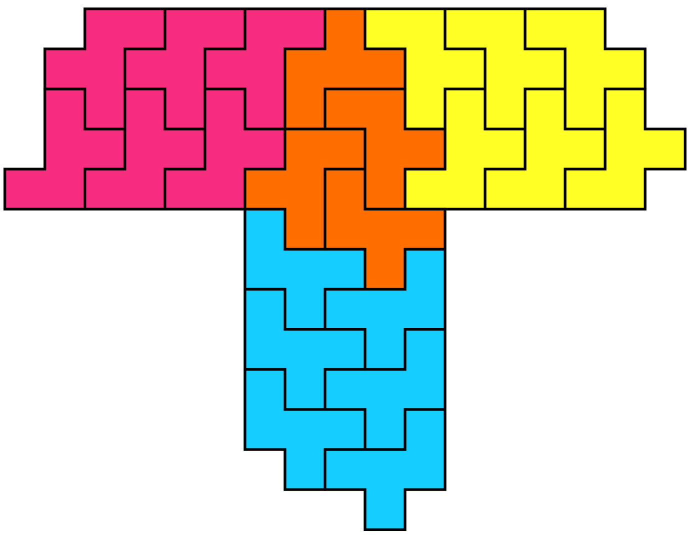


Figure 164: Branched strip for the F-pentomino.

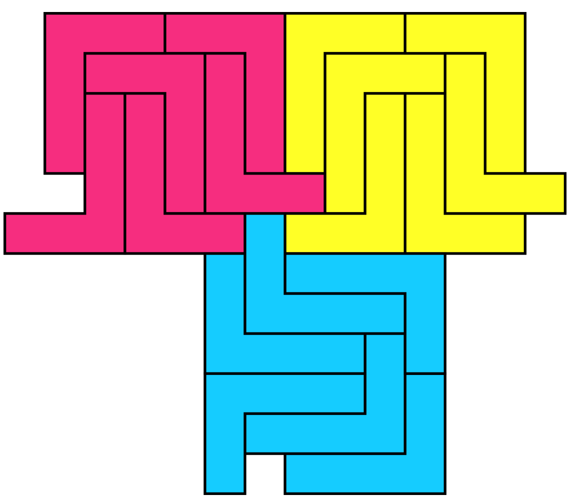


Figure 165: Branched strip.

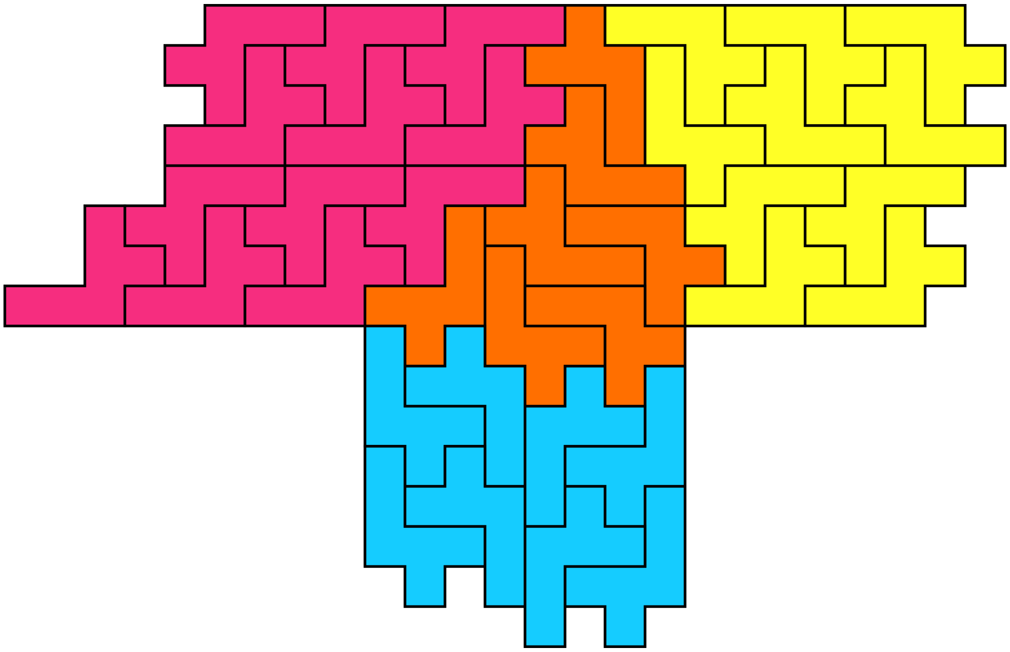
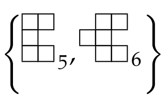
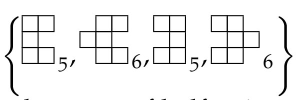
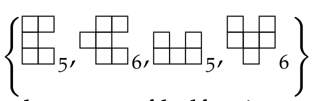
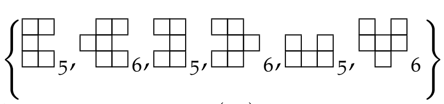


Figure 166: A branched strip

- If a set tiles any quadrant, it tiles the two half-planes that contains that quadrant.
- If a set tiles any half-plane, it tiles the entire plane.

We prove this in Theorem 311 and 312, after we looked a bit more carefully at plane tilings.

The following examples are from Nitica (2018a):

-  $\in$ **HS**(R) but it tiles no other half strip.
-  $\in$ **HS**(R) and **HS**(L), but it tiles no other type of half strip.
-  $\in$ **HS**(R) and **HS**(U), but it tiles no other type of half strip.
-  $\in$ **HS**(R), **HS**(L) and **HS**(U), but it is not in **HS**(D).

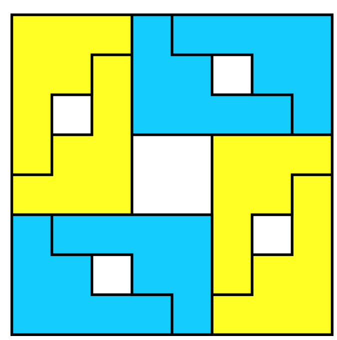


Figure 167: An example that shows a polyomino that tiles a non-square rectangle with k holes (1 in this case) also tiles a rectangle with $4k+1$ holes (5 in this case).

THE VERTEX-COUNT EXTENSION. The number of vertices a region has imposes a natural tiling hierarchy. Let $\mathbf{C}_n$ denote the polyominoes that can tile some region that is in $\mathcal{C}_n$. If $P \in \mathbf{C}_m$ and $P \in \mathbf{C}_n$, then $P \in \mathbf{C}_{m+n-2}$. If $P \in \mathbf{C}_n$, then $P \in \mathbf{C}_{kn-2k+2}$ for all $k > 1$.

A non-rectifiable L-shaped polyomino is a characteristic example of $\mathbf{C}_6$, but since it is itself L-shaped it is not such an interesting example. I do not know whether more interesting examples exist.

Problem* 55. *Are there polyominoes that can tile L-shaped regions, and cannot tile rectangles or are L-shaped themselves?*

THE TOPOLOGY EXTENSION.

Another natural extension is to consider the holes of a region. Let $\mathbf{R}_n$ denote polyominoes that can tile rectangles with n rectangular holes.

- $\mathbf{R} = \mathbf{R}_0 \subset \mathbf{R}_n$
- $\mathbf{R}_n \subset \mathbf{R}_{kn}$ for all k
- $\mathbf{R}_n \subset \mathbf{R}_{4n+1}$ provided the holy rectangles are not all squares, otherwise $\mathbf{R}_n \subset \mathbf{R}_{8n+1}$

All L-shaped polyominoes, and polyominoes that can tile L-shapes, can tile rectangles with rectangular holes. So if such a polyomino cannot tile a rectangle, it is characteristic of $\mathbf{R}_1$.

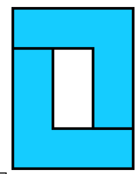
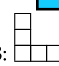


Figure 168: $_5$ can tile a rectangle with a rectangular hole but not a rectangle.

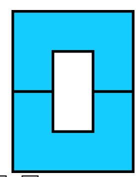
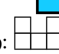


Figure 169: $_5$ can tile a rectangle with a rectangular hole but not a rectangle.

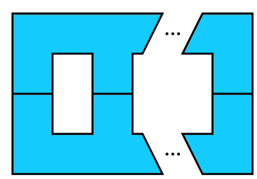
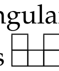


Figure 170: A general construction for figures that tile $\mathbf{R}_n$ but (probably) not $\mathbf{R}_{n-1}$.

All U-shaped polyominoes, and polyominoes that can tile U-shapes, can tile rectangles with one rectangular hole. So if such a polyomino cannot tile a rectangle, such as $_5$, it is characteristic of $\mathbf{R}_1$. In fact, we can generalize this idea: a *comb* with n teeth can tile a rectangle with $n-1$ holes.

It is easy to come up with polyominoes that require 4 copies to tile a rectangle with a hole; see Figure 171.

Here is a general way to construct a polyomino im $\mathbf{R}_n$ provided there is a tiling by the polyomino that use n tiles: take the polyomino, and double it in size. Remove a cell (a quarter of one the original cells) from a knob or anti-knob. Tile the polyomino as you would to form a rectangle if the cell was not removed. The resulting tiling has n holes.

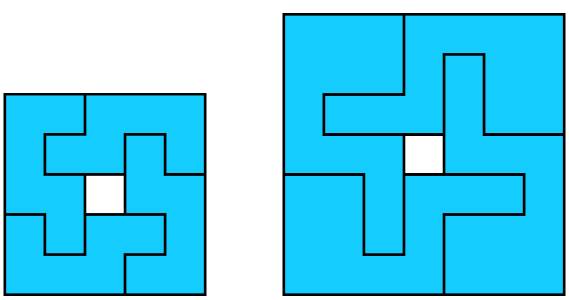
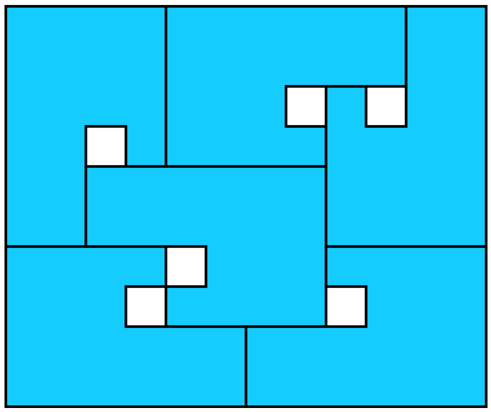


Figure 171: Polyominoes that belongs to $\mathbf{R}_1$

Note in a tiling of a rectangle, the anti-knobs and flats must all lie inside; only knobs can lie along the sides of the rectangle. Also notice that in the scaled tiling, if we apply the checkerboard coloring the hole has the same color in each tile. Therefore, the holes cannot be neighbors and therefore can combine to make bigger (and fewer) holes. Figure 4.1.3 shows an example with the P-pentomino.



Problem* 56.

(1) *Show that there is no polyomino that can tile a rectangle with a rectangular hole using three copies.*

(2) *Are there characteristic examples of* R_1 *that require more than four pieces to make the tiling?*

The Topology Vertex-count Extension. From here it is natural to define classes by the number of corners of the outer border, and the number and nature of the holes, also by their number of vertices, and derive some inclusions. For example, polyominoes that tile rectangular regions with one rectangular hole, also tile rectangular regions with 8 rectangular holes and one Z-shaped hole.

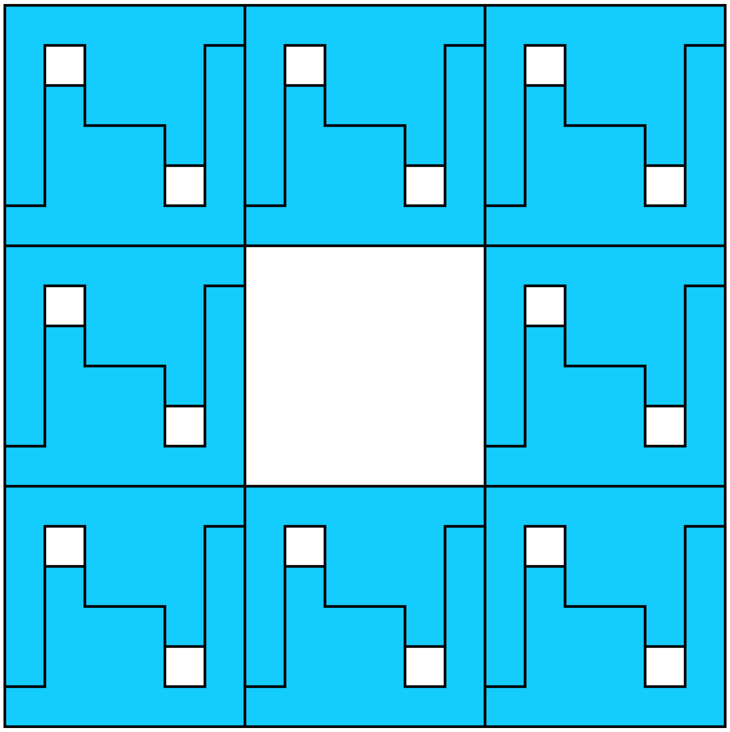


Figure 172: An example that shows a polyomino that tiles a square with k holes also tile a square with $8k+1$ holes.

Nitica (2018a) considers adding deficient regions—that is, regions with one cell removed, that is, regions with single-cell holes (possible at the edge). This is explored in detail in Nitica (2018b), where the finding is that most relationships between classes are not interesting. One mild exception is that a polyomino that tiles a quadrant can tile a deficient plane, as shown in Figure 173.

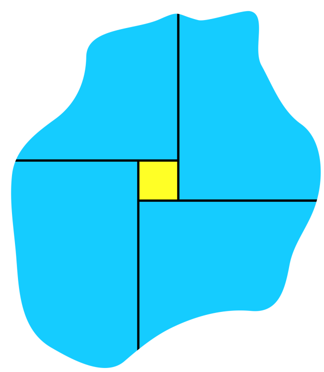


Figure 173: A polyomino that can tile a quadrant can tile a deficient plane, and indeed a plane with any size rectangular hole.

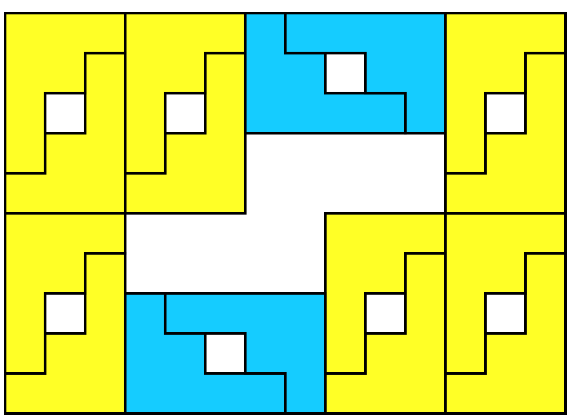


Figure 174: An example that shows a polyomino that tiles a rectangle with a rectangular hole can also tile a rectangle with 8 rectangular holes and one Z-shaped hole.

Irregular Tilings. If similar copies of a polyomino can tile a region, we say the region has an **irregular tiling** by that polyomino.

If we want to emphasize that all the copies of a tile in a tiling is the same size we say the tiling is a **regular** tiling (Reid, 2003c).

One polyomino that does not tile the plane regularly, but has an irregular tiling, is the polyomino with border $x^2y^{16}x^{-1}y^4x^{32}y^{-2}x^8y^{64}x^{-41}y^{-82}$ and area 2594. A piece of plane tiling is shown in Figure 175.

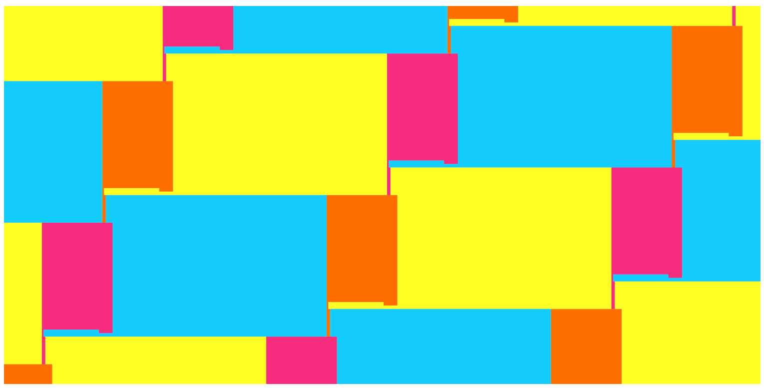


Figure 175: A tiling by a polyomino and one doubled in size. By itself it does not tile the plane. This example has been found by nickgard (un.) (2019).

Problem* 57. *Can you find other polyominoes that do not tile the plane regularly, but can irregularly?*

We consider a special class of this type, called irreptiles, in section 6.4.

THE PROBLEM WITH EXTENSIONS. The more classes there are, the more tedious it is to find the characteristic class of a polyomino or set of polyominoes. The notion of *prime shape* also becomes more complicated, although it may be interesting to explore. The classes also become less useful to use in shortcuts to investigate the tiling capabilities of a polyomino. The original classes proposed by Golomb are sufficiently simple and important to be useful in many tiling investigations.

4.1.4 *Further Reading*

We look at rectangles in more detail in Chapter 5; reptiles and irreptiles in Chapter 6; plane tilings in Chapter 7.

Irregular tilings by polyomino pairs are considered in Friedman (2008, https://erich-friedman.github.io/mathmagic/0908.html). More irregular tilings can be found in Kurchan (1997). Reid (2003c) names Scherer (1987) as inspiration.[4]

[4] Although I could not check this source myself.

Any region has irregular tilings by any rectangular polyomino (this follows from Theorem 136). This notion is extended in Su and Ding (2005) to non-polyomino rectangles, where they show what shapes such rectangles must have.

I mentioned the hierarchies studied in Nitica (2018a) and Nitica (2018b). These papers serve as the model for how to study extensions to the original hierarchy: defining new classes, proving which relationships hold, which fail, and organizing the final relationships in a new hierarchy. We cover the *mutually-tiling extension* in Section 6.3.

Strip tilings for polyominoes with 8 and less cells are given by Sicherman (2015a, http://www.recmath.org/PolyCur/nuniform/index.html).

4.2 *Colorings*

The checkerboard coloring played a fundamental role in our analysis of domino tilings. Other colorings can sometimes be used to analyze tilings too. Before looking at some other types of tilings, let's proof a useful result about the checkerboard coloring that is useful for tilings in general:

Theorem 143 (Generalized checkerboard criterion)**.** *Suppose a region R is tileable by tiles P_i. Then $d = \gcd_i |\Delta(P_i)| \mid |\Delta(R)|$.*

[Referenced on pages 364, 365 and 367]

Proof. $\Delta(R) = \sum_i k_i \Delta(P_i)$, where k_i is an integer, not necessarily positive. Now $d \mid |\Delta(P_i)|$, and therefore d also divides each term in the sum, so $d \mid |\Delta(R)|$. □

The theorem implies the following corollaries:

(1) Tilings by one tile with absolute deficiency d must have its deficiency divisible by d.

(2) Balanced tiles can only tile balanced regions.

(3) A region with absolute deficiency kd, tiled with tiles with deficiency d and 0, need at least k tiles with deficiency d (see Exercise 123).

One reason the checkerboard coloring works to analyze domino tilings is that it is an invariant: no matter how a domino is placed, it must always cover the same number of black and white squares. Therefore, the number of white and black cells in a tiling of any region is always the same, no matter how the region is tiled.

This type of coloring is quite special, and in this section we see when it exists for other polyominoes than dominoes. Essentially, we are asking this question: If we color the plane with k colors, how many colors do we need so that a polyomino does not cover two cells of the same color no matter how it is placed? When that number is the same as the number of cells, we have a coloring for that polyomino that can play a similar role as the checkerboard coloring plays for dominoes.

If we need k colors for a polyomino P, we call this the **chromatic number** of P and denote it by $C(P)$. We will use $C(m,n)$ as an abbreviation for $C(R(m,n))$. A coloring of the plane with $C(P)$ colors such that however we place P, it does not cover two cells of the same color is called an **sufficient coloring** for P. A sufficient coloring that minimizes the number of colors — and therefore use k colors — is called an **optimal coloring** for P.

We use the following notation for some common colorings. The first is a generalization of the checkerboard coloring, which we call a **flag coloring**.

$$F_{m,n}(x,y) = (x + yn) \bmod m. \tag{4.11}$$

In this coloring, there are m colors, each repeated in every row (in the same order), and each row offset by n cells from the previous row.

If $n = 1$, we simply write F_m. In this notation, the checkerboard coloring is F_2.

Note that $F_{m,n}$ is the same as $F_{m,m-n}$, but reflected around vertical axis and some colors interchanged.

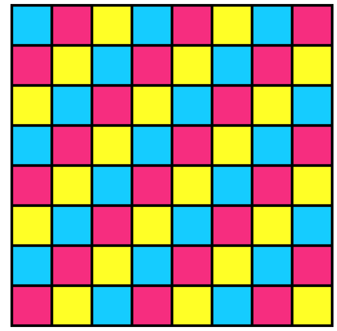


Figure 176: $R(8,8)$ colored with F_3. Red corresponds with color 0, yellow with color 1, and blue with color 2.

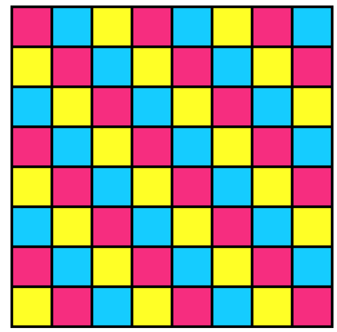


Figure 177: $R(8,8)$ colored with the reflected coloring.

Example 16. *(Golomb, 1996, p. 4)*

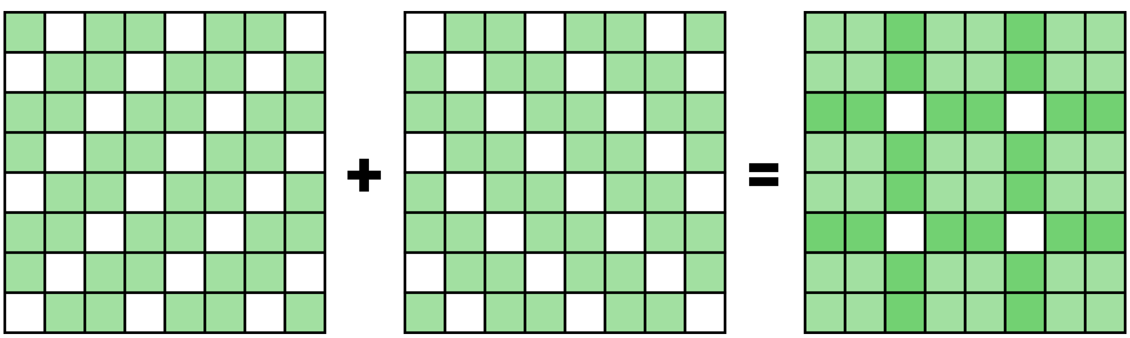


Figure 178: This shows the cells where no tiling would be possible if we removed it in green. On the left are the cells we found impossible under F_3, and in the center are the impossible cells under the reflected coloring. On the right are the two sets overlapped. The white cells are the only ones we can remove if the remaining figure is to be tileable.

Can $R(8,8)$ with one cell removed be tiled by straight trominoes?

Color the rectangle with F_3 (Figure 176). Note that no matter how we place a tromino, it always covers three different colors.

Now note that color 0 (red) occurs 22 times, but the other colors (yellow and blue) only occur 21 times. So the removed cell cannot be yellow or blue.

If we reflect the coloring, we can also eliminate the cells colored yellow or blue under the reflected coloring (Figure **??***). (This coloring is equivalent to F_{3_2}). This gives us only four possible locations from where the cell can be removed.*

It is not hard to find a tiling for such a region; an example is shown in Figure 179.

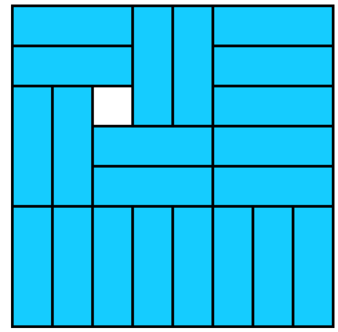


Figure 179: A tiling of $R(8,8)$ with a cell removed by straight trominoes.

The second is called a **square coloring**.

$$S_k(x,y) = (x \bmod k) + (y \bmod k)k. \tag{4.12}$$

This is the grid divided into $m \times m$ squares, each containing m^2 colors in the same pattern. Square colorings, when translated, rotated or reflected remain the same coloring with some colors permuted.

The final type of coloring is their product:

$$S_k \times F_{m,n}(x,y) = F_{m,n}(\lfloor x/k \rfloor, \lfloor y/k \rfloor)k^2 + S_k(x,y). \tag{4.13}$$

Problem† 58. *Show that the number of colors for*

(1) $F_{m,n}$ *is* m,

(2) S_k *is* k^2, *and*

(3) $S_k \times F_{m,n}$ *is* $k^2 m$.

Problem† 59. *Show that*

(1) $S_1 \times F_{m,n} = F_{m,n}$

(2) $S_k \times F_1 = S_k$

Theorem 144 (Yu (2020)[5])**.** *For any polyomino* P, $C(P) \geq |P|$.

[Referenced on page 164]

Proof. Obvious. □

Polyominoes for which $C(P) = |P|$ are called **efficient**. Otherwise they are **inefficient**.

Theorem 145 (Yu (2020), Implicit)**.** *Suppose the hull of* P *is* $R(m,n)$. *Then* $C(P) \leq C(m,n)$.

[Referenced on pages 163 and 170]

Proof. Suppose $C(P) > C(m,n)$. Color the plane with an optimal coloring for $R(m,n)$. Since this coloring requires less than $C(P)$ colors, some placement of P covers two cells of the same color. If we put $R(m,n)$ to cover P in such a placement, then two covered by $R(m,n)$ must have the same color, and therefore the coloring cannot be optimal for $R(m,n)$, a contradiction. Therefore $C(P) \leq C(m,n)$. □

[5] This was originally publish in Yu (2014), the newer publication adds a short conclusion.

If P can cover any two cells of a polyomino Q, we say P **protects** Q. A polyomino that cannot protect a polyomino larger than itself is called **weak**. Bars and squares are weak.

Problem* 60. *Which rectangles are weak?*

Theorem 146 (Yu (2020), Implicit)**.** *If P protects Q, then $C(P) \geq C(Q)$.*

[Referenced on pages 163, 170 and 356]

Proof. Suppose $C(Q) > C(P)$. Color the plane with an optimal coloring for P. This coloring requires less than $C(Q)$ colors, some placement of Q covers two cells of the same color. These cells can be covered by P, so the coloring cannot be optimal for P, a contradiction. Therefore, $C(Q) \leq C(P)$. □

Theorem 147. *All efficient polyominoes are weak.*

[Not referenced]

Proof. Suppose P is an efficient polyomino, and let Q be any polyomino that P protects. By Theorem 146 we must have $C(P) \geq C(Q)$, but $C(P) = |P|$ (because P os efficient). Thus

$C(Q) \leq |P|$, so if P protects Q, Q is the same size or smaller than P, so so P is weak. □

The converse does not hold. For example, below we will see that W-polyominoes with an odd number of cells are also weak, but they are not efficient.

Theorem 148 (Yu (2020), Implicit)**.** *If $Q \sqsubset P$ then $C(P) \geq C(Q)$.*

[Referenced on pages 168, 170 and 178]

Proof. P can cover the whole of Q, and in particular any two cells of Q. Therefore, by Theorem 146 $C(P) \geq C(Q)$. □

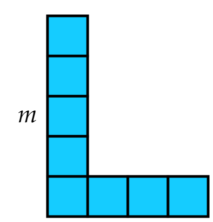


Figure 180: The shape of Theorem 149: $B(1^m \cdot (n-1))$.

Theorem 149. *Let $P = B(m \cdot 1^{n-1})$. Then $C(P) = C(m,n)$, with $F_{m(m+1)-1,m(m+2)}$ a sufficient coloring.*

[Referenced on pages 163, 170 and 356]

Proof. The hull of P is $R(m,n)$, so $C(P) \leq C(m,n)$ (Theorem 145).

P can cover any two cells of $R(m,n)$, so $C(P) \geq C(m,n)$ (Theorem 146).

Taking these together, we get $C(P) = C(m,n)$. □

Theorem 150. *Suppose that P has a optimal coloring with n colors, and the coloring is a square coloring. Then if we remove a cell from P and append it so that the removed cell has the same color to form Q, then $C(Q) = C(P)$.*

[Not referenced]

Problem† 61. *What are all the polyominoes that are equivalent under S_3 to the ?*

Theorem 151. *If P fits into Q in all orientations, and Q tiles the plane by translation[6], then $C(P) \leq |Q|$.*

[6] A polyomino that tiles the plane by translation is called an *exact polyomino*; exact polyominoes are discussed in Section 7.1.

[Referenced on pages 164 and 170]

Theorem 152. *If $P \in$ **All** tiles the plane, then $C(P) = |P|$, and so P is efficient.*

[Referenced on page 170]

Proof. P fits in itself in all orientations, so by Theorem 151 we have $C(P) \leq P$. But by Theorem 144, we have $C(P) \geq P$. Thus, $C(P) = |P|$. □

The next few Theorems establishes what is known of $R(m, n)$, an important case since the rectangular hull establishes an upper bound for the chromatic number of all polyominoes. One would imagine calculating the chromatic number of rectangles is easy; however, the problem is subtle and not completely solved.

	n																			
m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	2	4	8	8	12	12	16	16	20	20	24	24	28	28	32	32	36	36	40	40
3	3	8	9	15	18	18	24	27	27	33	36	36	42	45	45	51	54	54	60	63
4	4	8	15	16	24	32	32	32	40	48	48	48	56	64	64	64	72	80	80	80
5	5	12	18	24	25	35	45	50	50	50	60	70	75	75	75	85	95	100	100	100
6	6	12	18	32	35	36	48	60	72	72	72	72	84	96	108	108	108	108	120	132
7	7	16	24	32	45	48	49	63	77	91	98	98	98	98	112	126	140	147	147	147
8	8	16	27	32	50	60	63	64	80	96	112	128	128	128	128	128	144	160	176	192
9	9	20	27	40	50	72	77	80	81	99	117	135	153	162	162	162	162	162	180	198
10	10	20	33	48	50	72	91	96	99	100	120	140	160	180	200	200	200	200	200	200
11	11	24	36	48	60	72	98	112	117	120	121	143	165	187	209	231	242	242	242	242
12	12	24	36	48	70	72	98	128	135	140	143	144	168	192	216	240	264	288	288	288
13	13	28	42	56	75	84	98	128	153	160	165	168	169	195	221	247	273	299	325	338
14	14	28	45	64	75	96	98	128	162	180	187	192	195	196	224	252	280	308	336	364
15	15	32	45	64	75	108	112	128	162	200	209	216	221	224	225	255	285	315	345	375
16	16	32	51	64	85	108	126	128	162	200	231	240	247	252	255	256	288	320	352	384
17	17	36	54	72	95	108	140	144	162	200	242	264	273	280	285	288	289	323	357	391
18	18	36	54	80	100	108	147	160	162	200	242	288	299	308	315	320	323	324	360	396
19	19	40	60	80	100	120	147	176	180	200	242	288	325	336	345	352	357	360	361	399
20	20	40	63	80	100	132	147	192	198	200	242	288	338	364	375	384	391	396	399	400

Table 22: Computed values of $C(m, n)$, assuming that colorings are periodic. Colored cells are values that have been proven:

- Theorem 154
- Theorem 155
- Theorem 159

Table 22 shows some computed values. Only those marked are proved to be the lowest; I calculated the remainder of the values by exhaustively searching for the smallest periodic coloring. It is not established that there exist an optimal coloring that is not periodic (see Problem 66(1)).

	2	3	4	5	6	7	8	9	10
2	S_2	$F_{8,3}$	$S_2 \times F_2$	$F_{12,5}$	$S_2 \times F_3$	$F_{16,7}$	$S_2 \times F_4$	$F_{20,9}$	$S_2 \times F_5$
3	$F_{8,3}$	S_3	$F_{15,4}$	$S_3 \times F_2$	$S_3 \times F_2$	$F_{24,7}$	$S_3 \times F_3$	$S_3 \times F_3$	$F_{33,10}$
4	$S_2 \times F_2$	$F_{15,4}$	S_4	$F_{24,5}$	$S_6 \times F_{8,3}$	$S_4 \times F_2$	$S_4 \times F_2$	$F_{40,9}$	$S_{10} \times F_{12,5}$
5	$F_{12,5}$	$S_3 \times F_2$	$F_{24,5}$	S_5	$F_{35,6}$	$F_{45,19}$	$S_5 \times F_2$	$S_5 \times F_2$	$S_5 \times F_2$
6	$S_2 \times F_3$	$S_3 \times F_2$	$S_6 \times F_{8,3}$	$F_{35,6}$	S_6	$F_{48,7}$	$S_8 \times F_{15,4}$	$S_9 \times F_{8,3}$	$S_6 \times F_2$
7	$F_{16,7}$	$F_{24,7}$	$S_4 \times F_2$	$F_{45,19}$	$F_{48,7}$	S_7	$F_{63,8}$	$F_{77,34}$	$F_{91,27}$
8	$S_2 \times F_4$	$S_3 \times F_3$	$S_4 \times F_2$	$S_5 \times F_2$	$S_8 \times F_{15,4}$	$F_{63,8}$	S_8	$F_{80,9}$	$S_{10} \times F_{24,5}$
9	$F_{20,9}$	$S_3 \times F_3$	$F_{40,9}$	$S_5 \times F_2$	$S_9 \times F_{8,3}$	$F_{77,34}$	$F_{80,9}$	S_9	$F_{99,10}$
10	$S_2 \times F_5$	$F_{33,10}$	$S_{10} \times F_{12,5}$	$S_5 \times F_2$	$S_6 \times F_2$	$F_{91,27}$	$S_{10} \times F_{24,5}$	$F_{99,10}$	S_{10}
11	$F_{24,11}$	$S_3 \times F_4$	$S_4 \times F_3$	$F_{60,11}$	$S_6 \times F_2$	$S_7 \times F_2$	$F_{112,41}$	$F_{117,53}$	$F_{120,11}$
12	$S_2 \times F_6$	$S_3 \times F_4$	$S_4 \times F_3$	$F_{70,29}$	$S_6 \times F_2$	$S_7 \times F_2$	$S_{12} \times F_{8,3}$	$S_{12} \times F_{15,4}$	$S_{12} \times F_{35,6}$
13	$F_{28,13}$	$F_{42,13}$	$F_{56,13}$	$S_5 \times F_3$	$F_{84,13}$	$S_7 \times F_2$	$S_8 \times F_2$	$F_{153,35}$	$F_{160,49}$
14	$S_2 \times F_7$	$S_3 \times F_5$	$S_{14} \times F_{16,7}$	$S_5 \times F_3$	$S_{14} \times F_{24,7}$	$S_7 \times F_2$	$S_8 \times F_2$	$S_9 \times F_2$	$S_38 \times F_{45,19}$
15	$F_{32,15}$	$S_3 \times F_5$	$S_4 \times F_4$	$S_5 \times F_3$	$S_{15} \times F_{12,5}$	$F_{112,15}$	$S_8 \times F_2$	$S_9 \times F_2$	$S_{15} \times F_{8,3}$
16	$S_2 \times F_8$	$F_{51,16}$	$S_4 \times F_4$	$F_{85,16}$	$S_6 \times F_3$	$F_{126,55}$	$S_8 \times F_2$	$S_9 \times F_2$	$S_{10} \times F_2$
17	$F_{36,17}$	$S_3 \times F_6$	$F_{72,17}$	$F_{95,39}$	$S_6 \times F_3$	$F_{140,41}$	$F_{144,17}$	$S_9 \times F_2$	$S_{10} \times F_2$
18	$S_2 \times F_9$	$S_3 \times F_6$	$S_{18} \times F_{20,9}$	$S_5 \times F_4$	$S_6 \times F_3$	$S_7 \times F_3$	$S_{18} \times F_{40,9}$	$S_9 \times F_2$	$S_{10} \times F_2$
19	$F_{40,19}$	$F_{60,19}$	$S_4 \times F_5$	$S_5 \times F_4$	$F_{120,19}$	$S_7 \times F_3$	$F_{176,65}$	$F_{180,19}$	$S_{10} \times F_2$
20	$S_2 \times F_10$	$S_3 \times F_7$	$S_4 \times F_5$	$S_5 \times F_4$	$S_{20} \times F_{33,10}$	$S_7 \times F_3$	$S_{20} \times F_{12,5}$	$F_{198,89}$	$S_{10} \times F_2$

	11	12	13	14	15	16	17	18	19	20
2	$F_{24,11}$	$S_2 \times F_6$	$F_{28,13}$	$S_2 \times F_7$	$F_{32,15}$	$S_2 \times F_8$	$F_{36,17}$	$S_2 \times F_9$	$F_{40,19}$	$S_2 \times F_10$
3	$S_3 \times F_4$	$S_3 \times F_4$	$F_{42,13}$	$S_3 \times F_5$	$S_3 \times F_5$	$F_{51,16}$	$S_3 \times F_6$	$S_3 \times F_6$	$F_{60,19}$	$S_3 \times F_7$
4	$S_4 \times F_3$	$S_4 \times F_3$	$F_{56,13}$	$S_{14} \times F_{16,7}$	$S_4 \times F_4$	$S_4 \times F_4$	$F_{72,17}$	$S_{18} \times F_{20,9}$	$S_4 \times F_5$	$S_4 \times F_5$
5	$F_{60,11}$	$F_{70,29}$	$S_5 \times F_3$	$S_5 \times F_3$	$S_5 \times F_3$	$F_{85,16}$	$F_{95,39}$	$S_5 \times F_4$	$S_5 \times F_4$	$S_5 \times F_4$
6	$S_6 \times F_2$	$S_6 \times F_2$	$F_{84,13}$	$S_{14} \times F_{24,7}$	$S_{15} \times F_{12,5}$	$S_6 \times F_3$	$S_6 \times F_3$	$S_6 \times F_3$	$F_{120,19}$	$S_{20} \times F_{33,10}$
7	$S_7 \times F_2$	$S_7 \times F_2$	$S_7 \times F_2$	$S_7 \times F_2$	$F_{112,15}$	$F_{126,55}$	$F_{140,41}$	$S_7 \times F_3$	$S_7 \times F_3$	$S_7 \times F_3$
8	$F_{112,41}$	$S_{12} \times F_{8,3}$	$S_8 \times F_2$	$S_8 \times F_2$	$S_8 \times F_2$	$S_8 \times F_2$	$F_{144,17}$	$S_{18} \times F_{40,9}$	$F_{176,65}$	$S_{20} \times F_{12,5}$
9	$F_{117,53}$	$S_{12} \times F_{15,4}$	$F_{153,35}$	$S_9 \times F_2$	$S_9 \times F_2$	$S_9 \times F_2$	$S_9 \times F_2$	$S_9 \times F_2$	$F_{180,19}$	$F_{198,89}$
10	$F_{120,11}$	$S_{12} \times F_{35,6}$	$F_{160,49}$	$S_38 \times F_{45,19}$	$S_{15} \times F_{8,3}$	$S_{10} \times F_2$	$S_{10} \times F_2$	$S_{10} \times F_2$	$S_{10} \times F_2$	$S_{10} \times F_2$
11	S_{11}	$F_{143,12}$	$F_{165,76}$	$F_{187,67}$	$F_{209,56}$	$F_{231,43}$	$S_{11} \times F_2$	$S_{11} \times F_2$	$S_{11} \times F_2$	$S_{11} \times F_2$
12	$F_{143,12}$	S_{12}	$F_{168,13}$	$S_{14} \times F_{48,7}$	$S_{15} \times F_{24,5}$	$S_{16} \times F_{15,4}$	$F_{264,109}$	$S_{18} \times F_{8,3}$	$S_{12} \times F_2$	$S_{12} \times F_2$
13	$F_{165,76}$	$F_{168,13}$	S_{13}	$F_{195,14}$	$F_{221,103}$	$F_{247,77}$	$F_{273,64}$	$F_{299,116}$	$F_{325,51}$	$S_{13} \times F_2$
14	$F_{187,67}$	$S_{14} \times F_{48,7}$	$F_{195,14}$	S_{14}	$F_{224,15}$	$S_{16} \times F_{63,8}$	$F_{280,99}$	$S_68 \times F_{77,34}$	$F_{336,71}$	$S_54 \times F_{91,27}$
15	$F_{209,56}$	$S_{15} \times F_{24,5}$	$F_{221,103}$	$F_{224,15}$	S_{15}	$F_{255,16}$	$F_{285,134}$	$S_{18} \times F_{35,6}$	$F_{345,91}$	$S_{20} \times F_{15,4}$
16	$F_{231,43}$	$S_{16} \times F_{15,4}$	$F_{247,77}$	$S_{16} \times F_{63,8}$	$F_{255,16}$	S_{16}	$F_{288,17}$	$S_{18} \times F_{80,9}$	$F_{352,111}$	$S_{20} \times F_{24,5}$
17	$S_{11} \times F_2$	$F_{264,109}$	$F_{273,64}$	$F_{280,99}$	$F_{285,134}$	$F_{288,17}$	S_{17}	$F_{323,18}$	$F_{357,169}$	$F_{391,137}$
18	$S_{11} \times F_2$	$S_{18} \times F_{8,3}$	$F_{299,116}$	$S_68 \times F_{77,34}$	$S_{18} \times F_{35,6}$	$S_{18} \times F_{80,9}$	$F_{323,18}$	S_{18}	$F_{360,19}$	$S_{20} \times F_{99,10}$
19	$S_{11} \times F_2$	$S_{12} \times F_2$	$F_{325,51}$	$F_{336,71}$	$F_{345,91}$	$F_{352,111}$	$F_{357,169}$	$F_{360,19}$	S_{19}	$F_{399,20}$
20	$S_{11} \times F_2$	$S_{12} \times F_2$	$S_{13} \times F_2$	$S_54 \times F_{91,27}$	$S_{20} \times F_{15,4}$	$S_{20} \times F_{24,5}$	$F_{391,137}$	$S_{20} \times F_{99,10}$	$F_{399,20}$	S_{20}

Table 23: Colorings that realize the number of colors in Table 22.

Theorem 153. [7] *Assume $m \leq n$. Then*

$C(m,n) \leq m(2n-m)$.

[7] Yu (2020) gives an asymptotic upper bound of $(8/25)n^2$ for $C(P_n)$) without proof. With this theorem we can find a tighter upper bound.

[Not referenced]

Proof. All orientations of $R(m,n)$ fit in $B(n^m, m^{n-m})$. Therefore $C(m,n) \leq |B(n^m, n^{n-m})| = nm + m(n-m) = m(2n-m)$. □

Theorem 154 (Gregory J. Puleo (un.) (2018a))**.** *Suppose $m \leq n$. Then $C(m,n) = mn$ if and only if m divides n, and so*

(1) $C(m,1) = m$, *with coloring* F_m

(2) $C(m,m) = m^2$, *with coloring* S_m.

[Referenced on pages 164, 168, 170 and 346]

Proof.

If. If m divides n, then $S_m \times F_{1,n}$ is a sufficient coloring with mn colors, and therefore it is a sufficient coloring, and so $C(m,n) = mn$.

Only if. Assume then $m < n$, and that we have a sufficient coloring using mn colors. Consider a $(m+1) \times (n+1)$ rectangle.

Let's say the color in the top-left corner is purple. All the colors in the leftmost n columns of the top row will be called "shades of red", and all the colors in the top m columns of the left column will be called "shades of blue", as shown in Figure 181. Purple is both a shade of red and a shade of blue.

We now look at row $m+1$. The only colors available for the leftmost n columns are shades of red. Furthermore, as $m < n$, the leftmost square in row $m+1$ cannot be purple, as this would cause a vertical rectangle with the same upper-left corner to have two purple squares. With only the shades of red available for that row, purple must appear somewhere else among the leftmost n columns in row $m+1$.

On the other hand, in column $n+1$ we can only use shades of blue, among which there must be a purple square. If the circled square does *not* use the color purple, then the lower-right $(m \times n)$ rectangle has two purple squares. Hence the circled square must be purple. Thus two squares at distance n along the same row must have the same color. Repeating the argument with rows and columns interchanged shows that two square at distance n along a column have the same color.

We have shown that if two squares lie in the same row or the same column and are exactly n squares apart on that row or column, then they must both have the same color. Note that since no intervening squares on that row or column can also have the same color, this means that every row and every column is basically colored periodically, with period n.

We next prove that m divides n.

Suppose that m does not divide n, but we have an mn-coloring. This mn-coloring is determined by its values on an $n \times n$ square.

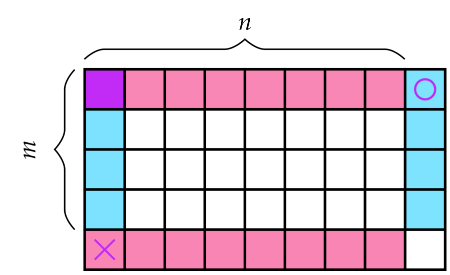


Figure 181: Proof

Let C_i be the set of colors used on the ith row of this square. We see that $C_1, \ldots, C_m$ are pairwise disjoint (these rows all being contained within an $m \times n$ rectangle), and that $C_i = C_{m+i}$ for all $i < n - m$, since C_{m+i} must be disjoint from $C_{i+1}, \ldots, C_{m+i-1}$, leaving only the n colors in C_i available. (Row $m + i$ and row i might have these colors in a different order, but they will be the same set of colors.)

If m divided n, then we'd get each of the sets $C_1, \ldots, C_m$ appearing exactly n/m times on the square. However, since m doesn't divide n, this repeating pattern of sets gets "cut off" at the bottom, and C_1 appears on some row C_{n-i} for $i < m$. Now a horizontal rectangle starting at row $n - i$ will contain two rows colored using colors from C_1 once the square repeats, contradicting the hypothesis that this is a sufficient coloring.

Therefore, if we have a sufficient coloring using mn colors, m must divide n. □

Theorem 155 (Gregory J. Puleo (un.) (2018b)[8]). *For $m > 1$, $C(m, m+1) = m(m+2)$.*

[Referenced on pages 164, 169 and 170]

Proof. Consider a $m \times (m+1)$ rectangle. Let all the top colors be *shades of red*, and the right-most color *crimson*.

Let all the colors *not* in the rectangle be shades of yellow. The cells below the rectangle must either be shades of yellow, or shades of red. But consider now a $m \times (m+1)$ rectangle with its top right corner just to the left of the crimson cell. We can see the cells must all be crimson or a shade of yellow, except the right-most one that must be purple. By moving this rectangle left, we can see the only one of those that can be crimson is the left most.

Now suppose we have less than m shades of yellow. This means neither the left-most nor right-most cell can be yellow. They must be crimson and purple respectively. This means we have a periodical coloring, with a cell the same color as those with offset $(\pm m, \pm m)$.

Therefore the tiling is periodic: every mth row consists of the $m+1$ reds and $m-1$ yellows shifted by m from the row m before. But none of the rows in between can contain any red or yellow, so we end up requiring $m(2m) = 2m^2$ colors. But if we have less than m shades of yellow, we have less than $m(m+1) + m$ colors (the colors inside the purple rectangle plus the yellows), i.e. we have less than $m^2 + 2m$ colors. But for $m > 1$, this is less than $2m^2$, which is a contradiction. Therefore we need at least m shades of yellow, and so we need at least $m(m+2)$ colors in total.

But we also have $C(m, m+1) \leq m(m+2)$ (Theorem 157), and since we need at least $m(m+2)$ colors, we need exactly $m(m+2)$ colors.

[8] Some details of the proof have been provided by Peter Taylor; see the reference for more information.

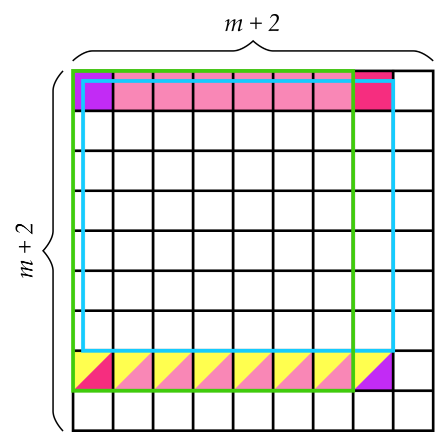


Figure 182: Proof

The coloring $F_{m(m+1)-1,m(m+2)}$ is sufficient, and since it uses $m(m+2)$ colors it is also optimal. $\square$

Theorem 156. *Suppose $P = aQ$. Then $C(P) = a^2C(Q)$. If K is the optimal coloring for Q, then $K \times S_a$ is the optimal coloring for P.*

[Not referenced]

Theorem 157. *Assume $m \leq n$. Then $C(m,n) \leq (m')^2k$ where $m' \geq m$ and $m'k \geq n$, with $S_{m'} \times F_k$ a sufficient coloring.*

[Referenced on page 167]

Proof. If $m' \geq m$ and $m'k \geq n$, then $R(m,n) \subset R(m', m'k)$, and so $C(m,n) \leq C(m', m'k)$ (Theorem 148), and so $C(m,n) = (m')^2k$ (Theorem 154). $\square$

Theorem 158 (Peter Taylor (un.) (2018)). $C(am, an) \leq a^2C(m,n)$

[Not referenced]

Proof. Let K be an optimal coloring for $R(m,n)$. Define $K'(x,y) = a^2K(\lfloor \frac{x}{a} \rfloor, \lfloor \frac{x}{a} \rfloor) + (x \bmod a) + a(y \bmod a)$. This coloring uses $a^2C(m,n)$ colors.

It is easy to see K' is sufficient for $R(am, an)$, and therefore $C(am, an) \leq a^2C(m,n)$. $\square$

Theorem 159. $C(k, 2k-1) = 2k^2$

[Referenced on pages 164 and 168]

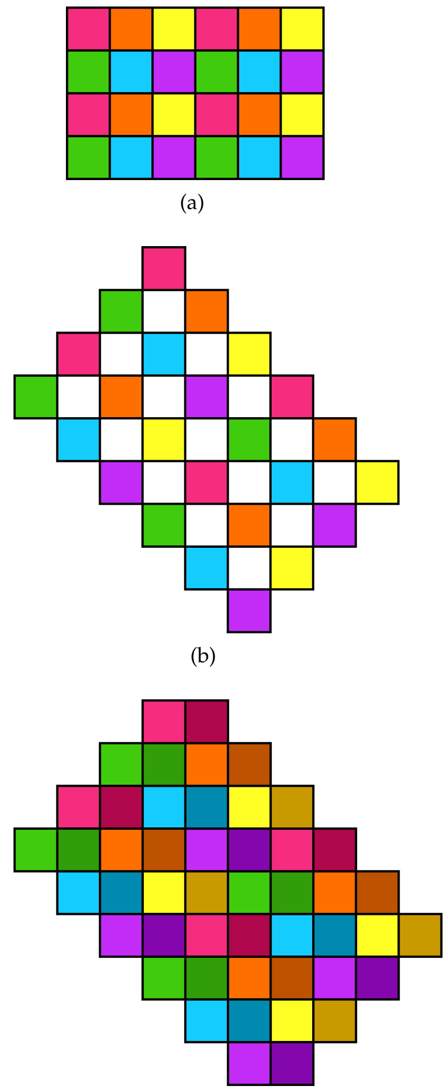
Proof. $R(k, 2k-1) \subset R(k, 2k)$, so $C(k, 2k-1) \leq C(k, 2k) = 2k^2$. So all we need to show is that $2k^2 - 1$ colors are not enough.

Towards a contradiction, suppose it *is* enough, and that K is an optimal coloring. Let $\delta_{xy} = x + y \bmod 2$, and define a new coloring as follows:

$$K'(x,y) = \begin{cases} K\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}\right) & \text{for } (x+y) \equiv 0 \pmod 2 \\ K'(x-1,y) + k^2 - 1 & \text{for } (x+y) \equiv 1 \pmod 2 \end{cases}$$

Notice that

- K' has twice the number of colors as K, so it has $4k^2 - 2$ colors.
- Two cells (x,y) and (x',y') can only have the same color if $x + y \equiv x' + y' \pmod 2$.



(a) (b) (c)

Figure 183: An example showing the coloring transformation used in Theorem 159. The first image show (an example) of original coloring K. The second image the new coloring for cells with $x + y$ (other cells are left blank). The final image shows the complete new coloring, with a darker shade of a color i used for color $i + k^2 - 1$.

- If two cells have the same color in K', then two corresponding cells have the same color in K.

We will now show K is an sufficient coloring for $R(2k, 2k-1)$. Suppose it is not; then there are two cells $a = (x, y)$ and $'a = (x', y')$ that can be covered with $R(2k, 2k-1)$. These cells must satisfy $x + y \equiv x' + y' \pmod 2$, and WLOG assume $x + y \equiv x' + y' \equiv 0 \pmod 2$.

Now if a and a' lie inside the same rectangle $R(2k, 2k-1)$, then a and a' lie inside the same rectangle $R(k, 2k-1)$, which means K cannot be sufficient, a contradiction. Therefore, K' must be sufficient.

Now K' uses $4k^2 - 2$ colors, but by Theorem 155 $C(2k, 2k-1) = 4k^2 - 1$; this is impossible (no sufficient coloring can use less colors than the chromatic number, which is minimal by definition). This means then that $2k^2 - 1$ colors is not enough for $R(k, 2k-1)$, which is what we wanted to show.

Therefore, $C(k, 2k-1) = 2k^2$. □

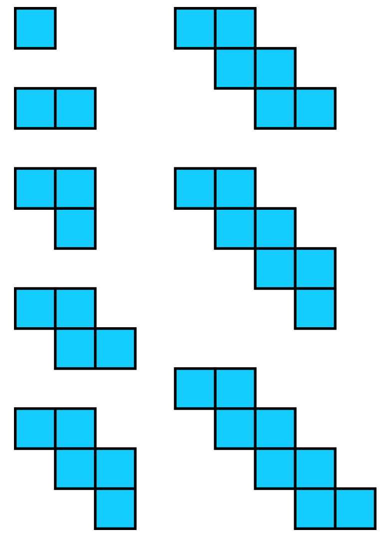


Figure 184: The first 8 W-polyominoes

Problem† 62. *The theorem above "scaled" the coloring above by a factor of two, so that we could deduce constraints on the optimal colorings for $R(k, 2k-1)$ from $R(2k, 2k-1)$ (rectangles double the size).*

It is worth investigating why this idea cannot be modified to work in other cases:

- *We can the double coloring not work to make deductions for $R(2k, 2k+1)$?*
- *Why can we not use the trick above twice, for example, to deduce something for $R(k, 4k+1)$ from $R(2k, 4k+1)$?*
- *The transformation preserved squares (in the sense that a square of colors from the same colors will transform into a square of same colors in the new coloring). Why is this necessary?*
- *There are not square-preserving transformations that are not simply scale-transformations for all factors. What factors are possible?*
- *Why can other factors than 2 not work; for example, to try to deduce something for $R(k, 5k-1)$ from $R(5k, 5k-1)$?*
- *We can use a forward and backward translation to get rational factors that does not have the problem of other integer factors. Why can this idea not work?*

Problem* 63 (Conjecture[9]). *Prove that for $m \le n$*

$$C(m, n) = \begin{cases} mn - m^2 & \text{if } m < \sqrt{2}n \\ m^2 \left\lceil \frac{n}{m} \right\rceil & \text{otherwise.} \end{cases}$$

[9] Since formulating this conjecture I discovered that it is not completely true. The first part seems true (when $m < \sqrt{2}n$, but the last part is false in between a third and a half of cases.

Note that we already proved it for when $m|n$, when $n = m - 1$, and when $n = 2m - 1$.

A general **W-polyomino**, which we will denote by W_n, is a snake with n cells that alternate in two directions.[10] W-polyominoes are weak. W polyominoes with odd area are unusual because they are not efficient.

[10] Scherphuis (2016) discovered that the set of tiles $W_1, W_2, \cdots W_8$ make an interesting puzzle. The set tiles many symmetric figures, including $R(6,6)$, $R(4,9)$, $B(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8)$, $B(3 \cdot 4 \cdot 5 \cdot 6^2 \cdot 5 \cdot 4 \cdot 3)$, $B(5 \cdot 6 \cdot 7^2 \cdot 6 \cdot 5)$. He calls these polyominoes *zig-zag polyominoes*.

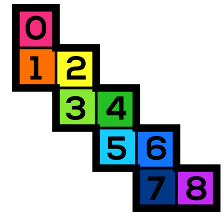


Figure 185: Coloring the first cells.

Theorem 160. *For W_n, and let $k = \lceil n/2 \rceil$. We need $2k$ colors, and an optimal coloring is given by $F_{2k,k}$.*

[Referenced on page 170]

Proof. It is clear that $F_{2k,k}$ has enough colors. Since it has $2k$ colors, and $2k = n$ for n even, it follows that $C(P) = n$ for n even. It remains to show that for odd n, when $2k = n + 1$, we need $n + 1$ colors.

Since $W_n \sqsubset W_{n+1}$, it follows that $C(W_n) \leq C(W_{n+1})$ (Theorem 148), so for odd n, $C(W_n) \leq C(W_{n+1}) = n + 1$. It is therefore enough to show that n colors are not enough.

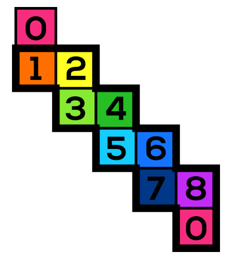


Figure 186: One color is forced.

Let's assume we need only n colors. Place the W-polyomino anywhere, and color the cells that it covers from 0 to $n - 1$.

Now place the W-polyomino so that it covers all the same cells except the one with color 0. The uncolored cell must be of color 0 (since we have only n colors and all the other colors are used already).

Repeat with color 1.

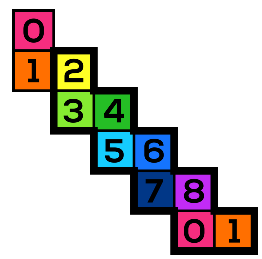


Figure 187: Another color is forced.

We can continue in this way, but for this proof we need to go only this far.

Now put the W-polyomino so that it covers the first few cells with even color, but none with odd colors. The cells that are not colored yet must be colored with the odd colors, one of each (again, because we only have n colors). In particular, one of the cells must be color 1. It cannot be the cell next to color 0, since it lies in the same W shape as the cell below which also has color 1. Therefore, one of the remaining cells must be color 1. Those cells are marked orange in Figure 188.

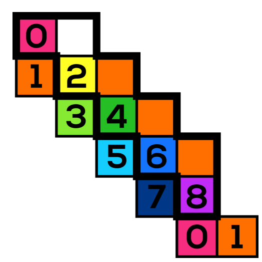


Figure 188: One of the unnumbered orange cells must be color 1.

Finally, move the W-polyomino one cell down and one to the right. It now covers all three orange cells (one of which must be color 1), and another cell with color 1. This violates the condition of this being a legal coloring, which means, we cannot get away with only n colors.

□

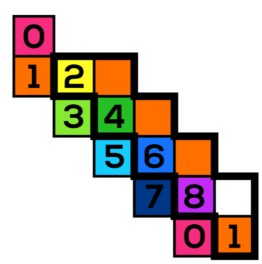
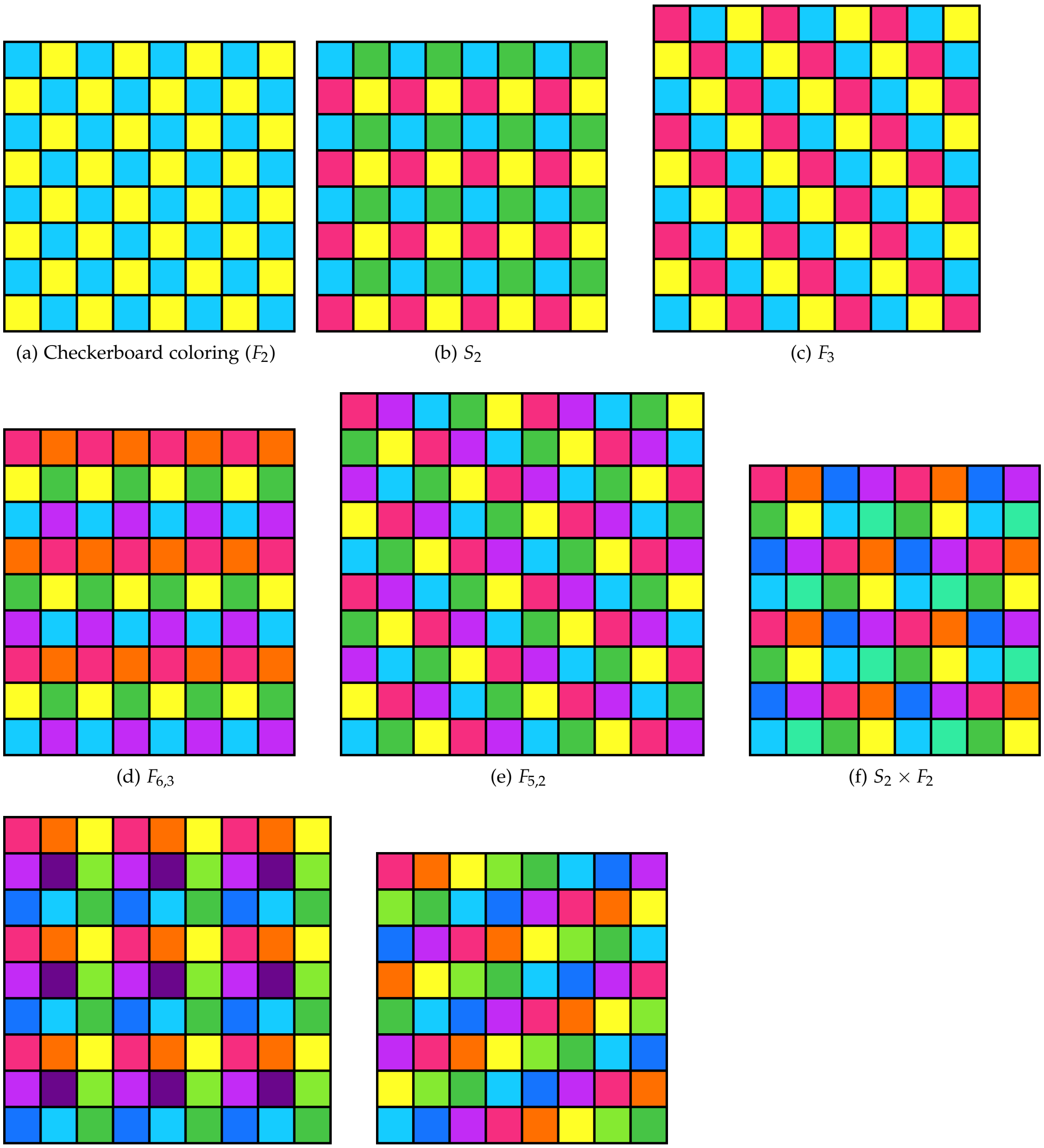


Figure 189: Two of the same colors (color 1) must occur in the same W. Therefore, we need at least one color more.



(a) Checkerboard coloring (F_2) (b) S_2 (c) F_3 (d) $F_{6,3}$ (e) $F_{5,2}$ (f) $S_2 \times F_2$ (g) S_3 (h) $F_{8,5}$

Figure 190: Some standard colorings.

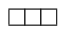
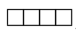
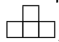
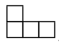
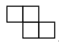
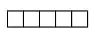
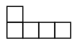
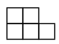
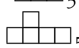
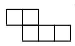
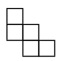
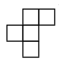
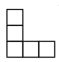
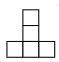
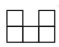
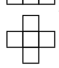
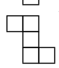
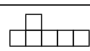
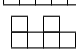
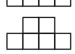
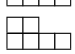
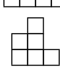
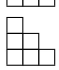
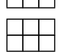
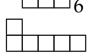
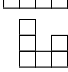
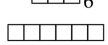
Polyomino P	$C(P)$		Theorems
$_1$	1	F_1	154(1)
$_2$	2	F_2	154(1)
$_3$	3	F_3	154(1)
$_3$	4	S_2	149
$_4$	4	F_4	154(1)
$_4$	4	S_2	154(2)
$_4$	5	$F_{5,2}$	146 ($Q = X_5$), 151 ($Q = X_5$)
$_4$	8	$S_2 \times F_2$	149 ($Q = R(3,2)$), 145
$_4$	4	$F_{4,2}$	160
$_5$	5	F_5	154(1)
$_5$	8	$S_2 \times F_2$	149
$_5$	8	$S_2 \times F_2$	148 ($Q = L_4$), 145
$_5$	8	$S_2 \times F_2$	148 ($Q = L_4$), 145
$_5$	8	$S_2 \times F_2$	148 ($Q = L_4$), 145
$_5$	6	$F_{6,3}$	160
$_5$	8	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
$_5$	9	S_3	149
$_5$	8	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
$_5$	8	$S_2 \times F_2$	148 ($Q = L_4$), 145
$_5$	5	$F_{5,2}$	152
$_5$	9	$S_2 \times F_2$	146 ($Q = R(3,3)$), 145
$_6$	[8*,10]	$F_{10,4}$	Search, coloring
$_6$	8	$S_2 \times F_2$	146 ($Q = L_4$), 145
$_6$	8	$S_2 \times F_2$	146 ($Q = L_4$), 145
$_6$	8	$S_2 \times F_2$	146 ($Q = L_4$), 145
$_6$	(8)	$S_2 \times F_2$	148 ($Q = T_5$)
$_6$	9	S_3	148 ($Q = V_5$), 145
$_6$	8	$S_2 \times F_2$	155
$_6$	12	$S_2 \times F_3$	149
$_6$	9	S_3	148 ($Q = V_5$), 145
$_6$	6	F_6	154(1)

Table 24: Chromatic numbers for small polyominoes. Intervals $[a, b]$ denote lower and upper bounds. Numbers in brackets show the upper bound is provided by an explicit coloring rather than a theorem.

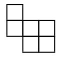
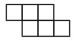
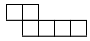
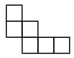
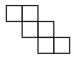
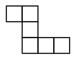
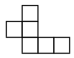
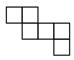
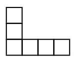
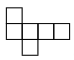
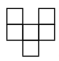
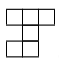
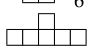
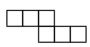
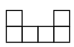
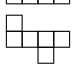
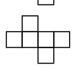
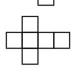
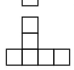
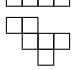
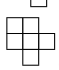
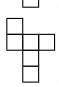
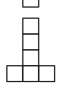
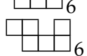
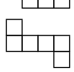
Polyomino P	$C(P)$		
 6	9	S_3	148 ($Q = Z_5$), 145
 6	8	$S_2 \times F_2$	148 ($Q = L_4$), 145
 6	8	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	(9)	S_3	146 ($Q = V_5$), Coloring
 6	6	$F_{6,3}$	160
 6	(9)	S_3	148 ($Q = V_5$), Coloring
 6	(9)	S_3	148 ($Q = V_5$), Coloring
 6	(9)	S_3	148 ($Q = Z_5$), Coloring
 6	15	$F_{15,11}$	149
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	9	S_3	148 ($Q = Z_5$), 145
 6	(8)	$F_{8,5}$	148 ($Q = L_4$), Coloring
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	8	$S_2 \times F_2$	148 ($Q = L_4$), 145
 6	[9*, 13]	$F_{13,5}$	Search, Coloring
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	[9*, 10]	($F_{10,3}$)	Search, Coloring
 6	(9)	S_3	146 ($Q = V_5$), Coloring
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	8	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 6	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring

Table 25: Chromatic numbers for small polyominoes.

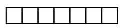
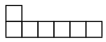
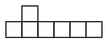
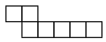
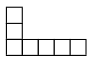
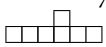
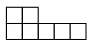
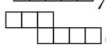
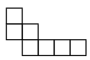
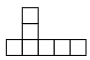
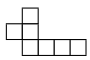
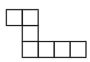
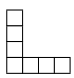
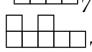
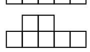
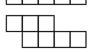
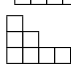
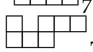
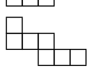
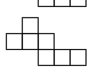
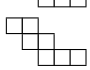
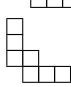
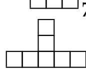
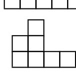
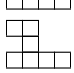
Polyomino P	$C(P)$		
 $_7$	7	F_7	154(1)
 $_7$	12	$S_2 \times F_3$	149
 $_7$	12	$S_2 \times F_3$	148 ($Q = L_6$), 145
 $_7$	12	$S_2 \times F_3$	148 ($Q = L_6$), 145
 $_7$	18	$S_3 \times F_2$	149
 $_7$	[8,* 12]	$F_{12,5}$	Search, Coloring
 $_7$	12	$S_2 \times F_3$	148 ($Q = L_6$), 145
 $_7$	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 $_7$	[8*, 16]	S_4	Search, Coloring
 $_7$	(15)	$F_{15,4}$	Coloring
 $_7$	(15)	$F_{15,4}$	148 ($Q = L(3,4)$), Coloring
 $_7$	(15)	$F_{15,4}$	148 ($Q = L(3,4)$), Coloring
 $_7$	16	S_4	149
 $_7$	12	$S_2 \times F_3$	148 ($Q = L(2,5)$), 145
 $_7$	[8*, 10]	$F_{10,4}$	Search, Coloring
 $_7$	(8)	$S_2 \times F_2$	148 ($Q = L(2,5)$), Coloring
 $_7$	15	$F_{15,4}$	148 ($Q = L(3,4)$), 145
 $_7$	[8*, 12]	$F_{12,5}$	Search, Coloring
 $_7$	[8*, 9]	S_3	Search, Coloring
 $_7$	[8*, 9]	S_3	Search, Coloring
 $_7$	[8*, 9]	S_3	Search, Coloring
 $_7$	[8*, 12]	$F_{12,4}$	Search, Coloring
 $_7$	[9*, 10]	$F_{10,3}$	Search, Coloring
 $_7$	[9*, 10]	$F_{10,5}$	Search, Coloring
 $_7$	[9*, 10]	$F_{10,3}$	Search, Coloring

Table 26: Chromatic numbers for small polyominoes.

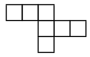
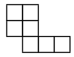
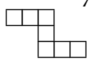
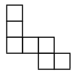
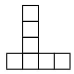
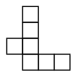
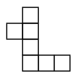
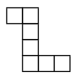
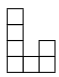
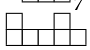
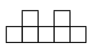
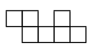
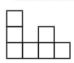
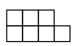
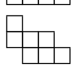
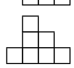
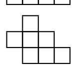
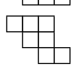
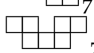
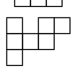
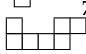
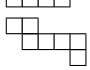
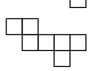
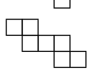
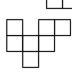
Polyomino	Class		
 $_7$	(9)	S_3	148 ($Q = V_5$), Coloring
 $_7$	(9)	S_3	148 ($Q = V_5$), Coloring
 $_7$	(9)	S_3	148 ($Q = V_5$), Coloring
 $_7$	[9*, 12]	$F_{12,4}$	Search, Coloring
 $_7$	(15)	$F_{15,4}$	Coloring
 $_7$	(15)	$F_{15,4}$	148 ($Q = L(3,4)$), Coloring
 $_7$	(15)	$F_{15,4}$	148 ($Q = L(3,4)$), Coloring
 $_7$	(15)	$F_{15,4}$	148 ($Q = L(3,4)$), Coloring
 $_7$	15	$F_{15,4}$	148 ($Q = L(3,4)$), 145
 $_7$	12	$S_2 \times F_3$	148 ($Q = L(2,5)$), 145
 $_7$	[8*, 10]	$F_{10,4}$	Search, Coloring
 $_7$	(8)		148 ($Q = L_4$), Coloring
 $_7$	15	$F_{15,4}$	148 ($Q = L(3,4)$), 145
 $_7$	8	$S_2 \times F_2$	148 ($Q = L_4$), 145
 $_7$	(9)	S_3	148 ($Q = V_5$), Coloring
 $_7$	[9*, 10]	$F_{10,3}$	Search, Coloring
 $_7$	(9)	S_3	148 ($Q = V_5$), Coloring
 $_7$	(9)	S_3	148 ($Q = V_5$), Coloring
 $_7$	[8, 10]	$F_{10,4}$	Search, Coloring
 $_7$	[8, 15]	$F_{15,4}$	Search, Coloring
 $_7$	[8, 12]	$F_{12,5}$	Search, Coloring
 $_7$	(8)	$S_2 \times F_2$	Coloring
 $_7$	[8*, 16]	$S_2 \times F_{4,2}$	Search, Coloring
 $_7$	(9)	S_3	148 ($Q = V_5$), Coloring
 $_7$	[8, 15]	$F_{13,5}$	Search, Coloring

Table 27: Chromatic numbers for small polyominoes.

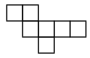
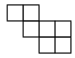
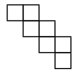
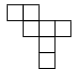
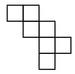
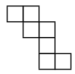
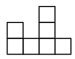
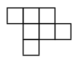
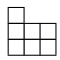
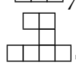
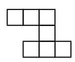
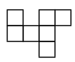
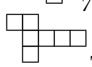
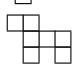
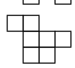
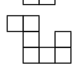
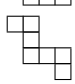
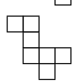
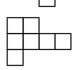
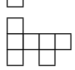
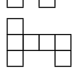
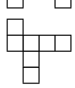
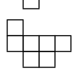
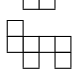
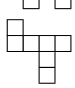
Polyomino P	$C(P)$		
 7	[8, 18]		
 7	(9)	S_3	148 ($Q = V_5$), Coloring
 7	8	$F_{8,4}$	160
 7	[8, 16]		
 7	[8, 16]		
 7	[9, 16]		
 7	[9, 15], 12?		
 7	[8, 15]		
 7	(9)	S_3	148 ($Q = V_5$), 145
 7	[9, 15]		
 7	(9)	S_3	148 ($Q = V_5$), Coloring
 7	[9, 15]		
 7	[8, 18]		
 7	(9)	S_3	148 ($Q = V_5$), Coloring
 7	(9)	S_3	148 ($Q = V_5$), Coloring
 7	(9)	S_3	148 ($Q = V_5$), Coloring
 7	[9, 16], 15?		
 7	(9)	S_3	148 ($Q = V_5$), Coloring
 7	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 7	(9, 15)		
 7	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 7	[9, 16]		
 7	[9, 15]		
 7	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 7	[11?, 16]		

Table 28: Chromatic numbers for small polyominoes.

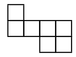
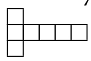
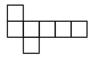
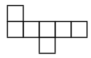
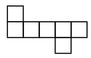
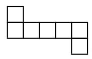
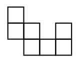
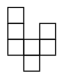
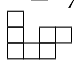
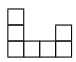
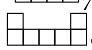
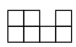
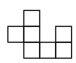
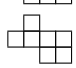
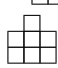
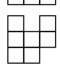
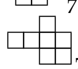
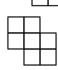
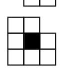
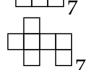
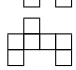
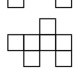
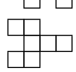
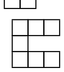
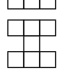
Polyomino	Class		
$_7$	[9, 15]		
$_7$	[12, 18]		
$_7$	[12, 18]		
$_7$	[12, 18]		
$_7$	[12, 18]		
$_7$	[12, 18]		
$_7$	[8, 15]		
$_7$	[8, 9]		
$_7$	[9, 15]		
$_7$	15	$F_{15,4}$	148 ($Q = L(3,4)$), 145
$_7$	12	$S_2 \times F_3$	148 ($Q = L(2,5)$), 145
$_7$	8	$S_2 \times F_2$	148 ($Q = L_4$), 145
$_7$	[9, 15]		
$_7$	[8, 15]		
$_7$	8	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
$_7$	9	S_3	148 ($Q = Z_5$), 145
$_7$	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
$_7$	9	S_3	148 ($Q = V_5$), 145
$_7$	9	S_3	148 ($Q = V_5$), 145
$_7$	[8, 15]		
$_7$	[9, 15]		
$_7$	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
$_7$	8	$S_2 \times F_2$	
$_7$	9	S_3	148 ($Q = V_5$), 145
$_7$	9	S_3	148 ($Q = V_5$), 145

Table 29: Chromatic numbers for small polyominoes.

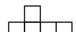
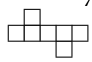
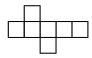
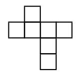
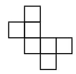
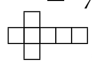
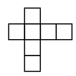
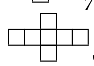
Polyomino P	$C(P)$		
 7	(8)	$S_2 \times F_2$	148 ($Q = L_4$), Coloring
 7	[9, 18]		
 7	[8, 18]		
 7	[9, 16]		
 7	[9, 16]		
 7	[8, 18]		
 7	[9, 16]		
 7	[8, 18]		

Table 30: Chromatic numbers for small polyominoes.

Problem† 64. *Show that*

(1) *the only efficient pentominoes are the I- and X-pentominoes;*

(2) *the only efficient hexominoes are the I- and W-hexominoes; and*

(3) *the only efficient heptomino is the I-heptomino.*

While efficient polyominoes are relatively scarce for polyominoes with 7 or less cells, there are *lots* with 8 or 9 colors; the problem below explores why.

Problem† 65.

(1) *Find a coloring $F_{n,k}$ that is not the optimal coloring of any efficient polyomino.*

(2) *Show that any polyomino formed from k^2 cells of different colors from a square coloring S_k has that square coloring as an optimal coloring, and hence such an polyomino is efficient.*

(3) *Show that any polyomino formed from $2k^2$ cells of different colors from a coloring $F_2 \times S_k$ has that coloring as an optimal coloring, and hence such an polyomino is efficient.*

(4) *Find examples that show the above in general does not hold for polyominoes with mk^2 cells of different colors from $F_{m,n} \times S_k$.*

(5) *How many octominoes are efficient?*

(6) *Can you say which decominoes are efficient?*

Problem* 66.

(1) *Is every optimal coloring periodic?*

(2) *Does every polyomino have an optimal coloring of the form $F_{m,n} \times S_k$?*

(3) *We considered colorings where each polyomino cover each color at most once. What about colorings where each polyomino covers each color at most k times? At least k times?*

(4) *Complete the tables by finding the exact chromatic number for the unknown hexominoes and heptominoes.*

The notion of chromatic number can be extended to tile sets. The chromatic number $C(\mathcal{T})$ of a tile set $\mathcal{T}$ is the minimum colors we need to color the plane so that no two cells have the same color when covered by any of the tiles.

The **taxicab** distance $d_1(p,q)$ between two points p and q is defined as

$$d_1(p,q) = |p_x - q_x| + |p_y - q_y|.$$

Theorem 161 (Milo Brandt (un.) (2019)). *The minimum taxicab distance between two different cells of the same color in a $F_{m,n}$ coloring is given by $d_1(m,n) \leq \sqrt{2m}$.*

[Not referenced]

Proof. See the reference for a sketch. □

In the proof of the next theorem, we make use of the following shape: a **diamond** $D(n)$ is all the squares within rook-distance $n-1$ from a central cell. The diamond of radius 1 has one cell, the diamond of radius 2 has 5 cells, and the diamond of radius 3 has 13 cells, and in general $1 + 2n(n-1)$ cells (Figure 191).

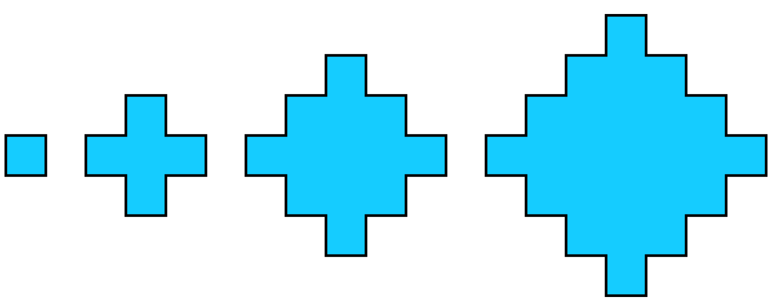


Figure 191: Diamonds $D(1)$, $D(2)$, $D(3)$ and $D(4)$.

We will also make use of **augmented diamonds** $D'(n)$, which is $D(n)$ but with one cell extra (on the right) in each row. $D'(n)$ has $2n^2$ cells. Examples are shown in Figure 192.

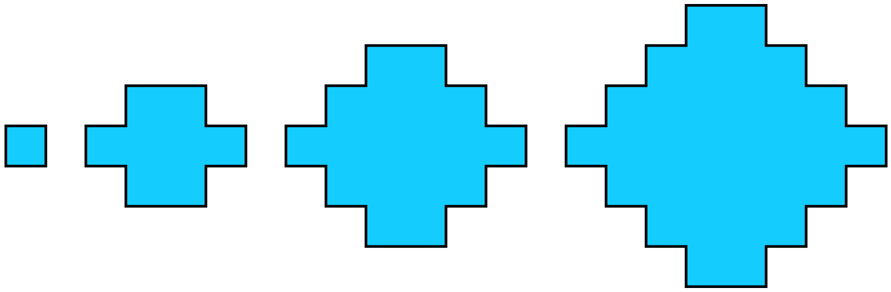


Figure 192: Diamonds $D'(1)$, $D'(2)$, $D'(3)$ and $D'(4)$.

Theorem 162 (joriki (un.) (2012) [11]).

$$C(\mathcal{P}_n) = \begin{cases} \dfrac{n^2+1}{2} & n \text{ odd} \\ \dfrac{n^2}{2} & \text{even} \end{cases}$$

[Not referenced]

[11] The question was posed by TROLL-HUNTER (un.) (2012).

Proof.

Case n odd. Consider $D((n+1)/2)$ with $(n^2+1)/2$ cells. There is a polyomino in $\mathcal{P}_n$ that cover any two cells in this diamond, so they must all have different colors. We therefore need at least $(n^2+1)/2$. To see that this is also enough, consider the coloring

$$F_{(n^2+1)/2,n}(x,y) = (x + ny) \bmod (n^2+1)/2.$$

The closest taxicab distance between cells of the same color is given by $d((n^2+1)/2, n) > n-1$, so these cells (or any two cells of the same color) cannot be covered by the same n-omino.

Case n even. Consider $D'(n/2)$ with $n^2/2$ cells. There is a polyomino that covers any two cells of the augmented diamond, so we need at least $n^2/2$ colors.

To see that this is also enough, consider the coloring

$$F_{n^2/2,n-1}(x,y) = (x + (n-1)y) \bmod (n^2/2).$$

The closest taxicab distance between different cells of the same color is $d(n^2/2, n-1) > n-1$, so two cells of the same color cannot be covered by the same polyomino. □

n	$C(\mathcal{P}_n)$	Coloring
1	1	$F_{1,1}$
2	2	$F_{2,1}$
3	5	$F_{5,3}$
4	8	$F_{8,3}$
5	13	$F_{13,5}$
6	18	$F_{18,5}$
7	25	$F_{25,7}$

Table 31: The chromatic numbers and optimal colorings for $\mathcal{P}_n$.

The idea in the proof above can be extended to other shapes:

(1) Find a shape with k cells such that any two cells can be covered by an element from the set.

(2) Find a coloring with distance d, larger than the taxicab distance between any two cells that has k' colors.

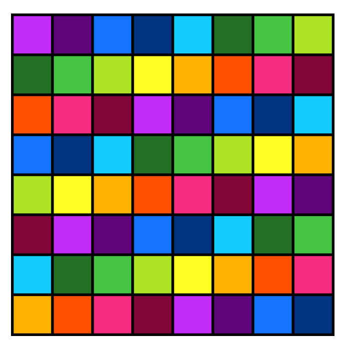


Figure 193: $F_{13,5}$ is the optimal coloring for pentominoes.

We then need at least k colors and k' is sufficient. If we get these to match we are done.

Example 17. *What is the minimum amount of colors we need for the P-pentomino and straight-tetromino?*

$D'(2)$ can be protected by them, so we need at least 8 colors. And indeed $F_{8,3}$ has distance 4, which is more than the distance between any two cells in either polyomino.

This technique does not always work.

Problem† 67. *What is the minimum amount of colors we need for the V-pentomino and straight tetromino?*

4.3 *Border Words*

In this section, definitions loosely follow those in Blondin-Massé et al. (2009).

Simply-connected polyominoes can be uniquely defined by their border; and the border can be described as a set of unit steps in one of four directions. Let's use x for horizontal steps, and y for vertical steps. And let's use an exponent for the amount and direction; positive for right and up, negative for left and down. Finally, let's decide to always go anticlockwise around the polyomino with the inside on the left. We call this sequence of steps a **word**.

With this, one way to describe a monomino is $xyx^{-1}y^{-1}$. We can also start with another vertex, giving us a cyclically shifted word such as $yx^{-1}y^{-1}x$. We will not distinguish between words cyclically shifted. Fig 32 shows the border words for the tetrominoes.

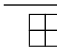
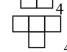
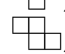
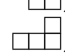
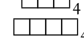
Polyomino	Border word
 4	$x^2y^2x^{-2}y^{-2}$
 4	$xyxyx^{-3}y^{-1}xy^{-1}$
 4	$x^2yx^{-1}yx^{-2}y^{-1}xy^{-1}$
 4	$x^3y^2x^{-1}y^{-1}x^{-2}y^{-1}$
 4	$x^4yx^{-4}y^{-1}$

Table 32: Border words for tetrominoes.

Problem† 68.

(1) *What is the border word of $R(m,n)$?*

(2) *What is the border word for $B(m^a \cdot n^b)$?*

(3) *Prove that sum of exponents of x equals 0, and that the sum of exponents of y equals 0.*

(4) *Suppose the word that describes a segment is $x^a y^b$; what is the word that describes the same segment traveled in reverse?*

The **reverse** of a word $W = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_n}y^{b_n}$ is defined as $\hat{W} = y^{-b_n}x^{-a_n}\cdots y^{-b_2}x^{-a_2}y^{-b_1}x^{-a_1}$; that is, we reverse the order of the steps and change the direction of all steps.

If W is the border word of a polyomino, then $\hat{W}$ is the word of an infinite region with a hole the same shape as the original polyomino.

If $W = \hat{W}$, then we call W a **palindrome**. Palindromes denote cyclosymmetric curves.

Problem† 69. *Suppose $W = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_n}y^{b_n}$ is a polyomino. What happens if we*

(1) *scale all exponents of x by a factor k?*

(2) *scale all exponents by a factor k?*

(3) *change the signs of all exponents of x?*

(4) *change the signs of all exponents of x and y?*

(5) *swap x and y?*

Can you write conditions on the border word of a polyomino for which type of symmetry the polyomino has?

4.3.1 Further Reading

The combinatorial aspects of words is treated thoroughly in Lothaire (1997); for a tutorial on the topic see Berstel and Karhumäki (2003). For applications to tiling, see for example Blondin-Massé et al. (2009) and Langerman and Winslow (2015). The paper Conway and Lagarias (1990) and surrounding work that I mentioned in the chapter on domino tilings also use border words, although the formulation there is algebraic.

5 Rectangles

We saw that determining whether a region can be tiled by dominoes can be determined efficiently (Theorem 96). But in general, this is not the case. In fact, it is hard to determine if a polyomino will even tile a rectangle.

Consider the right tromino. It is easy to see it will tile $R(2,3)$, and thus all rectangles $R(2m,3n)$ (by Theorem 22). Are these the only rectangles that it will tile? It turns out we can also tile $R(5,9)$ (see Figure 253), and so also $R(5m,9n)$. Have we found all the rectangles it will tile? Not yet! For we can stack $R(2,6)$ and $R(3,6)$ together to tile $R(5,6)$, which adds $R(5m,6n)$ to the mix. Combined with the $R(5m,9n)$, this allows us to tile $R(5m,3n)$ for $n \geq 2$. And combining this in turn with $R(2m,3n)$ allows us to tile $R(m,3n)$ for $m = 2$ and all $m > 3$ and $n > 2$. Thus, we can tile almost all rectangles with area divisible by three (the only exceptions are $R(m,3)$ for odd m — which are in fact not tileable as we will see later).

The tromino case was easy enough to analyze. We found two rectangles, called *prime rectangles*, from which all other tilings can be derived. But in other cases, the analysis is much more difficult. For example, the Y-pentomino requires a set of 40 rectangles before we can construct the set of all rectangles.

And what is really surprising, is how difficult it is to know which rectangles can be tiled by a set of rectangles. This is the topic of the first section.

The minimum number of polyominoes that can tile a rectangle is called the *order* of the polyomino. We know very little about what orders are possible. We know orders 2 and 4 are possible (and in fact ubiquitous), and we know order 3 is impossible. But we do not know whether 5, or 6, or 7 are possible. We know all orders of the form $4k$ are possible, but other than that, the highest order polyomino we know is 246. This is the topic of the second section.

The third section deals with prime rectangles and proofs of polyominoes that don't tile rectangles.

The fourth section deals with fault-free tilings of polyominoes. While we have a nice solution for when the polyomino is a rectangle, in general the picture is murky. The easy cases are usually those with a small number of prime rectangles.

The final section deals with simple tilings — tilings that have no subtilings with more than one tile that are rectangles. There are simple tilings of rectangles by any number of pieces larger than for, except[1] for 6.

[1] This is the second time 6 shows up out of the blue as an exception — recall that $R(m,n)$ has fault free tilings form $m,n > 4$, except for $R(6,6)$ (Theorem 125). This is probably a coincidence.

5.1 Tilings by Rectangles

5.1.1 Which rectangles can be tiled by a set of rectangles?

Many problems can be broken into simpler problems. One natural way to approach a polyomino tiling problem is to break regions into rectangles. Rectangles are the simplest figures, and therefore we can say a lot about them. This section is primarily about how rectangles can tile other rectangles. (So, if we know that a polyomino tiles certain rectangles, we can answer which other rectangles it may tile.) Unfortunately, this is a rather hard problem to solve in general.

In this section throughout $\mathcal{T} = \{R(p_i, q_i)\}$ is a set of rectangles, and $R = R(m,n)$ is some rectangle we wish to tile using tiles from the set *with translation only*. If you have a problem calling for both orientations of a rectangle, include a copy of both in the tile set. So in this section, the tile sets $\{R(m,n)\}$ and $\{R(m,n), R(n,m)\}$ are different in general.

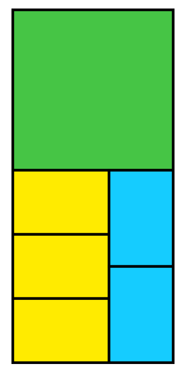


Figure 194: The smallest rectangle that requires all three tiles from the set $R(2,3)$, $R(3,2)$ and $R(5,5)$.

We will prove a bunch of theorems; Table 35 at the end of the section summarizes which theorem applies in each situation.

We begin by stating obvious necessary conditions. Let $\mathcal{T} = \{R(p_i, q_i)\}$. Then $\mathcal{T}$ can tile $R(m,n)$ when all the following conditions are met:

(1) $m = \sum p_i a_i$ for some integers $a_i \geq 0$ (Theorem 20)

(2) $n = \sum q_i b_i$ for some integers $b_i \geq 0$ (Theorem 20)

(3) $mn = \sum p_i q_i c_i$ for some integers $c_i \geq 0$ (Theorem 1)

Example 18.

- *If* $\mathcal{T} = \{R(2,3), R(3,2)\}$*, and* $R = R(5,5)$*, then conditions (1) and (2) hold, but not (3).*
- *If* $\mathcal{T} = \{R(2,3), R(3,3)\}$*, and* $R = R(3,5)$*, then conditions (1) and (3) hold, but not (2).*

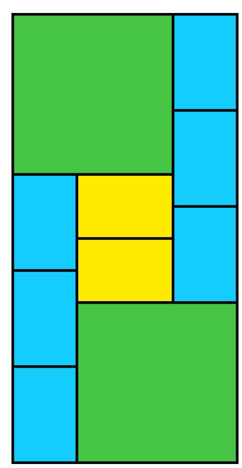


Figure 195: The smallest "interesting" rectangle that requires all three tiles from the set $R(2,3)$, $R(3,2)$ and $R(5,5)$.

We already proved the most basic rectangle tiling theorem (Theorem 22): A tiling of $R(m,n)$ by $R(p,q)$ exist if and only if $p \mid m$

and $q \mid n$. The rest of this section give theorems for larger sets of rectangles.

Theorem 163 (Klarner (1969), Theorem 5)**.** *Let $\mathcal{T} = \{R(p,1), R(1,p)\}$. A tiling exists if and only if p divides m or n.*

[Referenced on pages 185, 186 and 244]

Proof.

If. Suppose WLOG that p divides m, so that $m = pk$. A tiling exists by Theorem 22.

Only if. Apply the flag coloring with p colors (that is, each cell with coordinates x, y gets the color $(x + y) \bmod p$).

No matter how we place the bars, each covers one each of the p colors. So if the tiling requires k tiles, a tiling will have k of each color.

Now suppose $m \bmod p = m' > 0$ and $n \bmod p = n' > 0$, and WLOG that $m' \leq n'$. We can divide the rectangle into 4 quadrants: $R(m - m', n - n')$, $R(m - m', n')$, $R(m', n - n')$ and $R(m', n')$. The first three have tilings by the bars (by the first part of the theorem) and so each have the same number of each color. The last rectangle, however, has m' of color 0, but $m' - 1$ of color 1 (see Figure 196). Therefore, a tiling is impossible, and therefore either m' or n' must be 0, and so either m or n must be divisible by p. □

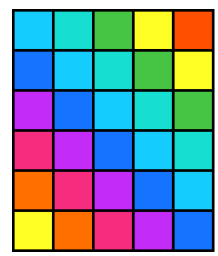


Figure 196: An example with $m' = 5$, $n' = 7$, and $p = 8$. There are 5 cells of light blue, but only 4 cells of green-blue.

Theorem 164 (Klarner (1969), Corollary of Theorem 5)**.** *Let $\mathcal{T} = \{R(p,q), R(q,p)\}$. A tiling exists if and only if each of p and q divide either m or n, and if pq divides one of side, the other side can be expressed as $px + qy$ with $x, y \geq 0$.*

[Referenced on pages 195, 196 and 227]

Proof.

If. To prove there is a tiling when the conditions hold, there are two cases to consider:

(1) $p|m$ and $q|n$

(2) $p|m$, $q|m$ and $n = px + qy$.

Suppose WLOG that $p < q$. In the first case a tiling exists by Theorem 22. In the second case, we can partition the rectangle in two rectangles: $R(m, px)$ and $R(m, nq)$. The first is tileable by $R(q, p)$ by Theorem 22; the second is tileable by $R(p, q)$ also by Theorem 22. The subregions of the partition is tileable, and therefore so is $R(m, n)$ (Theorem 2).

Only if. If $R(m,n)$ has a tiling by $\mathcal{T}$, then it must also have a tiling by $R(p,1)$ and $R(1,p)$, and so by Theorem 163 we must have p divides m or n.

Similarly, it must also have a tiling by $R(q,1)$ and $R(1,q)$, and so q divides m or n.

Each side is expressible in the form $px+qy$ by either Theorem 20 or 21.

□

Theorem 165 (Slightly generalized from Martin (1991), p.43)**.** *$R(m,n)$ is tileable by $R(p,1)$ and $R(1,q)$ iff p divides m or q divides n.*

[Referenced on pages 186 and 187]

Proof.

If. WLOG suppose p divides m. A tiling exists by Theorem 22.

Only if. WLOG, assume p does not divide m. We will show q divides n.

Color the columns of $R(m,n)$ cyclically with p colors. Denote the number cells of color i by c_i. Since p does not divide m, we can write $m = dp + r$, where $0 < r < p$. There is $d+1$ columns of the first color, and d columns of the second color; therefore $c_0 = n(d+1)$, and $c_{p-1} = nd$.

Consider now a tiling by the two bars. The horizontal bar covers one of each color; the vertical bar covers q cells of the same color. Let x be the number of horizontal bars, and y_i the number of vertical bars that cover cells of color i.

Then $c_0 = x + qy + 0$, and $c_{p-1} = x + qy_{p-1}$, so $c_0 - c_{p-1} = q(y_0 - y_{p-1})$.

Now

$$\begin{aligned} n &= n(d+1) - hd \\ &= c_0 - c_{p-1} \\ &= q(y_0 - y_{p-1}), \end{aligned}$$

which means q divides n, which is what we wanted to show. □

Theorem 166 (Divisibility Lemma, Bower and Michael (2004), Section 3)**.** *Suppose we have a set of rectangles partitioned into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that all rectangles in $\mathcal{T}_1$ have widths with a factor r and all rectangles in $\mathcal{T}_2$ have heights with a factor s. Then if $\mathcal{T}$ tiles a $R(m,n)$, then $r \mid m$ or $s \mid n$.*

[Referenced on pages 187, 192 and 196]

Proof. [2] The tiles in $\mathcal{T}_1$ are all tileable by $R(r, 1)$, and the tiles in $\mathcal{T}_2$ are all tileable by $R(1, s)$. Therefore, if a tiling of $R(m, n)$ by $\mathcal{T}$ exists, it also has a tiling by $\mathcal{T}' = \{R(r, 1), R(1, s)\}$ (Theorem 3).

But by Theorem 165, this is possible only if $r \mid m$ or $s \mid n$. □

[2] The original proof uses the idea of *charges*. The argument here hinges on Theorem 165, where we adapted an argument Martin originally used to prove Theorem 163 (and is in fact very similar to the argument we used there from Golomb). Clearly these three theorems are closely related.

Theorem 167 (de Bruijn (1969), Theorem 1)**.** *Suppose a number k divides one side of each rectangle in $\mathcal{T}$. If $\mathcal{T}$ tiles $R(m, n)$, then either $k \mid m$ or $k \mid n$.*

[Referenced on pages 191, 192, 196 and 278]

Proof. This follows immediately from 166 by setting $r = s = k$. □

One consequence of this theorem is that we cannot tile a 10×10 with horizontal and vertical bars of length 4.

Theorem 168 (Bower and Michael (2004), Theorem 5)**.** *Let p_1, p_2, q_1, and q_2 be positive integers with $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$. Then $R(m, n)$ can be tiled by translates of $R_1 = R(p_1, q_1)$ and $R_2 = R(p_2, q_2)$ rectangular bricks if and only if one of the following holds:*

(1) $p_1 \mid m$ *and* $q_1 \mid n$

(2) $p_2 \mid m$ *and* $q_2 \mid n$

(3) $p_1 p_2 \mid m$ *and* $n = aq_1 + bq_2$ *for some integers* a, b

(4) $q_1 q_2 \mid n$ *and* $m = ap_1 + bp_2$ *for some integers* a, b.

[Referenced on pages 189, 192 and 227]

Proof. [3]

If. In case (1), a tiling of $R(m, n)$ by $R(p_1, q_1)$ exists by Theorem 22; similarly in case (2), a tiling of$R(m, n)$ by $R(p_2, q_2)$ exists.

In case (3), suppose $m = m' p_1 p_2$. We can split the rectangle in $R(m' p_1 p_2, aq_1)$ and $R(m' p_1 p_2, bq_2)$. From Theorem 22, the first is tileable by R_1 and the second by R_2.

In case (4), suppose $n = n' q_1 q_2$. We split the rectangle in $R(ap_1, nq_1 q_2)$ and $R(bp_2, nq_1 q_2)$, which by Theorem 22 is tileable by R_1 and R_2 respectively.

Only if. If R is tileable by $R(p_1, q_1)$ and $R(p_2, q_2)$, then $m = ap_1 + bp_2$ for some integers a, b and $n = a' q_1 + b' q_2$ for some integers a', b' (Theorem 20 or 21). Since R_1 has width p_1, and R_2 has height q_2, by the divisibility lemma (Theorem 166) we have p_1 divides m or q_2 divides n. Similarly, R_2 has width p_2 and R_1 has height q_1, so p_2 divides m or q_1 divides n. Taken together, one of the four conditions must hold. □

[3] Kolountzakis (2004) gives an alternative proof using the Fourier transform. Fricke (1995) showed the special case when the two rectangles are squares.

Example 19. *Let $\mathcal{T} = \{R(6,6), R(10,10), R(15,15)\}$. There are 6 different ways to partition this set so that the conditions for Theorem 166 hold. These give the following set of conditions, which must be satisfied for any tileable rectangle $R(m,n)$:*

(1) $2 \mid m$ *or* $15 \mid n$

(2) $3 \mid m$ *or* $10 \mid n$

(3) $5 \mid m$ *or* $6 \mid n$

(4) $15 \mid m$ *or* $2 \mid n$

(5) $10 \mid m$ *or* $3 \mid n$

(6) $6 \mid m$ *or* $5 \mid n$

From this we can make inferences such as the following:

- *If $m = 6$, it already satisfies 1, 2, and 6. Since none of 5, 15, or 10 divides 6, the only way to satisfy conditions 4, 5 and 6 is for $6 \mid n$.*
- *For a tileable rectangle $R(m,n)$, we have one of 6, 10 or 15 must divide either m or n.*
- *If m is prime, then $30 \mid n$.*

It is not hard to see the conditions are sufficient for a tiling to exist provided the sides satisfy Theorem 20.

Example 20. *Let $\mathcal{T} = \{R(3,2), R(2,3), R(9,5), R(5,9)\}$. These rectangles have a common factor 3, so we know they will only tile rectangles with a factor of 3.*

We can stack three of the smaller rectangles to make $R(9,2)$ and $R(2,9)$. By combining these with $R(9,5)$, we $R(5,9)$, we can tile $R(9,k)$ or $R(k,9)$ for any $k \neq 1,3$.

We can also tile $R(6,2), R(2,6)$, and we can also tile $R(6,3)$ and $R(3,6)$, so we can tile any $R(6,k)$ and $R(k,6)$ for any $k > 1$. (Because any $k >$ can be written as $2x + 3y$ for nonnegative integers x and y.)

Finally, we can tile all $R(3,2k)$ and $R(2k,3)$ for $k \geq 1$. Putting this together, we have the following:

(1) $R(3m,n)$ *for all* $m > 1$, $n > 3$

(2) $R(3,2n)$ *for all* $n \geq 1$

(3) $R(2m,3)$ *for all* $m \geq 1$

Theorem 169 (Fricke (1995))**.** *Let m,n, x and y be positive integers with $\gcd(x,y) = 1$. Then, $R(m,n)$ can be tiled with $R(x,x)$ and $R(y,y)$ if and only if either*

(1) *m and n are both multiple of x, or*

(2) *m and n are both multiple of y, or*

(3) *one of the numbers m, n is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.*

[Referenced on page 189]

Proof. Apply Theorem 168 by setting $p_1 = q_1 = x$ and $p_2 = q_2 = y$. □

Theorem 170. *Let x and y be relatively prime, and let r be the smallest non-negative integer such that $ry \equiv m \pmod{x}$. Suppose we have a tiling of $R(m, n)$ by $R(x, x)$ and $R(y, y)$. We need at least $r \left\lceil \frac{n}{y} \right\rceil$ of $R(y, y)$.* [4]

[4] This is a slight generalization of the lemma by SmileyCraft (un.) (2019).

[Referenced on page 189]

Proof. In each row, we need $px + qy = m$ for some integers p and q (Theorem 21). This implies $qy \equiv m \pmod{x}$, and therefore and $q \geq r$. In each row must overlap with at least r squares of type $R(y, y)$. Each of these squares can overlap at most y rows, so we need $r \left\lceil \frac{n}{y} \right\rceil$ in total. □

Conjecture 171. *Let x and y are relatively prime, and suppose we have a tiling of $R(m, n)$ by $R(x, x)$ and $R(y, y)$.*

(1) *If $x \mid m$ and $x \mid n$, we need 0 of $R(y, y)$.*

(2) *Otherwise, if $xy \mid m$ and $n = px + qy$ for some p, q, then we need $r \left\lceil \frac{n}{y} \right\rceil$, where r is the smallest non-negative integer such that $ry \equiv m \pmod{x}$.*

(3) *Otherwise, if $xy \mid n$ and $m = px + qy$ for some p, q, then we need $r \left\lceil \frac{m}{y} \right\rceil$, where r is the smallest non-negative integer such that $ry \equiv n \pmod{x}$.*

(4) *Otherwise, if $y \mid m$ and $y \mid n$, then we need $\frac{rn+mr'-rm}{y} = \frac{rn+mr'-r'n}{y}$, where r is the smallest non-negative integer such that $ry \equiv m \pmod{x}$, and r' is the smallest non-negative integer such that $r'y \equiv n \pmod{x}$.*

(5) *Otherwise, no tiling is possible.*

Partial Proof. In case 1, $R(m, n)$ is tileable by $R(x, x)$ (by Theorem 22) and we do not need any $R(y, y)$. In case 2, we need at least $r \left\lceil \frac{n}{y} \right\rceil$, where r is the smallest non-negative integer such that $ry \equiv m \pmod{x}$, by Theorem 170. A tiling that realizes this number is given by splitting the rectangle in $R(ry,n)$, which is tileable by $R(y,y)$, and $R(m-ry,n)$, which is tileable by $R(x,x)$. Case 3 is symmetrical to case 2 (apply Theorem 170 with m and n swapped). In case 4, a tiling is given by splitting the rectangle into three rectangles $R(ry,n-r'y)$ (tileable by $R(y,y)$, $R(m,r'y)$ (tileable by $R(y,y)$, and $R(m-ry,n-r'y)$, tileable by $R(x,x)$. See for example Figure 197. It remains to show this is optimal (i.e. we don't need fewer $R(y,y)$. This is the missing piece making this a conjecture. Case 5 follows from Theorem 169.

□

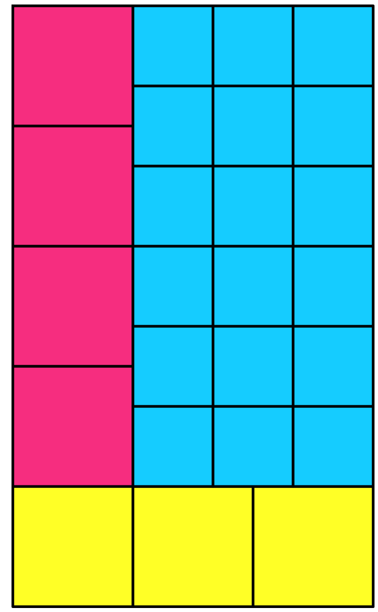


Figure 197: This tiling of $R(9,15)$ by $R(2,2)$ and $R(3,3)$ uses 7 of $R(3,3)$. We get this tiling by splitting the rectangle into three smaller rectangles: $R(3,12)$, $R(9,3)$, and $R(6,12)$. The conjecture states this is optimal.

Problem† 70. *Which rectangles can be tiled by $R(2,2)$ and $R(3,3)$?*

Problem† 71 (proton (un.) (2019))**.** *Suppose $R(m,n)$ can be tiled by $R(2,2)$ and $R(3,3)$. What is the minimum number of $R(3,3)$ we need? (A solution is given by SmileyCraft (un.) (2019).)*

Things get more interesting when $\mathcal{T}$ has more than two rectangles. For rectangles satisfying certain conditions, they can tile any sufficiently large rectangle.

This is somewhat similar to what happens to integers: given two integers that are relatively prime[5], we can write any big-enough integer as a positive combination of the two. For example, given integers 3 and 5, we can write any integer larger than 7 as the sum of multiples of 3 and 5. For example, $8 = 3+5$, $13 = 3+2\cdot 5$, $11 = 2\cdot 2+5$.

Of course, we can do the same when we have a larger set of integers. For example, given 3, 4 and 5, we can write any number larger than 3 as a positive combination. We call the minimum number that cannot be written as a positive combination of the set the set's **Frobenius number**, and we write it $g(a,b,c,\cdots)$. For example, $g(3,5)=7$, and $g(3,4,5)=2$.

[5] Recall, two integers are realtively prime when they don't have any common divisors other than 1

Theorem 172 (Labrousse and Ramírez Alfonsín (2010), Theorem 5)**.** *Suppose we have $k \geq 3$ rectangles R_i, where is R_i is a $p_i \times q_i$ rectangle, and*

(1) $\gcd(p_i,p_j)=1$, *for any $i \neq j$,*

(2) $\gcd(q_i,q_j)=1$, *for any $i \neq j$,*

Let g_1 be the maximum Frobenius number of p_i taken $(k-1)$ at a time. Let g_2 be the maximum Frobenius number of products $\frac{q_1\cdots q_k}{q_i q_j}$ for some j and $i \neq j$.

Then if we have $p_i, q_i \geq \max(g_1,g_2)$, the rectangle is tileable by R_i.

[Referenced on page 192]

The proof is quite technical, and I omit it. For details, see the reference. There is also a courser version of the theorom above, that is much easier to calculate:

Theorem 173 (Labrousse and Ramírez Alfonsín (2010), Corollary 1)**.** *Suppose we have $k \geq 3$ rectangles R_i, where is R_i is a $p_i \times q_i$ rectangle, and*

(1) $\gcd(p_i, p_j) = 1$, *for any* $i \neq j$,

(2) $\gcd(q_i, q_j) = 1$, *for any* $i \neq j$,

If $r = \max_i(p_i, q_i)$, then the rectangle is tileable by this set if $m, n > r^4$.

[Not referenced]

Rectangles	Minimum Length
$2 \times 3, 3 \times 2, 5 \times 5$	30

Theorem 174. *For a set of three rectangles, one of the following holds:*

(1) $\gcd(q_i, p_j) = \gcd(q_i, q_j) = 1$

(2) *We can partition the rectangles as in Theorem 167 (possibly, with the two factors equal).*

[Not referenced]

Theorem 175. *Let $R_1, \cdots, R_4$ be four rectangles that satisfy the following conditions:*

(1) $\gcd(p_1, p_2) = r > 1$

(2) $\gcd(p_3, p_4) = s > 1$

(3) $\gcd(q_i, q_j) = 1$, *for* $i, j = 1, 2, 3, 4$, $i \neq j$

(4) $\gcd(r, s) = 1$

The there exist C such that these rectangles can tile any $R(m, n)$ if $m, n > C$.

[Referenced on page 192]

Proof. Let $u = \operatorname{lcm}(p_1, p_2)$ and $v = \operatorname{lcm}(p_3, p_4)$. We can then build these rectangles:

(1) $u \times q_1$

(2) $u \times q_2$

(3) $v \times q_3$

(4) $v \times q_4$

Using the first two of these, we can build any $u \times x$ rectangle for large enough x (say $x > C_1$) and using the last two rectangles, we can build any $v \times y$ rectangle for large enough y (say $y > C_2$). Since $\gcd(u, v) = 1$, we can, when $x = y$, build any $z \times x$ rectangle for large enough z (say $z > C_3$, and of course we already have $x, y > \max(C_1, C_2)$. So we can tile any $z \times x$ rectangle with $z, x > \max(C_1, C_2, C_3)$.

And of course,

- $C_1 = g(q_1, q_2) = q_1 q_2 - q_1 - q_2$
- $C_2 = g(q_3, q_4) = q_3 q_4 - q_3 - q_4$
- $C_3 = g(u, v) = uv - u - v = \frac{p_1 p_2 p_3 p_4 - p_1 p_2 s - p_3 p_4 r}{rs}$

and so

$$C = \max \left\{ q_1 q_2 - q_1 - q_2, q_3 q_4 - q_3 - q_4, \frac{p_1 p_2 p_3 p_4 - p_1 p_2 s - p_3 p_4 r}{rs} \right\}$$

$\square$

Theorem 176. *For a set of four or more rectangles, one of the following holds:*

(1) $\gcd(p_i, p_j) = \gcd(q_i, q_j) = 1$ *for* $i \neq j$.

(2) *We can partition the rectangles as in Theorem 167 (possibly, with the two factors equal).*

(3) *We can select 4 rectangles that can tile a sufficiently large rectangle.*

[Not referenced]

It can be tricky to keep track of the theorems and know which one applies in a given situation. Table 35 summarizes to conditions and the theorems that apply.

We conclude this section with two special cases where the rectangles in the tileset are all squares.

Theorem 177 (Labrousse and Ramírez Alfonsín (2010), Theorem 6)**.** *All sufficiently large rectangles can be tiled with three squares whose side-lengths are pairwise relatively prime.* $R(a, a)$ *can be tiled with* $R(a_1, a_1)$, $R(a_2, a_2)$, *and* $R(a_3, a_3)$ *when*

$$a > \left(2 - \sum_1^3 \frac{1}{a_i}\right) \prod_1^3 a_i$$

or

$$a > 2a_1 a_2 a_3 - a_1 a_2 - a_1 a_3 - a_2 a_3$$

$\lvert\mathcal{T}\rvert$	Conditions	Theorem
1		22
2	$\gcd(p_1, p_2) = 1$ $\gcd(q_1, q_2) = 1$	168
2	$\gcd(p_1, p_2) = k > 1$	167
≥ 3	$\gcd(p_i, p_j) = 1, i \neq j$ $\gcd(q_i, q_j) = 1, i \neq j$	172
3	$\gcd(p_1, p_2) = k > 1$ $\gcd(p_i, p_3) = 1, i \neq 3$	166
4	$\gcd(p_1, p_2) = r > 1$ $\gcd(p_3, p_4) = s > 1$ $\gcd(q_i, q_j) = 1, i \neq j$ $\gcd(r, s) = 1$	175
≥ 4	$\gcd(p_1, p_2 \cdots, p_k) = r$ $\gcd(q_k, p_{k+1} \cdots p_{\lvert\mathcal{T}\rvert}) = s$	166
≥ 4	$\gcd(p_i, p_j) = 1, i \neq j, i, j = 1, 2, 3, 4$ $\gcd(q_i, q_j) = 1, i \neq j, i, j = 1, 2, 3, 4$	172

Table 33: Characterization of tilings of rectangles by rectangles.

[Not referenced]

For example, for squares with length 2, 3, and 5, this theorem gives us $a > 2 \cdot 2 \cdot 3 \cdot 5 - 2 \cdot 3 - 3 \cdot 5 - 2 \cdot 5 = 60 - 6 - 15 - 10 = 29$.

Theorem 178 (Labrousse and Ramírez Alfonsín (2010), Theorem 7). *Let $p > 4$ be relatively prime to 2, and 3. Then $R(a, a)$ can be tiled with $R(2, 2)$, $R(3, 3)$ and $R(p, p)$ if $a \geq 3p + 2$.*

[Not referenced]

Problem† 72. *Let*

- *$\mathcal{S}(R_1, R_2, \cdots, R_n)$ denote the smallest square that is tileable by each of R_i, in either rotations,*
- *$A = \{R(2,5), R(5,2), R(3,7), R(7,3)\}$, and*
- *$B = \{R(2,3), R(3,2), R(5,7), R(7,5)\}$.*

(1) *Find a rectangle that is tileable by all rectangles in A but not by one of B.*

(2) *Find $\mathcal{S}(A)$.*

(3) *Find $\mathcal{S}(B)$.*

(4) *Show that $\mathcal{S}(R(m, n)) = R(mn/d, mn/d)$, where $d = \gcd m, n$.*

(5) *Show* $\mathcal{S}(R(m,m), R(n,n)) = R(mn/d, mn/d)$.

*(6) *Is the following identity true?*

$$\mathcal{S}(R_1, R_2) \stackrel{?}{=} \mathcal{S}(\mathcal{S}(R_1), \mathcal{S}(R_2))$$

5.1.2 The gap number of rectangular tilings

Recall that the *gap number* of a region R is the minimum number of monominoes of a tiling by $\mathcal{T}^+$, which is the tileset $\mathcal{T}$ plus a monomino. One would think that determining the gap number for a set of rectangles and a rectangular region is easy; however, it is not the case. We will give some partial results in this section.

Theorem 179. *Suppose* $R(m,n)$ *with* $m, n < p$ *and* $m + n < p$ *is a rectangle with the flag coloring* F_p *applied. Then one color does not occur at all.*

[Referenced on page 194]

Proof. The first row goes from color 0 to color $m-1$; the second row from color $p-1$ to color $m-2$; and so on, until the last row, which goes from $p-n+1$ to $m-n$.

In row i, we have the first $n-i$ colors and the last i colors; since i ranges from 0 to $n-1$, all colors satisfy $0 \le c \le m-1$, or $0 \le c < m$ and $p-(n-1)+1 \le c \le p-1$, or $p-n \le c \le p-1$. From the last inequality and $m+n < p$, we get $m < c \le p-1$. So all color used are smaller than m or bigger than m; not are equal to m. So color m is not used. □

Theorem 180. *Suppose* $R(m,n)$ *with* $m \bmod p + n \bmod p < p$. *Then* $G_p(m,n) \ge (m - m \bmod p)(n - n \bmod p)$.

[Not referenced]

Proof. Apply the coloring F_p. Partition the rectangle in four rectangles R_1, R_2, R_3, R_4 such that $R_4 = R(m \bmod p, n \bmod p)$. Each color occurs the same number of times in each of R_1, R_2, R_3; each color occurs $(m-m')(n-n')/p$ in R_1, $m'(n-n')/p$ in R_2, and $n'(n-n')/p$ in R_3. One color is missing in R_4 (Theorem 179), so that color occurs this many times in total:

$$\begin{aligned}
&(m-m')(n-n')/p + m'(n-n')/p + n'(m-m')/p \\
=&(mn - m'n - mn' + m'n' + m'n - m'n' + mn' - m'n')/p \qquad (5.1)\\
=&(mn + m'n')/p.
\end{aligned}$$

So we can fit at most that many bars, which means $G(m,n) \ge mn - p(mn - m'n')/p$, or $G(m,n) \ge m'n'$. □

Theorem 181 (Barnes (1979), Lemma 1). *Let $\mathcal{T} = \{R(1,p), R(p,1)\}$. Then*

$$G_{\mathcal{T}}(R(m,n)) = \begin{cases} (m \bmod p)(n \bmod p) & \text{if } m < p \text{ or } n < p \text{ or } m \bmod p + n \bmod p < p \\ (m - m \bmod p)(n - n \bmod p) & \text{otherwise} \end{cases}$$

[Referenced on page 279]

Theorem 182. *Exactly $\left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor$ squares $R(p,p)$ can fit in $R(m,n)$.*

[Referenced on pages 195 and 232]

Proof. (Adapted from JimmyK4542 (un.) (2017).) Apply the square coloring S_p to $R(m,n)$. No matter how we place a $p \times p$ square, it always covers p of each color. Let c be the color at coordinates $(p-1, p-1)$. This color occurs $\left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor$ times, so we can fit at most this number of squares. □

Theorem 183. *Let $\mathcal{T} = \{R(p,p)\}$. Then*

$$G_{\mathcal{T}}(R(m,n)) = mn - (m - m \bmod p)(n - n \bmod p).$$

[Referenced on page 29]

Proof. By Theorem 182 we can fit at most $\left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor$ squares, which means the gap number satisfies $G \geq G_{\mathcal{T}}(R(m,n)) = mn - (m - m \bmod p)(n - n \bmod p)$. The naive tiling has a gap of this size, and so is optimal, and therefore the gap number is given by $G_{\mathcal{T}}(R(m,n)) = mn - (m - m \bmod p)(n - n \bmod p)$. □

Theorem 184. *$\left\lfloor \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor / q \right\rfloor$ rectangles $R(p,pq)$, and no more, can fit in $R(m,n)$.*

[Referenced on page 195]

Proof. By Theorem 182 exactly $k = \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor$ squares $R(p,p)$ can fit in $R(m,n)$, this means at most $\frac{k}{q}$ of $R(p,pq)$ can fit in $R(m,n)$.

All we need to show is a tiling that uses $\left\lfloor \frac{k}{q} \right\rfloor$ tiles.

Consider the rectangle $R(p \lfloor m/p \rfloor, p \lfloor n/p \rfloor)$ that fits in $R(m,n)$. For this rectangle, a tiling exists by Theorem 164. This tiling uses $\left\lfloor \frac{k}{q} \right\rfloor$ tiles. Thus we can fit $\left\lfloor \frac{k}{q} \right\rfloor$ of $R(p,pq)$ in $R(m,n)$. □

Theorem 185. *Let $\{R(p,pq), R(pq,p)\}$. Then*

$$G_{\mathcal{T}}(R(m,n)) = mn - \left\lfloor \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor / q \right\rfloor p^2 q.$$

[Not referenced]

Proof. From Theorem 184 we can fit $\left\lfloor \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor / q \right\rfloor$ rectangles, and since each has an area of p^2q, the total covered area is at most $\left\lfloor \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor / q \right\rfloor p^2q$ and therefore the gap is at least $G_{\mathcal{T}}(R(m,n)) = mn - \left\lfloor \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{n}{p} \right\rfloor / q \right\rfloor p^2q$. □

5.1.3 *Tilings of arbitrary regions by rectangles*

In this section, we extend some of the results of the previous section to tiling of arbitrary simple figures by rectangles.

Theorem 186 (Csizmadia et al. (2004), Corollary 2.4)**.** *If a simply-connected region is tileable by bars of length k, the region has at least one side divisible by k.*

[Referenced on pages 196 and 341]

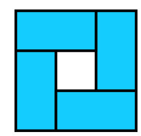


Figure 198: A region tileable by dominoes where all sides are not divisible by two. This is only possible for regions with holes.

Figure 198 shows an example of how the theorem fails for regions with holes.

Theorem 187 (Csizmadia et al. (2004), Corollary 2.3)**.** *If a simply-connected region is tileable by $m \times n$ and $n \times m$ rectangles, then it must have at least one edge whose length is divisible by m, and at least one edge whose length is divisible by n.*

[Not referenced]

Proof. Note that the region is tileable by both bars of length m, and by bars of length n. Therefore, by Theorem 186 one side is divisible by m, and one side is divisible by n. □

For example, a region is not tileable by dominoes if all the edges have odd length.

These theorems are generalizations of Theorem 164 and Theorem 167.

Problem† 73.

(1) *Show the following: If a simply-connected figure is tileable by a set of rectangles $\mathcal{T}$ that can be partitioned into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$ such that all the rectangles in $\mathcal{T}_1$ has widths with factor r and all rectangles in $\mathcal{T}_2$ has heights with factor s, then the figure has either an edge with factor r, or an edge with factor s. (This a generalization of the Divisibility Lemma, Theorem 166.)*

(2) *Show that when the conditions above hold, we need two edges with factors r or s.*

(3) *Find a tiling by dominoes that don't have any knobs or flats that are divisible by 2.*

(4) *Find a tiling by straight trominoes that don't have any knobs or flats that are divisible by 3.*

Theorem 188 (Diagonal criterion for rectangles[6]). *In a tiling of a region, the number of lower-left tile-corners on diagonals (going top left to bottom right) are constant, regardless of the way in which the region is tiled.*

All terms in the diagonal series of a region (defined below) must be positive for a tiling to exist.

[Referenced on pages 128 and 197]

[6] The theorem for bars was first conjectured in Saturday (un.) (2022), and a proof given by RavenclawPrefect (un.) (2022). The same idea works for general rectangles.

Figure 199 illustrates the theorem for a tiling by straight trominoes.

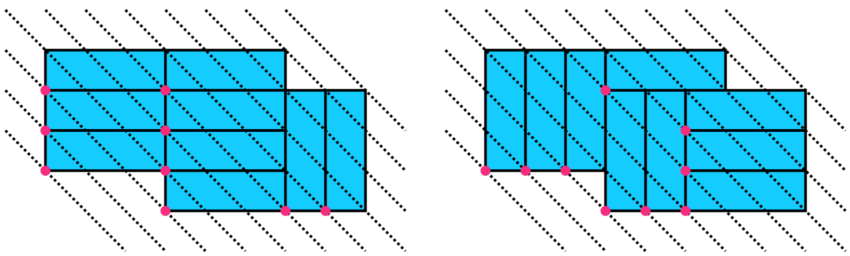


Figure 199: The number of bottom-left vertices on corresponding diagonals are equal in the two tilings.

Problem† 74. *Use Theorem 188 to show the mutilated chessboard cannot be tiled by dominoes.*

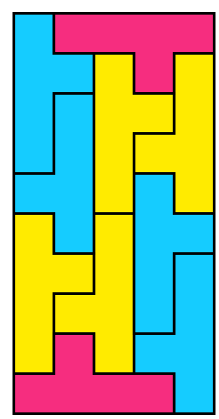


Figure 200: Polyomino with order 10.

5.2 Order

A polyomino that can tile a rectangle is called **rectifiable**.

The **rectangular order** of a polyomino is the smallest number k such that k copies of the polyomino tiles a rectangle. The **odd rectangular order** of a polyomino is the smallest odd number k such that k copies of the polyomino can tile the rectangle, if such a tiling exists. Polyominoes that has an odd order are called **odd**, otherwise they are called **even** (Klarner, 1965).[7] The **square order** is the minimum number of tiles required to tile a square. The notation used to denote some order of polyominoes is shown in Table 34, although we will not be using it often.

[7] This notion of even is different from the one in Section 3.3.4.

(1) In a tiling of a rectangle, only knobs can lie on the border.

(2) Each side of the rectangle must be expressible as a combination of the knobs.

(3) If knobs of a polyomino is divisible by k, then so is each side of any rectangle it can tile.

RO	Rectangular order
SO	Square order
OO	Odd order
IO	Reptile order
HO	Hexagonal order

Table 34: The symbols we use to denote various orders.

5.2.1 Order Theorems

Theorem 189. *If a polyomino P has order n, then a scaled copy of the polyomino P' order has order n.*

[Not referenced]

Proof. Suppose P' has order $n' < n$, and that the associated rectangle is R'. If we divide each tile into $S = k \times k$ squares, we have a tiling of R' by S. But this tiling is unique (Theorem 4), and is the trivial tiling with all squares lying in a grid. But if this is the case, we can scale the tiling by k, and so find a valid polyomino tiling of a rectangle using n' tiles, which means the order of P is smaller than n, a contradiction.

It follows that the order of P' must be larger than or equal to n. But by scaling a tiling of R by P, we can find a tiling of R' by P'. Therefore, the order of P is n. □

Theorem 190. *Suppose the smallest rectangle a polyomino can tile is $R(m, n)$, and let $d = \gcd(m, n)$. If P is not a square, then*

$$\mathrm{IO}(P) \leq \mathrm{SO}(P)\,|P| \leq (\mathrm{RO}(P)k/d)^2.$$

If P is a square, them $\mathrm{IO}(P) = 4$ and $\mathrm{SO}(P) = \mathrm{RO}(P) = 1$.

[Referenced on page 340]

Theorems like the one above is helpful when doing computer searches. Theorem 327 is a similar theorem.

Problem† 75. *We cannot define the concept of order for the strip and plane directly, since these regions require an infinite number of tiles to tile. However, we showed that if a polyomino tiles a strip, it tiles a cylinder (Theorem 137); we will also see if a polyomino tiles the plane periodically, it tiles some pseudo-hexagon.*

We can then define the ***hexagonal order*** *of a polyomino $\mathrm{HO}(P)$ as the minimum number of copies required to tile a pseudo-hexagon. Similarly the* ***cylindrical order*** *$\mathrm{CO}(P)$ of a polyomino as the minimum number of copies required to tile a cylinder.*

Prove that

$$\mathrm{HO}(P) \leq \mathrm{CO}(P) \leq \mathrm{RO}(P).$$

Theorem 191 (Klarner (1969), Theorem 2)**.** *An unbalanced polyomino with $2n$ cells is even.*

[Referenced on pages 205, 228 and 234]

Proof. Since the polyomino has $2n$ cells, the rectangle must have an even area (Theorem 1), and is therefore balanced (see Theorem 56).

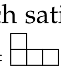
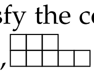
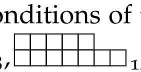
But an odd number of unbalanced polyominoes with an even number cannot have an equal number of white and black cells, so a region tiled by these cannot be balanced. Therefore, the only possibility for a balanced region be tileable by these polyominoes must be tileable by an even number of them. □

Theorem 192 (Klarner (1969), Theorem 4)**.** *Color the plane with $F_{2,0}$. If a polyomino with an even number of cells always covers k black cells or k white cells regardless of its position and orientation and the polyomino tiles a rectangle, it is even.*

[Not referenced]

Proof. Suppose R is a rectangle tiled by the polyomino. Since the area of the polyomino is even, so must the are of the rectangle (Theorem 1). WLOG assume then the orientation of the rectangle is such that the rectangle has the same number of black and white cells; that is, the width is even.

Then form two sets, B with the copies of the polyomino colored with k black cells, and W with the copies of the rectangle with k white cells. Clearly, $|B| = |W|$, and therefore the total number of polyominoes must be even. □

Polyominoes which satisfy the conditions of the theorem include $B((k+1)\cdot(k+3)) =$ $_4$, $_8$, $_{12}, \cdots$.

Theorem 193. *A rectifiable polyomino must cover at least one corner of its hull.*

[Referenced on pages 199, 231 and 232]

Proof. There is no way to place the polyomino to cover a corner cell of the rectangle. □

This theorem is also a consequence Theorem 134, which states that any *reptile* must cover at least one corner of its hull (and remember, rectifiable polyominoes are reptiles by Theorem 136).

Theorem 194 (Klarner (1965), (i) p. 18)**.** *If a polyomino has symmetry index 2 or less, and is rectifiable, it is a rectangle.*

[Not referenced]

Proof. The polyomino must cover at least one corner of its hull (Theorem 193).

Suppose the polyomino is in class **All**, **Rot2** or **Axis2**. Then, by symmetry, must cover all corners of its hull. If we place the polyomino in a corner of a rectangle, we cannot have gaps between the rectangle edge and the polyomino edge, therefore, all the cells between two adjacent corners must be part of the polyomino. This must be true for all edges of the polyomino, so we have a rectangle, possibly with some holes. But if there is a hole, it is too small to fit even one polyomino, so we cannot have holes. Therefore, the polyomino is a rectangle.

Suppose the polyomino is in class **Diag2**. If it covers two adjacent corners of its hull, by symmetry it covers 4 corners of the hull and the same argument as above applies.

If it does not cover two adjacent corners (Figure 201), by symmetry it must cover two opposite corners. If this polyomino tiles a rectangle, the tiles in the bottom corners must have different orientations (there is only one way to place a tile in each corner). If we consider the bottom edge, then there must be at least a pair of adjacent tiles with opposite orientations also adjacent to the rectangle edge. The hulls of these tiles cannot overlap, and so the cells corresponding to the hull corners that are not covered cannot be covered by another tile (See Figure 202). Therefore, this arrangement is impossible, and therefore this case cannot occur.

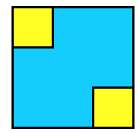


Figure 201: A polyomino (in blue) in **Diag2** that covers only 2 corners of its hull.

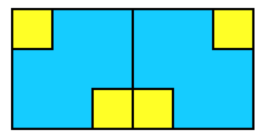


Figure 202: A polyomino (in blue) in **Diag2** that covers only 2 corners of its hull.

In all cases, the only possibilities are for polyominoes that are rectangles. □

Theorem 195 (Klarner (1965), (ii) p. 18)**.** *If a polyomino fits inside a rectangle, and covers two diagonally opposite corners of this rectangle, and it tiles this rectangle, it is a rectangle.*

[Not referenced]

Proof. We have two cases: either the polyomino demarcates two disconnected sections, or it also covers a third corner of the hull. In both cases, we cannot fit the polyomino in the remaining regions to be tiled; so therefore, the only possibility is that the polyomino is a rectangle. □

Theorem 196 (Reid (2014), p. 117)**.** *If a polyomino tiles a (different) odd polyomino with an odd number of tiles, it is odd.*

[Not referenced]

This follows essentially from transitivity of tilings (Theorem 3).

Problem* 76. *Does the converse hold?*

Theorem 197. *An order-2 polyomino must cover at least 2 adjacent corners of its hull.*

[Referenced on page 202]

Proof. Any dissection of a rectangle into two connected pieces must have at least one tile with two adjacent corners of the rectangle. Because if this is not the case, then pairs of opposite corners must belong to the same tile (Figure 5.2.1). But there is no way that the two blue corners can be connected *and* the two pink corners can be connected. Furthermore, all the corners of the rectangle of a piece must lie on that piece's hull.

□

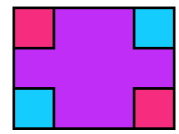


Figure 203: No dissection of a rectangle into two pieces can make pairs of opposite corners belong to the same pieces.

Figure 5.2.1 shows it is possible for a order-2 polyomino to only cover 2 corners of its hull.

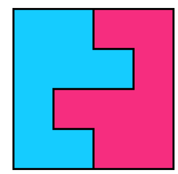


Figure 204: An example of a order-2 polyomino that covers only two of its hull's corners.

Theorem 198. *A polyomino that covers exactly two opposite corners of its hull, with the bottom one on the left, can only tile a rectangle if it and a copy rotated* $90°$ *counter-clockwise can tile a straight edge.*

[Referenced on pages 228, 231 and 232]

Proof. There are up to eight possible orientations for a polyomino. Let's call a polyomino with a covered corner on the bottom left *left*, and *right* otherwise. Suppose the polyomino tiles a rectangle, and consider the polyominoes making the bottom edge of the rectangles. Since the corners of this edge must be filled, we must have a left polyomino on the left and a right polyomino on the right. And so, on this edge, we must have at least one pair of left and right polyominoes adjacent.

The right polyomino cannot be the left polyomino reflected about the Y-axis. If this was the case, the top covered corners would be adjacent, and so we could not covered the uncovered corners, and so there would be a gap on the edge.

By similar reasoning, the right polyomino cannot be the left polyomino reflected about the Y-axis and then rotated $180°$.

Next, suppose the right polyomino is the left polyomino rotated $90°$ clockwise.

If they are to pack an edge, their convex hulls must overlap, because if they don't, there is no way to tile the uncovered hull corners.

The center of rotation must lie within this overlapping hull.

Now consider the two corners of the overlapping rectangle along the edge. The left one must belong to L, (otherwise it would cover R's hull), and therefore the right cell must belong to R. If we rotate the left cell by $90°$ counter clockwise about the center of rotation, it overlaps with the right cell, which cannot be by Theorem 24. Therefore, this configuration is impossible.

That leaves only one possibility, namely, the right polyomino is rotated $90°$ counter-clockwise. Figure 5.2.1 shows that in this configuration it is possible for the two polyominoes to pack an edge, and indeed, the polyomino can tile a rectangle.

□

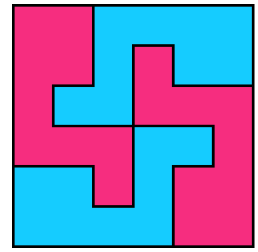


Figure 205: The only way in which a polyomino that covers only 2 opposite corners of it's hull can pack an edge with a copy. If such a polyomino can tile an edge, it has order 4.

Theorem 199. *If a n-omino and a $90°$ rotated copy pack an edge of length $k \leq \sqrt{n}$, the polyomino has order at most 4. And the order is exactly four when the polyomino covers exactly two opposite corners of the hull.*

[Referenced on page 202]

Proof. By Theorem 24 we can add two more copies, rotated $180°$ and $270°$degrees around the same center of rotation, and they won't overlap. In this compound figure, the packed edge must also exist in all four rotations; and together these form the border of a tiled rectangle (possibly with holes). So the four copies pack a rectangle (possibly with holes). Since the edge is of length k, the rectangle must have area $k^2 \leq n$. So there cannot be any holes, and in fact $k^2 = n$. Therefore, four copies tile a rectangle (with no holes), and so the order of the polyomino is at most 4.

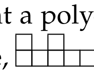
If the polyomino covers exactly two corners of its hull, it cannot be a rectangle, so its order cannot be 1. It can also not have order 2 (since a order 2 polyominoes must at least cover two adjacent corners of its hull by Theorem 197). □

A set of polyominoes **pack a corner** of size k if it can cover all cells that satisfy $x + y < k$ for some k without overlap and without covering any cells $x, y < 0$.

Theorem 200. *A set of polyominoes that cannot pack a corner of size k, for any k, cannot tile a rectangle.*

[Referenced on page 202]

Proof. A set that cannot pack a corner, can clearly not tile a quadrant, and therefor not a rectangle (Theorem 137). □

The theorem above is sometimes useful to prove that a polyomino is non-rectifiable, especially by computer. For example, $_7$ cannot pack a corner of size 17 (Dahlke, 2020).

A **gun** is a bar graph with height 2, which we will draw upside down so that it resembles a gun (Figure 207).[8] A gun has a left and right **end**. If the corners are not missing from an end, it is called a **blunt end**. A gun with one blunt end is called a **blunt gun** (Figure

[8] The terminology is from Dahlke (2001), http://www.eklhad.net/polyomino/gun.html. He does not give this exact definition.

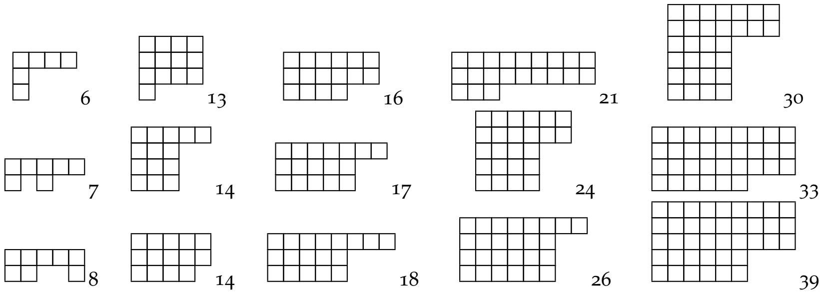


Table 35: Polyominoes that can be proven to be non-rectifiable using Theorem 200 ((Dahlke, 2020, https://github.com/eklhad/trec).

206); a gun with two blunt ends is a **degenerate gun** (Figure 208). The longest non-blunt end is called the **barrel**. If the other end is also non-blunt, it is called the **tab**. The side with the contiguous row of cells is called the **back** of the gun; the opposite side is called the **face**. A contiguous band of cells of the face is called a **tooth**, and the space between two teeth is called a **gap**. The number of cells in the tooth at the blunt end of a gun is the **thickness** of the blunt end. A gun that is a rectangle with two corners removed is called a **solid gun** (Figure 209).

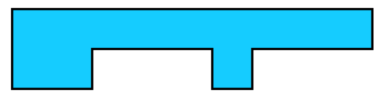


Figure 206: A blunt gun.

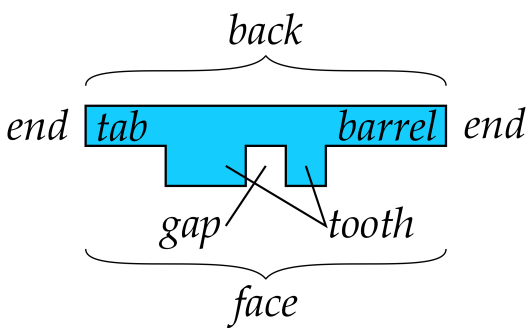


Figure 207: A gun is a bar polyomimo with height 2.

Theorem 201 (Dahlke (2001))**.** *If a polyomino with height 2 is rectifiable, it is a gun.*[9]

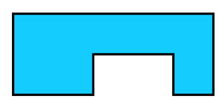


Figure 208: A degenerate gun.

Figure 209: A solid gun.

[9] This is (1) from the *Gun Theorem*, http://www.eklhad.net/polyomino/gun.html.

[Referenced on pages 204 and 232]

Proof. A polyomino of width 2 must cover at least two corners of its hull.

If there are exactly two corners covered, and they are opposite, then there are only two ways to put the polyomino in a corner. Either way, we have gaps along the two rectangle edges that can only be filled in one of up to two ways; an all cases new gaps form. No matter how we continue along the edge, there are gaps, and so we can never fill an adjacent corner. Therefore, the polyominoes cannot tile a rectangle, so this case is not possible.

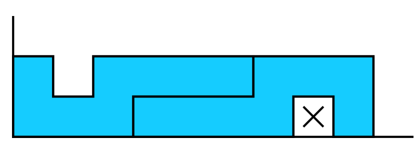


Figure 210:

If these are the only two, or all four corners are covered, then the cells between them must be covered, as can be seen by placing the tile in the corner of the rectangle. We have a bar of height 2.

Suppose there are three corners covered, with the bottom right corner missing, and some squares from the top row. The first tile must cover the origin in exactly this orientation. The barrel points to the right. If it has more than two cells, it forces a second copy, rotated 180°, to fit in the space under the barrel. Since there are squares missing from the top, this isolates a hole that cannot be tiled (Figure 210).

If the barrel has length 1, and the gun is missing the second cell from the top row, the second gun can fill the hole under the barrel with its blunt end.

The gun at $(0,2)$ must be vertical, otherwise it creates holes with the first tile. If the barrel is up, and the third cell in the top row of the gun is covered, this causes an untileable hole (Figure 211). If the third cell is not missing, we can only cover $(1,1)$ by a vertical gun reflected and moved down and right. But both copies occupy $(1,n)$(Figure 212). Whatever the case, the barrel must point down and fill the gap at (1, 1).

But then either $(2,1)$ or $(2,2)$ cannot be covered without isolating untilable holes (Figures 213 and 214).

Therefore, if three corners are covered, there cannot be any cells missing from the top row, and so we have a bar. □

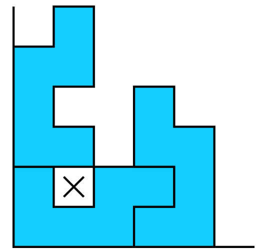


Figure 211:

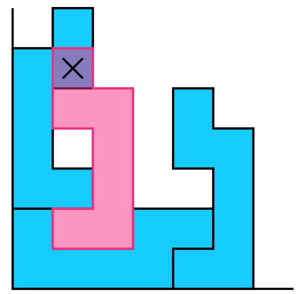


Figure 212:

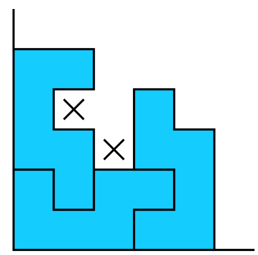


Figure 213:

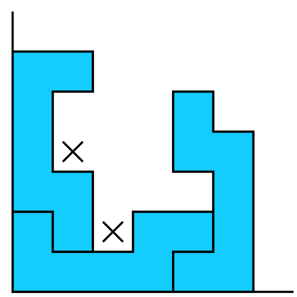


Figure 214:

Theorem 202 (The Gun Theorem, Dahlke (2001)). *The only rectifiable polyominoes with width 2 and order higher than 2 are the T-tetromino, the Y-pentomino, the Y-hexomino, the D-hexomino, and the heptomino* $B(1 \cdot 2^2 \cdot 1^2)$*, except perhaps* $B(1 \cdot 2^3 \cdot 1^3)$.

[Referenced on page 232]

Proof. The proof is long. For full details, see the reference. Here is an outline:

In this proof, with "gun" we mean a gun that does not have order 2. Let n be the length of the gun.

(1) Theorem 201.

(2) If two guns are adjacent with their ends on the floor, they must be back-to-back.

(3) Three guns cannot be adjacent if they are all vertical. Therfore, we cannot fill 3 adjacent cells with guns with tabs. We can at most fill 4 cells with blunt guns.

(4) Gaps with width $n > k \geq 3$ and height 2 cannot be filled with guns with a tab, and gaps with width $n > k \geq 5$ and height 2 cannot be filled with blunt guns.

(5) Degenerate guns with two thick ends are not rectifiable.

(6) No thick ends can cover the origin.

(7) Degenerate guns with one thick end are not rectifiable.

(8) Degenerate guns with no thick ends are not rectifiable.

(9) Two guns aimed at the origin must cover $(1, 1)$.

(10) Guns with tabs of length larger than one are not rectifiable.

(11) In a tiling of a rectangle by blunt guns, the barrel cannot touch the blunt end,

(12) the blunt ends cannot touch, and

(13) the blunt edge cannot touch the barrel.

(14) Blunt guns that have barrels with more than one cell are not rectifiable.

(15) Blunt guns are not rectifiable.

(16) Solid guns are not rectifiable, except for the T-tetromino and the D-hexomino.

(17) A gun with a barrel of length 1 and teeth of lenth more than one are not rectifiable.

(18) A gun with a barrel of length 1 is not rectifiable.

(19) Guns with gaps are not rectifiable.

(20) Guns with a face of one cell are not rectifiable, except for the T-tetromino, Y-pentomino and Y-hexomino.

(21) Guns with a barrel with two cells are not rectifiable, except for the D-hexomino.

(22) Guns with a barrel with more than two cells are not rectifiable, except perhaps for $B(1 \cdot 2^3 \cdot 1^3)$ shown in Figure 215.

(23) This covers all guns, except for $B(1 \cdot 2^3 \cdot 1^3)$. It is not known whether this gun is rectifiable.

□

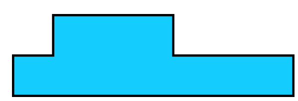


Figure 215: $B(1 \cdot 2^3 \cdot 1^3)$, the only polyomino with width 2 that we do not know whether it tiles a rectangle. By Theorem 191, if this polyomino is rectifiable, it is even. Therefor, the area must divide 20. One side must be divisible by 5; if one side is odd, the other must be divisible by 4.

Problem 77.** *Determine whether $B(1 \cdot 2^3 \cdot 1^3)$ (Figure 215) tiles a rectangle.*

5.2.2 *L-shaped polyominoes*

Recall that a polyomino is called an *L-shape* if it has 6 corners; that is, if it belongs to $\mathcal{C}_6$. After Reid (2014), we make the following definitions: Let $L(a,b,c) = B(a^c \cdot b^c)$, shown in Figure 216.

Two copies of these tiles the rectangle $R(a+b, 2c)$, called the **basic rectangle**. We will assume that $\gcd(a,b,c) = 1$. This does not lead to any loss of generality since a $L(ad, bd, cd)$ is similar to $L(a,b,c)$. Note that $L(a, 2a, b) = L(b, 2b, a)$.

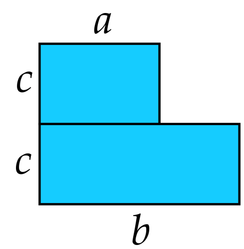


Figure 216: The polyomino $L(a,b,c)$ is made from $R(a,c)$ and $R(b,c)$.

Theorem 203 (Klarner (1969), Theorem 1)**.** *$L(a, 2a, b)$ is odd with odd order at most 15.*

[Not referenced]

Proof. Scale the tiling of $R(5,9)$ (Figure 253) by the right tromino by (a,b) to find a new tiling of $R(5a, 9b)$ or $R(9a, 5b)$ by $L(a, 2a, b)$. $\square$

This gives us an infinite family of polyominoes with odd rectangular order at most 15 (Golomb, 1966, p. 104). Figure 217 shows an example where the tiling has been scaled by a factor of 3 horizontally and 2 vertically. It is not necessarily true that these polyominoes have odd order 15. For example, $L(2,4,1)$ has odd order 11 (Figure 229).

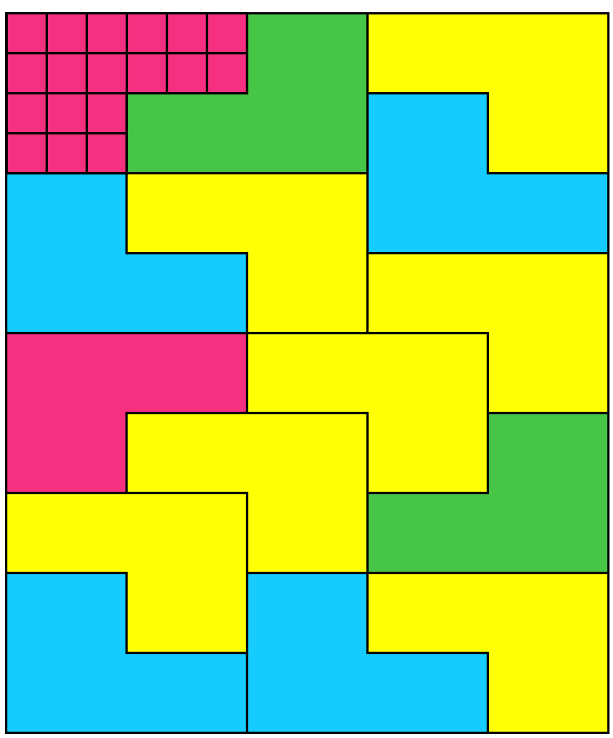


Figure 217: A infinite family of polyominoes with odd order at most 15 can be found by stretching the right tromino.

Theorem 204 (Fletcher (1996) via Reid (2014), p. 342). *$L(2s, 3s, t)$ is odd if s is odd.*

[Not referenced]

Theorem 205 (Fletcher (1996) via Reid (2014), p. 342). *$L(2s, 3s, 1)$ is odd if s is even.*

[Not referenced]

Theorem 206 (Fletcher (1996) via Reid (2014), p. 343). *$L(s, 4s, t)$ is odd if s is odd.*

[Not referenced]

Theorem 207 (Fletcher (1996) via Reid (2014), p. 343). *$L(s, 4s, 1)$ is odd for all s.*

[Not referenced]

Theorem 208 (Fletcher (1996) via Reid (2014), p. 343). *$L(s, 4s, 3)$ is odd for all s.*

[Not referenced]

Theorem 209 (Jepsen et al. (2003) via Reid (2014), p. 343; Marshall (1997), p. 188; Reid (1997), Proposition 3.3). *For odd n, $L(1, n-1, 1)$ tiles $R(n+2, 3n)$, and thus is odd with odd order at most $3(n+2)$.*

[Not referenced]

Theorem 210 (Jepsen et al. (2003) via Reid (2014) p. 343). *For $n \equiv 2 \pmod 4$, $L(1, n-1, 1)$ is odd.*

[Not referenced]

Theorem 211 (Reid (2014), Proposition 2.3). *Suppose a is odd, 2c divides b, and $\gcd(a, b, c) = 1$. Then $L(a, b, c)$ tiles $R((a+b)(a+2b)+b, 5c(a+b))$, and so is odd with order at most $5((a+b)(a+2b)+b)$.*

[Not referenced]

Theorem 212 (Reid (2014), Theorem 2.2). *If $\gcd(a+b, 2c) = 1$, then $L(a, b, c)$ is odd.*

[Not referenced]

Theorem 213 (Reid (2014), Proposition 3.4). *If $L(1, k-1, 1)$ tiles $R(m, n)$, then either m is even, or $m \geq k-1$, and the same for n.*

Polyomino	OO	Rect
$L(1,5,1)$	21[ab]	9×14[c]
$L(3,4,1)$	33[ab]	11×21[c]
$L(4,5,1)$	49[ab]	21×21[c]
$L(4,6,1)$	49[b]	14×35[c]
$L(2,3,2)$	33[ab]	15×22[c]
$L(1,4,2)$	45[ab]	18×25[c]
$L(1,5,2)$	35[ab]	20×21[a]
$L(2,5,1)$	57[a]	9×21[c]
$L(2,5,2)$	63[a]	18×49[a]
$L(2,3,3)$	55[a]	11×75[a]
$L(2,7,3)$	133[d]	57×63[d]
$L(4,5,3)$	119[d]	51×63[d]
$L(1,8,3)$	125[d]	45×45[d]

Table 36: Odd orders for various L-polyominoes. [a]Marshall (1997) [b]Reid (1997) [c]Reid (2003a) [d]Reid (2014)

[Not referenced]

Theorem 214. *The polyomino $B(2, 1^{n-2})$, with n even, has square order n.*

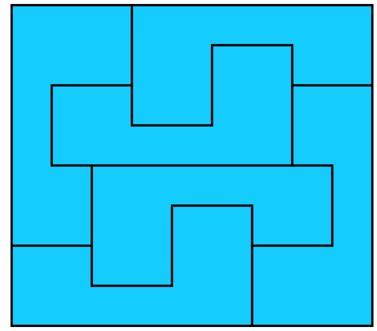
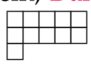


Figure 218: A rectangular tiling using six 12-ominoes. The polyomino has order 2.

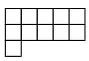
[Not referenced]

Proof. The polyomino tiles $R(2, n)$, and because n is even, this rectangle can tile $R(n, n)$. No smaller square can be tiled. If the square is smaller $n-1$, the polyomino cannot fit in it. If the square is $n-1$, the polyomino cannot tile it because n is not a factor of $(n-1)^2$. □

Theorem 215 (The L Theorem, Dahlke (2020)[10]). *s All rectifiable L-shaped polyominoes — except*  $_{11}$ *— are of order 2, 4, 28, 50, 60, 270, or 396.*

[10] I included this theorem, because if true it would be very interesting. The proof by Dahlke is extremely long (60 pages in an average font) and is very hard to check.

[Not referenced]

Problem* 78. *Does*  $_{11}$ *tile a rectangle, and if so what is its order?*

5.2.3 *Known Orders*

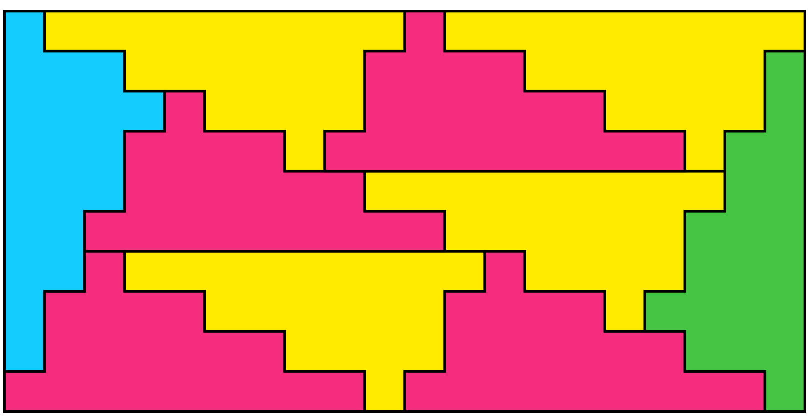


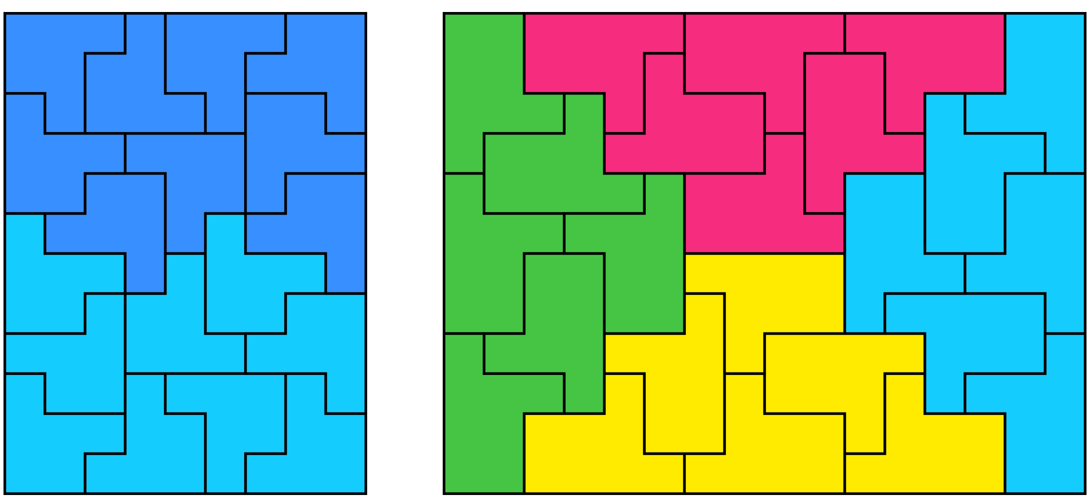
Figure 219: Marshall (1997).

There are polyominoes of order $4k$ for any integer k. Table 37 summarizes results for other orders. There are no known polyominoes with rectangular order 6, but the tilings in Figures 218 and 259 suggest that such polyominoes may exist.

Order	Tiling
1	All rectangles
2	Figure 227
4	Figure 233
8	Figure 234
10[a]	Figure 200
12[b]	http://polyominoes.org/data/20L1
18[a]	Figure 220(a)
24[a]	Figure 220(b)
28[a]	Figure 221
50[a]	Figure 222
60[a]	Figure 223
76[a]	Figure 224
92[a]	Figure 225
96[a]	Figure 226
138[c]	https://www.cflmath.com/Polyomino/10omino26_rect.html
180[c]	http://www.cflmath.com/Polyomino/8omino10_rect.html
270[c]	http://www.cflmath.com/Polyomino/11omino7_rect.html
246[c]	http://www.cflmath.com/Polyomino/8omino11_rect.html
360[c]	https://www.cflmath.com/Polyomino/14omino01_rect.html
396[c]	https://www.cflmath.com/Polyomino/14omino02_rect.html

Table 37: Examples of orders. From [a]Golomb (1996, p. 97–100), [b]Grekov (2013), http://polyominoes.org/rectifiable and [c]Reid (2003a), http://www.cflmath.com/Polyomino/rectifiable_data.html.

Golomb (1996) gives a tiling of an octomino with 312 copies, but this polyomino was later found to have order 246 (Reid, 2003a, http://www.cflmath.com/Polyomino/8omino11_rect.html).



(a) Polyomino of order 18.

(b) Polyomino of order 24Figure 220: Golomb (1996).

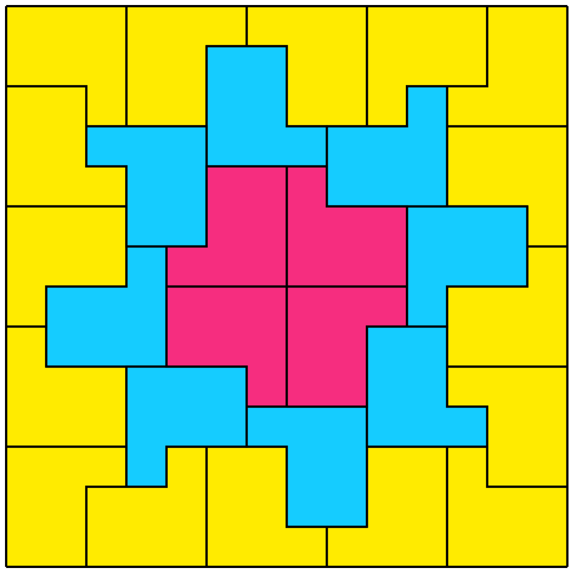


Figure 221: Polyomino with order 28.

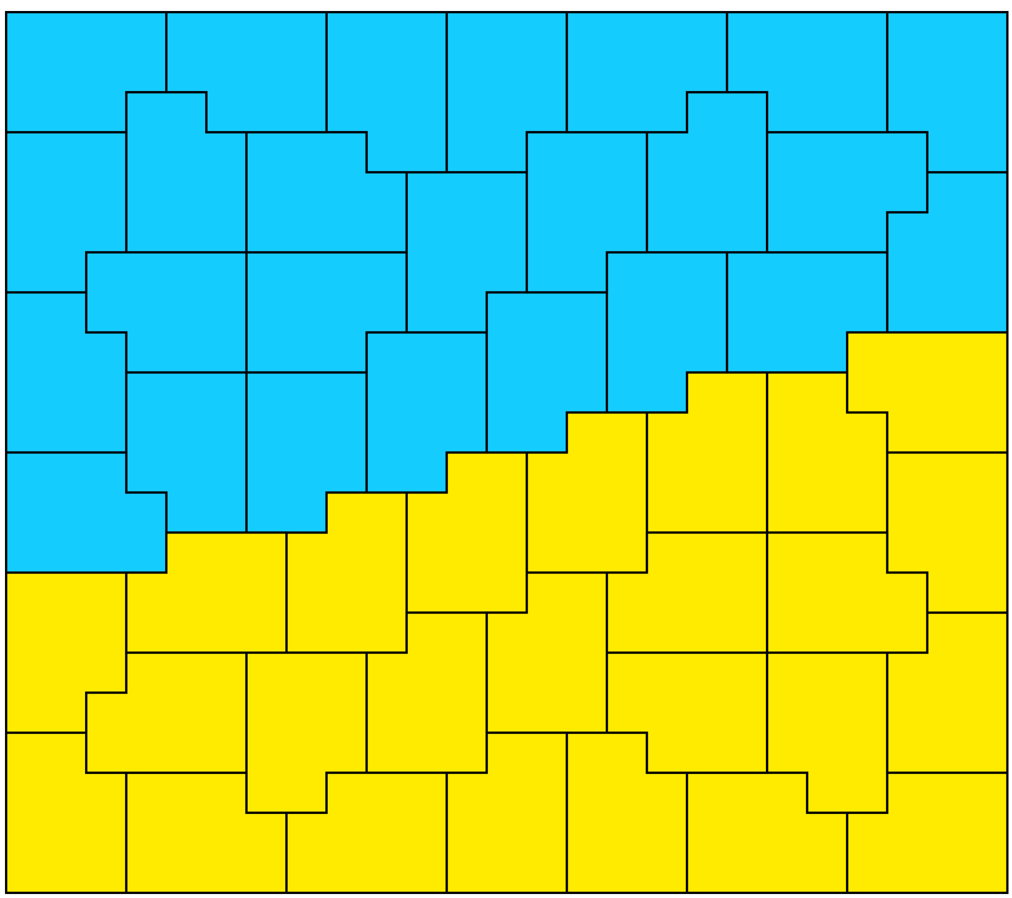


Figure 222: Polyomino with order 50.

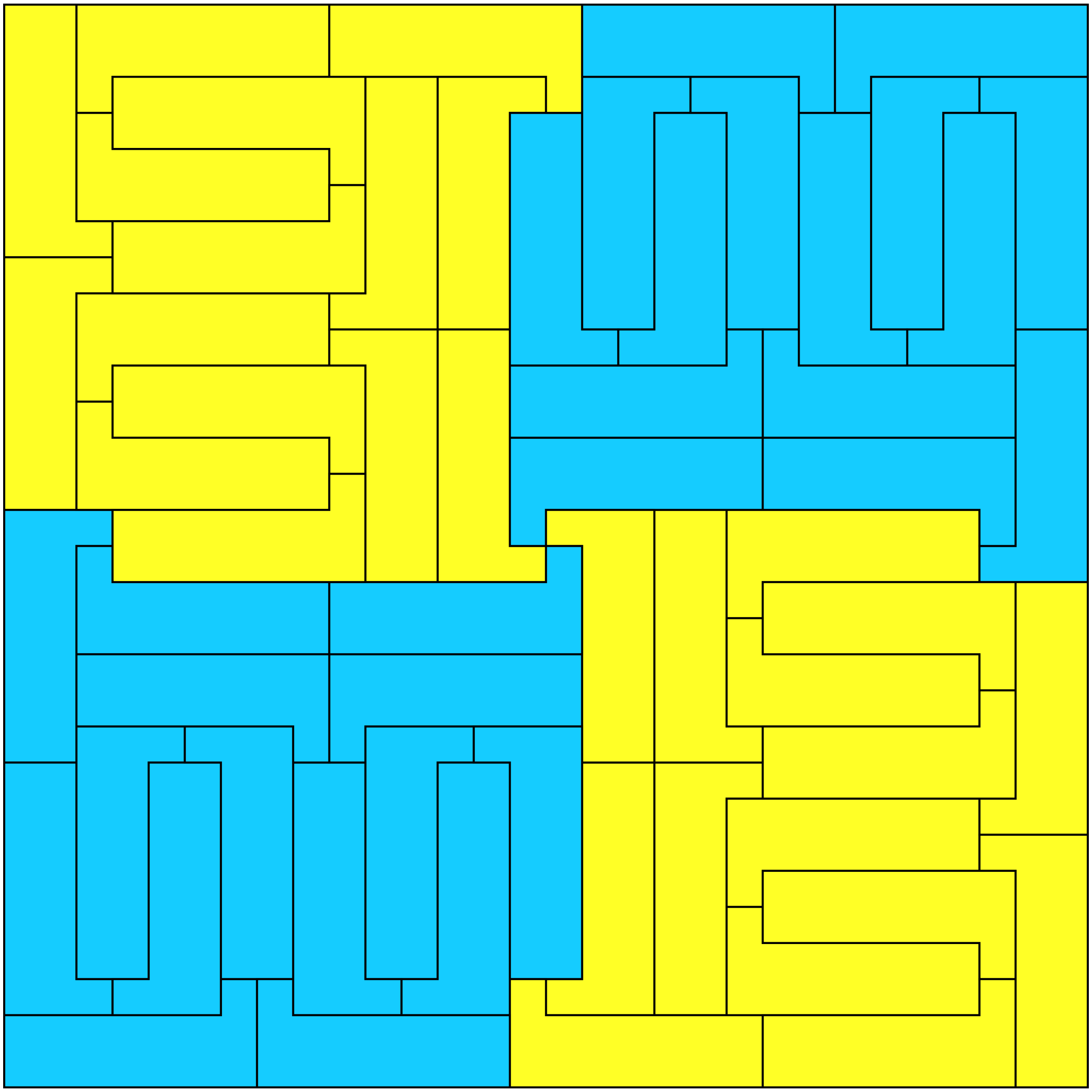


Figure 223: Polyomino with order 60.

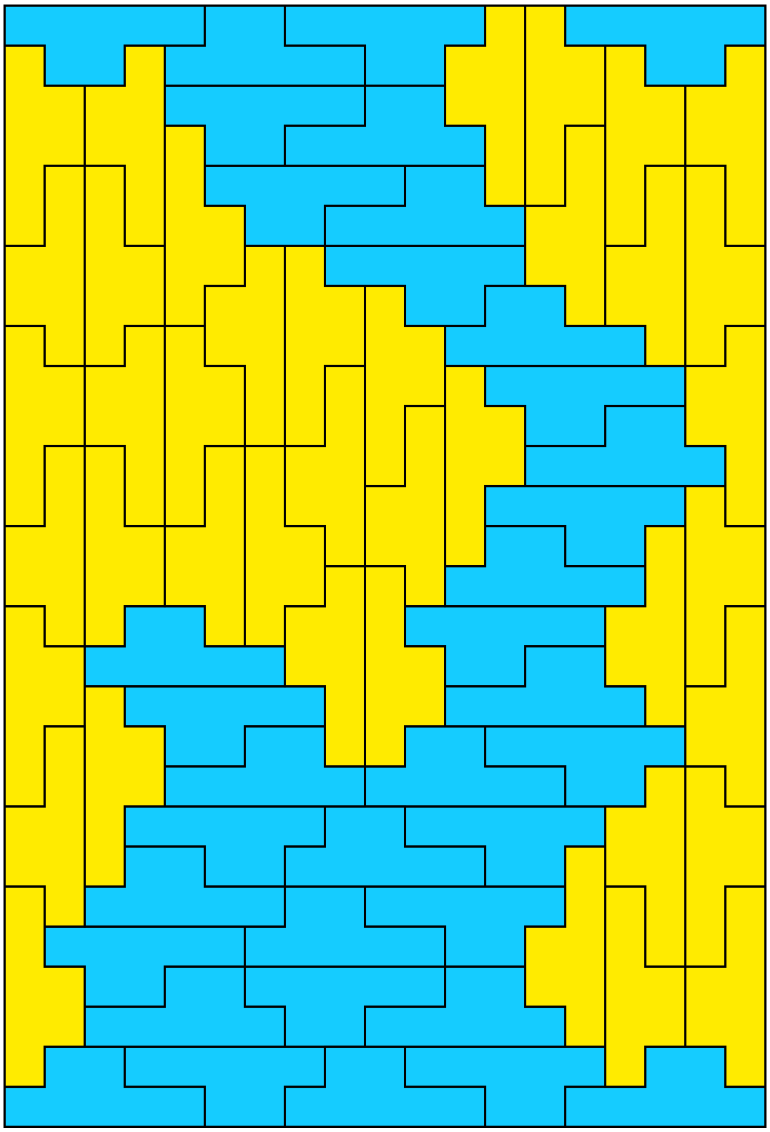


Figure 224: Polyomino with order 76.

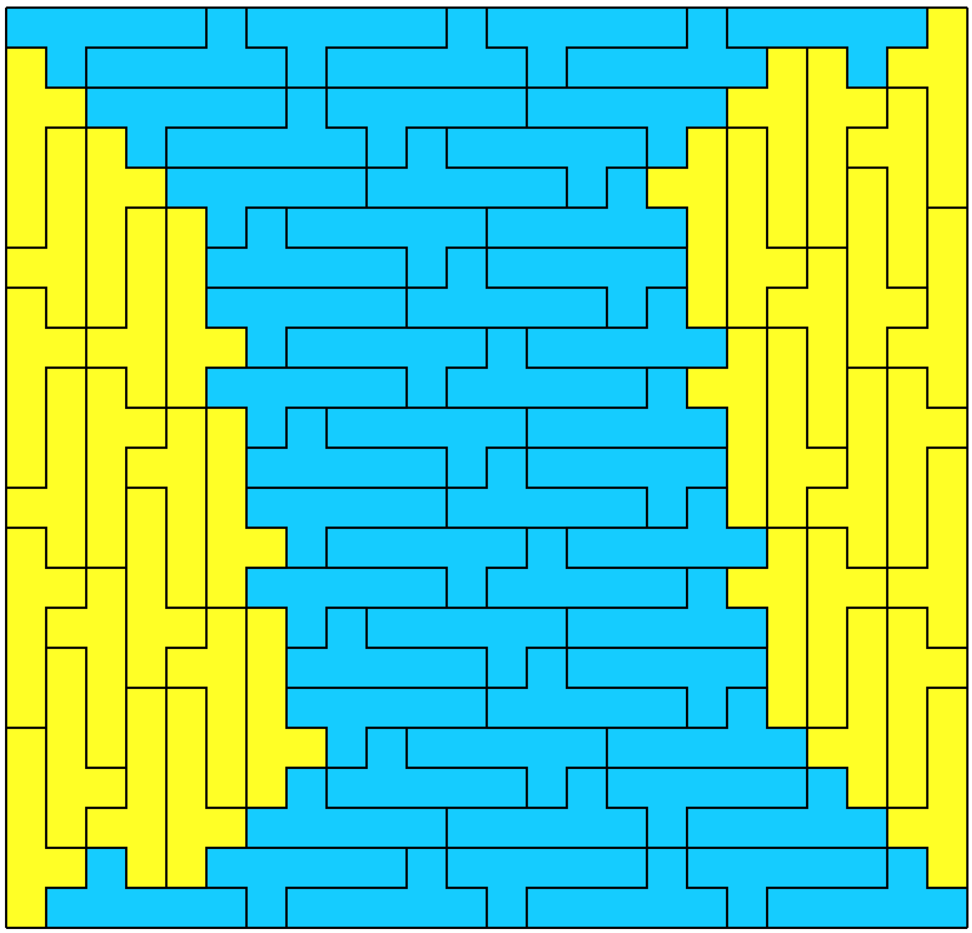


Figure 225: Polyomino with order 92.

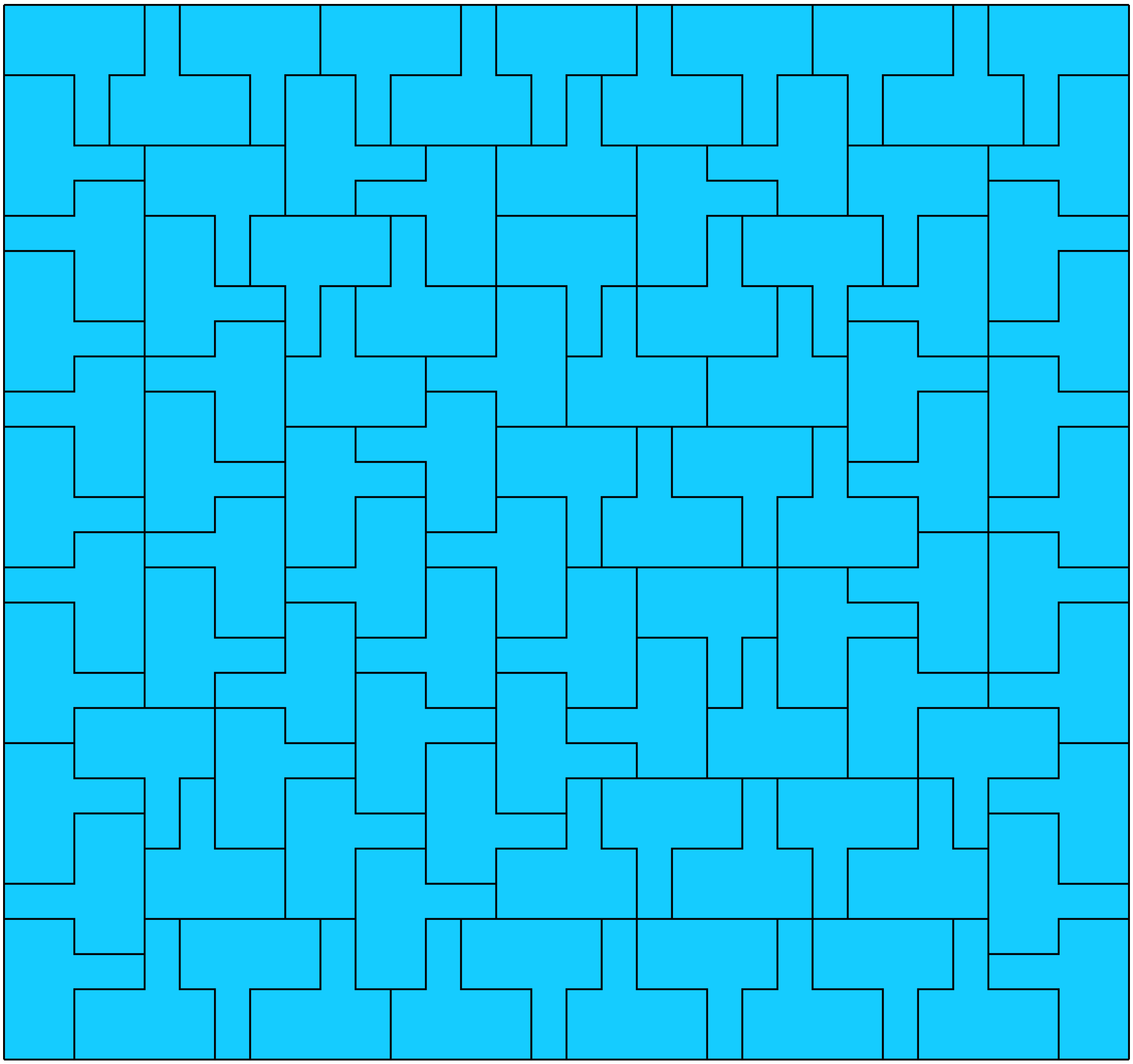


Figure 226: Polyomino with order 96.

Currently, we do not know which other orders greater than 4 are possible, and in particular, we do not know if there are any polyominoes with an order that is odd other than 1 (Winslow, 2018, Open Problem 5).

Only two known polyominoes has order equal to 6 modulo 8: (Reid (2003a), http://www.cflmath.com/Polyomino/11omino7_rect.html and http://www.cflmath.com/Polyomino/8omino11_rect.html).

It is easy to come up with polyominoes of order 2: simply take a rectangle with even area, and divide it into two with a lattice curve that is centrosymmetric.

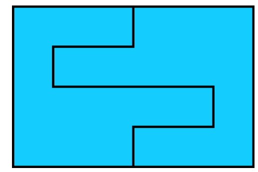
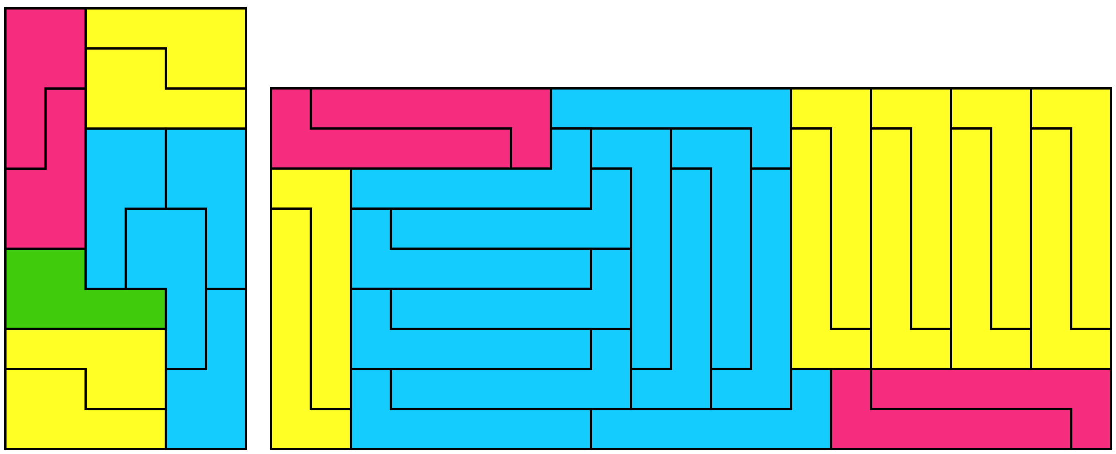


Figure 227: An example of a polyomino with rectangular order 2.

Table 38 shows results for known odd orders. There are also some families, for example odd orders of the form $3(p+2)$ occur for all odd prime p (Reid, 1997).

Figure 228:



(a) Polyomino with odd order 11.

(b) A polyomino with odd order 27.

Figure 229: A polyomino with odd order 27.

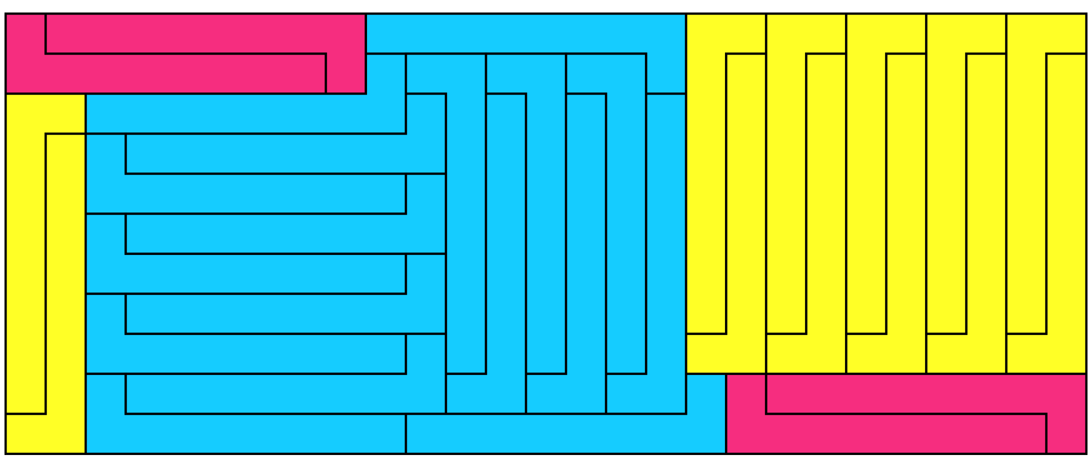


Figure 230: A polyomino with odd order 33.

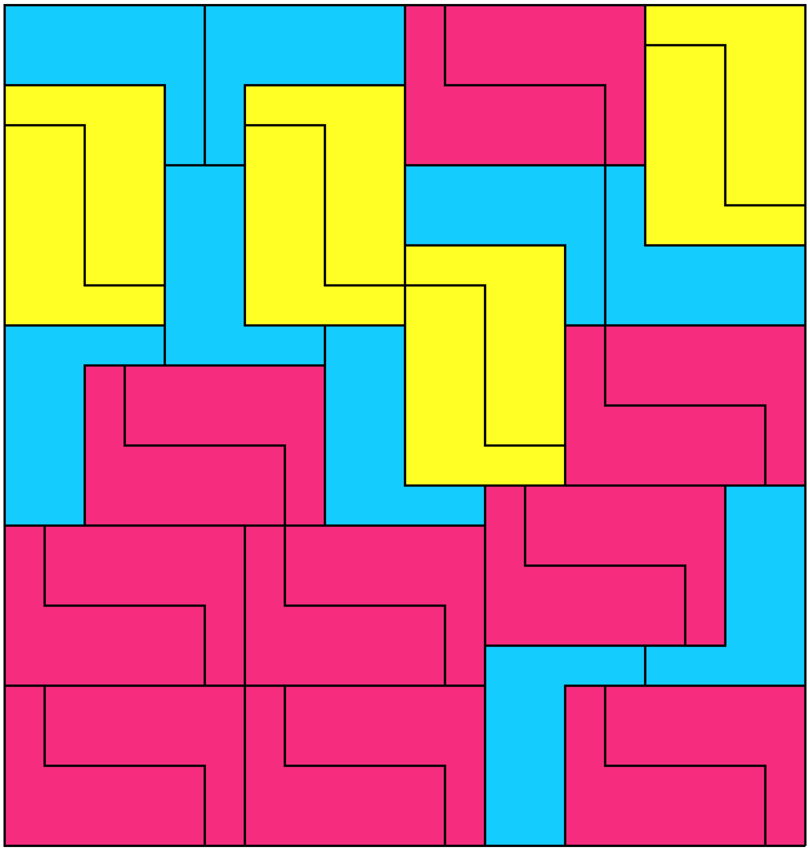


Figure 231: A polyomino with odd order 35.

Odd Order	Tiling
1	All rectangles
11	Figure 229
15	$R(5,9)$ in Figure 253
21	$R(7,15)$ in Figure 259 and 263
27	Figure 229
33	Figure 230
35	Figure 231
45	http://www.cflmath.com/Polyomino/y5_rect.html
	http://www.cflmath.com/Polyomino/7omino4_rect.html

Table 38: Examples of odd orders. From Golomb (1966), Grekov (2013), Reid (2014).

We do know polyominoes cannot have order 3.

Theorem 216 (Stewart and Wormstein (1992))**.** *If three congruent copies of a connected polyomino P tile a rectangle, then P is itself a rectangle and the tiling can only be one of the two shown in Figure 232.*

[Referenced on page 279]

The proof of this theorem is long, but not very difficult. See the reference for details.

Because rectangles have order 1, this theorem implies that *no* polyomino of order 3 exists.

There are 3 known ways in which polyominoes of order 4 can fit together (Golomb, 1996, p. 89-99):

(1) In a construction with symmetry class **Rot2**.

(2) In a construction with symmetry class **Axis2**.

(3) In a construction with symmetry class **Rot**.

Problem 79.**

(1) *Are all of the order 4 polyominoes one of these three types?*

5.2.4 Families

Most polyominoes that tile rectangles occur as **families**. Here are some known families:

(1) Families of order $4s$ (Golomb, 1996, pp. 102–204).

(2) Family of order 8 (Reid, 1998)[11]. The first three polyominoes is shown in Figure 234.

(3) Families of order $2(a^2+b^2)$ with $\gcd(a,b) = 1$ (Marshall, 1997). This family is constructed from a right triangle with sides a and b. Take b copies of a centrosymmetric curve , and replace side of length a with b copies, and side of length b with a copies. Figures 235-237 shows some examples. Compare this construction with the construction of Reid polyominoes discussed in Section 4.1.1.

(4) Families with odd order $3n+6$ for odd primes n (Marshall, 1997). The polyomino is $B(2 \cdot 1^{n-2})$, and tilings are shown in Figure 238. If n is prime, the $R(n+2, 3n)$ is minimal rectangle with a tiling. It is not know whether the rectangles are minimal for composite n, except for $n = 9, 15, 21$.

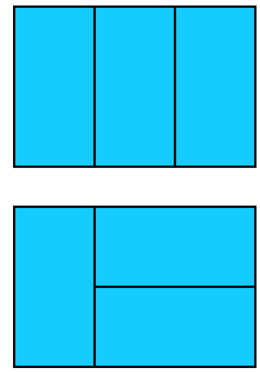


Figure 232: The two ways in which a rectangle can be disected into three congruent shapes.

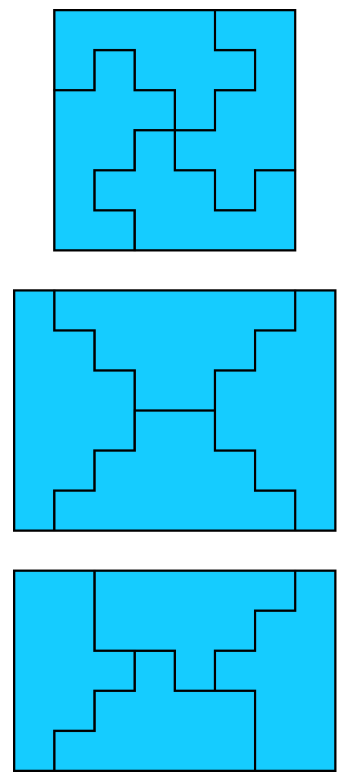


Figure 233: Examples of order 4 polyominoes.

[11] The first polyomino in this family was discovered by Marshall (1997), but Reid (1998) discovered the family it is part of.

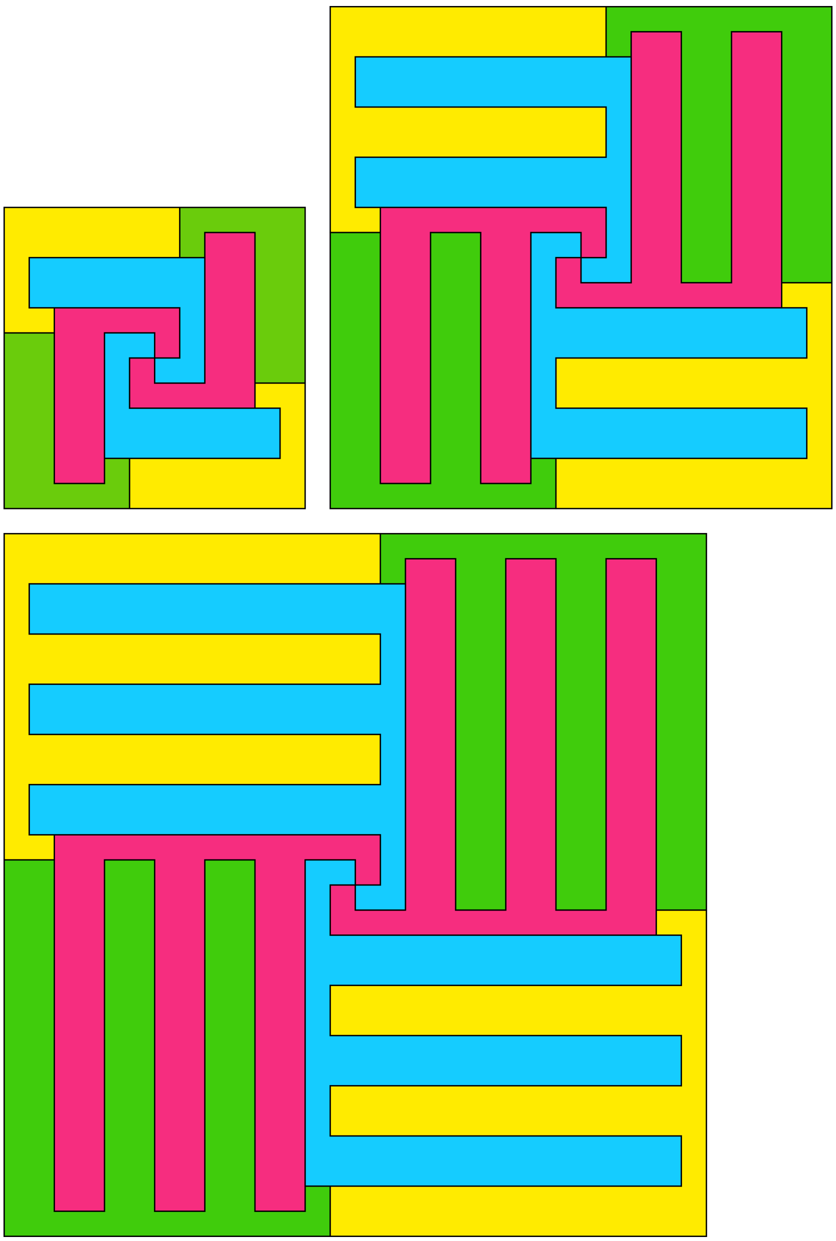


Figure 234: A family of polyominoes of order 8.

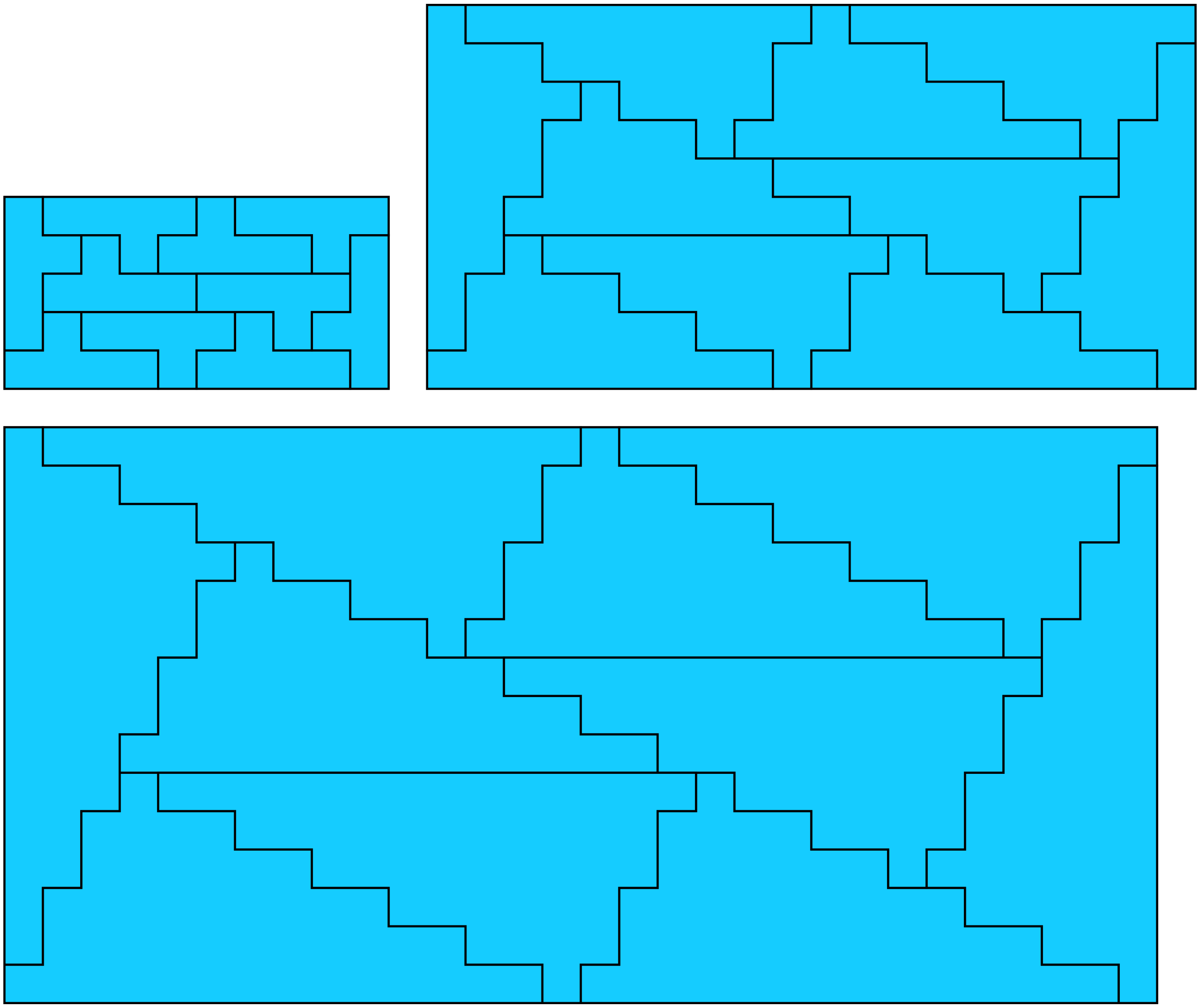


Figure 235: Order 10

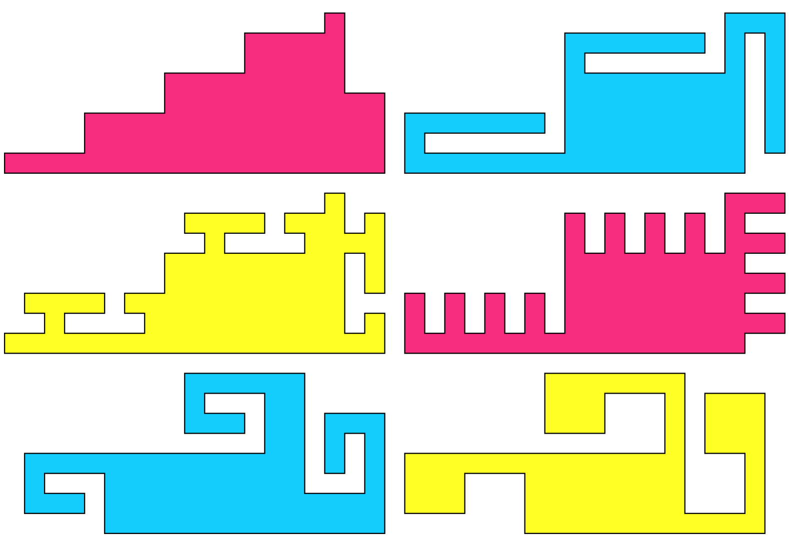


Figure 236: Order 10

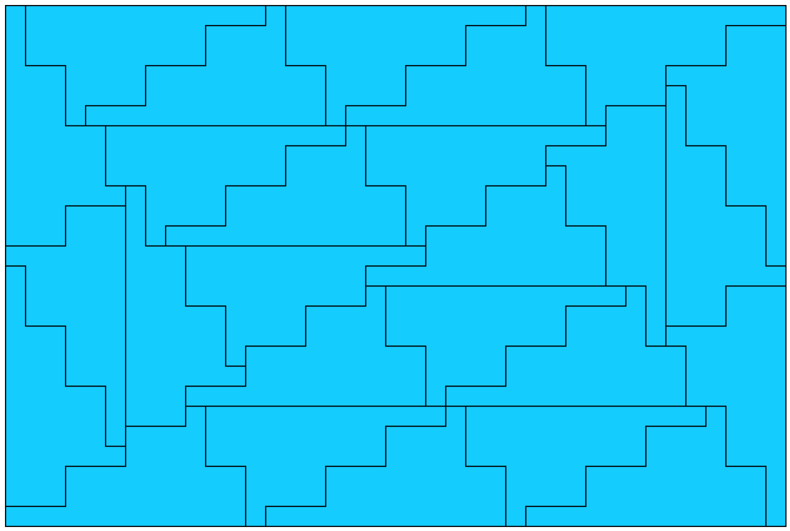


Figure 237: Order 26

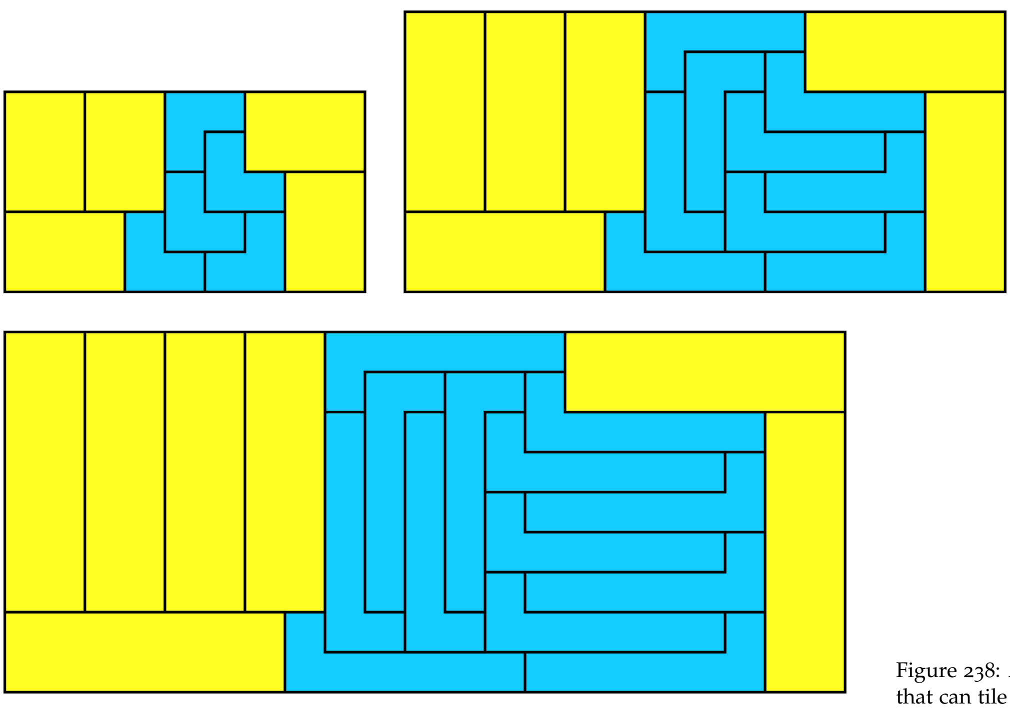


Figure 238: A family of polyominoes that can tile a rectangle with an odd number of pieces.

5.2.5 *The orders of small polyominoes*

The orders of small polyominoes is given in Tables 39–43.

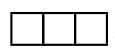
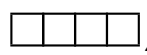
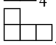
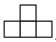
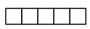
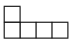
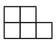
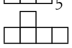
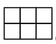
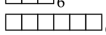
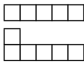
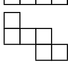
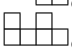
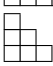
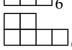
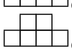
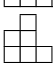
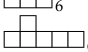
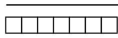
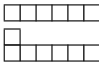
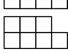
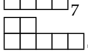
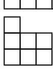
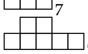
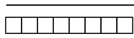
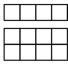
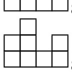
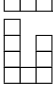
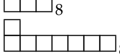
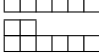
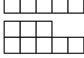
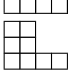
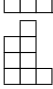
Polyomino	O	PR	SO	OO	$\sqrt{\text{IO}}$
 $_1$	1	1	1	1	2
 $_2$	1	1	2	1	2
 $_3$	1	1	3	1	2
 $_3$	2	2	12	15	2
 $_4$	1	1	4	1	2
 $_4$	1	1	1	1	2
 $_4$	2	2	4	∞	2
 $_4$	4	1	4	∞	4
 $_5$	1	1	5	1	2
 $_5$	2	2	20	21	4
 $_5$	2	2	20	21	2
 $_5$	10	40	20	45	9
 $_6$	1	1	6	1	2
 $_6$	1	1	6	1	2
 $_6$	2	≥ 8	6	21	?
 $_6$	2	≥ 1	24	?	?
 $_6$	2	≥ 1	24	?	?
 $_6$	2	≥ 1	24	?	?
 $_6$	2	5	6	11	?
 $_6$	4	2	24	?	?
 $_6$	18	≥ 26	96	∞	?
 $_6$	92[a]	≥ 45	≤ 384	?	?
 $_7$	1	1	7	1	2
 $_7$	2	2	28	27	6
 $_7$	2	≥ 3	28	33	4
 $_7$	2	≥ 4	28	≤ 57	?
 $_7$	28	≥ 37	28	45	?
 $_7$	76[b]	≥ 26	?	≤ 153	?
 $_8$	1	1	8	1	2
 $_8$	1	1	2	1	2
 $_8$	2	≥ 1	2	?	4
 $_8$	2	≥ 1	2	?	?
 $_8$	2	4	8	∞	4
 $_8$	2	≥ 5	8	?	4
 $_8$	2	≥ 5	8	?	4
 $_8$	2	≥ 4	2	?	4
 $_8$	≤ 180	≥ 2	?	?	?

Table 39: Table giving order information about polyominoes. Adapted from Grekov (2013). See http://polyominoes.org/rectifiable.

O = Order
PR = Prime Rectangles
SO = Square Order
OO = Odd Order
IO = Reptile order

A dash in the OO column indicates the polyomino is even.

[a]Dahlke (1989b) [b]Dahlke (1989a)

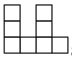
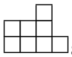
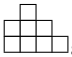
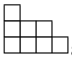
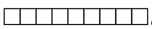
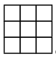
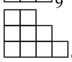
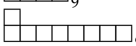
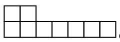
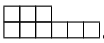
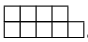
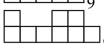
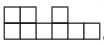
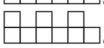
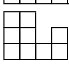
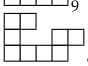
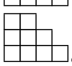
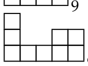
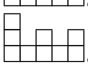
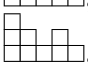
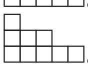
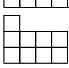
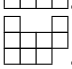
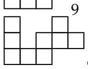
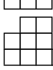
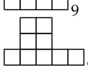
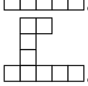
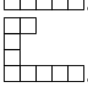
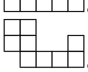
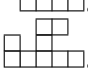
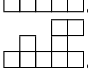
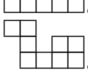
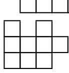
Polyomino	O	PR	SO	OO	$\sqrt{\text{IO}}$
 8	2	1	2	-	≤ 4
 8	24	11	32	-	≤ 16
 8	246	≥ 167	≤ 288	?	
 8	2	1	2	-	
 9	1	1	9	1	2
 9	1	1	1	1	2
 9	2	≥ 17	4	≤ 247	6
 9	2	≥ 12	36	33	8
 9	2	≥ 9	?	≤ 95	?
 9	2	≥ 4	4	15	2
 9	2	13	36	49	?
 9	2	1?	4	-	?
 9	2	1	4	-	?
 9	2	1	4	-	?
 9	2	1	4	-	?
 9	2	1	4	-	?
 9	2	24	4	115	?
 9	2	1?	4	-	?
 9	2	1	4	-	?
 9	2	1	4	-	?
 9	2	1?	4	-	?
 9	4	8	4	45	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?

Table 40: Table giving order information about polyominoes. Adapted from Grekov (2013) and Reid (2003a). See http://polyominoes.org/rectifiable.

O = Order

PR = Prime Rectangles

SO = Square Order

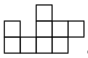
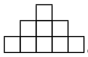
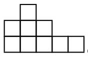
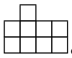
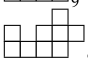
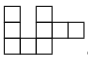
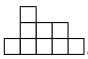
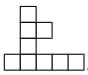
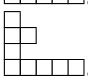
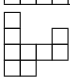
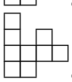
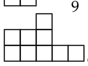
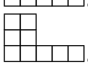
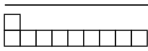
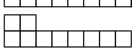
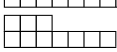
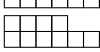
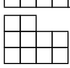
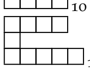
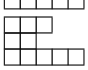
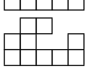
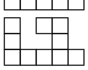
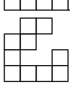
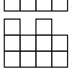
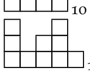
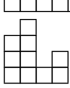
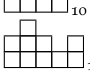
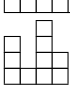
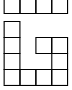
OO = Odd Order

IO = Reptile order

A dash in the OO column indicates the polyomino is even.

[a] http://www.cflmath.com/Polyomino/14omino01_rect.html [b] http://www.cflmath.com/Polyomino/14omino02_rect.html

Table 41: Table giving order information about polyominoes.

Polyomino	RO	PR	SO	OO	$\sqrt{IO}$
 9	4	1	4	-	?
 9	4	1?	4	-	?
 9	4	1?	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	1	4	-	?
 9	4	57	4	221	?
 9	4	59	4	≤ 87	6
 10	2	≥ 2	10	?	?
 10	2	≥ 15	10	99	?
 10	2	1?	10	?	?
 10	2	25	10	49	?
 10	2	8	≤ 40	33	
 10	2	≥ 7	≤ 40	?	?
 10	2	1	40	-	?
 10	2	1	40	-	?
 10	2	1	40	-	?
 10	2	1	40	-	?
 10	2	1	40	-	?
 10	2	1	40	-	?
 10	2	1	40	-	?
 10	2	1	40	-	?
 10	2	1	40	-	?
 10	2	1	40	-	?

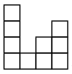
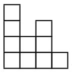
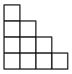
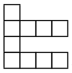
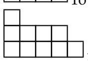
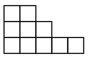
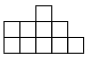
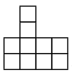
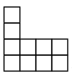
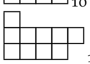
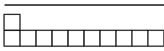
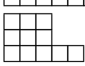
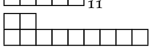
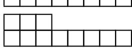
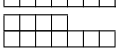
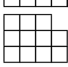
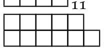
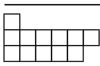
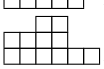
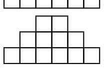
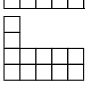
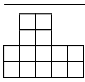
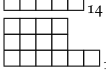
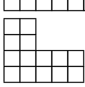
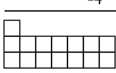
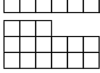
Polyomino	RO	PR	SO	OO	$\sqrt{IO}$	Reference
 10	2	1	40	-	?	
 10	2	1	40	-	?	
 10	2	1	40	-	?	
 10	2	1	40	-	?	
 10	2	1	40	-	?	
 10	2	1	40	-	?	
 10	4	2	≤ 160	-	?	
 10	96	255	≤ 160	117	?	
 10	2	10	40	45	?	
 10	138	330	250	221	?	
 11	2	2	?	39	?	
 11	≤ 270	≥ 233	?	≤ 423	?	
 11	2	≥ 1	?	?	?	
 11	2	≥ 1	?	?	?	
 11	2	≥ 1	?	?	?	
 11	50	127	141	≤ 176	?	
 11	2	≥ 5	44	≤ 175	?	
 12	8	≥ 2	12	?	12	
 12	180	≥ 304	?	-	?	
 12	4	≥ 1	≤ 288	?	?	http://tilingsearch.org/HTML/data113/G08B.html
 12	2	≥ 2			?	http://tilingsearch.org/HTML/data170/M13D.html
 14 [a]	360	≥ 274	?	≤ 402609	?	
 14 [b]	396	?	?	?	?	
 14	2	?	≤ 14	≤ 63	?	http://tilingsearch.org/HTML/data170/M13I.html
 15	60	≥ 1	60	?	?	
 15	2	?				http://tilingsearch.org/HTML/data170/M13H.html

Table 42: Table giving order information about polyominoes.

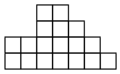
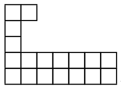
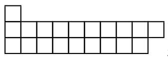
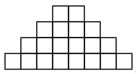
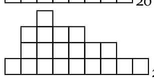
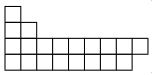
Polyomino	RO	PR	SO	OO	$\sqrt{IO}$	Reference
 18	4					http://tilingsearch.org/HTML/data113/G09B.html
 18	8					http://tilingsearch.org/HTML/data170/M9A.html
 20	12	≥ 3	20	?	?	
 20	4					http://tilingsearch.org/HTML/data113/G08C.html
 20	10					http://tilingsearch.org/HTML/data113/K15.html
 20	12					http://tilingsearch.org/HTML/data113/K9.html

Table 43: Table giving order information about polyominoes.

5.3 *Prime Rectangles*

A **prime rectangle** of a polyomino is a rectangle that can be tiled by that polyomino, and cannot be split into any number of smaller rectangles that can be tiled by that polyomino. A **strong prime rectangle** is a rectangle that can be tiled by the polyomino, but cannot split into *two rectangles* that can each be tiled by that polyomino (Reid, 2005, 3.1 – 3.5).[12]

It should be clear that every strong prime rectangle is a prime rectangle. Is the converse true? For sets of polyominoes, it is not. For example, I_3 and I_4 tile $R(5,5)$ (Figure 239), but cannot do so with a fault. However, for single polyominoes we do not know whether there are primes that are not also strong primes.

[12] Some authors use the word "prime rectangle" for what we refer to as a "strong prime rectangle" ((Klarner, 1981)). Unfortunately and confusingly, the definitions in Reid (2005) for prime rectangle and strong prime seems to be swapped. Earlier versions of this book made the same mistake.

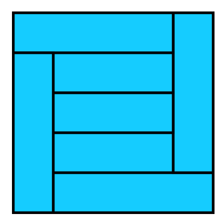


Figure 239: $R(5,5)$ can be tiled by $R(3,1)$ and $R(4,1)$, but it cannot be split into two rectangles that can both be tiled by these tiles. For these tiles, $R(5,5)$ is a prime rectangle, but not a strong prime rectangle.

Problem 80** (Reid (2005), Question 3.9)**.** *Is there a polyomino with a prime rectangle that is not a strong prime rectangle?* .

Theorem 217 (Reid (2005), Theorem 3.6)**.** *The set of prime rectangles of a polyomino is finite.*

[Not referenced]

See the reference for a proof. For two alternative proofs, see de Bruijn and Klarner (1975).

Theorem 218 (Reid (2005), Proposition 3.10)**.** *A rectangle has only one prime rectangle: itself.*[13]

[13] Mentioned without proof in Klarner (1981).

[Not referenced]

Proof. We need to show that any rectangle $R(m,n) \neq R(p,q)$ that is tileable by a rectangle $P = R(p,q)$ can be split into two rectangles that can each be split into two rectangles, each of which is tileable by P.

From Theorem 168 we know that one of the following is true:

(1) $p \mid m$ and $q \mid n$

(2) $pq \mid m$ and $n = px + qy$ for $x, y > 0$.

In the first case, we can divide the rectangle into either

- $R(m-p,n)$ and $R(p,n)$ if $m > p$
- $R(m,n-q)$ and $R(m,q)$ if $n > q$

In the second case, we can divide the rectangle in $R(m,px)$ and $R(m,qy)$, two rectangles of case (1).

So no other rectangle other than $R(p,q)$ can be a prime of $R(p,q)$.

□

5.3.1 *Trominoes*

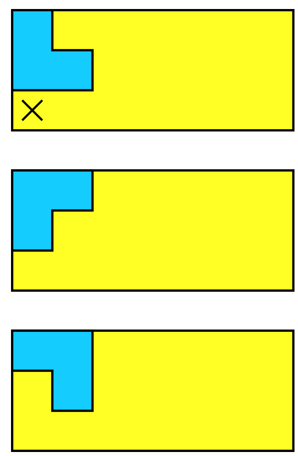


Figure 240: The three ways a tromino can fit into the top left corner of a rectangle.

Theorem 219 (Chu and Johnsonbaugh (1985)). *The right tromino tiles a rectangle iff* $3|mn$, *and when one side is* 3 *the other is even.*

[Referenced on page 228]

Proof. (The proof follows the outline in Ash and Golomb (2004), p. 48.)

(1) $R(3m,2n)$ is tileable by Theorem 22.

(2) $R(6k,2n+3)$ os tileable by Theorem 164.

(3) $R(9+6k,2n+5)$ can be split into four rectangles: $R(9,5)$ (see Figure 253), $R(6k,2n+5)$ (tileable by (2)), $(9,2n)$ (tileable by (1)) and $R(6k,n2)$ (tileable by (1)).

(4) $R(3,2n+1)$ is not tileable. There are only three ways to place a tromino in the top left corner. Only two of them can work; for both these cases the only tiling option is to complete the $R(3,2)$ rectangle. This reduces the tiling to that of $R(3,2n-1)$. We continue this, until a single row of three cells is left, which is untileable.

□

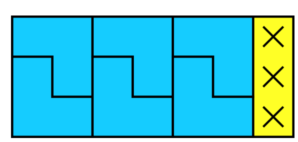


Figure 241: Trying to tile $R(3,2n+1)$ with right trominoes always result in a column that cannot be tiled.

5.3.2 Tetrominoes

Theorem 220 (Martin (1991), p. 50)**.** *The skew tetromino does not tile any rectangle.*

[Not referenced]

Proof. The proof follows directly from Theorem 198.

Here is an alternative proof given in Martin (1991): The polyomino can be placed in the corner in one of two ways, which are equivalent. The placement produces two notches which can fit the polyomino only one way each, so that both neighbors are determined. This produces new notches, which eventually result in a untileable cell in the adjacent corner of the rectangle. □

Theorem 221 (Golomb and Klarner (1963))**.** *The L-tetromino tiles $R(m, n)$ iff $8 \mid mn$ and $m, n > 1$.*

[Not referenced]

Proof. The L-tetromino satisfies the conditions of Theorem 191, and hence it is even. Therefore, the area must be divisible by 8. The proof that all these rectangles have tilings uses the same ideas as used in Theorem 219. Figure 255 shows a tiling for $R(8, 3)$. □

We next want to prove that if a rectangle is tileable by T-tetrominoes, both sides are divisible by four. To do this, we will show that any tiling of the quadrant satisfy certain properties, and if we stack rectangles together to form a quadrant tiling, these properties can only be satisfied if both sides are divisible by 4.

We introduce some terminology from (Walkup, 1965, Definition 1). An edge of the quadrant is called a **cut** if it coincides with an outer edge of some T-tetromino in every tiling of the quadrant. A point[14] in the quadrant is called **cornerless** if it does not coincide with an convex corner[15] of a T-tetromino in any tiling of the quadrant. A point is a **type-A point** if it is congruent to $(0, 0)$ or $(2, 2)$ modulo 4. A point is a **type-B point** if it is congruent to $(0, 2)$ or $(2, 0)$ modulo 4.

[14] Walkup (1965) calls this a *vertex*.

[15] Walkup (1965) calls this an *outside vertex* of the tetromino.

Theorem 222 (Walkup (1965), Lemma 1)**.** *In a tiling of the quadrant by T-tetrominoes, every type-B point is cornerless, and every edge incident with a type-A point is a cut.*

[Referenced on page 230]

Proof. For integers $k \geq 0$, let $P(k)$ be the proposition that the theorem holds for all type-A and type-B points on or below the line $x + y = 4k$.

$P(0)$ is true, because the two edges incident on the origin are cuts. We will prove that $P(k)$ implies $P(k+1)$, and will use the Figure 242 as reference, which shows the situation for $k = 2$. The cuts and cornerless points required for $P(k)$ are shown by heavy lines and dots.

(1) We show that edges a and b, and their translates, are cuts.

If a tiling of the quadrant contains the tetromino $1-2-3-4$, it will also contain $8-10-11-12$, and so all upwards translates of $1-2-3-4$, because no other arrangement can cover cells 8 and 9. But this leads to two cells $(13, 14)$ at the y-axis that cannot be covered. Therefore, the tiling cannot contain $1-2-3-4$. By symmetry, it can also not contain $1-5-6-7$.

There are four other ways to cover 1; all of them has a and b as outer edges, and therefore a and b are cuts, and so are their translates. These are marked in Figure 2.

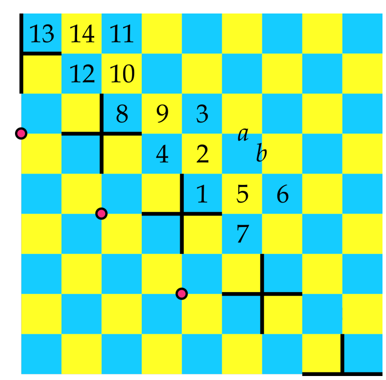
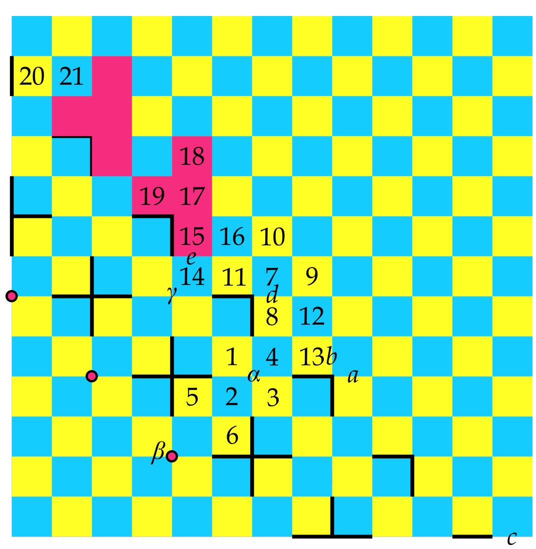


Figure 242:

(2) We now show that point α is cornerless. Suppose, to the contrary, that there is a tiling containing a T-tetromino having α as an convex corner.



In this tetromino the cell forming the outside corner at α cannot be 1, 2, or 3 because of the nearby cut segments. Consequently, either $1-2-3-5$ or $1-2-3-6$ is a tetromino of the supposed dissection.

But no tetromino could then contain square 6 or 5 respectively, as A is a cornerless point. This proves that α is cornerless. By the same or similar arguments all translates of α, either inside the quadrant or on the axes, are cornerless.

(3) Finally, we show edges a and b, and their translates, are cuts.

By symmetry, it is enough to show a is a cut, and since c is a cut, it is enough to show that if an edge a is a cut, so is its translate d.

Suppose, a is a cut, and suppose, for a contradiction, that cells 7 and 8 lie in the same tetromino. The tetromino cannot contain 4, since α is cornerless. It must contain two of the three cells 9, 10, and 11. If it contained 9, there would be no way to cover 12 and 13. The tetromino must thus be $7-8-10-11$. The cells 14 and 15 must be covered by different tetrominoes, since γ is cornerless. Then the only way to cover 15 and 16 is for $15-17-18-19$ to be part of the tiling. Then upwards translates of $15-17-18-19$ must be part of the tiling, which means that there is no way to cover 20 and 21, a contradiction. Therefore, 7 and 8 must lie in different tetrominoes and so d is a cut.

We have shown a and b and their translates are cuts.

Taken together, we have shown that $P(k)$ implies $P(k+1)$, and the theorem follows from induction. □

Theorem 223 (Walkup (1965), Theorem 1)**.** *The T-tetromino tiles $R(m,n)$ iff $4 \mid m$ and $4 \mid n$.*

[Referenced on page 247]

Proof. If. The T-tetromino tiles $R(4,4)$; and so tiles $R(m,n)$ by Theorem 22.

Only if. We cannot have m congruent to 2 module 4, because this puts a corner of R at a type-B point, which is impossible since type-B points are cornerless (Theorem 222). Moreover, m cannot be congruent to 1 or 3 module 4. If the edge of the rectangle is placed at either P or Q, there is no way to cover cells 22 or 23. Thus, m and by symmetry, must be congruent 0 modulo 4. □

Problem† 81 (Walkup (1965), Definition 2, Theorem 2)**.** *Let $R = R(m,n)$ be a rectangle with a tiling by T-tetrominoes. A block is a 2×2 square whose points have even coordinates. A chain of R is any minimal subset of R that is both a union of blocks and a union of T-tetrominoes from the tiling.*

Show that if we color every other block of R in the checkerboard fashion, then each tetromino has three cells in one block and one in an adjacent block. Every chain consists out of an even number of blocks, which may be cyclically ordered in such a way that the blocks are alternately colored black and white, and the T-tetrominoes of the chain contain three cells of the one block and one cell of the succeeding block.

5.3.3 Pentominoes

We state a useful result for V-like n-ominoes. Let $V_n = B((n-2) \cdot 1^2)$ (Saxton Jr, 2015).

Theorem 224 (Saxton Jr (2015), Theorem 2.3, p. 10)**.** *For $n \geq 5$, V_n does not tile a quadrant.*

[Referenced on pages 231 and 232]

The proof is long and tedious with many cases. See the reference for details.

Theorem 225 (Cibulis and Liu (2001))**.** *The F-, S-, T-, U-, V-, W-, X-, and Z-pentominoes do not tile a rectangle.*

[Referenced on page 261]

Proof. The X-pentomino does not cover any corner of its hull, so it cannot tile a rectangle (Theorem 193).

Placing the F-, T- and Z-pentominoes in any position in a corner leaves one cell that cannot be tiled.

The U-pentomino can cover the corner in only one way without cutting off any cells. But then the gap in the U can be covered only one way, which leaves one cell that cannot be covered.

The S- and W-pentominoes are not rectifiable by Theorem 198.

The V-pentomino cannot tile a quadrant (Theorem 224) and thus not a rectangle (Theorem 137).

An alternative proof from Cibulis and Liu (2001): The V-pentomino can be placed in the corner in two ways. The first creates an inaccessible pair. The second, forces a second to be placed as shown (otherwise it is not tileable for the same reason as before), and a third. This configuration leads to an inaccessible pair. □

Theorem 226. *The L-pentomino tiles:*

(1) $R(5m, 2n)$

(2) $R(10m, 2)$, $R(10m, n+4)$

(3) $R(15m, 7+2n)$

[Not referenced]

Theorem 227. *The P-pentomino tiles these and only these rectangles:*

(1) $R(5m, 2n)$

(2) $R(10m, 2)$, $R(10m, n+4)$

(3) $R(15m, 7+2n)$

[Referenced on page 249]

Proof. Figure 259 gives tilings of $R(2,5)$ and $R(15,7)$. From this we can construct $R(5m, 2n)$ which proves (1). In particular, we can construct $R(15,2)$, so together with $R(15,7)$ we can construct all $R(15, 7+2k)$, which proves (3). From (1) we can construct $R(10,5)$ and $R(10,2)$, putting these together proves (2).

Since $5|mn$ (Theorem 1), 5 divides either m or n. The only remaining rectangles we have not given tilings for are $R(1,5m)$, $R(5,2n+1)$, and $R(10,3)$. We prove now these are impossible.

The pentomino cannot fit $R(5,1)$, so $m, n > 1$.

Any rectangle that fits k P-pentominoes must also fit k $R(2,2)$ squares.

$R(5, 2n+1)$ fits $\left\lfloor \frac{5}{2} \right\rfloor \left\lfloor \frac{2n+1}{2} \right\rfloor = 2n$ squares (Theorem 182), but $2n+1$ are required; therefore no tiling of $R(5, 2n+1)$ is possible.

Similarly, $R(10,3)$ fits only 5 squares, but 6 are required. So $R(10,3)$ cannot be tiled. □

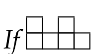
Theorem 228 (Sillke (1992)[16]). *The complete list of 40 prime-rectangles of the Y-pentomino is given in Tables 45–46.*

[Not referenced]

Proof. See Reid (2005, Example 5.2) for an outline. □

5.3.4 Hexominoes

Proving which hexominoes cannot tile rectangles is a bit tedious. In most cases we find an inaccessible cell within two steps, in other cases we can use one of the theorems to show the polyomino is untilable. This is summarized in Table 44.

Theorem 229 (Reid (2003a)[17]). *If*  $_6$ *tiles* $R(m,n)$, *then*

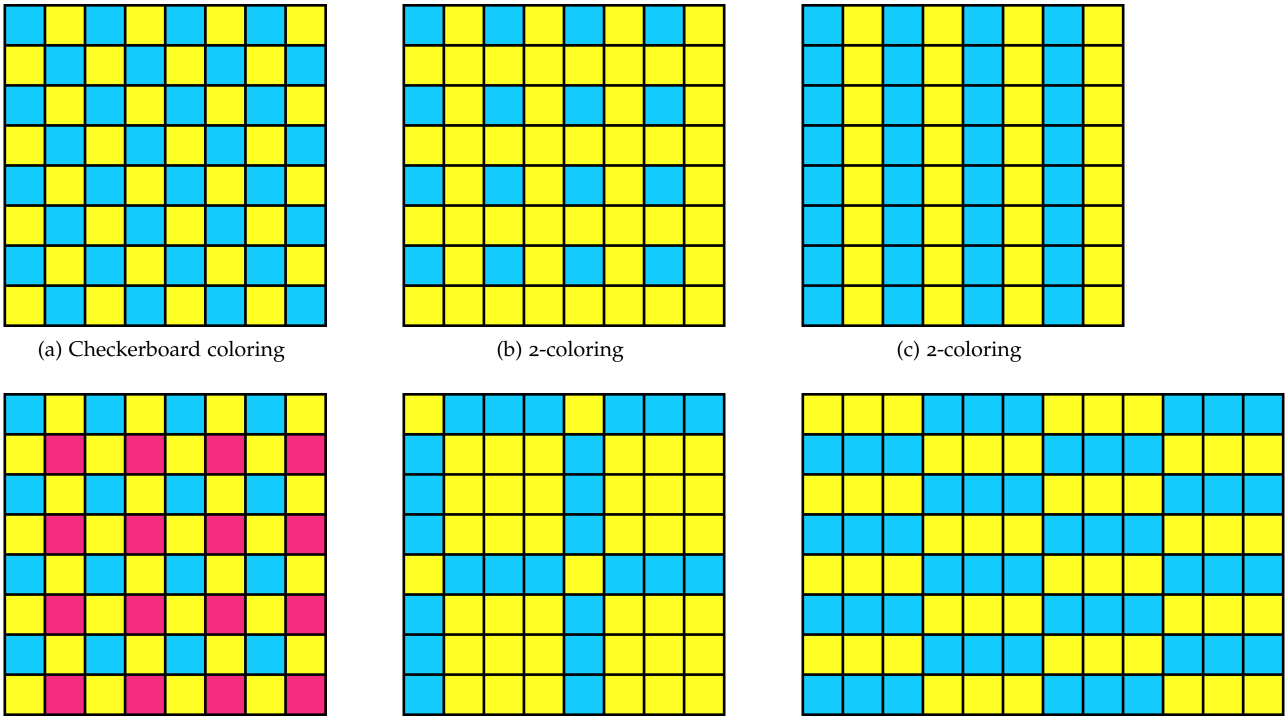
(1) $12 \mid mn$

(2) $4 \mid m$ *or* $4 \mid n$

[16] According to Reid (2003a) (http://www.cflmath.com/Polyomino/y5_rect.html) the first complete list was published in Fogel et al. (2001), but I could not locate the reference. The thesis by one of the authors Goldenberg (2001) also give all 40 prime rectangles; presumably the paper was based on the thesis. The complete list was identified before then by Sillke (1992). Reid (2005, Example 5.2) gives an overview of the publishing history of the prime rectangles.

[17] http://www.cflmath.com/Polyomino/f6_rect.html).

Note that Reid (2003a) calls this hexomino the F-hexomino.



(a) Checkerboard coloring (b) 2-coloring (c) 2-coloring

(d) 3-coloring (e) 2-coloring (f) 2-coloring

Figure 243: Various coloring used for proofs in this section.

Hexomino	Reason	Hexomino	Reason	Hexomino	Reason	Hexomino	Reason
A	✓	J	✓	R	✓	X	Thm. 193
C	Thm. 202	K	×	Long S	Thm. 198	Italic X	Thm. 193
D	✓	L	✓	Long T	×	High Y	✓
E	×	M	×	Short T	××	Low Y	Thm. 202
High F	×	Long N	Thm. 198	U	✓	Short Z	Thm. 198
Low F	×	Short N	Thm. 198	V	Thm. 224	Tall Z	×
G	Thm. 201	O	✓	Wa	××	High 4	×
H	×	P	✓	Wb	Thm. 198	Low 4	×
I	✓	Q	Thm. 198	Wc	×		

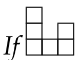
Table 44: Summary of proofs for pentominoes that do not tile rectangles.

.

[Not referenced]

Proof. The hexomino is unbalanced, so the first part follows from Theorem 191.

This implies that all rectangles are of the form $R(4m, n)$ or $R(2m + 2, 2n + 2)$. We prove that the $R(2m + 2, 2n + 2)$ rectangle cannot be tiled: Color the rectangle such that (x, y) is black if x and y are both even. No matter how the polyomino is placed, it covers an even number of black cells. However, $R(2m + 2, 2n + 2)$ has an odd number of black cells. Therefore, it cannot be tiled. □

Theorem 230 (Reid (2003a)[18]). *If*  $_6$ *tiles* $R(m, n)$*, then* $4 \mid m$ *or* $4 \mid n$.

[18] http://www.cflmath.com/Polyomino/j6_rect.html

[Not referenced]

Proof. The polyomino always tiles in pairs, as shown in Figure 244, so we only need to show the tiling capability of these shapes. The area is therefore divisible by 12, and so it suffices to prove that we cannot tile $R(4m + 2, 4n + 2)$.

Apply this coloring: a cell is black if and only if x or y, but not both, are divisible by 4 (Figure 243(e)). No matter how we place one of the pairs, the pair always cover an odd number of black cells. $R(4m + 2, 4n + 2)$ has an even number of black cells; however, it must be tiled by an odd number of pairs, which will cover an odd number of black cells, which is a contradiction. Therefore, a tiling of $R(4m + 2, 4n + 2)$ is impossible, and so one side must be divisible by 4. □

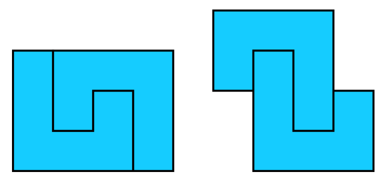
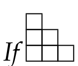


Figure 244: J-hexominoes always tile in pairs.

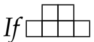
Theorem 231 (Reid (2003a)[19]). *If*  $_6$ *tiles* $R(m, n)$*, then* $4 \mid m$ *or* $4 \mid n$.

[19] http://www.cflmath.com/Polyomino/a6_rect.html

[Referenced on page 365]

Proof. The hexomino is unbalanced with an even number if cells, and so is even (Theorem 191). Thus the area is divisible by 12, and it suffices to show that we cannot tile $R(4m + 2, 4n + 2)$.

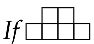
Apply this coloring: a cell is black if and only if x and y are both even (Figure 243(b)). No matter how we place the polyomino, it always cover an odd number of black cells. $R(4m + 2, 4n + 2)$ has an even number of black cells; however, it must be tiled by an odd number of pairs, which will cover an odd number of black cells, which is a contradiction. Therefore, a tiling of $R(4m + 2, 4n + 2)$ is impossible, and so one side must be divisible by 4. □

Theorem 232 (Reid (2003a)[20]). *If*  $_6$ *tiles* $R(m,n)$*, then* $6 \mid m$ *or* $6 \mid n$*.*

[20] http://www.cflmath.com/Polyomino/d6_rect.html

[Not referenced]

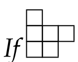
Proof. Apply this coloring: a cell is black if and only if $x + \lfloor \frac{y}{3} \rfloor$ is even (Figure 243(f)). No matter how we place the polyomino, it always cover the same number of black and white squares. For rectangles of the forms $R(6m+2, 6n+3)$ and $R(6m+4, 6n+3)$, we can be colored so that the number of black and white cells are different; which means they cannot be tiled. The only other possibility is $R(6m, n)$ or $R(m, 6n)$, which means 6 divides on of the sides. $\square$

Theorem 233 (Reid (2003a)[21];Reid (2005), Theorem 5.12). *If*  $_6$ *tiles* $R(m,n)$*, then* $4 \mid m$ *or* $4 \mid n$*.*

[21] http://www.cflmath.com/Polyomino/d6_rect.html

[Not referenced]

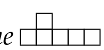
The proof is not very difficult, but requires some concepts that we have not covered. See the second reference for details.

Theorem 234 (Reid (2003a)[22]; Reid (2003b), Theorem 5.4). *If*  $_6$ *tiles* $R(m,n)$*, then* $4 \mid m$ *or* $4 \mid n$*.*

[22] http://www.cflmath.com/Polyomino/g6_rect.html

[Not referenced]

As with the previous theorem the proof requires concepts we have not covered. See the second reference for details.

Theorem 235 (Reid (2003a)[23]). *If the*  $_6$ *tiles a rectangle with an odd side, then the other side is divisible by 8.*

[23] http://www.cflmath.com/Polyomino/y6_rect.html

[Not referenced]

Proof. $R(2m+1, 8n+k)$ is not tileable for odd k, since at least one side must be even (Theorem 1). If $R(2m+1, 8n+2)$ is tileable, then so is $R(2m+1, 8n'+4)$; if $R(2m+1, 8n+6)$ is tileable, then so is $R(2m+1, 8n'+2)$., then so is $R(2m+1, 8n''+4)$.

Therefore, it is enough to show $R(2m+1, 8n+4)$ is not tileable for any integers m and n.

Consider the numbering

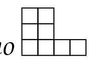
$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are both even} \\ -1 & \text{if } x \text{ and } y \text{ are both odd} \\ 0 & \text{otherwise.} \end{cases}$$

No matter how the tile is placed, it covers ± 2, or $2 \pmod 4$. $R(2n+1, 8n+4)$ also covers a total that is $2 \pmod 4$. However, it would require an even number of tiles, which would cover a total divisible by 4; a contradiction.

Therefore, $R(2m+1, 8n+4)$ is not tileable by Y-hexominoes, and so neither is $R(2m+1, 8n+k)$ for any k not divisible by 8.

Therefore, if $R(2m+1, n)$ is tileable by Y-hexominoes, n is divisible by 8. □

5.3.5 *Other polyominoes*

Theorem 236 (Klarner (1969), Theorem 6)**.** *The polyomino* $_8$ *tiles* $R(m,n)$ *iff* $m, n > 3$ *and* $16 \mid mn$.

[Not referenced]

Proof.

If. If $m, n > 3$ and $16 \mid mn$, then $R(m,n)$ can be cut into rectangles with dimensions $R(4,4)$, $R(5,16)$, $R(6,8)$, and $R(7,16)$, which are all tileable as shown in Figure 245 (See Problem 82).

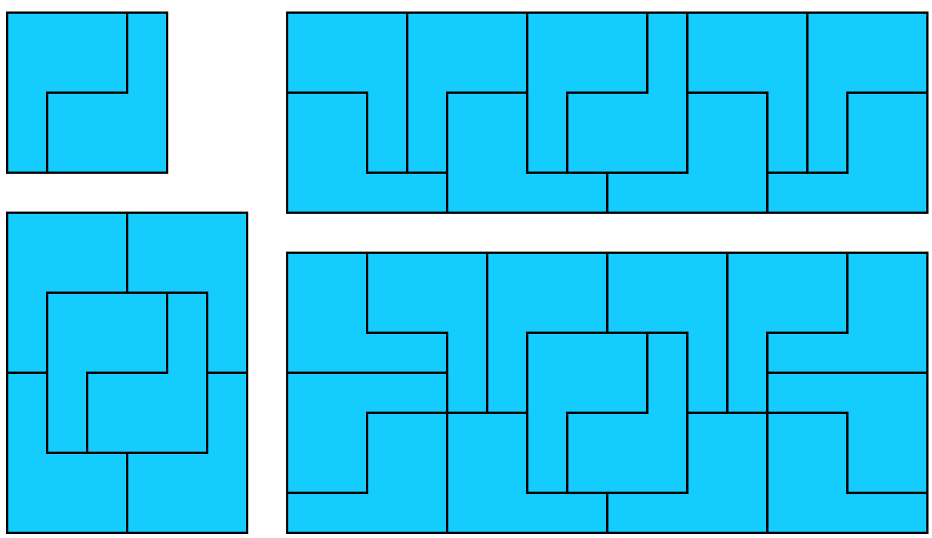


Figure 245: The prime rectangles of the P-octomino.

Only if. If $R(m,n)$ is tileable, it is obvious $m, n > 3$. To show that mn is divisible by 16, we will show the P-octomino is even.

Since 8 divides mn, 4 divides some side, say m. We apply the coloring in shown in Figure 246 on the rectangle. The polyomino is of type (a, b, c) if it covers a amber cells, b blue cells and c cherry cells.

Let $k_{1\ldots6}$ denote the number of tiles in the tiling with types $(3,4,1)$, $(1,3,4)$, $(4,2,2)$, $(2,6,0)$, $(2,2,4)$, $(0,6,2)$ respectively. It is easy to check no other types are possible.

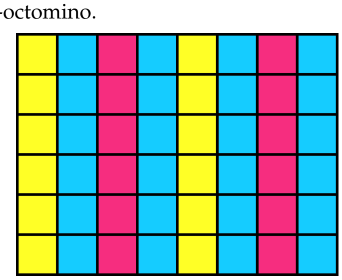


Figure 246: $F_{0,3}$, a 3-coloring.

Since the total number of blue and cherry cells are equal, we have

$$3k_1 + k_2 + 4k_3 + 2k_4 + 2k_5 = k_1 + 3k_2 + 2k_3 + 4k_5 + 2k_6,$$

from which we get

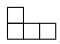
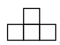
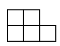
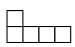
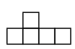
$$k_1 + k_3 + k_4 = k_2 + k_5 + k_6.$$

The total number of tiles is $k_1 + k_2 + \cdots k_6 = 2(k_1 + k_3 + k_4)$, which means the total is even. Hence the P-octomino is even, and hence any rectangle $R(m, n)$ that is tileable by this polyomino must have mn divisible by 16. $\square$

Problem† 82. *Show that any rectangle $R(m, n)$ with mn divisible by 16 and $m, n > 3$, can be divided into the four rectangles $R(4, 4)$, $R(5, 16)$, $R(6, 8)$, and $R(7, 16)$.*

5.3.6 Prime rectangles of small polyominoes

The tables in this section lists the prime rectangles of small polyominoes.

Polyomino	PR	$m \times n$
 3		2×3
		5×9
 4		2×4
		3×8
 4		4×4
 5		2×5
		7×15
 5		2×5
		7×15
 5		5×10
		$9 \times 20\ 30\ 45\ 55$
		$10 \times 14\ 16\ 23\ 27$
		$11 \times 20\ 30\ 35\ 45$
		$12 \times 50\ 55\ 60\ 65\ 70\ 75\ 80\ 85\ 90\ 95$
		$13 \times 20\ 30\ 35\ 45$
		14×15
		$15 \times 15\ 16\ 17\ 19\ 21\ 22\ 23$
		$17 \times 20\ 25$
		$18 \times 25\ 35$
		22×25

$\times$

Table 45: Prime rectangles of polyominoes. Adapted from Reid (2003a).

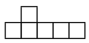
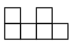
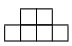
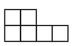
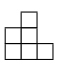
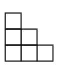
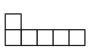
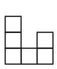
Polyomino	PR	$m \times$	n
		23 ×	24
		24 ×	29 35 41 47 53 59 63 65 71 77 83 89 95 101 102 103
		×	107 108 113 114 119 120
 6		30 ×	64 68 72 80 88 92 96 100 104 106 108 112 116 120 124 ...
		32 ×	36 42 48 54 60 66 ...
		48 ×	48 ...
		... ×	
 6		3 ×	4
 6		4 ×	6
		5 ×	12
		2 ×	7
 6		11 ×	49
		13 ×	35
		19 ×	21
		9 ×	12 20 28
		12 ×	13 14 17 19 21 24 25 29
 6		15 ×	28 32 36 40 44 48 52
		16 ×	18 27 30 33 39 42
		20 ×	21 24
 6		3 ×	4
		2 ×	6
		7 ×	12
 6		8 ×	15
		9 ×	14 16 34
		10 ×	15
		11 ×	18
 6		3 ×	4

Table 46: Prime rectangles of polyominoes. Adapted from Reid (2003a).

5.4 *Fault-free Tilings*

Recall that a *fault* in a tiling is a line on the grid (horizontal and vertical) that goes through the region and is not crossed by any tile.

We are mostly interested in fault-free tilings because a fault-free tiling cannot be broken down into two (or more) tilings by smaller rectangles.

Any prime rectangle is fault-free by definition. To find all possible rectangles of a polyomino that has fault-free tilings, we make use of following

- The prime rectangles we looked at in the previous section.
- The theorems describing which rectangles have fault-free tilings by rectangles (described later in this section).
- Tiling extensions: systematic ways of building bigger rectangles from smaller ones. We will see several ways that fault-free rectangles can be extended.
- Finding more **basic** tilings of rectangles not covered by any of the above or proving that they don't exist. Once we have a basic rectangle, we also apply the three methods above to extend the set of rectangles with fault-free tilings.

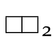
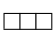
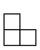
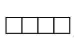
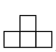
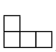
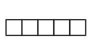
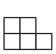
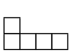
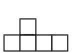
Polyomino	Fault-Free Rectangles				Reason
$_2$	$m \times n$	$2 \mid mn,$	$m, n > 5,$	$(m, n) \neq (6, 6)$	Thm. 128
$_3$	$m \times n$	$3 \mid mn$	$m, n > 6$		Thm. 240
$_3$	$m \times n$	$3 \mid mn,$	$m, n > 3$		Thm. 242
$_4$	$m \times n$	$4 \mid m$ or $4 \mid n$	$m, n > 8$		Thm. 240
$_4$	$4m \times 4n$				Thm. 243
$_4$	$4k \times 4k$	$(2 + 4k)k$	$(3 + 4k) \times 8k$	$(5 + 4k) \times 8k$	Thm. 244
$_5$	$m \times n,$	$5 \mid mn,$	$m, n > 10$		Thm. 240
$_5$	$m \times n$	$5 \vert mn$	if $m = 2$ then $n = 5,$	if $n = 2$ then $m = 5$	Thm. 245
$_5$	?				
$_5$	?				

Table 47: Fault-free Rectangles of polyominoes.

5.4.1 Rectangles

Theorem 237. *Tilings of a rectangle by a square cannot be fault-free.*

[Referenced on page 246]

Proof. Suppose the square has side k. Then the only rectangles tileable are $R(mk, nk)$ for some integers m and n (Theorem 22). A tiling is given by putting m squares in each of n rows. By Theorem 4 this tiling is unique, and since it has a fault, no fault-free tilings of the rectangle exists. □

Theorem 238 (Robinson (1982), Theorem 4)**.** *Let* $1 < m < n$ *and* $\gcd(m, n) = 1$*. The only rectangles with fault-free tilings of a* $m \times n$ *polyomino are these:*

(1) $(kmn + pm + qn) \times \ell mn$*, with* $k \geq 1, \ell \geq 3, 0 < p < n,$ $0 < q < m$ *(Figures 247 and 248)*

(2) $(kmn + pm) \times (\ell mn + qn)$*, with* $k, \ell \geq 2, 0 < p < n, 0 < q < m$ *(Figure 249)*

(3) $(kmn + pm) \times (\ell mn)$*, with* $k \geq 2$*,* $\ell \geq 3, 0 < p < n$ *(Figures 249 and 250)*

(4) $(kmn) \times (\ell mn + qn)$*, with* $k \geq 3$*,* $\ell \geq 2, 0 < q < m$ *(Figure 248)*

(5) $(kmn) \times (\ell mn)$*, with* $k, \ell \geq 3$ *(Figure 248)*

[Referenced on page 250]

To get some intuition for this theorem, a useful exercise is to construct one of the minimal rectangles in Figures 247 –250 for a given tile, and see if you can find the ways to extend it (without referring to the figure). After this, you will be able to be able to see more easily how the diagrams can be stretched for different tilings.

The following theorem is equivalent to the above, but often easier to work with if we don't need to produce the fault-free tiling.

Theorem 239 (Graham (1981), Theorem p.125)**.** *Let* $1 \leq m < n$ *and* $\gcd(m, n) = 1$*. Then a rectangle* p, q *has a fault free tiling if and only if:*

(1) *Each of* m *and* n *divides* p *or* q*,*

(2) p *and* q *can each be expressed as* $xm + yn$ *in at least two ways, with* $x, y > 0$*, and*

(3) *for* $m = 1$ *and* $n = 2$*,* $p, q \neq 6$*.*

[Referenced on pages 244 and 246]

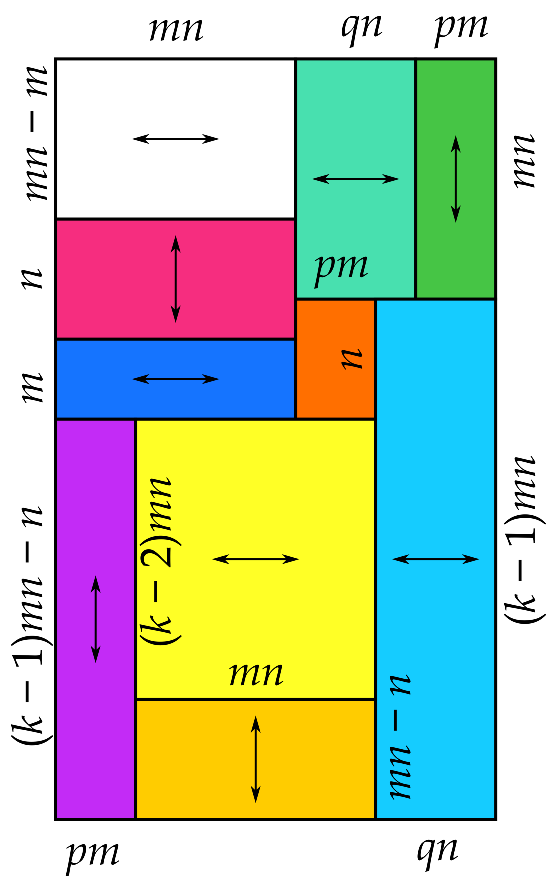


Figure 247: $kmn \times (mn + pm + qn)$, $k \geq 3$, $0 < p < n$, $0 < q < m$.

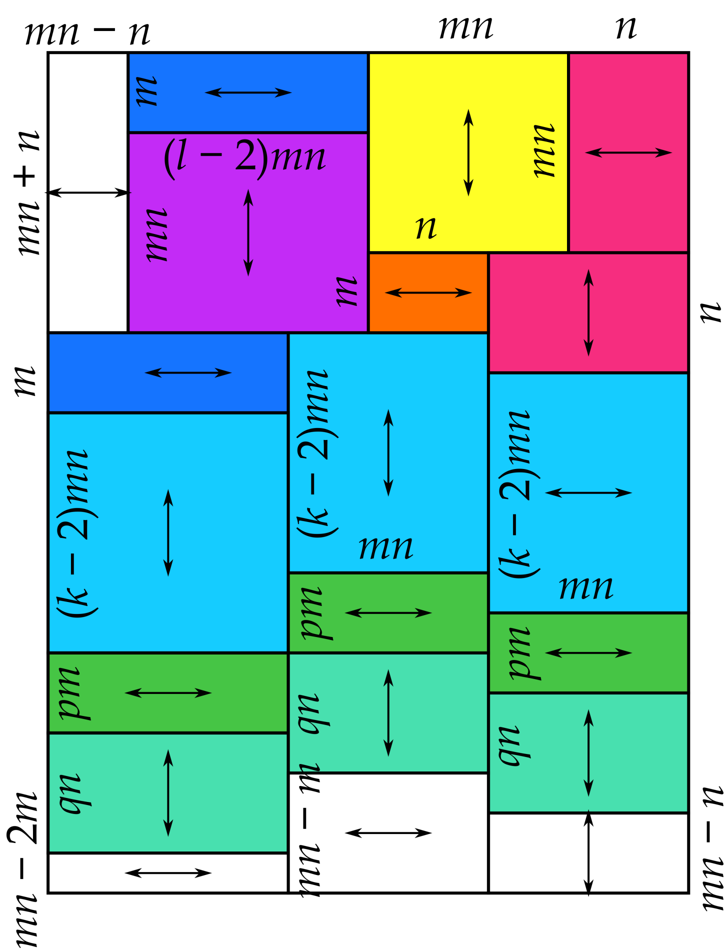


Figure 248: $kmn \times (mn + pm + qn)$, $k, l \geq 3$, $0 \leq p < n$, $0 \leq q < m$. If $q > 0$, k may be 2.

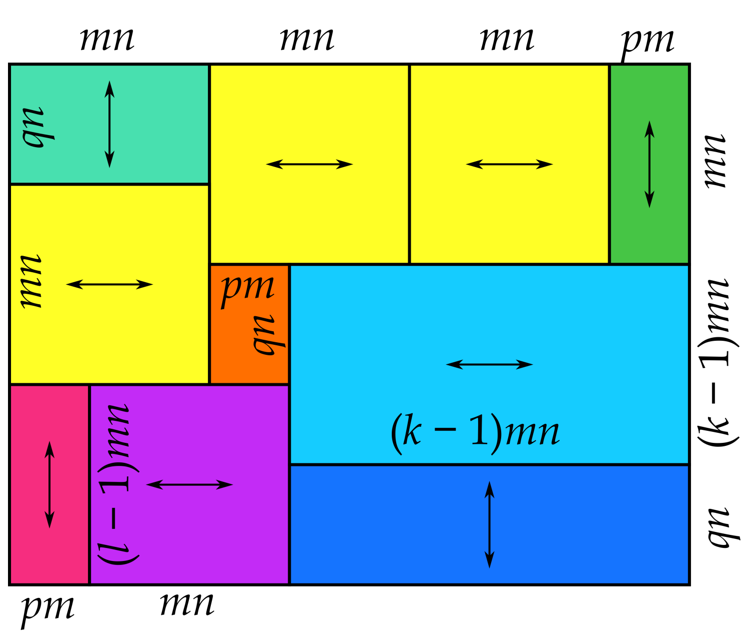


Figure 249: $(kmn + pm) \times (lmn + qn)$, with $k, l \geq 2$, $0 < p < n$, $0 < q < m$.

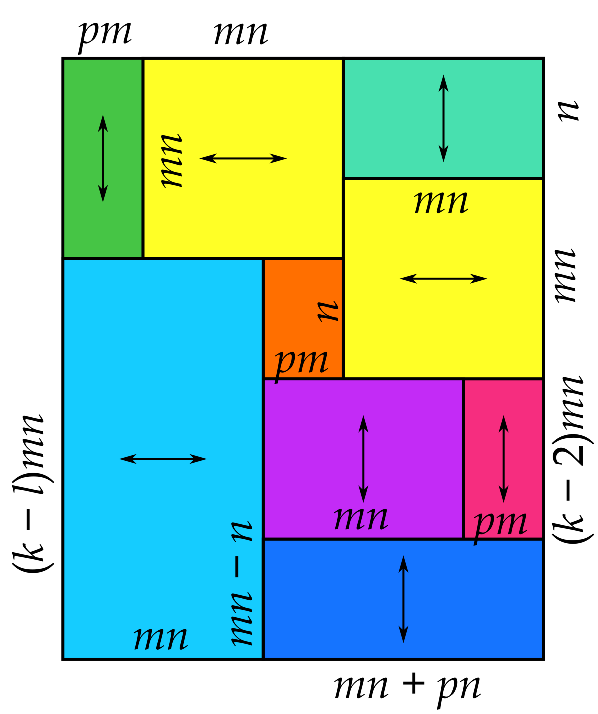


Figure 250: $kmn \times (2mn + pm)$, with $k \geq 3, 0 < p < n$.

We state a special case for bars:

Theorem 240. *A rectangle $R(p, q)$ has a fault-free tiling by bars $R(1, m)$ if and only if both the following hold*

(1) $m \mid p$ *or* $m \mid q$

(2) $p, q > 2m$.

[Referenced on pages 240, 244 and 246]

Proof.

If. Figure 252 gives a tiling for rectangles that satisfy the conditions.

Only if. The first condition is necessary by Theorem 163.

By Theorem 239, we must have $p = x + ym$ in at least two ways. This is possibly only if $p > 2m$, in which case we can write $p = x + m = x' + 2m$. The same holds for q. Therefore, we must have $p, q > 2m$. □

Theorem 241 (Robinson (1982), Corollary 5)**.** *The minimal rectangle with a fault-free tiling is a $3mn \times (mn + m + n)$ rectangle*[24].

[Not referenced]

[24] This was first proven for bars in Scherer (1980).

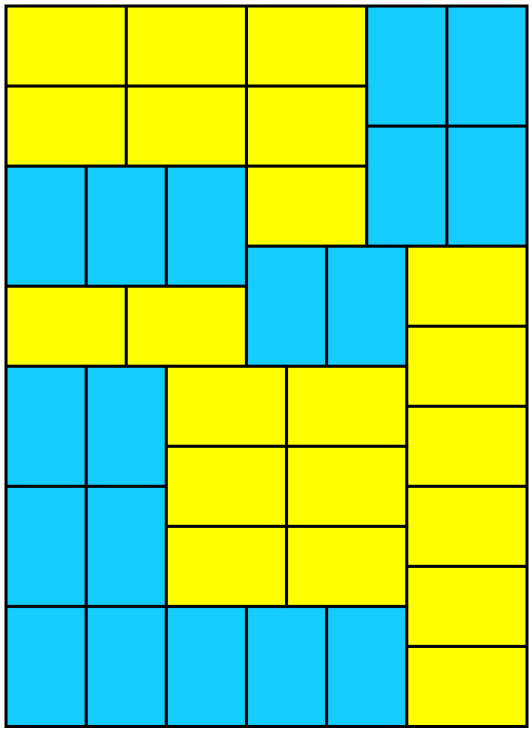


Figure 251: A fault-free tiling of 13×18 rectangle by 2×3 rectangles. The tiling was obtained from Figure fig:ff-rect1 with $k = 3$, $p = 2$, $q = 1$. Can you see how we can reduce the width of this rectangle and keep it fault-free? Why can we not reduce the height?

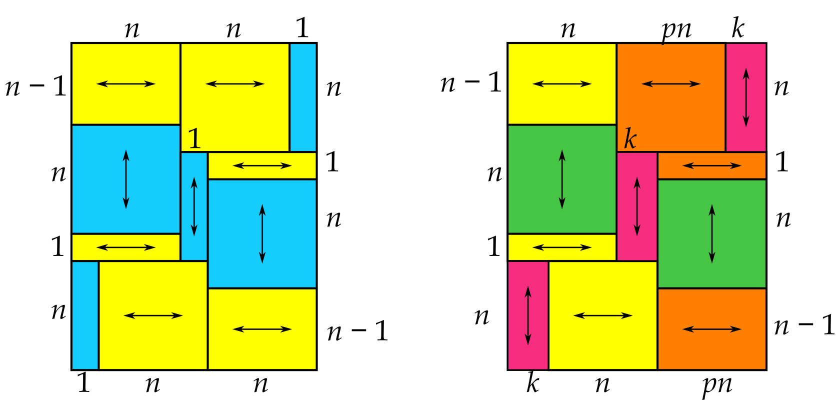


Figure 252: A fault-free tiling of $3n \times (2n+1)$ rectangle by $n \times 1$ rectangles, and an extension.

5.4.2 Trominoes

The straight tromino is covered by Theorem 240, so we only deal with the right tromino.

Theorem 242 (Aanjaneya and Pal (2006), Theorem 2)**.** *All $p \times q$ rectangles with $p, q \geq 4$, $3 \mid pq$, admit a fault-free tiling with right trominoes.*

[Referenced on page 240]

See the reference for a proof.

Since two trominoes can be put together to form a 2×3-rectangle, any rectangle that has a fault-free tiling with 2×3 tiles will also have a fault-free tiling with right trominoes. From Theorem 239 we know this is possible for rectangles with p, q either an odd number greater than 11 or an even number greater than 14 provided at least one is divisible by three.

There are also rectangles with fault-free tilings not decomposable into 2×3 rectangles. The basic tromino fault-free tilings shown in Figure 253. These can be extended (similar to the extensions we have for dominoes) to for all other rectangles that satisfy the conditions.

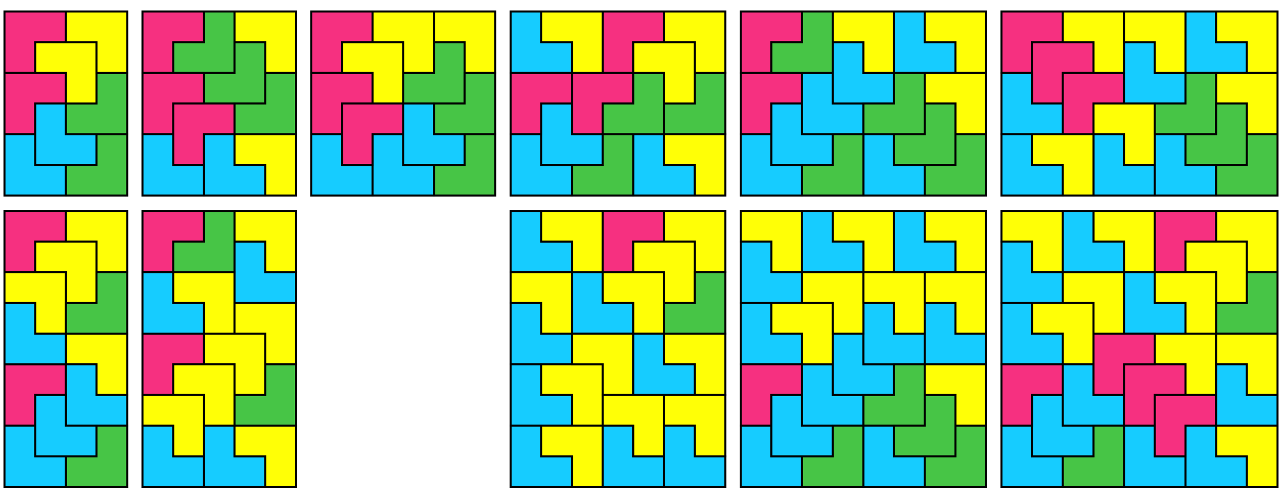


Figure 253: Basic fault free tilings using trominoes. Note that the gap is not really a gap — the missing $R(6,9)$ is present in the top-right as $R(9,6)$. Image adapted from Aanjaneya and Pal (2006, Figure 4).

5.4.3 Tetrominoes

All the tetrominoes except the square admit fault-free tilings. By Theorem 239 The I-tetromino has fault-free tilings for rectangles p, q with $p, q \geq 9$ and $4|(p \text{ or } q)$. The smallest can be derived from Figure 252.

The bar (Theorem 240) and square tetromino (Theorem 237) have already been dealt with. The skew tetromino does not tile a rectangle, so here we will only consider the L- and T- tetrominoes.

Theorem 243. *There exist fault-free tilings by the T tetromino for any rectangle that can be tiled by the T-tetromino (that is, for all* $4m \times 4n$ *rectangles).*

[Referenced on page 240]

Proof. Figure 254 shows a generic fault-free tiling. The blue sections can be repeated any number of times (including 0), leading to a ring that can be filled with 4×4 squares. This accounts for all $4m \times 4n$ rectangles, which are the only ones that the T-tetromino can tile (Theorem 223).

□

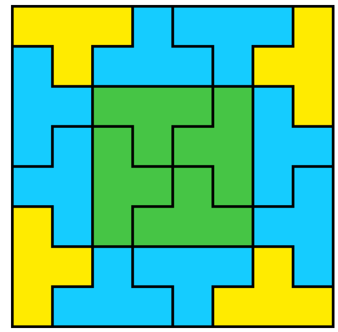


Figure 254: A generic fault-free tiling with T-tetrominoes.

Theorem 244. *Fault-free tilings by L-tetrominoes exist for all rectangles that can be tiled by L-tetrominoes, except when* $m = 2$ *and* $n > 4$ *or* $n = 2$ *and* $m > 4$*.*

[Referenced on page 240]

Proof. Fault-free tilings exist for the following rectangles (Figure 255):

(1) 4×4

(2) 6×4

(3) 3×8

(4) 5×8

These basic tilings can all be extended to give fault free tilings of the following:

(1) $4k \times 4k$

(2) $(2 + 4k) \times 4k$

(3) $(3 + 4k) \times 8k$

(4) $(5 + 4k) \times 8k$

Together, these classes account for all rectangles that can be tiled by L-tetrominoes except for rectangles with width or height 2. Of these, only the 2×4 and 4×2 are fault free.

The extensions of the first two classes can be done by inserting appropriate rectangles as shown in Figure 256.

Vertical extensions of the other two classes is done in the same way.

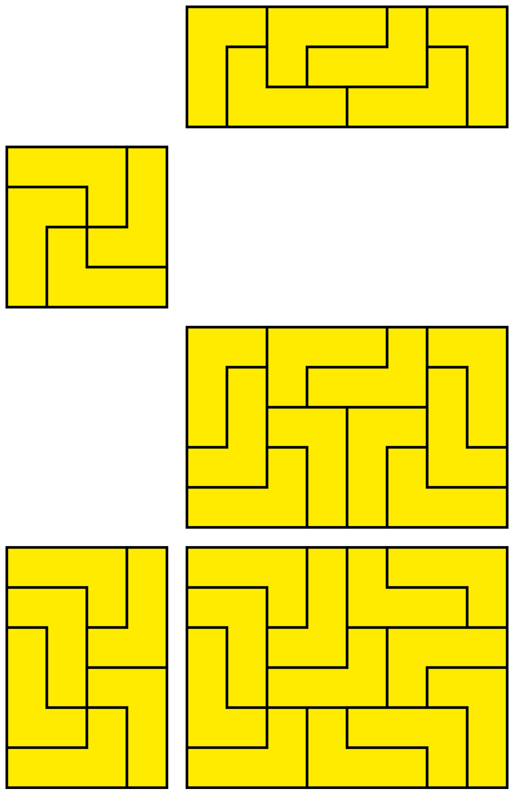


Figure 255: The basic fault-free tilings of rectangles by the L-tetromino.

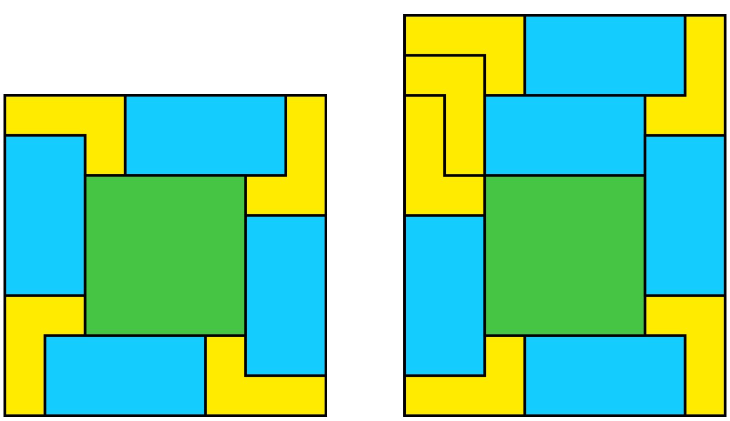


Figure 256: The standard extensions of the $4m \times 4n$ and $2 + 4m \times 4n$ rectangles.

To extend the $3 \times 8k$ rectangle horizontally: take off one tile at the corner, and insert the shape shown in figure, and add the tile in the opening (you will need to rotate it). Doing this twice is equivalent to inserting a cylinder; we therefore call this inserting a **half cylinder**.

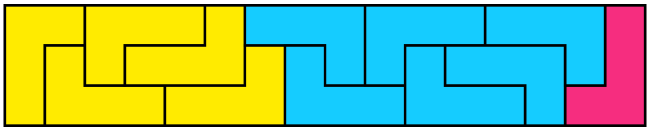
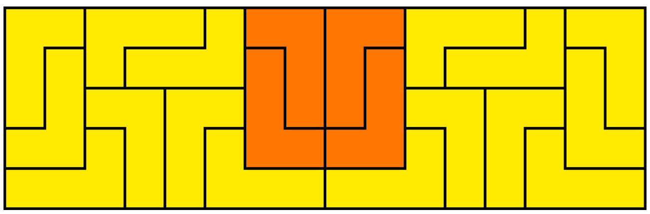
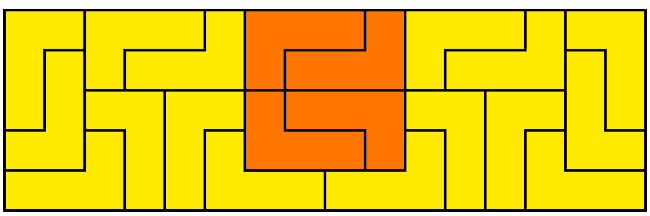


Figure 257: To extend the $3 \times 8k$ rectangle, remove the pink corner tile. Add the blue shape shown to fill the gap, and finally add the pink tile back to complete the rectangle.

To extend the $5 \times 8k$ rectangle horizontally, add a basic 5×8 rectangle, and flip the square shown in the figure.



(a)



(b)

Figure 258: To extend the $5 \times 8k$ rectangle, add another 5×8 rectangle, and then flip the orange square to remove the fault.

□

5.4.4 *Pentominoes*

There are only 3 pentominoes to consider: the L-pentomino, the P-pentomino, and the Y-pentomino.

Theorem 245. *The P pentomino have fault-free tilings of all rectangles that it can tile, except for rectangles with one dimension 2 when the other is larger than 5.*

[Referenced on page 240]

Proof. We know that one of the dimensions must be divisible by 5 (Theorem 1). It is not hard to show $5 \times m$ rectangles only have tilings if m is even (this is done in the proof of Theorem 227).

The P-pentomino can tile the following basic rectangles fault-free:

- 2×5, 4×5, 6×5
- 10×7, 10×9, 15×7, 15×9

- 15×11, 15×11, 15×13, 15×15

See Figures 259 and 260.
These can be extended to tile the following rectangles:

(1) $2m \times 5n$ (if $m = 1$, then $n = 1$ too.)

(2) $2m + 5k \times 10n$

(3) $2m + 5 \times 5n + 5$

Extension strategies are shown in Figure 261.

This list make all rectangles with one side divisible by 5, except $5 \times m$ when m is odd, and rectangles of the form $2 \times 5n$ when $n > 1$, as is required. □

Other strategies can also be used to extend basic rectangles. Figure 262 shows an example where we put a 7×15 rectangle next to a 2×15 rectangle, and then eliminate the fault by retiling the orange tiles.

Theorem 246. *The L-pentomino has these basic fault-free tilings (incomplete):*

- 10×7
- 15×7
- 15×9

[Referenced on page 250]

Theorem 247. *The L-pentomino has these fault-free tilings (incomplete):*

(1) $10 \times (7 + 2k)$

(2) $(5k + 2p) \times 5\ell$

[Referenced on page 250]

Proof.

(1) See Figure 264.

(2) The 2×5 rectangle has a fault free-tiling, and so any fault-free tiling by $R(2,5)$ will also be fault-free if replace the rectangles with the tilings by the pentomino. The fault-free tilings of $R(2,5)$ is given by Theorem 238.

□

Problem* 83. *Complete the lists of Theorems 246 and 247.*

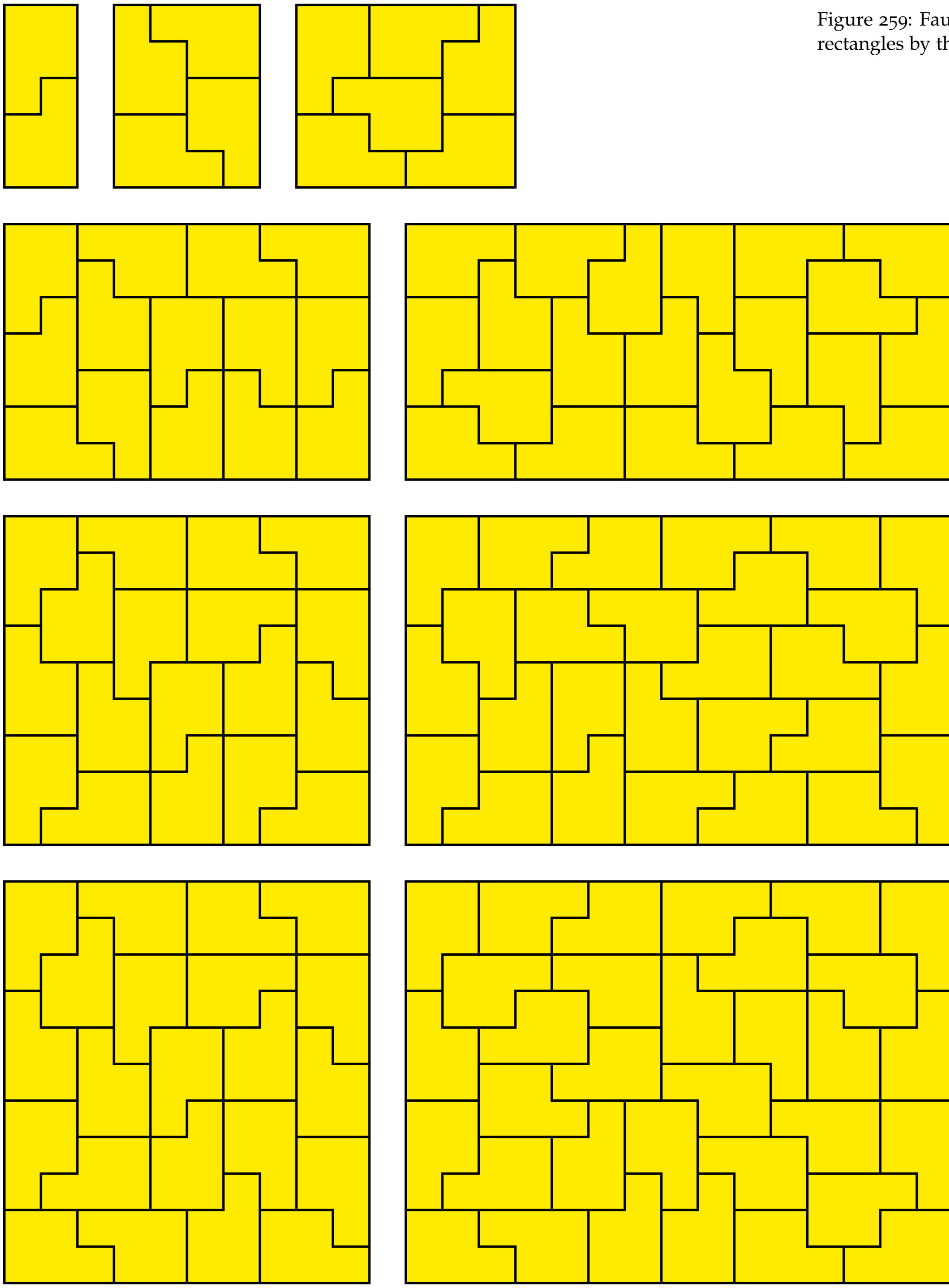


Figure 259: Fault-free tilings of basic rectangles by the P-pentomino.

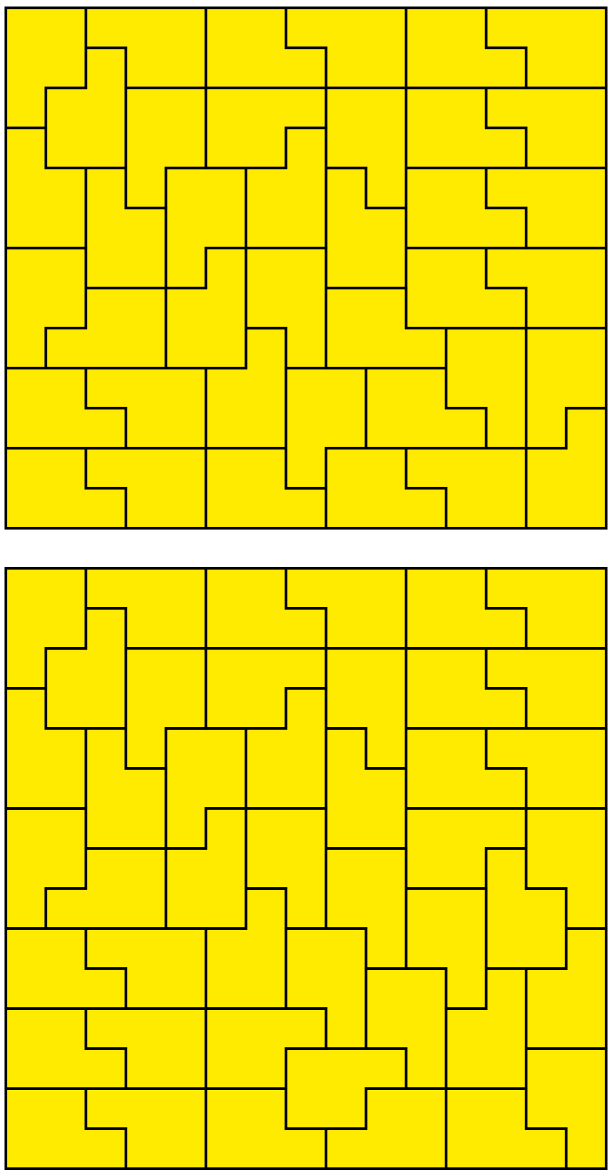


Figure 260: Fault-free tilings of basic rectangles by the P-pentomino.

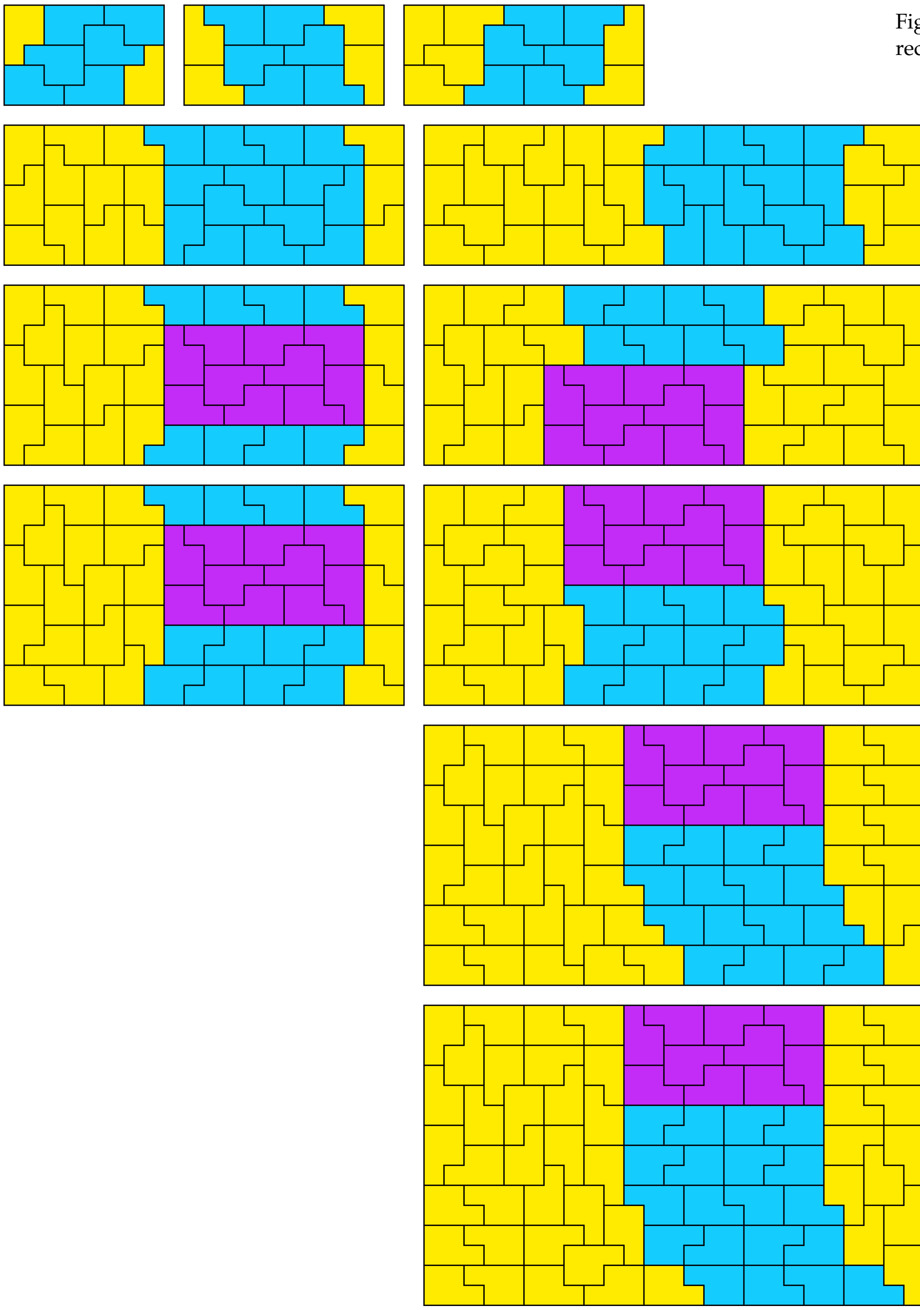


Figure 261: Fault-free tilings of basic rectangles by the P pentomino.

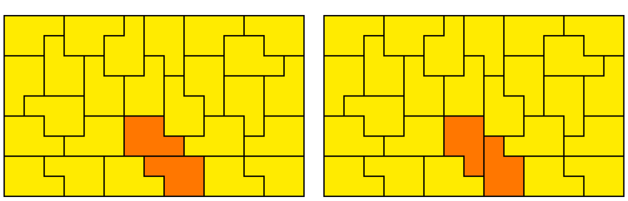


Figure 262: Fault-free tilings of basic rectangles by the P pentomino.

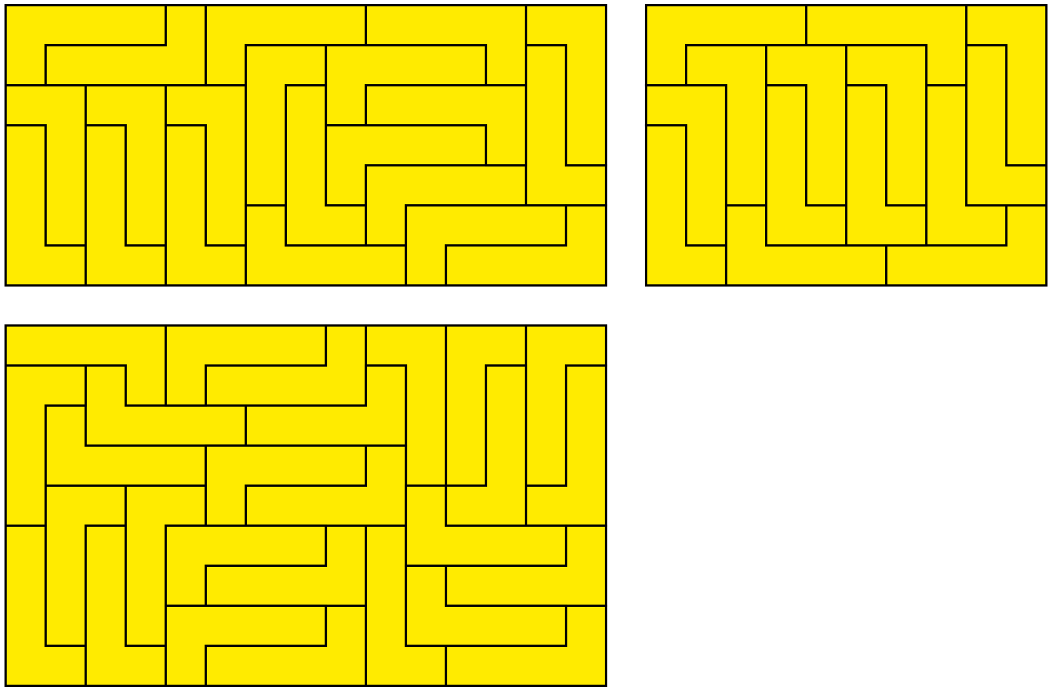


Figure 263: Basic fault-free rectangles.

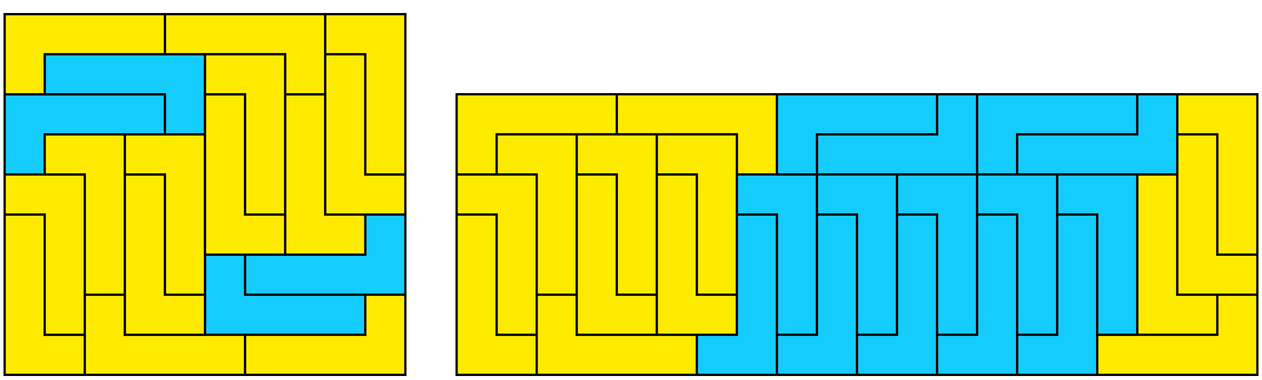


Figure 264: Fault-free extensions.

Finding all the rectangles with fault-free by Y-pentominoes is more tricky. Some extensions is shown in Figures: 265–272. Extension like these are also given in Cibulis and Mizniks (1998).

Problem* 84. *Find all the fault-free tilings of rectangles by the Y-pentomino.*

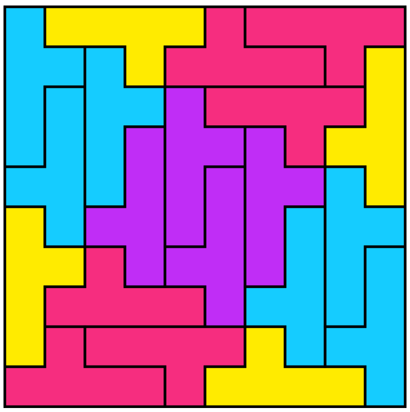


Figure 265: Example of an extension of a fault-free rectangle. We can duplicate the blue-purple region *n* times (including 0) to extend the rectangle vertically; as shown in the next three figures. We can also repeat the pink-purple region to extend the rectangle horizontally.

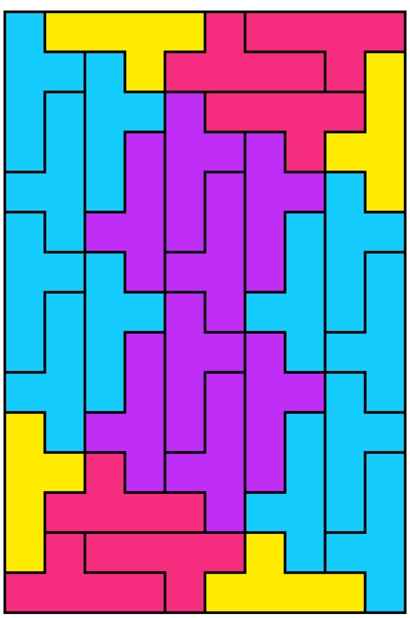


Figure 266: Example of an extension of a fault-free rectangle.

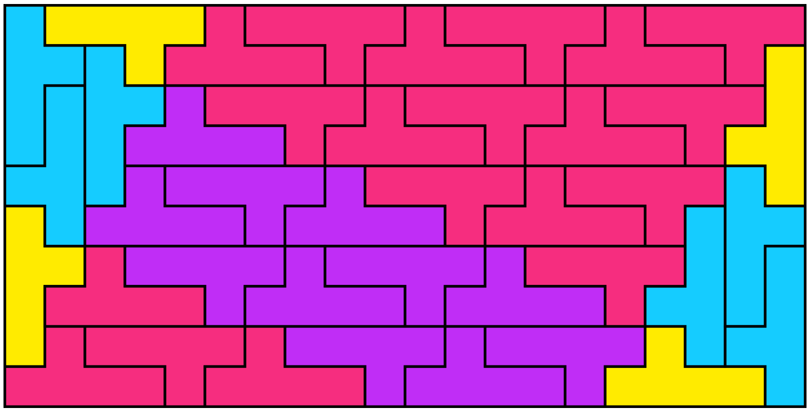


Figure 267: Example of an extension of a fault-free rectangle.

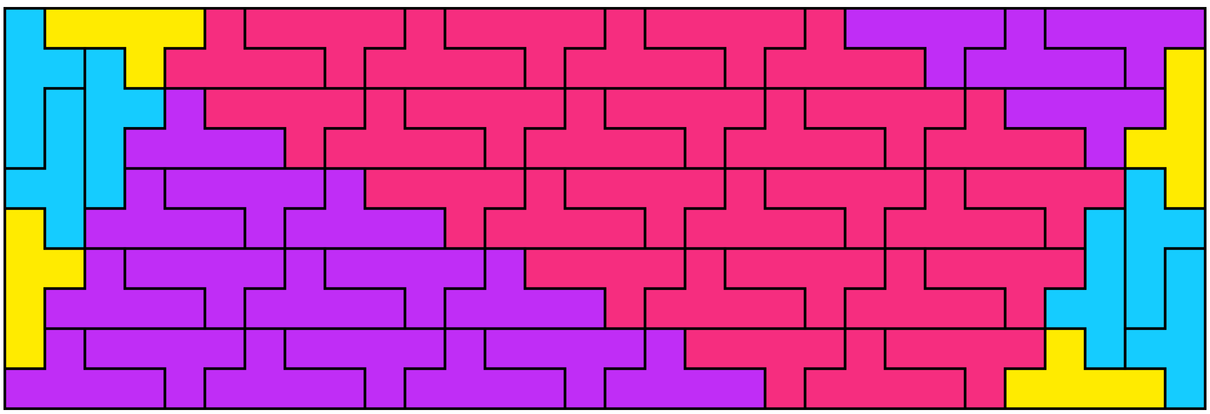


Figure 268: Example of an extension of a fault-free rectangle.

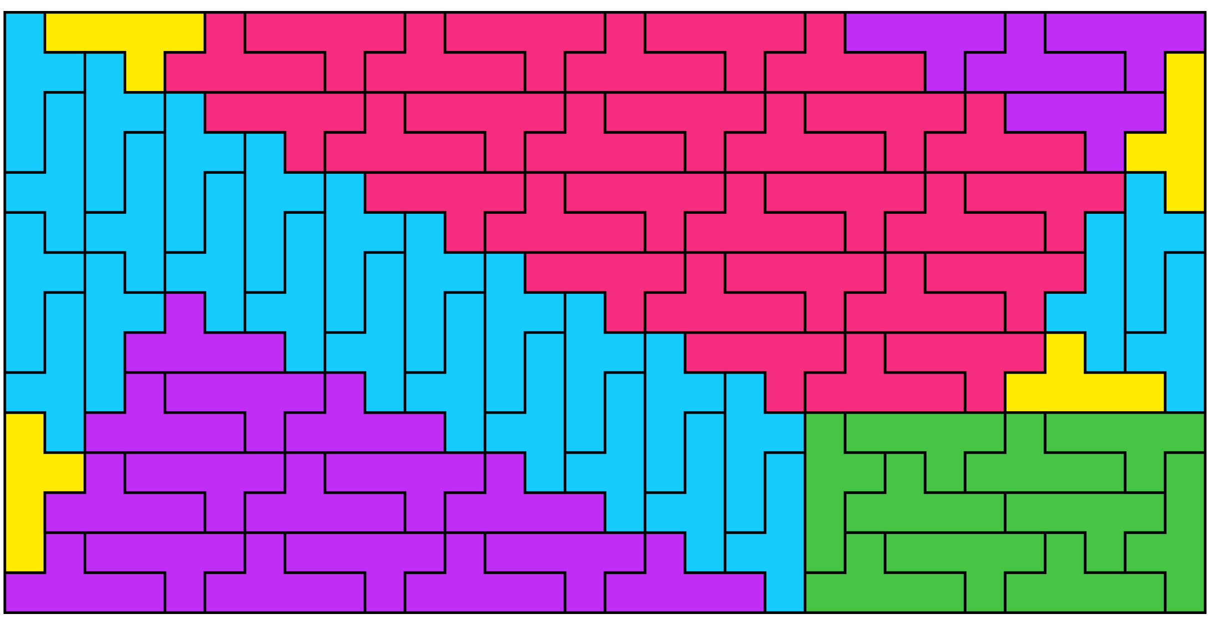


Figure 269: Example of an extension of a fault-free rectangle.

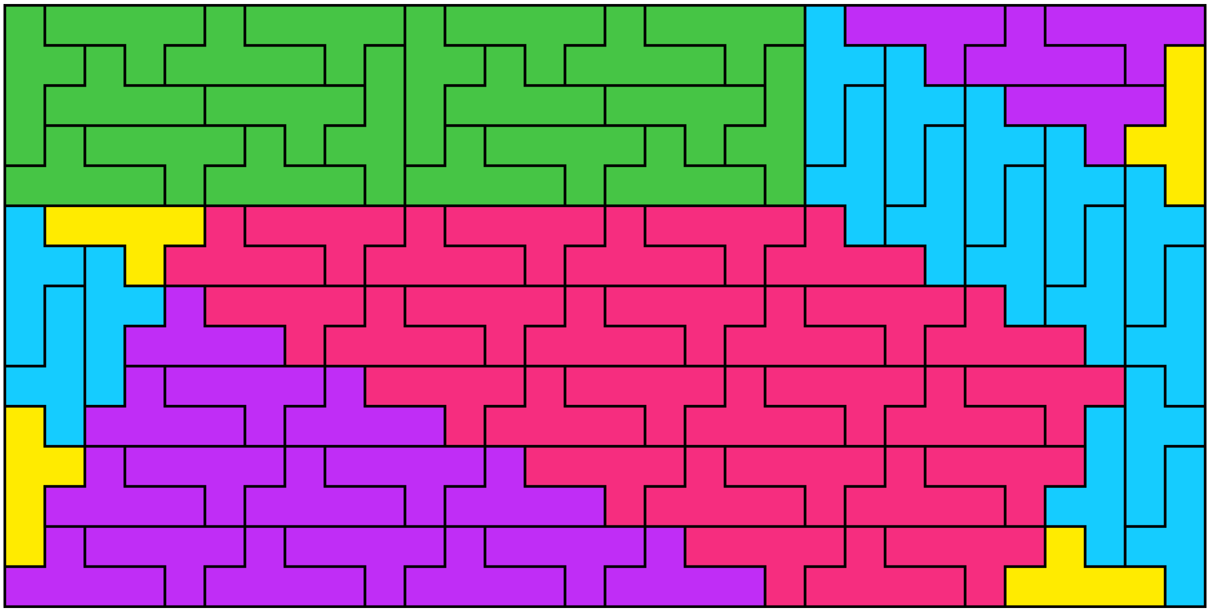


Figure 270: Example of an extension of a fault-free rectangle.

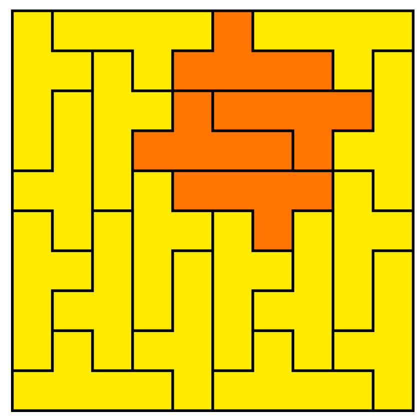


Figure 271: Another way to extend the 5×10 rectangle. Place two copies next to each other, and rotate the orange tiles as shown to eliminate the fault.

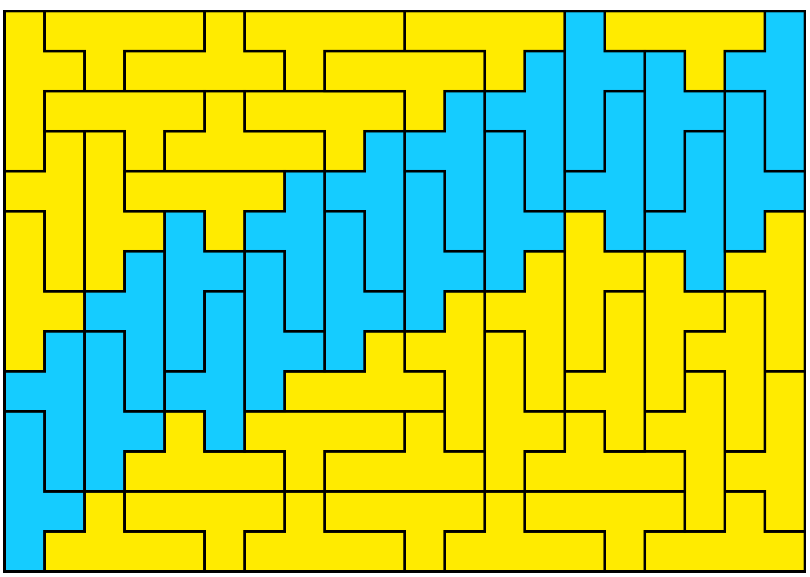


Figure 272: Example of an extension of a fault-free rectangle.

5.4.5 *Extensions*

We saw a few ways that rectangles can be extended:

- Through cylinder insertion, for example Figure 254.
- Through half-cylinder insertion, for example Figure 257.

- Through concatenation combined with fault-elimination (by retiling a subregion), for example Figures 271 and 258.

5.5 Gap Numbers

In this section we consider the gap number for rectangular regions tile by the various polyominoes. We already considered the gap number of dominoes on Section 3.3.1, and for some rectangles in Section 5.1.2

5.5.1 Trominoes

Theorem 248. *The gap number for the right tromino is given by*

$$G(m,n) = \begin{cases} m & \textit{if } n = 1 \\ n & \textit{if } m = 1 \\ 3 & \textit{if } m = 3 \textit{ and } n \textit{ is odd} \\ 3 & \textit{if } n = 3 \textit{ and } m \textit{ is odd} \\ mn \bmod 3 & \textit{otherwise.} \end{cases}$$

[Not referenced]

5.5.2 Tetrominoes

Theorem 249. *The gap number for the L-tetromino is given by*

$$G(m,n) = \begin{cases} m & \textit{if } n = 1 \\ n & \textit{if } m = 1 \\ 4 & \textit{if } mn \equiv 4 \pmod 8 \\ mn \bmod 4 & \textit{otherwise.} \end{cases}$$

[Not referenced]

Theorem 250. *The gap number for the skew-tetromino is given by*

$$G(m,n) = \begin{cases} 4 & \textit{if } mn \equiv 0 \pmod 4 \\ mn - (m - m \bmod 2)(n - n \bmod 2) & \textit{otherwise.} \end{cases}$$

[Not referenced]

The gap number of T-tetromino tilings is considerably harder to analyze. I summarize what is known, for details see Hochberg (2015).

Theorem 251. *For the T-tetromino, the following is know about the gap number. The functions e_i are bounded.*

(1) $G(1,n) = n$

(2) $G(3,n) = \begin{cases} 5 & n = 3 \\ \lfloor n/3 \rfloor & n \equiv 0 \pmod 3, n \neq 3 \\ \lfloor n/3 \rfloor + 3 & n \equiv 1 \pmod 3, n \neq 3 \\ \lfloor n/3 \rfloor + 2 & n \equiv 2 \pmod 3, n \neq 3 \end{cases}$ *(Hochberg, 2015, Theorem 1).*

(3) $G(5,n) = \lfloor n/5 \rfloor + e_5(n)$ *(Hochberg, 2015, Theorem 5).*

(4) $G(7,n) = \lfloor n/7 \rfloor + e_7(n)$ *(Hochberg, 2015, Theorem 5).*

(5) $G(9,n) = \lfloor n/17 \rfloor + e_9(n)$ *(Hochberg, 2015, Theorem 4).*

(6) $G(11,n) = \lfloor n/35 \rfloor + e_11(n)$ *(Hochberg, 2015, Theorem 5).*

(7) $G(13,3) = 7$, $G(13,n) = 6$ *for* $n = 14, 18, 22, 30, 34, 38$ *(Hochberg, 2015, Theorem 7).*

(8) $G(13,n) = k$, *where* $k \geq 2$ *is the smallest integer such that* $k \equiv 13n \pmod 4$ *(except for the cases above) (Hochberg, 2015, Theorem 7).*

(9) $G(15,5) = 7$, $G(15,n) = 6$ *for* $n = 10, 14, 18, 22, 26$ *(Hochberg, 2015, Theorem 7).*

(10) $G(15,n) = k$, *where* $k \geq 2$ *is the smallest integer such that* $k \equiv 15n \pmod 4$ *(except for the cases above) (Hochberg, 2015, Theorem 7).*

(11) $G(10,n) = 6$ *for* $n = 3, 7, 11, 15, 19, 23$ *(Hochberg, 2015, Theorem 6).*

(12) $G(m,n) = k$ *for* m *even and* $m \leq 12$ *and* $n \geq 2$ *(except for the cases in the line above), where* $k = 4$ *if* $m \equiv n \equiv 2 \pmod 4$, *or* $k = mn \bmod 4$ *otherwise. (Hochberg, 2015, Theorem 6).*

(13) $G(m,n) \leq 9$ *for* $m, n \geq 12$ *(Hochberg, 2015, Theorem 8).*

[Not referenced]

The proof is not very difficult, but a bit long and tedious; the ideas resemble those used for tiling half-strips and rectangles (see Reid (1997)). The basic idea is to construct optimal tilings from cylinders and partial tilings of rectangles for a few basic widths, and then to stack these together to form additional rectangles.

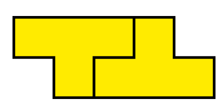


Figure 273: Cylinder of height 2.

We can construct cylinders of even height by stacking copies of the height 2 cylinder shown in Figure 273. The smallest cylinder with odd height is the cylinder with height 13 shown in Figure 274. We can stack height-2 cylinders to get cylinders with any odd height for $n > 13$. An alternative cylinder with height 15 is shown.

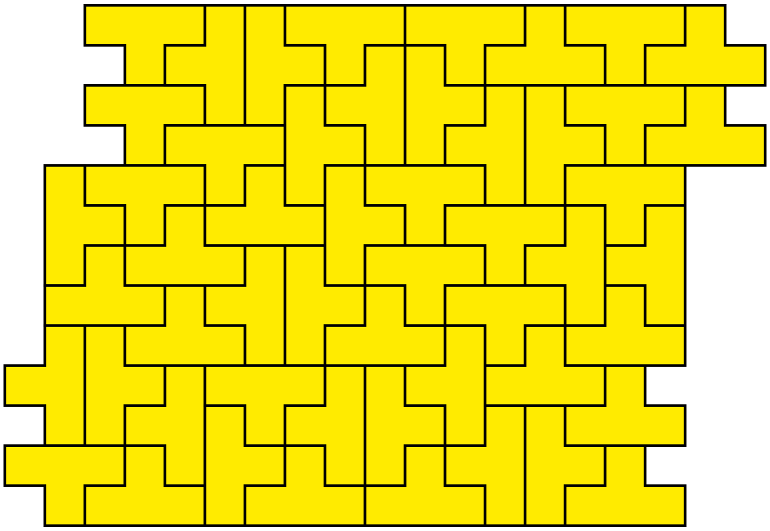


Figure 274: Cylinder with height 13 (Hochberg, 2015, Fig. 12).

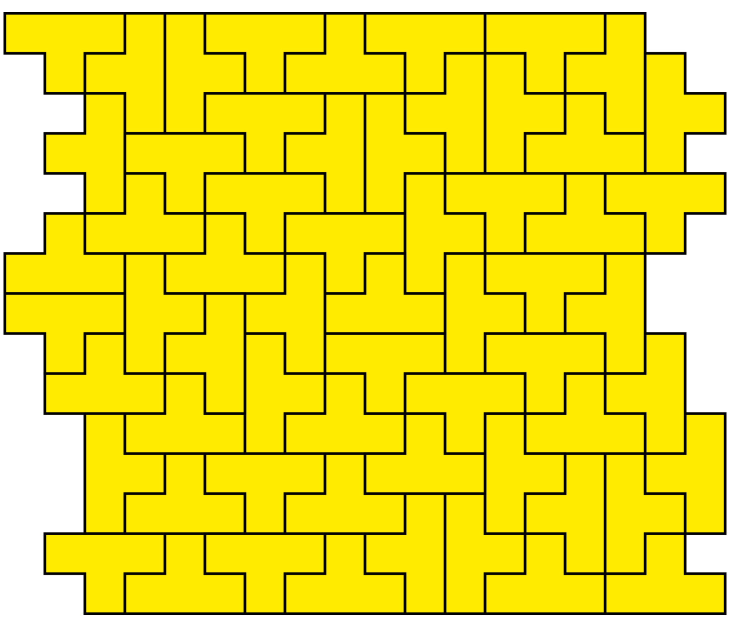


Figure 275: Cylinder with height 15 (Hochberg, 2015, Fig. 12).

Conjecture 252 (Hochberg (2015), Section 2.2, Section 2.3)**.**

(1) $e_5 \leq 4$ *for* $n > 15$

(2) $e_7 \leq 4$ *for* $n > 10$

(3) $e_9 \leq 6$ *for* $n > 2$

(4) $e_1 1 \leq 7$ *for* $n > 1$

5.6 *Rectifiable Sets*

Trivially, if a polyomino in a set is rectifiable, then so is the set. How do rectifiable sets differ from single rectifiable polyominoes then?

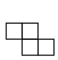
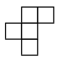
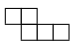
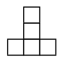
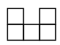
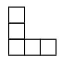
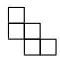
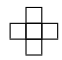
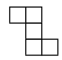
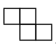
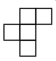
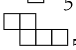
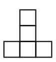
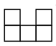
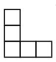
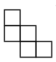
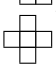
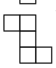
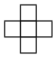
	$_4$	$_5$	$_5$	$_5$	$_5$	$_5$	$_5$	$_5$	$_5$
$_4$	H	▷	▷	✓	✓	✓	▷	▷	▷
$_5$	▷	H	▷	×	✓	✓	▷, ×	×	×
$_5$	▷	▷	H	✓	✓	✓	▷	▷	▷
$_5$	✓	×	✓	H	✓	?	✓	×	×
$_5$	✓	✓	✓	✓	H	✓	▷	✓	?
$_5$	✓	✓	✓	?	✓	H	✓	?	✓
$_5$	▷	▷, ×	▷	✓	▷	✓	H	▷	▷
$_5$	▷	×	▷	×	✓	?	▷	H	▷
$_5$	▷	×	▷	×	?	✓	▷	▷	H

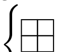
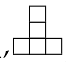
Table 48: Pairs of pentominoes that are rectifiable.
×All placements in the corners lead to untileable cells.
▷ All placement in the corner leads to infinite repeating patterns that cannot close another corner.
? The pair probably does not tile a rectangle but no proof is available. See prob 87.
H Used for a pentomino with itself; non of which are rectifiable as shown in Theorem 225.

In general, a set can tile more than each of its members. For example, a set may tile a rectangle when none of its members can. Table 48 shows several pairs of polyominoes that can tile rectangles together, but not by themselves (the smallest such tilings appear in Figures 276–277).

Even if some members of the set are rectifiable, the set may tile *more* rectangles than the rectifiable members alone or by themselves can. The set $\{R(2,2), R(3,3)\}$ can tile any rectangle bar a few small ones; but individually they can only tile rectangles of the form $R(2m, 2n)$ and $R(3m, 3n)$ respectively.

This means some rectangles require all tiles from a subset for a tiling.

Sometimes a set can *not* tile more rectangles than one of its members by itself, even if we would expect it to. For example, I_3 tile only rectangles with one side divisible by 3. And so does the set $\{I_3,$  $_5\}$, even though the cross has 5 cells which is not divisible by 3 (see Theorem 254.) This means if the number of crosses in the tiling is always a multiple of 3.

When we looked at individual tiles, the number required is determined by the size of the tiling. We did see that in some cases we could determine relationships between the number of tiles of different orientations. When we have more than one element in a tile set, we may similarly determine certain relationships among the number of each piece. For example, in a tiling of a rectangle by $\{$  $_4,$  $_5\}$, one side is always even, and therefore the number of pentominoes is even.

Order—the number of pieces used— is not such a useful concept for tile sets. When the tile set have tiles of different sizes, it is not clear that the number of pieces is anything interesting. Even when the size are the same size, the order is not interesting; it is easy to construct a tile set that has any order.

When are there tilings by a set that requires at least one of each piece? Of course, when no member of the set tiles a rectangle by themselves, all tilings are of this type; so interesting cases are when one (or more) of the members are rectifiable.

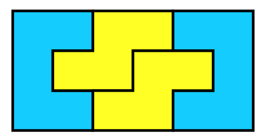
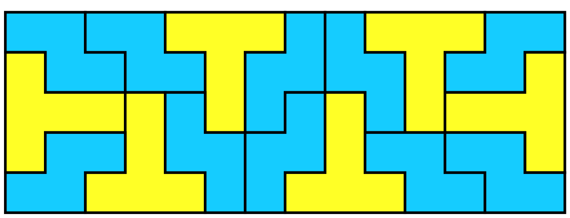
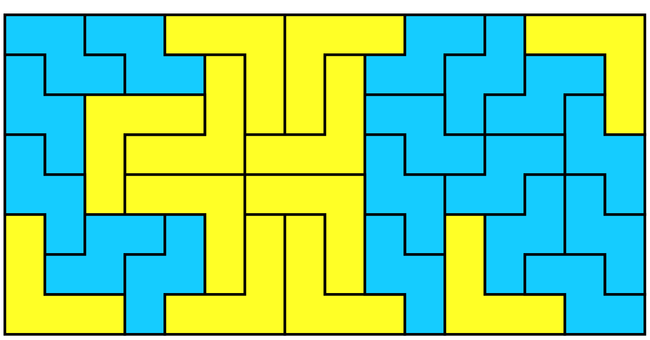
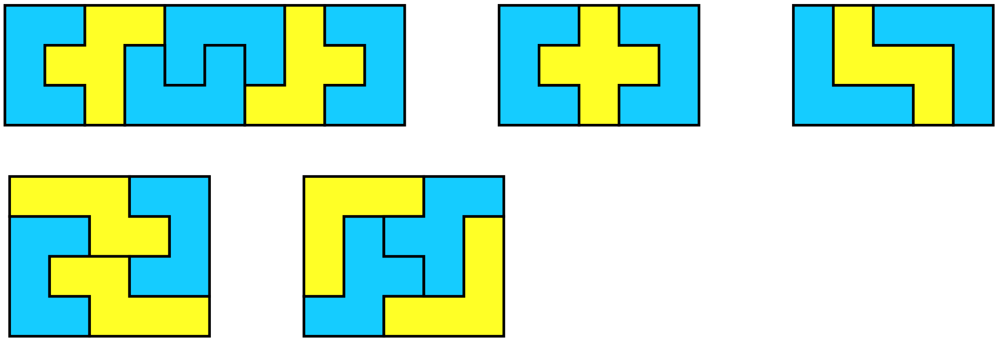


Figure 276: The smallest rectangle tileable by the skew tetromino and the U-pentomino.





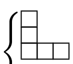
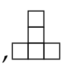
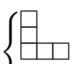
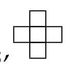
Problem† 85. *Construct a tile set that can tile a rectangle with three tiles. Can you construct a tile set with 2 tiles that require three to tile a rectangle?*



Problem† 86. *Shows for any P that is not rectifiable, there is an infinite number of polyominoes Q that is not rectifiable, but $\{P, Q\}$ is rectifiable.*

(This is one way to show the set of all simply-connected polyominoes can tile the plane.)

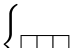
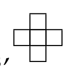
Problem* 87. *Can the following sets tile a rectangle?*

(1) $\{$  $_5$,  $_5\}$ (2) $\{$  $_5$,  $_5\}$

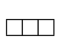
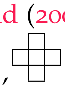
Theorem 253. *In a tiling of a balanced region, there must be an even number of tiles with odd deficiency.*

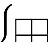
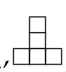
[Not referenced]

What follows is a collection of theorems for specific sets; most proofs require algebra beyond the scope of this book and are omitted. The paper where most of them come from—Reid (2003b)—give details on the general method with some worked examples.

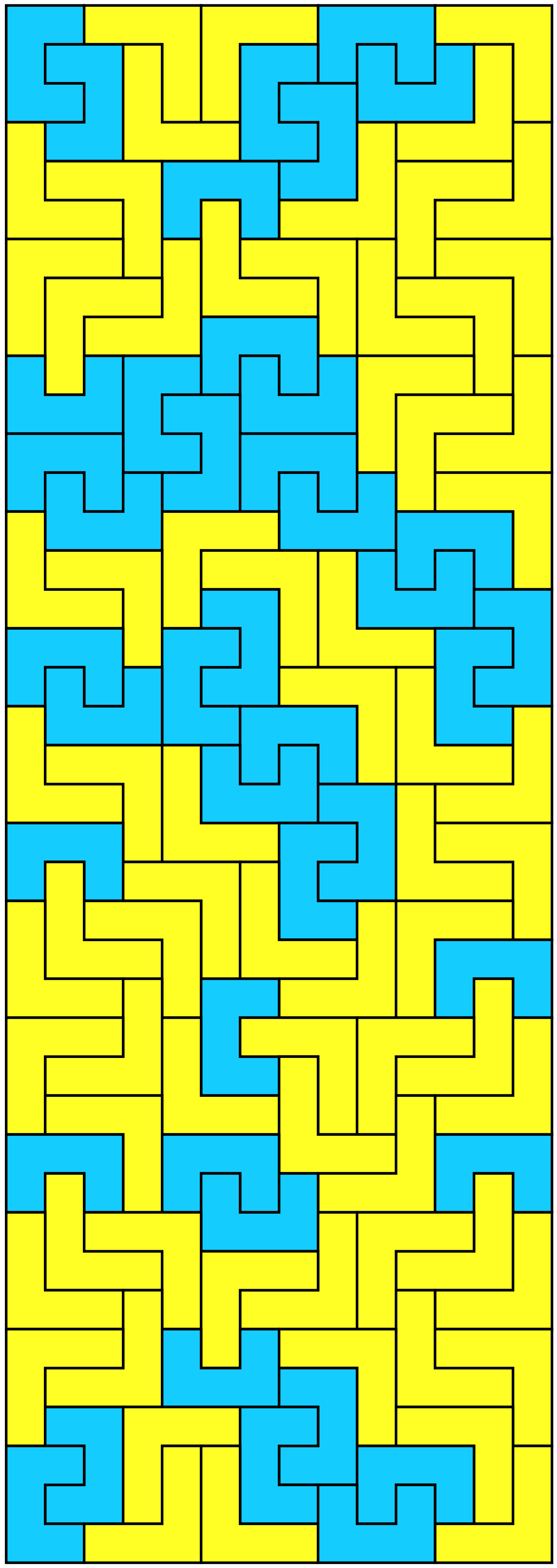
Theorem 254 (Reid (2003b), Theorem 3.9)**.** *If $\mathcal{T} = \{$  $_3$,  $_5\}$ tiles a rectangle, one side is divisible by 3.*

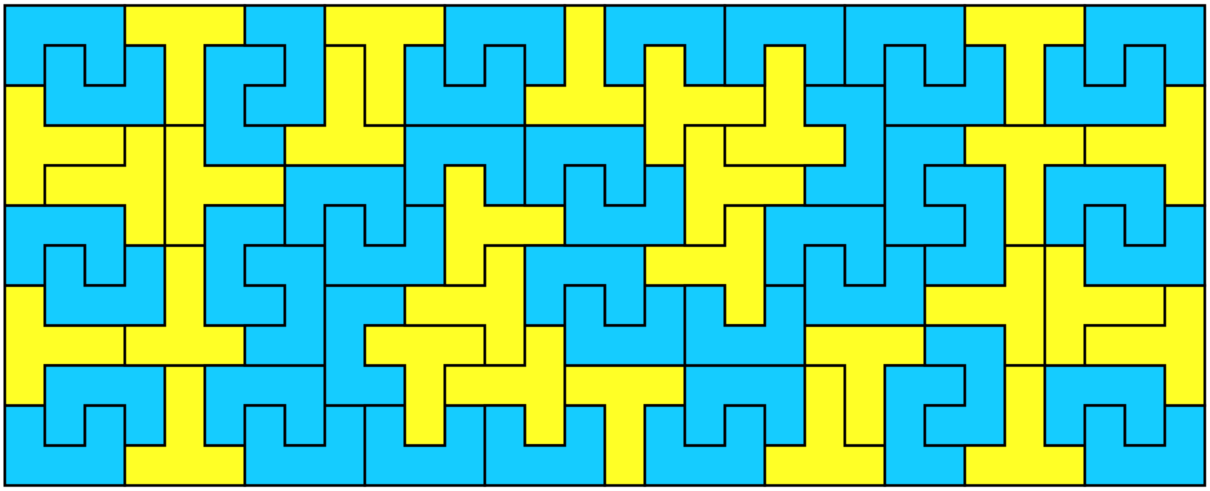
[Referenced on page 261]

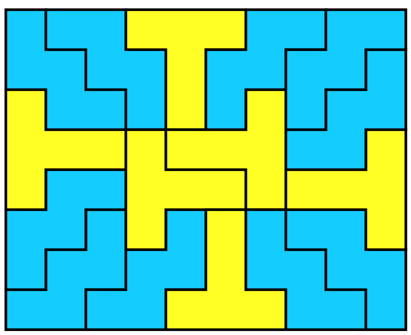
Problem 88** (Reid (2003b), Question 3.14)**.** *Is there a tiling of a rectangle by $\mathcal{T} = \{$  $_3$,  $_5\}$ that uses exactly 3 X-pentominoes?*

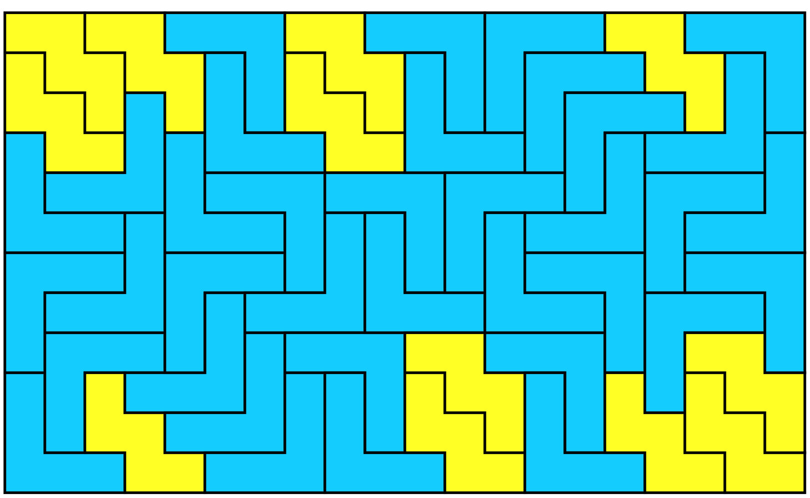
Theorem 255 (Reid (2003b), Theorem 5.7)**.** *If $\mathcal{T} = \{$  $_4$,  $_5\}$ tiles a rectangle, one side is even, and therefore so are the number of pentominoes.*

[Not referenced]









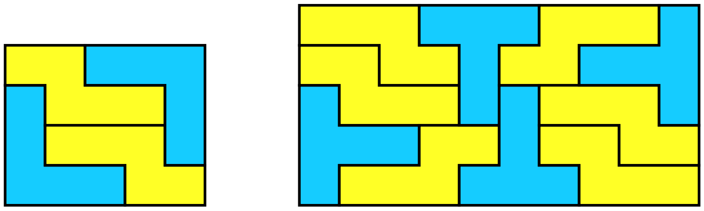
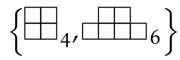
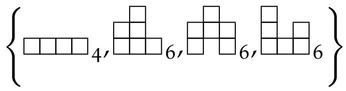


Figure 277: Minimum rectangles.

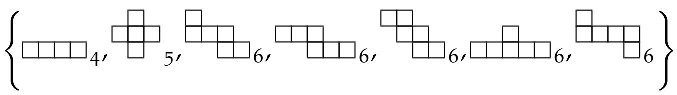
Theorem 256 (Reid (2003b), Theorem 7.2). *If* $\mathcal{T} =$  *tiles* $R(m, n)$, *mn is divisible by 4.*

[Not referenced]

Theorem 257 (Reid (2003b), Theorem 7.5). *If* $\mathcal{T} =$  *tiles a rectangle, one side is divisible by 4.*

[Not referenced]

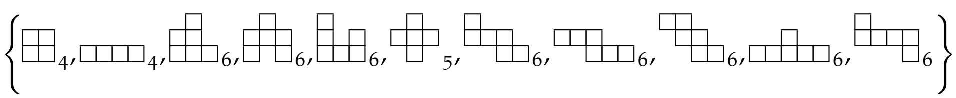
Theorem 258 (Reid (2003b), Theorem 7.6). *If*

$\mathcal{T} =$ 

tiles a rectangle, one side is divisible by 4.

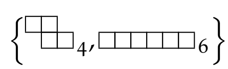
[Not referenced]

Theorem 259 (Reid (2003b), Theorem 7.7). *If*

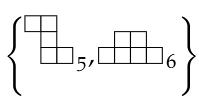
$\mathcal{T} =$ 

tiles a rectangle, one side is divisible by 4.

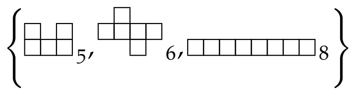
[Not referenced]

Theorem 260 (Reid (2003b), Theorem 7.8). *If* $\mathcal{T} =$  *tiles a rectangle, one side is divisible by 6.*

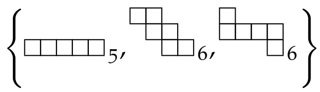
[Not referenced]

Theorem 261 (Reid (2003b), Theorem 7.9). *If* $\mathcal{T} =$  *tiles a rectangle, one side is divisible by 3.*

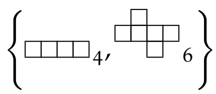
[Not referenced]

Theorem 262 (Reid (2003b), Theorem 7.10). *If $\mathcal{T} =$*  *tiles a rectangle, one side is even.*

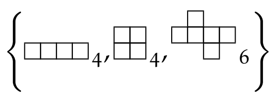
[Not referenced]

Theorem 263 (Reid (2003b), Theorem 7.11). *If $\mathcal{T} =$*  *tiles a rectangle, one side is even.*

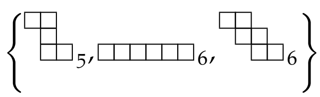
[Not referenced]

Theorem 264 (Reid (2003b), Theorem 7.12). *If $\mathcal{T} =$*  *tiles a rectangle, one side is divisible by 4.*

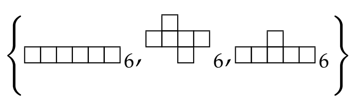
[Not referenced]

Theorem 265 (Reid (2003b), Theorem 7.13). *If $\mathcal{T} =$*  *tiles a rectangle, one side is divisible by 4.*

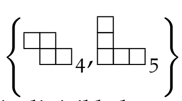
[Not referenced]

Theorem 266 (Reid (2003b), Theorem 7.14). *If $\mathcal{T} =$*  *tiles a rectangle, one side is divisible by 6.*

[Not referenced]

Theorem 267 (Reid (2003b), Theorem 7.15). *If $\mathcal{T} =$*  *tiles a rectangle, one side is divisible by 6.*

[Not referenced]

Theorem 268. *Let $\mathcal{T} =$* *. In a tiling of a rectangle, if one side is odd, the other side is divisible by 3.*

[Not referenced]

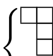
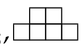
Proof. WLOG lets suppose n is odd; we will prove m is divisible by 3.

Color rows alternately black and white, starting with black. Let $N_{b,w}$ be the number of polyominoes in the tiling covering b black and w white cells. There are only three types; $N_{2,2}$ tetrominoes, $L_{4,1}$ pentominoes and $L_{1,4}$ pentominoes.

The number of black cells in the rectangle is $B = (n+1)m/2$, and the number of white cells is $W = (n-1)m/2$. Looking at the tiling by polyominoes, we also find the black cells is $B = 2N_{2,2} + 4N_{4,1} + N_{1,4}$, and the number of white cells $W = 2N_{2,2} + N_{4,1} + 4N_{1,4}$. Putting this together, we find:

$$(n+1)m/2 = 2N_{2,2} + 4N_{4,1} + N_{1,4} \tag{5.2}$$
$$(n-1)m/2 = 2N_{2,2} + N_{4,1} + 4N_{1,4} \tag{5.3}$$

Subtracting the last equation from the first, gives us $m = 3(N_{4,1} - N_{1,4})$, from which we can conclude that $3 \mid m$. □

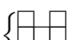
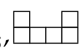
Theorem 269 (Reid (2008), Theorem 4.13)**.** *Let* $\mathcal{T} = \{$  $_5$,  $_6\}$. *Then if* $m, n \geq 16$, *then* $\mathcal{T}$ *packs a rectangle* $R(m, n)$ *if and only if:*

(1) *either 3 divides m or 6 divides n and*

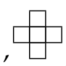
(2) *either 6 divides m or 3 divides n.*

Moreover, the prime rectangles are 4×6, 5×12, 10×78, 11×30, 13×42, 14×18, $15 \times \{18, 21, 27\}$, 17×18, *and* 21×21.

[Not referenced]

Theorem 270 (Reid (2008), Theorem 5.3)**.** *Let* $\mathcal{T} = \{$  $_5$,  $_6\}$. *If* $\mathcal{T}$ *tiles* $R(m, n)$, *then mn is even.*

[Not referenced]

Theorem 271. *There is no tiling of a rectangle by* $\{R(m, n),$  $_5\}$ *that requires at least one of each where* $m, n > 1$.[25]

[Referenced on page 342]

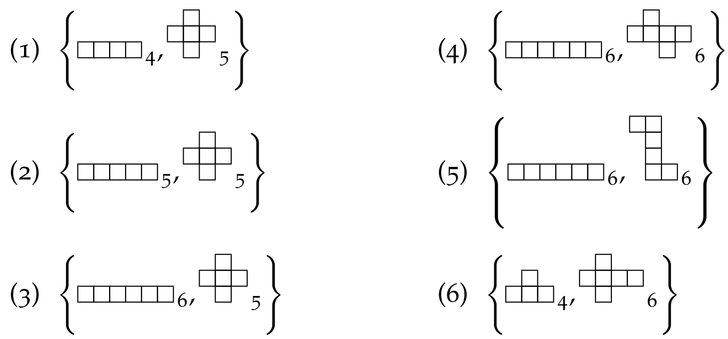
Proof. Suppose there is a tiling of rectangle R which use at least one piece of each. Find a cross, and let S be the biggest connected subregion of R that is covered by this cross, and only crosses. This region must have a peak (Problem 24), and WLOG let this peak be a horizontal peak with the cross below.

This peak cannot lie on the side of R; if it does the cells to the left and right of the peak cell are not tileable.

So there must be a cell C to the top of the peak inside R. The cell C must be covered by a rectangle (it cannot be covered by a cross; otherwise it would be part of S). However we do it, there is a non-tileable cell between the rectangle and the cross (either to the left or the right of the peak cell). Therefore, the C is not tileable, and therefore, no tiling of a rectangle that use both squares and crosses exist. □

[25] Friedman (2008), August 2010, Nr. 1 https://erich-friedman.github.io/mathmagic/0810.html gives a few specific instances without proof.

Problem 89** (Friedman (2008), August 2010, Nr. 1[26]). *Is there a tiling of a rectangle by the following sets that requires at least one tile of each?*



[26] https://erich-friedman.github.io/mathmagic/0810.html

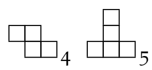
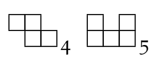
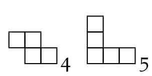
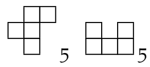
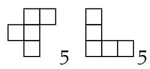
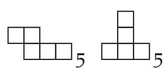
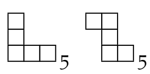
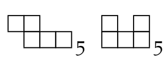
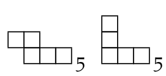
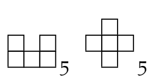
Pair	PR	×
	41	5 × 14 20 26 32 38 44 50
		6 × 6 8 10 11 13 15
		7 × 10 12 13 14 16 17 18 19 21
		8 × 8 10 11 13 15
		9 × 10 11 12 13 14 15 16 17 18 19
		10 × 10 11
		11 × 11 13
	43	3 × 6
		5 × 10 18 22 26 34
		6 × 16 17
		7 × 10 16 18 22 23 24 25 27 29 31
		8 × 11 12 13 14 15 16 17 19 20 21
		9 × 10 11 13 14 15
		10 × 11 13
		11 × 11 13 14 15
		13 × 13 14 15
		14 × 14
	40	8 × 16 18 20 22 24 26 28 30
		9 × 18 21 24 27 30 33
		10 × 14 16 18 20 22 24 26
		11 × 12 18
		12 × 12 13 14 15 16 17 18 19 20 21
		13 × 18
		14 × 14 16 18
		15 × 15 18 21

Table 49: Prime Rectangles for pairs of polyominoes. From Friedman (2008, https://erich-friedman.github.io/mathmagic/0101.html).

Pair	PR	×	
	12	3 × 10	
		6 × 15	
		7 × 10	15
		8 × 25	35 40 45 55
		9 × 15	
		10 × 11	
		11 × 15	
	10	4 × 5	
		6 × 10	
		7 × 20	30 45 55
		9 × 10	15
		10 × 11	
		11 × 15	
	12	5 × 10	16 18 22 24
		6 × 10	15
		7 × 10	15
		9 × 10	15
		11 × 15	
	6	4 × 5	
		5 × 6	
		7 × 10	25
		9 × 15	
		11 × 15	
	16	4 × 5	
		7 × 40	50 55 60 65 75 85
		9 × 15	25
		10 × 13	14 15
		11 × 15	20 25
	6	4 × 5	
		5 × 6	
		7 × 10	15
		9 × 15	
		11 × 15	
	13	3 × 5	
		17 × 85	110
		19 × 40	65
		20 × 70	71
		22 × 50	
		23 × 35	40
		25 × 28	38
		28 × 35	

×

Table 50: Prime Rectangles for pairs of polyominoes. From Friedman (2008, https://erich-friedman.github.io/mathmagic/0101.html).

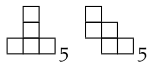
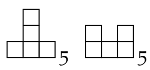
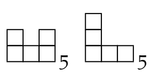
Pair	PR	×
	16	8 × 10 15
		9 × 10 15
		10 × 10 11 12 13 15
		11 × 15
		12 × 15
		13 × 15
		14 × 15
		15 × 15 17
		12 × 30 40 45 50 55 65
		13 × 25 30 35 40 45
		14 × 35 40 45 50 55 60 65
		15 × 24 28 29 31 32 33 34 35 36 37 38 39 40 41 42
		× 43 44 45 46 47 49 51 59
		16 × 30 35 40 45 50 55
		17 × 20 25 30 35
		18 × 20 25 30 35
		19 × 25 30 35 40 45
		20 × 20 21 22 23 24 25 26 27 28 29 30 31 32 33
		21 × 25 30 35
		22 × 25 30
		14 × 40 45 50 55 65 70 75
		15 × 28 32 36 38 40 41 42 44 45 46 47 48 49
		× 50 51 52 53 54 55 57 58 59 61 62 63 67
		16 × 30 35 40 45 50 55
		17 × 30 35 40 45 50 55
		18 × 30 35 40 45 50 55
		19 × 30 35 40 45 50 55
		20 × 20 21 22 23 25 26 27 28 29 30 31 32 33 34 35
		× 36 37 38 39
		21 × 25 30 35
		22 × 25 30 35
		23 × 35 30 25
		24 × 25 30 35 40 45
		25 × 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39
		26 × 30 35
		27 × 30 35
		28 × 30 35
		29 × 30 35
		30 × 30 31

Table 51: Prime Rectangles for pairs of polyominoes. From Friedman (2008, https://erich-friedman.github.io/mathmagic/0101.html).

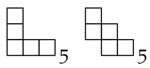
Pair	PR	×
		12 × 20 25 30 35
		13 × 35 40 45 50 55 60 65
		14 × 30 35 40 45 50 55
		15 × 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37
		× 38 39 40 41 42 43 44 45
		16 × 25 30 35 40 41 42 43 44 45
		17 × 25 30 35 40 45
		18 × 20 25 30 35
		19 × 20 25 30 35
		20 × 20 21 22 23 25 26 27 28 29
		21 × 25 30 35
		22 × 25 30 35
		25 × 25 26

Table 52: Prime Rectangles for pairs of polyominoes. From Friedman (2008, https://erich-friedman.github.io/mathmagic/0101.html)

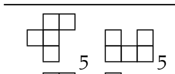
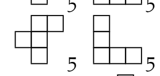
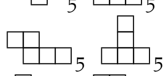
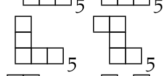
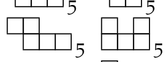
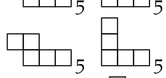
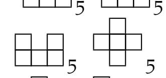
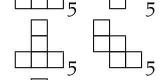
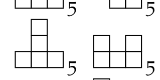
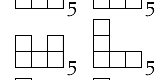
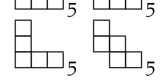
Pair	RO
	6
	4
	10
	4
	4
	4
	3
	16
	65
	80
	48

Table 53: Orders for pairs of polyominoes.

5.7 *Simple Tilings*

A tiling of a region with more than two tiles is **simple** if no subset of tiles (with more than one tile) form a rectangle strictly inside the region. For rectangles, simple tilings are also fault free (otherwise, the rectangle will split into two rectangles at the fault, at least one of which must have more than one tile).

Theorem 272 (Martin (1991), p. 15)**.** *No rectangle has a simple tiling with dominoes.*

[Not referenced]

Proof. It is obvious for $R(m,1)$ or $R(1,n)$.

A rectangles $R(m,n)$ with $m,n > 1$ have no peaks or holes, and therefore any tiling of it has a flippable pair (which is a rectangle) as subtiling (Theorem 69), and is therefore not simple. □

Theorem 273 (Martin (1991), p. 15)**.** *Any simple tiling of the plane with dominoes is fault-free.*

[Not referenced]

Proof. Suppose there is a fault, WLOG let it be a horizontal fault. The pattern on the fault must be alternating vertical and horizontal dominoes, since any two adjacent dominoes of the same orientation will form a rectangle (and therefore the tiling would not be simple.) Therefore, we can find an LR-configuration, which means there is a forced flippable pair inside the induced pyramid (Theorem 100), which means the tiling cannot be simple. Therefore, there can be no fault-line. □

Theorem 274 (Martin (1991), p. 16)**.** *The plane has a unique (up to reflection) simple tiling with dominoes.*

[Not referenced]

Proof. WLOG, suppose the tiling contains a vertical domino, with 6 neighbors, labeled 1–6, starting from the bottom, going anticlockwise.

(1) In position 1, we cannot fit a vertical domino, so it must fit a horizontal domino. Let's say it lies to the right.

(2) Similarly, in position 4, we must fit a horizontal domino. It cannot also lie to the right, since otherwise we will have an LR configuration, which will force a flippable pair (Theorem 100), therefore it must lie to the left.

(3) In position 2, we cannot fit a vertical domino (since it will either overlap with the domino on 1, or form a rectangle), therefore it must be horizontal.

(4) A similar argument applies to position 5.

(5) In position 3, a horizontal domino will form a rectangle with the domino at position 2, therefore it must be vertical.

(6) A similar argument applies to position 6.

We can now repeat the same argument for the new vertical dominoes, except that because we have some neighbors placed, we do not have a choice we had in step (1). And a symmetrical procedure can be applied to horizontal dominoes; the entire tiling is forced.

If we made a different choice in step (1) above, we will construct the same tiling, but reflected.

The pattern is called a *herringbone pattern*.

□

If we now consider general polyominoes, we can find simple tilings with 5 (Figure 286) and 7 (Figure 287) tiles, and as the following theorem shows, for all numbers of tiles 7 or bigger.

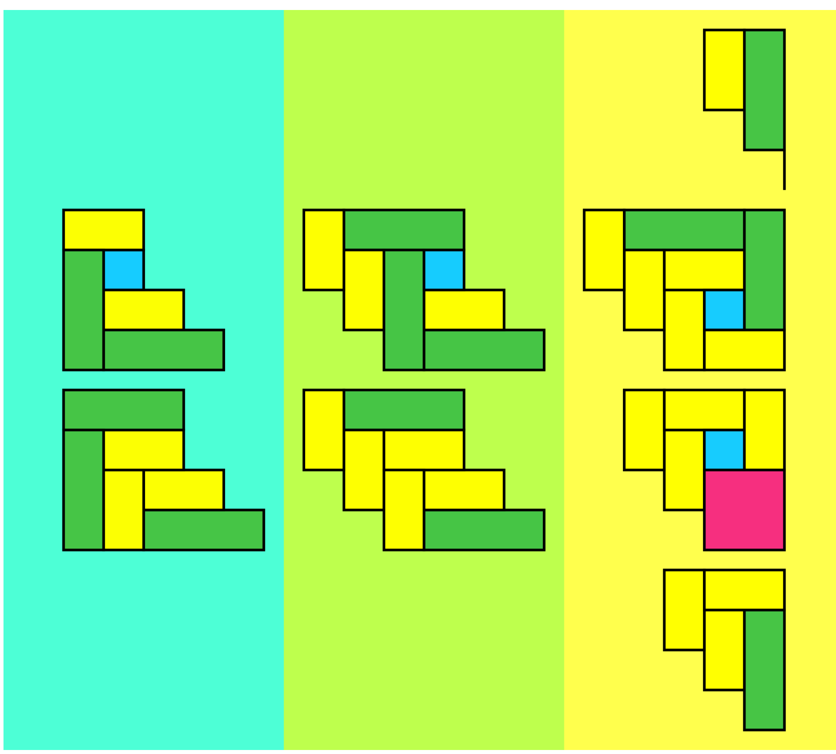


Figure 278: Blocks that can be used to construct simple tilings using n pieces for all $n \geq 7$.

Theorem 275 (Los et al. (1960))**.** *There exists a simple tiling with n rectangles for all $n \geq 7$.*

[Not referenced]

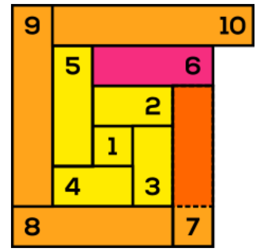


Figure 279: A construction showing how we can get a simple tiling for all $n \geq 5, n \neq 6$. For example, to get a simple tiling with 7 tiles, place the first 7 tiles in the spiral as shown, and truncate it at the dotted line.

Proof. Consider the construction shown in Figure 279. The rectangles are placed in a spiral as shown. To get a simple tiling with n tiles, place the first n tiles in the spiral as shown, but truncate the last tile so that it does not extend beyond the rectangle. This works for all $n \geq 5, n \neq 6$. (It does not work for $n = 6$, because truncating the sixth tile will make it the same size and aligned with the second tile.) □

Proof. (Alternative proof (Chung et al., 1982, Theorem 1).) Using the blocks in Figure 278 we can construct simple tilings using any number of tiles greater than 7 by choosing an appropriate left and right block, and any number (including 0) of center blocks. An example construction is shown in Figure 280. □

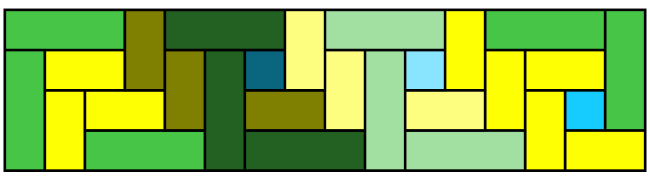


Figure 280: An example construction of a simple tiling with 28 tiles.

Theorem 276. *There is no simple tiling with 6 rectangles.*

[Not referenced]

Proof. (Adapted from Parcly Taxel (un.) (2016).)

(1) In a simple tiling, no rectangle can cover two corners of the rectangular region. If there were such a piece, the remaining pieces would form a rectangle. Therefore, the four corners must be covered by different rectangles.

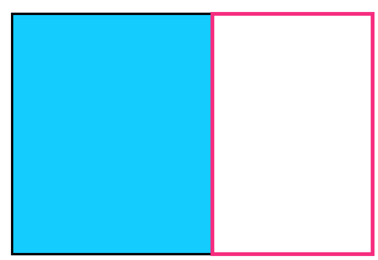
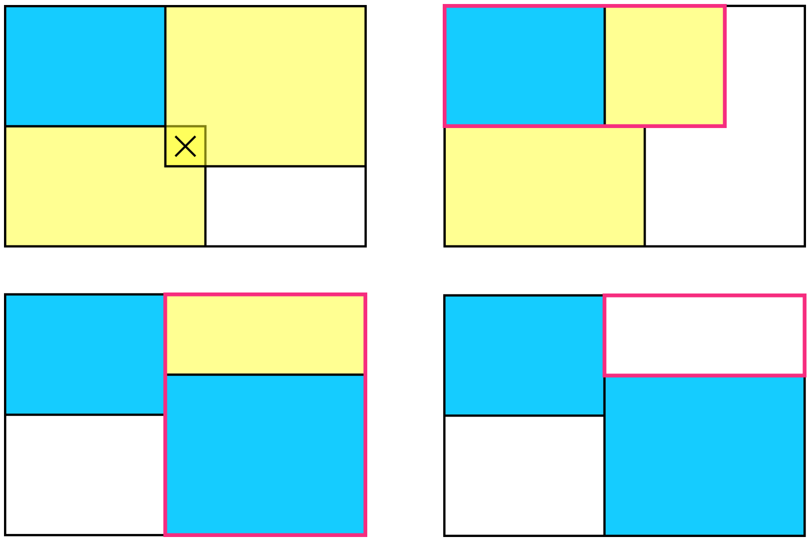


Figure 281:

(2) A corner piece must have at least 3 neighboring pieces. If a corner piece had two neighbors, they have to lie on opposite sides. If both neighbors were longer than the side, they must overlap. Therefore, at least one is the same size. But then it makes a rectangle with the corner piece, which is not possible since the tiling is simple.

(3) Diagonally opposite corner pieces cannot touch. If they did, the remaining region would be two rectangles. If either rectangle is a single piece, it forms a rectangle with one of the corner pieces. Otherwise, it is tiled by smaller pieces, which violates the tiling being simple.



Now lets consider $n = 6$ specifically. Four of those six pieces must be in the four corners of the final figure (1).

Suppose the remaining two pieces are fully internal to the rectangular region. Each of the outside pieces can only expose one side to the internal area, so the internal area is rectangular. Filling it with the remaining two pieces creates a sub-rectangle with 2 pieces.

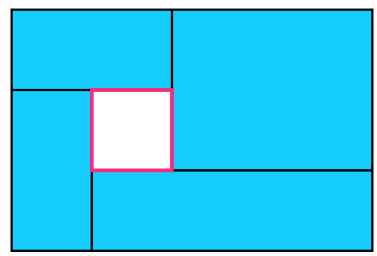


Figure 282:

Suppose on the other hand that all 6 pieces are on the boundary of the rectangular region, i.e. there are no internal pieces. So we have 4 corner pieces and 2 edge pieces.

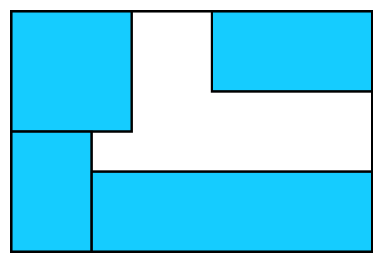
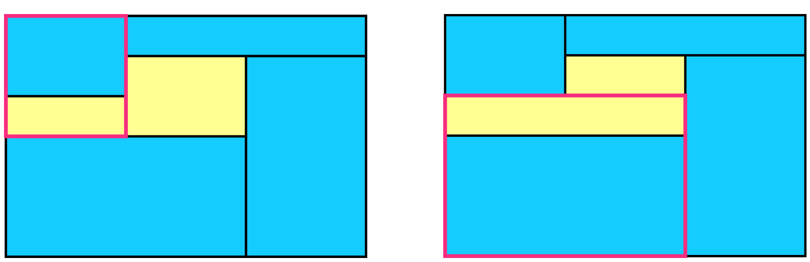


Figure 283:

- Suppose a corner piece lies between two edge pieces. It must have a third neighbor (2), but the only candidate is the diagonally opposite corner, which violates (3).
- Suppose the two edge pieces are adjacent. The four corner pieces must be arranged so that the opening is an L-shape. There are two ways to fill the L-shape with rectangles; in each case one rectangle must be interior so both cannot be edge pieces and therefore this arrangement is impossible.[27]

[27] The original proof omits this case.



- The only other arrangement for the edge pieces is on opposite sides of the rectangular region, say the left and right sides. The two top corners are adjacent, cannot touch either of the bottom corners, so the only way for them to have 3 neighbors is for both corners to be adjacent to both edge pieces. This is not possible.

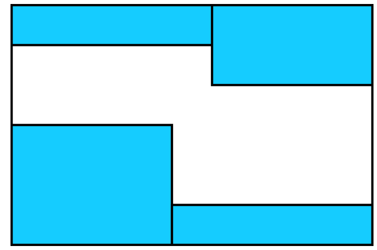


Figure 284:

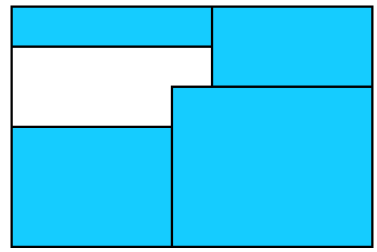


Figure 285:

The last possibility is that we have 4 corner pieces, 1 edge piece, and 1 internal piece. The two corners next to the edge piece must have the internal piece as their third neighbor. The edge piece has three internal sides and so must have at least three neighbors. The only possibility is that it is also adjacent to the internal piece. In a similar argument to (2), the corners cannot be the same length as the edge piece, and if both were longer then the edge piece and the internal piece have matching lengths and form a rectangle.

All possibilities lead to failure, so $n = 6$ is impossible. □

The **average area** of a tile in a tiling of a region R is given by $|R|/n$, where R is scaled so that the smallest tile has area 1, and n is the number of tiles[28].

[28] We always stretch the rectangle so that the smallest tile is a square.

Theorem 277 (Chung et al. (1982), Theorem 2)**.**

(1) *Except for the simple tiling in Figure 286 that has average area* $9/5$*, all other simple tilings of rectangles has average area stricter greater than* $11/6$*.*

(2) *There is a unique simple tiling of the plane with average area of* $11/6$*. This tiling is shown in Figure 289.*

[Not referenced]

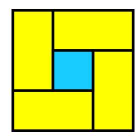


Figure 286: A simple tiling with 5 tiles.

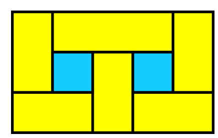


Figure 287: A simple tiling with 7 tiles.

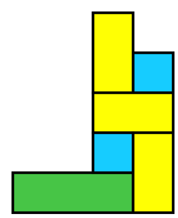


Figure 288: A tile that can be used to construct the tiling shown in Figure 289.

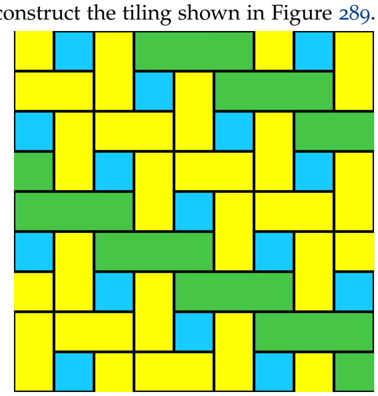


Figure 289: A tiling of the plane with average area is $11/6$.

5.8 Further Reading

Theorem 167 is given in a slightly more general form in the original paper (Bower and Michael, 2004), where it applies to n-dimensional bricks with arbitrary lengths, not necessarily integers. Wagon (1987) gives and discuss 14 proofs of this theorem, including some generalizations.

Aamand et al. (2020) gives an algorithm for determining whether a region is tileable by square tiles or dominoes that is linearithmic-time[29] in the number of corners of the region.

[29] An algorithm with time complexity $O(n \log n)$.

Some problems relating to tiling rectangles by rectangles:

(1) *Finding square tilings of squares.* When each square has a different area, the tiling is called *perfect*. Anderson (2013) gives a chronology of squaring the square discoveries and is a good introduction to the problem.

(2) *Finding whether a set of rectangles will pack a given rectangle.* A survey of this is given by Lodi et al. (2002).

The question to show that there exists an integer C such that $R(m,n)$ is tileable by $R(4,6)$, $R(6,4)$, $R(5,7)$ and $R(7,5)$ for all $m,n > C$ was asked in the *The Fifty-Second William Lowell Putman mathematical competition* (Klosinski et al., 1992, Problem B-3). Narayan and Schwenk (2002) finds the minimum value C, and the thesis of Dietert (2010) considers which rectangles can be tiled by this set.

Barnes (1982a) and Barnes (1982b) considers tilings of bricks by bricks from a algebraic viewpoint. The n-dimensional version of Theorem 181—the gap number bricks packed by rods—is proven in Barnes (1995). In Brualdi and Foregger (1974) the problem is discussed for harmonic bricks, and in Barnes (1979) for sufficiently large rectangles .

Frobenius numbers and related problems are covered in detail in Ramírez-Alfonsín (2005).

Theorem 216 has been generalized by Maltby (1994) by showing that trisecting a rectangle into 3 congruent pieces of any shape (not necessarily polyominoes), implies that the pieces are rectangles. A different proof is given in Dumitru et al. (2018). Yuan et al. (2016) showed that if a square is dissected into 5 convex congruent pieces, there is a unique solution with rectangles only. Rao et al. (2020) deals with dissecting squares into convex polygons, and show among other things that certain shapes can only tile a square if the number of tiles is even. (Unfortunately, these last two papers are not helpful for polyominoes, since the only polyominoes that are convex in the normal geometric sense are rectangles. See Problem 115(1))

Kenyon (1996) gives an algorithm for determining whether a rectilinear region (not necessarily with all integer sides) is tileable with rectangles each of which has at least one integer side. One drawback of this algorithm is that the tiling rectangles cannot be specified as part of the input.

For more on tilings by bars, see Beauquier et al. (1995).

Bloch (1979) contains tilings by rectangles for small rectangles. Mitchell et al. (1976) describe how to generate ones with distinct topologies and also list a few such generated ones. Tables with the number of tilings for small polyominoes can be found in Grekov (2013) (http://polyominoes.org/data).

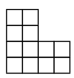
For more on prime rectangles see de Bruijn and Klarner (1975), Klarner (1973), and Reid (2005). The latter also gives many examples of finding prime rectangle for polyominoes and polyomino sets. Reid (2008) discusses tiling theorems for "sufficiently large" rectangles.

Clarke (2006a) gives a large (but incomplete) lists of prime rectangles for pentominoes, hexominoes, and heptominoes.

Korn and Pak (2004) consider T-tetromino chains and height functions of T-tetromino tilings. The number of tilings of rectangles by T-tetrominoes is considered in Merino (2008). Tilings of deficient rectangles by L-tetrominoes are discussed in Nitica (2004), and by T-tetrominoes in Zhan (2012). Gap numbers for the T-tetromino on rectangles is covered in Hochberg (2015).

6
Reptiles and Irreptiles

This chapter covers some details on reptiles and irreptiles that I introduced in section 4.1. Recall that a *irreptile* is a polyomino for which a scaled copy has a non-trivial tiling by scaled copies of the polyomino, potentially by different factors. If a tiling exist where all the factors are the same, the polyomino is a *reptile*.

The notation $k \cdot P$ is used to denote P scaled by k. For example, $2 \cdot$  $_3$ is the polyomino  $_{12}$.

6.1 Reptiles

We will now prove a bunch of theorems that state what types of reptile tilings is possible for various polyominoes. Most proofs use the idea of tiling extensions that we used in previous chapters. I only treat the first one in detail; thereafter I only give the tilings, with how they should be extended[1].

[1] In the current version this chapter still omits most of the figures that make out the proofs.

Theorem 278. *If a polyomino is rep-k^2 and rep-ℓ^2, is is also rep-$(k\ell)^2$.*

[Not referenced]

Proof. In the rep-k^2 tiling, replace each tile with the rep-ℓ^2 tiling; the result is a rep $(k\ell)^2$ tiling. □

This theorem is useful when we are trying to determine all k for which a polyomino is rep-k^2; for example, if we know a polyomino is rep-2^2 and rep-k^2 for odd k, then it is rep-k^2 for all k.

Theorem 279. *Rectangles are rep-k^2 for all k.*

[Not referenced]

Proof. By Theorem 22 $R(m,n)$ tiles $R(mk,nk)$, and this requires k^2 tiles; therefore $R(m,n)$ is rep-k^2. □

Theorem 280 (Nitica (2002), Proposition 4). $\square_3$ *is rep-k^2 for all $k \geq 1$.*

[Referenced on page 283]

Proof. The proof is by induction. For the base cases $k = 2, 3$, see Figure 290.

We now show that if there is a tiling for k^2, there is a tiling for $(k+2)^2$. We divide the proof into 3 cases, depending on the value of $k \pmod 3$.

Case: $k \equiv 0 \pmod 3$: The figure can be partitioned into six pieces:

- $k \cdot \square_3$ (tileable because of induction hypothesis),
- $2 \cdot \square_3$ (tileable as shown in Figure 290),
- 2 of $R(k, 2)$ (tileable by Theorem 22, since 3 divides k),
- 2 of $R(2k, 2)$ (tileable by Theorem 22, since 3 divides k)

All partitions are tilable, therefore $(k+2) \cdot \square_3$ is tileable.

Case: $k \equiv 1 \pmod 3$: The figure can be partitioned into six pieces:

- $k \cdot \square_3$ (tileable because of induction hypothesis),
- $2 \cdot \square_3$ (tileable as shown in Figure 290),
- 4 of $R(k+2, 2)$ (tileable by Theorem 22, since 3 divides $k+2$),

All partitions are tilable, therefore $(k+2) \cdot \square_3$ is tileable.

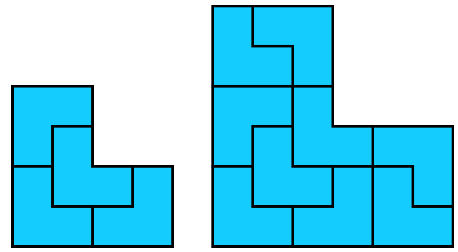


Figure 290: Base cases.

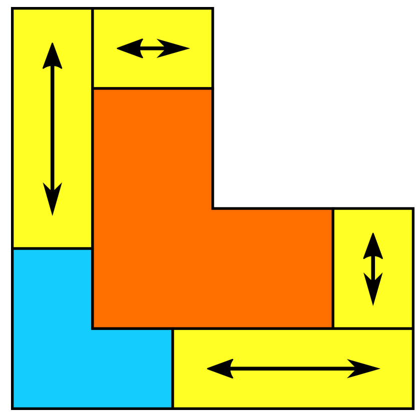


Figure 291: Case $k \equiv 0 \pmod 3$.

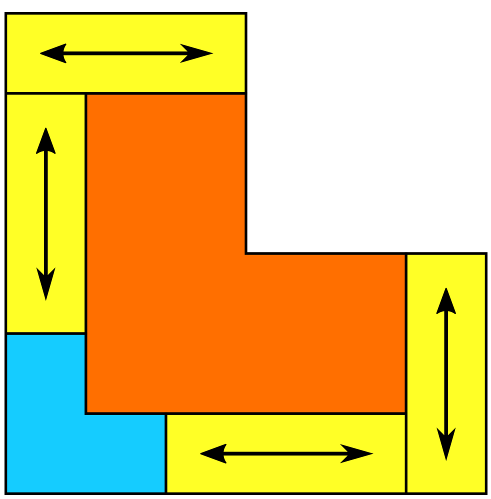


Figure 292: Case $k \equiv 1 \pmod 3$.

Case: $k \equiv 2 \pmod 3$: The figure can be partitioned into eight pieces:

- $k \cdot \square_3$ (tileable because of induction hypothesis),

- 3 of $2 \cdot$ ▙$_3$ (tileable as shown in Figure 292),
- 2 of $R(k-2,2)$ (tileable by Theorem 22, since 3 divides $k-2$),
- 2 of $R(2k-4,2)$ (tileable by Theorem 22, since 3 divides $2k-4$)

All partitions are tilable, therefore $(k+2) \cdot$ ▙$_3$ is tileable.

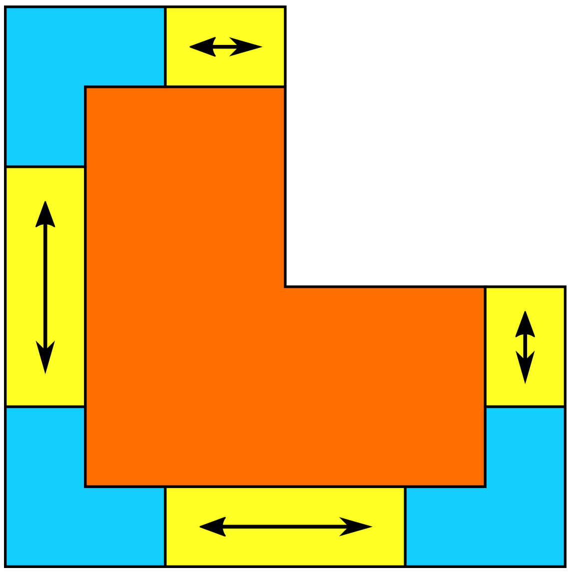


Figure 293: Case $k \equiv 2 \pmod 3$.

In all three cases, if $k \cdot$ ▙$_3$ is tileable, then so is $(k+2) \cdot$ ▙$_3$, and therefore, by induction, $k \cdot$ ▙$_3$ is tileable for all $k \geq 2$. □

Theorem 281. *The polyomino $B(2m)^n \cdot m^n)$ is rep-k^2 for all $k \geq 1$.*

[Not referenced]

Proof. $P = B((2m)^n \cdot m^n)$ is the right tromino stretched by a factor m vertically, and by a factor n horizontally. Any tiling of a region R by right trominoes can be transformed into a tiling of $(m,n) \cdot R$ by $B((2m)^n \cdot m^n)$; in particular , a tiling of $k \cdot$ ▙$_3$ can be transformed into a tiling $(mk, nk) \cdot$ ▙$_3$. This means P is a reptile, and P is rep$k$ whenever ▙$_3$ is rep-k. By Theorem 280, ▙$_3$ is rep-k for all $k \geq 1$; therefore P is rep-k for all $k \geq 1$.

□

Theorem 282 (Nitica (2002), Proposition 4)**.** ▙▄$_4$ *is rep-k^2 for all $k \geq 1$:*

[Not referenced]

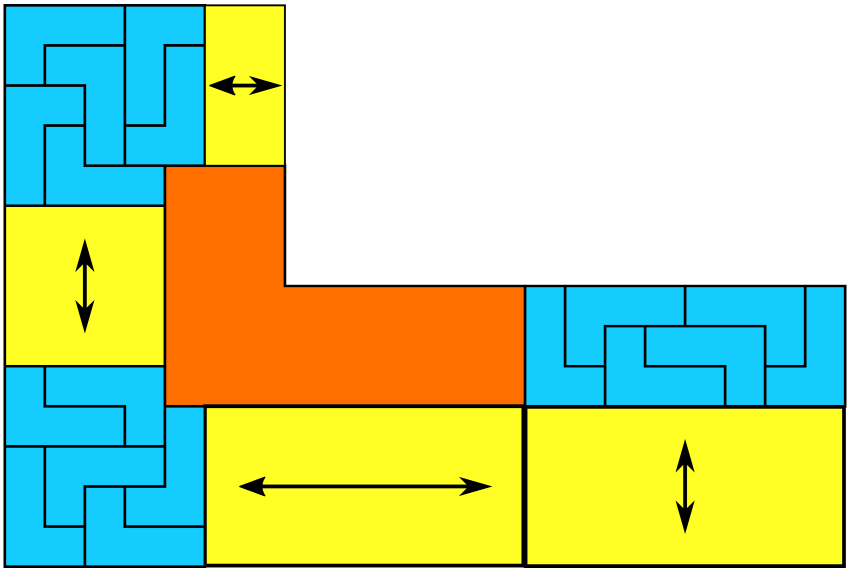


Figure 294: How to extend a reptile tiling of the L-tetromino.

Proof. Note that $k \cdot {}_4$ can be divided into two rectangles: $R(k, 2k)$ and $R(2k, k)$, which are both tileable by $R(2, 4)$ when k is even, and therefore tileable by ${}_4$.

Assume then k is odd.

We can partition the region into 8 pieces:

- $k \cdot {}_4$ (tileable because of induction hypothesis)
- 2 of $B(5^4 \cdot 4)$ (tileable as in Figure 295)
- $R(k-1, 4)$
- $R(4, 2k-2)$
- $R(3k-1, 4)$
- $R(8, k+1)$
- $R(8, 3)$ (tileable as in Figure 296)

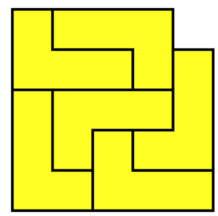


Figure 295: $B(5^4, 4)$

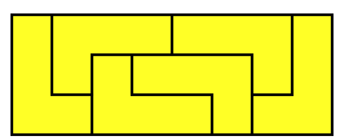


Figure 296: $R(8, 3)$

$\square$

Theorem 283 (Nitica (2002), Proposition 4). ${}_5$ *is rep-k^2 for all* $k \geq 1$*:*

[Not referenced]

Theorem 284 (Clarke (2006b)). *The L-pentomino is rep-k^2 for $k \geq 4$.*

[Not referenced]

For pentominoes, the analysis of the rectangle tilings by the Y-pentomino are the most complicated; for reptilings, we have a similar situation.

Theorem 285 (Clarke (2006b)). *The Y-pentomino is rep-k^2 for $k \geq 9$.*

[Not referenced]

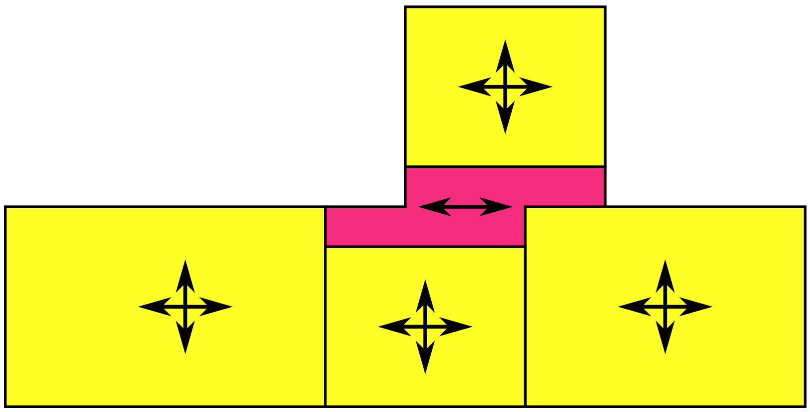


Figure 297: Construction for rep $(5k)^2$.

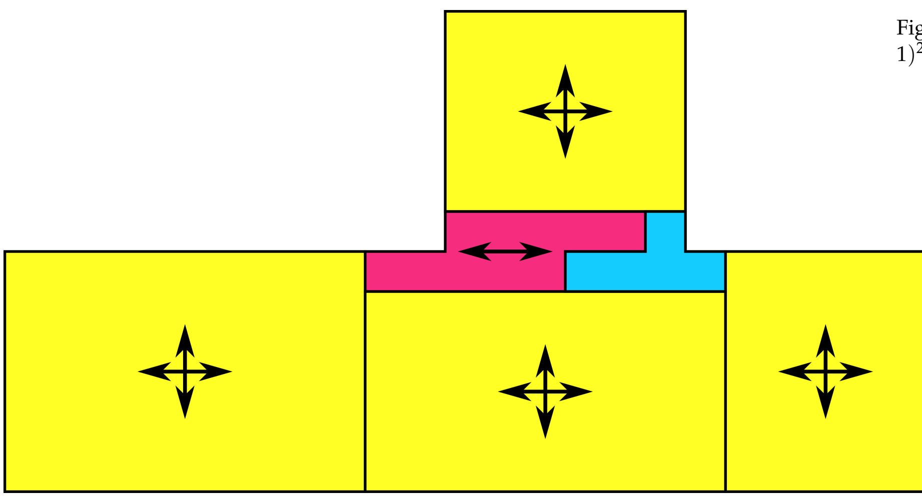
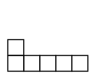
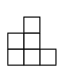


Figure 298: Construction for rep $(5k+1)^2$.

Theorem 286 (Clarke (2006b)). $_6$ *is rep k^2 for all $k \geq 6$.*

[Not referenced]

Theorem 287 (Clarke (2006b)). $_6$ *is rep k^2 for $k = 9$ and all $k \geq 12$.*

[Not referenced]

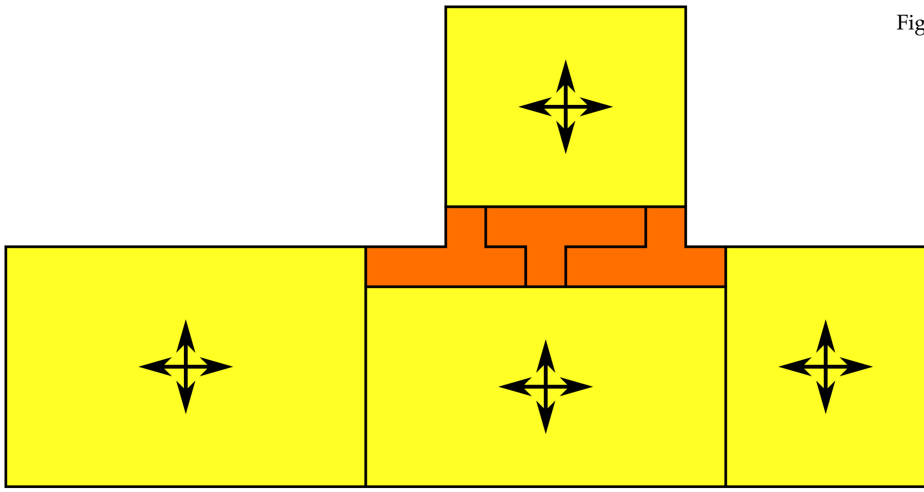


Figure 299: Construction for rep $(6k^2)^2$.

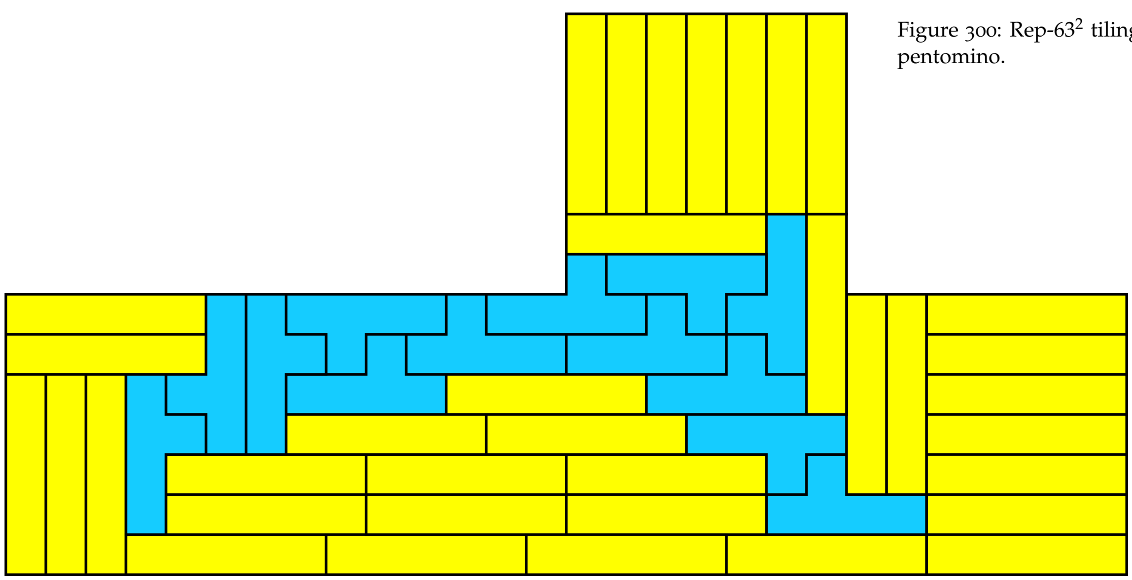


Figure 300: Rep-63^2 tiling by the y-pentomino.

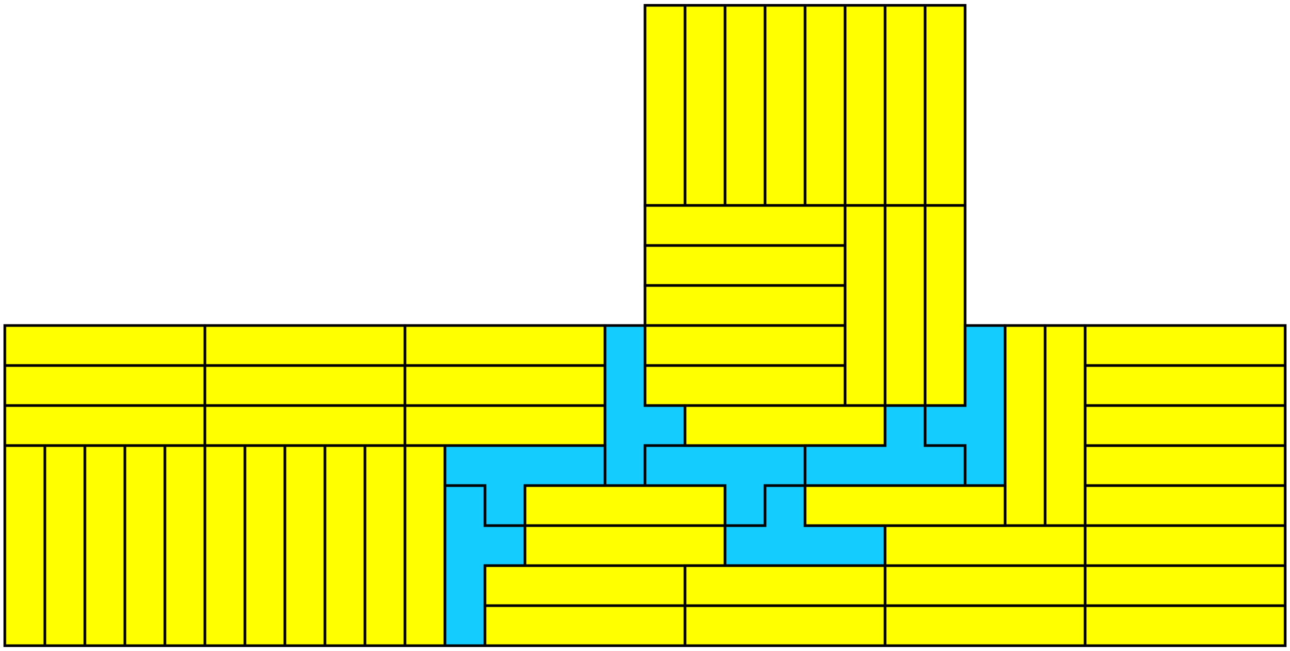


Figure 301: Rep-72^2 tiling by the y-pentomino.

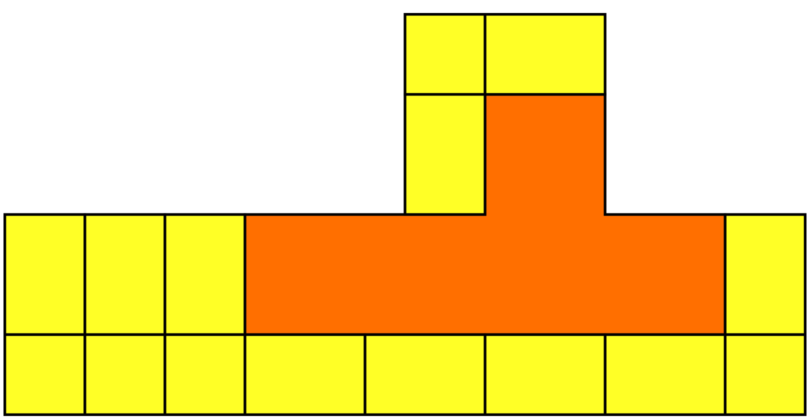
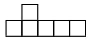
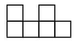


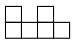
Figure 302: How to extend rep-k^2 to rep-$(k+10)^2$ or rep-$(k+15)^2$.

Theorem 288 (Clarke (2006b)). $_6$ *is rep* $(24k)^2$ *for all* $k \geq 2$ *and rep* $(24k+1)^2$ *for* $k \geq 5$.

[Not referenced]

Theorem 289 (Clarke (2006b)). $_6$ *is rep-*$(12k+\ell)^2$ *for* $\ell = 0, 8, 9, 13, 15, 16, 17, 19, 23$.

[Not referenced]

Reptilings for $_6$ are usually made from the one of 3 shapes made from a pair of polyominoes, and a lone polyomino. Clarke (2006b)

Problem* 90.

(1) *Construct a rep-*$(8k+15)^2$ *tiling.*

(2) *Use the tiling in Figure 304 and construct a generic tiling for rep-*$(12k+9)^2$.

(3) *Prove that there are no rep-*$(12k+10)^2$ *tilings.*

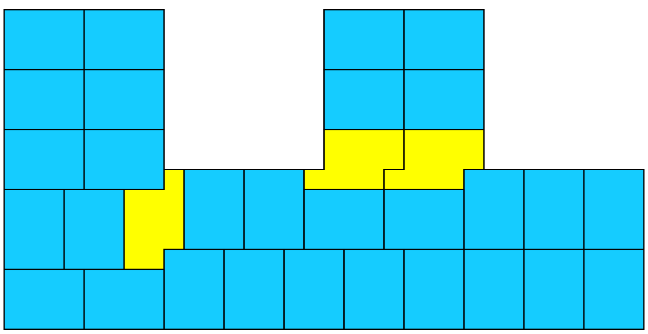


Figure 303: The rep-8^2 tiling.

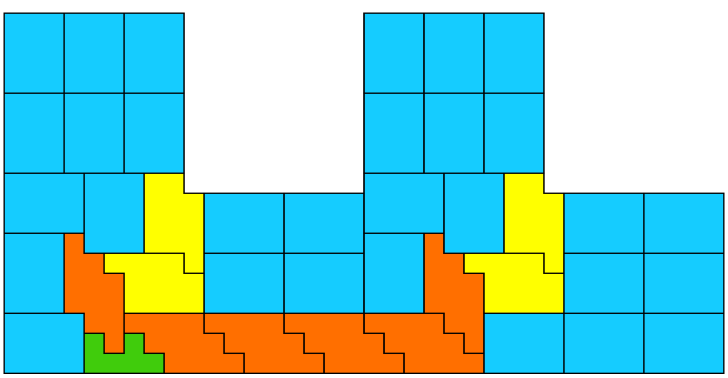
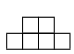


Figure 304: The rep-9^2 tiling.

Theorem 290 (Clarke (2006b)). $_6$ *is rep* $(6k)^2$ *and rep* $(12k \pm 1)$.

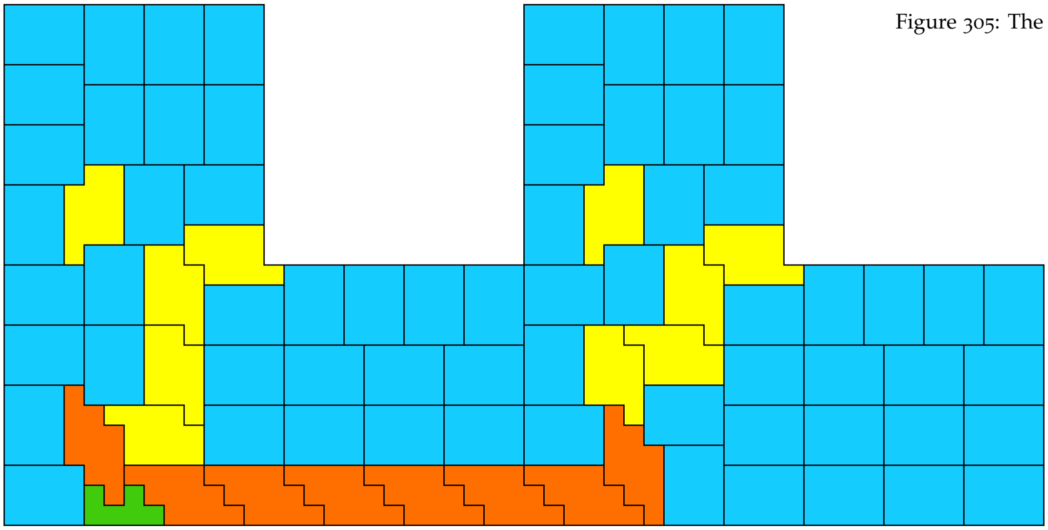
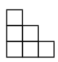


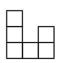
Figure 305: The rep-13^2 tiling.

[Not referenced]

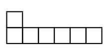
Theorem 291 (Clarke (2006b)). $_6$ *is rep* $(12k+\ell)^2$ *for* $\ell = 0, 1, 11$.

[Not referenced]

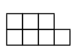
Problem* 91. *Is* $B(6^3, 4^3, 3^3)$ *a reptile? For what values of* k *is it rep-*k*?*

Theorem 292 (Clarke (2006b)). $_6$ *is rep* $(6k)^2$.

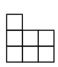
[Not referenced]

Theorem 293 (Clarke (2006b)). $_7$ *is rep* n^2 *for* $n = 7k, 7k \pm 1$.

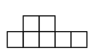
[Not referenced]

Theorem 294 (Clarke (2006b)). $_7$ *is rep* k^2 *for* $k = 4$ *and all* $k \geq 6$.

[Not referenced]

Theorem 295 (Clarke (2006b)). $_7$ *is rep* k^2 *for* $k = 11, 12$ *and all* $k \geq 14$.

[Not referenced]

Theorem 296 (Clarke (2006b)). $_7$ *is rep* n^2 *for* $n = 7k$, $k = 4, 6, 9, 3r; 5, 10, 5r; 10, 12, 2r$ *and* $n = 7k+1$ *with* $k = 12, 14, 2r$.

[Not referenced]

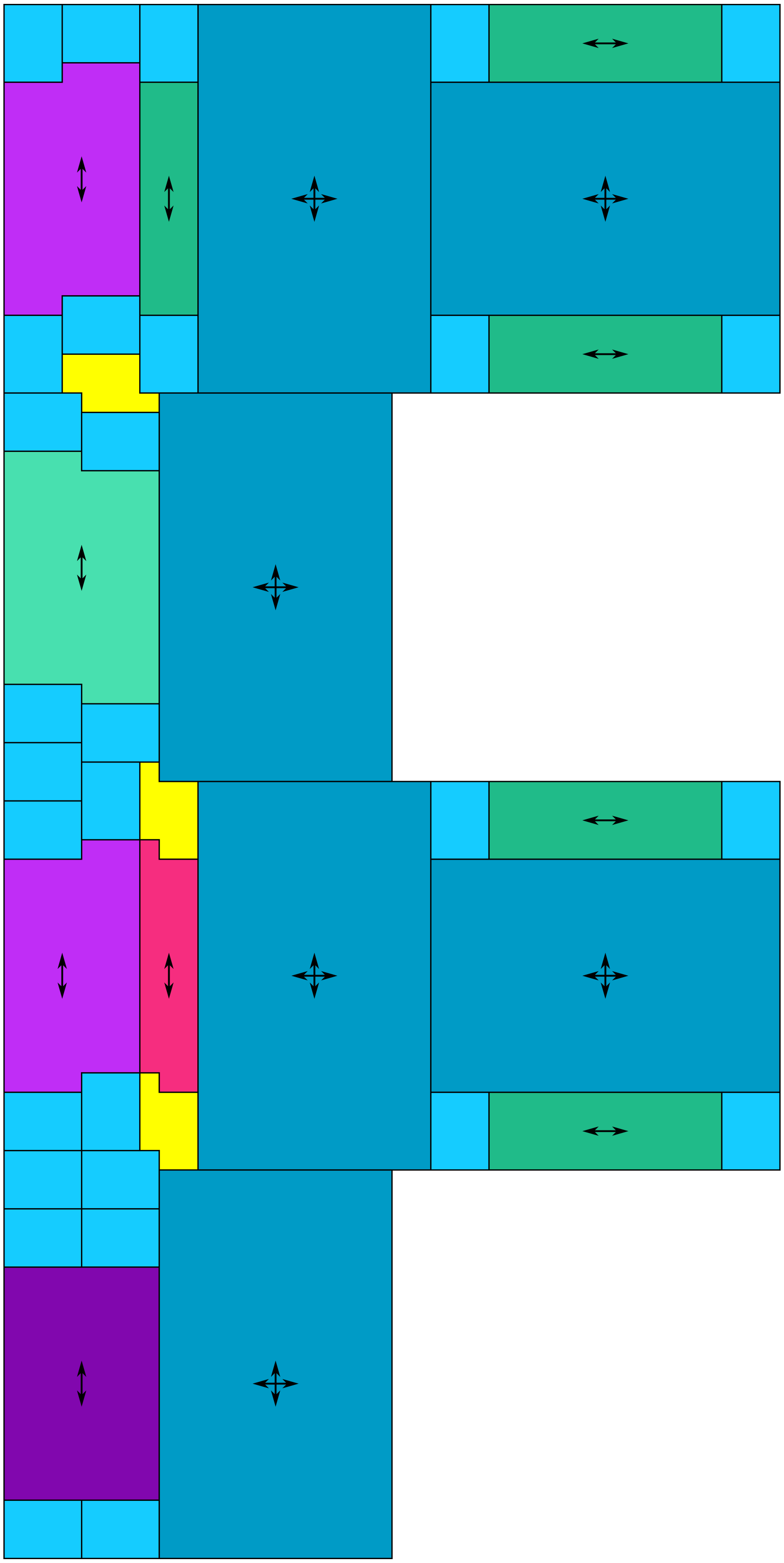
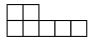
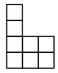


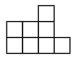
Figure 306: A generic rep-$(12k+8)^2$-tiling. This generic tiling has been constructed from the specific tiling in Figure 303

Theorem 297 (Clarke (2006b)). $_7$ *is rep* n^2 *for* $n = 7k, 7k+1$ *with* $(k > 1)$.

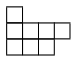
[Not referenced]

Theorem 298 (Clarke (2006b)). $_8$ *is rep* k^2 *for* $k \geq 2$.

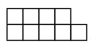
[Not referenced]

Theorem 299 (Clarke (2006b)). $_8$ *is rep* k^2 *for* $k = 12, 16, 20$.

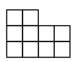
[Not referenced]

Theorem 300 (Clarke (2006b)). $_8$ *is rep* k^2 *for* $k = 32, 36, 40$.

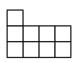
[Not referenced]

Theorem 301 (Clarke (2006b)). $_9$ *is rep* n^2 *for* $n = 18k, 21k, 15k,$ $(k \geq 2)$, $24k$.

[Not referenced]

Theorem 302 (Clarke (2006b)). $_{10}$ *is rep* k^2 *for* $k \geq 4$.

[Not referenced]

Theorem 303 (Clarke (2006b)). $_9$ *is rep* k^2 *for* $k = 6, 7, 8, 9, 10,$ $k \geq 12$.

[Not referenced]

6.2 *Reptile Sets*

Theorem 304. *If any subset of a set of polyominoes is rectifiable, the set satisfies the strong reptile property.*

[Not referenced]

Proof. From Theorem 136 we know that rectifiable set can tile any polyomino kP for some k; and therefore so can the entire set. This is exactly the strong reptile property. □

We do not know any sets that satisfy the strong reptile property that are not also rectifiable, although some reptile tilings are smaller than can be achieved by tilings of the minimal squares tileable by these sets. Examples are shown in Figures 308–310.

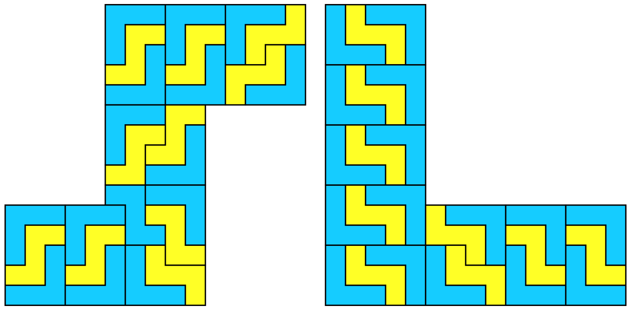


Figure 307: The V- of Z-pentominoes together are rep-5^2. From Clarke (2006b).

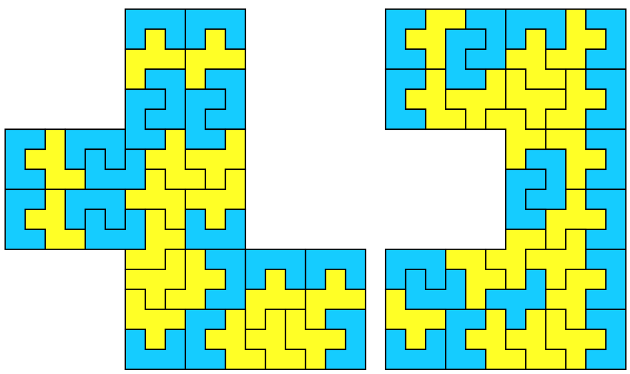


Figure 308: The U- of F-pentominoes together are rep-6^2, but their minimum rectangle is $R(3, 10)$, and their minimum square is $R(10, 10)$. From Clarke (2006b).

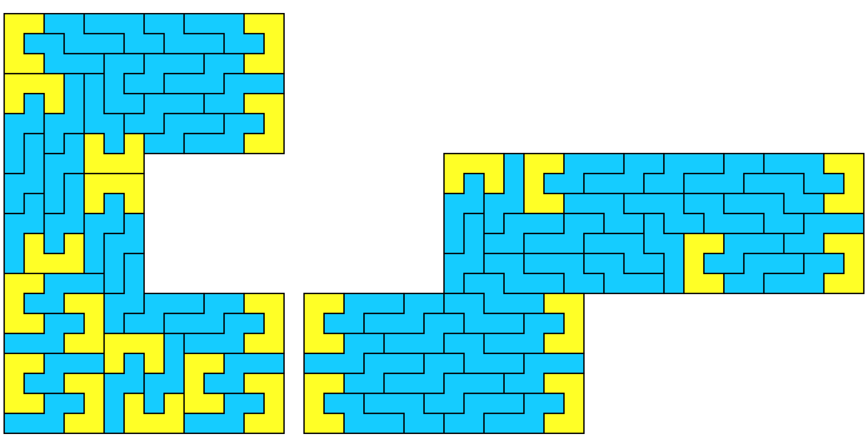


Figure 309: The N- and U-pentominoes together are rep-7^2. From Clarke (2006b).

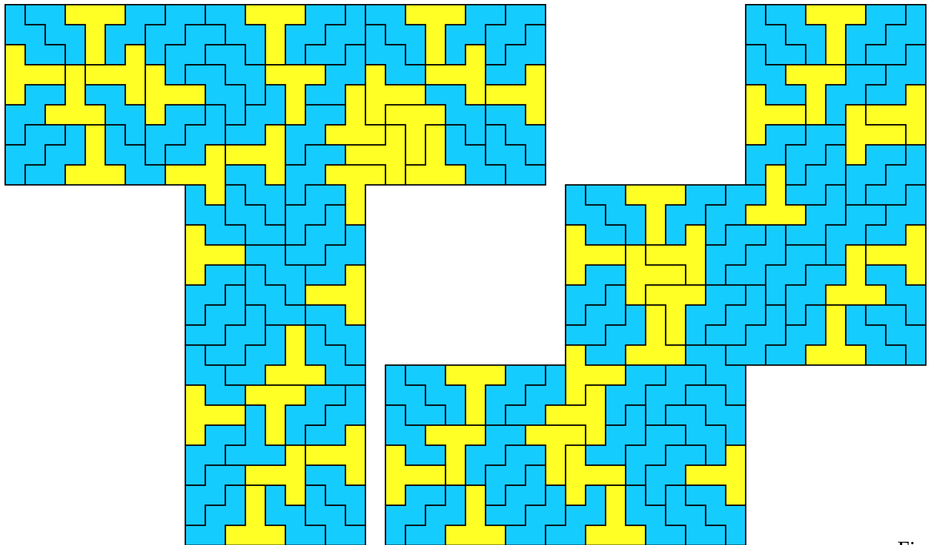


Figure 310: The T- and W-pentominoes together are rep 9. From Clarke (2006b).

6.3 *The Mutually-tiling Extension*

Reptiles can be generalized in another way, proposed in Sallows (2012) and expanded in Sallows (2014). This gives as another way to extend the tiling hierarchy (see Section 4.1.3).

Suppose we have a sequence of n tile sets $\mathcal{T}_i$ all with the same number of tiles, and suppose we can put the tiles in T_i together to form scaled copies of elements from T_{i+1}, (we let $\mathcal{T}_n = \mathcal{T}_0$). These sets of tiles are called **mutually-tiling sets**.

An example of two sets with 4 tiles in each is shown in Figure 311.

A set can also tile scaled copies of itself in this way; this is called a **self-tiling** set.[2]

[2] Golomb (1970, Section 6) says sets that can tile themselves in this way have the *regular reptile property*. Since we use regular to mean something else, we will not use this term.
Sallows (2012) uses the shortened term *setiset*, from which we could derive the term mutiset.

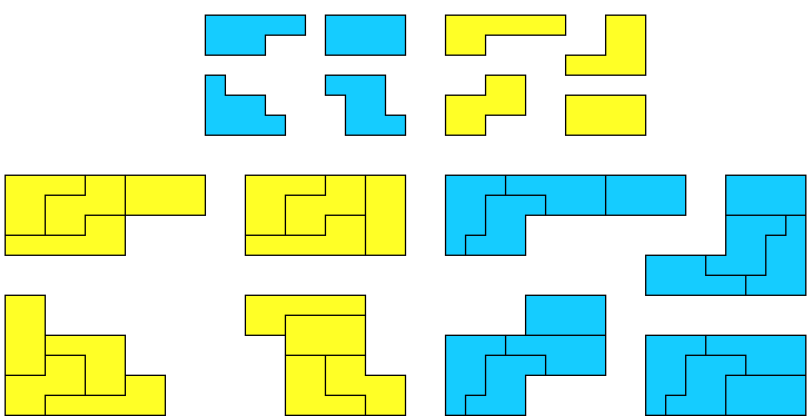


Figure 311: Two mutually self-tiling sets.

Tile sets can be part of different loops. The longest loop length is

14; these are by no means uncommon. Among octominoes, there are 46,080 such loops!

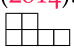
Problem† 92.

(1) *Among four tiles in a mutual tiling set, there must be four hull corners covered among them. Is it possible for one tile of such a set to have no hull corners covered?*

(2) *Show that a set of mutually tiling polyominoes can tile a quadrant.*

The number of tiles in mutually tiling sets must be a square.

Problem† 93. *Suppose $\mathcal{T}_i$ are mutually tiling sets. Show that each set has the strong reptile property. In particular, if there is one tile in each set, each tile is a reptile.*

There is a way to use reptiles to construct self-tiling set, described in Sallows (2014). Suppose we have a reptile of order 4, such as the polyomino $_6$. A reptiling of this polyomino is shown in Figure 312. Notice that we can dissect the larger region into two smaller regions in two different ways such that each region can be tiled by the smaller tile.

These four regions form a self-tiling set.

The same idea can be used to construct more complicated self-tiling sets. For example, Figure 314 shows an example of a set with 9 pieces, and Figure 315 shows a set with 16 pieces.

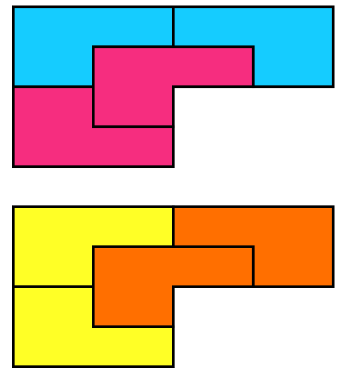


Figure 312: A self-tiling set with 4 tiles formed by disecting an order-4 reptile in two different ways. The bug polyomnoes can be combined to form scaled copies of each tile in the set, as shown in Figure 313.

6.4 Irreptiles

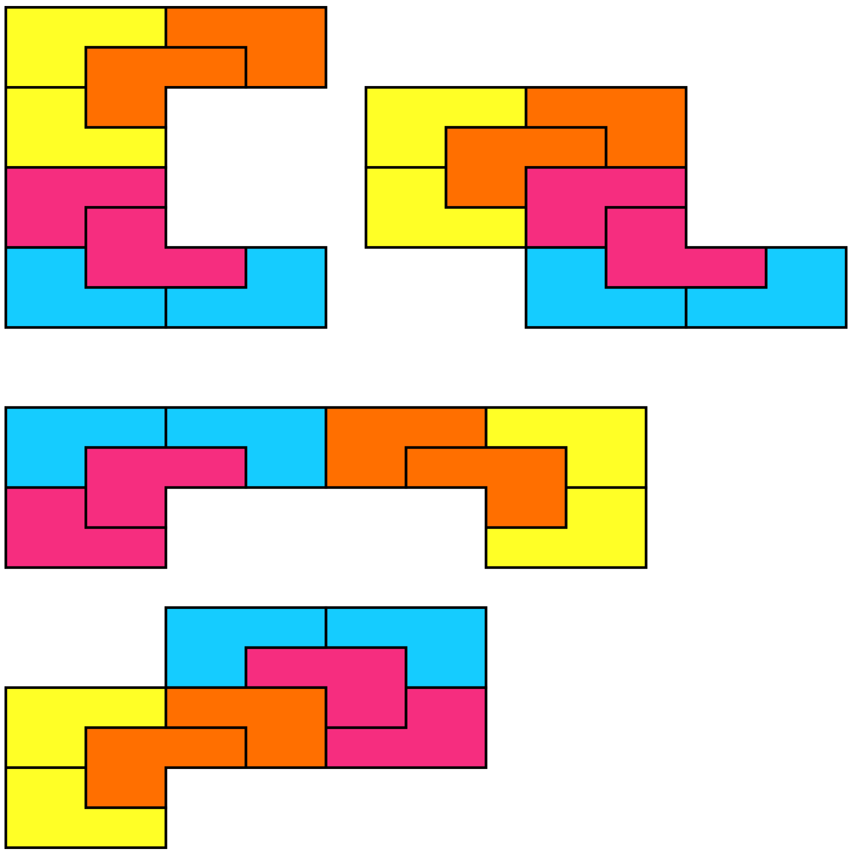


Figure 313: How to put the tiles in Figure 312 together to form bigger copies.

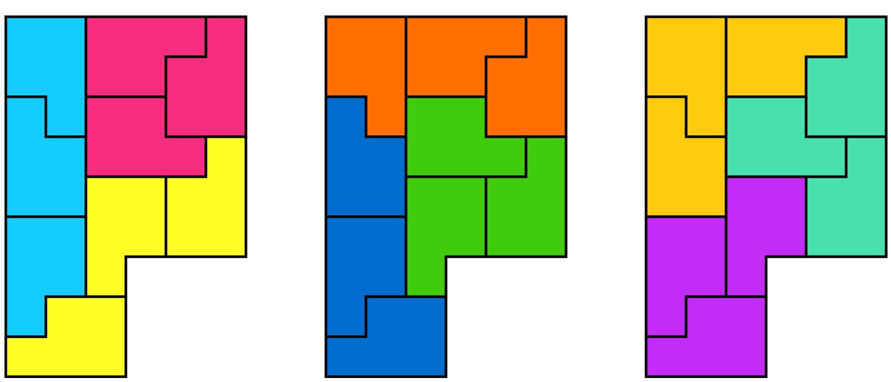


Figure 314: A self-tiling set with 9 pieces. Each peice is made from 3 P-pentominoes and is shown in the same color. The three big P-pentominoes can be put together to form a larger copy of any piece in the set.

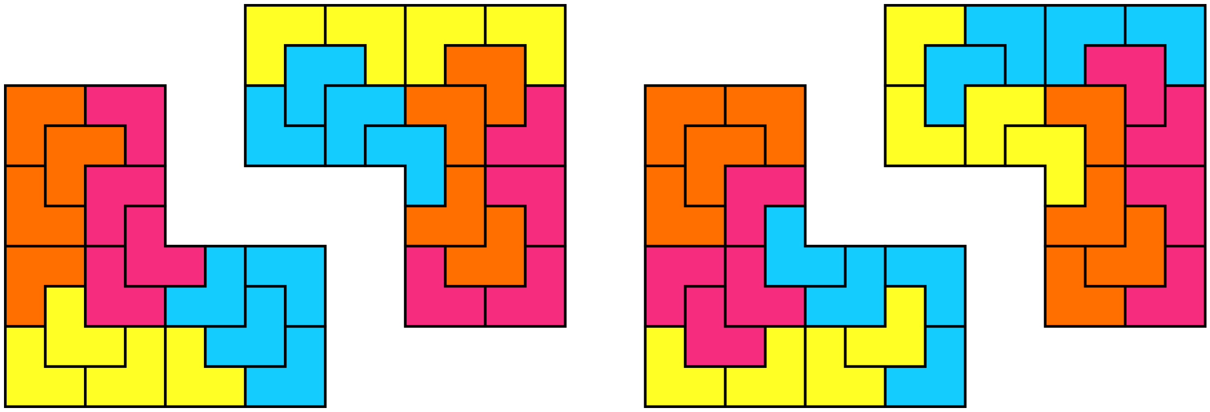
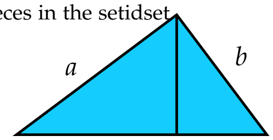
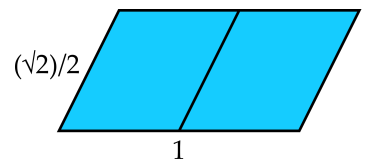


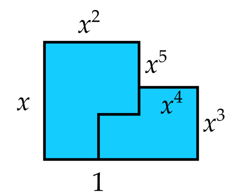
Figure 315: A self-tiling set with 16 pieces. The 4 big triominoes can combine to form a scaled copy of any of the 16 pieces in the setidset.



(a) Right triangle



(b) Parallelogram



(c) Golden bee
($x = (1+\sqrt{5})/2$)

Figure 316: The only irreptile shapes of order 2.

An **irreptile** is a polyomino that can tile a scaled copy of itself irregularly. (A reptile is a special reptile that can also tile a scaled copy of itself regularly.) The idea of irreptiles is easy to extend to more general polygons. In this setting, there are only three types of irreptiles with order 2 Schmerl (2011):

(1) Any right triangle.

(2) A parallelogram with one side $\sqrt{2}/2$ times the other.

(3) A *golden bee*, a hexagon that looks like a letter bee and involves the golden ratio. The side proportions are 1, x, x^2,x^3, x^4, and x^5 and with $x = (1+\sqrt{5})/2$.

None of these shapes can be polyominoes, and therefore there are no polyominoes that are irreptiles of order 2.

Theorem 305. *There are no irreptiles of order 2.*

[Not referenced]

Proof. This follows from the main theorem in Schmerl (2011). See the reference for details. □

Theorem 306. *If a polyomino can tile a rectangle irregularly, the polyomino is a irreptile.*

[Not referenced]

Proof. From Theorem 136 the scaled copies can tile the polyomino scaled, and so the polyomino is an irreptile. □

Problem† 94.

(1) *Show that the irreptile order of rectangles cannot be 2 or 3, and so the the irreptile order for all rectangles is 4.*

(2) *Show that irreptile tilings of squares cannot use 5 tiles.*

(3) *Give an upper-bound for the irreptile order of rectifiable polyominoes, given the size of one rectangle that the polyomino can tile.*

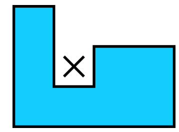
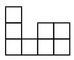


Figure 317: The only way to cover the marked space is for tiles to interlock, forming a scaled skew tetromino.

While there are no examples of reptiles that are not also rectifiable, there are examples of irreptiles that are not rectifiable. One example is shown in Figure 318. Since scaled copies tile a rectangle, the polyomino is an irreptile (Theorem 136). Note that the only way to tile the space marked in Figure 317 is for two tiles to interlock. In this configuration, we have the skew tetromino which does not tile a rectangle; therefore $_8$ does not tile a rectangle.

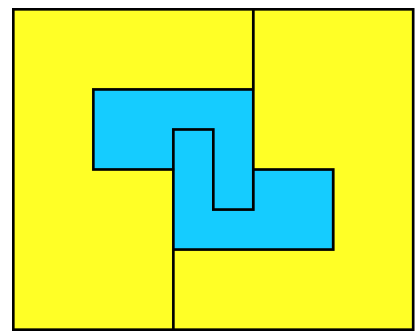


Figure 318: An example of an irreptile that is not rectifiable.

Problem 95** (Reid (2003c)). *If scaled copies of a polyomino can tile a rectangle, we call the polyomino **irrectifiable**.*

Are all irreptiles irrectifiable?

Problem† 96.

(1) *Prove that an irreptile must cover one corner of its hull.*

(2) *Find an irreptile that covers exactly one corner of its hull.*

(3) *Show that an irreptile can tile a quadrant irregularly.*

In table 55 we show what number of tiles is possible. Note that if p and q or possible, then so is $p - 1 + q$ (if we remove one tile and replace it by a tiling). So since 4 is possible, it follows that $4 - 1 + 4 = 7$ is possible too. In fact, from 4, 6, and 8 we can show all other numbers are possible.

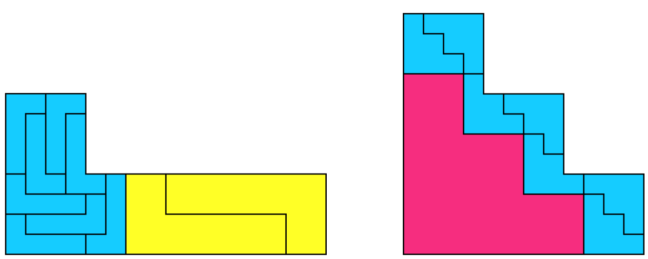


Figure 319: Irreptiles.

We denote the irreptile order of a polyomino P by $\text{IIO}(P)$.

Theorem 307 (Andrew Bayly, via Friedman (2008)[3]). *The irreptile-order of polyominoes can be arbitrarily large.*

[3] https://erich-friedman.github.io/mathmagic/1002.html. He gives a different polyomino for the sequence but the principle is the same.

[Referenced on page 301]

Proof. The polyomino $B(1 \cdot (k2+1)^k \cdot k \cdot 1^k \cdot (2k+1)$ tiles the square $R(2k+2, 2k+2)$. We can use $\cdot(4k+2)$ squares and 2 squares k times

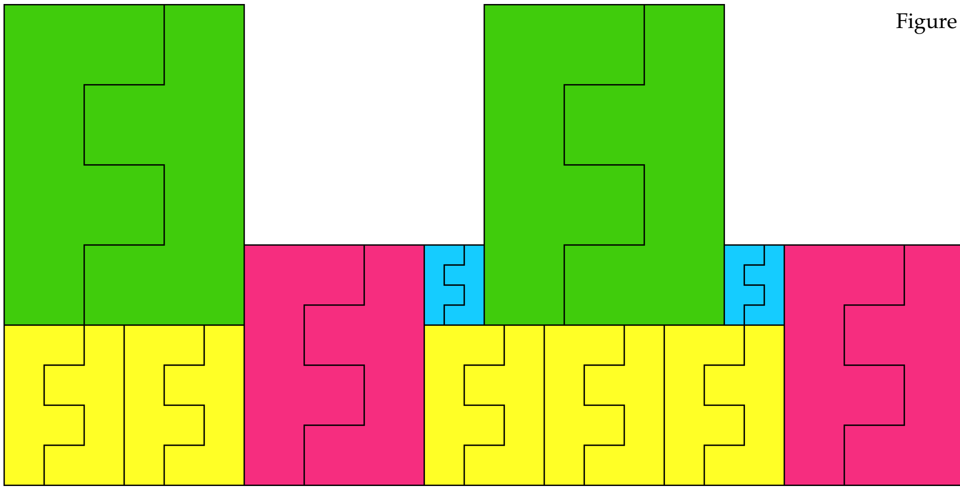


Figure 320: Irreptiles.

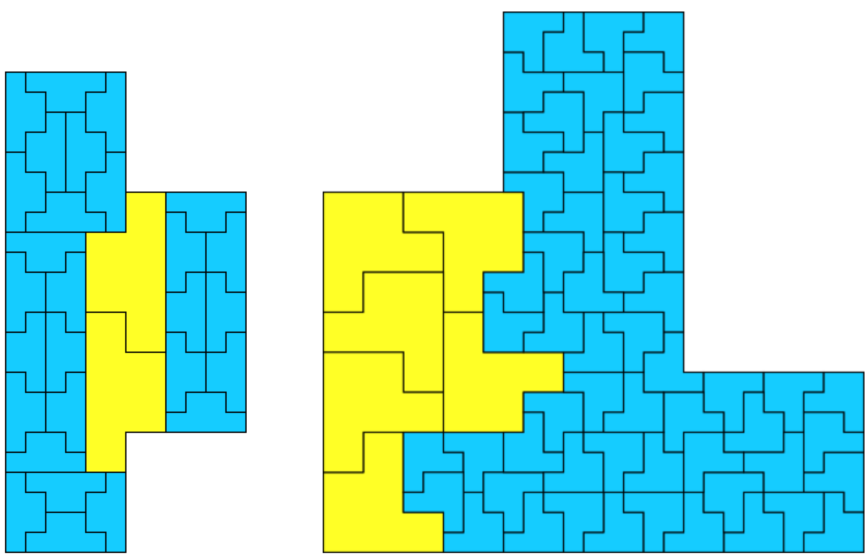


Figure 321: Irreptiles.

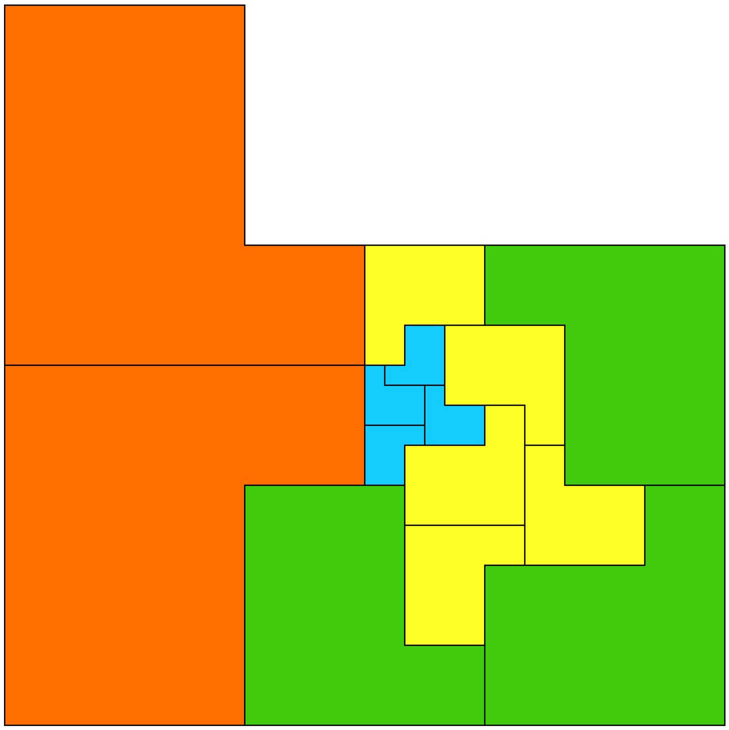


Figure 322: Irreptiles.

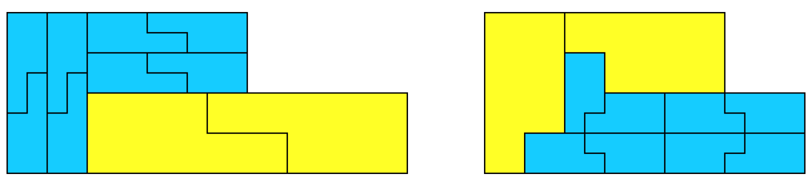


Figure 323: Irreptiles.

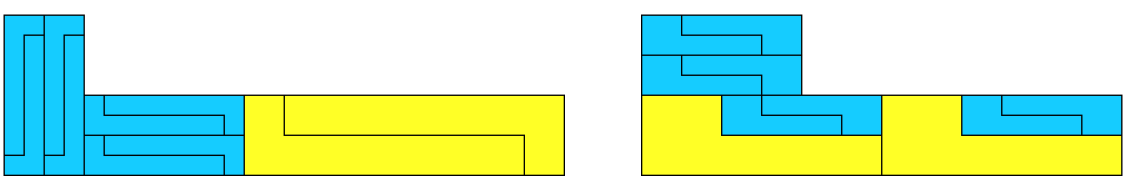
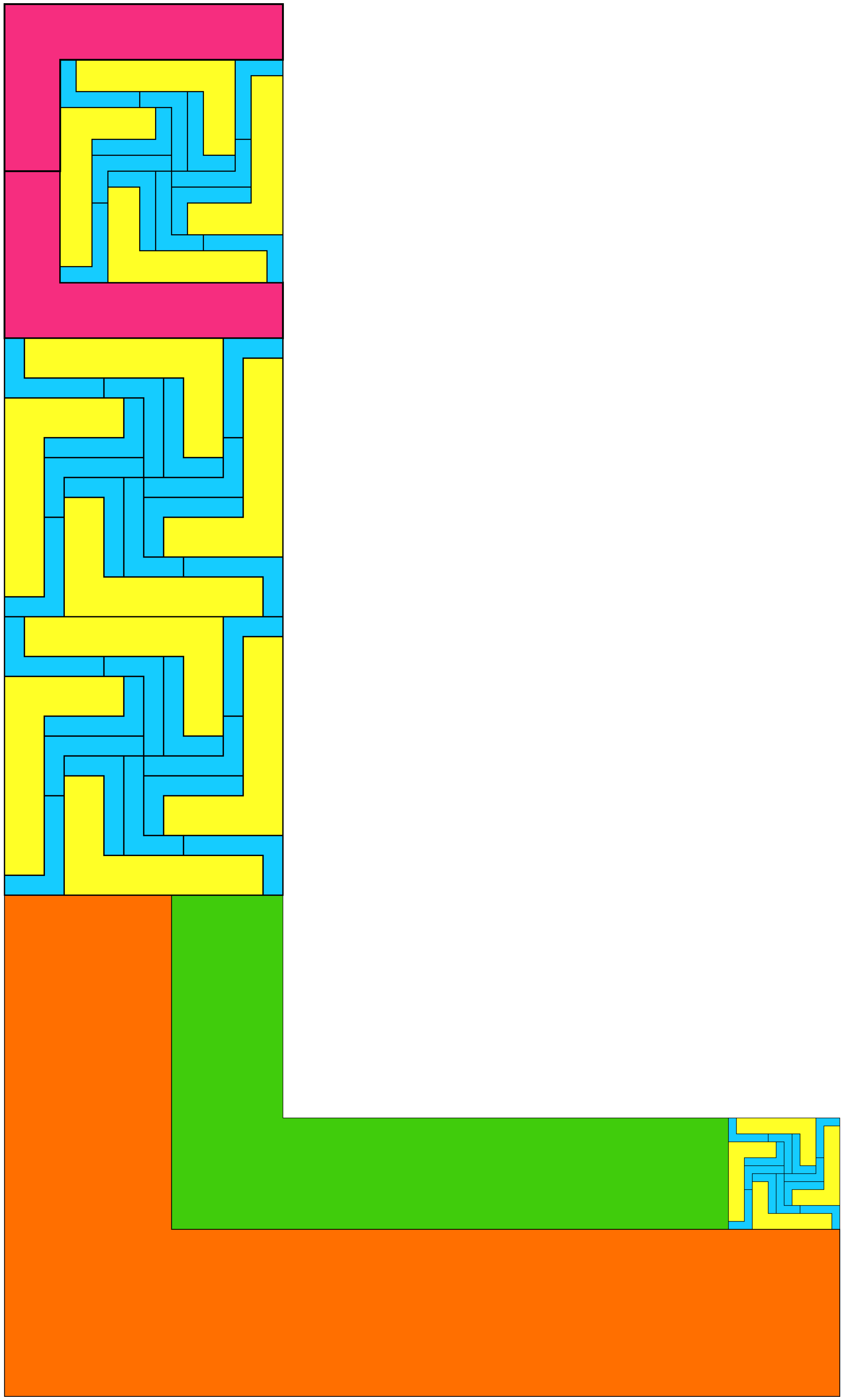


Figure 324: Irreptiles.



Polyomino	IR	Polyomino	IR	Polyomino	IR	Polyomino	IR	Polyomino	IR
$_3$	4	$_6$	9	$_7$	14	$_8$	10	$_9$	12
$_4$	4	$_6$	12	$_7$	68	$_8$	40	$_9$	12
$_4$	16	$_6$	18	$_8$	10	$_8$	43	$_9$	14
$_5$	4	$_6$	22	$_8$	10	$_9$	12	$_{10}$	17
$_5$	10	$_6$	30	$_8$	10	$_9$	12	$_{10}$	9
$_5$	40	$_6$	63	$_8$	10	$_9$	12	$_{10}$	7
$_6$	8	$_7$	10	$_8$	10	$_9$	12	$_{10}$	62

Table 54: The irreptile orders for small polyominoes. Most of the entries are from Friedman (2008, https://erich-friedman.github.io/mathmagic/1002.html)

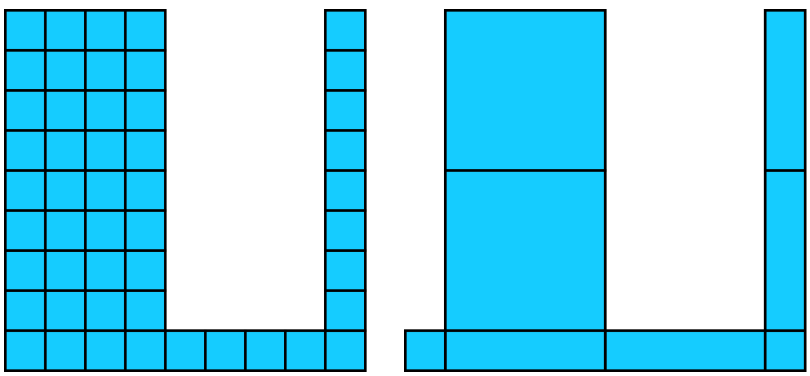


Figure 325: The polyomino $B(1 \cdot 9^4 \cdot 1^4 \cdot 9)$ and how it can be dissected into squares to get a irreptile tiling. Note that no tiling is possible that uses fewer tiles.

the size to form a larger copy of the polyomino. So in total we need $4k + 4$ squares, or $8k + 8$ tiles.

□

Problem* 97.

(1) *Are there irreptiles of order 3, 5 and 6?*

(2) *Can you find families of irreptiles that show there are irreptile orders for any integer (larger or equal to 4, 5, 6, or 7)?*

Theorem 308. *For a reptile, the following holds:*

$$\mathrm{IIR}(P) \le \mathrm{IR}(P)$$

[Not referenced]

Proof. This follows simply from the fact that every irreptile tiling is also a reptile tiling, but not vice versa. □

The last two theorems shows that the reptile order of a polyomino can be arbitrarily large (although this is also obvious directly from the proof of 307).

Theorem 309. *Suppose we have a irreptiling that uses k tiles, and we have irreptilings that use* $m, m+1, m+2, \cdots, m+k-2$ *tiles. Then there exist an irreptiling that uses n tiles for all* $n \geq m$*.*

[Not referenced]

Proof. We can replace one tile in any tiling that uses ℓ tiles with a scaled tiling that uses k tiles, to yield a new tiling that uses $\ell - 1 + k$ tiles. So from the sequence of tiles we can form a new sequence that uses from $m - 1 + k$ to $m + k - 2 - 1 + k = m + 2k - 3$ tiles. We can repeat this, and so find a tiling that uses any number of tiles m or larger. □

Here is an example to illustrate the theorem. For a square, there is an irreptiling that uses 4 squares. We can also find a sequence of tilings that use 6, 7, or 8 tiles, as shown in Figure 326. Replacing one square with 4, we can get tilings for 9, 10, or 11 tiles, as shown in Figure 327. Repeating this we can get a tiling that use any number of tiles larger than 5.

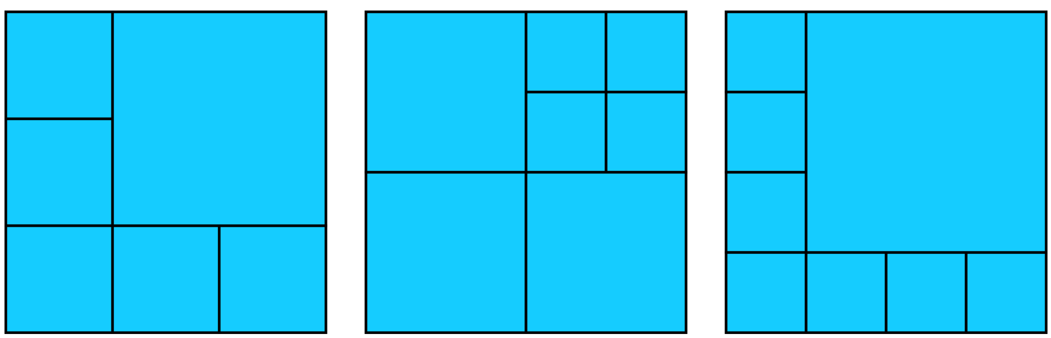


Figure 326: Tilings that use 6, 7, and 8 tiles.

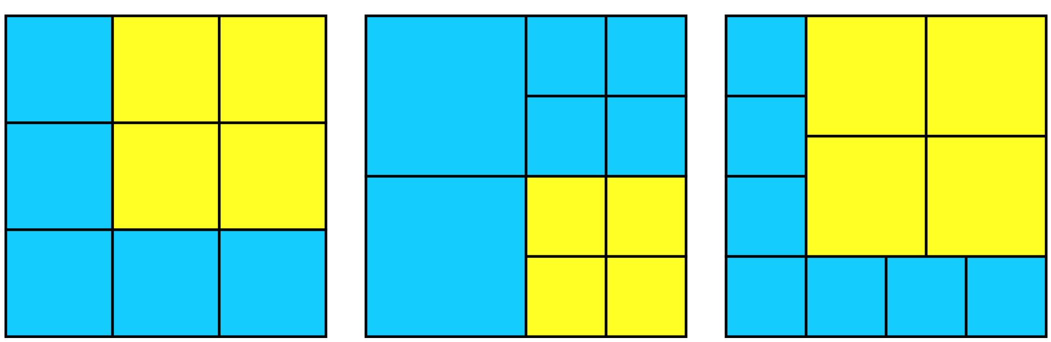


Figure 327: By replacing one square in the tilings above with four squares, we can find new tilings that use 9, 10, or 11 tiles.

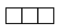
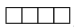
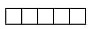
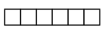
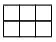
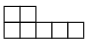
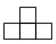
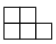
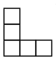
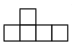
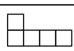
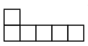
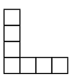
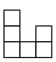
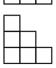
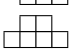
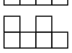
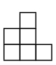
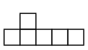
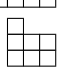
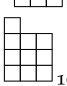
Polyomino	Impossible	Prime Tilings
 $_1$  $_3$  $_4$  $_4$  $_5$  $_6$  $_6$	2, 3, 5	4, 6, 8
 $_2$	2, 3	4–6
 $_3$  $_7$	2,3,5,8	4, 6
 $_4$	2–15,17, 81, 21	6, 19, 20, 22, 24, 25, 29
 $_5$	2,3,5,8	4, 6, 11
 $_5$	2,3,5,6	4, 8, 9
 $_5$	2–39, 41, 42, 45, 47, 48, 50, 51, 53, 54, 57	40, 43, 44, 46, 56, 59, 60, 62, 63, 71, 81
 $_5$	2–9,11,12	10, 13–15, 17, 18
 $_6$	2–11	12–17
 $_7$	2–8,10, 12	9, 11, 13, 14, 16, 18
 $_6$	2–17,20,21,25	18, 19, 22, 23
 $_6$	2–7,9–13,16,17	8, 19, 23
 $_6$	2–29,33,35,39,42	30, 31–34, 36, 37, 38, 40, 41, 45, 47
 $_6$	2–21, 23–25, 27, 29, 31, 35	22, 26, 30, 33
 $_6$	2–62, 64, 65, 67, 68, 70, 71, 73, 74, 76, 77, 79, 80, 82–124, 126, 127, 129, 130, 132, 133, 135, 136, 138, 139, 142, 145, 148, 151, 154, 160, 173, 175, 179, 185, 187, 191, 193, 197	63, 66, 69, 72, 75, 78, 81, 141, 144, 146, 147, 162, 163, 164, 165, 167, 169
 $_6$	?	888
 $_7$	2-16,18, 21	17,19,20,22–32,34
 $_{10}$	2-16, 18, 19, 21, 24	17, 20, 22, 23, 25–32, 34, 35, 37, 40

Table 55: Impossible and possible number of tiles for irreptile tilings of polyominoes (Friedman, 2008, https://erich-friedman.github.io/mathmagic/0810.html). See also the reference for actual tilings. The results are not complete for all polyominoes.

6.5 *Further Reading*

Reptiles—including reptiles of more general polygons—were introduced in Golomb (1964); they are also discussed in Gardner (2001, Chapter 5), Martin (1991), Martin (2012, Section 12.2).

7
Plane Tilings

When and how polyominoes tile the plane are central questions of this section.

In many ways, plane tilings are more fundamental than tilings of rectangles; for one thing — any tile set that tiles a rectangle can also tile the plane.

A tiling is called **monohedral** if all the tiles in the tilesets are congruent[1] (Grünbaum and Shephard, 1987, p. 20).

(From Beauquier and Nivat (1991):) Suppose there are two linearly independent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that the tiling offset by $k\boldsymbol{u} + \ell\boldsymbol{v}$ is the same tiling for all $k, \ell \in \mathbb{Z}$, then the tiling is **periodic**. If there is only one such vector, the tiling is called **half-periodic**. If there is no such vector, the tiling is **non-periodic**. If there are three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ such that the tiling offset by $\boldsymbol{w} + k\boldsymbol{u} + \ell\boldsymbol{v}/\gcd(k, \ell)$ is the same tiling, the tiling is called **regular**.[2]

Example 21. *We can easily form half-periodic and non-periodic tilings of the plane from tiles using irrational numbers. Here are some examples using dominoes:*

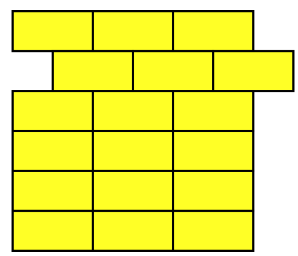
- *Form a half-strip using vertical and horizontal flippable pairs, using alternate digits of π of each; that is, 3 vertical, 1 horizontal, 4 vertical, 1 horizontal, etc. Reflect the strip and form a complete strip; now stack these together to tile the plane. The resulting tiling is half-periodic.*
- *Form a strip using the same scheme as above; form another strip reversing the roles of vertical and horizontal flips. Now stack these together using alternate digits of π for each.*

Another method is to use a tiling with all dominoes vertical (and aligned). Flip a single flippable pair; this tiling is non-periodic. If we flip all the dominoes in a row, we get a half-periodic tiling.

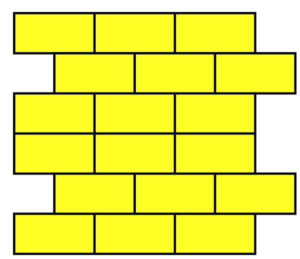
Some tile-sets can only tile the plane non-periodically. Such a tile-set is called **aperiodic**. If the set is a single tile, we call the tile

[2] When not talking about polyominoes, the term *regular tiling* also implies that the tiling is monohedral and that the tile is a regular polygon.

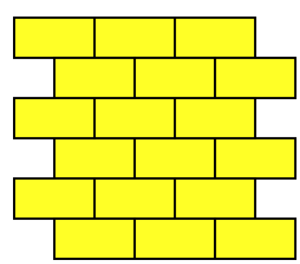
[1] The terminology makes sense when applied to tilings of other regions, but is generally only used to discuss plane tilings.



(a) Half periodic tiling with $\boldsymbol{u} = (2, 0)$. This tiling is not periodic.



(b) A periodic tiling with $\boldsymbol{u} = (2, 0)$ and $\boldsymbol{v} = (0, 3)$. This tiling is not regular.



(c) A regular tiling with $\boldsymbol{u} = (2, 0)$ and $\boldsymbol{v} = (0, 2)$.

Figure 328: Types of plane tilings, adapted from Beauquier and Nivat (1991).

aperiodic. Currently, we don't know if connected aperiodic tiles (polyominoes or non-polyominoes) exist[3] (Winslow, 2018, Open Problem 5).

Informally, a **fundamental region** of a tiling is the smallest part of the tiling that we can translate to form the entire tiling. More formally, the fundamental region of a periodic tiling with period $\boldsymbol{u}, \boldsymbol{v}$ is a set F of connected cells such that if $x \in F$, then $x + m\boldsymbol{u} + n\boldsymbol{v} \neq F$ for all $m \neq 0, n \neq 0$ (Rhoads, 2005, p. 330). [4]

[3] See Socolar and Taylor (2011) for an example of a disconnected aperiodic tile.

[4] The fundamental region is also called a *fundamental domain*. Our definition is different from the definition given in for example Grünbaum and Shephard (1987, p. 55), and (Kaplan, 2009, Definition 3.1, p. 17) that is defined in terms of symmetry groups.

Theorem 310 (The Extension Theorem, Grünbaum and Shephard (1987), Theorem 3.8.1[5])**.** *Suppose we have a sequence of squares R_i, with $R_i \subset R_{i+1}$, and suppose we have a cover for each such that the cover of R_i is contain in the cover of R_{i+1}. Then this sequence of covers define a tiling of the plane.*

[5] We use squares instead of circles this version.

[Referenced on pages 306 and 310]

We can use this theorem to prove the relations in Nitica's hierarchy extension we discussed in Section 4.1.3.

Theorem 311 (Nitica (2018a), Theorem 6 (18))**.** *If a set tiles any of the four half-planes by translation only, it tiles a plane by translation only.*

[Referenced on page 155]

Proof. Suppose $\mathcal{T}$ is a finite tile set that tiles the right half plane. Let n be a positive integer. Consider an infinite sequence of disjoint $n \times n$ squares positioned with their centers on the positive X-axis.

Each square is covered by tiles (some tiles extend beyond the boundary of the square), and there can only be a finite number of such arrangements (since both $\mathcal{T}$and n are finite). But since there are an infinite number of such squares, there must be at least two with identical covers.

We can now find a subsequence of $(n+2) \times (n+2)$ squares, each of which has the center of one of the $n \times n$ squares, and has the same cover.

By induction then, we can find a sequence of squares $R(1,1)$, $R(3,3)$, $R(5,5)$, such that the center of $R(k,k)$ has the same cover as $R(k-2, k-2)$. Thus, by Theorem 310, gives us a tiling of the plane. □

Theorem 312 (Nitica (2018a), Theorem 6 (12–15))**.** *If a set tiles one of the four quadrants by translation only, it tiles the two half-planes that contains the quadrant by translation only.*

[Referenced on page 155]

Proof. The proof is similar than the one above, except that we place the squares on the side of the quadrant instead of on the x-axis. □

7.1 *Tilings by Translation*

A polyomino that can tile the plane by translation alone is called **exact** (Beauquier and Nivat, 1990, Section 1). The smallest tiles that are not exact are the pentominoes: F, U and T.

A **pseudo-hexagon** is a region whose border can be divided into six segments, A, B, C, D, E and F such that the pairs $A - D$, $B - E$ and $C - F$ are translates of each other. Both edges in one of these pairs may be empty; if this is the case the region is called a **pseudo-square**[6].

If a word can be written in the form[7] $ABC\hat{A}\hat{B}\hat{C}$ (possibly with one pair of factors empty), we call that form the **BN-factorization** of the word[8]. We do not consider $CAB\hat{C}\hat{A}\hat{B}$ a distinct factorization from $ABC\hat{A}\hat{B}\hat{C}$. Even so, a word can have more than one BN-factorization. For example, X_5 has the following factorizations:

$$(xyx)(yx^{-1}y) \cdot (x^{-1}y^{-1}x^{-1})(y^{-1}xy^{-1})$$
$$(yxy)(x^{-1}yx^{-1}) \cdot (y^{-1}x^{-1}y^{-1})(xy^{-1}x)$$

[6] Some authors drop the "pseudo" and simply use *hexagon* and *square*; some use *parallelogram* instead of square; and some do not include pseudo-squares in the class of pseudo-hexagons.

[7] Recall that $\hat{A}$ is A in reversed; that is, the order of the symbols are reversed and the exponents' signs are flipped. See section 4.3.

[8] BN stands for the authors of Beauquier and Nivat (1991), used first in Brlek et al. (2009).

Theorem 313 (Translation Criterion, Beauquier and Nivat (1991), Theorem 3.2)**.** *The following statements are equivalent*[9]*:*

- *A polyomino is exact.*
- *A polyomino is a pseudo-hexagon.*
- *A polyomino has a BN-factorization.*

[Referenced on pages 307 and 310]

[9] See also Winslow (2015, Lemma 1), Rhoads (2005, Theorem 2). Winslow (2015) calls it the *Beauquier-Nivat criterion*.

Problem† 98. *Show that the only Young diagrams that are exact are rectangles and L-shaped polyominoes. That is, exact Young diagrams belong to* $\mathcal{C}_6$.

Problem† 99.

(1) *Find a square that is not similar to a polyomino in Table 56.*

(2) *Suppose P is a polyomino with border word* $x^{\pm 1}y^{\pm 1}x^{\pm 1}y^{\pm 1} \cdot x^{\pm 1}y^{\pm 1}$. *Characterize the ones with a BN-factorization.*

Theorem 314. *The families of polyominoes listed in table 56 are all exact.*

[Not referenced]

Polyomino		BN-factorization
X_5		$(xyx)(yx^{-1}y)\cdot(x^{-1}y^{-1}x^{-1})(y^{-1}xy^{-1})$
Z_5		$(x)(y)(x^{-1}y^2x^{-1})\cdot(x^{-1})(y^{-1})(xy^{-2}x)$
N_5		$(xy^{-1}x^2)(x)(y)\cdot(x^{-2}yx^{-1})(x^{-1})(y^{-1})$
$R(m,n)$	Rectangle	$(x^m)(y^n)\cdot(x^{-m})(y^{-n})$
$A(n)$	Aztec diamond	$(xy)^n(yx^{-1})^n\cdot(x^{-1}y^{-1})^n(y^{-1}x)^n$
$B(a^m\cdot b^n)$	L-shaped polyomino	$(y^{n-m}x^b)x^ay^n\cdot(x^{-b}y^{m-n})x^{-a}y^{-n}$
W_{2k}	Even-area W-polyomino	$x\left[(xy)^{k-1}x\right]y\cdot x^{-1}\left[(x^{-1}y^{-1})^{k-1}x^{-1}\right]y^{-1}$
W_{2k+1}	Odd-area W-polyomino	$x(xy)^ky\cdot x^{-1}(y^{-1}x^{-1})^ky^{-1}$
Y_n	General Y-polyomino	$x(yx^{n-3})(yx^{-1})\cdot x^{-1}(x^{3-n}y^{-1})(xy^{-1})$
$B(1^k\cdot 2^m\cdot 1^n)$		$x^m(yx^n)(yx^{-k})\cdot x^{-m}(x^{-n}y^{-1})(x^ky^{-1})$
$C_k(a_1\cdot a_2\cdots a_m)$	Cylinder	$x^kA\cdot x^{-k}\hat{A}$, where $A=x^ky^{a_1}x^ky^{a_2}\cdots x^ky^{a_m}$

Table 56: BN-factorizations for various polyominoes and families of polyominoes.

Proof. The BN-factorization is given in the table for each family, which makes it exact (Theorem 313). □

Note that this proves exactness for all n-ominoes with $n \leq 5$, except for the F-, T- and U-pentominoes. It is easy to prove these three polyominoes are not exact.

Theorem 315 (Wijshoff and van Leeuwen (1984), Theorem 6.1)**.** *There is an algorithm to determine whether a polyomino can tile the plane by translation. The algorithm is polynomial in the size of the polyomino.*

[Not referenced]

Theorem 316 (Massé et al. (2012), Theorem 1)**.** *A pseudo-square admits at most two regular tilings.*

[Not referenced]

A pseudo-square that admits exactly two tilings is called a **double square**. Figure 329 shows an example.

Below, we define $\mathsf{FST}()$ as follows:

$$\mathsf{FST}(x^k\cdots)=\begin{cases}x & k>0\\ x^{-1} & k<0\end{cases} \tag{7.1}$$

$$\mathsf{FST}(y^k\cdots)=\begin{cases}y & k>0\\ y^{-1} & k<0\end{cases} \tag{7.2}$$

Similarly, $\mathsf{LST}()$ is defined as

$$\mathsf{LST}(\cdots x^k) = \begin{cases} x & k > 0 \\ x^{-1} & k < 0 \end{cases} \tag{7.3}$$

$$\mathsf{LST}(\cdots y^k) = \begin{cases} y & k > 0 \\ y^{-1} & k < 0 \end{cases} \tag{7.4}$$

Theorem 317. *A double square P with factorization $AB\hat{A}\hat{B}$ has these properties:*

(1) *A and B are palindromes (Massé et al., 2013, Theorem 1).*

(2) *$P \in$ **Rot** (Massé et al., 2013, Theorem 1).*

(3) $\mathsf{FST}(A) = \mathsf{LST}(A)$ *and* $\mathsf{FST}(B) = \mathsf{LST}(B)$ *(Massé et al., 2013, Proposition 4).*

(4) $\mathsf{FST}(A)$, $\mathsf{FST}(\hat{A})$, $\mathsf{FST}(B)$, $\mathsf{FST}(\hat{B})$ *are all distinct (Massé et al., 2013, Proposition 4).*

[Not referenced]

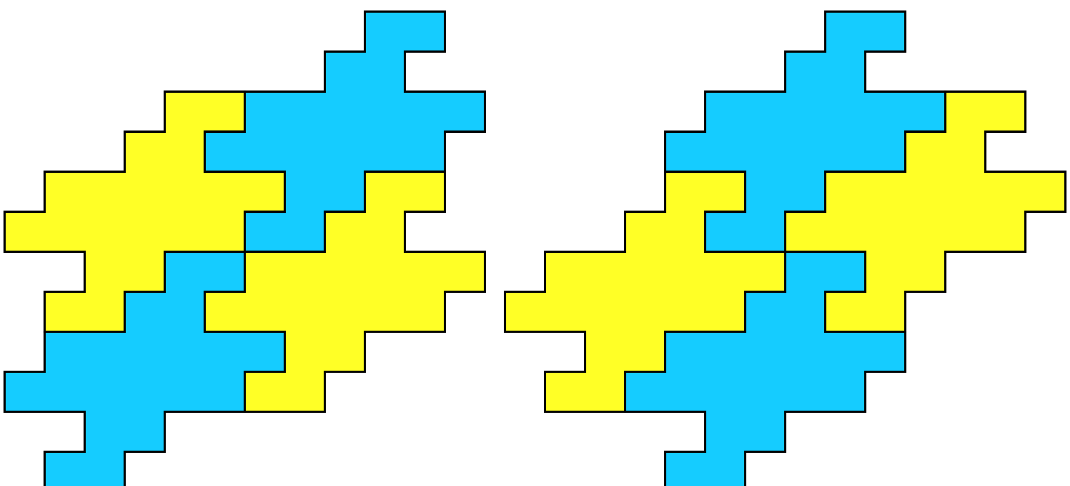


Figure 329: Two tilings of this pseudo-square.

7.2 Conway's Criterion

Theorem 318 (Conway's Criterion, Schattschneider (1980))**.** *A polygon admits a periodic tiling of the plane using only translations and $180°$ rotations if it has 6 points A, B, C, D, E, and F (at least three of which are distinct), that satisfy these conditions:*

(1) *The boundary between A and B is congruent by translation to the boundary between E and D.*

(2) *The boundaries BC, CD, EF and FA are all centrosymmetric.*

[Referenced on page 310]

To see why this works, rotate a copy $A'B'C'D'E'F'$ of the tile 180°and join them together along their borders so that B coincides with C' and C coincides with B'. In this combined tile $ABD'E'F'A'CDEF$, the borders $ABD'E' \equiv EDCA'$, $FA \equiv A'F'$ and $EF \equiv F'E$, so this shape tiles the plane as shown in Figure 330. (If the original tile is a polyomino, the combined tile is a pseudo-hexagon that tiles the plane by Theorem 313.) The reference proves the theorem more rigorously.

All polyominoes with 6 cells or less satisfy Conway's Criterion. See Table 59 for a break-down for small polyominoes.

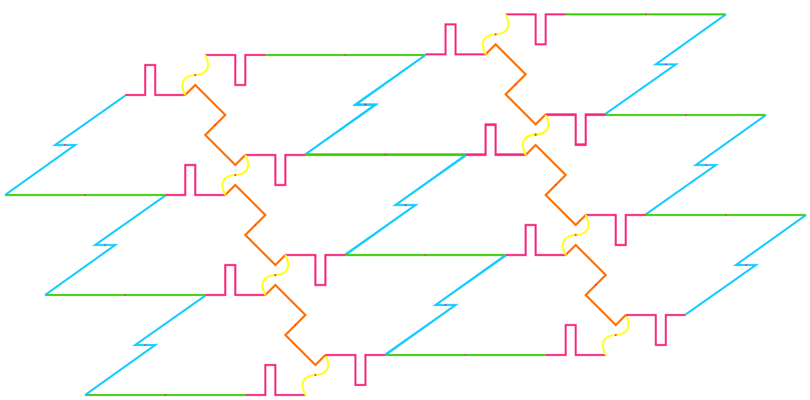


Figure 330: A tile that satisfies the Conway criterion tiles the plane.

Problem† 100. *State Conway's criterion as a condition on the border word of a polyomino.*

Problem† 101. *Show that the golygon with 8 sides (Figure 15) satisfies the Conway criterion.*

7.3 *Summary of Tiling Criteria*

7.4 *Isohedral and k-Isohedral Tilings*

For two congruent tiles in a tiling there must be at least one isometry of the plane that maps the one tile to the other. If this isometry is also a symmetry of the tiling, then this motion maps the whole tiling to itself, and the two tiles are **transitively equivalent**. Transitive equivalence is a equivalence relation, that divides the tiles in

Condition	Thm.
$P \in$ **HP**, **Q**, **S**, **HS**, **BS**, **I**, **R**	137
P is exact, a pseudo-hexagon or has a BN-factorization	313
P satisfies the Conway Criterion	318
There is an infinite sequence of squares $R(k_i, k_i)$, with $k_{i+1} > k_i$ that can be covered by P.	310

Table 57: Criteria for a polyomino P to tile the plane.

the tiling into **transitivity classes**. If there are k classes, we call the tiling **k-isohedral**[10]; if $k = 1$, we simply call the tiling **isohedral**[11]. (Adapted from (Grünbaum and Shephard, 1987, p. 31).)

[10] Also *tile-k-transitive* (Huson, 1993, Section1, p.272).

[11] Also *tile-transitive* (Huson, 1993, Section1, p.272).

There are 93 kinds of isohedral tilings of the plane with marked tiles, distinguished by how a tile relates to its neighbors. 81 of these can be realized with tiles without markings. (Grünbaum and Shephard (1977), Grünbaum and Shephard (1987, 6.2.1)) 67 of the 93 tilings can be realized with polyominoes; 60 can be realized with polyominoes without markings. The 81 kinds of isohedral tilings can all be described by 9 rules given in Table 58.

Theorem 319 (Grünbaum and Shephard (1987), Chapter 3)**.** *In a tiling by any tile set $\mathcal{T}$, there must be a tile with at most 6 neighbors.*

[Referenced on page 311]

Theorem 320. *Each tile in an isohedral tiling has at most 6 neighbors.*

[Not referenced]

Proof. By Theorem 319 there must be at least one tile with at most 6 neighbors. But since transformations that map any other tile to this one maps the tiling to itself, it means all tiles must have at most 6 neighbors. □

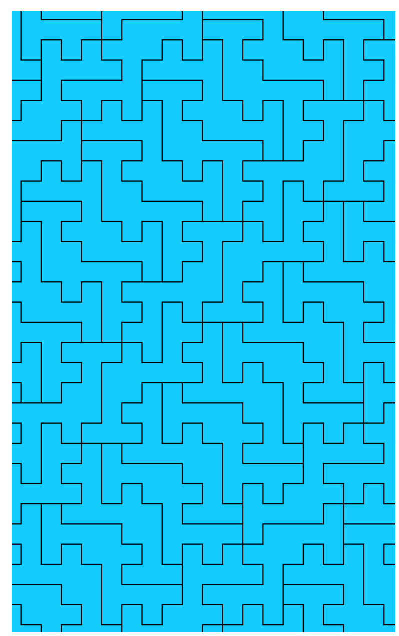


Figure 331: How may transitivity classes does this tiling have?

In general, a single tile can belong to different transitivity classes in a single tiling. Some examples are shown. Therefore, being k-isohedral is a property of the tiling and not the tile set. However, a tile set that admits k-isohedral tilings, but no m-isohedral tilings for any $m < k$, is called **k-anisohedral** (Berglund, 1993).

To summarize: tilings can be k-isohedral; tile sets can be k-anisohedral.

There is k-anisohedral polyominoes for all k between 1 and 6 (shown in Figures 332-337). We do not know whether polyominoes exist for other k (Winslow, 2018, Problem 6).[12]

[12] We *do* know that other tiles (non-polyominoes) can be k-anisohedral for $k = 8, 9, 10$. See for example Myers (2019).

Problem† 102. *Find the number of transitivity classes for the tiling in Fig 331.*

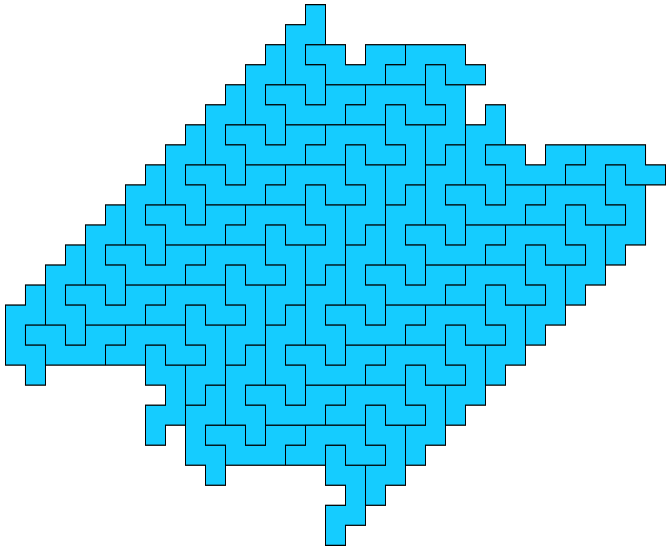


Figure 332: An isohedral tile that does not satisfy Conway's Criterion. (Myers, 2019)

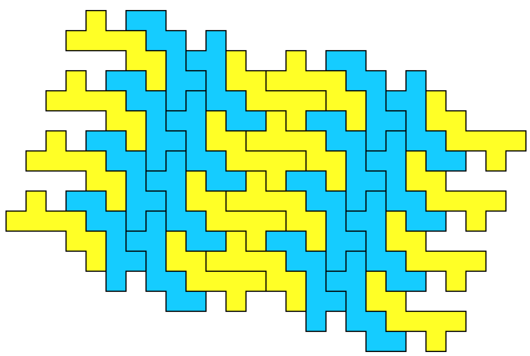


Figure 333: An 2-anisohedral tile. Transitivity classes are shown in the same color. (Myers, 2019)

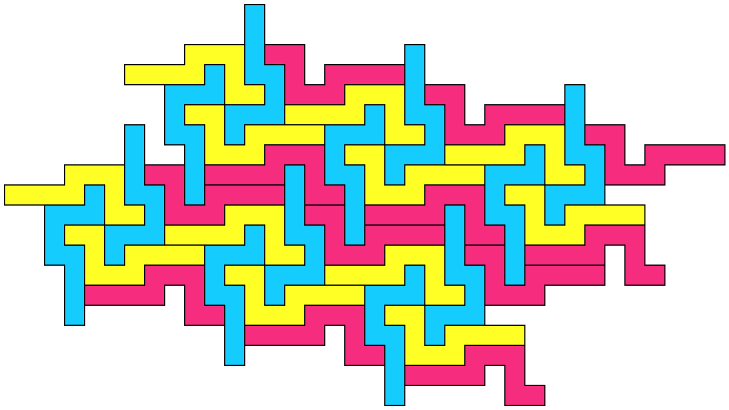


Figure 334: An 3-anisohedral tile. Transitivity classes are shown in the same color. (Myers, 2019)

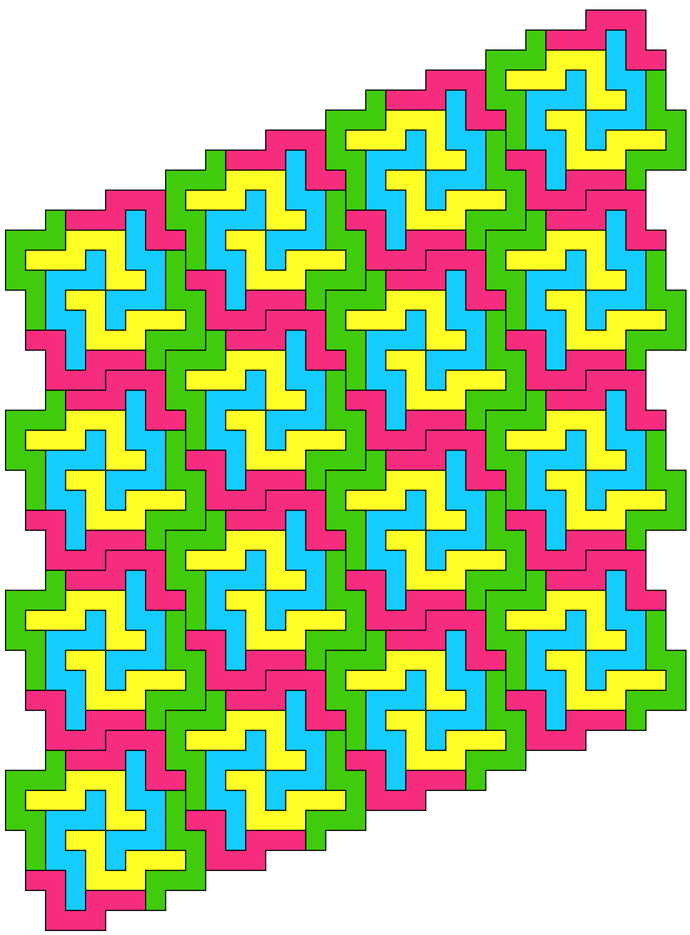


Figure 335: An 4-anisohedral tile. Transitivity classes are shown in the same color. (Myers, 2019)

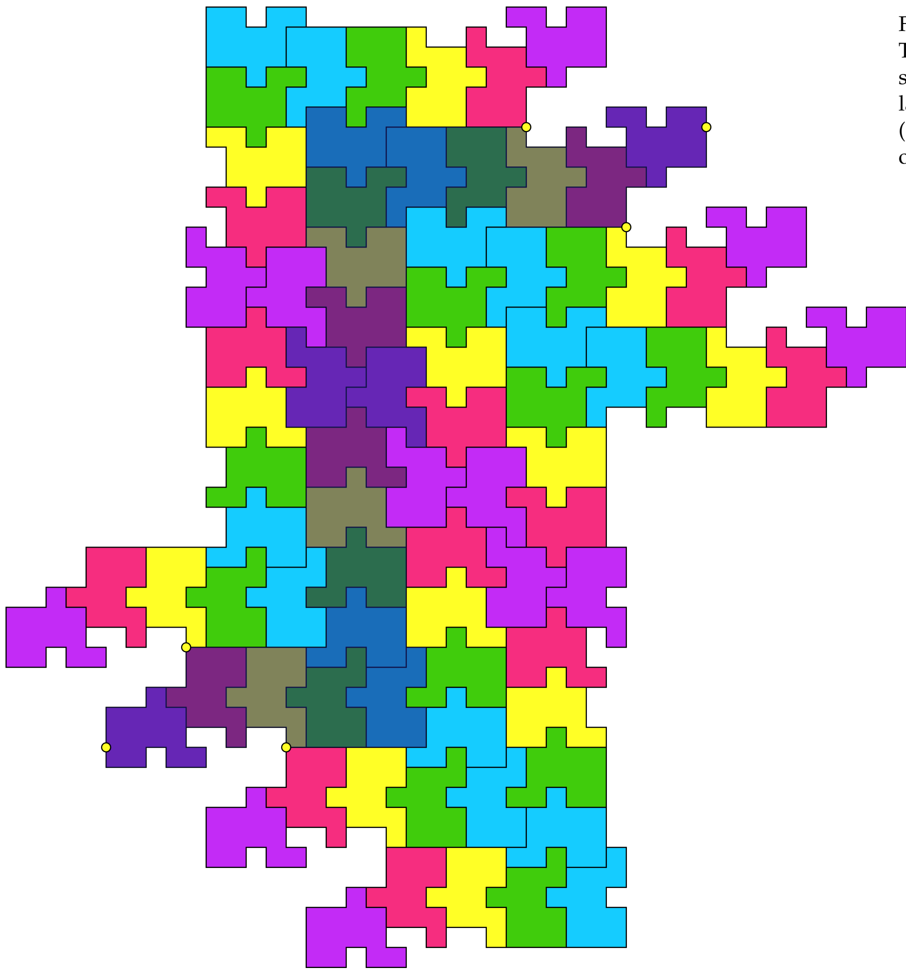


Figure 336: An 5-anisohedral tile. Transitivity classes are shown in the same color. This polyomino has the largest known fundamental region (among polyominoes) that consists out of 20 tiles. (Myers, 2019)

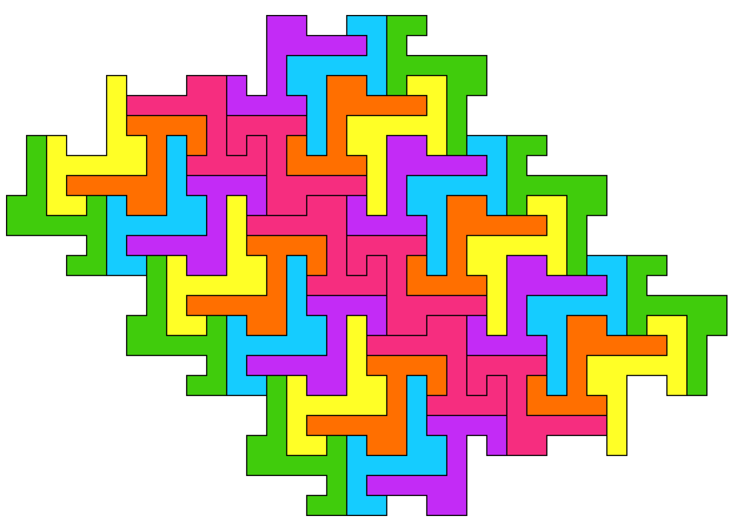


Figure 337: An 6-anisohedral tile. Transitivity classes are shown in the same color. (Myers, 2019)

In the vast majority of known tilings, there is one tile of each transitivity class in the fundamental region, however Church (2008, pp. 14–17) has found four counterexamples among polyominoes, shown in Figure 338. Only one other example that is not a polyomino is known (shown in Figure 339).

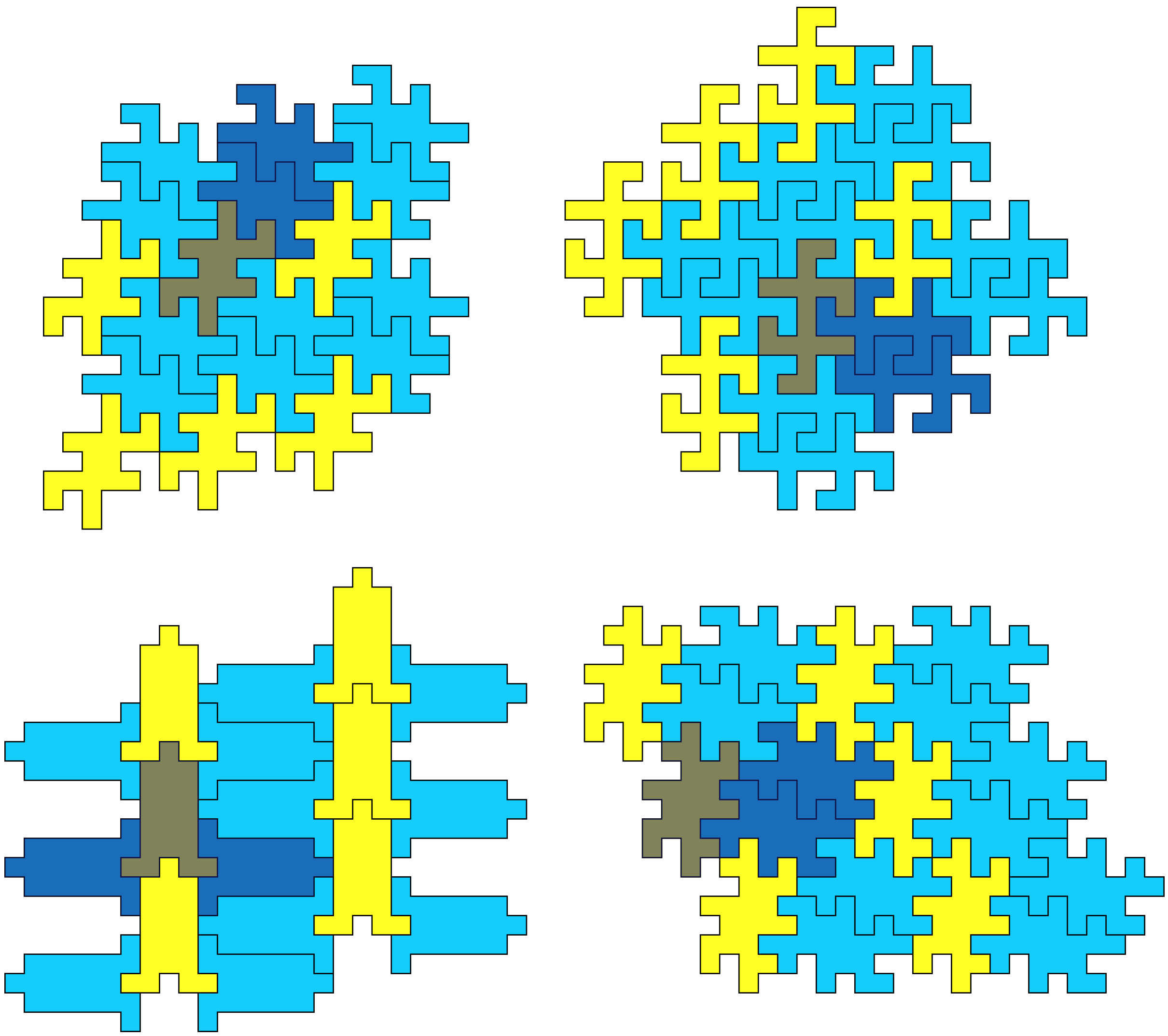


Figure 338: The only known polyominoes with no balanced tilings. From Church (2008, p. 16).

There are essentially 9 ways a arbitrary tile can tile the plane, but two of these — Criteria 7 and 8 — cannot be realized with polyominoes, as the following theorem shows.

Theorem 321. *An equilateral triangle cannot have all it's vertices on lattice points.*

[Not referenced]

Proof. The area A of an equilateral triangle is given by $A = \frac{\sqrt{3}}{4}s$, where s is the length of the side and given by $s = \sqrt{(a-a')^2 + (b-b')^2}$. Therefore, the area is given by

$$A = \frac{\sqrt{3}\sqrt{(a-a')^2 + (b-b')^2}}{4}.$$

This value is irrational if a, a', b and b' are all integers (see Problem 103).

But by Pick's Theorem (Theorem 14), the area of any polygon with its vertices on the lattice is rational. This is a contradiction, and therefore the equilateral triangle cannot have all its vertices on lattice points. □

Problem† 103. *Prove that if the sum of two squares is divisible by 3, it is divisible by 9. (It follows that $\sqrt{3} \cdot \sqrt{x^2 + y^2}$ cannot be rational if x and y are rational.)*

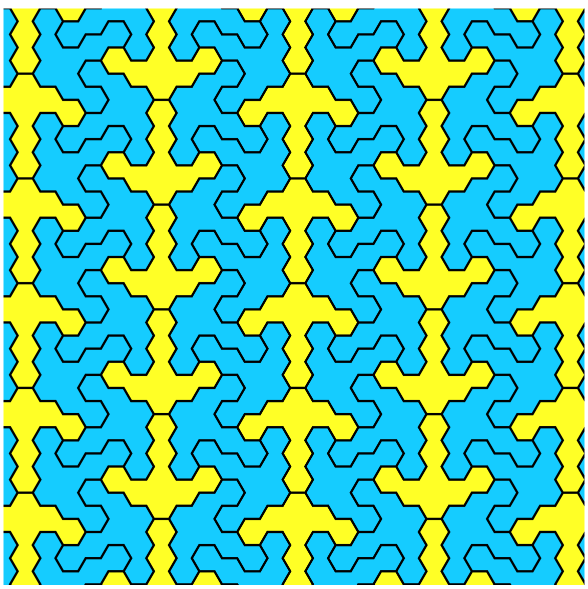


Figure 339: An example of a 2-anisotropic tile where the transitivity classes are not equally big. The tiling was discovered by Myers (2019), and Berglund (2002), and also shown in Church (2008, p. 16).

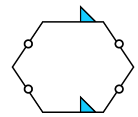
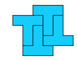
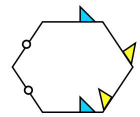
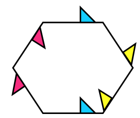
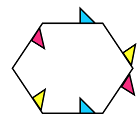
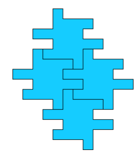
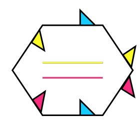
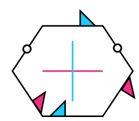
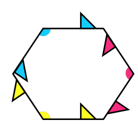
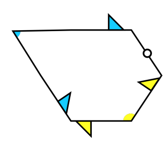
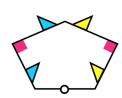
Criterion	Rules	Conceptual Diagram	Polyomino Example
Criterion 1: Conway's Criterion	(1) Sides a and d will be translates of each other. (2) Sides b, c, e, f will be centrosymmetric.		
Criterion 2	(1) Sides a and d will be translates of each other. (2) Sides b and c will be centrosymmetric. (3) Sides e and f will glide reflect to each other.		
Criterion 3	(1) Sides a and d will be translates of each other. (2) Sides b and c will glide reflect to each other. (3) Sides e and f will glide reflect to each other.		
Criterion 4: Translation Criterion	(1) Sides a and d will be translates of each other. (2) Sides b and e will be translates of each other. (3) Sides c and f will be translates of each other.		
Criterion 5	(1) Sides a and d will be translates of each other. (2) Sides b and f will glide reflect to each other. (3) Sides c and e will glide reflect to each other. (4) The glide reflections above will be parallel.		
Criterion 6	(1) Sides a and d will glide reflect to each other. (2) Sides b and f will glide reflect to each other. (3) Sides c and e will be centrosymmetric. (4) The glide reflections above will be perpendicular.		
Criterion 7	(1) Side c will rotate $120°$ around D to side d. (2) Side e will rotate $120°$ around F to side f. (3) Side a will rotate $120°$ around B to side b.		Impossible.
Criterion 8	(1) Side c will rotate $120°$ around D to side d. (2) Side e will be centrosymmetric. (3) Side a will rotate $60°$ around B to side b.q		Impossible.
Criterion 9	(1) Side b will rotate $90°$ around C to side c. (2) Side a will be centrosymmetric. (3) Side d will rotate $90°$ around E to side e.		

Table 58: The 9 ways a tile can tile the plane isohedrally. According to Langerman and Winslow (2015, Section 3) these are first shown in Heesch and Kienzle (2013, Table 10), and reproduced in Schattschneider and Escher (1990, p. 326).The final rules in Criteria 5 and 6 were added by Church (2008, p. 28), who discovered counterexamples that show the extra rules are needed.

n	n-ominoes A000105	Holes A001419	Translation A075198	180° only A075201	Isohedral A075204	Anisohedral A075206	Non-tilers A054361
1	1	0	1	0	0	0	0
2	1	0	1	0	0	0	0
3	2	0	2	0	0	0	0
4	5	0	5	0	0	0	0
5	12	0	9	3	0	0	0
6	35	0	24	11	0	0	0
7	108	1	41	60	3	0	3
8	369	6	121	199	22	1	20
9	1285	37	213	748	80	9	198
10	4655	195	522	2181	323	44	1390
11	17073	979	783	5391	338	108	9474
12	63600	4663	2712	17193	3322	222	35488
13	238591	21474	3179	31881	3178	431	178448
14	901971	96496	8672	85942	13590	900	696371
15	3426576	425449	16621	218760	43045	1157	2721544
16	13079255	1849252	37415	430339	76881	2258	10683110
17	50107909	7946380	48558	728315	48781	1381	41334494
18	192622052	33840946	154660	2344106	551137	7429	155723774
19	742624232	143060339	185007	3096983	93592	5542	596182769
20	2870671950	601165888	573296	9344528	2190553	18306	2257379379
21	11123060678	2513617990	876633	17859116	3163376	22067	8587521496
22	43191857688	10466220315	1759730	31658109	3542450	47849	32688629235
23	168047007728	43425174374	2606543	49644736	1065943	10542	124568505590
24	654999700403	179630865835	8768743	172596719	39341178	202169	475147925759
25	2557227044764	741123699012	10774339	228795554	31694933	28977	1815832051949

Table 59: Table showing counts for the number of various polyominoes that tile a certain way. From Myers (2019).

	k				
n	2	3	4	5	6
8	1	0	0	0	0
9	8	0	1	0	0
10	41	3	0	0	0
11	89	18	1	0	0
12	214	6	2	0	0
13	406	24	0	1	0
14	874	24	1	0	1
15	1107	49	1	0	0
16	2210	46	1	0	1
17	1316	60	2	0	3
18	7380	42	7	0	0
19	5450	85	2	0	5
20	18211	86	5	0	4
21	21866	199	2	0	0
22	47702	135	9	1	2
23	10390	149	3	0	0
24	201834	324	11	0	0
25	28784	182	8	1	2

Table 60: Table showing number of polyominoes that are k-anisohedral Myers (2019).

n	180° as well	Translation only
1	1	0
2	1	0
3	2	0
4	5	0
5	9	0
6	24	0
7	41	0
8	121	0
9	212	1
10	520	2
11	773	10
12	2577	135
13	3037	142
14	8081	591
15	13954	2667
16	32124	5291
17	41695	6863
18	118784	35876
19	150188	34819
20	411484	161812
21	604304	272329
22	1305265	454465
23	1954823	651720
24	5326890	3441853
25	7331606	3442733

Table 61: Polyominoes that tile the plane by translation Myers (2019).

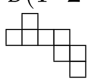
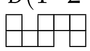
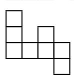
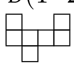
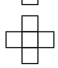
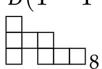
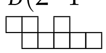
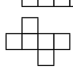
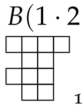
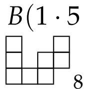
7.5 m-morphic and p-poic Polyominoes

We may also ask in how many ways can a tile tile the plane. If we allow all transformations of a tile, a tile that is m-**morphic** tiles the plane in m distinct ways. If we only allow translations and rotations, we tile is p-**poic** if it tiles the plane in p distinct ways. Table 62 introduced additional terminology. We know tiles for all values of $m \leq 11$ (Martin (1991, p. 96), taking into account the 11-morphic polyomino given in Myers (2019).)

$m = 0$	*gymnomorphic*
$m > 0$	*polymorhic*
$m = \infty$	*hypermorphic*
$p = 0$	*gymnopoic*
$p > 0$	*multipoic*
$p = \infty$	*hyperpoic*

Table 62: Terminology for how tiles can tile the plane. From Martin (1991).

Table 63 gives m and p values for the various polyominoes that tile the plane.

Tile	m	t	p	h
$B(1 \cdot 2 \cdot 1^2 \cdot 2)$	1	0	0	12
 8	1	0	0	14
$B(1 \cdot 2 \cdot 1^2 \cdot 2 \cdot 1)$	1	1	1	16
 8	1	1	1	18
 8	2	1	1	∞
$B(1 \cdot 2 \cdot 1^3 \cdot 2)$	2	1	1	10
 7	∞	1	1	12
 5	1	2	1	13
$B(1^2 \cdot 4 \cdot 1)$	2	2	2	14
 8	3	2	2	∞
$B(2 \cdot 1^2 \cdot 3)$	3	2	2	6
 8	4	2	2	12
 6	∞	2	2	13
$Z(2,1,1,2)$	2	3	2	16
$B(1 \cdot 2 \cdot 1 \cdot 3)$	3	3	3	19
 11	4	3	3	∞
$Z(2,3,1,1)$	∞	3	3	14
$B(1 \cdot 5 \cdot 1)$	2	4	2	18
 8	4	4	4	∞
$B(3 \cdot 2 \cdot 10)$	5	4	4	8

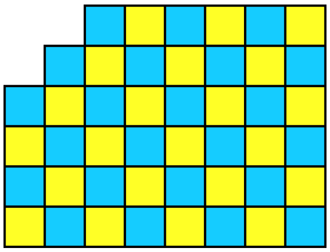
Tile	m	t	p	h
$B(2 \cdot 3 \cdot 4 \cdot 5^5)$	6	4	4	12
$B(4 \cdot 5 \cdot 6^6)$	7	4	4	14
$B(4 \cdot 5 \cdot 6^7)$	8	4	4	16
$B(4^3 \cdot 2^3 \cdot 8^3)$	9	4	4	18
$B(1 \cdot 2 \cdot 4 \cdot 5^2)$	∞	4	4	∞
$B(1 \cdot 8^5 \cdot 4^5)$	5	5	5	10
$B(4^3 \cdot 1^4 \cdot 9^4)$	6	5	5	12
$B(4 \cdot 21 \cdot 6)$	7	5	5	13
$B(2^2 \cdot 1^2 \cdot 7^2)$	7	5	5	14
$B(2 \cdot 1^2 \cdot 4^2)$	∞	5	5	∞
$B(3 \cdot 2 \cdot 3)$	3	6	3	6
$B(4^4 \cdot 28^4 \cdot 21^4 \cdot 14^4 \cdot 7^4)$	6	6	6	12
$B(4^4 \cdot 15^4 \cdot 10^4 \cdot 5^4)$	7	6	6	13
$B(2^2 \cdot 10^2 \cdot 5^2)$	8	6	6	16
$B(3^3 \cdot 8^3 \cdot 4^3)$	10	6	6	19
$Z(22,1,1)$	∞	6	6	∞
$B(1^2 \cdot 2 \cdot 3 \cdot 4)$	7	7	7	14
$B(7^2 \cdot 14^2 \cdot 8^2)$	9	7	7	18
$B(4^3 \cdot 1^4 \cdot 6^4)$	∞	7	7	∞
$B(4 \cdot 5 \cdot 4)$	4	8	4	8

Table 63: Examples of m-morphic and p-poic polyominoes. From Martin (1986).

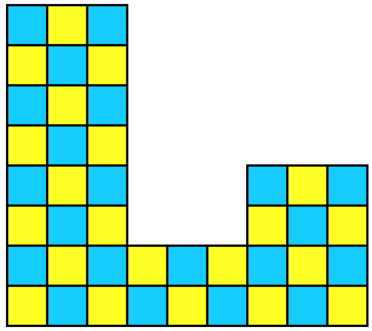
Theorem 322. *A polyomino is hypermorphic when it tiles two prime strips of different width, or one strip in two different ways, or the period of the strip is larger than 1.*

[Not referenced]

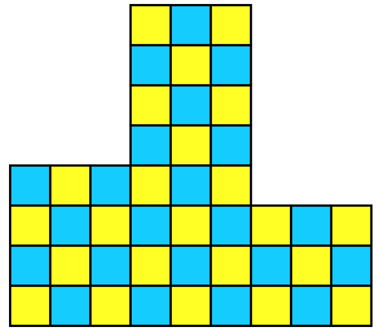
This means interesting cases involve polyominoes that belong characteristically to **P**.



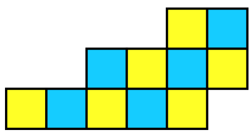
(a) An 8-morphic polyomino.



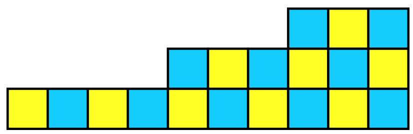
(b) A 9-morphic polyomino.



(c) A 10-morphic polyomino.



(d) A 10-morphic polyomino.



(e) A 11-morphic polyomino.

Figure 340: Examples of polymorphic polyominoes. The first three is from Winslow (2018, Figure 8), the last two from Myers (2019).

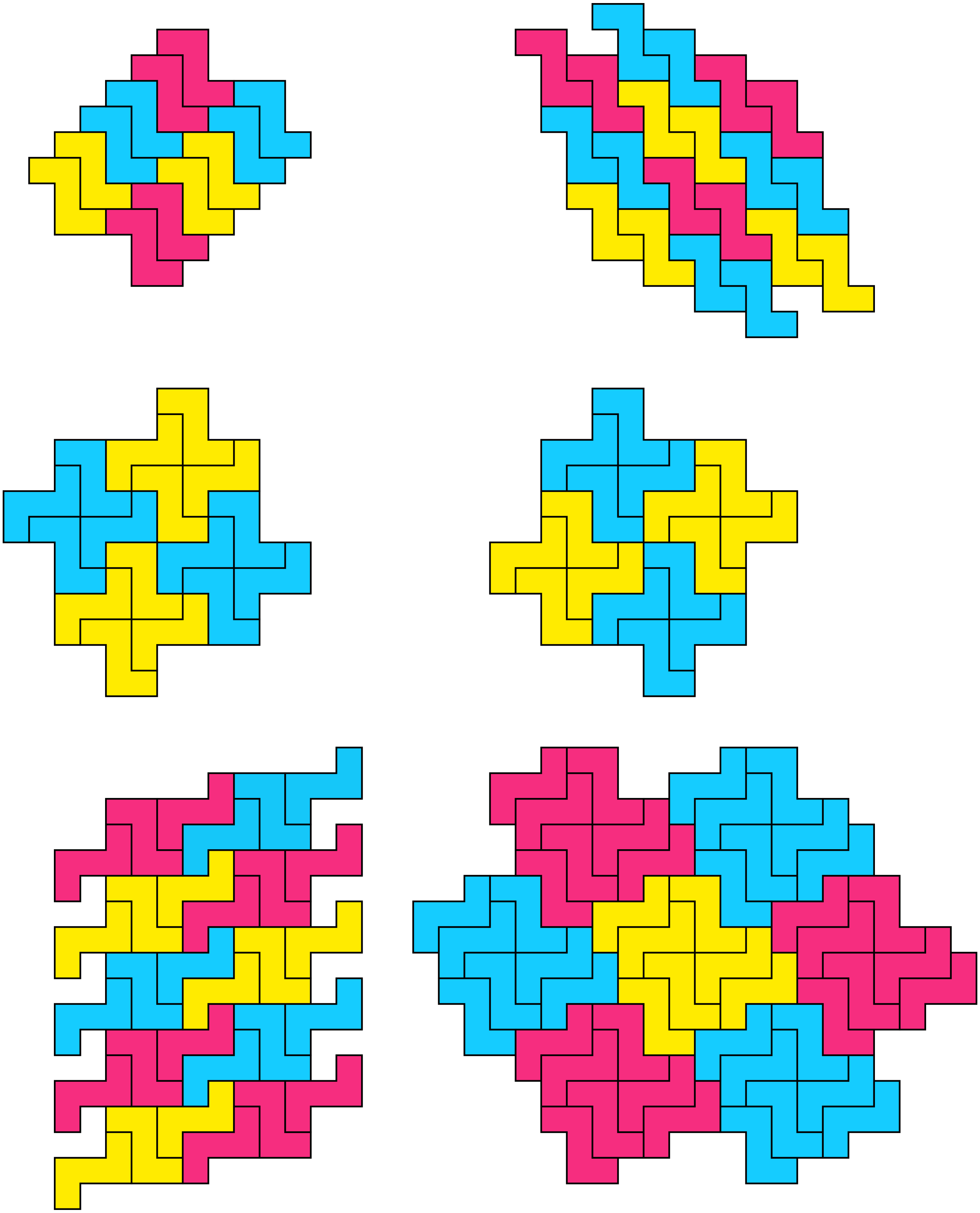


Figure 341: The 6 tilings of the Z-pentomino if reflection is not allowed.

					m								
n	1	2	3	4	5	6	7	8	9	10	11	∞	Skipped
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	0	0	0	0	0	2	0
4	0	0	0	0	0	0	0	0	0	0	0	5	0
5	1	0	0	0	0	0	0	0	0	0	0	11	0
6	0	0	0	0	0	0	0	0	0	0	0	35	0
7	6	6	3	0	0	0	0	0	0	0	0	89	0
8	20	18	4	2	0	0	1	0	0	0	0	298	0
9	193	84	14	6	1	0	0	0	0	0	0	752	0
10	749	257	41	2	2	1	0	0	0	0	0	2018	0
11	3222	809	148	31	12	3	2	0	0	1	0	2392	0
12	9026	1440	153	22	4	0	0	0	0	0	0	12803	1
13	25090	3645	435	94	25	4	3	3	0	0	0	9368	2
14	63746	5681	416	56	17	2	1	0	0	0	0	39176	9
15	180669	11842	665	85	13	1	0	1	0	0	0	86306	1
16	366557	16758	1128	142	37	6	2	0	0	0	0	162257	6
17	683157	30733	1473	264	63	9	2	1	0	0	0	111321	12
18	2192816	65557	2226	238	31	1	1	1	0	0	0	796444	17
19	2936540	72811	2130	360	96	7	2	1	1	1	1	369160	14
20	9444080	126363	3550	322	68	42	1	1	0	0	0	2552238	18
21	18299457	221446	4767	458	90	14	0	0	2	0	0	3394939	19

Table 64: The number of polyominoes that are m-morphic (Myers, 2019). (Meyers originally classified the monomino as hypermorphic—presumably because he allows non-lattice tilings.)

7.6 Colorable Tilings

Problem 104. *A tiling is* ***colorable*** *by n colors if each tile in the tiling is colored with one of the colors, and no two tiles that share an edge is the same color. Because of the four color theorem, we know that all tilings are 4-colorable.*[13] *No tiling is 1-colorable.*

(1) *Which tilings are 2-colorable?*

(2) *Which tilings are 3-colorable?* [14]

[13] Informally, the four color theorem states that any planer map can be colored with 4 colors so that no two countries that share an edge have the same color. This was a longstanding conjecture first proved in Appel and Haken (1976). The *Further Reading* section provides more references.

[14] Friedman (2008, https://erich-friedman.github.io/mathmagic/1009.html) give a large number of 3-colorable tilings.

7.7 Non-tilers

We call a polyomino that does not tile the plane a **non-tiler**. The smallest non-tilers have 7 cells; there are only 3 without holes, shown in Figure 342. The 20 simply-connected octominoes that don't tile the plane are shown in Figure 343. Numbers for non-tiles of polyomninoes with 25 cells or less is shown in Table 59.

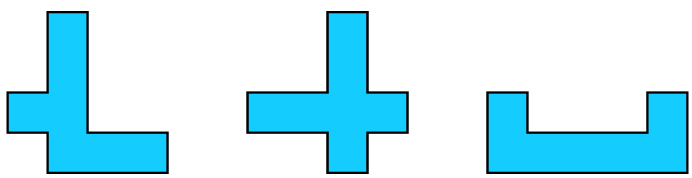


Figure 342: Heptominoes without holes that don't tile the plane.

When a tile does not tile the plane, we may wonder: how close can it come to tiling it?

There are different ways to look at it:

(1) What is the highest tiling density we can achieve?

(2) What is the biggest square we can cover?

(3) How many layers can we build around a polyomino with copies so that there are no holes?

We will not consider the first two questions here.

The answer to the last question is called a polyomino's **Heesch number**. We denote the Heesch number of polyomino P by $H_c(P)$. Each layer around the polyomino must cover all vertices on the border of the interior region. Some authors allow holes in the last corona, which gives slightly higher values in some cases. We will denote this by $H_h(P)$. The notation is from Kaplan (2021a).

Currently, we don't know of any polyomino that does not tile the plane with a Heesch number greater than three; in fact, we do not know of *any* tile with Heesch number bigger than 6. See (Bašić, 2021, Figure 1) for an example of a tile with Heesch number 6 and Mann (2004, Figure 6) for example of a tile with Heesch number 5.

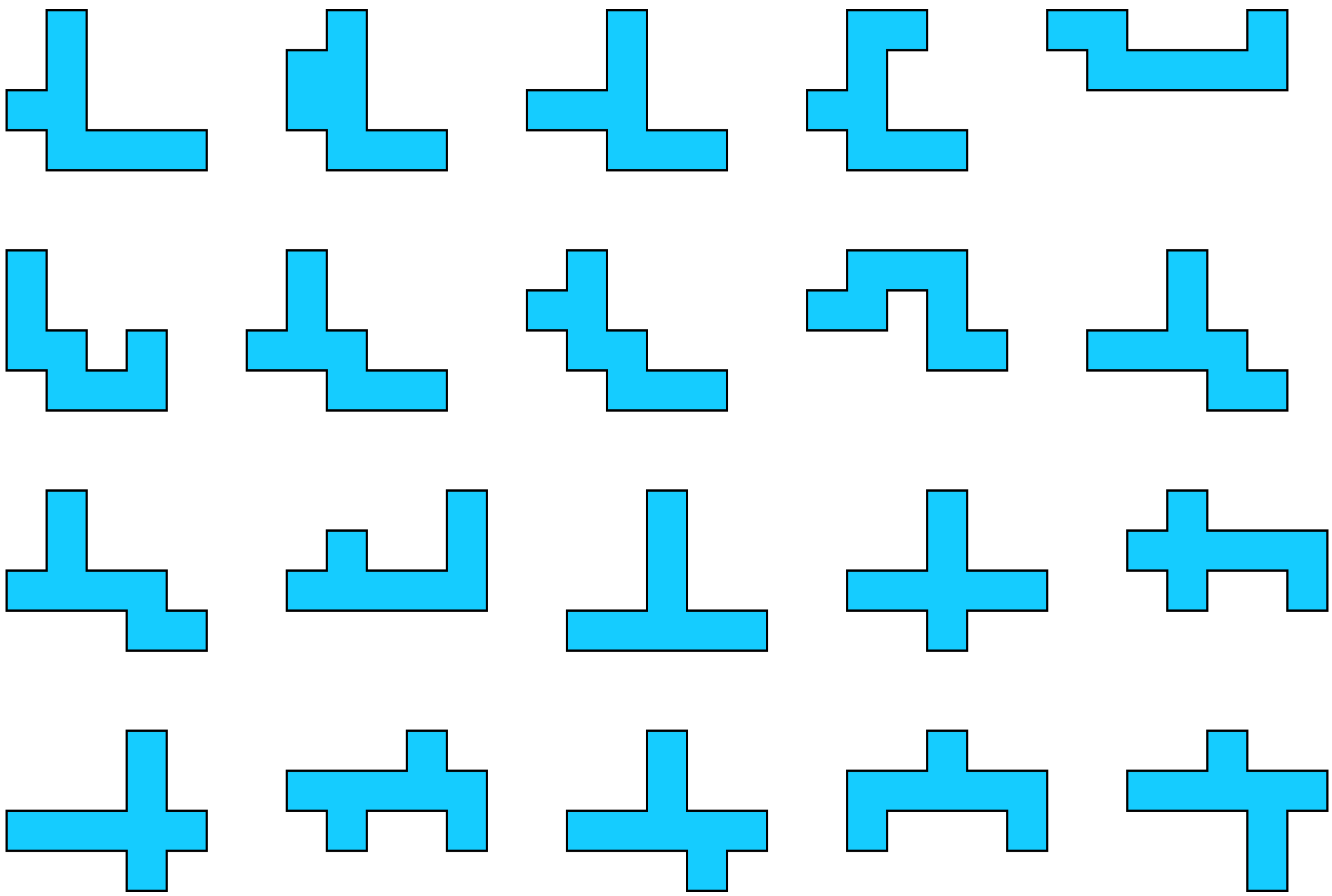


Figure 343: Octominoes without holes that don't tile the plane.

Some polyominoes with Heesch number 2 are shown in Figure 345. Some example tilings are shown in in Figure 344. The polyomino in Figure 346 has Heesch number 3; it is derived from the *4-polypillar* in Mann (2004, Figure 8) and (Mann and Thomas, 2016, Figure 12).

The Heesch number for all polyominoes up to 19 cells are shown in Table 66.

		Heesch number H_c			
n	Hole-free non-tilers	0	1	2	3
7	3	1	2		
8	20	6	14		
9	198	75	122	1	
10	1390	747	642	1	
11	9474	5807	3628	39	
12	35488	28572	6906	10	
13	178448	149687	28694	67	
14	696371	635951	60362	58	
15	2721544	2598257	123262	25	
16	10683110	10397466	285578	66	
17	41334494	40695200	639162	130	2
18	155723774	154744331	979375	68	
19	596182769	593856697	2325874	198	

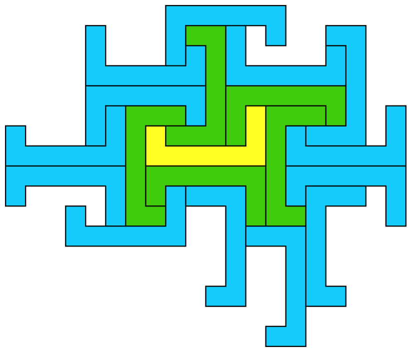
Table 65: Heesch numbers for polyominoes (Kaplan, 2021a, Table 1) with no holes allowed in any of the coronas.

		Heesch number H_h			
n	Hole-free non-tilers	0	1	2	3
7	3	0	3		
8	20	0	19	1	
9	198	36	157	5	
10	1390	355	1020	15	
11	9474	2820	6544	109	1
12	35488	17409	18038	41	
13	178448	100180	78048	219	1
14	696371	485807	210362	202	
15	2721544	2185656	535724	164	
16	10683110	9300840	1381965	305	
17	41334494	37932265	3401701	525	3
18	155723774	148955184	6768266	324	
19	596182769	580412188	15769814	767	

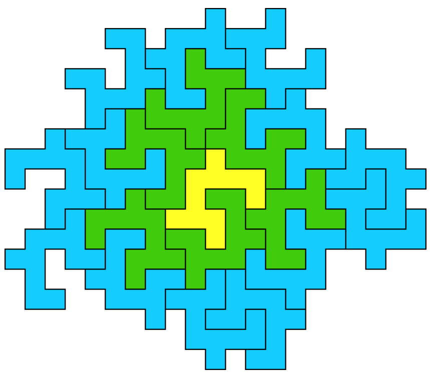
Table 66: Heesch numbers for Polyominoes (Kaplan, 2021a, Table 2) with holes allowed in the outer corona.

A **U-frame polyomino** is a polyomino of the form $x^{-h}y^{g}x^{-f}y^{-e}x^{d}y^{c}x^{-b}y^{-a}$ (Fontaine, 1991). See Figure 347.

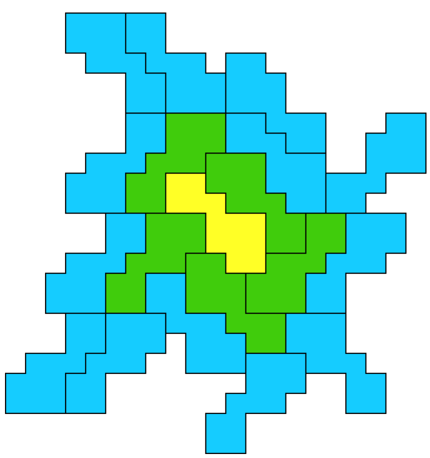
Figure 344: Some almost-tilings of polyominoes with Heesch number 2



(a) Kaplan (2017)



(b) Rhoads (2003)



(c)

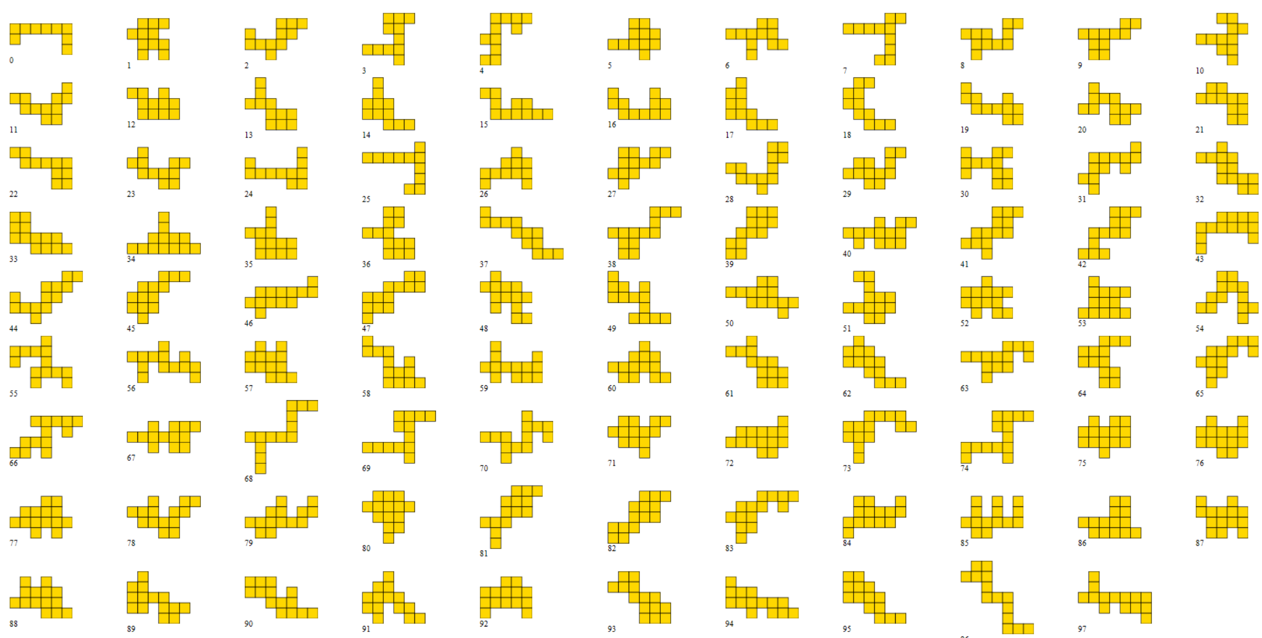


Figure 345: Polyominoes with Heesch number 2. List obtained from Kaplan (2017).

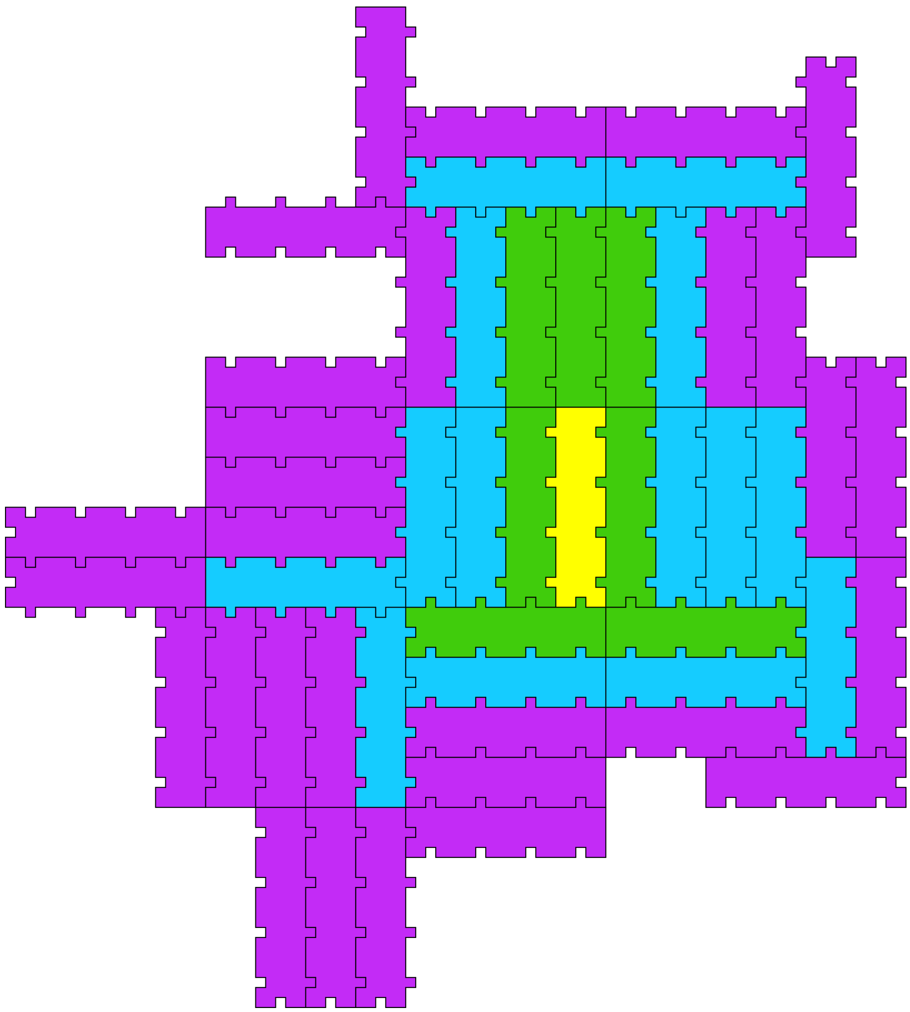


Figure 346: A 99-omino with Heesch number 3.

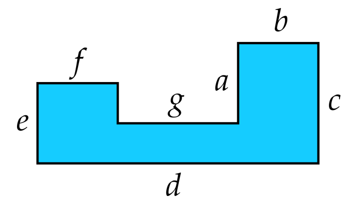


Figure 347: A u-frame polyomino.

Theorem 323 (Fontaine (1991)). *A U-frame polyomino has Heesch number 2 if:*

(1) $a = g + h - e$

(2) $b = g + h$

(3) $c = h$

(4) $d = 2g + 3h$

(5) $f = g + h$,

where $0 < g < e < h < 2e$,

[Not referenced]

7.8 *Aperiodict Tilings*

While it is easy to find tilings that are non-periodic, it is not so easy to find aperiodic tile sets. Two examples of aperiodic sets are shown in Figures 348 and 350. The tiling of the first set is shown in Figure 349.

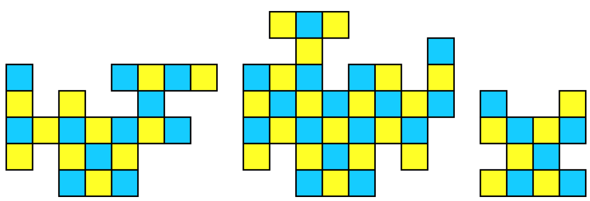


Figure 348: An aperiodic tile set discovered by Roger Penrose, derived from a set by Robert Amman (Penrose, 1994, Figure 1.3, p. 32). Also shown in Sequeiros (2009, Figure 3).

A **Wang tile**[15] is a square with colored edges; in a legal tiling of Wang tiles colors must match. Despite their simplicity, there are sets of Wang tiles that tile the plane aperiodically. Such a set must use at least four colors (Chen et al., 2014), and have at least 11 tiles (Jeandel and Rao, 2015). An example is shown in Figure 357.

There is a way to convert between polyomino problems and Wang tile problems (Golomb, 1970). To convert from Wang tiles to polyominoes, the basic idea is to use a base tile that is monomorphic, and encode the colors in binary by perforating the edges (by adding and removing cells). For this to work, the base tile must be big enough. Figure 353 shows the sequence of G_i of base tiles introduced. To encode 4 colors, we need the perforated edge to be 2 cells wide, so we can use G_2. The set of 11 Wang tiles realized as polyominoes is shown in Figure 358.

Another encoding scheme (from Yang (2014)) is shown in Figure 354.

[15] Golomb (1970) calls this a *MacMahon tile*, after Major P.A. MacMahon who studied their properties in MacMahon (1921). The question whether aperiodic sets of Wang tiles exist was first asked in Wang (1961). The first aperiodic tile set is a set of 20,426 Wang tiles, described in Berger (1966), where the term *Wang tile* is also used for the first time.

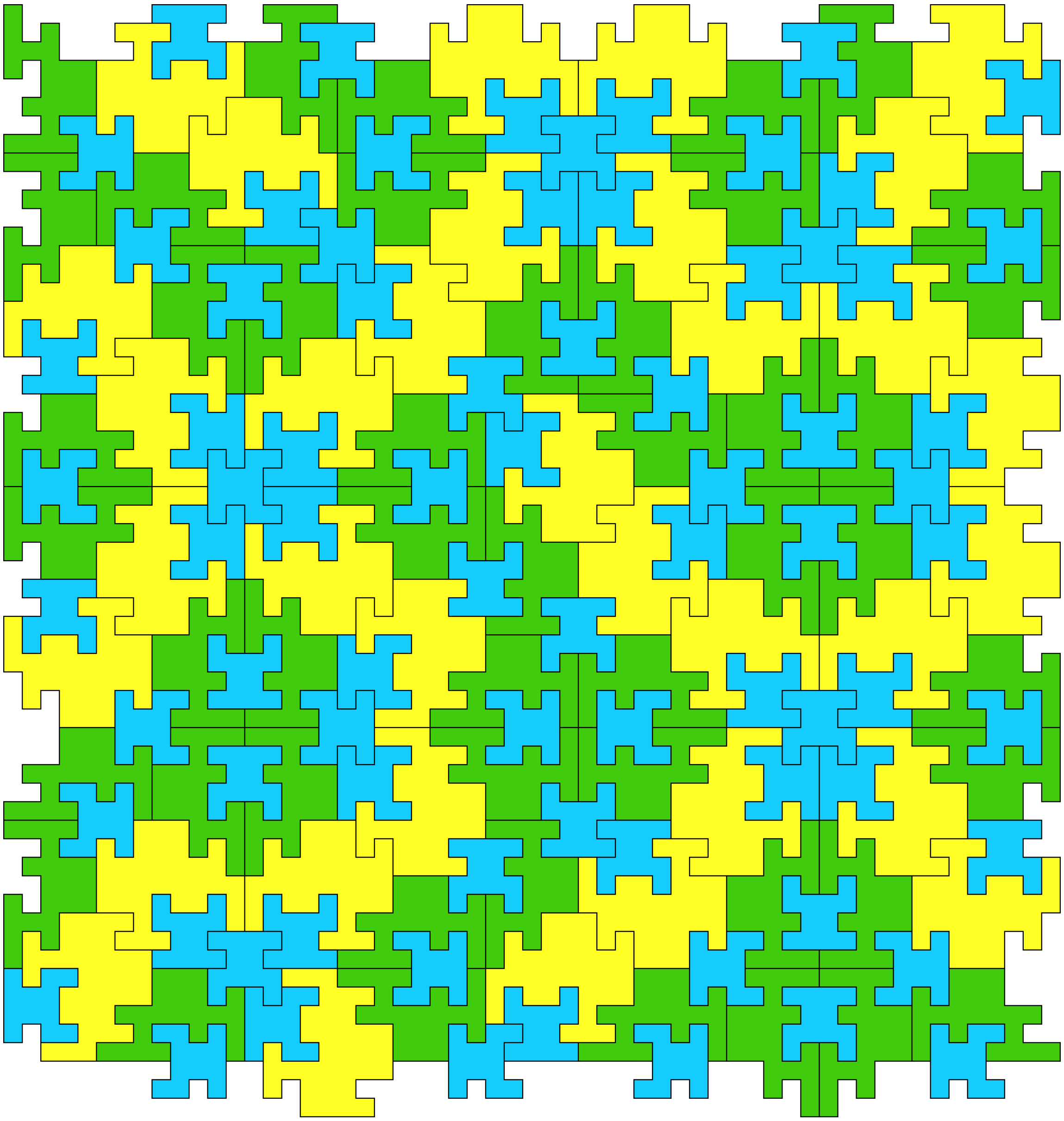


Figure 349: A patch of a plane tiling by the Penrose set. Also shown in Sequeiros (2009, Figure 4).

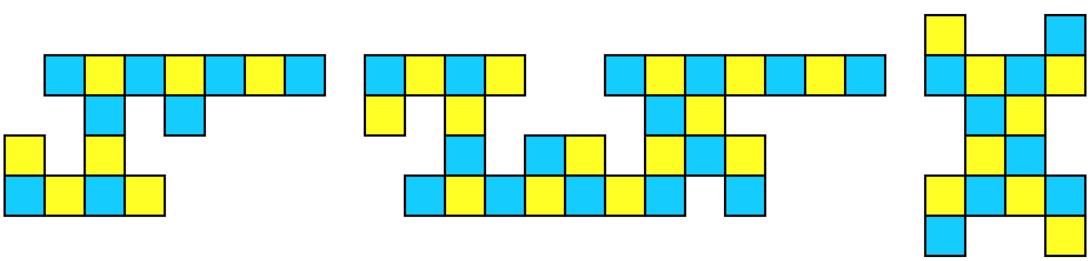


Figure 350: An aperiodic set discovered by Matthew Cook (Wolfram, 2002, p. 943).

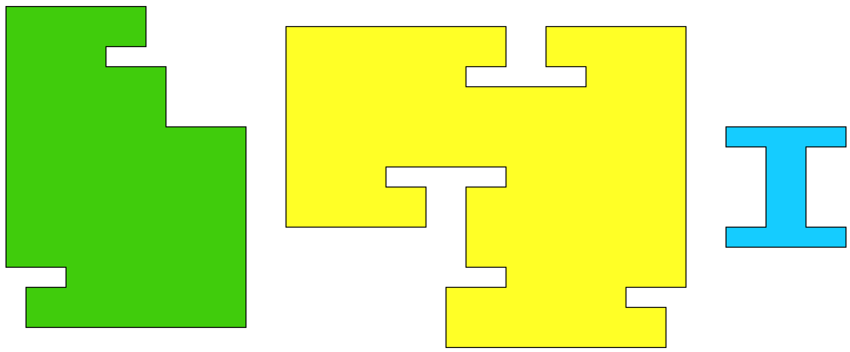


Figure 351: An aperiodic set given in (Winslow, 2015, Figure 5), modified from a similar set in Ammann et al. (1992, Figure 1).

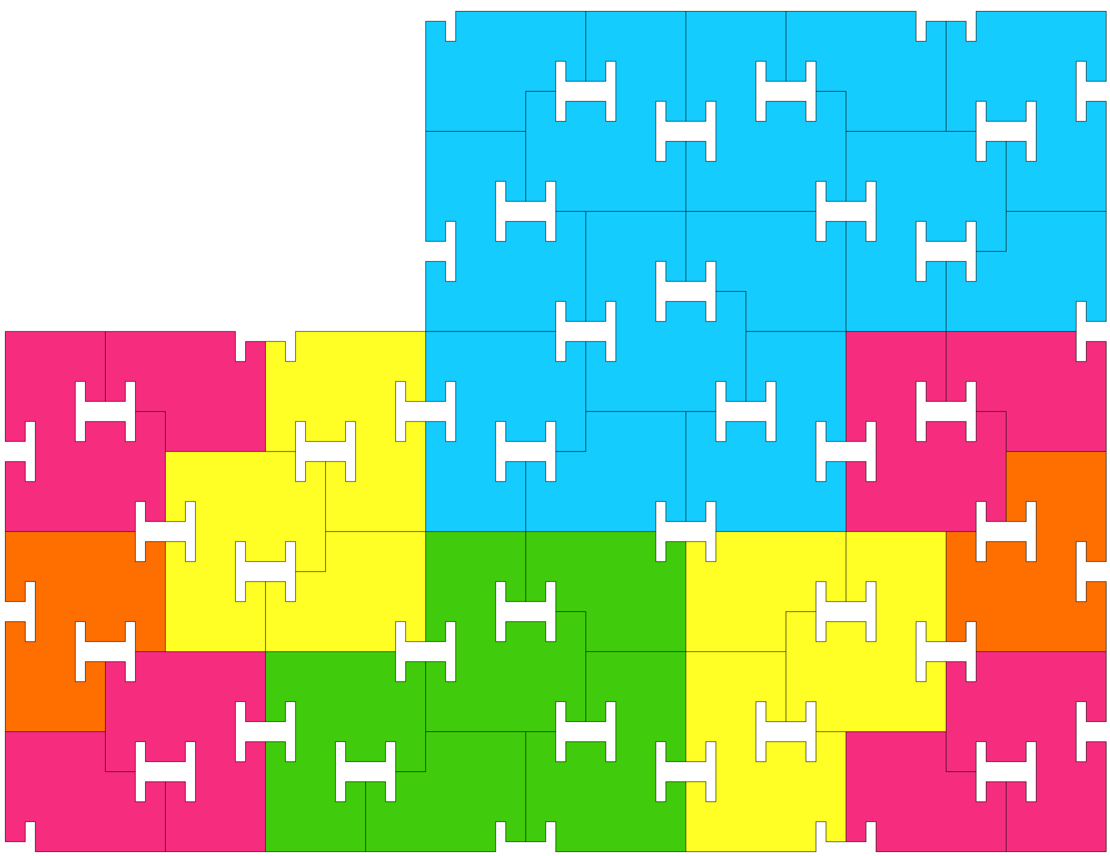


Figure 352: A tiling by Wilson's tiles.

A different scheme (from Beauquier and Nivat (2003)) is to use tiles with hooks whose length represents the color. Examples for 4 colors are shown in Figure 355.

Yet another encoding is given in Walzer et al. (2014) and shown in

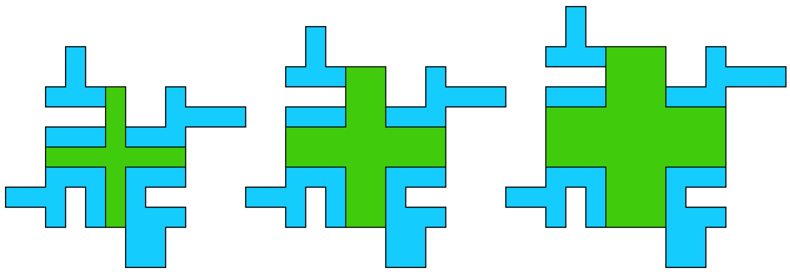


Figure 353: Golomb's Encoding of Wang tiles.

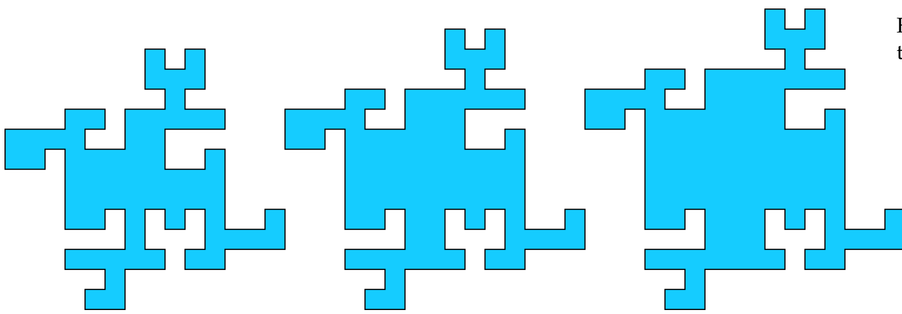


Figure 354: Yang's encoding of Wang tiles.

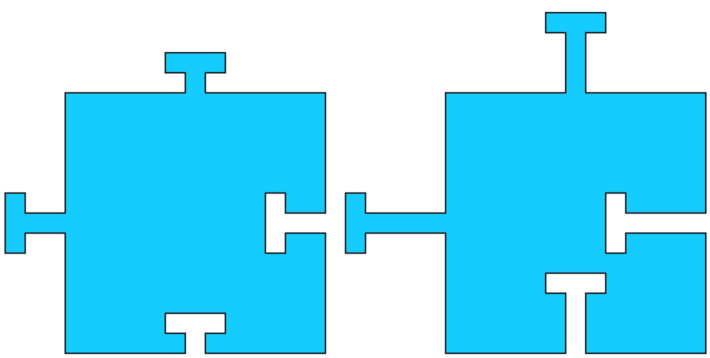


Figure 355: Beauquier and Nivat's encoding of Wang tiles.

To convert from polyominoes to Wang tiles, we will have a unique Wang tile for each cell in the set. We choose one color for outer edges, and one color for each internal edge. This way, the only tilings will match a polyomino tiling exactly. For this to work, rotations and reflections are not allowed; otherwise we can always match any tile with a reflected or rotated copy with itself to make a domino or a tile bigger than that was was in the original set.

Problem† 105. *How many Wang tiles do we need to represent tilings by tetrominoes?*

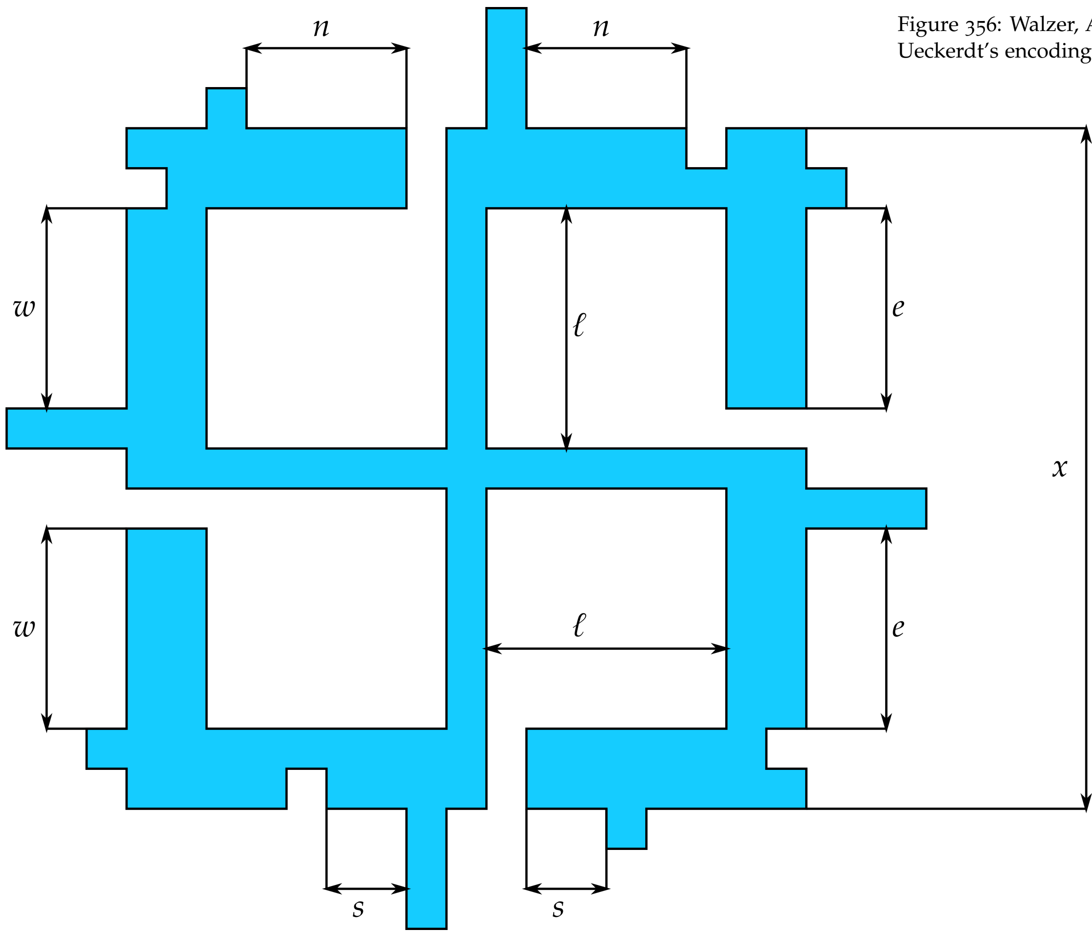


Figure 356: Walzer, Axenovich and Ueckerdt's encoding of Wang tiles.

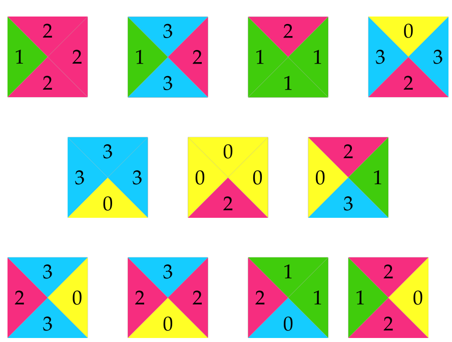


Figure 357: The smallest aperiodic set of Wang tiles.

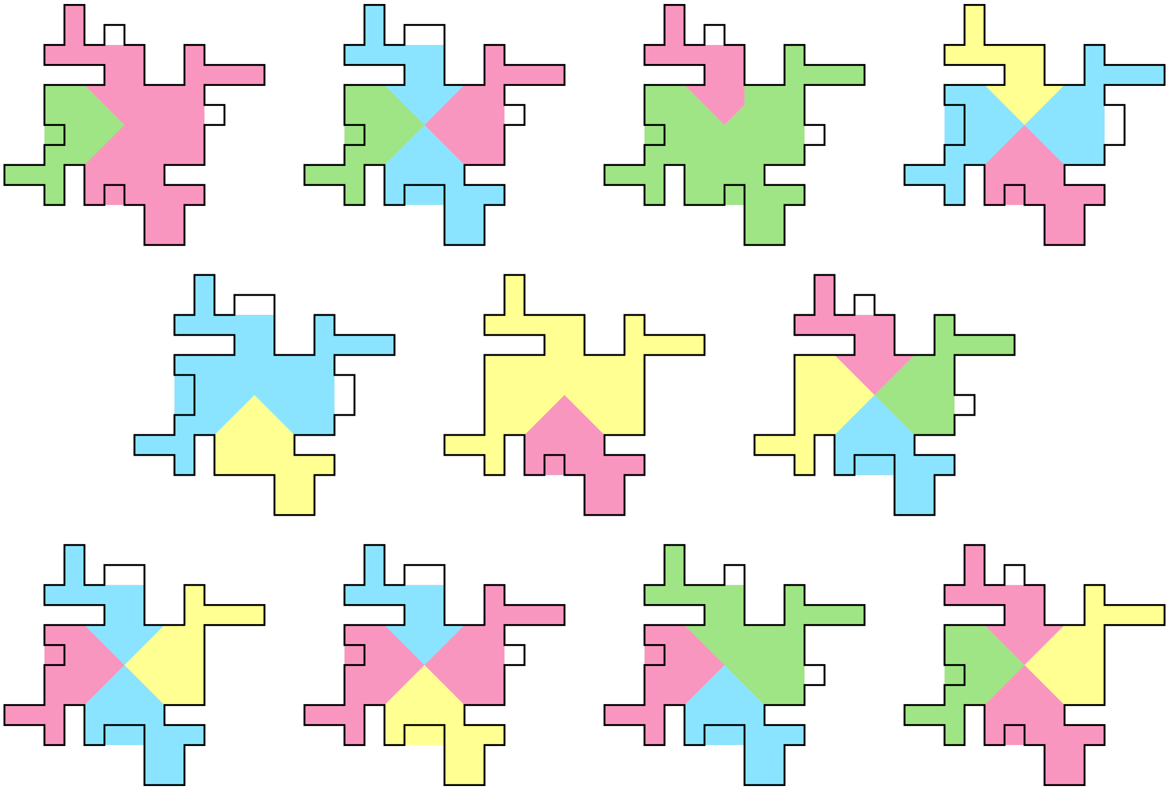


Figure 358: Encoded as polyominoes.

Because of their simple structure, Wang tiles are much easier to work with. Because of the conversion, some general theorems that can be proven for Wang tiles also apply to Polyominoes.

Here are some examples:

(1) Whether a set of Wang tiles (polyominoes) can tile the plane is undecidable.

(2) Whether a set of Wang tiles (polyominoes) can tile an infinite strip is decidable.

(3) There are no aperiodic tilings of a infinite strip.

7.9 *Further Reading*

Plane tilings for more general tiles than polyominoes is a widely covered topic. Toth et al. (2017, Chapter 3) is a handy reference for terminology, main results and open problems. I already mentioned Grünbaum and Shephard (1987) in the book's introduction, which is the most comprehensive source on the topic. Adams (2022) and Fathauer (2020) are lighter reads (both books are beautifully illustrated in full-color), and so is Kaplan (2009), a book especially suitable if you want to use a computer to experiment. Conway et al. (2016) is a very interesting book, focusing on symmetry, algebra, and topology, not only on the plane, but also on the sphere and hyperbolic plane. Other resources include Horne (2000), and a short survey Goodman-Strauss (2016). Open problems is discussed in Goodman-Strauss (2000), and listed and briefly discussed in Brass et al. (2005).

The 1270 types of 2-isohedral tilings of the plane is given in Delgado et al. (1992). Huson (1993) contains information about classification of isohedral, 2-isohedraL and 3-isohedral tilings of the plane, sphere, and hyperbolic plane.

For more on double squares, see Massé et al. (2013). For more families of polymorphic tiles, see Fontaine and Martin (1984). Martin (1991, Chapter 6, 9) also discusses polymorphic and polypoic polyominoes.

In Kaplan (2021a) several polyominoes with Heesch number 3 are given. The paper also describes algorithms, and give other polyforms with Heesch numbers 3 and 4. The dataset that the paper is based on is also shared (Kaplan, 2021b), and gives the Heesch numbers for all polyominoes between 7 and 9 cells, and a patch for each to show the coronas.

The four color theorem is treated in many books on graph theory. See for example Harris et al. (2008, Section 1.6.3), Bondy and Murty (2008, Chaper 11). For historical and mathematical background, see Dohas et al. (1998).

The web site Wichmann (2018) is a tiling database of tilings (not necessarily using polyomino tiles). In addition to information on over 2000[16] tilings, it also contains information such as statistics on the tilings in the database, and various sets. Of particular note are the rectangles tiled by dominoes http://www.tilingsearch.org/special/special1.html and regular colorings: http://www.tilingsearch.org/special/special2.html. Tilings of all rectilinear shapes can be found from http://www.tilingsearch.org/general.htm by ticking "All angles of the pattern a multiple of: 90 degrees".

[16] As of 2 December 2019.

8
Selected Topics

8.1 Compatibility

Two polyominoes P and Q are **compatible** if there is a finite[1] region that they both can tile (Cibulis et al., 2002). For example, the X- and N-pentominoes are compatible, as shown by the tilings in 361.

We will see below that any two rectifiable polyominoes are compatible. The monomino is compatible with any polyomino, and probably the only polyomino with this property (see Problem 106(3)). The domino, for example, is not compatible with the polyominoes shown in Figure 359. Minimal incompatible regions can be found in Sicherman (2015b). Incompatible polyominoes allow us to construct arbitrary untileable regions (perhaps that satisfy additional constraints) that are useful to test tiling algorithms..

A region that is tileable by all polyominoes in a set is called a **common multiple** of the set; the smallest region (not necessarily unique) is called a **smallest common multiple** (abbreviated **LCM**) of the set. The LCM does not exist for all sets. Similarly, if a polyomino tiles all regions in a set, it is called a **common divisor** of the set, and the biggest (not necessarily unique) is called the **greatest common divisor** (abbreviated **GCD**). The GCD always exist since, the monomino at least tiles everything (Cibulis et al., 2002).

The smallest common multiple of the tetrominoes is shown in Figure 360.

[1] There are additional interesting problems if we remove this restriction: What polyominoes are infinitely compatible, but not finitely? Can the infinite regions that serve as common multiples be characterized?

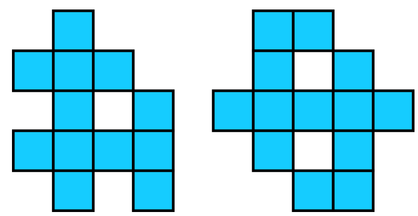


Figure 359: Polyominoes that are not compatible with dominoes (Sicherman, 2015b).

Problem† 106.

(1) *Prove that the domino is not compatible with the polyominoes in Figure 359.*

*(2) *If we close the holes of the polyominoes in Figure 359, are they still incompatible with dominoes?*

*(3) *Find more examples of polyominoes that are not compatible with dominoes. Can you characterize them?*

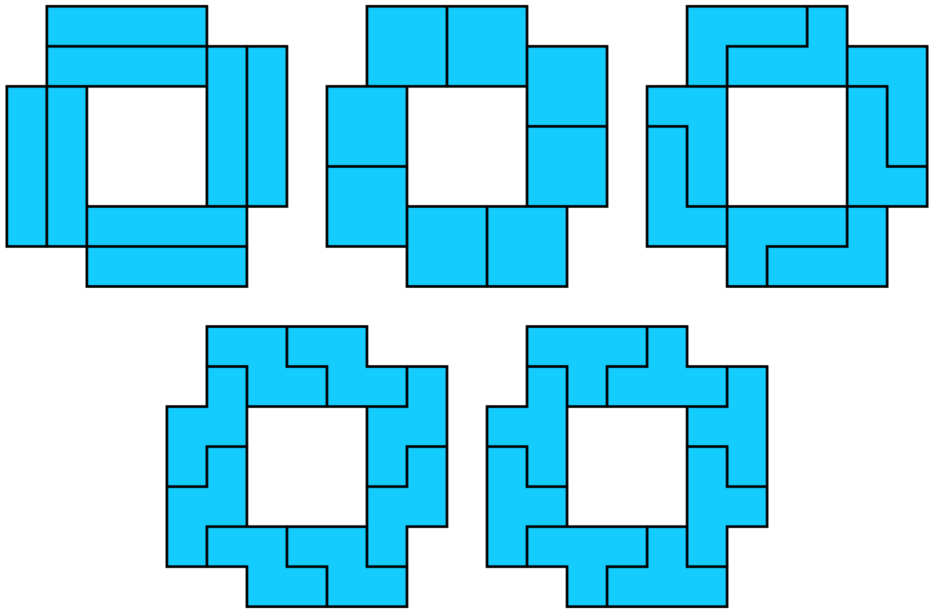


Figure 360: The smallest (known) common multiple of the tetrominoes (Sicherman, 2015b, http://www.recmath.org/PolyCur/nmultcom/index.html).

(4) *Find an example of polyominoes P, Q and R, such that P and Q are compatible, and Q and R are compatible, but P and R are not compatible; that is, compatibility is not transitive, and therefore also not an equivalence relation.*

**(5) *Can you find a general strategy for constructing a polyomino that is not compatible with a given polyomino? (It is an open problem whether the monomino is the only polyomino compatible with all polyominoes, although it seems likely that it is true.)*

Problem† 107.

(1) *What is the LCM of two arbitrary rectangles?*

(2) *What is the GCD of two arbitrary rectangles?*

(3) *Are the LCM and GCD distributive lattices over the set of rectangles?*

Theorem 324. *Any two rectangles are compatible.*

[Referenced on page 340]

Proof. $R(m, n)$ and $R(m', n')$ both tile $R(mm', nn')$ (Theorem 22). □

Theorem 325. *If P tiles Q, and P' tiles Q', and Q and Q' are compatible, then so are P and P'.*

[Referenced on page 340]

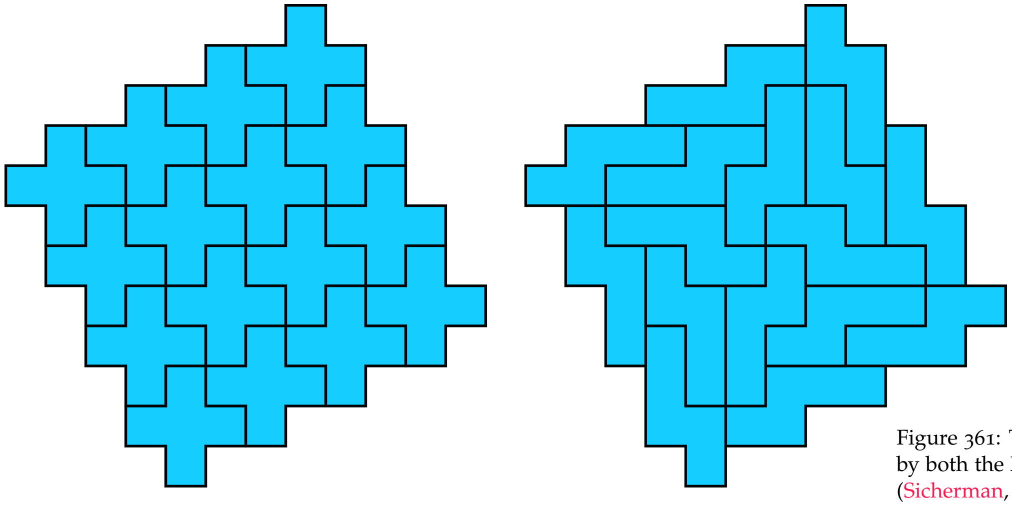


Figure 361: The smallest region tileable by both the X- and N-pentominoes (Sicherman, 2015b, http://www.recmath.org/PolyCur/n5com/n5com.html).

Proof. Let R be a region tileable by both Q and Q'.

Now P tiles Q, and Q tiles R, so P tiles R (Theorem 3). Similarly P' tiles R. Since P and P' tile a common region, they are compatible. □

Theorem 326. *Any two rectifiable polyominoes are compatible.*

[Not referenced]

Proof. Suppose P and P' are rectifiable polyominoes that tile rectangles R and R'. Then, R and R' are compatible by Theorem 324, and so P and P' are compatible by Theorem 325. □

It follows that the LCM always exist for a set of rectifiable polyominoes.

Suppose P and Q are compatible polyominoes. The number of P we need to tile their smallest common region is denoted by $\mathrm{CO}_P(P,Q)$. If $|P| = |Q|$, then $\mathrm{CO}_P(P,Q) = \mathrm{CO}_Q(P,Q)$ and we simply write $\mathrm{CO}(P,Q)$ and called it their **compatibility order**.

Theorem 327. *If P and Q are rectifiable polyominoes with minimal rectangles $R(m,n)$ and $R(k,\ell)$, then*

$$\mathrm{CO}_P(P,Q) \le \frac{mnk\ell}{d|P|},$$

where $d = \gcd m,n,k,\ell$.

[Referenced on page 198]

This theorem is similar to Theorem 190 and can be used to constrain computer searches.

Theorem 328. *The X-pentomino is not compatible with any rectangle except the domino.*

[Not referenced]

Proof. Any region tileable by the X-pentomino has at least one peak (in fact there are 4, see Problem 24). This peak cannot be tiled by any rectangle that is not a bar.

In any tiling by bars there must be two exposed long sides (see Problem 23). But the longest side in a tiling by X-pentominoes is 3. Therefore, no tiling is possible by bars of length 4 or higher. We could also use Theorem 186 to show we need at least one side whose length as long as the length of the bar.

That leaves only the bar of length 3. We use the proof in Barbans et al. (2003, p. 93–95).

Suppose there is a region that is the LCM of the X-pentomino and the straight tromino. Let the lowest cell that touches the left side of the hull be R, and the left most cell that touches the bottom side of the hull be G. For each cell on the border between R and G, color the cell red if it is covered by a horizontal bar in the tiling by bars, and green when it is covered by a vertical bar. Note that R is red and G is green.

The number of color changes must be odd, say $2n - 1$. We claim that cells must be monotone lower from R to the first green cell. For if they were not, we would have a knob of length $3k > 1$ formed by one or more horizontal dominoes, as shown in Figure 362. But knobs of length more than 1 cannot be tiled by the X-pentomino, so this situation is impossible, and therefore the cells must be monotone lower.

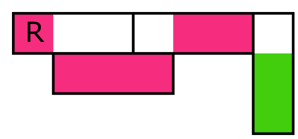


Figure 362: The border must be monotone between R and the first color change.

Next, we show by induction that no region is possible for any n.

For the base case, we prove the common multiple cannot exist when $n = 1$. At the point where the color change takes place, a horizontal bar meets a vertical bar. All possible configurations are shown in Figure 363.

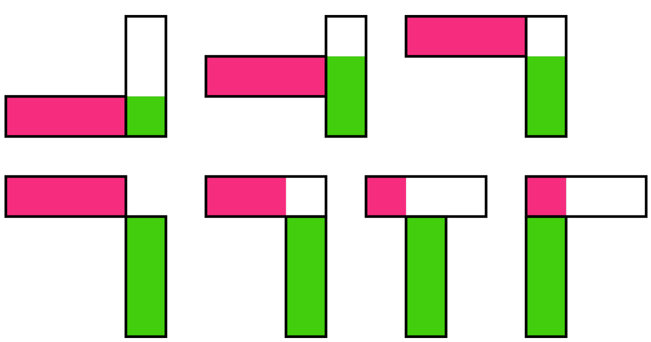


Figure 363: All the cases of a color change.

Each of these leads to a contradiction when we try to tile it with X-pentominoes, an example is shown in Figure 364.

Suppose the common multiple cannot exist if the number of color-changes of the boundary squares between R and G is at most $2n-1$ for some $n \geq 1$. Suppose there exist a common multiple such that the number of color-changes is $2n+1$ for some $n \geq 1$. Translate the left edge of the minimal rectangle inward until it touches for the first time a red boundary square R' which occurs after the first color-change. Now the number of color-changes of the boundary squares between R' and G is at most $2n-1$ for some $n \geq 1$. The desired result follows from the induction hypothesis. □

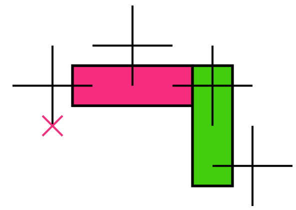


Figure 364: An example of a contradiction, showing this situation is impossible. The cross marks a cell that is violates monotonicity.

Compare the first part of this theorem with Theorem 271, where we showed that tilings with a rectangle and cross using at least one of each is impossible (if the rectangle is not a bar).

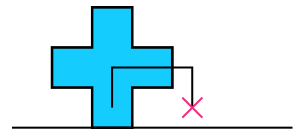


Figure 365: The second arm of the U-pentomino cannot be tiled by an X.

Theorem 329. *The X-pentomino is not compatible with the U- or V-pentominoes.*

[Not referenced]

Proof. We say that the cells of a polyomino in the first and last rows and columns **touch the hull**. In any tiling of X-pentominoes there must be a peak that touches the hull.

Consider such a peak.

First, for the U-pentomino. This pentomino can essentially fit only one way in the peak, and leads to a cell that cannot be covered by an X. Therefore, The X- and U-pentominoes are incompatible.

Second, for the V-pentomino (Figure 366). This pentomino can fit only one way in the peak. This forces a second V. There are two ways to place an X to cover the next uncovered cell of the second V. One way leads to a two cells that cannot be both covered by Xes, and therefore is impossible. The second method forces us to place another X, with a peak on the hull. This is the situation we started with, and must therefore continue forever. Therefore, no finite tiling is possible, and hence the X- and V-pentominoes are not compatible. □

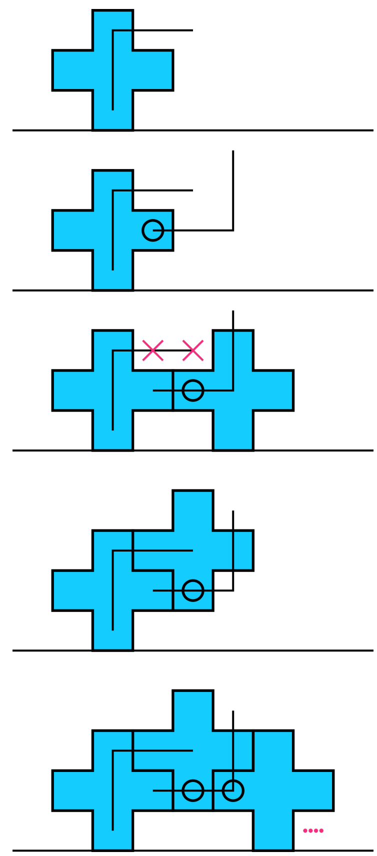


Figure 366: The tiling can never be completed.

There are many unproven conjectures on compatibility. What we know of the compatibility of pentominoes is shown in Tables 68 and 69.

Problem† 108. *Find tilings that realize the numbers shown in Table 67.*

Theorem 330 (Aydar Sayranov (un.) (2018))**.** *A region tileable by both the square and skew tetromino has a hole.*

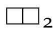
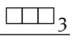
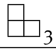
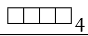
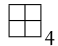
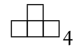
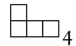
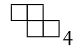
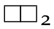
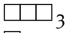
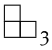
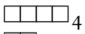
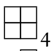
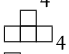
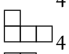
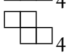
								
	1	3	3	2	2	4	2	2
	3	1	2	4	4	4	4	4
	3	2	1	4	4	4	4	4
	2	4	4	1	2	4	2	2
	2	4	4	2	1	4	2	4
	4	4	4	4	4	1	4	2
	2	4	4	2	2	4	1	2
	2	4	4	2	4	2	2	1

Table 67: Compatibility for small polyominoes. The number shows the number of tiles needed of the smallest tile. From Cibulis et al. (2002, p. 150).
Self

	F	I	L	N	P	T	U	V	W	X	Y	Z
F	1	10	2	2	2	2	4	4	2	2	2	2
I	10	1	2	2	2	4	12	4	10	×	2	20
L	2	2	1	4	2	2	2	2	2	44	2	2
N	2	2	4	1	2	2	2	2	2	16	2	2
P	2	2	2	2	1	2	2	2	2	4	2	2
T	2	4	2	2	2	1	4	2	14	4	2	2
U	4	12	2	2	2	4	1	2	2	×	2	4
V	4	4	2	2	2	2	2	1	6	×	2	4
W	2	10	2	2	2	14	2	6	1	?	2	4
X	2	×	44	16	4	4	×	×	?	1	2	?
Y	2	2	2	2	2	2	2	2	2	2	1	2
Z	2	20	2	2	2	2	4	4	4	?	2	1

Table 68: Table showing the least number the number of tiles necessary to tile the LCM (as far as is known). From Sicherman (2015b).
Self
Overall compatibility unknown
Overall incompatible

	F	I	L	N	P	T	U	V	W	X	Y	Z
F	1	10	2	2	2	2	4	6	2	2	2	2
I	10	1	2	2	2	32	?	10	10	×	2	?
L	2	2	1	4	2	2	2	2	2	×	2	2
N	2	2	4	1	2	2	2	2	2	16	2	2
P	2	2	2	2	1	2	2	2	2	4	2	2
T	2	32	2	2	2	1	?	2	16	4	2	30
U	4	?	2	2	2	?	1	?	2	×	2	?
V	6	10	2	2	2	2	?	1	6	×	2	4
W	2	10	2	2	2	16	2	6	1	?	2	10
X	2	×	×	16	4	4	×	×	?	1	2	?
Y	2	2	2	2	2	2	2	2	2	2	1	2
Z	2	?	2	2	2	30	?	4	10	?	2	1

Table 69: Table showing the least number the number of tiles necessary to tile the holeless LCM (as far as is known). From Sicherman (2015b).
Self
Overall compatibility unknown
Holeless compatibility unknown
Overall incompatible
Holeless incompatible
Overall best solution has holes

[Referenced on page 32]

Proof. [2] Let the common multiple be the region R, and apply the checkerboard coloring. Then in either of the two tilings (by squares or skew tetrominoes), every tetromino has exactly 2 white and 2 black cells. Let's connect cells of the same color covered by the same tetromino with an edge, these edges will form 2 perfect matchings. Let's take their symmetric difference. Obviously it's not empty. Let's take a cycle such that there is no other cycle completely inside it (but there can be some cycles intersecting it). Without loss of generality let's assume that this cycle is white. Then all cycles intersecting it are black. Notice that if there are two intersecting edges, then they correspond to some square tetromino. That means that if some black cycle intersects our cycle, then both intersecting edges are from the same matching. All the cycles are alternating, so we conclude that the number of vertices of that black cycle inside our cycle is even.

[2] This proof use some ideas from graph theory that fall outside the framework we set up. However, the result is important and the proof does not appear in print, so I think it is necessary to provide a version of it here.

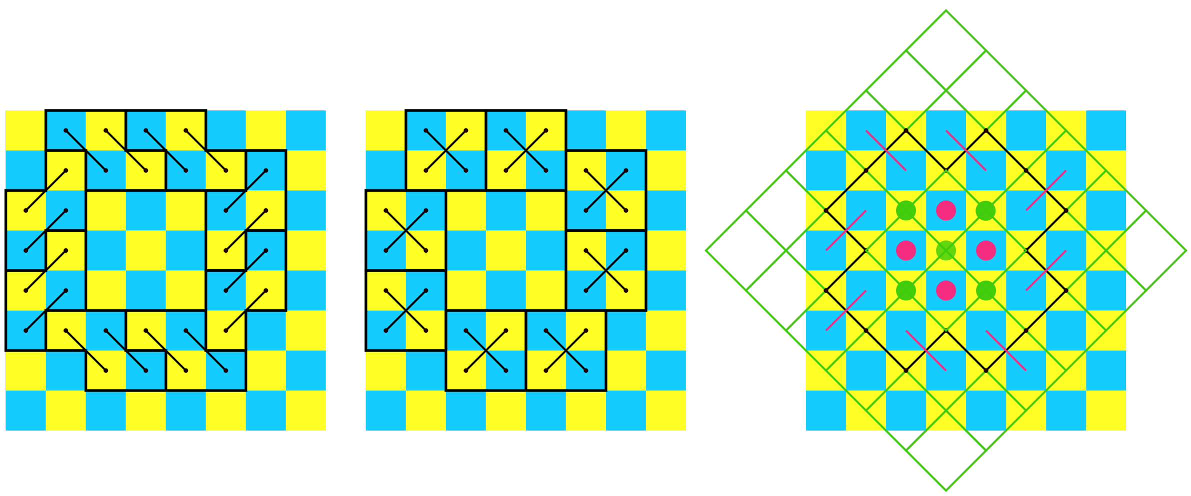


Figure 367: On the left two images we show the matching induced by the tetrominoes. On the right, we take the symmetric difference. Special cells are marked red, and interior vertices with green.

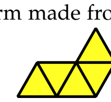
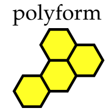
Now, let's change the coordinate system: rotate everything by 45° and let integer lattice points correspond to the white cells and unit squares correspond to black cells. Now our white cycle became a polygon with integer coordinates. Let the length of cycle be $2k$. Every second edge on the cycle is from 2×2-square covering. For every such edge there is corresponding black edge that connects 2 cells on which that edge lies. One of these cells lies inside the cycle, the other outside. So, we found k cells inside our cycle. Let's call them *special*. Let the number of non-special cells inside our cycle be x and the number of integer points inside the cycle be i. By Pick's theorem (Theorem 14), we have: $x + k = i + 2k/2 - 1 \Leftrightarrow i = x + 1$. This means that either i or x is odd. If i is odd, then we will not be able to find a perfect matching on those i vertices. If x is odd, we may match some of the cells with special cells. If we do that, it will mean that those cells lie on some black cycle that intersects our cycle. But we know that all such cycles have even number of vertices inside our cycle. So we will have some unmatched cell. In both cases we can't cover everything inside our cycle, and so there will be a hole. □

8.1.1 Further Reading

The basic idea of compatibility was introduced in a slightly more general setting in Golomb (1981). For more, see Cibulis et al. (2002), Barbans et al. (2003), Barbans et al. (2004), and Wainwright (2009). In Liu (2018b), the author generalizes the concept so that tilings may involve more than one polyomino, and he calls this *Tetris algebra*. This is also related to ideas in Reid (2003b), in particular, the restrictions on how many of each type of tile in a tiling of rectangle. See for example sections 3.8–3.14.

There are a few web sites that have large databases of compatibility and related problems:

- Resta (2009) contains a large number of solutions to the compatibility problem.
- Sicherman (2015b) contains solutions to pairwise and multiple compatibility for polyominoes and other polyforms[3]. It gives minimum solutions, minimum solutions with an odd number of tiles, minimum-width solutions, and minimum holeless solutions. It also gives minimal incompatibility regions.
- Zucca (1999) contains solutions to the compatibility problem using an odd number of tiles, and the *triple compatibility* problem — that is, finding the LCM of triplets of polyominoes. He also give some solutions on cylinders where planar solutions don't exist.

[3] A *polyform* is a shape made from polygons of the same type joined edge-to-edge; polyominoes are polyforms made from squares. A *polyiamond* is a polyform made from equilateral triangles , and a *polyhex* is a polyform made from regular hexagons . Polyforms are also called *animals*, although that term is also used for subsets of $\mathbb{Z}^2$ which are not necessarily connected.

8.2 Exclusion and Clumsy Packings

What is the least number of monominoes we can place inside $R(m,n)$ so we cannot place a polyomino P anywhere? What fraction of cells of the plane must be covered by monominoes so that a polyomino P cannot be placed anywhere?

These problem are introduced in Golomb (1996, Chapter 3), and we will call it **exclusion by monominoes**.

Theorem 331. *If P is an efficient polyomino, it can be excluded from the plane with $1/|P|$ monominoes.*

[Not referenced]

Proof. Suppose K is an optimal coloring for P. Then put a monomino wherever $K(x,y) = 0$. Since however we place a polyomino it must cover each color once, if there is no cells with color 0 available, it is impossible to place P anywhere. □

Among others, this gives us solutions for various rectangles (all those whose one side divides the other, including squares and bars) by Theorem 154.

Problem† 109. *Is the converse true; that is: if we can exclude a polyomino P from the plane with $1/|P|$ monominoes, is P efficient?*

The following allow us to get a lower bound on the exclusion fraction. Suppose a certain arrangement excludes P, and $P \subset Q$. Then Q is also excluded by the same arrangement.

Problem* 110. *Find exclusion patterns for the hexominoes.*

	n								
m	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	5	8	8	12	12	16	16
3	3	5	9	10	13	18	18	18	27
4	4	8	10	16	17	17	25	32	32
5	5	8	13	17	25	26	26	26	41
6	6	12	18	17	26	36	37	37	37
7	7	12	18	25	26	37	49	50	50
8	8	16	18	32	26	37	50	64	65
9	9	16	27	32	41	37	50	65	91

Table 70: Inverse frequency for rectangles. (Adapted from Klamkin and Liu (1980))

Next we look at excluding a polyomino by placing the same polyomino. We restate this problem in slightly different language used in Walzer et al. (2014). A **packing** of a region by set of polyominoes is a placement of copies from the set onto the region so that no polyominoes overlap, they are all contained within the region, and no more copies can be placed.

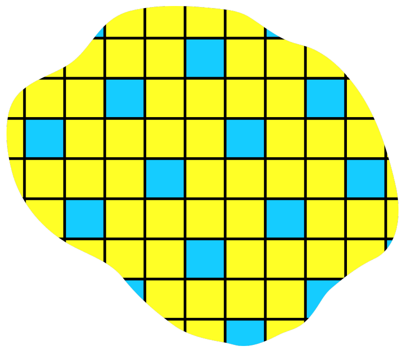
The **density** of a packing is the fraction of cells from the region that are covered. A (finite) packing with density 1 is a tiling. A packing of a region by a tileset with the lowest density is called a **clumsy tiling**[4], and the density of a clumsy tiling is called the **clumsiness**[5] of the region by the tile set, denoted $\mathrm{clum}_{\mathcal{T}}(R)$.[6]

The definition of density is intuitively defined as a limit. I will leave out the details (see Walzer et al. (2014)), but here are a few remarks:

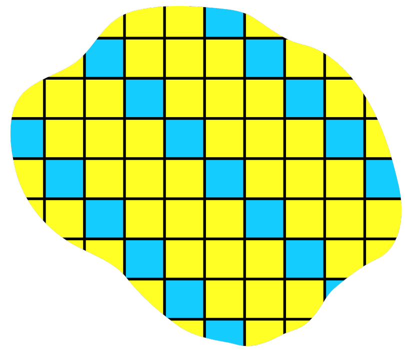
[4] Also called a *mistiling* in Goddard (1995). Finding a clumsy tiling of dominoes is called the *gunport problem* in Sands (1971), where the concept was introduced for the first time.

[5] Goddard (1995) call this *mistiling ratio*.

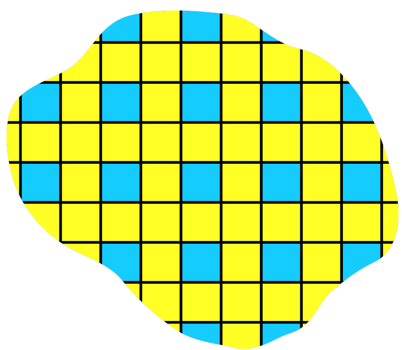
[6] Authors that study clumsy tilings of the plane by $\mathcal{T}$ use the notation $\mathrm{clum}(\mathcal{T})$. I use the notation here so that tile sets are consistently the subscript, as I have done throughout this book, and denote clumsy tilings of the plane by $\mathcal{T}$ by $\mathrm{clum}_{\mathcal{T}}(\mathbb{Z}^2)$.



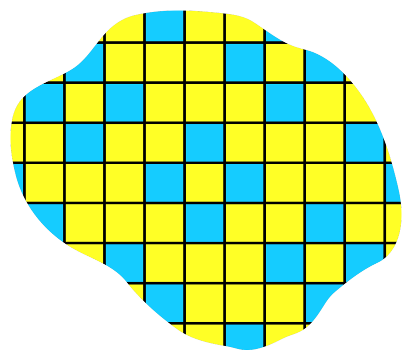
(a) Excludes X, I ($F_{5,2}$)



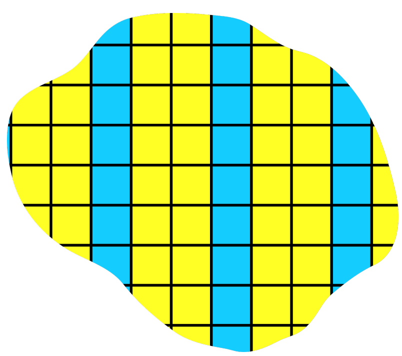
(b) Excludes L, T, U, Y (F_4)



(c) Excludes F, N, P, W (S_2)



(d) Excludes V ($F_{13,8}$)



(e) Excludes Z($F_{3,0}$)

Figure 368: Exclusion patterns for pentominoes. Except for the pattern for the V-pentomino, the pattern can be obtained from the coloring in brackets by putting a monomino at each cell with color 0. For the V-petomino, put a monomino where the color equals 0, 4, 6, and 10. From Golomb (1996, Fig. 67), Klamkin and Liu (1980, Fig. 3).

- The limit always exist for all periodic packings.
- The limit may not exist for aperiodic packings, and in this case the density is not defined.
- It is possible for a non-clumsy packing to have the same density as a clumsy packing. For example, remove a finite number of tiles from a clumsy packing. The resulting packing is not clumsy, but it still has the same density.
- For any aperiodic clumsy packing with density d, and any real $\epsilon > 0$, there is a periodic packing with density $d + \epsilon$.

Theorem 332. *Suppose we have a region R, a tileset $\mathcal{T}$, and that all placements of monominoes that exclude $\mathcal{T}$ from R have a density of d or higher. Then* $\mathrm{clum}_{\mathcal{T}}(R) \geq d$.

[Not referenced]

Proof. Take a clumsy tiling of R by $\mathcal{T}$, and replace each tile by monominoes. The density of the monominoes is $\mathrm{clum}_{\mathcal{T}}(R)$, but since it excludes $\mathcal{T}$, we must have $\mathrm{clum}_{\mathcal{T}}(R) \geq d$. □

Theorem 333 (Gyárfás et al. (1988), Proposition 2.1)**.** *In any packing of a rectangle by dominoes, there are at least as many dominoes as uncovered cells, and so the density of a clumsy packing is at least* $2/3$.

[Not referenced]

Theorem 334 (Goddard (1995), Theorem 2)**.** *The clumsiness of the right-tromino is* $6/11$.

[Not referenced]

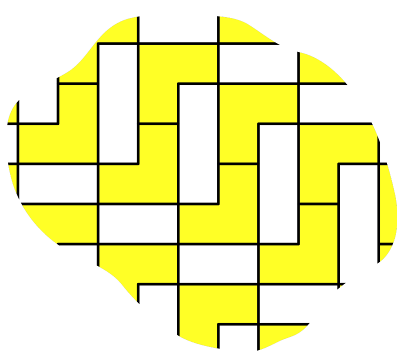


Figure 369: A clumsy packing of the plane by right trominoes (Goddard (1995)).

Theorem 335 (Goddard (1995), Theorem 3). *The density of clumsy tiling of the plane by $R(m,m)$ is given by*

$$\text{clum}_{R(m,n)}(\mathbb{Z}^2) = \frac{m^2}{(2m-1)^2}.$$

[Not referenced]

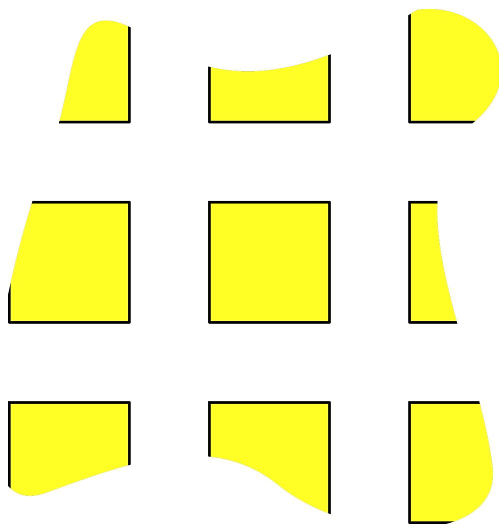


Figure 370: A clumsy packing of the plane by square nonominoes (Goddard (1995)).

Theorem 336 (Goddard (1995), Theorem 4). *The density of clumsy packings by a bar with k cells satisfies*

$$\text{clum}_{B_k}(\mathbb{Z}^2) \geq \frac{2}{k+1}.$$

[Not referenced]

Goddard (1995) (Conjecture 1) conjectures that for $k \geq 3$, the clumsiness of bars with k cells is given by:

$$\text{clum}_{B_k}(\mathbb{Z}^2) = \frac{2k}{k^2+1}.$$

Figure 371 shows the packing that realizes this density.

Theorem 337 (Walzer et al. (2014), p. 41). *A square ring P with side m and thickness 1, has clumsiness given by*

$$\text{clum}_P(\mathbb{Z}^2) = \frac{4(k-1)}{(2k-1)^2}.$$

[Not referenced]

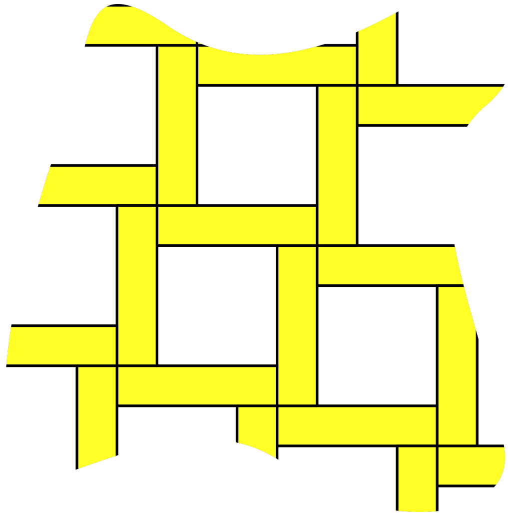


Figure 371: A conjectured clumsy packing by bars (Goddard (1995)).

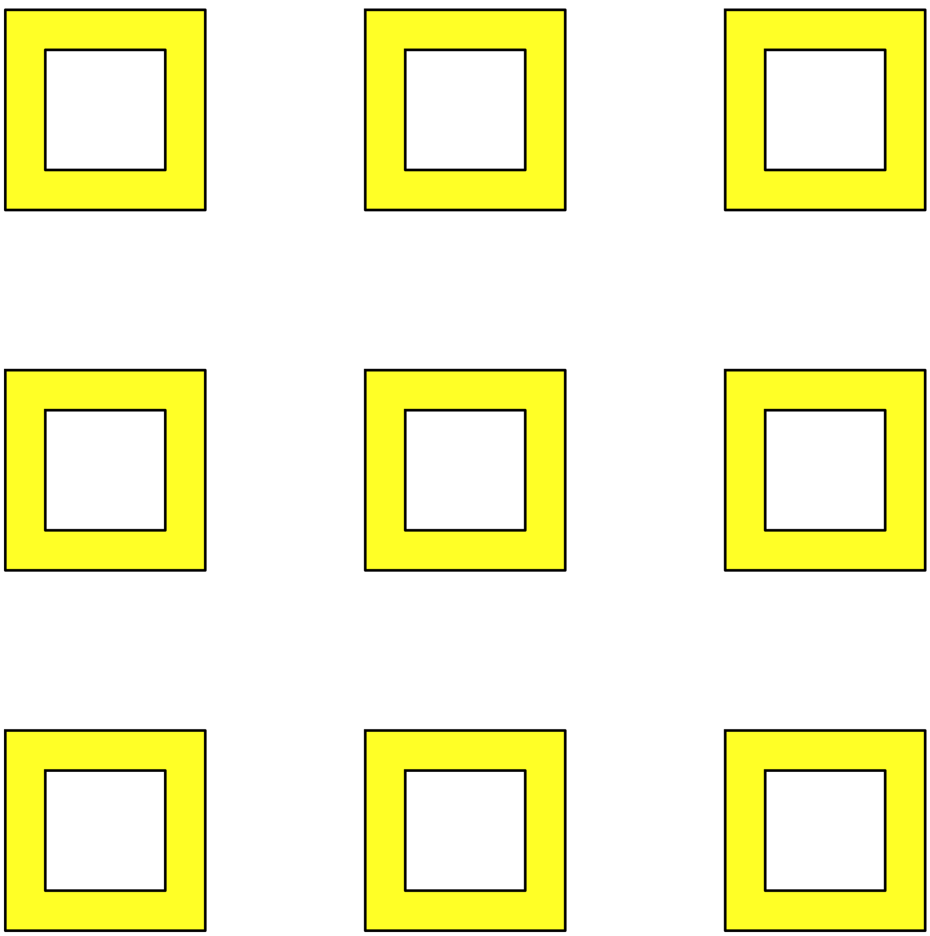


Figure 372: A clumsy packing by rings.

Theorem 338 (Walzer et al. (2014), Theorem 4)**.** *If $\mathcal{T}$ is a single polyomino with k cells, then*

$$\text{clum}_{\mathcal{T}}(\mathbb{Z}^2) \geq \frac{k}{k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2}.$$

[Not referenced]

Problem 111** (Walzer et al. (2014) Problem 3 from Section 7.5)**.** *What is the clumsiest polyomino with k cells?*

8.2.1 Further Reading

Exclusion by monominoes for pentominoes on rectangles has been discussed in Gravier and Payan (2001), Gravier et al. (2007) (which extends the concept to other types of graphs) and Zerovnik (2006). Klamkin and Liu (1980), Barnes and Shearer (1982), Shearer (1982b) and Shearer (1982a) treat the exclusion problem on the plane.

The gunport problem is briefly discussed in (Gardner and Richards, 2006, Problem 7.8), (reproduced from (Gardner, 1974, Problem 1),). Algorithms for solving it is given in Chlond and Bosch (2006) and Kind (2020). Friedman (2008, https://erich-friedman.github.io/mathmagic/0316.html) considers a slightly different formulation of clumsy tilings and gives many empirical results.

Miller et al. (2022) analysis clumsy packings for rectangles and other simply polyominoes in a finite space.

8.3 Fountain Sets

The monomino tiles any polyomino. What is the smallest set that can tile all polyominoes except the monomino?

We can form such a set as follows: add the domino. The domino cannot tile either tromino, so add those too. The tetrominoes can all be tiled by dominoes, except for the T-tetromino. Add that. All pentominoes can be cut into a domino and tromino, except for the X-pentomino. Add that. We now see all hexominoes can be cut into smaller polyominoes that are in our set; and so we are done. The set is shown in in Figure 8.3.

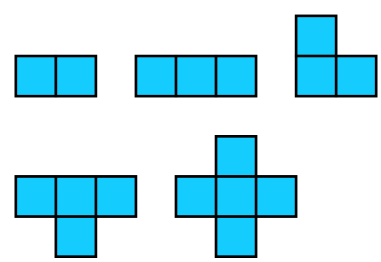


Figure 373: A fountain set. These polyominoes can tile all polyominoes, except the monomino.

In general, a **fountain set** is a set of polyominoes that can tile any polyomino formed from appending a monomino to a member of the set (Kenyon and Tassy, 2015). [7]

[7] As far as I can tell, this idea was first introduced in Mackinnon (1990), where these polyominoes are called *prime polyomino*es, and the problem is to find the set of polyominoes that cannot be decomposed into smaller polyominoes where each piece has at least $n = 2, 3, \cdots$ cells. The notion is explored in Rinaldi and Rogers (2008). There, these polyominoes are called *n-indecomposable*, and these are precisely the polyominoes of $\mathcal{F}_n$. Note that the first paper mistakenly gives $|\mathcal{F}_n| = 42$ instead if 32 (as the second paper points out).

Problem† 112 (Kenyon and Tassy (2015), Lemma 3.2)**.** *Prove that the set in Figure 8.3 is a fountain set.*

Fountain sets are often infinite. In fact, any fountain set that contains no bars is infinite.

Here is an algorithm for generating fountain sets from an given input set $\mathcal{T}_1$ (Kenyon and Tassy (2015), Algorithm 2) .

(1) Let $i = 1$

(2) For each polyomino that is a polyomino from the set with an appended monomino:

  (a) If it is not tileable by $\bigcup_{j<i} \mathcal{T}_j$, add it to $\mathcal{T}_{i+1}$.

(3) If $\mathcal{T}_{i+1}$ is not empty, $i \leftarrow i+1$, and repeat from step 2.

The set $\bigcup_j \mathcal{T}_j$ is a fountain set. We will use the notation $\mathcal{F}_n$ to denote the fountain set with input $\mathcal{P}_n$. $\mathcal{F}_1 = \{\square_1\}$, and $\mathcal{F}_2$ is the set of polyominoes in Figure 8.3.

Example 22 (Essentially also given in Rinaldi and Rogers (2008), Section 2)**.** *Suppose we start with $\mathcal{T}_1 = \mathcal{P}_3$. No tetromino is tileable by $\mathcal{P}_3$ (Theorem 1), and so we have $\mathcal{T}_2 = \mathcal{P}_4$, and similarly $\mathcal{T}_3 = \mathcal{P}_5$. Most hexominoes are tileable by trominoes; there are 5 that are not and these make $\mathcal{T}_4$. Notice that these all have a junction with 3 or 4 arms that are each less than three. This is in fact the characteristic of the remainder of the elements of the fountain set: they all have this property. There are 5 such heptominoes (that make $\mathcal{T}_5$), one octomino (that makes $\mathcal{T}_6$) and one nonomino $\mathcal{T}_7$.*

Let's prove the claim that the only polyominoes not tileable by the set are those described. Since all trominoes, tetrominoes, and pentominoes are in the set, lets assume the polyomino has at least 6 cells.

If no cell in a polyomino has more than two neighbors (i.e. it's a snake polyomino) then we can partition the polyomino into smaller snakes such that each part has between 3 and 5 cells. These snakes are in one of $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$, and therefore tileable.

Suppose then we do not have a snake. We can make a cut to split the polyomino into two. It is either possible to make a cut so that each piece is bigger than two cells, or it is not. The only polyominoes with 6 or more cells that cannot be split into pieces all larger than 2 cells are the ones shown in Figure 374.

Assume then we can make a cut so that the two pieces are have more than two cells.

If both pieces are smaller than 6 cells, they are in $\mathcal{F}_3$ and we are done.

Otherwise, we repeat the procedure with each that has 6 cells or more.

In the case where every cut leads to one component smaller than three cells, the piece must be one of the junction pieces in $\mathcal{T}_4, \cdots, \mathcal{T}_7$, and therefore it must be tileable.

The size of this fountain set is given by $|\mathcal{F}_3| = 2+5+12+6+5+1+1 = 32$. The polyominoes from this set with 6 or more cells are shown in Figure 374.

See also Example 2.

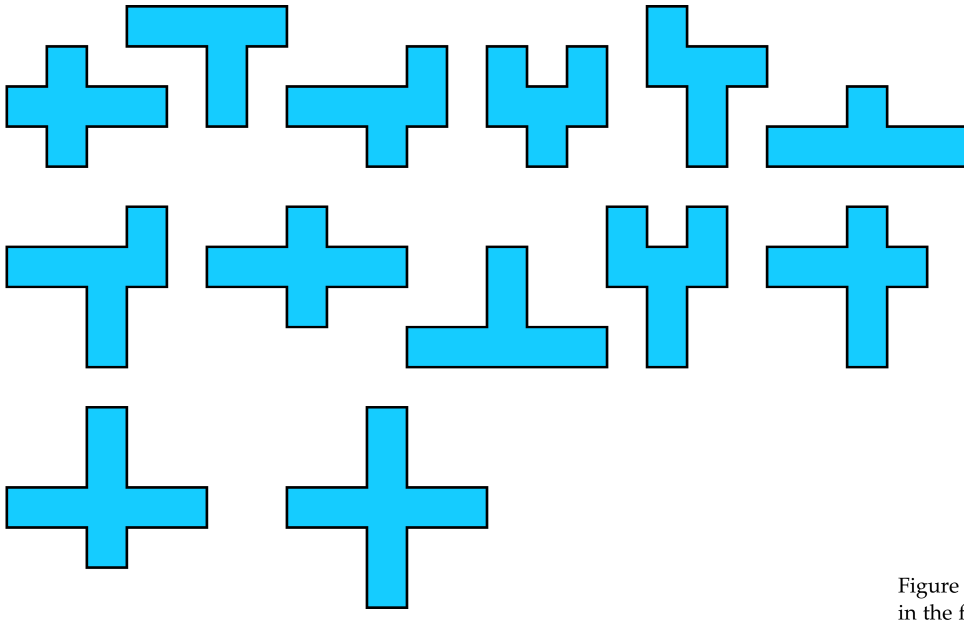


Figure 374: These are the polyominoes in the fountain set $\mathcal{F}_3$ with 6 or more cells.

Problem† 113.

(1) *Determine the fountain set that results when we start with the right tromino.*

(2) *Determine the fountain set that results when we start with the straight tromino.*

(3) *Determine $\mathcal{F}_4$.*

Table 71 shows how the standard fountain sets are composed.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	$\lvert\mathcal{F}_n\rvert$ A125709
1	1																	1
2	0	1	2	1	1													5
3	0	0	2	5	12	6	5	1	1									32
4	0	0	0	5	12	35	108	73	76	80	25	15	15					444
5	0	0	0	0	12	35	108	369	1285	1044	1475	2205	2643	983	1050	1208	958	13375

Table 71: This table shows how many polyominoes of each cell count make out fountain sets $\mathcal{F}_n$. This is sequence A125709.

We can extend the idea of fountain sets in several ways, as explored in the following problems:

Problem* 114.

(1) *What is the smallest set of polyominoes can tile all*

  (a) *but a finite set of polyominoes? (Pick some small sets to explore.)*

  (b) *polyominoes that have area divisible by n?*

(c) *polyominoes that are balanced for some coloring K?*

(2) *What sets can tile all elements of the set appended with*

(a) *a domino?*

(b) *some other polyomino?*

(c) *some other set of polyominoes?*

8.4 *Types of Convex Polyominoes*

We have seen many problems that are unsolved for polyominoes in general. Problems can sometimes be solved for restricted classes of polyominoes. The polyominoes in this section make it possible to solve enumeration (discussed in Chapter 9) and tomography problems[8].

[8] Essentially, determining the shape of a polyomino, or the structure of a tiling, from information of the projections. See for example Chrobak and Dürr (1999).

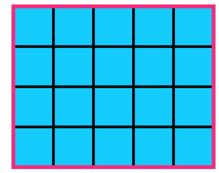
A polyomino is **row convex**[9] if there are no gaps between the endpoints of each row; that is, there is a horizontal path between endpoints of each row. Similarly, a polyomino is **column convex**[10] if there are no holes between end points of a column; that is, there is a vertical path between end points in each column. (Delest and Viennot, 1984)

[9] Also *horizontally convex* or *h-convex*.

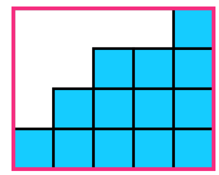
[10] Also *vertically convex*or *v-convex*.

A polyomino that is both row and column convex is called **convex**[11] (Delest and Viennot, 1984). This notion is convexity is different from classical convexity: a region is **classically convex** if the line segment between any two points inside the region is completely contained in the figure.

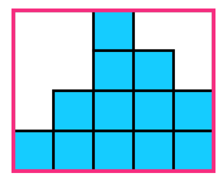
[11] Also *hv-convex*.



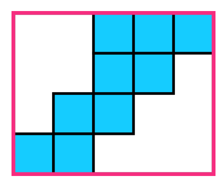
(a) A rectangle



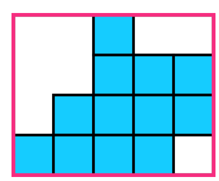
(b) A Young diagram



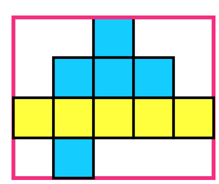
(c) A stack polyomino



(d) A parallelogram polyomino



(e) A directed polyomino



(f) An h-centered polyomino

Figure 375: The way various polyomino types touch their hull.

Problem† 115.

(1) *Prove that if a polyomino is classically convex, it is a rectangle.*

(2) *Show that row-convex (or column-convex) polyominoes cannot have holes.*

(3) *What is the smallest simply-connected polyomino that is neither row nor column convex?*

(4) *Show that a polyomino is convex if and only if it does not have any anti-knobs.*

(5) *How can you recognize a convex polyomino from its border word?*

A path P_i is called **monotone** if all differences $P_{i+1} - P_i$ are only two of the four possibilities $\{(0,1),(1,0),(0,-1),(-1,0)\}$ (Castiglione and Restivo, 2003).

Theorem 339 (Castiglione and Restivo (2003), Proposition 1)**.** *A polyomino is convex if and only if there is a monotone path between each two cells of the polyomino.*

[Not referenced]

Proof.

If. Suppose there is a monotone path between each pair of cells. Then there is also a monotone path between the end points of each row. Suppose this path is not completely horizontal. Then it must either go up or down; WLOG let us say it goes up. But the only way it goes to the other endpoint if it goes down again. But then it goes right, up and down, so it is not monotone. If follows that the path must be horizontal, which means it must contain all the cells between the endpoints, and therefore the polyomino is row convex. A similar argument shows it is also column convex. Therefore, it is convex.

Only if. Suppose a polyomino is convex. Since the polyomino is connected, there is a path between any two cells. If the path is monotone, we are done. If it is not monotone, there is a section of the path that connects two cells in a row or column. We can replace this section with a completely horizontal or vertical path, since the polyomino is convex. We can repeat this, until we have a monotone path between the two cells. □

If a polyomino has a single cell that is connected by a monotone path to every other cell (using the same two directions), the polyomino is a **directed polyomino**. A **parallelogram polyomino** is polyomino that lies between two non-intersecting monotone paths between two cells (Delest, 1991).

Problem† 116. *Show that*

(1) *Young diagrams, stack polyominoes, and parallelogram polyominoes are directed polyominoes.*

(2) *(All but the first from Castiglione and Restivo (2006))*

   (a) *A rectangle is a convex polyomino that contains all four corners of its hull.*

   (b) *A Young diagram is a convex polyomino that contains three corners of its hull.*

   (c) *A stack polyomino is a convex polyomino that contains two adjacent corners of its hull.*

   (d) *A parallelogram polyomino is a convex polyomino that contains two opposite corners of its hull.*

   (e) *A directed polyomino is a convex polyomino that contains one corner of its hull.*

Problem† 117.

(1) *Show that a simple polyomino is convex if and only if it has four knobs.*

(2) *Show the perimeter of polyomino is the same as that of its hull if and only if it is convex (Bousquet-Mélou et al. (1999) use this to define convex polyominoes).*

(3) *Suppose the hull of a convex polygon has width w and height h, with $w < h$. Show that the number of flats F satisfy:*

  (a) $F \leq 2w - 2$, *if the polyomino is a Young diagram. Notice that this implies that if $w < h$, then two rows must be equal. This overlaps with Theorem 44; in fact we have two rows or two columns equal unless the bounding box is a square and we have a triangle.*

  (b) $F \leq 2w$, *if the polyomino is a stack polyomino with vertical base.*

  (c) $F \leq 2w - 2$, *if the polyomino is a stack polyomino with a horizontal base.*

  (d) $F \leq 4w - 4$, *otherwise.*

In a simple path, if $P_{i+1} - P_i \neq P_{i+2} - P_{i+1}$, we say there is a **direction change** at node $i + 2$. If there is a monotone path between each two cells with at most k direction changes, we say the path is ***k*-convex**.

If $k = 0$, the polyomino is a bar. If $k = 1$, we call the polyomino **L-convex** (Castiglione and Restivo, 2003). If $k = 2$, we call the polyomino **Z-convex** (Duchi et al., 2008, Sec. 1.3).

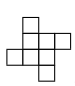
Problem† 118.

(1) *Classify pentominoes according to their convexity, that is: which pentominoes are k-convex, and what are their k values?*

(2) *Show that polyominoes that belong to $\mathcal{C}_6$ (polyominoes with 6 corners) are L-convex.*

(3) *What polyominoes in $\mathcal{C}_8$ (polyominoes with 8 corners) are not Z-convex?*

For any L-convex polyomino P, there is a smallest polyomino $Q = B(1^m, (n-1))$ that covers every path between any two points, and therefore protects P, and so $C(P) \leq C(Q) = C(m, n)$ by Theorems 146 and 149.

For example, the polyomino  $_8$ is protected by $B(1^3 \cdot 3)$, and therefore gives an upper bound on chromatic number as $C(3, 4) = 15$, which is better than we get by the hull (which is $C(4, 4) = 16$).

Each extreme row or column of a convex polyomino is contiguous, and is called a **foot** (Tawbe and Vuillon, 2011, Definition 2 (modified)). Note that a cell touches the hull if and only if it is part of a foot.

Two rows of a polyomino **intersect** if they have two cells in the same column. Two columns intersect if they have two cells in the same row.

If the horizontal feet intersect, the polyomino is **v-centered** (see Fig 376). If vertical feet intersect, the polyomino is **h-centered** (Tawbe and Vuillon, 2011, Definition 3).[12]

[12] These notions where defined slightly differently in Chrobak and Dürr (1999) that first introduced them. There, a polyomino is called centered if it is convex and has a row with the same number of cells as the width of the bounding box. The two definitions are equivalent.

Problem† 119.

(1) *Show that h-centered polyominoes are stack polyominoes, or can be partitioned into two stack polyominoes.*

(2) *Are balanced centered polyominoes tileable by dominoes?*

Theorem 340. *Any L-convex polyomino is centered.*

[Not referenced]

Theorem 341 (Tawbe and Vuillon (2011), Proposition 2)**.** *If a polyomino is h- or v-centered, the polyomino is Z-convex.*

[Not referenced]

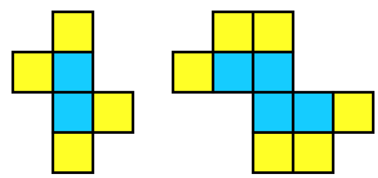
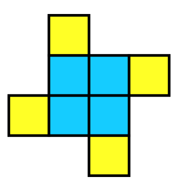
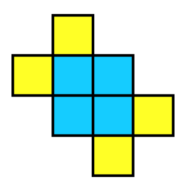
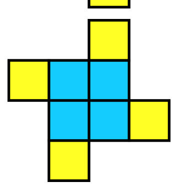
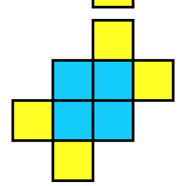


Figure 376: Two examples of h-centered convex polyominoes.

Problem† 120. *Show the converse does not hold.*

When a convex polyomino is not centered, it can be one of essentially two classes, depending on how the feet are situated relative to each other: **windmill** and **bone** polyominoes. In the table we we use N, E, etc. for the northern foot, eastern foot, etc., and $N < S$ means all the cells of the northern foot are to the left of the southern foot.

Windmill		Bone	
$N < S$ and $W < E$		$N < S$ and $W > E$	
$N > S$ and $W > E$		$N > S$ and $W > E$	

Theorem 342 (Tawbe and Vuillon (2011), Proposition 3 and 4)**.** *Windmill polyominoes are Z-convex.*

[Not referenced]

The **horizontal projection** $\mathcal{H}(P)$ of a polyomino P is a vector that counts the number of cells in each column. Similarly, the **vertical projection** $\mathcal{V}(P)$ is a vector counting the number of cells in each row.

Theorem 343 (Castiglione et al. (2005b), Lemma 1)**.** *There is a unique L-convex polyomino with given projections.*

[Not referenced]

Theorem 344 (Castiglione and Restivo (2004), Theorem 4)**.** *The set of L-convex polyominoes form a well-order under inclusion.*

[Not referenced]

Theorem 345 (Castiglione et al. (2005a), Equation 9)**.** *The number $f(n)$ of fixed L-convex polyominoes with perimeter $2n+4$ satisfies the recurrence $f(n) = 4f(n-1) + 2f(n-2)$ for $n \geq 3$, with $f(0) = 1$, $f(1) = 2$, $f(2) = 7$.*

[Not referenced]

Two rows are **comparable** when the set of y-coordinates of one row is contained in the set of y-coordinates of the other. A similar definition can be made for rows. A polyomino where every two rows and every two columns are comparable is called **intersection-free**[13] (Jonsson, 2005, Definition 2).

[13] This terminology is unfortunate; it does not mean rows or columns do not intersect in the sense of intersection we defined earlier.

8.5 *Signed Tilings and n-fold Tilings*

Our notion of what it means for a region to be tileable by a set of tiles does not allow tiles to overlap.

We now introduce a different notion of tiling that assigns weights to tiles, and allows overlaps. Each point in the lattice now has a value, which is the sum of the weights of all tiles that cover it. A region is said to have a **signed tiling** if the value of each point inside the region is 1, and 0 for each point outside it (Conway and Lagarias, 1990, p. 186). This notion of tiling corresponds with our earlier notion if we do not allow negative weights for tiles.

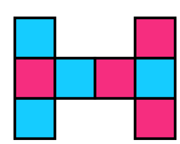
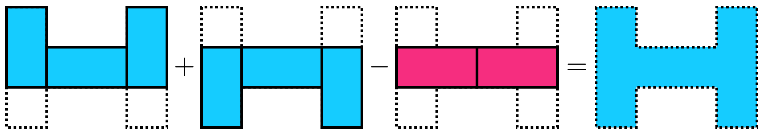


Figure 377: A region with no conventional tiling by dominoes.

For example, consider the region shown in Figure 377. We saw that this region is not tileable by dominoes (p. 67). However, it does have a signed tilling, as shown below:



Theorem 346. *The following criteria hold for signed tilings:*

(1) *(Area criterion for signed tilings) If we have a tile set where all tiles have area n, then we can only tile regions whose area is divisible by n.*

(2) *(Row criterion for signed tilings) Suppose our region have e_i cells in each row, and our tiles have e_i' cells in each row. If no rotation is allowed, then a tiling can exist only if $e_i = \sum_j n_j e_j'$ for some $n_i \in \mathbb{Z}$.*

[Referenced on page 359]

Problem† 121.

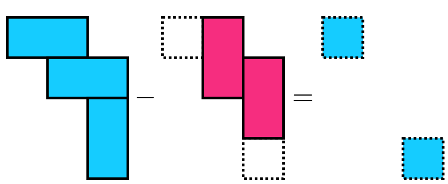
(1) *Prove the theorem above.*

(2) *Formulate an knob criterion for signed tilings.*

Theorem 347. *There is a signed tiling for any any two cells of opposite color in the plane by dominoes.*

[Not referenced]

Proof. Choose any path that starts with the one cell and ends with the other. This path has an even number of cells (since its ends are opposite colors, by Theorem 54), and hence it can be tiled (Theorem 55). The path without the two cells is also even (by the same argument) and can therefore also be tiled. A signed tiling is given by tiling the path *with* end points with positive dominoes, and the path without endpoints with negative dominoes. □



Theorem 348. *A region has a signed tiling by dominoes if and only if it is balanced.*

[Not referenced]

Proof.

If. Since the region is balanced, we can pair black and white squares. By Theorem 346 each pair has a signed tiling, and therefore the entire region has a signed tiling.

Only if. Each domino in a signed tiling contributes contributes an equal amount of back and white cells to the the tiling, whether that amount is positive or negative 1. Therefore, the total of black and white cells in the tiling must be equal. □

Problem† 122. *Find signed tilings for all the balanced regions in Figure 30.*

Example 23. *There are regions with signed tilings by dominoes that require more than two tiles per cell. One such region is shown in Figure 378. Note that no matter how we place a domino in the center tile, it must also cover one of its neighbors. Since these are not part of the region, we must cover this neighbor with a negative tile. But then we cover one of* its *neighbors, and since this* is *part of the region, we need at least two dominoes on top of the negative one. The region shown is not connected, but it's not difficult to construct a connected region using the same principle.*

There is another way to view this. Given a balanced set of points, we cannot always construct a path from between each white cell to a black cell without some of the paths overlapping. Otherwise, it would always be possible to use the construction of the previous theorem to get a signed tiling, and since we can make it so the paths don't cross, we would need at most two tiles to cover a cell.

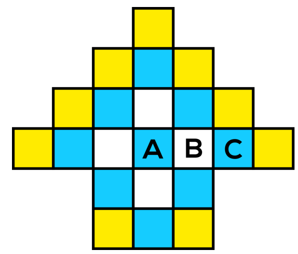


Figure 378: A region with a signed tiling by dominoes where at least one cell will have more than three dominoes covering it. To cover cell A, we need to cover one of its neighbors too (say cell B for example). But B is not part of the region, so we need a negative domino to cover it. This domino will also cover one of Bs neighbors (say cell C for example). This will give C a negative score, so we need at least two dominoes to cover it, giving us at least three dominoes in total to cover C.

Since signed tilings include unsigned tilings, it follows that if a region has no signed tilings, it also has no unsigned tilings.

Theorem 349. *If R has a signed tiling, then so does $\mathbb{Z}^2 - R$, that is, the entire plane with a hole the same shape as R.*

[Not referenced]

Theorem 350. *If two non-overlapping regions have signed tilings, so does their union.*

[Not referenced]

Theorem 351. *If two regions have signed tilings, and their intersection has a signed tiling, then so does their union.*

[Not referenced]

Let's fix a tile set $\mathcal{T}$. Two cells C and C' are equivalent (under $\mathcal{T}$) if there is a signed region R tileable by $\mathcal{T}$such that $C + R = C'$.

Example 24. *If $\mathcal{T}$ is the domino, and we apply the checkerboard coloring to the plane, then all white cells are equivalent, and all black cells are equivalent. No black cell is equivalent to any white cell.*

Theorem 352. *The equivalence we defined above is indeed an equivalence relation.*

[Not referenced]

Proof. $C + \emptyset = C$, so $C \sim C$. If $C + R = C'$, then $C' - R = C$, and so $C \sim C'$ implies $C' \sim C$. If $C + R = C'$ and $C' + R' = C''$, then $C + R + R' = C''$, so $C \sim C'$ and $C' \sim C''$ implies $C \sim C''$.

Thus, the relation $\sim$ is reflexive, symmetric and transitive, and therefore an equivalence relation. □

It follows that the equivalence relation induced by a tile set partitions any set into equivalence classes.

Theorem 353. *Suppose a region R is composed from distinct cells $C_1 \cdots C_n$, and F' is a region composed from distinct cells $C'_1 \cdots C'_n$, and for all i $C_i \sim C'_i$. If R has a signed tiling, so does R'.*

[Not referenced]

Theorem 354. *A region has a signed tiling if and only if it is equivalent to a region with a non-signed tiling.*

[Not referenced]

Theorem 355. *Suppose $C \sim C'$, and that $C + (m, n) = C'$. Then for any k, the following cells are also equivalent to C.*

- $C + k(m, n)$
- $C + k(-m, n)$
- $C + k(m, -n)$
- $C + k(-n, m)$
- $C + k(-m, -n)$
- $C + k(n, -m)$

[Not referenced]

Theorem 356. *If $C \sim C'$ and C and C' are neighbors, then $C \sim D$ for any $D \in \mathbb{Z}^2$.*

[Not referenced]

Theorem 357. *If $m, n \neq 0$ and $\gcd(m, n) = 1$, and $C \sim C + (m, n)$, then $C \sim D$ for any $D \in \mathbb{Z}^2$.*

[Not referenced]

Theorem 358. *If a tile set only has one induced equivalence class, all regions that satisfy the area criterion (Theorem 1) have signed tilings by the set.*

[Referenced on page 362]

Proof. Consider a region R. Arrange the tiles in any region R' with the same area as R. Now since all cells are equivalent, the regions are equivalent, and since R' has a tiling, so does R. □

Example 25. *Any region with area divisible by 3 has a signed tiling by right trominoes.*

To see why, note that neighbors are equivalent (see Figure), and therefore all cells in the plane is equivalent, and so by Theorem 358 all regions whose area is divisible by 3 can be tiled.

Theorem 359 (Reid (2003b)). *Let R be a region that does not have a signed tiling by a tileset $\mathcal{T}$. Then there is a rational coloring of all cells with such that:*

(1) *Any placement of a tile covers a total that is an integer, and*

(2) *The total covered by the region is not an integer.*

[Not referenced]

8.6 *Tiling Triangles*

[14] Recall that a *triangle* $T(n)$ is the bar graph $B(n \cdot (n-1) \cdot (n-2) \cdots 1)$. Triangles can be dissected into smaller triangles, and this reminds of tilings of rectangles by rectangles (discussed in Section 5.1). They are more difficult to tile (any set that tiles a triangle tiles a rectangle, but not vice versa), and have additional restrictions that allows us to rule out certain possibilities. Yet, it is still hard to find whether a set can tile any triangle. For example, we do not know whether $T(4)$ tiles any larger triangle.

[14] This section is based on Friedman (2006).

$\lvert T(n)\rvert$	$\dfrac{n(n+1)}{2}$	$1+2+\cdots+n=\dfrac{n(n+1)}{2}$
$\lvert\Delta T(n)\rvert$	$\left\lceil\dfrac{n}{2}\right\rceil$	Theorem 43

Table 72: Useful formulas for triangles.

We call the cells along the diagonal edge of the triangle—those with coordinates $(i, n-i-1)$— **diagonal cells**, and their neighbors that fall inside the triangle **subdiagonal cells**. In a triangle there is one diagonal cell more than subdiagonal cells.

The cells on the bottom row are called bottom cells, and the cells inside the triangle above them are called **sub-bottom**. In a triangle, there is one more bottom cell than sub-bottom cells.

A polyomino has the **diagonal property** if it can cover more diagonal cells than subdiagonal cells. The monomino, the right tromino, and the W-pentomino are the only polyominoes with 5 cells or less that satisfy the diagonal property.

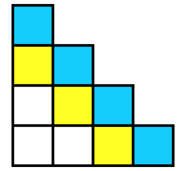


Figure 379: If a polyomino can cover more blue cells than yellow cells it has the diagonal property.

A polyomino has the **bottom property** if it can cover more bottom cells than sub-bottom cells. Examples include the T-tetromino, L-pentomino, and P-pentomino .

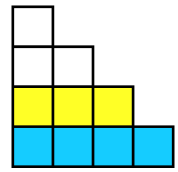


Figure 380: If a polyomino can cover more blue cells than yellow cells it has the bottom property.

Theorem 360 (The diagonal criterion, Friedman (2006)). *If a tile set tiles a triangle, at least one polyomino must satisfy the diagonal property.*

[Referenced on page 367]

Proof. Since a triangle have more diagonal cells than subdiagonal cells, it will be impossible to cover all diagonal cells if all subdiagonal cells are covered only with tiles without the diagonal property. □

Theorem 361 (The bottom criterion, Friedman (2006)). *If a tile set tiles a triangle, at least one polyomino must satisfy the bottom property.*

[Referenced on page 367]

Proof. Since a triangle have more bottom cells than sub-bottom cells, it will be impossible to cover all bottom cells if all sub-bottom cells are covered only with tiles without the bottom property. □

If the the hull of a polyomino is not square, we call the polyomino **non-squarish**. Non-squarish polyominoes can be oriented horizontally or vertically depending on whether the longest dimension of the hull is horizontal or vertical.

Theorem 362 (The orientation criterion, Friedman (2006)). *In a tiling of a triangle by a set $\mathcal{T}$, the non-squarish polyominoes along the diagonal, from top to bottom, start vertical and end horizontal, and there must be an odd number of flips between vertical and horizontal. In particular, there must be at least one flip.*

[Referenced on pages 363 and 367]

Proof. The top corner must be filled with a vertically oriented polyomino; the other corner must be filled by a horizontally oriented polyomino. Since the orientations are different, the number of transitions along the diagonal from one to the other must be odd. □

We can use this to show that a set cannot tile a triangle.

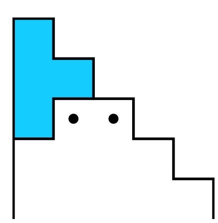


Figure 381: All the T-tetrominoes on the diagonal must be vertically oriented, proving it cannot tile any triangle.

Example 26. *The T-tetromino can only fit one way in the top corner, in a vertical orientation. The resulting gap in turn can only be tiled one way, with both tetrominoes vertical. The new gap is identical to the first, and we conclude that all tetrominoes on the diagonal must be vertical. But if a tiling exists, this is impossible (Theorem 362), and so no tiling exists.*

Problem† 123 (Friedman (2006))**.** *Suppose all the polyominoes in a set but one are balanced; and the unbalanced polyomino has absolute deficiency k.*

Show that if a tiling by the set exists for $T(n)$, then

(1) $k \mid |\Delta T(n)|$

(2) *The tiling require at least* $\dfrac{|\Delta T(n)|}{k}$ *copies of the unbalanced polyomino.*

(Hint: use Theorem 143.)

Only three examples of polyominoes are known that can tile triangles (with more than one copy): $T(1)$ (the monomino), $T(2)$ (Fig. 382) and $T(3)$ (Fig. 384). It seems likely that all triangles can tile triangles larger than themselves.

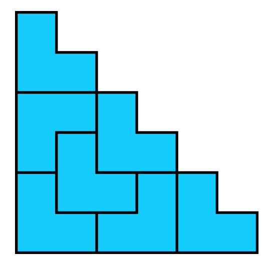


Figure 382: A tiling of $T(6)$ by $T(2)$.

Problem* 124.

(1) *Are all polyominoes that can tile triangles themselves triangles?*

(2) *What polyominoes can tile triangles?*

Problem† 125.

(1) *Show that if a set tiles $T(n)$, then it tiles $R(n, n+1)$, and therefore also all rectangles that are large enough.*

(2) *Show if a set tiles $T(n)$ and $T(n+1)$, it tiles $T(k(n+1))$ for all k.*

(3) *Give examples of polyominoes that are rectifiable, but cannot tile triangles. Are there examples where at least one knob has length 1?*

It is possible to extend triangle tilings, similar to how tilings can be extended for rectangles.

Theorem 363.

(1) *If a set tiles $T(m)$, $T(n)$ and $R(m,n)$, it also tiles $T(m+n)$.*

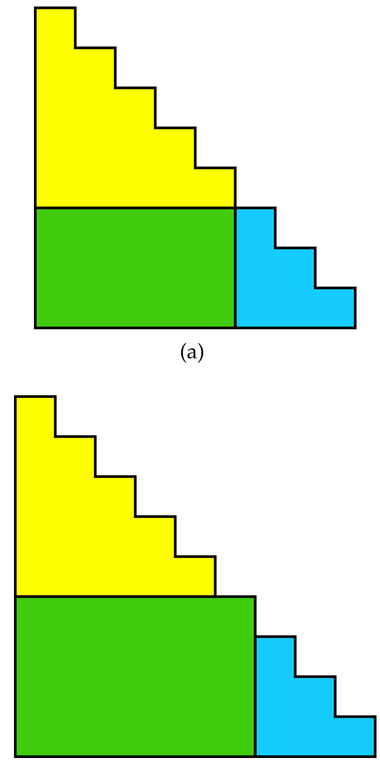
(2) *If a set tiles $T(m)$, $T(n)$ and $R(m+1, n+1)$, it also tiles $T(m+n+1)$.*

[Referenced on page 364]

Proof. The image in Fig. 383 shows the arrangement that proves the result. Triangle tilings can be extended similar to rectangle tilings. □

A triangle is called a **prime triangle** (with respect to a tiling set) if it does not have a tiling that can be decomposed as described in Theorem 363. Prime triangles of sets of triangles have exactly two faults, created by the two tiles in each of the diagonal corners.

Problem* 126. *Can you find tilings of any non-prime triangles with exactly two faults (where we know tilings to exist)?*



(a)

(b)

Figure 383: These arrangements show that if $T(m)$ (blue), $T(n)$ (yellow), and a $R(m,n)$ (green) is tileable, then so is $T(m+n)$; and similarly if $T(m)$ (blue), $T(n)$ (yellow), and a $R(m+1,n+1)$ (green) is tileable, then so is $T(m+n+1)$.

8.6.1 *Triangles tiled by triangles*

Theorem 364. *Except for the two triangles in diagonal corners, all tiles that cover an edge cell are part of a rectangular subtiling.*

[Referenced on page 365]

Theorem 365. *In a prime triangle, all the rectangular subtilings along the edges cannot all be the same orientation.*

[Referenced on page 365]

Proof. If they were, they would make an additional fault, and the triangle would not be prime. □

In a tiling of a triangle by triangles, we call a tile on the diagonal **aligned** if all the diagonal cells of the tile are on the diagonal of the region; a tile on the diagonal with only one cell on the diagonal of the region is called **misaligned**.

Theorem 366. *In a tiling of $T(n)$ by $T(m)$, misaligned tiles lie multiples of m apart, and they cannot lie on the corners, and therefore the number of them is no larger than* $\left\lfloor \frac{n-m}{m+1} \right\rfloor$.

[Referenced on page 365]

Proof. Misaligned tiles cannot lie next to each other; therefore they must be separated by aligned tiles, which each covers m diagonals cells. Thus, the distance between the misaligned tiles must be a multiple of m. □

Problem* 127. *Can you use the coloring and argument of Theorem 231 (and generalizations) to deduce further constraints on triangular tilings by triangles?*

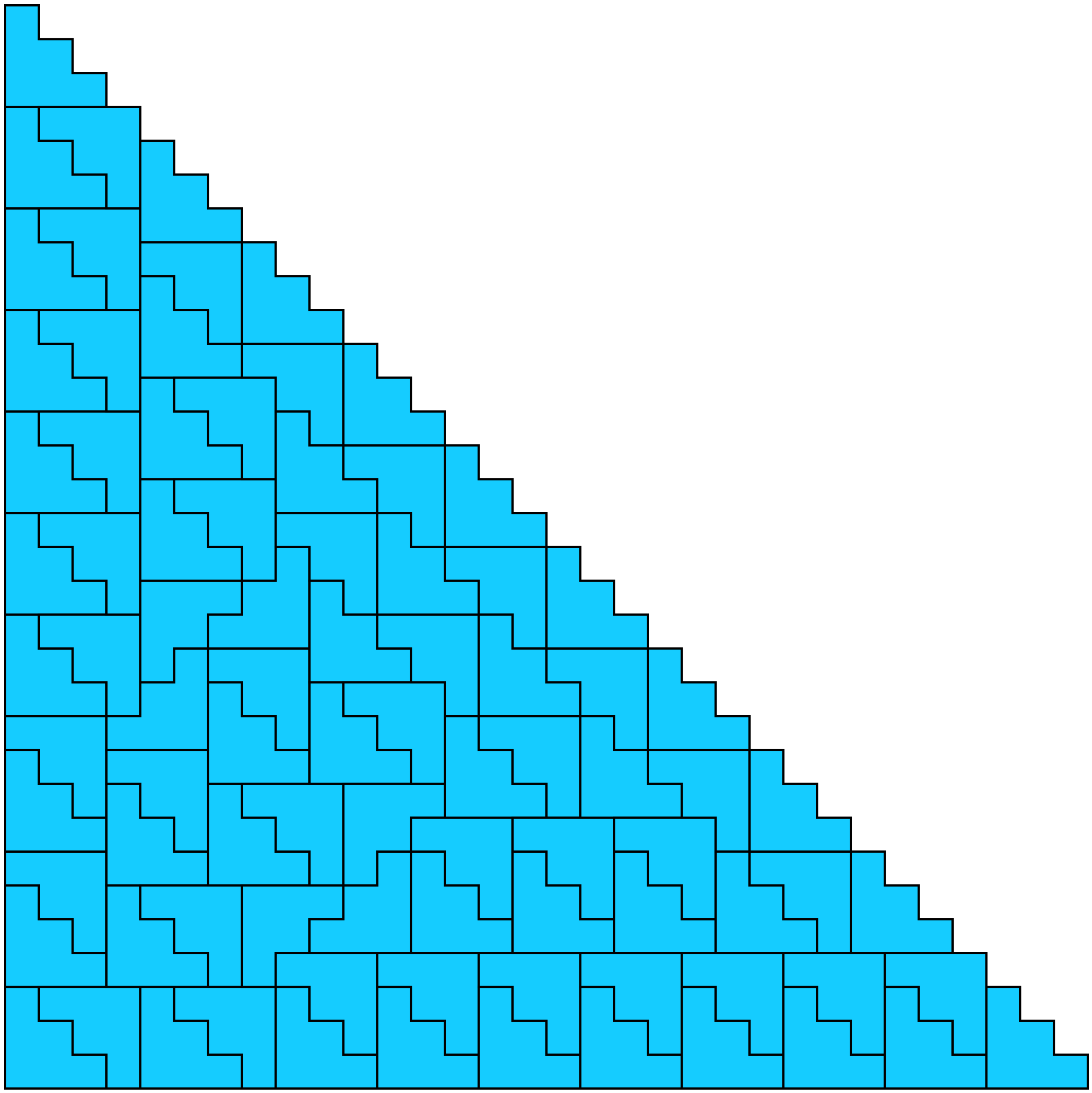


Figure 384: A tiling of $T(32)$ by $T(3)$.

Constraint	Conditions	Theorem
All Triangles		
$m(m+1) \mid n(n+1)$		1
$\left\lceil \frac{m}{2} \right\rceil \mid \left\lceil \frac{n}{2} \right\rceil$		143
$2m + km + \ell(m+1) = 2m + 1 + k'm + \ell'(m+1) = n$	$k, \ell, k, \ell' \geq 0$	364, 20
$km + \ell = n$	$k \geq 2, \ell \leq \left\lfloor \frac{n-m}{m+1} \right\rfloor$	366
Prime Triangles		
$4m + 1 + km + \ell(m+1) = 4m + 2 + k'm + \ell'(m+1) = n$	$k, \ell, k, \ell' \geq 0$	365

8.6.2 *Triangles tiled by pairs of polyominoes*

We won't include $T(1)$, $T(2)$ or $T(3)$ in our analysis here, since they already tile triangles by themselves.

FOUR CELLS OR LESS. The only polyomino with the diagonal property is the T-tetromino, it must be part of any pair that tiles a triangle (by Theorem 360). We can eliminate the straight tromino, straight tetromino, L-tetromino using the orientation criterion (Theorem 362). The remainder have tilings.

Theorem 367. *If a tiling of a region R by a set $\mathcal{T}$ use all tiles $P_i \in \mathcal{T}$, then*

$$|R| \geq \sum_i \|P_i\| .$$

[Referenced on page 367]

FIVE CELLS OR LESS. Only four pentominoes have the diagonal property, therefore they need to be one if the pair of the other polyomino is not the T-tetromino. All combinations of the T-tetromino and a pentomino can be eliminated with the orientation criterion, except for the P- and W-pentominoes, for which tilings exist.

OTHER KNOWN CASES. Table 74 shows other pairs known to tile triangles.

8.6.3 *Further reading*

More tilings of triangles by pairs of polyominoes are shown at Friedman (2008, https://erich-friedman.github.io/mathmagic/0506.html).

	4	5	5	5	5
2	$T(3)$ A′	$T(5)$ C	$T(6)$ C	$T(6)$ C	$T(8)$ C, A′
3	$\times$ O	$\times$ O	$\times$ O	$\times$ O	$T(9)$ A, C
4	$\times$ O	$\times$ O	$\times$ O	$\times$ O	$T(19)$ A, C, S
4	$\times$ O	$\times$ O	$\times$ O	$\times$ O	$T(10)$ C, S
4	$T(7)$ C, A′	$\times$ F	$\times$ F	$T(13)$ C, A, S	$\times$ B
4	$T(8)$ C, S	$\times$ B	$\times$ B	$T(10)$ C, A, S	$\times$ B
5	$T(7)$ A, A′, C	$\times$ D	$\times$ D	$\times$ O	$\times$ O
5	$T(8)$ A, A′, C, S	$\times$ D	$\times$ D	$\times$ O	$\times$ B

Table 73: Triangular tilings by pairs of polyominoes.
A = Area Criterion (Theorem 1)
A′ = Theorem 367
B = Bottom Criterion (Theorem 361)
C = Checkerboard Criterion (Theorem 143)
D = Diagonal Criterion (Theorem 360)
F = No tile fits in corner
O = Orientation Criterion 362
S = Computer search

5, 5	$T(9)$	5, 5	$T(4)$	5, 5	$T(5)$	5, 5	$T(4)$
5, 5	$T(9)$	5, 5	$T(14)$	5, 5	$T(15)$	5, 5	$T(24)$
5, 5	$T(29)$	6, 6	$T(20)$	6, 6	$T(20)$	6, 6	$T(23)$
6, 6	$T(17)$	6, 6	$T(32)$	6, 6	$T(23)$	7, 7	$T(6)$
7, 7	$T(7)$	7, 7	$T(7)$	7, 7	$T(7)$	7, 7	$T(7)$
7, 7	$T(13)$	7, 7	$T(13)$	7, 7	$T(21)$	7, 7	$T(21)$
7, 7	$T(21)$	7, 7	$T(20)$	7, 7	$T(21)$	7, 7	$T(27)$
7, 7	$T(27)$	7, 7	$T(27)$	7, 7	$T(27)$	9, 9	$T(8)$
9, 9	$T(8)$	9, 9	$T(8)$	9, 9	$T(9)$	10, 9	$T(9)$
9, 9	$T(17)$	9, 9	$T(18)$	9, 9	$T(18)$		

Table 74: Pairs of polyominoes known to tile triangles. From Friedman (2006).

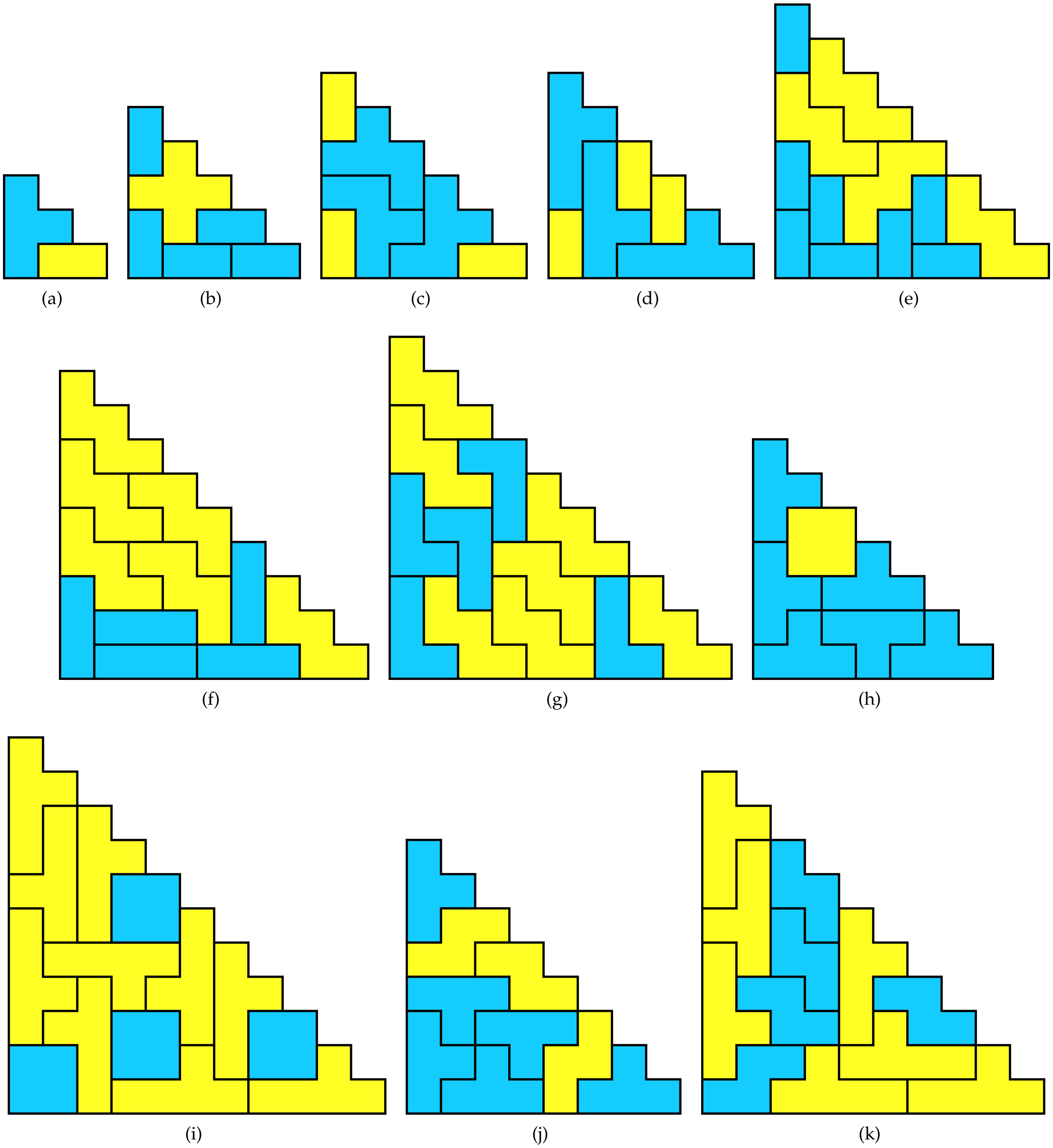


Figure 385: Triangular tilings by pairs of polyominoes. From Friedman (2006).

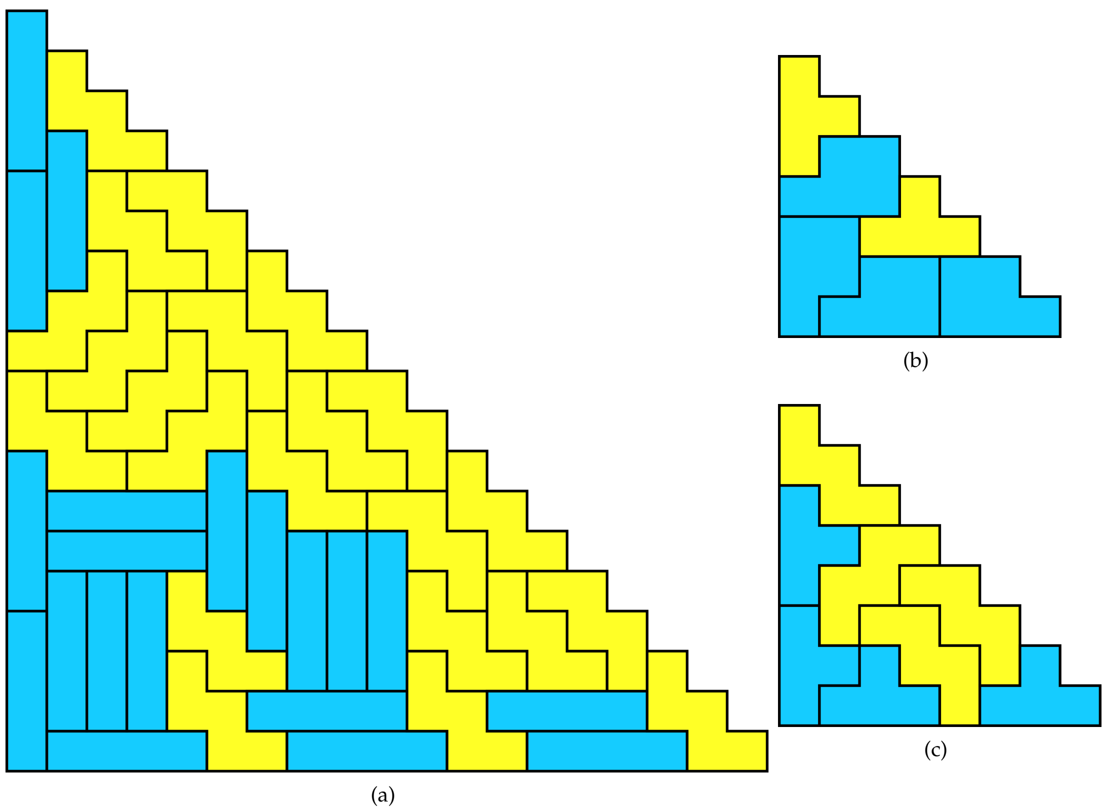


Figure 386: Triangular tilings by pairs of polyominoes.

9 Enumeration

How many polyominoes are there? In general, we do not know; in fact we only know counts for free polyominoes up to 48 cells (**A000105**), and for fixed polyominoes, up to 56 cells (**A001168**).[1]

[1] The purpose of this short chapter is to include the tables after I realized I constantly needed to consult them. I may expand it in a future version.

Area	Fixed A001168	Free A000105	**None** A006749	**Rot** A006747	**Axis** A006746	**Diag** A006748	**Rot2** A144553	**Axis2** A056877	**Diag2** A056878	**All** A142886
	56	48	48	48	48	48	95	81	87	163
1	1	1								1
2	2	1						1		
3	6	2				1		1		
4	19	5	1	1	1			1		1
5	63	12	5	1	2	2		1		1
6	216	35	20	5	6	2		2		
7	760	108	84	4	9	7		3	1	
8	2725	369	316	18	23	5	1	4	1	1
9	9910	1285	1196	19	38	26		4		2
10	36446	4655	4461	73	90	22		8	1	
11	135268	17073	16750	73	147	91		10	2	
12	505861	63600	62878	278	341	79	3	15	3	3
13	1903890	238591	237394	283	564	326	2	17	3	2
14	7204874	901971	899265	1076	1294	301		30	5	
15	27394666	3426576	3422111	1090	2148	1186		35	6	
16	104592937	13079255	13069026	4125	4896	1117	12	60	14	5
17	400795844	50107909	50091095	4183	8195	4352	7	64	9	4
18	1540820542	192622052	192583152	15939	18612	4212		117	20	
19	5940738676	742624232	742560511	16105	31349	16119		128	20	
20	22964779660	2870671950	2870523142	61628	70983	15849	44	236	56	12
21	88983512783	11123060678	11122817672	62170	120357	60174	25	241	32	7
22	345532572678	43191857688	43191285751	239388	271921	60089		459	80	
23	1344372335524	168047007728	168046076423	240907	463712	226146		476	64	
24	5239988770268	654999700403	654997492842	932230	1045559	228426	165	937	224	20
25	20457802016011	2557227044764	2557223459805	936447	1792582	854803	90	912	114	11
26	79992676367108	9999088822075	9999080270766	3641945	4034832	872404		1813	315	
27	313224032098244	39153010938487	39152997087077	3651618	6950579	3247207		1789	217	
28	1228088671826973	153511100594603	153511067364760	14262540	15619507	3342579	603	3706	863	45

Table 75: Enumerations of polyominoes. From Oliveira e Silva (2015). More terms of these sequences have been calculated by Iwan Jensen, Toshihiro Shirakawa, John Mason, Andrew Howroyd, and Robert A. Russell. See the OEIS links for details. The top line shows how many terms we know for each sequence.

n	A001168	
1	1	1.00×10^{0}
2	2	2.00×10^{0}
3	6	6.00×10^{0}
4	19	1.90×10^{1}
5	63	6.30×10^{1}
6	216	2.16×10^{2}
7	760	7.60×10^{2}
8	2725	2.73×10^{3}
9	9910	9.91×10^{3}
10	36446	3.64×10^{4}
11	135268	1.35×10^{5}
12	505861	5.06×10^{5}
13	1903890	1.90×10^{6}
14	7204874	7.20×10^{6}
15	27394666	2.74×10^{7}
16	104592937	1.05×10^{8}
17	400795844	4.01×10^{8}
18	1540820542	1.54×10^{9}
19	5940738676	5.94×10^{9}
20	22964779660	2.30×10^{10}
21	88983512783	8.90×10^{10}
22	345532572678	3.46×10^{11}
23	1344372335524	1.34×10^{12}
24	5239988770268	5.24×10^{12}
25	20457802016011	2.05×10^{13}
26	79992676367108	8.00×10^{13}
27	313224032098244	3.13×10^{14}
28	1228088671826973	1.23×10^{15}
29	4820975409710116	4.82×10^{15}
30	18946775782611174	1.89×10^{16}
31	74541651404935148	7.45×10^{16}
32	293560133910477776	2.94×10^{17}
33	1157186142148293638	1.16×10^{18}
34	4565553929115769162	4.57×10^{18}
35	18027932215016128134	1.80×10^{19}
36	71242712815411950635	7.12×10^{19}
37	281746550485032531911	2.82×10^{20}
38	1115021869572604692100	1.12×10^{21}
39	4415695134978868448596	4.42×10^{21}
40	17498111172838312982542	1.75×10^{22}
41	69381900728932743048483	6.94×10^{22}
42	275265412856343074274146	2.75×10^{23}
43	1092687308874612006972082	1.09×10^{24}
44	4339784013643393384603906	4.34×10^{24}
45	17244800728846724289191074	1.72×10^{25}
46	68557762666345165410168738	6.86×10^{25}
47	272680844424943840614538634	2.73×10^{26}
48	1085035285182087705685323738	1.09×10^{27}
49	4319331509344565487555270660	4.32×10^{27}
50	17201460881287871798942420736	1.72×10^{28}
51	68530413174845561618160604928	6.85×10^{28}
52	273126660016519143293320026256	2.73×10^{29}
53	1088933685559350300820095990030	1.09×10^{30}
54	4342997469623933155942753899000	4.34×10^{30}
55	17326987021737904384935434351490	1.73×10^{31}
56	69150714562532896936574425480218	6.92×10^{31}

Table 76: The number of fixed polyominoes; also expressed in scientific notation to make it easier to see how large the numbers get.

	$k=0$ A000104	1 A057418	2 A089454	3 A089455	4 A089456	5 A089457	6 A089458	7	8
1	1								
2	1								
3	2								
4	5								
5	12								
6	35								
7	107	1							
8	363	6							
9	1248	37							
10	4460	195							
11	16094	975	4						
12	58937	4622	41						
13	217117	21128	346						
14	805475	94109	2384	3					
15	3001127	410820	14560	69					
16	11230003	1766591	81863	798					
17	42161529	7506179	433172	7021	8				
18	158781106	31596240	2192752	51775	179				
19	599563893	131991619	10726252	339958	2509	1			
20	2269506062	547992183	51094203	2053872	25609	21			
21	8609442688	2263477612	238259280	11665593	214932	573			
22	32725637373	9309386178	1092053068	63192945	1578984	9140			
23	124621833354	38150082057	4934652298	329798278	10536260	105417	64		
24	475368834568	155859235424	22034767837	1670466031	65411645	982767	2131		
25	1816103345752	635067478628	97407519119	8256762365	383981499	7919375	38022	4	
26	6948228104703	2581737704039	426916828181	39992202198	2156114468	57387444	480713	329	
27	26618671505989	10474587325120	1857253575577	190431003084	11678888362	383757344	4872679	10332	
28	102102788362303	42422970467980	8027749130623	893726550231	61413242603	2410292366	42360239	188221	37

Table 77: Number of polyominoes with k holes. From Oliveira e Silva (2015).

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