Coalitional Games

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1 Basics

A **normal form** game is a tuple $(P, \mathcal{S}, \mathcal{F})$, where $P = \{1, 2, \ldots, n\}$ is a set of players, $\mathcal{S} = (S_1, S_2, \ldots, S_n)$ is a tuple of strategy sets, and $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ is a sequence of payoff functions, with $F_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ representing the utility for player i.

2 Finite Coalitional Games

We'll assume that utility can be freely exchanged between players by side-payments.

A **coalitional game** is a pair (P, v), where $P = \{1, 2, \ldots, n\}$ is a set of players and v is a real-valued function on 2^P describing the value attainable when each subset of the players acts as a coalition. v is called the **characteristic function** of the game.

We usually consider coalitional games for which v is **superadditive**: for each disjoint pair of subsets $S, T \subseteq P$, $v(S \cup T) \geq v(S) + v(T)$. This is pretty natural, since one strategy for the coalition $S \cup T$ could be to have S and T act like separate coalitions as before, guaranteeing at least $v(S) + v(T)$.

An **imputation** of an n-player game (P, v) is a payoff vector $(x_1, \ldots, x_n)$ satisfying the following:
(i) Individual rationality: $x_i \geq v(\{i\})$ for all i
(ii) Pareto-optimality: $\sum_{i=1}^{n} x_i = v(P)$.

For a subset $S \subseteq P$, we say that imputation x **S-dominates** imputation y if $x_i > y_i$ for all $i \in S$ and $\sum_{i \in S} x_i \leq v(S)$.

In a normal game $(P, \mathcal{S}, \mathcal{F})$, the **security level** of a coalition $C \subseteq P$ is the minimum total payoff which the members of C can guarantee themselves assuming all players in $P \backslash C$ are acting as a coalition to minimize the payoff of C. For any normal-form game $(P, \mathcal{S}, \mathcal{F})$, we can let $v(C)$ be the security level of C for each coalition C to produce a coalitional game (P, v) with superadditive utility function.

2.1 Examples

1. A 3 player game in normal form:

   $n = 3$: Players Alice (1), Bob (2), Charlie (3)

   $P = \{1, 2, 3\}$, $S_i = \{A, B\}$ for each i, payoff functions shown below:

   Charlie A:

	Bob A	Bob B
Alice A	(1,1,-2)	(-4,3,1)
Alice B	(2,-4,2)	(-5,-5,10)

   Charlie B:

	Bob A	Bob B
Alice A	(3,-2,-1)	(-6,-6,12)
Alice B	(2,2,-4)	(-2,3,-1)

2. 3 player "Divide the Dollar"

2.2 Additional Desirable Properties of an Imputation

1. Reasonableness from Above: For each player i,

$$x_i \leq \max_{S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S).$$

   Player i should not get more than the most they contribute to any coalition.

2. Reasonableness from Below: For each player i,

$$x_i \geq \min_{S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S).$$

   Player i should get at least the smallest amount they contribute to any coalition.

3. Reduced-Game Property on a set G of games: If $(P, v) \in G$, then an imputation x has the RGP if for any nonempty subset $S \subseteq P$, the game (S, v') is in G, where the characteristic function v' for the restricted game is defined by $v'(T) = \max_{Q \subseteq P \setminus S} v(T \cup Q) - \sum_{i \in Q} x_i$ for $\emptyset \subsetneq T \subsetneq S$, $v'(\emptyset) = 0$, and $v'(S) = v(N) - \sum_{i \in N \setminus S} x_i$.

3 The Core

For a coalitional game (P, v), the **core**, denoted $\mathcal{C}(P, v)$, is the set of all imputations $x = (x_1, \ldots, x_n)$ such that $\sum_{i \in S} x_i \geq v(S)$ for each $S \subseteq P$. Equivalently, the core is the set of all imputations that are not S-dominated by another imputation for any nonempty subset S of the players.

If an imputation is in the core, then no subset of the players can guarantee that they will do better by deviating from the agreement that produced the imputation. Thus, if we can find the core for a game, we have an idea of a potentially stable outcome.

We care about the core since it satisfies the nice properties we want:

Theorem 3.1. *Any imputation in the core is reasonable from above and below, and satisfies the reduced-game property on the set of all games with non-empty core.*

Proof. By individual rationality, any imputation x is reasonable from below since $x_i \geq v(\{i\}) - v(\emptyset) = v(\{i\})$.

Now let x be an imputation in the core $C(P, v)$ of a game (P, v). By collective rationality, $\sum_{i \in P} x_i = v(P)$, and by the core property, $\sum_{j \in P, j \neq i} x_j \geq v(P \setminus \{i\})$, so we can get that $x_i \leq v(P) - v(P \setminus \{i\})$, and so x is reasonable from above.

Finally, let (P, v) be a game with nonempty core, and let $x \in C(P, v)$; take any nonempty $S \subseteq P$ and any nonempty subset $T \subseteq S$. Two cases: $T = S \implies v'(S) - \sum_{i \in S} x_i = 0$, as desired. Otherwise,

$$v'(T) - \sum_{i \in T} x_i = \left(\max_{Q \subseteq P \setminus S} v(T \cup Q) - \sum_{i \in Q} x_i \right) - \sum_{j \in T} x_j = \max_{Q \subseteq P \setminus S} v(T \cup Q) - \sum_{i \in T \cup Q} x_i \leq 0,$$

so the restriction of x to S is also in the core of (S, v') and so that game has nonempty core. $\square$

It turns out that there's also a pretty clean criterion for when there's an imputation in the core!

Theorem 3.2 (Duality Theorem)**.** *If there's an optimal vector $\vec{x}$ that maximizes $\vec{c}^T \vec{x}$ given $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq \vec{0}$, and an optimal vector $\vec{y}$ that minimizes $\vec{b}^T \vec{y}$ given $A^T \vec{y} \geq \vec{c}$ and $\vec{y} \geq \vec{0}$, then $\vec{c}^T \vec{x} = \vec{b}^T \vec{y}$.*

We'll use the Duality Theorem for linear programs to prove the following criterion, which will depend on this definition:

Definition 3.3. *A collection of nonempty subsets of players $B \subseteq 2^P, \emptyset \notin B$ is **balanced** if there exist positive weights w_S for each subset S in B with $\sum_{\{i\} \subseteq S \in B} w_s = 1$ for each $i \in P$. We say that the weights $w_S, S \subseteq P$ are a system of balancing weights.*

Theorem 3.4 (Bondareva-Shapley Theorem). *The game (P, v) has a nonempty core if and only if for each balanced collection B and each corresponding system of balancing weights (w_S for each $S \in B$), we have that*

$$v(P) \geq \sum_{S \in B} w_S v(S).$$

Proof. (Sketch) We'll reformulate the condition that the core be nonempty as a series of constraints.

Let $x = (x_1, ..., x_n)$ be some payoff vector such that each coalition gets at least its value: $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq P$.

The core is the set of all *imputations* satisfying these constraints, so we also need that $\sum_{i=1}^{n} x_i = v(P)$ for x to be in the core. Thus if we take the minimum value of $\sum_{i=1}^{n} x_i$ over all x, the core is nonempty iff it is $v(P)$.

So now we have a linear program. We can convert it to its dual now: given that for every nonempty subset $S \subseteq P$, we have that $w_S \geq 0$, and also for any $i \in P$, $\sum_{\{i\} \subseteq S \subseteq P} w_S \leq 1$, maximize $\sum_{S \subseteq P, S \neq \emptyset} w_S v(S)$.

By the Duality Theorem, we then have that the maximum value of $\sum_{S \subseteq P, S \neq \emptyset} w_S v(S)$ equals the minimum value of $\sum_{i=1}^{n} x_i$. Therefore, we have that the core is nonempty if and only if for all sequences of nonnegative weights w_S satisfying the conditions above, we have that

$$\sum_{S \subseteq P, S \neq \emptyset} w_S v(S) \leq v(P).$$

We can now let $B = \{S \subseteq P : w_S > 0\}$, ignoring zero-weight subsets, to obtain that the original condition in the theorem is necessary and sufficient. □

4 The Shapley Value

We would like to find an imputation $\phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ for any coalitional game (P, v) such that the imputation satisfies the following axioms:

Axiom 1. For any two players $i, j \in P$, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq P \backslash \{i, j\}$, then $\phi_i(v) = \phi_j(v)$.

Axiom 2. For any player $i \in P$, if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq P \backslash \{i\}$, then $\phi_i(v) = 0$.

Axiom 3. For any two characteristic functions v, w on 2^P, we have $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$.

Note: Axiom 3 is convenient but also much less natural and more controversial than the other two axioms: there are some other ways to construct imputations designed to be fair using different axiom systems, including the *nucleolus* and *Gately point*. These 3 axioms are the most commonly-used though.

It turns out that these axioms completely determine a unique imputation!

Theorem 4.1 (Shapley). *For each game (P, v) such that v is superadditive, there is a unique imputation $\phi(v)$ satisfying Axioms 1 through 3. $\phi(v)$ is called the* ***Shapley value****.*

Proof. (Sketch.) First, we'll prove that this value $\phi(v)$ exists. Let $\pi \in S_n$ be an arbitrary permutation and let $i \in P$ be an arbitrary player. We're going to have the players all join a grand coalition of everyone in the order π and each get the value they contribute to the coalition. Let $a_\pi^i(v) = v(\{j \in P : \pi(j) \leq \pi(i)\}) - v(\{j \in P : \pi(j) < \pi(i)\})$ be the difference between what the coalition can get before i joins and after i has just joined. Then we just average over all possible orderings to define $\phi_i(v)$ for each i:

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in S_n} a_\pi^i(v).$$

We claim that this is an imputation and that it satisfies all the axioms.

Since for any π we have $\sum_{i=1}^{n} a_i(\pi) = v(P)$, it satisfies collective rationality, and by superadditivity of v we have that for any π, $a_\pi^i(v) = v(\{j \in P : \pi(j) < \pi(i)\} \cup \{i\}) - v(\{j \in P : \pi(j) < \pi(i)\}) \geq v(\{i\})$, so we also have individual rationality. Thus, $\phi(v)$ is an imputation.

To show axiom 1, for each pair $i, j \in P$ of players we can consider the map taking each permutation π to the permutation $\pi' = \pi \circ (ij)$ swapping the positions of i and j, and note that $a^i_\pi(v) = a^j_{\pi'}(v)$. The Shapley value works because this is a bijection.

Then, if the condition $v(S \cup \{i\}) = v(S)$ in axiom 2 holds, then $a^i_\pi(v)$ must be 0 for every $\pi \in S_n$, so we get $\phi_i(v) = 0$ and axiom 2 holds.

Finally, for any π we have that $a^i_\pi(v + w) = a^i_\pi(v) + a^i_\pi(w)$, so taking the sum we get axiom 3.

Next, we will show that this is unique. We're going to put together a basis for the vector space of possible characteristic functions v, and then use Axiom 3 to reduce the problem to the basis vectors.

For any subset $T \subseteq P$, let $u_T(S)$ be 1 if $S \subseteq T$ and otherwise 0. We can check that the u_T's are linearly independent. Since there are $2^n - 1$ of them (ignoring $u_\emptyset$) and the space of characteristic functions has dimension $2^n - 1$ (because $v(0) = \emptyset$), this implies that $\{u_T \mid T \subseteq P\}$ is a basis for the desired space.

Let ϕ be some value, and let $T \subseteq P$ be nonempty. For any $\alpha \in \mathbb{R}$, consider the game $(P, \alpha u_T)$. Any player $i \in P/T$ has $\alpha u_T(S \cup \{i\}) = \alpha u_T(S)$, so by Axiom 2 we need to give them a value $\phi_i(\alpha u_T) = 0$. Also, any two players in T contribute the same to any coalition, so Axiom 1 forces $\phi_i(\alpha u_T) = \phi_j(\alpha u_T)$ for any $i, j \in T$. Thus, since $\phi(\alpha u_T)$ is an imputation, we have that $\phi_i(\alpha u_T) = \alpha/|T|$ for each $i \in T$ and 0 for each $i \in P \setminus T$. Therefore, $\phi(\alpha u_T)$ is uniquely determined.

Since the set $\{u_T | T \subseteq P, T \neq \emptyset\}$ is a basis, any characteristic function v can be written as $v = \sum_{T \subseteq P} \alpha_T u_T$, so by Axiom 3 the value of $\phi(v)$ is uniquely determined for any v. □

There's a nice combinatorial formula for the Shapley value with a number of terms in the sum that's much smaller than $n!$ – see if you can find it! (See problem 4.)

(Also, if you're curious how the Shapley value relates to the core, see problem 2!)

5 Games with infinitely many players: Aumann-Shapley value

The notion of a coalitional game (P, v) on a finite set P of players can be generalized to an infinite game $(I, \mathscr{C}, v)$, where $(I, \mathscr{C})$ is a measurable space. We normally consider spaces isomorphic to $([0, 1], \mathscr{B})$, where $\mathscr{B}$ is the set of Borel subsets of $[0, 1]$. Then, $v : \mathscr{B} \to \mathbb{R}$ is a real-valued (not necessarily positive) function such that $v(\emptyset) = 0$.

We will call a function $v : \mathscr{B} \to \mathbb{R}$ a **game**, where we actually mean the game $([0, 1], \mathscr{B}, v)$ with player set $[0, 1]$, coalition set $\mathscr{B}$, and characteristic function v.

Definition 5.1. *A game $v : \mathcal{B} \to \mathbb{R}$ is **monotonic** if $V(E \cup F) \geq v(E)$ for all $E, F \in \mathcal{B}$.*

Definition 5.2. *A game $v : \mathcal{B} \to \mathbb{R}$ has **bounded variation** if $v = w - u$ for monotonic w, u.*

We define the variation v by $||v|| = \inf\{w(I) + u(I) | u, w \text{ monotonic }, v = w - u\}$.

Definition 5.3. *Let BV be the space of all games with bounded variation.*

Theorem 5.4. *The variation is a norm on BV, and every sequence of games in BV has a limiting game, so BV is a Banach space.*

Definition 5.5. *Let FA be the space of all bounded finitely additive games, those for which $v(E_1 \cup E_2 \cup \cdots \cup E_n) = v(E_1) + ... + v(E_n)$ for $E_1, ..., E_n$ pairwise disjoint and $v(E) < +\infty$ for all E.*

Definition 5.6. *A measure μ on a measure space $(X, \mathcal{X}, \mu)$ is **non-atomic** if for all $S \in \mathcal{X}$ with $\mu(S) > 0$, there exists $T \subseteq S$ such that $0 < \mu(T) < \mu(S)$.*

(A measure on $([0, 1], \mathscr{B})$ is a function $\mu : \mathscr{B} \to \mathbb{R}$ with $\mu(E) \geq 0$ for all $E \in \mathscr{B}$, $\mu(\emptyset) = 0$, and for any pairwise disjoint $E_1, E_2, \ldots$, $\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j)$.)

Definition 5.7. *Let NA be the subspace of all games v in FA for which v is a non-atomic measure.*

Definition 5.8. *Let pNA be the space of all games v such that there are non-atomic measures $\mu_1, \mu_2, \cdots \in NA$ with $\mu_i([0, 1]) = 1$ for each i, positive integers $k_1, k_2, \cdots \in \mathbb{N}$, and real numbers $\alpha_1, \alpha_2, \ldots$ for which*

$$v(E) = \sum_{j=1}^{\infty} \alpha_j (\mu_j(E))^{k_j}.$$

For any subspace Q of BV, we define the following axioms that a value on Q must satisfy. These are the infinite analogous of Axioms 1 through 3 that specify Shapley values for finite games.

A value here will be a generalization of the idea of an imputation.

Definition 5.9. *A* ***value*** *is a map ϕ from a space of games $Q \subseteq BV$ to FA satisfying the following four axioms.*

Axiom 1: Positivity. For any monotonic game $v \in Q$, $\phi(v)$ is monotonic. - corresponds to individual rationality

Axiom 2: Efficiency. For every game $v \in Q$, $(\phi(v))([0,1]) = v([0,1])$. - corresponds to Pareto-efficiency

Axiom 3: Symmetry. For any bijective map $f : [0,1] \to [0,1]$, such that $f(S) \in \mathscr{B}$ if and only if $S \in \mathscr{B}$, we define fv for a game v by $(fv)(S) = v(f(S))$. Then, ϕ is symmetric if $f(\phi(v)) = \phi(f(v))$ for all games $v \in Q$. - corresponds to Axiom I for finite games, the one where we swap players i, j

Axiom 4: Linearity. For any $v, w \in Q$, we have $\phi(v + w) = \phi(v) + \phi(w)$. - corresponds to Axiom III for finite games

Definition 5.10. *Let $X \subseteq \mathbb{R}^n$ be convex, and let $f : X \to \mathbb{R}$. Then, for any $\vec{x}, \vec{z} \in \mathbb{R}^n$, let the value of the derivative of f in the direction $\vec{z}$ at $\vec{x}$ be $f_{\vec{z}}(x) = g'(0)$, where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(t) = f(\vec{x} + t\vec{z})$, if g is differentiable.*

Now we can generalize the value to a class of infinite games!

Theorem 5.11 (Aumann, Shapley)**.** *There is a unique value ϕ on pNA, and the variation $||\phi|| = 1$. For any game v of the form $v = f \circ \mu$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n) : \mathscr{B} \to \mathbb{R}^n$ is a vector of non-atomic measures and $f : \mathrm{Range}(\mu) \to \mathbb{R}$ is continuously differentiable with $f(\vec{0}) = 0$, we have that $f \circ \mu \in pNA$ and the value ϕ satisfies*

$$\phi(f \circ \mu)(S) = \int_0^1 f_{\mu(S)}(t\vec{a})\, dt,$$

where $\vec{a} = \mu([0,1])$.

It would be nice if we could define a value on the entire space BV of games of bounded variation rather than just games that are polynomials in probability measures, but unfortunately no such value exists. (See problem 5!)

6 References

This handout uses material from the following sources:

R.J. Aumann and L.S. Shapley. *Values of Non-Atomic Games.* (This is where the material on the generalization to games with an infinite continuum of players came from.)

B. Peleg and P. Sudholter. *Introduction to the Theory of Cooperative Games.* (This one covers cooperative/coalitional game theory in detail and is pretty rigorous; most of the material on finite coalitional games came from here.)

A.E. Roth. *The Shapley value: Essays in honor of Lloyd S. Shapley* (This had some helpful applications of the Shapley value and also gave a second perspective on generalizing to infinitely many players that was easier to follow.)

P.D. Straffin. *Game Theory and Strategy.* (This one presents cooperative games in a very elementary and intuitive way, which really helped me to understand them at first. Several of the problems came from here.)

"In war, resolution; in defeat, defiance; in victory, magnanimity; in peace, goodwill."
– Winston Churchill, on tit-for-tat strategies in iterated games (probably)

7 Problems

1. The *Glove Market* game is the game $(\{1, 2, 3, 4, 5\}, v)$, where v is defined as follows:

   Players $1, 2$, and 3 each have a left-hand glove, and players 4 and 5 each have a right-hand glove. Define the characteristic function v so that for any subset S of players, $v(S)$ is equal to the number of complete pairs of gloves S has (e.g. $v(S) = \min(|S \cap \{1, 2, 3\}|, |S \cap \{4, 5\}|)$).

   (a) Find an imputation that dominates the equal split $x = (2/5, 2/5, 2/5, 2/5, 2/5)$.

   (b) Find all imputations in the core of this game.

2. Let (P, v) be an n-player coalitional game for which v satisfies $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq P$.

   It is known that if v satisfies the condition above, then for every permutation $\pi \in S_n$, we have that $a_\pi(v)$ is in the core $C(P, v)$, where $a_\pi(v)$ is the imputation with a payoff to the ith player of

   $$a^i_\pi(v) = v(\{j \in P \mid \pi(j) \leq \pi(i)\}) - v(\{j \in P \mid \pi(j) < \pi(i)\}).$$

   Using this, prove that the Shapley value φ of v is in the core $C(P, v)$.

3. Let (P, v) be an n-player coalitional game satisfying $v(S) + v(P \backslash S) = v(P)$ for all $S \subseteq P$ and $v(P) > v(\{1\}) + v(\{2\}) + \cdots + v(\{n\})$. Is it possible for the core $C(P, v)$ to be nonempty?

4. In the proof of Shapley's theorem, we constructed the Shapley value for a coalitional game (P, v) using the formula

   $$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in S_n} a^i_\pi(v),$$

   where $a^i_\pi(v) = v(\{j \in P \mid \pi(j) \leq \pi(i)\}) - v(\{j \in P \mid \pi(j) < \pi(i)\})$ for each $i \in P$.

   Using this formula, derive an equivalent formula for the Shapley value $\phi_i(v)$ for each player $i \in P$ in a game (P, v) such that the formula involves a summation of at most 2^n terms.

5. This problem is to prove that there does not exist a value ϕ on the entire space BV of bounded variation games.

   (a) Construct a game v such that for any automorphism $f : [0, 1] \to [0, 1]$, such that $f(S) \in \mathscr{B}$ if and only if $S \in \mathscr{B}$, we have that $v(fS) = v(S)$ for all $S \in \mathscr{B}$.

   (b) If ϕ is a value on BV, what must $(\phi(v))([0, 1])$ be? Use this to get a contradiction. (Hint: ϕv must be finitely additive.)