A Mechanism Design Approach towards Equitable School Choice Policy

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1 Introduction

In the United States, school choice policies are still at the forefront of debate amongst state legislators, educators and families. One of the central issues faced in designing these policies is to allow lower-income students to get access to higher quality schools. However, public school allocation is still largely based on a students' neighborhood which exacerbates income inequalities in access to quality education, as higher-income neighborhoods are likely to have better funded schools. Some of this has been combated in big cities across the country and the world by designing better allocative mechanisms. Abdulkadiroglu and Sonmez (2003) laid the foundation to think about school choice under a mechanism design framework, which is essentially a more generalized auction-like framework. The goal of this paper is to extend a mechanism of school choice to consider how these income inequalities in education access could potentially be solved, by presenting a theoretical framework of allocative mechanisms.

This paper will first provide a quick primer of the theory of mechanism design followed by a literature review. I will then consider two extensions to the basic model of school choice, namely Controlled Choice and Information Asymmetry. Within these settings, I attempt to consider allocative mechanisms that could improve school allocations for lower-income students.

2 Main Body

2.1 Theory of Mechanism Design and School Choice

Traditionally, economics takes institutions as given and analyzes the outcomes of these institutions. However, mechanism design is a normative economic concept whereby a social planner decides on the outcome that is desirable, possibly based on some measure of welfare. The planner then designs a system that most effectively gets that desired outcome. The theoretical framework is as follows.

- **Set of agents**: $N = \{1, 2, \ldots, n\}$,
- **Set of outcomes**: X,
- **Type of agent** i: $\theta_i \in \Theta_i$, where Θ_i is the set of possible types for agent i,
- **Type profile**: $\theta = (\theta_1, \ldots, \theta_n) \in \Theta$, with $\Theta = \Theta_1 \times \cdots \times \Theta_n$,
- **Probability distribution over type profiles**: $\phi \in \Delta\Theta$, where $\Delta\Theta$ is the set of all probability distributions over Θ
- **Utility function for agent** i: $u_i : X \times \Theta_i \to \mathbb{R}$,
- **Mechanism**: $M = ((S_1, S_2, \ldots, S_n), g(\cdot))$, where $g : C \to X$ and $C = S_1 \times S_2 \times \cdots \times S_n$. S_i is defined as the action or message given by agent i

The probability density function $\phi(\cdot)$, the type sets $\Theta_1, \ldots, \Theta_n$, and the utility functions $u_i(\cdot)$ are assumed to be common knowledge among the agents.
We can extend a similar idea to think about a theoretical framework for school choice. Instead of a planner making a decision $x \in X$, we instead have schools making a decision of who to admit based on a priority ordering of students. So now the agents that we modeled above have an interaction with another set of agents instead of an ”inactive” social planner. A general theoretical framework for school choice is as follows.

Consider a set of students $N = \{1, 2, \ldots, |N|\}$, and a set of schools $S = \{s_1, s_2, \ldots, s_{|S|}\}$. Each school $s \in S$ has a capacity of q_s. Each student $i \in N$ has a strict preference relation P_i over $S \cup \{i\}$, where $\{i\}$ denotes the possibility of being unmatched possibly due to an outside option like a private school. Moreover, each school $s \in S$ has a weak priority ordering $\succeq_s$ over $N \cup \{s\}$, where $\{s\}$ represents keeping a seat empty. A mechanism φ determines a matching for any tuple $(N, S, q, P, \succeq)$. In some school choice problems we can simply reduce the mechanism to $\varphi(P)$, as every other element can be considered constant. We apply the mechanism to P, but students give a signal defined by some preference relation P', which may not be equal to P. For the sake of keeping consistent with the mechanism design framework I presented, we can instead consider $\varphi(P')$.

2.2 Background and Literature Review

The foundations of thinking about school choice as a mechanism design problem were first discussed in Abdulkadiroglu and Sonmez (2003). They claim that the immediate acceptance or Boston mechanism that was in place at the time was inefficient and did not incentivize truth-telling, as a school would drop student priority if a student did not list the school as their top choice. They then propose two different school choice algorithms; namely, the Gale-Shapley deferred acceptance (DA) mechanism and the Top Trading Cycles (TTC) mechanism, which they claim have more desirable properties than the Boston mechanism. The Gale-Shapley deferred acceptance mechanism is modeled after Gale and Shapley (1962) as an application of the stable matching problem to school choice. Under this mechanism, students apply for their first choice school. Each school then provisionally assigns a seat to a student based on their randomized priority orders until capacity, and the rest are rejected. The rejected students then move down to their next preferred school and apply there. Schools revise their provisional acceptances and will drop a student admitted in a previous round if another student with a higher priority applies. Once again, this is done till capacity is met. The algorithm terminates when every student is matched with a school. Gale and Shapley showed that this algorithm is a stable solution to the matching problem; that is, there is no pair (i,s) such that student i prefers school s over his/her assignment and i has higher priority in school s than some other student who was assigned a seat at s. Roth (1982) showed that the Gale-Shapley matching problem that gave the optimal stable matching for one set of agents has truthful revelation as a weakly-dominant strategy for all agents in that set. Essentially, this means that the deferred acceptance mechanism is strategy-proof, i.e, no student can do any better by misrepresenting their preferences over schools. However, this mechanism is not necessarily Pareto optimal[1]

The other algorithm that Abdulkadiroglu and Sonmez propose for the design of school choice mechanisms is the Top-Trading Cycles mechanism, which was first developed in Shapley and Scarf (1974). Under this mechanism, we allocate a counter for the capacity of each school. Each student "points" to her most preferred school and each school "points" to their highest priority student. This forms a cycle of schools and students[2]. Allocate each student to their most preferred school, drop the student from the mechanism and reduce the capacity of each school accordingly. If the capacity drops to 0, a school is also removed from the mechanism.

[1]Illustration in Appendix

[2]This is a well known result in Graph Theory. Since every student points to a school and every school points to a student, and there are finitely many students and schools, we have an undirected graph with minimum vertex degree of 2, which necessarily has a cycle. See Proposition 1.3.1 in "Graph Theory" by Reinhard Diestel

In the next round, students then point to their most preferred school amongst the remaining options and schools go down their priority list and point to their highest priority student who has not been assigned a school in the previous round. Once all students are assigned a seat, the algorithm terminates. Unlike the Gale-Shapley mechanism, assignments here are permanent. In the same paper, Abdulkadiroglu and Sonmez, show that the TTC mechanism is both strategy-proof and Pareto efficient. However, TTC is not necessarily stable[3]

The immediate acceptance (IA) mechanism works the same as the deferred acceptance mechanism except that assignments are permanent at every round. It is exactly this permanent assignment that makes IA susceptible to strategizing. In 2005, the Boston school district switched from this mechanism to the deferred acceptance mechanism for exactly this reason.

The rest of the paper attempts to develop more realistic mechanisms to address income inequalities in school choice, by looking at Controlled Choice and Information Asymmetry settings.

2.3 Controlled Choice

We now move towards theoretical mechanisms involving controlled choice, which is an extension to encompass a slightly more realistic approach to the theory of school choice. Controlled Choice policies in the United States originated in March 1981, when the Cambridge Public School (CPS) district implemented a school choice policy that strove to keep a racial and ethnic balance in schools while empowering parents with school choice. In December 2001, upon review of the Controlled Choice policy, CPS found high concentrations of students in poverty in some schools and overhauled their policy to now distribute low-income students throughout the district. Now, they included considerations of socioeconomic status (SES) based on voluntary disclosures on whether or not a child qualifies for the Free and Reduced Lunch program. In many school districts, such as in Columbus and Minneapolis, controlled choice was implemented by imposing type-specific quotas. When considering SES, a school district might consider two types of students; namely, those who qualify for the Free and Reduced Lunch program and those who do not. We can extend our theoretical school choice models to include both quotas and reserves.

[3] Example in Appendix

2.3.1 Controlled Choice Theory

Formally, we can describe a controlled-choice problem by considering choice functions, which are mappings that consider which students a school would choose from a given subset of students. By adding this complexity, we move towards a slightly more realistic look at how schools might prioritize students. A choice function $C : 2^N \to 2^N$ is a mapping that takes a subset of N students to another subset of N students. We assume that $C_s(I) \subseteq I$ and $|C_s(I)| \leq q_s$ for all $I \subseteq N$; which means that schools select students from a subset of the total without exceeding their capacity q_s. The remaining $I \setminus C_s(I)$ are rejected.

Now consider a set of mutually exclusive finite types represented by T. This set could be T = {Low Income, Middle Income, High Income} to represent mutually exclusive income classes. Let $\tau : N \to T$ be a type function where $\tau(i) \in T$ is a student $i's$ type. For a set of students $I \subseteq N$, let $I_t = \{i \in I \mid \tau(i) = t\}$ be the set of type-t students.

For modeling reserves, we consider r_t to be the number of reserved seats for students of type t. Let $R = (r_t)_{t \in T}$ be a vector such that $\sum_{t \in T} r_t \leq q$. The vector R is the type-specific reserves. Then a choice function is reserves-respecting for R if for all $I \subseteq N$ and $t \in T$, $|C(I)_t| \geq \min(|I_t|, r_t)$. Similarly, $Q = (q_t)_{t \in T}$ is the vector of type-specifc quotas such that $\sum_{t \in T} q_t \geq q$. Then a choice function is quotas-respecting for Q if for all $I \subseteq N$ and $t \in T$, $|C(I)_t| \leq q_t$. We now define some other properties of choice functions that allow us to achieve a desirable Deferred Acceptance Algorithm under Controlled Choice.

- **Nonwastefulness**: A choice function C is non-wasteful if for all $I \subseteq N$, and $i \in I, I \setminus C(I)$ and $|C(I)| \leq q$, then $\left|C(I)_{\tau(i)}\right| = q_{\tau(i)}$. Essentially, this says that if student i is rejected and the school still has place, it must be the case that the quota for student i's type has been met.
- **Within-type priority compatible**: A choice function C is within-type priority compatible if, for all $I \subseteq N$, whenever student i is chosen from I, student j is rejected and they are of the same type, then i must be ranked higher that j by s, that is, $i \in C(I)$, $j \in I \setminus C(I)$ and $\tau(i) = \tau(j)$, then $i \succ j$.
- **Across-type priority compatible**: A choice function C is across-type priority compatible if, for all $I \subseteq N$, whenever i is chosen and j is rejected and they are of different types, it must be that case that i is ranked higher than j by school s, when the chosen number of students in student i's type exceeds total reserves for that type and the quota for student j's type has not been met

yet. Formally for all $i, j \in I$ such that $i \in C(I)$ and $j \in I \setminus C(I)$, and $\left|C(I)_{\tau(i)}\right| > r_{\tau(i)}$, $\left|C(I)_{\tau(j)}\right| < q_{\tau(j)}$ then it must be the case that $i \succ j$.

Inamura(2020) defines a reserves and quotas choice rule (essentially a choice function) as one that is reserves and quotas respecting, non-wasteful and within and across type priority compatible.

2.3.2 Allocative Mechanisms Under Controlled Choice

Abdulkadiroglu and Sonmez (2003) propose an extension of the Gale-Shaley deferred acceptance mechanism and the Top-Trading Cycles mechanism under Controlled Choice with rigid type-specific quotas. The deferred acceptance mechanism works the same way except that at each round schools reject students of a particular type once the type quota has been met. Similarly, the Top-Trading Cycles mechanism also works the same way except that there is also a counter for the type-specific quota which reduces by 1 whenever an assignment for a student of that type is made. When the type-specific quota goes to 0, any applicants of that type are rejected. Both mechanisms still keep their strategy-proofness. The TTC mechanism with type-specific quotas is also constrained efficient, meaning that there is no other matching that satisfies all the controlled choice requirements and assigns every student at least a weakly better school.

Specifically, under the conditions of choice functions defined by Imamura (2020), we have the result that the Deferred Acceptance (DA) Algorithm is stable, strategy-proof and Pareto dominates any other stable algorithm. According to Hatfield and Milgram (2005), we only require that the choice function be substitutable and satisfy the law of aggregate demand. I could not find a direct proof that links these conditions to the conditions defined by Imamura (2020)[4].

We have previously discussed that the DA is not necessarily Pareto efficient and the TTC is not necessarily stable. Hence, the choice between the two boils down to a discussion between efficiency and fairness. This is the basic idea behind Pathak (2016). In the context of controlled choice however, it seems that complete fairness is not possible for all types of students. Both reserves and quotas are methods of controlled choice, but they have differing welfare outcomes. Kojima(2012) shows that quotas may hurt intended beneficiaries, while Hafalir et al.(2013) show that reserves outperform quotas with regards to student welfare. Stroh-Maraun (2024) proposed a Weighted Top-Trading Cycles (WTTC) mechanism where they adjust the TTC mechansim to give weights to each student. Essentially, they give every

[4]I attempt a sketch of a proof in Appendix 4.3 and 4.4

student with type t the same weight. They prove that the TTC mechanism leads to Pareto efficiency if all schools have the same priority over types. The problem here is that a student's weight might be too large and may be rejected for a student with a smaller weight. The author also shows that some students may be left unmatched. Aziz, Gaspers, and Sun (2020) propose a class of Generalized Deferred Acceptance for Type Combinations (GDA-TC) algorithms which yield outcomes that are non-wasteful and fair at least for students of the same type. GDA-TC is just a small extension to Controlled Choice models, where we consider a set $U = T_1 \times \cdots \times T_N$, which corresponds to all possible type combinations, where each student has exactly one type combination, essentually ensuring there is no type overlap. They find that it is strategy-proof for students. Upon running simulations they also find that it outperforms a version of the Deferred Acceptance algorithm in terms of achieving diversity goals. It seems that the GDA-TC might be a better mechansim to allocate better schools to lower-income students. Deferred acceptance is more also more widely used in practice due to its simplicity to understand, whereas it may be hard to reliably communicate information about the TTC mechanism to parents and students.

2.4 Information Asymmetry

In the discussion of Controlled Choice, we focused on policies that affect a school's choice of its students. Instead, we now focus on the student side of this mechanism problem. One of the issues that exacerbates income inequalities in school choice is that lower income students and parents are more likely to lack effective information about available educational opportunities. This could be partially because school quality information is difficult to measure and thus hard to communicate or because poorer households might lack the time to effectively research options for their children as they spend more of their time working.

Even though I have not explicitly mentioned it, we have so far only considered school choice when students and parents have perfect information about school quality but that is of course not the case. We consider then a framework analogous to the common values setting we discussed in class. Kloosterman and Troyan (2020) describe a situation where they examine school choice in an interdependent values framework, where student preferences might be correlated in some way possibly due to location or perception of quality. This quality perception may be the same for all students but perception of quality differs due to differences in information. They introduce a concept called the curse of acceptance, similar to the winner's curse in common value auctions. The idea is that more informed students will submit a preference ranking that lists high-quality schools on top, which leaves lower-quality schools with open spots. Less informed students are aware that others have more information than them, and so once assignment happens, they know that they have likely been assigned to a school of lower quality. So these students would prefer not to enroll in a school that accepts them.

2.4.1 Theoretical Model of Information Asymmetry

Once again, we have a set of students $J = \{j_1, \ldots, j_N\}$ and schools $S = \{s_1, \ldots, s_M\}$. Each school s has a capacity q_s. The preference interdependencies in this setting are modeled by considering an underlying state ω. The state space Ω is finite with an associated probability distribution. Every student has a state-dependent utility function $u_j^\omega(s)$, which reflects student j's utility for school s in state ω. We denote strict ordinal preferences for student j in state ω as $P_j(\omega)$. Agents evaluate lotteries using von Neumann-Morgenstern preferences. In particular we denote such preferences by an expected utility specification defined as

$$E(u_j^\omega(s) \mid \mathcal{I}') = \sum_{\omega \in \Omega} u_j^\omega(s) \Pr(\omega \mid \mathcal{I}')$$

where $\mathcal{I}' \subseteq \Omega$. Each student then receives a signal $\mathcal{I}_j(\omega) \subseteq \Omega$, where $\mathcal{I}_j(\omega)$ denotes the possible subset of states given j's signal when true state is ω. We can partition $J = I \cup U$, where for $j \in I$, we have that:

$$\mathcal{I}_j(\omega) = \{\omega' : P_j(\omega) = P_j(\omega')\}, \text{for all } \omega \in \Omega.$$

For $j \in U$, we have that:

$$\mathcal{I}_j(\omega) = \Omega \text{ for all } \omega \in \Omega.$$

Consider a mechanism $\psi : \mathcal{P}^N \to \mathcal{M}$, where $\mathcal{P}$ is the set of all strict ordinal preferences over S, and $\mathcal{M}$ is the set of all possible matches. A profile of strategies $\sigma = \{\sigma_1, \ldots, \sigma_N\}$ is a Bayesian Nash equilibrium of the game induced by the mechanism , if:

$$E(u_j^\omega(\psi_j(\sigma(\omega)))|\mathcal{I}_j(\omega)) \geq E(u_j^\omega(\psi_j(\sigma_j'(\omega), \sigma_-j(\omega)))|\mathcal{I}_j(\omega))$$

Consider a full matching that is just a function $\mu : \Omega \to \mathcal{M}$. Let $\mathcal{A}_j(s') = \{\omega \in \Omega : \mu_j^\omega = s'\}$, and $\mathcal{B}_j(s) = \Big\{\omega \in \Omega : \mu_j^\omega \neq s \text{ and } |\mu_s^\omega| < q_s \text{ or } j \succ_s j' \text{ for some } j' \in \mu_s^\omega\Big\}$. We define $\mathcal{C}_j(s', s) = \mathcal{A}_j(s') \cap B_j(s)$. Essentially, $\mathcal{C}_j(s', s)$ reflects student j's matching to s' that he could block with s. So a full matching μ is stable if there is no (j,s) blocking pair.

In order to depict the idea of stability, the researchers make some assumptions about what exactly is known by students. Students are assumed to know their own match, and they have knowledge of the equilibrium mapping from states to matches. A mechanism ψ is stable if there exists an equilibrium σ such that full matching $\mu(\sigma)$ is stable. This is similar to a rational bidder in the common values auction case. Here, a student decides whether to block ex-ante by updating his beliefs through considering ex-post realizations of the equilibrium matching that the mechanism induces[5]. They claim that even with these conditions, Deferred Acceptance will not be stable due to the curse of acceptance, that is, uninformed students will prefer higher quality schools after ex-post realizations of the mechanism.

2.4.2 Priority Design

So, now because of the impossibility of stable matching under DA with information asymmetries, we now switch to thinking about a class of priority structures that has desirable properties for DA. We say that school s is a secure school for student j if $|\{j' \in J : j' \succeq_s j\}| \leq q_s$, so it is a school with enough seats for j and every student who was priority over j. We could think of a secure school as simply being one's neighborhood school.

We now define a strategy profile $\sigma^*_j(\omega)$. For all informed students $j \in I$, $\sigma^*_j(\omega) = \omega$, and for all uninformed students $j \in U$, $\sigma^*_j(\omega) = \tilde{P}_j$ for all ω. Here $\tilde{P}_j$ is any preference ranking that lists one of j's secure schools first. For the informed students, a dominant ω is a dominant strategy as they are aware of the quality of schools and their decision is simply that signal. Consider than a market with common ordinal preferences and suppose every student has a secure school, then the equilibrium strategy $\sigma^*_j(\omega)$ induces a stable Deferred Acceptance matching.

We now consider a slight extension of correlated preferences under a continuum economy. Essentially, the state ω no longer determines a common ordinal ranking and students' may differ from the ranking of the intrinsic quality of schools. For each state ω, we have λ_ω over $\mathcal{P}$, where $\sum_{P \in \mathcal{P}} \lambda_\omega(P) = M$, where M is the total mass of students. We use this to essentially generalize an exchange economy under perfect competition in the context of school choice. Suppose each student j is endowed with priority $l_j(s) \in [0, 1]$, one for each school s. Essentially, this is their secure school. They also have endowment $l_j(s) + 1$ for any school that is not their secure school. Here we also define cardinal preferences as a function over students' ordinal preference rankings over schools. The Deferred Acceptance under secure school (DA-SS) is a stable matching induced by the equilibrium strategy σ^*.

[5]Example in Appendix 4.5

We now compare welfare outcomes under three conditions: DA-SS, DA-NSS, and No Choice. DA-NSS is Deferred Acceptance under no secure school option, which is just the standard DA algorithm in the case of ordinal preference rankings of schools. Under No Choice, students do not submit any information about their preferences and they are simply assigned a school which we assume to be their secure school. Kloosterman and Troyan show that under the continuum economy described above, No Choice strictly Pareto dominates DA-NSS for uninformed students U. This is largely because, students in U are prone to the curse of acceptance under DA-NSS. However, DA-SS in equilibrium σ^* strictly Pareto dominates No Choice for all students J. Hence, when we have heterogeneity in student information, the Deferred Acceptance Mechanism when students are given a secure school option is a stable matching that Pareto dominates any other DA mechanism.

3 Conclusion

We have explored extensions to mechanism design approaches in the context of school choice in order to devise equitable outcomes for lower-income students. My exploration largely leads me to consider Deferred Acceptance mainly due to its simplicity to both explain and implement, although we might lose some efficiency by using DA over TTC.

I consider two extensions, namely: Controlled Choice and Information Asymmetry. In Controlled Choice, we considered general setting of how schools might make their choice by introducing choice function. We considered some conditions for the choice function under which DA Pareto dominates any other stable mechanism. A simple extension to consider a unique type combination for a student is shown to outperform with regards to meeting diversity goals under reserves. We could consider a student to have essentially a low-income type combination and consider this algorithm. We then consider that different families could have different information about school quality, and it is likely that low-income students are less informed. We then extend school choice here to include a secure school option for each student. Essentially, this is a school that a student would have an option of attending if he were to drop out of the game altogether. With this option, we have the result that the equilibrium stable matching under DA Pareto dominates both DA without this option and a trivial match to the secure school. So, for urban school choice settings, a Deferred Acceptance with Type Combinations combined with secure school options for students would likely be a Pareto dominant outcome relative to having neither of these.

Of course, school choice in the real world is a lot more complicated than the simple application of any mechanism. The policy debates surrounding school choice involve school vouchers, magnet schools and charter schools. It would be interesting to see how mechanism design approaches can accommodate these complications. It is my understanding that most of the literature on school choice mechanisms largely focus on public school allocation in urban areas. Hence school choice in rural areas is another complication to consider. Rural areas are more likely to have the lowest income students but the mechanisms I consider here largely do not apply to rural school choice. This is because in rural areas, students largely only really have one option as there likely only one school that serves students from multiple counties, so an allocative mechanism would be trivial.

4 Appendix

4.1 Pareto Inefficiency of the Student Optimal Gale-Shapley Match

There are three students i1, i2, i3, and three schools s1, s2, s3, each of which has only one seat. The schools' priorities and student preferences are as follows.
$s1 : i1 \succ i3 \succ i2 \quad i1 : s2 \succ s1 \succ s3$
$s2 : i2 \succ i1 \succ i3 \quad i2 : s1 \succ s2 \succ s3$
$s3 : i2 \succ i1 \succ i3 \quad i3 : s1 \succ s2 \succ s3$
Let us interpret the school priorities as school preferences and consider the associated school admissions problem. In this case there is only one stable matching:

$$\begin{pmatrix} i1 & i2 & i3 \\ s1 & s2 & s3 \end{pmatrix}$$

But this matching is Pareto-dominated by

$$\begin{pmatrix} i1 & i2 & i3 \\ s2 & s1 & s3 \end{pmatrix}$$

The second match here Pareto dominates the first because now both i1 and i2 are better off because they both get their first choice schools. However, this match is unstable.

4.2 Instability of the TTC Mechanism

Under TTC, $i1$ points to his first choice which is $s2$, and both $i2$ and $i3$ point to thier first choice which is $s1$. Now, $i1$ is allocated to $s2$ and since $s1$ prefers $i3$, $i3$ is allocated to $s1$. $i2$ moves down his list an applies to $s2$. Since $s2$ already has full capacity, $i2$ is then allocated to $s3$. So our TTC matching gives:

$$\begin{pmatrix} i1 & i2 & i3 \\ s2 & s3 & s1 \end{pmatrix}$$

This is not a stable match because $i2$ prefers $s2$ over $s3$ and $s2$ prefers $i2$ over their assigned student $i1$. So there is a school-student pair that has a different preference over their current assignment. However, this is Pareto efficient for students because I can only make $i2$ better off but cannot do so without making $i1$ or $i3$ worse off.

4.3 Sketch of Proof that Non-Wastefulness implies Substitutability

A choice function is substitutable if for all $i \in I$, $I' \subseteq I \subseteq N$:

$$i \in C_s(I) \implies i \in C_s(I')$$

A choice function C is non-wasteful if for all $I \subseteq N$, and $i \in I$,

$$I \setminus C(I) \quad \text{and} \quad |C(I)| \leq q,$$

then

$$\left|C(I)_{\tau(i)}\right| = q_{\tau(i)}.$$

Consider a proof by contradiction. So, assume a choice function is not substitutable. Then student i is not necessarily chosen from a subset I' of I even if i is chosen in I. Since i is chosen in I, it must mean that there is that for students of type $\tau_(i)$, the quota has not been met yet, therefore it cannot be met when considering $I' \subseteq I$. So i being rejected in I' when the quota has not been met yet suggests wastefulness. So, we have that non-wastefulness implies substitutability.

4.4 Sketch of Proof that Non-Wastefulness satisfies the Law of Aggregate Demand

A choice function satisfies the law of aggregate demand if it chooses weakly more students when it considers additional students, that is, for all $I' \subseteq I \subseteq N$:

$$|C_s(I')| \leq |C_s(I)|$$

If the law of aggregate demand is not met, then that means that when it considers weakly more students, it chooses strictly less. Suppose when faced with options in I', we allocate $|I'|$, so everyone is chosen. Now, when faced with I, assume that $|I| = |I'| + 1$, so I has only one more student than I', but violation of law of aggregate demand suggests that $|C(I)| < |I'|$. What this means is that with I, I now have to drop one student out of the whole set of I'. Under I', since I took everyone, it must the case that I have not met my total quota and there is no individual who has had the quota of his type met yet. The fact that I drop from my choice $|I'|$ when faced with an additional student means that I do not satisfy non-wastefulness. So, we have that Non-Wastefulness must satisfy the Law of Aggregate Demand.

4.5 Matching and Blocking under Asymmetric Information

Suppose that there are three students $J = \{j_1, j_2, j_3\}$, with only j_1 uninformed and three schools $S = \{A, B, C\}$, each with capacity 1. A has priorities $j_3 \succ_A j_1 \succ_A j_2$. There are 4 states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, that are all equally likely. Then a matching is given by:

	$\boldsymbol{\mu}^{\omega}_{j_1}$	$\boldsymbol{\mu}^{\omega}_{j_2}$	$\boldsymbol{\mu}^{\omega}_{j_3}$
ω_1	C	A	B
ω_2	C	A	B
ω_3	C	B	A
ω_4	A	B	C

Figure 1: Matching example based on different states

Now suppose that j_1 is matched to C and is considering blocking with A. As j_1 observes that she is matched to C, she rules out ω_4 as a possible state. As j_1 has lower priority than j_3 at A but higher priority than j_2, so a blocking pair can only form in states ω_1 and ω_2. So j_1 will condition on the state being either of those two. Ex-ante, states are equally likely, so j_1 proposes a block if and only if $\frac{1}{2}u^{\omega_1}_{j_1}(A) + \frac{1}{2}u^{\omega_2}_{j_1}(A) > \frac{1}{2}u^{\omega_1}_{j_1}(C) + \frac{1}{2}u^{\omega_2}_{j_1}(C)$.

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