A study of some practical impacts of twisted embeddings in lattice-based cryptography

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Basic definitions

Lattices

A **lattice** $\Lambda \subset \mathbb{R}^n$ is a subgroup of the additive group $\mathbb{R}^n$.

Lattices

In other words, given m linear independent vectors in $\mathbb{R}^n$, the set $\{v_1, v_2, ..., v_m\}$ is called a **basis** for Λ and the lattice may be defined by:

$$\Lambda := \left\{ x = \sum_{i=1}^{m} \lambda_i v_i \in \mathbb{R}^n \mid \lambda_i \in \mathbb{Z} \right\}.$$

That is, any $\lambda \in \Lambda$ can be written as $\lambda = Mv$, where M is the **generator matrix** of Λ where each row is a vector from the basis and $v \in \mathbb{Z}^n$.

Lattices and cryptography

In the last two decades, lattice-based cryptosystems have become an important field in the cryptography community, since these cryptosystems rely on mathematical problems we believe are hard and quantum-resistant, such as the Shortest Vector Problem and the Shortest Independent Vectors Problem.

Lattices problems

Gap Shortest Vector Problem
For an approximation factor $\gamma = \gamma(n) \geq 1$, the *GapSVP*$_\gamma$ is: given a lattice Λ and length $d > 0$, output **YES** if $\lambda_1(\Lambda) \leq d$ and **NO** if $\lambda_1(L) > \gamma d$.

Shortest Independent Vectors Problem
For an approximation factor $\gamma = \gamma(n) \geq 1$, the *SIVP*$_\gamma$ is: given a lattice Λ, output n linearly independent lattice vectors of length at most $\gamma(n) \cdot \lambda_n(\Lambda)$.

The H space

Let $r, s, n \in \mathbb{Z}_+$ such that $n = r + 2s > 0$. The space $H \subset \mathbb{C}^n$ is defined as:

$$H = \{(a_1, \ldots, a_r, b_1, \ldots, b_s, \overline{b_1}, \ldots, \overline{b_s}) \in \mathbb{C}^n\},$$

where $a_i \in \mathbb{R}$, $\forall i \in \{1, \ldots, r\}$ and $b_j \in \mathbb{C}$, $\forall\, j \in \{1, \ldots, s\}$.

The H space

For all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in H$ the space H is endowed with inner product $\langle x, y \rangle_H$ defined as:

$$\langle x, y \rangle_H = \sum_{i=1}^{n} x_i \overline{y_i} = \sum_{i=1}^{r} x_i y_i + \sum_{i=1}^{s} x_{i+r} \overline{y_{i+r}} + \sum_{i=1}^{s} \overline{x_{i+r}} y_{i+r}.$$

The ℓ_2-norm and infinity norm of any $x \in H$ are defined as $\|x\| = \sqrt{\langle x, x \rangle_H}$ and $\|x\|_\infty = \max\{|x_i|\}_{i=1}^{n}$.

Number Fields

For K, L two fields, we denote by L/K a **field extension** if $K \subseteq L$. Then L is said to be an **extension field** over K, or just an **extension** over K. In a field extension L/K, L has the structure of a vector space over K.

A field extension is called a **number field** when it is over the rational field $\mathbb{Q}$.

Twisted embeddings

Let K and L be two field extensions and a homomorphism $\phi : K \to L$. ϕ is said to be a $\mathbb{Q}$-**homomorphism** if $\phi(a) = a, ; \forall a \in \mathbb{Q}$.
A $\mathbb{Q}$-homomorphism $\phi : K \to \mathbb{C}$ is called an **embedding**.

Twisted embeddings

Theorem
If K is a number field with degree n then there are exactly n embeddings $\sigma_i : K \to \mathbb{C}$ where by $\sigma_i(\theta) = \theta_i$ where $\theta_i \in \mathbb{C}$ is a distinct zero of K's minimum polynomial.

Twisted embeddings

The homomorphism $\sigma : K \to \mathbb{R}^r \times \mathbb{C}^s$, where (r, s) is the signature of K, is the **canonical embedding** and is defined by:

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_r(x), \sigma_{r+1}(x), \ldots, \sigma_{r+s}(x))\,.$$

Note that we could rewrite the canonical embedding as $\sigma : K \to \mathbb{R}^n$,

$$\begin{aligned}\sigma(x) = (&\sigma_1(x), \ldots, \sigma_r(x),\\ &\Re(\sigma_{r+1}(x)), \Im(\sigma_{r+1}(x)), \ldots, \Re(\sigma_{r+s}(x)), \Im(\sigma_{r+s}(x))).\end{aligned}$$

Learning problems

Learning from Parity

Given m vectors uniformly chosen $a_i \leftarrow \mathbb{Z}_2^n$ and some $\epsilon \in [0, 1]$, we define the problem **Learning from Parity (LFP)** as:

Find $s \in \mathbb{Z}_2^n$ such that, for $i \in \{1, \dots, m\}$

$$\langle s, a_i \rangle \;\approx_\epsilon\; b_i \;\; (mod\ 2).$$

In other words, the equality holds with probability $1 - \epsilon$.

Learning with Errors

Learning with Errors (LWE) is a generalization of LFP with two new parameters $p \in \mathbb{P}$ and χ a probability distribution on $\mathbb{Z}_p$ so that we have:

$$< s, a_i > \approx_\chi \; b_i \pmod{p} \quad \text{or} \quad < s, a_i > + e_i = b_i \pmod{p},$$

where $a_i \leftarrow \mathbb{Z}_p^n$ uniformly and $e_i \leftarrow \mathbb{Z}$ according to χ.

Ring-LWE search

Let K be a number field, $R = \mathcal{O}_K$ its ring of integers and $R^\vee$ the codifferent ideal of K. Also let $K_\mathbb{R}$ be the tensor product $K \otimes_\mathbb{Q} \mathbb{R}$.

Let Ψ be a family of distributions over $K_\mathbb{R}$. The **search version of the** *$ring - LWE$* **problem**, denoted $R - LWE_{q,\Psi}$, is defined as follows: given access to arbitrarily many independent samples from $A_{s,\psi}$ for some arbitrary $s \in R_q^\vee$ and $\psi \in \Psi$, find s.

Twisted Ring-LWE

For a totally positive element $\tau \in F$, let ψ_τ denote an error distribution over the inner product $\langle \cdot, \cdot \rangle_\tau$ and $s \in R_q^\vee$ (the "secret") be an uniformly randomized element. The *Twisted Ring-LWE distribution* $\mathcal{A}_{s,\psi_\tau}$ produces samples of the form

$$a, b = a \cdot s + e \pmod{qR^\vee} \in R_q \times K_\mathbb{R}/qR^\vee.$$

Twisted Ring-LWE hardness

Solving the Twisted Ring-LWE is as hard as solving the usual Ring-LWE.

Theorem

Let K be an arbitrary number field, and let $\tau \in F$ be totally positive. Also, let (s, ψ) be randomly chosen from $(U(R_{\boldsymbol{q}}^{\vee}) \times \Psi)$ in $(K_{\mathbb{R}}, \langle \cdot, \cdot \rangle_{\tau=1})$. Then there is a polynomial-time reduction from Ring-LWE$_{\boldsymbol{q},\psi}$ to Ring-LWE$^{\tau}_{\boldsymbol{q},\psi_\tau}$.

Twisted R-LWE cryptosystem

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Objectives

Main goal

- Validate the idea of using twisted embeddings in cryptography
- Explore the theoretical and the practical aspects of this proposal

Practical aspects

- Compare implementations and instances of the Twisted Ring-LWE and Ring-LWE
- Maximum realsubfield versus the cyclotomic power-of-tw
- Search for proper sizes of keys and messages

Theoretical aspects

- Study the polynomial arithmetic of themaximal real subfield
- Study the relation between the orthonormal basis and the efficient conversion between latticepoints and elements of number field
- Examine if it is possible toachieve a satisfactory efficiency with non-orthonormal basis

Methodology and timeline

Methodology

- **Literature Review:** review proposals of new cryptosystems, such as *NTTRU*.
- **Theoretical experiments:** perform experiments using algebra libraries to discover twist factors and to discover orthonormal bases.
- **Experimental outcome:** to calculate the expansion factor of the polynomial $f(x)$ that defines the ring $\mathbb{Z}[x]/f(x)$. Adapt or develop algorithms for polynomial multiplication.
- **Implementation:** implement a Twisted Ring-LWE based cryptosystem.
- **Practical experiments:** to estimate the cost in terms of clock cycles, also key and message sizes.

Timeline

- First and second semesters of 2021
  - Study the Twisted Ring LWE problem and implementation.
  - Perform theoretical experiments with number fields, twist factors and lattices.
  - Calculate the expansion factor and adapt/develop algorithms for polynomial multiplication.
- First and second semesters of 2022
  - Implement a Twisted Ring-LWE based cryptosystem.
  - Compare instances of Ring LWE and Twisted Ring LWE, *i.e.*, analyze the cryptosystem in both terms of clock cycles and key sizes.
  - Defense of dissertation.

Thank you!