Universidade Estadual de Campinas
Instituto de Computação

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The Dissertation or Thesis Title in English

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Dissertação apresentada ao Instituto de Computação da Universidade Estadual de Campinas como parte dos requisitos para a obtenção do título de Mestra em Ciência da Computação.

Dissertation presented to the Institute of Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Master in Computer Science.

Supervisor/Orientador: Prof. Dr. Ricardo Dahab

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Chapter 1

Introduction

Chapter 2

Mathematical Background

2.1 Groups

Definition 2.1.1. A **group** is a set G closed under a binary operation $\cdot$ defined on G such that:

- **Associativity:** $\forall a, b, c \in G,\ a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- **Identity element:** $\exists e \in G\ ;\ \forall a \in G,\ a \cdot e = e \cdot a = a$
- **Inverse element:** $\forall a \in G,\ \exists b \in G\ ;\ a \cdot b = b \cdot a = e$

And it is denoted by $\langle G, \cdot \rangle$, or simply G if the operation is implied.

Definition 2.1.2. A group is said to be **commutative** or **abelian** if $\forall a, b \in G,\ a \cdot b = b \cdot a$

A group is called **additive** or **multiplicative** if its operation is addition or multiplication, respectively.

Definition 2.1.3. A subset H of G is a **subgroup** of $\langle G, \cdot \rangle$ if it is closed under $\cdot$ induced by $\langle G, \cdot \rangle$.

Definition 2.1.4. The **order** of a group $\langle G, \cdot \rangle$ is the cardinality of the set G.

Definition 2.1.5. A subgroup H of G can be used to decompose G in uniform sized and disjoints subsets called **cosets**. Given an element $g \in G$:

- A **left coset** is defined by $gH := \{g \cdot h\ ;\ h \in H\}$
- A **right coset** is defined by $Hg := \{h \cdot g\ ;\ h \in H\}$

2.2 Rings and Fields

Definition 2.2.1. A **ring** is a set together with two binary operations, we will note by $+$ and $*$ and call it addition and multiplication, respectively, such that:

- $\langle R, + \rangle$ is an abelian group.
- $*$ is associative
- $*$ is distributive over $+$

And it is denoted by $\langle R, +, *\rangle$, or simply G if the operations are implied.

Definition 2.2.2. A ring is said to be **commutative** if its $*$ operation is commutative.

Definition 2.2.3. A ring is said to be **with unity** if $*$ has a identity element. We shall note it by 1 and it is called **unity**.

Definition 2.2.4. A **division ring** is a ring R where $\forall r \in R,\ \exists s \in R\ ;\ r * s = 1$.

Definition 2.2.5. A **field** is a commutative division ring.

2.3 Lattices

Definition 2.3.1. A Lattice $\Lambda \subset \mathbb{R}^n$ is a subgroup of the additive group $\mathbb{R}^n$

In other words, given m linear independent vectors in $\mathbb{R}^n$, the set $\{v_1, v_2, ..., v_m\}$ is called a **basis** for Λ and the Lattice may defined by:

Definition 2.3.2.

$$\Lambda := \left\{ x = \sum_{i=1}^{m} \lambda_i v_i \in \mathbb{R}^n \mid \lambda_i \in \mathbb{Z} \right\}$$

I.e., any $\lambda \in \Lambda$ can be written as $\lambda = Mv$ where M is the **generator matrix** of Λ where each row is a vector from the basis and $v \in \mathbb{Z}^n$.

2.4 Number Fields

Definition 2.4.1. Let K and L be two fields, L is said to be a **field extension** of K if $L \subseteq K$ and we denote it by L/K

Note that in a field extension L/K, L has a structure of a vector space over K, where vector addition is in L and scalar multiplication $a \in K,\ v \in L \implies av \in L$. The dimension of L as a vector space is called **degree** and it is denoted by $[L : K]$.

Definition 2.4.2. A field extension is called **number field** when it is over $\mathbb{Q}$.

Definition 2.4.3. Let $\alpha \in L$ where L/K is a field extension. We say that α is **algebraic over** K if $\exists p \in K[X]\ ;\ p(\alpha) = 0$. p is said to be **the minimal polynomial of** α **over** K denoted by p_α. If $\alpha \in L = \mathbb{Q}[\theta]$, we simply call α an **algebraic number**.

Example 2.4.1. It is known that $\mathbb{Q}$ is a field. If we add $\sqrt{2}$ to the set, we can build a new field adding also all the powers and multiples of $\mathbb{Q}$. This new field is denoted by $\mathbb{Q}[\sqrt{2}]$, note that $\sqrt{2}$ is algebraic and its minimal polynomial $p_{\sqrt{2}} = x^2 - 2$. All elements of $\mathbb{Q}[\sqrt{2}]$ are in the form $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ and one of its basis is $\{1, \sqrt{2}\}$, so it has degree is 2.

Example 2.4.2. If we add $\sqrt[3]{2}$ to $\mathbb{Q}$ instead, its elements would have the form $\{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$, so one of its basis is $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$, $p_\alpha = x^3 - 2$ and its degree is 3.

Theorem 2.4.1 (add font 45 p.40)**.** *If K is a number field, then $K = \mathbb{Q}[\theta]$ for some algebraic number $\theta \in K$, called primitive element.*

Then we conclude that $\{1, \theta, \theta^2, ..., \theta^{n-1}\}$ is a basis for the vector space $K = \mathbb{Q}[\theta]$ over $\mathbb{Q}$. Note that we can represent an number $a \in K$ as a linear combination of θ, *i.e* $a = \sum_{i=0}^{n} a_i\theta^i$ or as a polynomial $a(x) = \sum_{i=0}^{n} a_i x^i$.

Definition 2.4.4. A number α is said to be an **algebraic integer** if $p \in \mathbb{Z}[X]$; $p(\alpha) = 0$. The set of all algebraic integers of K forms a ring called **ring of ingegers** of K and is denoted by $\mathcal{O}_K$.

Definition 2.4.5. An **integral basis** is a basis for a ring of integers.

2.5 Twisted Embeddings

2.5.1 Embeddings

Definition 2.5.1. Let K and L be two field extensions and a homomorphism $\phi : K \to L$. ϕ is said to be a $\mathbb{Q}$**-homomorphism** if $\phi(a) = a, ; \forall a \in \mathbb{Q}$

Definition 2.5.2. A $\mathbb{Q} - homomorphism; \phi : K \to \mathbb{C}$ is callend an **embedding**.

Theorem 2.5.1 (inserir fonte 45, p.41)**.** *If K is a number field with degree n then there are exactly n embeddings $\sigma_i : K \to \mathbb{C}$ where by $\sigma_i(\theta) = \theta_i$ where $\theta_i \in \mathbb{C}$ is a distinct zero of the K's mininum polynomial.*

Definition 2.5.3. Let $\{\sigma_i\}_n$ the possible embeddings of a number field K. Let r the number of embeddings with real images and $2s$ the complex ones, then $r + 2s = n$. The pair (r, s) is called **signature** of K.

Definition 2.5.4. The homomorphism $\sigma : K \to \mathbb{R}^r \times \mathbb{C}^s$, where (r, s) is the signature of K, is said to be the **canonical embedding** and is defined by:

$$\sigma(x) = (\sigma_1(x), ..., \sigma_r(x), \sigma_{r+1}(x), ..., \sigma_{r+s}(x))$$

Note that we could rewrite the canonical embedding as $\sigma : K \to \mathbb{R}^n$

$$\sigma(x) = (\sigma_1(x), ..., \sigma_r(x), \Re(\sigma_{r+1}(x)), \Im(\sigma_{r+1}(x)), ..., \Re(\sigma_{r+s}(x)), \Im(\sigma_{r+s}(x)))$$

For now on we will denote it simply by:

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_r(x), \sigma_{r+1}(x), \ldots, \sigma_{r+2s}(x))$$

2.5.2 Algebraic Lattices

Theorem 2.5.2 (adicionar citaçao 45, p155)**.** *Let $\{\omega_1, ..., \omega_n\}$ be an integral basis of K, The n vectors $v_i = \sigma(\omega_i) \in \mathbb{R}^n$ are linearly independent, so thety define a full rank algebraic lattice $\Lambda = \Lambda(\mathcal{O}_K) = \sigma(\mathcal{O}_K)$.*

The generator matrix of $\Lambda = \sigma(\mathcal{O}_K)$ is defined by:

$$\begin{pmatrix} \sigma_1(\omega_1) & \ldots & \sigma_{r+2s}(\omega_1) \\ & \vdots & \\ \sigma_1(\omega_n) & \ldots & \sigma_{r+2s}(\omega_n) \end{pmatrix} \tag{2.1}$$

Remark 2.5.1. An embedding creates the correspondence between a point $\lambda \in \Lambda \subset \mathbb{R}^n$ of an algebraic lattice (Theo. 2.5.2) and an integer in $\mathcal{O}_K$:

Let λ be a point of a lattice Λ:

$$\begin{aligned}\lambda &= (\lambda_1, \ldots, \lambda_{r+2s}) \in \Lambda \\ &= \left(\sum_{i=1}^{n} z_i \sigma_1(\omega_i), \ldots, \sum_{i=1}^{n} z_i \sigma_{r+2s}(\omega_i)\right) \\ &= \left(\sigma_1\left(\sum_{i=1}^{n} z_i \omega_i\right), \ldots, \sigma_{r+2s}\left(\sum_{i=1}^{n} z_i \omega_i\right)\right)\end{aligned}$$

where $z_i \in \mathbb{Z}$. Since any element $x \in \mathcal{O}_K$ has the form $x = \sum_{i=1}^{n} \lambda_i \omega_i$, we can conclude that:

$$\lambda = (\sigma_1(x), \ldots, \sigma_{r+2s}(x)) = \sigma(x)$$

2.5.3 Twisted Embeddings

Definition 2.5.5. Let K be a number field with degree n and σ an embedding. We say that a number $\tau \in K$ is **totally positive** if $\forall i \in 1, \ldots, n,\ \sigma_i(\tau) \in \mathbb{R}_+^*$

Definition 2.5.6 (Twisted Embedding)**.** Given τ a totally positive number, the **τ-twisted embedding**, or simply twisted embedding, is the monomorphism defined as:

$$\sigma_\tau(x) = (\sqrt{\tau_1}\sigma_1(x), \ldots, \sqrt{\tau_{r+2s}}\sigma_{r+2s}(x))$$

where $\tau_i = \sigma_i(\tau)$.

Chapter 3

Objectives

Chapter 4

Methodology

Bibliography