Properties of nano semi pre-θ-closed set in nano topological spaces

P. Sathishmohan, V. Rajendran and C. Vignesh Kumar*
Department of Mathematics,
Kongunadu Arts and Science College, Coimbatore−641 029.
Email :[1] iiscsathish@yahoo.co.in,[3] vigneshkumarcvk@gmail.com

Mathematics Subject Classification: 54B05
Keywords: nano β-θ-interior, nano β-θ-frontier, nano β-θ-exterior and nano β-θ-derived sets.

1. Introduction

Lellis Thivagar and Richard [1] established the notion of nano topology in terms of approximations and boundary region of a subset of an universe using an equivalence relation on it and also make known about nano-closed sets, nano-interior, nano-closure and weak form of nano open sets namely nano semi-open sets, nano pre-open, nano α-open sets and nano β-open sets. Nasef et.al.[2] make known about some of nearly open sets in nano topological spaces. Revathy and Gnanambal Illango [4] gave the idea about the nano β-open sets. Sathishmohan et.al.[5] brings up the idea about nano neighourhoods in nano topological spaces. This motivates the author to induct and study the properties of nano β-θ-closed sets in nano topological spaces.

The structure of this manuscript is as follows:

In section 2, we recall some existing definitions and remarks which are more important to prove our main results.

In section 3, we induct and study some theorems which satisfies the conditions of nano β-θ-interior points and nano β-θ-derived sets.

In section 4, we brings up the concept of nano β-θ-frontier of a sets and nano β-θ-exterior of a sets and also proved some the theorems which satisfies some existing properties.

2. Preliminaries

Definition 2.1. [3] *Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$. Then,*

(i)The lower approximation of x with respect to R is the set of all objects, which can be for certain classified as X with respect to R and is denoted by $L_R(X)$.

$L_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \subseteq X\}$ where $R(x)$ denotes the equivalence class determined by $x \in U$.

(ii)The upper approximation of x with respect to R is the set of all objects which can be possibly classified as X with respect to R and is denoted by $U_R(X)$.

$U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$

(iii)The boundary region of x with respect to R is the set of all objects which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$.

$B_R(X) = U_R(X)$ - $L_R(X)$.

Definition 2.2. [3] *Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms*

(i) U and $\phi \in \tau_R(X)$.

(ii) The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.
(iii) The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.
Then $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X. We call $(U, \tau_R(X))$ as nano topological space. The elements of $\tau_R(X)$ are called as nano-open sets. The complement of the nano-open sets are called nano-closed sets.

Remark 2.3. [3] *If $\tau_R(X)$ is the nano topology on U with respect to X , then the set $B = \{U, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.*

Definition 2.4. [1] *If $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then*
(i)The nano interior of A is defined as the union of all nano-open subsets of A is contained in A and is denoted by $Nint(A)$. That is, $Nint(A)$ is the largest nano-open subset of A.
(ii)The nano closure of A is defined as the intersection of all nano-closed sets containing A and is denoted by $Ncl(A)$. That is, $Ncl(A)$ is the smallest nano-closed set containing A.

Definition 2.5. [5] *For a subset A of U, $N\beta cl(A) - N\beta int(A)$ is said to be nano semi pre-frontier of A and is denoted by $N\beta Fr(A)$.*

Definition 2.6. [5] *The set of all $N\beta$-limit points of A is said to be the nano semi pre derived set and is denoted by $N\beta D(A)$.*

Definition 2.7. [5] *A point $x \in U$ is called nano semi pre-exterior point of a subset A of U if x is $N\beta$-interior point of $U - A$ and set of all $N\beta$-exterior points of A is called $N\beta$-exterior of A and denoted by $N\beta Ext(A)$. Therefore $N\beta Ext(A) = N\beta int(U - A)$.*

Definition 2.8. [6] *A space U is said to be nano semi pre-regular if for each $N\beta$-closed set F and for each $x \in U - F$ there exists nano-open sets G and H such that $F \subset G$ and $x \in H$ and it is denoted by $N\beta$-regular space.*

3. Nano β-θ-interior

Definition 3.1. *A point $x \in U$ is called a nano semi pre-θ cluster (briefly, $N\beta$-θ-cluster) point of A if $N\beta cl(G) \cap A \neq \phi$ for every $G \in N\beta O(U, x)$. The set of all nano β-θ-cluster points of A is called nano semi pre-θ-closure of A and it is denoted by $N\beta$-$\theta cl(A)$. If $N\beta$-$\theta cl(A) = A$, then A is said to be nano semi pre-θ-closed (briefly, $N\beta$-θ-closed). The complement of a nano semi pre-θ-closed set is said to be nano semi pre-θ-open (briefly, $N\beta$-θ-open).*

Definition 3.2. *A point $x \in U$ is called a nano β-θ-interior (briefly, $N\beta$-θ-interior) point of A if there exists $N\beta$-open set G containing x such that $G \subset N\beta cl(G) \subset A$. The set of all $N\beta$-θ-interior points of A is called $N\beta$-θ-interior of A and is denoted by $N\beta$-$\theta int(A)$. Thus $N\beta$-$\theta int(A) = \{x \in U / x \in G \subset N\beta cl(G) \subset A, \forall\ G \in N\beta O(U, x)\}$.*

Example 3.3. *Let $U = \{1, 2, 3, 4\}$, $U/R = \{\{1\}, \{2, 4\}, \{3\}\}$ and $X = \{1, 2\}$. Then $\tau_R(X) = \{U, \phi, \{1\}, \{1, 2, 4\}, \{2, 4\}\}$ be a nano topology on U. We have $N\beta O(U, \tau_R(X)) = \{U, \phi, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.*
Let $x = \{1\}$ and $G = \{1, 2\}$ then $N\beta$-$\theta int(A) = \{1, 2\}$.

Theorem 3.4. *A subset G of a space is $N\beta$-θ-open in U if and only if for each $x \in G$, there exists a $N\beta$-open set W containing x such that $N\beta cl(W) \subset G$.*

Proof : Suppose that G be $N\beta$-θ-open set of U implies $U-G$ is $N\beta$-θ-closed. Let $x \in G$, then $x \notin N\beta$-$\theta cl(U-G)$ and so there exists a $N\beta$-open set W containing x such that $N\beta cl(W) \cap (U-G) = \phi$ implies $N\beta cl(W) \subset G$.
Conversely, assume that G is not $N\beta$-θ-open set in U, then $U-G$ is not $N\beta$-θ-closed, so there exists $x \in N\beta$-$\theta cl(U-G)$ but $x \notin (U-G)$. Since $x \in G$, by hypothesis there exists a $N\beta$-open set W containing x such that $N\beta cl(W) \subset G$. Thus $N\beta cl(W) \cap (U-G) = \phi$. This is a contradiction to $x \in N\beta$-$\theta cl(U-G)$. Therefore G is $N\beta$-θ-open set in U.

Theorem 3.5. *For subsets A and B of a space U the following statements are true,*

(1) $U - N\beta\text{-}\theta int(A) = N\beta\text{-}\theta cl(U-A)$.
(2) $U - N\beta\text{-}\theta cl(A) = N\beta\text{-}\theta int(U-A)$.
(3) $N\beta\text{-}\theta int(A)$ *is* $N\beta$-θ-*open iff* $A = N\beta\text{-}\theta int(A)$.
(4) $N\beta\text{-}\theta int(N\beta\text{-}\theta int(A)) \subset N\beta\text{-}\theta int(A)$.
(5) $A \subset B \Rightarrow N\beta\text{-}\theta int(A) \subset N\beta\text{-}\theta int(B)$.
(6) $N\beta\text{-}\theta int(A) \cup N\beta\text{-}\theta int(B) \subset N\beta\text{-}\theta int(A \cup B)$.
(7) $N\beta\text{-}\theta int(A \cap B) \subset N\beta\text{-}\theta int(A) \cap N\beta\text{-}\theta int(B)$

Proof

(1) $x \in U - N\beta\text{-}\theta int(A) \Rightarrow x \notin N\beta\text{-}\theta int(A)$ therefore for each $N\beta$-open set G containing x, $x \in G \subset N\beta cl(W) \not\subset A \Rightarrow N\beta cl(W) \cap (U-A) = \phi$ for every $N\beta$-open set G containing x, therefore $x \in N\beta\text{-}\theta cl(U-A)$ that implies $U - N\beta\text{-}\theta int(A) = N\beta\text{-}\theta cl(U-A)$.

(2) It is obvious from (1).

(3) Let A be $N\beta$-θ-open $\Rightarrow U-A$ is $N\beta$-θ-closed. Therefore $N\beta\text{-}\theta cl(U-A) = U-A$ i.e $U - N\beta\text{-}\theta int(A) = U-A \Rightarrow N\beta\text{-}\theta int(A) = A$. Converse is obvious.

(4) Since $N\beta\text{-}\theta int(B) \subset B$ therefore $N\beta\text{-}\theta int(N\beta\text{-}\theta int(A)) \subset N\beta\text{-}\theta int(A)$.

(5) Let $x \in N\beta\text{-}\theta int(A) = \cup\{G/x \in G \subset N\beta\ cl(G) \subset A \subset B\} = N\beta\text{-}\theta int(B)$ implies $N\beta\text{-}\theta int(A) \subset N\beta\text{-}\theta int(B)$.

(6) $A \subset A \cup B$ and $B \subset A \cup B \Rightarrow N\beta\text{-}\theta int(A) \subset N\beta\text{-}\theta int(A \cup B)$ and $N\beta\text{-}\theta int(B) \subset N\beta\text{-}\theta int(A \cup B) \Rightarrow N\beta\text{-}\theta int(A) \cup N\beta\text{-}\theta int(B) \subset N\beta\text{-}\theta int(A \cup B)$.

(7) Clearly $N\beta\text{-}\theta int(A \cap B) \subset N\beta\text{-}\theta int(A) \cap N\beta\text{-}\theta int(B)$ but $N\beta\text{-}\theta int(A) \cap N\beta\text{-}\theta int(B) \not\subset N\beta\text{-}\theta int(A \cap B)$.

Note that $N\beta\text{-}\theta int(A \cup B) \not\subset N\beta\text{-}\theta int(A) \cup N\beta\text{-}\theta int(B)$, it can be shown by the following example.

Example 3.6. *From the Example 3.3, let $A = \{3\}$ and $B = \{4\}$ then $N\beta\text{-}\theta int(A \cup B) = \{3,4\}$ but $N\beta\text{-}\theta int(A) \cup N\beta\text{-}\theta int(B) = \{4\}$.*

Theorem 3.7. *A subset A of a space U is $N\beta$-θ-open iff $N\beta\text{-}\theta int(A) = A$.*

Proof : Let A be a $N\beta$-θ-open set therefore $U-A$ is $N\beta$-θ-closed. That implies $N\beta\text{-}\theta cl(U-A) = U-A$ i.e $U - N\beta\text{-}\theta int(A) = U-A$ that implies $N\beta\text{-}\theta int(A) = A$ and other part is obvious.

Theorem 3.8. *For a subset A of a nano topological space U, following are true*

(1) $N\beta cl(A) \subset N\beta\text{-}\theta cl(A)$ *and* $N\beta\text{-}\theta int(A) \subset N\beta int(A)$
(2) $N\beta cl(A) = N\beta\text{-}\theta cl(A)$ *if* $A \in N\beta O(U)$

(3) $N\beta int(A) = N\beta\text{-}\theta int(A)$ *if* A *is* $N\beta$*-closed*

Proof

(1) Let $x \in U - (N\beta\text{-}\theta cl(A)) = N\beta\text{-}\theta int(U - A)$, therefore there exists a $N\beta$-open set G such that $x \in G \subset N\beta cl(G) \subset U - A$ implies that $x \in U - A$ $\Rightarrow x \notin A \subset N\beta cl(A)$ i.e. $x \notin N\beta cl(A)$. Therefore $N\beta cl(A) \subset N\beta\text{-}\theta cl(A)$. Similrly, let $x \in N\beta\text{-}\theta int(A)$ implies there exists $G \in N\beta O(U)$ such that $x \in G \subset N\beta cl(W) \subset A$, implies that $x \in G \subset A$ i.e $x \in N\beta int(A)$. Thus, $N\beta\text{-}\theta int(A) \subset N\beta int(A)$.

(2) From (1) $N\beta cl(W) \subset N\beta\text{-}\theta cl(A)$. Let $A \in N\beta O(U)$ and $x \notin N\beta cl(A)$ then there exists $N\beta$-open set G containing x such that $G \cap A = \phi$, this shows that $N\beta cl(W) \cap A = \phi$, as $A \in N\beta O(U)$, $x \notin N\beta\text{-}\theta(A)$. Therefore $N\beta\text{-}\theta cl(A) \subset N\beta cl(A)$ and hence $N\beta\text{-}\theta cl(A) = N\beta cl(A)$ if $A \in N\beta O(U)$.

(3) We have $N\beta\text{-}\theta int(A) \subset N\beta int(A)$. Let A be $N\beta$-closed and let $x \in N\beta int(A)$. Therefore there exists $N\beta O(U)$ set G such that $x \in G \subset A$ but $G \subset N\beta cl(W)$ and $A = N\beta cl(A)$ as it is $N\beta$-closed. Therefore, $x \in G \subset A \Rightarrow N\beta cl(W) \subset N\beta cl(A) = A$ this shows that $x \in N\beta cl(G) \subset A$ that is $x \in G \subset N\beta cl(W) \subset A$ for $G \in N\beta O(U)$. Hence, $x \in N\beta\text{-}\theta int(A)$. Therefore $N\beta int(A) \subset N\beta\text{-}\theta int(A)$ if A is $N\beta$-closed. Hence $N\beta$ int(A) = $N\beta\text{-}\theta int(A)$.

We define $N\beta\text{-}\theta$ derived set in the following.

Definition 3.9. *A point* $x \in U$ *is called* $N\beta\text{-}\theta$*- limit point of a subset* A *of a space* U *if every* $N\beta\text{-}\theta$*-open set* G *containing* x *contains points of* A *other than* x*, i.e* $\{G - \{x\}\} \cap A \neq \phi$ *and the set of all* $N\beta\text{-}\theta$*-limit points of* A *is called* $N\beta\text{-}\theta$ *derived set of* A *and is denoted by* $N\beta\text{-}\theta D(A)$

Theorem 3.10. *For subsets* A *and* B *of* U *the following are true:*

(1) $N\beta D(A) \subset N\beta\text{-}\theta D(A)$ *where* $N\beta D(A)$ *is set of* $N\beta$*-limit points of* A
(2) *If* $A \subset B$ *then* $N\beta\text{-}\theta D(A) \subset N\beta\text{-}\theta D(B)$
(3) $N\beta\text{-}\theta D(A) \cup N\beta\text{-}\theta D(B) \subset N\beta\text{-}\theta D(A \cup B)$
(4) $N\beta\text{-}\theta D(A \cap B) \subset N\beta\text{-}\theta D(A) \cap N\beta\text{-}\theta D(B)$

Proof

(1) Let $x \in N\beta D(A)$ therefore every $N\beta$-open set G containing x contains points of A other than x implies every $N\beta\text{-}\theta$-open set G containing x contains points of A other than x as every $N\beta\text{-}\theta$-open set is $N\beta$-open. Therefore $N\beta D(A) \subset N\beta\text{-}\theta D(A)$.

(2) Let $A \subset B$ and $x \in N\beta\text{-}\theta D(A)$ implies $\{G - \{x\}\} \cap A \neq \phi$ for every $N\beta$-open set G containing x therefore $\{G - \{x\}\} \cap B \neq \phi$ for every $N\beta$-open set G containing x which implies $N\beta\text{-}\theta D(A) \subset N\beta\text{-}\theta D(B)$.

(3) $A \subset A\cup B$, $B \subset A\cup B = N\beta\text{-}\theta D(A) \cup N\beta\text{-}\theta D(B) \subset N\beta\text{-}\theta D(A \cup B)$.

(4) We know that $A \cap B \subset A$ and $A \cap B \subset B$ implies from (2) $N\beta\text{-}\theta D(A \cap B) \subset N\beta\text{-}\theta D(A)$ and $N\beta\text{-}\theta D(A \cap B) \subset N\beta\text{-}\theta D(B)$ and hence $N\beta\text{-}\theta D(A \cap B) \subset N\beta\text{-}\theta D(A) \cap N\beta\text{-}\theta D(B)$.

Theorem 3.11. *For a subset* A *of a space* U*,* $x \in N\beta cl(A)$ *iff* $G \cap A \neq \phi$ *for any* $N\beta$*-open set* G *containing* x*.*

Proof: Recall that $N\beta cl(A)$ is intersection of all $N\beta$- closed sets containing A. i.e $N\beta cl(A) = \cap F$ where F is $N\beta$-closed set containing A. Let us assume that $x \notin N\beta cl(A) \Rightarrow x \notin \cap F$, Therefore there exists a $N\beta$-closed set F_x such that $x \notin F_x \Rightarrow x \in (U - F)$ and $(U - F)$ is a $N\beta$-open set such that $A \cap (U - F) \subset N\beta cl(A) \cap (U - F) = \phi$. Therefore there exists a $N\beta$-open set such that $(U - F) \cap A = \phi$, contradiction to hypothesis. Therefore $x \in N\beta cl(A)$.
Conversely, Let $x \in N\beta cl(A)$ and assume that $G \cap A = \phi$ for some $N\beta$-open set G containing x. Therefore $A \subset (U - G)$ and $(U - G)$ is a $N\beta$-closed set not containing x i.e there exists a $N\beta$-closed set $(U - G)$ contained in A but not containing $x \Rightarrow x \notin N\beta cl(A)$ which is contradiction to hypothesis. Therefore $G \cap A \neq \phi$. Thus $x \in N\beta cl(A)$ iff $G \cap A \neq \phi$ for every $N\beta$- open set G containing x.

Theorem 3.12. *In a nano topological space $(U, \tau_R(X))$ for a subset A, $N\beta D(A) \subset N\beta\text{-}\theta cl(A)$.*

Proof: Let $x \in N\beta D(A)$, therefore every $N\beta$-open set G containing x contains points of A other than x i.e $\{G - \{x\} \cap A\} \neq \phi \Rightarrow G \cap A \neq \phi$ then by Theorem 3.11, $x \in N\beta cl(A) \subset N\beta\text{-}\theta cl(A)$ therefore $N\beta D(A) \subset N\beta\text{-}\theta cl(A)$.

Theorem 3.13. *A nano topological space U is $N\beta$-regular iff $N\beta cl(A) = N\beta\text{-}\theta cl(A)$ for any subset A of U.*

Proof: Let U be $N\beta$-regular and $x \in N\beta\text{-}\theta cl(A)$. As U is $N\beta$-regular, for any $N\beta$-open set G containing x, there exists $N\beta$-open set V such that $x \in V \subset N\beta cl(V) \subset G$. On the other hand, as $x \in N\beta\text{-}\theta cl(A)$, $N\beta cl(V) \cap A \neq \phi \Rightarrow G \cap A \neq \phi$ as $N\beta cl(V) \subset G$ for all $N\beta O(U)$ set G containing $x \Rightarrow x \in N\beta cl(A)$. Therefore, $N\beta\text{-}\theta cl(A) \subset N\beta cl(A)$ and $N\beta cl(A) \subset N\beta\text{-}\theta cl(A)$ is obvious. Hence, $N\beta\text{-}\theta cl(A) = N\beta cl(A)$ if U is $N\beta$- regular.
Conversely $N\beta\text{-}\theta cl(A) = N\beta cl(A)$. Let F be $N\beta$-closed set such that $x \notin F \Rightarrow x \notin N\beta cl(F)$ and hence $x \notin N\beta\text{-}\theta cl(F)$. Therefore there exists a $N\beta$-open set G containing x such that $N\beta cl(G) \cap F = \phi$. This implies that $F \subset U - N\beta cl(G)$ which is $N\beta$-open. Therefore for a nano semi pre-closed set F not containing x, there exists two disjoint $N\beta$-open sets G and $U - N\beta cl(G)$ such that $x \in G$ and $F \subset U - N\beta cl(G)$. Therefore U is a $N\beta$-regular space.

Theorem 3.14. *If U is $N\beta$-regular then*

(1) *every $N\beta$-closed subset A of U is $N\beta$-θ-closed.*
(2) *$N\beta\text{-}\theta cl(A)$ is a $N\beta$-θ-closed set and $N\beta\text{-}\theta int(A)$ is a $N\beta$-θ-open set.*

Proof:

(1) Let U be $N\beta$-regular. Let A be $N\beta$-closed then $N\beta cl(A) = A$. By Theorem 3.13 , $N\beta\text{-}\theta cl(A) = N\beta cl(A)$ as U is $N\beta$-regular, implies that $N\beta\text{-}\theta cl(A) = A$. Therefore A is $N\beta$-θ-closed.
(2) From (1), $N\beta\text{-}\theta cl(N\beta\text{-}\theta cl(A)) = N\beta\text{-}\theta cl(A)$ which implies that $N\beta\text{-}\theta cl(A)$ is $N\beta$-θ-closed. In other words, we have $N\beta\text{-}\theta cl(A) = N\beta cl(A)$ and $N\beta cl(A)$ is $N\beta$-closed set implies $N\beta\text{-}\theta cl(A)$ is $N\beta$-closed by (1) above , since every $N\beta$-closed is $N\beta$-θ-closed. Therefore $N\beta\text{-}\theta cl(A)$ is $N\beta$-θ-closed. Similarly $N\beta\text{-}\theta int(A)$ is $N\beta$-θ-open.

Next, we define and study the concept of $N\beta$-θ-frontier of a set in the following.

4. Nano semi pre-θ-frontier and nano semi pre-θ-exterior of a set

Definition 4.1. *For a subset A of U, $N\beta\text{-}\theta cl(A) - N\beta\text{-}\theta int(A)$ is called nano semi pre-θ-frontier of A and it is denoted by $N\beta\text{-}\theta Fr(A)$. Therefore $N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta cl(A) - N\beta\text{-}\theta int(A)$.*

Theorem 4.2. *For a subset A of a space U, the following statements are true*

(1) $N\beta Fr(A) \subset N\beta\text{-}\theta Fr(A)$
(2) $N\beta\text{-}\theta cl(A) \subset N\beta\text{-}\theta int(A) \cup N\beta\text{-}\theta Fr(A)$
(3) $N\beta\text{-}\theta int(A) \cap N\beta\text{-}\theta Fr(A) = \phi$
(4) $N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta cl(A) \cap N\beta\text{-}\theta cl(U - A)$
(5) $N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta Fr(U - A)$
(6) $N\beta\text{-}\theta Fr(A)$ *is* $N\beta$*-closed.*

Proof :

(1) Let $x \in N\beta Fr(A) \Rightarrow x \in N\beta cl(A) - N\beta int(A \Rightarrow x \in N\beta cl(A)$ and $x \notin N\beta int(A) \Rightarrow x \notin N\beta\text{-}\theta cl(A)$ and $x \notin N\beta\text{-}\theta int(A)$ i.e $N\beta Fr(A) \subset N\beta\text{-}\theta Fr(A)$
(2) Now $N\beta\text{-}\theta int(A) \cup N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta int(A) \cup [N\beta\text{-}\theta cl(A) - N\beta\text{-}\theta int(A)] = N\beta\text{-}\theta cl(A)$
(3) $N\beta\text{-}\theta int(A) \cap N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta int(A) \cap [N\beta\text{-}\theta cl(A) - N\beta\text{-}\theta int(A)] = [N\beta\text{-}\theta int(A) \cap N\beta\text{-}\theta cl(A)] - N\beta\text{-}\theta int(A) = \phi$ as $(A \cap B) - A = \phi$
(4) $N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta cl(A) - N\beta\text{-}\theta int(A) = N\beta\text{-}\theta cl(A) \cap [U - N\beta int(A)] = N\beta\text{-}\theta cl(A) \cap N\beta\text{-}\theta cl(U - A)$
(5) $N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta cl(A) \cap N\beta\text{-}\theta cl[(U - A)] = N\beta\text{-}\theta Fr(U - A)$.
(6) Is obvious.

Example 4.3. *Let* $U = \{a, b, c, d\}$, $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ *and* $X = \{a, d\}$ *then we have* $\tau_R(X) = \{\{d\}, \{a, b, d\}, \{a, b\}\}$.

(1) *Let* $A = \{a, b, d\}$ *then* $N\beta Fr(A) = \{c\}$ *and* $N\beta\text{-}\theta Fr(A) = U$. *Therefore* $c \subset U \Rightarrow N\beta Fr(A) \subset N\beta\text{-}\theta Fr(A)$.
(2) *Let* $A = \{c\}$ *then* $N\beta\text{-}\theta cl(A) = \{c\}$, $N\beta\text{-}\theta int(A) = \phi$ *and* $N\beta\text{-}\theta Fr(A) = \{c\}$. *Therefore* $N\beta\text{-}\theta cl(A) \subset N\beta\text{-}\theta int(A) \cup N\beta\text{-}\theta Fr(A)$.
(3) *Let* $A = \{c\}$ *then* $N\beta\text{-}\theta int(A) = \phi$ *and* $N\beta\text{-}\theta Fr(A) = \{c\}$. *Therefore* $N\beta\text{-}\theta int(A) \cap N\beta\text{-}\theta Fr(A) = \phi$.
(4) *Let* $A = \{a, b, d\}$ *then* $N\beta\text{-}\theta Fr(A) = \{c\}$, $N\beta\text{-}\theta cl(A) = U$ *and* $N\beta\text{-}\theta cl(U - A) = \{c\}$. *Therefore* $N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta cl(A) \cap N\beta\text{-}\theta cl(U - A)$.
(5) *Let* $A = \{c\}$ *then* $N\beta\text{-}\theta Fr(U - A) = \{c\}$ *and* $N\beta\text{-}\theta Fr(A) = \{c\}$. *Therefore* $N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta Fr(U - A)$
(6) *Let* $A = \{c\}$ *is in* $N\beta$*-closed.*

Theorem 4.4. *In a nano topological space U, a subset A of U is both $N\beta$-θ-closed and $N\beta$-θ open iff* $N\beta\text{-}\theta Fr(A) = \phi$.

Proof. : Let A be $N\beta$-θ-closed then $N\beta\text{-}\theta cl(A) = A$ and if A is $N\beta$-θ-open then $U - A$ is $N\beta$-θ-closed therefore $N\beta\text{-}\theta cl(U - A) = U - A$. We have $N\beta\text{-}\theta Fr(A) = N\beta\text{-}\theta cl(A) \cap N\beta\text{-}\theta cl(U - A) = A \cap (U - A) = \phi$
Conversely : Let $N\beta\text{-}\theta Fr(A) = \phi \Rightarrow N\beta\text{-}\theta cl(A) - N\beta\text{-}\theta int(A) = \phi$ that implies $N\beta\text{-}\theta cl(A) = N\beta\text{-}\theta int(A)$. But we have also the relation $N\beta\text{-}\theta int(A) \subset A \subset N\beta\text{-}\theta cl(A)$ since $A \subset N\beta cl(A) \subset N\beta\text{-}\theta cl(A)$ and $N\beta\text{-}\theta int(A) \subset N\beta int(A) \subset A$. Hence it follows A $= N\beta\text{-}\theta int(A) = N\beta\text{-}\theta cl(A)$ which it means A is both $N\beta$-θ-closed and $N\beta$-θ-open.

We define $N\beta$-θ-exterior of a set in the following.

Definition 4.5. *In a nano topological space $(U, \tau_R(X))$, $N\beta$-θ- exterior of a subset A is $N\beta$-θ-interior of $(U - A)$ and is denoted by $N\beta\text{-}\theta Ext(A)$. Therefore* $N\beta\text{-}\theta Ext(A) = N\beta\text{-}\theta int(U - A)$

Theorem 4.6. *In a nano topological space U the following statements are true where A is a subset of a space U. Then,*

(1) *$N\beta$-$\theta Ext(A) \subset N\beta Ext(A)$ where $N\beta$-$Ext(A)$ is nano semi pre exterior of A.*
(2) *$N\beta$-$\theta Ext(A)$ is $N\beta$-open.*
(3) *$N\beta$-$\theta Ext(A) = N\beta$-$\theta Ext(U-A) = U - N\beta$-$\theta cl(A)$.*
(4) *$N\beta$-$\theta Ext[N\beta$-$\theta Ext(A)] = N\beta$-$\theta int[N\beta$-$\theta cl(A)]$.*
(5) *If $A \subset B$ then $N\beta$-$\theta Ext(B) \subset N\beta$-$\theta Ext(A)$.*
(6) *$N\beta$-$\theta Ext(A \cup B) \subset N\beta$-$\theta Ext(A) \cup N\beta$-$\theta Ext(B)$.*
(7) *$N\beta$-$\theta Ext(A) \cap N\beta$-$\theta Ext(B) \subset N\beta$-$\theta Ext(A \cap B)$.*
(8) *$N\beta$-$\theta Ext(\phi) = U$ and $N\beta$-$\theta Ext(U) = \phi$.*

Proof:

(1) Now $N\beta$-$\theta Ext(A) = N\beta$-$\theta int(U-A) \subset N\beta int(U-A) = N\beta Ext(A)$.
(2) Since $N\beta$-$\theta Ext(A) = N\beta$-$\theta int(U-A)$ and is $N\beta$-open.
(3) $N\beta$-$\theta Ext(A) = N\beta$-$\theta int(U-A) = U - N\beta$-$\theta cl(A)$.
(4) $N\beta$-$\theta Ext[N\beta$-$\theta Ext(A)] = N\beta$-$\theta Ext[N\beta$-$\theta int(U-A)] = N\beta$-$\theta int[(U - N\beta$-$\theta int(U-A)] = N\beta$-$\theta int[N\beta$-$\theta cl(A)]$.
(5) $A \subset B$ that implies $U - B \subset U - A \Rightarrow N\beta$-$\theta int(U - B \subset N\beta$-$\theta int(U-B)$ that is $N\beta$-$\theta Ext(B) \subset N\beta$-$\theta Ext(A)$.
(6) clearly $N\beta$-$\theta Ext(A) \cup N\beta$-$\theta Ext(B) \subset N\beta$-$\theta Ext(A \cup B)$
On the other hand $N\beta$-$\theta Ext(A \cup B) \subset N\beta$-$\theta int[U - A \cup B] \subset N\beta$-$\theta int[(U - A) \cap (U-B)] \subset N\beta$-$\theta int(U-A) \cap N\beta$-$\theta int(U-B) \subset N\beta$-$\theta int(U-A) \cup N\beta$-$\theta int(U-B) \subset N\beta$-$\theta Ext(A) \cup N\beta$-$\theta Ext(B)$
Therefore $N\beta$-$\theta Ext(A \cup B) = N\beta$-$\theta Ext(A) \cup N\beta$-$\theta Ext(B)$.
(7) and (8) are obvious.

Example 4.7. *Let $U = \{a,b,c,d\}$, $U/R = \{\{a,b\},\{c\},\{d\}\}$ and $X = \{a,d\}$ then we have $\tau_R(X) = \{\{d\},\{a,b,d\},\{a,b\}\}$.*

- *Let $A = \{a\}$ then $N\beta Ext(A) = \{b,c,d\}$ and $N\beta$-$\theta Ext(A) = \{b,c,d\}$. Therefore $\{b,c,d\} \subset \{b,c,d\} \Rightarrow N\beta$-$\theta Ext(A) \subset N\beta Ext(A)$.*
- *Let $A = \{c,d\}$ then $N\beta$-$\theta Ext(A) = \{a,d\}$ is $N\beta$-open.*
- *Let $A = \{a\}$ then $N\beta$-$\theta Ext(A) = N\beta$-$\theta Ext(\{b,c,d\}) = U - N\beta$-$\theta cl(A) = \{b,c,d\}$*
- *Let $A = \{a,b\}$ then $N\beta$-$\theta Ext(A) = \{c,d\}$, $N\beta$-$\theta Ext(\{c,d\}) = \{a,d\}$ and $N\beta$-$\theta cl(A) = \{a,b\}$, $N\beta$-$\theta int(\{a,b\}) = \{a,b\}$. Therefore $N\beta$-$\theta Ext[N\beta$-$\theta Ext(A)] = N\beta$-$\theta int[N\beta$-$\theta cl(A)]$.*
- *Let $A = \{a\}$ and $B = \{a,d\}$, $A \subset B$ then $N\beta$-$\theta Ext(A) = \{b,c,d\}$, $N\beta$-$\theta Ext(B) = \{b,c\}$, $\{b,c\} \subset \{b,c,d\}$. Therefore $N\beta$-$\theta Ext(B) \subset N\beta$-$\theta Ext(A)$.*
- *Let $A = \{a,b\}$ and $B = \{b,c\}$, then $\{a,b\} \cap \{b,c\} = \{b\}$, $N\beta$-$\theta Ext(\{b\}) = \{a,c,d\}$, $N\beta$-$\theta Ext(\{a,b\}) = \{c,d\}$, $N\beta$-$\theta Ext(\{b,c\}) = \{a,d\}$, $\{c,d\} \cap \{a,d\} = \{d\} \subset \{a,c,d\}$. Therefore $N\beta$-$\theta Ext(A) \cap N\beta$-$\theta Ext(B) \subset N\beta$-$\theta Ext(A \cap B)$.*
- *Let $A = \{a,b\}$ and $B = \{b,c\}$, then $\{a,b\} \cup \{b,c\} = \{a,b,c\}$, $N\beta$-$\theta Ext(\{a,b,c\}) = \{d\}$, $N\beta$-$\theta Ext(\{a,b\}) = \{c,d\}$, $N\beta$-$\theta Ext(\{b,c\}) = \{a,d\}$, $\{c,d\} \cup \{a,d\} = \{a,c,d\}$. Therefore $N\beta$-$\theta Ext(A \cup B) \subset N\beta$-$\theta Ext(A) \cup N\beta$-$\theta Ext(B)$.*

References

[1] Lellis Thivagar and Richard.C, On nano forms of weekly open sets, *International Journal of Mathematics and Statistics Invention.* 1 (1)(2013) 31 - 37.
[2] Nasef.A, Aggour.A.I and Darwesh.S.M, On some classes of nearly open sets in nano topological space, *Journal of Egyptian Mathematical Society.* 24 (2016) 585 - 589
[3] Pawalk.Z, Rough sets,Theoretical Aspects of Reasoning about Data, *Kluwer Academic Publishers*, Boston, 1991
[4] Revathy.A and G.Illango, On nano β-open sets, *International Journal of Engineering, Contemporary Mathematics and Science.*, 1(2)(2015)1 - 6.
[5] Sathishmohan.P, Rajendran.V, Vignesh Kumar.C and Dhanasekaran.P.K, On nano semi pre neighbourhoods on nano topological spaces, *Malaya Journal of Mathematik*, 6(1), 2018, 294 -298

[6] Sathishmohan.P, Rajendran.V and Vignesh Kumar.C, On nano semi pre-regular spaces and nano semi pre-normal spaces (*Communicated*)