:- Unit - II :-

:- Analytic function:-

(a) CR equations

** (b) Milne - Thomson method (construction of Analytic function)

(c) 12 - standard results

** (d) z-plane to w-plane

(e) Bilinear Transformation.

* continous functions:-

A function $f(z)$ is said to be continous if its limit value is equal to its function value.

In other words, a function whose graph is continous is said to be continous function.

* Differentiable function:-

A function $f(z)$ is said to be differentiable at the point z_0 and it is denoted by $f'(z_0)$ and it is given by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$\Downarrow$$

$$\frac{d}{dz}\left[f(z)\right]\Big|_{z=z_0}$$

Note: Differentiable function $\Rightarrow$ continous but the vice versa is not true.

Notation: In z-plane $z = x+iy$, $x = \text{Re part}(z)$, $y = \text{Im part}(z)$, $i^2 = -1$, $i = \sqrt{-1}$.

In w-plane $w = f(z) = u + iv$

$\downarrow$

$u(x,y) + i\, v(x,y)$

for each value of z, there corresponds a value in w-plane i.e., Necessary and sufficient conditions for analytic function

(a) Cauchy Riemann Equation

(a) cartesian form : (x,y) $z = x+iy$

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$$\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} \quad \frac{\partial v}{\partial x}$$

To find $f'(z)$,

Take $z = x+iy$

$\Delta z = \Delta x + i\Delta y$

$w = f(z)$

$f(z) = u(x,y) + iv(x,y)$ —(1)

$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$ —(2)

(2) − (1),

$f(z+\Delta z) - f(z) = [u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]$

$= [u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]$

Taking $\lim_{\Delta z \to 0}$ and ÷ by $\Delta z = \Delta x + i\Delta y$

$$\Rightarrow \lim_{\Delta z \to 0} = \lim_{\Delta x + i\Delta y \to 0} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y} \quad \text{—(*)}$$

case(i): $\Delta y = 0$ and $\Delta x \to 0$

$f'(z) = \lim \; [u(x+\Delta x, y) - u(x,y)] + i[v(x+\Delta x, y) - v(x,y)]$

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Case (ii): Take $\Delta x = 0$ and $\Delta y \to 0$.

$$f'(z) = \lim_{\substack{i\Delta y \to 0 \\ \Delta y \to \frac{0}{i} \\ \Delta y \to 0}} \frac{u(x, y+\Delta y) - u(x,y)}{i\,\Delta y} + i \lim_{\substack{i\Delta y \to 0 \\ \Delta y \to \frac{0}{i} \\ \Delta y \to 0}} \frac{v[x, y+\Delta y) - v(x,y)}{i\,\Delta y}$$

$$= \frac{1}{i} U_y + V_y$$

$$f'(z) = (-i) U_y + V_y \quad — (4)$$

from (3) & (4),

$$\boxed{\begin{matrix} Re(z) \Rightarrow U_x = V_y \\ Im(z) \Rightarrow V_x = -U_y \end{matrix}} \Longrightarrow \text{cartesian form}$$

CR equation in Polar form:-

$$U_r = \frac{1}{r} \cdot V_\theta \quad , \quad U_\theta = \frac{-1}{r} \cdot V_r$$

Take, $z = r \cdot e^{i\theta}$ (or) $r(\cos\theta + i\sin\theta)$

$$w = f(z) = u + iv$$

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \to (*)$$

Part. Diff w.r.t r,

$$f'(r \cdot e^{i\theta}) \cdot (e^{i\theta}) = U_r + iV_r \longrightarrow (1)$$

Part. Diff w.r.t θ,

$$f'(re^{i\theta})(r \times e^{i\theta} \times i) = U_\theta + iV_\theta$$

$$f'(re^{i\theta})(e^{i\theta}) = \frac{U_\theta + iV_\theta}{ir} = \frac{U_\theta}{ir} + \frac{V_\theta}{r}$$

$$= -i\frac{U_\theta}{r} + \frac{V_\theta}{r} \to (2)$$

Comparing ① and ②,

Real part, $\boxed{u_r = \frac{v_\theta}{r}}$

Ima part, $\boxed{v_r = \frac{u_\theta}{r}(-1) = -\frac{u_\theta}{r}}$ → Polar form.

Sufficient conditions for Analytic functions:-

* Sufficient condition is u, v are continous and partial derivative u_x, u_y, v_x, v_y exist such that $u_x = v_y$ and $v_x = -u_y$. then $f(z)$ is analytic.

Pr: Test the Analticity of $w = e^z$

$$w = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$$= e^x(\cos y + i\sin y)$$

$$u + iv = e^x \cos y + i e^x \sin y.$$

$$u = e^x \cos y \; ; \quad v = e^x \sin y$$

$$u_x = (\cos y) e^x \; ; \quad v_x = (\sin y)(e^x)$$

$$u_y = e^x(-\sin y) \; ; \quad v_y = e^x(\cos y)$$

$\therefore$ $\boxed{u_x = v_y}$ $\boxed{v_x = -u_y}$

$\boxed{e^x \cos y = e^x \cos y}$ $\boxed{e^x \sin y = -e^x \sin y.}$

$\therefore$ CR equations are satisfied.

$\therefore$ $f(z)$ is analytic.

Pr: Show that $f(z) = |z|$ is analytic at origin but nowh
(differentiable)

differentiable.

$$w = |z| = \sqrt{z \cdot \bar{z}}$$

$$z = x+iy \qquad \bar{z} = x-iy.$$

$$z\bar{z} = (x+iy)(x-iy)$$

$$= x^2+y^2$$

$$u+iv = \sqrt{x^2+y^2} + i(0)$$

hence, $u = \sqrt{x^2+y^2}$ $\qquad V = 0$

$$u_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} \qquad V_x = 0$$

$$u_y = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} \qquad V_y = 0$$

$$\boxed{u_x \neq V_y} \quad \& \quad \boxed{-V_x \neq U_y}$$

$\therefore$ $f(z) = |z|$ is analytic only at origin.

$\therefore$ CR equation is satisfied at origin.

Pr: Show that $f(z) = z - \bar{z}$

$$z = x+iy \quad ; \quad \bar{z} = x-iy$$

$$z - \bar{z} = \cancel{x}+iy-\cancel{x}+iy. = 2iy$$

$$u+iv = 0+i2y$$

$u = 0$ $\qquad V = 2y.$

$u_x = 0$ $\qquad V_x = 0$

$u_y = 0$ $\qquad V_y = 2.$

$$\boxed{u_x \neq V_y.} \quad \& \quad \boxed{u_y = -V_x}$$

$\therefore$ CR equation is not satisfied.

$\therefore$ It is not differentiable.

Pr: Verify whether CR equation is satisfied (or) not.

(a) $f(z) = \sin z$ (b) $f(z) = \cos z$ (c) $f(z) = \cosh z$ (d) $f(z) = \sinh z$.

(a) $f(z) = \sin z$

$u + iv = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + \cos x \cdot i \sinh y.$

$u + iv = \sin x \cosh y + i \cos x \sinh y.$

$u = \sin x \cosh y$ $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ $v_x = (-\sin x)(\sinh y)$

$u_y = \sin x \sinh y$ $v_y = \cos x \cosh y.$

$\boxed{u_x = v_y}$ $\boxed{u_y = -v_x}$

$\therefore$ CR equation is satisfied.

(b) $f(z) = \cos z.$

$u + iv = \cos(x+iy)$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos x \cosh y - i \sin x \sinh y$

$u = \cos x \cosh y$ $v = -\sin x \sinh y.$

$u_x = (-\sin x)(\cosh y)$ $v_x = (-\cos x)\sinh y.$

$u_y = (\cos x)(\sinh y)$ $v_y = (-\sin x)(\cosh y).$

$\boxed{u_x = v_y}$ $\boxed{u_y = -v_x}$

$\therefore$ CR equation is satisfied.

$\therefore \cos z$ is analytic

(c) $f(z) = \cosh(z) = \cos(iz)$

$u + iv = \cosh(x+iy). = \cos(i(x+iy))$

$= \cos(ix - y)$

$= \cos ix \cos y + \sin ix \sin y.$

$u+iv = \cosh x \cos y + i\sinh x \sin y.$

$u = \cos y \cosh x \qquad v = \sin y \sinh x$

$u_x = (\cos y)(\sinh x) \qquad v_x = (\sin y)(\cosh x)$

$u_y = (\cosh x)(-\sin y) \qquad v_y = (\cos y)(\sinh x).$

$\therefore \boxed{u_x = v_y}$ & $\boxed{u_y = -v_x}$

$\therefore$ CR equation is satisfied

$\therefore \cosh z$ is analytic

(d) $f(z) = \sinh(z)$

$= \dfrac{\sin(iz)}{i} = \dfrac{\sin(i(x+iy))}{i} = \dfrac{\sin(ix-y)}{i}.$

$u+iv = \dfrac{\sin ix \cos y}{i} - \dfrac{\cos ix \sin y}{i}$

$= \dfrac{i\sinh x \cos y}{i} - \dfrac{\cosh x \sin y}{i}$

$= \cancel{-\cosh x \sin y + i \sinh x \cos y} = \sinh x \cos y + i\cosh x \sin y$

$\cancel{u = -\cosh x \sin y \qquad v = \sinh x \cos y.}$

$\cancel{u_x = -\sinh x \sin y \qquad v_x = \cosh x \cos y}$

$\cancel{u_y = -\cosh x \cos y \qquad v_y = \sinh x(-\sin y)}$

$\cancel{\boxed{u_x = v_y} \text{ \& } \boxed{u_y = -v_x}.}$

$\cancel{\therefore \text{ CR equation is satisfied.}}$

$u = \sinh x \cos y \qquad v = \cosh x \sin y.$

$u_x = \cos y \cdot \cosh x \qquad v_x = \sinh x \cdot \sin y$

$u_y = \sinh x(-\sin y) \qquad v_y = \cosh x \cdot \cos y.$

$\boxed{u_x = v_y}$ & $\boxed{-v_x = u_y}$

$\therefore$ CR equation is satisfied

$\therefore \sinh z$ is analytic.

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Note:-

* In cartesian form, after CR equations are satisfied,

$f'(z) = u_x + i v_x$.

* In polar form, after CR equations are satisfied,

$$f'(z) = \frac{u_r + i v_r}{e^{i\theta}} \quad ; \quad f'(z) = \frac{dw}{dz}$$

Pr: Test the analyticity of function $f(z) = z^n$ and also find the first derivative.

$f(z) = z^n$

In polar form, $z = r.e^{i\theta}$

$w = (r.e^{i\theta})^n$

$= r^n . e^{in\theta}$

$u + iv = r^n[\cos n\theta + i\sin n\theta]$.

$u = r^n \cos n\theta \quad ; \quad v = r^n \sin n\theta$

$u_r = n.r^{n-1}. \cos n\theta \qquad v_r = n.r^{n-1}.(\sin n\theta)$

$u_\theta = r^n.(-\sin n\theta)(n) \qquad v_\theta = r^n (\cos n\theta).n.$

$u_r = \frac{1}{r} . n r^n \cos\theta \qquad v_\theta = r^n n\cos n\theta$

$$\therefore \boxed{u_r = \frac{1}{r}.v_\theta.}$$

$v_r = \frac{n.r^n \sin n\theta}{r} \qquad u_\theta = -n.r^n \sin n\theta.$

$$\therefore \boxed{v_r = \frac{-1}{r}.u_\theta}$$

$\therefore$ CR equation is satisfied & it is analytic.

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{n.r^{n-1}\cos n\theta + i(n.r^{n}\sin n\theta)}{e^{i\theta}}$$

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$$= \frac{\frac{n.r^n \cos n\theta}{r} + i\frac{n.r^n \sin n\theta}{r}}{e^{i\theta}}$$

$$= \frac{n.r^n}{r}\left[\frac{\cos n\theta + i\sin n\theta}{e^{i\theta}}\right]$$

$$= \frac{n.r^n}{r} \frac{[\cos\theta + i\sin\theta]^n}{e^{i\theta}}$$

$$= \frac{n.r^n}{r} . [e^{i\theta}]^{n-1}$$

$$= n.r^{n-1} \times \left[\frac{z}{r}\right]^{n-1}$$

$$= \frac{n.\cancel{r^{n-1}} \times z^{n-1}}{\cancel{r^{n-1}}} = n.z^{n-1}.$$

Pr: Test the Analyticity of $f(z) = \frac{1}{z}$ and also find $\frac{dw}{dz}$. is analytic everywhere except at $z=0$.

$$f(z) = \frac{1}{z} = z^{-1}$$

In polar form, $z = r.e^{i\theta}$ = $(re^{i\theta})^{-1} = f(z)$.

$$u+iv = r^{-1}.e^{-i\theta} = r^{-1}[\cos(-\theta)+i\sin(-\theta)]$$

$$= r^{-1}[\cos\theta - i\sin\theta].$$

$$u = \frac{1}{r}\cos\theta ; \quad v = -\frac{1}{r}\sin\theta$$

$$u_r = (-1)\frac{1}{r^2}\cos\theta \qquad v_r = (+1)\frac{1}{r^2}\sin\theta$$

$$u_\theta = \frac{1}{r}(-\sin\theta) \qquad v_\theta = \frac{-1}{r}\cdot\cos\theta$$

$$\boxed{u_r = \frac{1}{r}v_\theta} \qquad \boxed{v_r = -\frac{1}{r}v_\theta}$$

$\therefore f(z)$ is analytic

$$f'(z) = \frac{u_r+iv_r}{e^{i\theta}} = \frac{\frac{-1}{r^2}\cos\theta + i\left(\frac{1}{r^2}\sin\theta\right)}{e^{i\theta}}$$

$$= \frac{-\frac{1}{r^2}(+\cos\theta \mp i\sin\theta)}{e^{i\theta}}$$

$$= \frac{-1}{r} , \frac{e^{-i\theta}}{re^{i\theta}}$$

$$= -\frac{1}{z^2}$$

$$= (-1)\, z^{-2}.$$

11. Test the analticity of $f(z) = \log z$ and also obtain $f'(z)$.

(a) polar form:

$$w = \log z$$
$$= \log(r \cdot e^{i\theta})$$
$$= \log r + \log e^{i\theta}$$
$$u + iv = \log r + i\theta$$

$u = \log r$ $\quad$ $v = \theta$

$u_r = \frac{1}{r}$ $\quad$ $v_r = 0$

$u_\theta = 0$ $\quad$ $v_\theta = 1$

$\therefore \boxed{u_r = \frac{1}{r} \cdot v_\theta}$ & $\boxed{u_\theta = -\frac{1}{r} \cdot v_r}$

$\therefore$ CR equations are satisfied.

$f(z) = \log z$ is analytic

(b) cartesian form:-

$$w = \log(x+iy)$$
$$= \frac{1}{2} \cdot \log(x^2+y^2) + i\tan^{-1}\left[\frac{y}{x}\right].$$

$u = \frac{1}{2}\log(x^2+y^2)$ $\quad$ $v = \tan^{-1}\left(\frac{y}{x}\right)$

$u_x = \frac{1}{\cancel{2}} \times \frac{1}{(x^2+y^2)}(\cancel{2}x) = \frac{x}{x^2+y^2}$ $\quad$ $v_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$

$u_y = \frac{1}{\cancel{2}} \times \frac{1}{(x^2+y^2)}(\cancel{2}y) = \frac{y}{x^2+y^2}$ $\quad$ $v_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x} \times 1\right)$

$$= \frac{x}{x^2+y^2}$$

$\therefore \boxed{u_x = v_y}$ & $\boxed{u_y = -v_x}$

$\therefore$ CR equation is satisfied

$\therefore f(z)$ is analytic

$$f'(z) = u_x + iv_x = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} \times \frac{x+iy}{(x+iy)}$$
$$= \frac{\cancel{x^2+y^2}}{\cancel{x^2+y^2}} \times \frac{1}{(x+iy)} = \frac{1}{z}.$$

$$f'(z) = \frac{\frac{1}{r} + i(0)}{e^{i\theta}}$$

$$= \frac{1}{re^{i\theta}} = \frac{1}{z}$$

$\therefore \frac{dw}{dz}$ is found on both cases

Ans: $f(z) = z^3 + z$

$= (x+iy)^3 + (x+iy)$

$u+iv = (x^3 - 3xy^2) + i(3x^2y - y^3) + x + iy.$

$= (x^3 - 3xy^2 + x) + i(3x^2y - y^3 + y)$

$u = x^3 - 3xy^2 + x \qquad v = 3x^2y - y^3 + y$

$u_x = 3x^2 - 3y^2 + 1 \qquad v_x = 6xy$

$u_y = -6xy \qquad v_y = 3x^2 - 3y^2 + 1$

$\boxed{u_x = v_y}$ & $\boxed{u_y = -v_x}$

$f'(z) = u_x + iv_x$

$= (3x^2 - 3y^2 + 1) + i(6xy)$

$= (3x^2 - 3y^2) + i(6xy) + 1$

$= 3(x^2 - y^2) + i(6xy)$

$= 3(x^2 + i^2y^2 - 2xiy) + i6xy + 1$

$= 3[x^2 + (iy)^2 + 2ixy] + 1$

$= 3[(x+iy)^2] + 1$

$= 3z^2 + 1.$

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Q. find the values of constant A, B, C such that $f(z)$ is analytic.

(i) $f(z) = (x+ay) - i(bx+cy)$ (ii) $f(z) = ax^2 - by^2 + icxy$.

$f(z)$ is satisfied,

CR equation is satisfied.

$u = x+ay$ $v = -(bx+cy)$

$u_x = 1$ $v_x = -b$

$u_y = a$ $v_y = -c$

$u_x = v_y \Rightarrow \boxed{1 = -c}$ $u_y = -v_x$

$\boxed{c = -1}$ $a = -(-b)$

$\boxed{a = b}$

(ii) $f(z) = ax^2 - by^2 + icxy$.

CR equation is satisfied.

$u = (ax^2 - by^2)$ $v = cxy$.

$u_x = 2ax$ $v_x = cy$

$u_y = -2by$. $v_y = cx$

$u_x = v_y \Rightarrow 2a\cancel{x} = c\cancel{x}$ $u_y = -v_x \Rightarrow \cancel{-}2by = \cancel{-}cy$

$\boxed{a = \frac{c}{2}}$ $\boxed{b = \frac{c}{2}}$

$\boxed{a = b} = c/2$

Note:-

For some cases it is not possible to obtain u_x & u_y, v_x, v_y directly

$$u_x = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} \qquad u_y = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$v_x = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} \qquad v_y = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y}$$

* Analytic function $\rightarrow$ Regular function (or) Holomorphic function

* A function which is Analytic everywhere is called as Entire function.

Pr: Show that $f(z) = \sqrt{|xy|}$ is not regular at the origin eventhough CR equations are satisfied at the origin.

$$f(z) = \sqrt{|xy|}$$

$$\Downarrow$$

$$w = u + iv = \sqrt{|xy|}$$

$$u(x,y) = \sqrt{|xy|}$$

$$v(x,y) = 0.$$

$$u_x = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \frac{0-0}{0} = 0$$

$$u_y = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \frac{0-0}{0} = 0$$

$$v_x = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = 0$$

$$v_y = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = 0$$

$$\boxed{u_x = v_y} \quad \& \quad \boxed{u_y = -v_x}$$

$\therefore$ CR equation is satisfied at origin (0,0)

* $$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt{|xy|} - 0}{x+iy} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt{|xy|}}{x+iy}.$$

Take * $y = mx$ (straight line)

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt{|x \cdot mx|}}{x + imx}$$

$\therefore$ for every value 'm', line value vali

$$\lim \frac{\sqrt{\cancel{x^2}} \times \sqrt{|m|}}{} \qquad \sqrt{|m|}$$

$\therefore f'(z)$ does not exis

Pr. $w = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $f(0)=0$. Prove that $f(z) = w = u+iv$ at $z \neq 0$. given conditions.

is continous and the CR equations is satisfied at the origin is $f(z)$ is ~~and~~ holomorphic at origin.

$$u = \frac{x^3-y^3}{x^2+y^2} \quad ; \quad v = \frac{x^3+y^3}{x^2+y^2}$$

$$u_x = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{\left(\frac{x^3-0}{x^2}\right)}{x} = 1$$

$$u_y = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{(-y^3/y^2)}{y} = -1$$

$$v_x = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{(x^3/x^2)}{x} = 1$$

$$v_y = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{(y^3/y^2)}{y} = 1$$

$$\boxed{u_x = v_y} \quad \& \quad \boxed{u_y = -v_x}$$

$\therefore$ CR equation is satisfied.

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$$

Put $y = mx$,

$$= \lim_{x\to 0} \frac{x^3(1+i) - (mx)^3(1-i)}{(x^2+m^2x^2)(x+imx)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}[(1+i) - m^3 + im^3]}{\cancel{x^2}[1+m^2]\cancel{x}[1+im]}$$

$$= \lim_{x\to 0} \frac{(1+i) - m^3(1-i)}{(1+m^2)(1+im)}$$

$\therefore$ ~~for~~ every value of m, limit value varies.

$\therefore f'(z)$ does not exist $\therefore f(z)$ is not holomorphic.

Formula:-

To find f(z) fails to (ceases to) analytic at some point ⊛ $w = f(z) \rightarrow \frac{dz}{dw} = 0$

⊛ $z = \phi(w) \rightarrow \frac{dw}{dz} = 0$.

Pr: find out the points for which, the function ceases to be analytic.

(i) $w = \frac{1}{z^2 - 1}$ (ii) $\frac{z^2 - 4}{z^2 + 1}$ (iii) $z^3 - 4z - 1$

(i) $w = \frac{1}{z^2 - 1} \Rightarrow \frac{dw}{dz} = \frac{-1}{(z^2-1)^2}(2z) = \frac{-2z}{(z^2-1)^2}$

$$\frac{dz}{dw} = \frac{(z^2-1)^2}{-2z} = 0$$

$$(z^2-1)^2 = 0.$$

$$z^2 - 1 = 0$$

$$z^2 = 1$$

$$\boxed{z = \pm 1}$$

(ii) $w = \frac{z^2 - 4}{z^2 + 1} \Rightarrow \frac{dw}{dz} = \frac{(z^2+1)(2z) - (z^2-4)(2z)}{(z^2+1)^2}$

$$= \frac{2z(\cancel{z^2}+1-\cancel{z^2}+4)}{(z^2+1)^2} = \frac{10z}{(z^2+1)^2}$$

$$\frac{dz}{dw} = \frac{(z^2+1)^2}{10z} = 0$$

$$(z^2+1)^2 = 0.$$

$$z^2 + 1 = 0$$

$$z^2 = -1$$

$$z = \pm\sqrt{-1} = \pm i$$

$f(z) = z^3 - 4z - 1 = \omega$

$$\frac{d\omega}{dz} = 3z^2 - 4$$

$$\frac{dz}{d\omega} = \frac{1}{3z^2-4} = 0$$

$\therefore$ function is ~~not~~ analytic anywhere.

(ii) $z = e^{-v}(\cos u + i\sin u)$

$$= e^{-v} \cdot e^{iu}$$

$$z = e^{iu-v}$$

$$= e^{i(u - \frac{v}{i})} = e^{i(u+iv)} = e^{i\omega}$$

$$z = e^{i\omega}$$

$$\frac{dz}{d\omega} = (e^{i\omega})(i)$$

$$\frac{d\omega}{dz} = \frac{1}{i(e^{i\omega})}$$

$\therefore$ function is analytic everywhere

Construction of Analytic function (or) Milne-thomson method:-

(a) Real part given

$$f_1 = (U_x)_{(z,0)}$$

$$f_2 = (U_y)_{(z,0)}$$

$$\therefore f(z) = \int f_1 dz - i\int f_2 dz + c$$

(b) Imaginary part given

$$f_1 = (V_y)_{(z,0)}$$

$$f_2 = (V_x)_{(z,0)}$$

$$\therefore f(z) = \int f_1 dz + i\int f_2 dz + c$$

Pr: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ → Determine the analytic function whose real part is given by.

$$f_1 = (u_x)_{(z,0)} = (3x^2 - 3y^2 + 6x + 0 + 0)_{(z,0)}$$

$$= (3z^2 + 6z)$$

$$f_2 = (u_y)_{(z,0)} = (0 - 3x(2y) - 6y)_{(z,0)}$$

$$= (-6xy - 6y)_{(z,0)}$$

$$= 0.$$

$$\therefore f(z) = \int f_1 dz - i \int f_2 dz + c$$

$$= \int (3z^2 + 6z)\, dz + c$$

$$= \cancel{3}\left[\frac{z^3}{\cancel{3}}\right] + \overset{3}{\cancel{6}}\left[\frac{z^2}{\cancel{2}}\right] + c$$

$$= z^3 + 3z^2 + c$$

Pr: $V = x^4 - 6x^2y^2 + y^4$

$$f_1 = (V_y)_{(z,0)} = 0 - 6x^2(2y) + 4y^3$$

$$= (0 - 12x^2y + 4y^3)_{(z,0)}$$

$$= 0.$$

$$f_2 = (V_x)_{(z,0)} = (4x^3 - 6y^2(2x) + 0)_{(z,0)}$$

$$= 4z^3.$$

$$\therefore f(z) = \int f_1 dz + i \int f_2 dz + c.$$

$$= 0 + i \int 4z^3 dz + c$$

$$= i4\left[\frac{z^4}{\cancel{4}}\right] + c$$

$$= iz^4 + c.$$

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$$f_1 = (u_x)_{(z,0)} = e^x \cdot \cos y$$

$$= e^z$$

$$f_2 = (u_y)_{(z,0)} = e^x(-\sin y)$$

$$= 0.$$

$$\therefore f(z) = \int f_1 dz - i\int f_2 dz + c$$

$$= \int e^z dz$$

$$= e^z + c.$$

Pr. $u = e^{2x}(x\cos 2y - y\sin 2y)$

$$u = e^{2x} \cdot x\cos 2y - e^{2x} \cdot y\sin 2y.$$

$$f_1 = (u_x)_{(z,0)} = \cos 2y\,(e^{2x}(1) + 2(e^{2x})) - e^{2x}(2)y\sin 2y.$$

$$= (3e^{2x}\cos 2y - 2e^{2x}y\sin 2y)_{(z,0)}$$

$$= (3e^{2z})$$

$$f_2 = (u_y)_{(z,0)} = (x \cdot e^{2x})(-\sin 2y) - e^{2x}(1 \cdot \sin 2y + y(2\cos 2y)).$$

$$= [0 - e^{2x}(\sin 2y + 2y\cos 2y)]_{(z,0)}$$

$$= -e^{2z}[0+0] = 0.$$

$$f_1 = (u_x)_{(z,0)} = \left[e^{2x}[\cos 2y(1) - 0] + (x\cos 2y - y\sin 2y)(e^{2x})(2)\right]_{(z,0)}$$

$$= e^{2z}[1] + 2e^{2z}[z - 0].$$

$$= e^{2z} + 2z\,e^{2z} = e^{2z}[1 + 2z]$$

$$f_2 = (u_y)_{(z,0)} = \left[e^{2x}\left[x(-\sin 2y)(2) - [y \cdot \cos 2y \times 2 + (\sin 2y)]\right]\right]_{(z,0)}$$

$$= e^{2z}[0 - [0+0]] = 0.$$

$$\therefore f(z) = \int f_1 dz - i\int f_2 dz + c.$$

$$= \int e^{2z}[1 + 2z]\, dz - i\int 0 + c.$$

$$= \int e^{2z}[1+2z]\,dz + c$$

$$\int uv\,dz = uv_1 - u'v_2 + u''v_3 - \dots$$

$$u = 1+2z \qquad v = e^{2z}$$

$$u' = 2 \qquad v_1 = e^{2z}/2$$

$$u'' = 0 \qquad v_2 = e^{2z}/4$$

$$f(z) = (1+2z)\frac{e^{2z}}{2} - 2\cdot\frac{e^{2z}}{4} + c$$

$$= \frac{e^{2z}}{2}\left[\cancel{1}+2z-\cancel{1}\right] + c$$

$$= \frac{e^{2z}}{\cancel{2}} \times \cancel{2}z + c = ze^{2z} + c$$

$$\therefore f(z) = z\cdot e^{2z} + c$$

Take $z = x+iy$, $w = u+iv = f(z)$

$$\therefore u+iv = (x+iy)\,e^{2(x+iy)} + c.$$

$$= (x+iy)\cdot e^{2x}\cdot e^{2iy} + c$$

$$= (x+iy)\cdot e^{2x}(\cos 2y + i\sin 2y) + c$$

$$= x\cdot e^{2x}\cos 2y + ix\cdot e^{2x}\sin 2y + iy\cdot e^{2x}\cos 2y - ye^{2x}\sin 2y + c$$

$$= (xe^{2x}\cos 2y - ye^{2x}\sin 2y) + i(xe^{2x}\sin 2y + ye^{2x}\cos 2y) + c$$

$$u+iv = e^{2x}(x\cos 2y - y\sin 2y) + i\,e^{2x}(x\sin 2y + y\cos 2y) + c$$

$$\therefore v = e^{2x}(x\sin 2y + y\cos 2y).$$

Pr: (a) $u = e^{(x^2-y^2)}\cos(2xy)$ (b) $v = -e^{-2xy}\cdot\cos(x^2-y^2)$

(b) $f_1 = (v_y)_{(z,0)} = -\left[e^{-2xy}\left[-\sin(x^2-y^2)\right]\left[0-2y\right]\right] + \cos(x^2-y^2)\,(e^{-2xy})(-2x)\Big]_{(z,0)}$

$$= -\left[e^0\left[\sin(z^2-0)\left[0-0\right]\right] + \cos(z^2-0)\left[e^0\right]\left[-2z\right]\right]$$

$$= -\left[-2z\cos z^2\right] = 2z\cos z^2$$

$f_2 = (V_x)_{(z,0)}$

$= -\left[e^{-2xy}\left(-\sin(x^2-y^2)(2x-0)\right) + \cos(x^2-y^2)(e^{-2xy})(-2y)\right]_{(z,0)}$

$= -\left[e^0(-\sin(z^2-0)(2z) + \cos(z^2-0)(e^0)(0)\right]$.

$= +2z\cdot\sin z^2$

$\therefore f(z) = \int f_1 dz + i\int f_2 dz + c$.

$= \int 2z\cos z^2 + i\int 2z\sin z^2 dz + c$.

Put $z^2 = t$

$2z\,dz = dt \Rightarrow dz = \frac{dt}{2z}$

$= \int \cancel{2z}\cos t\frac{dt}{\cancel{2z}} + i\int \cancel{2z}\sin t\frac{dt}{\cancel{2z}} + c$.

$= \int \cos t\,dt + i\int \sin t\,dt + c$

$= \sin t - i\cos t + c$

$\therefore f(z) = \sin z^2 - i\cos z^2 + c$

$= -i\left[-(-i)\sin z^2 + \cos z^2\right] + c$.

$= -i\left[i\sin z^2 + \cos z^2\right] + c$.

$= -i \times e^{iz^2} + c$.

(a) $e^{(x^2-y^2)}\cdot\cos(2xy)$.

$f_1 = (u_x)_{(z,0)} = \left[e^{(x^2-y^2)}\left[-\sin(2xy)(2y)\right] + \cos 2xy\cdot(e^{(x^2-y^2)})(2x)\right]_{(z,0)}$

$= \left[e^{(z^2-0)}[0] + \cos 0\,(e^{(z^2-0)})(2z)\right]$.

$= 2z\cdot e^{z^2}$.

$f_2 = (u_y)_{(z,0)} = \left[e^{(x^2-y^2)}\left[-\sin(2xy)(2x)\right] + \cos 2xy\,(e^{x^2-y^2})(2y)\right]_{(z,0)}$

$= \left[(e^{z^2-0})\left[-\sin(2z0)(2z)\right]\right]$.

$= 0$.

$$\therefore f(z) = \int f_1 dz - i\int f_2 dz + c$$

$$= \int 2z \cdot e^{z^2} dz + c.$$

$z^2 = t \qquad 2z\,dz = dt$

$$= \int e^t dt + c$$

$$= e^t + c \quad = e^{z^2} + c$$

Pr:-

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x} \quad (\text{or}) \quad \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$$

$$f_1 = (u_x)_{(z,0)} = \left[\frac{\overset{v}{(\cosh 2y - \cos 2x)}\ \overset{u'}{(2\cos 2x)} - \overset{u}{(\sin 2x)}\ \overset{v'}{(0 + 2\sin 2x)}}{[\cosh 2y - \cos 2x]^2}\right]_{(z,0)}$$

$$= \frac{(\cosh 0 - \cos 2z)(2\cos 2z) - (\sin 2z)(2\sin 2z)}{[\cosh 0 - \cos 2z]^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(1)}{(1 - \cos 2z)^2}$$

$\cosh 0 = \cos(i0)$

$= \cos(i0)$

$= 1.$

$$= \frac{-2(\cancel{1 - \cos 2z})}{(1 - \cos 2z)^{\cancel{2}}}$$

$$= \frac{-2}{1 - \cos 2z} = \frac{-\cancel{2}}{\cancel{2}\sin^2 z} = -\operatorname{cosec}^2 z$$

$$f_2 = (u_y)_{(z,0)} = \left[\frac{\underbrace{[\cancel{\cosh 2y - \cos 2x}]}_{0}(0) - (\sin 2x)(2\sinh 2y)}{[\cosh 2y - \cos 2x]^2}\right]_{(z,0)}$$

$$= \left[\frac{-(\sin 2z)(2\sinh(2y))}{[\cosh 2y - \cos 2x]^2}\right]_{(z,0)}$$

$\sinh z = i\sin(i\ldots$

$$= 0.$$

$\therefore f(z) = \int f_1 - i\int f_2 + c$

$= -\int \operatorname{cosec}^2 z \, dz + c$

$$\boxed{f(z) = +\cot(z) + c}$$

Homework :-

(1) $v = 3x^2y - y^3$

$f_1 = (v_y)_{(z,0)} = (3x^2 - 3y^2)_{(z,0)}$

$= 3z^2$

$f_2 = (v_x)_{(z,0)} = (6xy)_{(z,0)}$

$= 0$

$\therefore f(z) = \int f_1 dz + i\int f_2 dz + c$

$= \int 3z^2 dz + i\int 0 + c$

$= \cancel{3}\frac{z^3}{\cancel{3}} + c$

$= z^3 + c$

(2) $v = e^{2x}[x\cos 2y - y\sin 2y]$

$f_1 = (v_y)_{(z,0)} = \left[e^{2x}\left(x(-\sin 2y)(2) - (y\cos 2y(2) + \sin 2y)\right)\right]_{(z,0)}$

$= \left[e^{2z}\left(2z(-\sin 2(0)) - (0+0)\right)\right]$

$= 0$

$f_2 = (v_x)_{(z,0)} = \left[e^{2x}(\cos 2y - 0) + (x\cos 2y - y\sin 2y)(e^{2x})(2)\right]_{(z,0)}$

$= \left[e^{2z}(\cos 0^\circ) + (z\cos 0^\circ - 0)(e^{2z})(2)\right]$

$= \left[e^{2z} + z \cdot 2e^{2z}\right] = e^{2z}[1+2z]$

$\therefore f(z) = \int f_1\, dz + i\int f_2\, dz + c$

$= 0 + i\int e^{2z}[1+2z] + c$

$=$ $u = 1+2z$; $v = e^{2z}$

$u' = 2$ $\quad v_1 = e^{2z}/2$

$u'' = 0.$ $\quad v_2 = e^{2z}/4$

$f(z) = i\left[(1+2z)\frac{e^{2z}}{2} - 2\times\frac{e^{2z}}{4}\right]$

$= i\left[\frac{e^{2z}}{2}(1+2z-1)\right]$

$= i[e^{2z}] \cdot z + c$

$z = x+iy$; $u+iv = f(z)$

$u+iv = i\left[(x+iy)\, e^{2(x+iy)}\right] + c$

$= i\left[(x+iy) \cdot e^{2x} \cdot e^{2iy}\right] + c$

$= i\left[(x+iy) \cdot e^{2x}(\cos 2y + i\sin 2y)\right] + c$

$= i\left[x \cdot e^{2x}\cos 2y + x \cdot e^{2x} i\sin 2y + i\, ye^{2x}\cos 2y - ye^{2x}\sin 2y\right].$

$= i\,xe^{2x}\cos 2y - xe^{2x}\sin 2y - ye^{2x}\cos 2y - i\,ye^{2x}\sin 2y$

$= -(xe^{2x}\sin 2y + ye^{2x}\cos 2y) + i\,(xe^{2x}\cos 2y - ye^{2x}\sin 2y)$

$u = -(xe^{2x}\sin 2y + ye^{2x}\cos 2y).$

Note:- In Milne-Thompson method, if we are having combination of u and v like $u+v$, $u-v$, $3u+2v$, $5u+3v$ then take $f(z) = u+iv \rightarrow (1)$

$$if(z) = iu - v \rightarrow (2)$$

Pr: Determine the analytic function for which $u+v =$

(a) $u+v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

(b) $2u+3v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

(a) $f(z) = u+iv \rightarrow (1)$

$if(z) = iu - v \rightarrow (2)$

$(1) \times 1 \Rightarrow f(z) = u+iv.$

$(2) \times -1 \Rightarrow -if(z) = -iu+v$

$$f(z) - if(z) = (u+v) + i(v-u)$$

$$f(z)[1-i] = (u+v) + i(v-u).$$

$$F[z] = U + iV$$

$$\therefore F[z] = U + iV$$

$f_1 = (U_x)_{(z,0)} = -\operatorname{cosec}^2 z.$

$f_2 = (U_y)_{(z,0)} = 0.$

from the before problem solution.

$$F(z) = \int f_1 dz - i\int f_2 dz + c.$$

$$F(z) = \int -\operatorname{cosec}^2 dz + c = \cot z + c.$$

$$f(z)(1-i) = \cot z + c$$

$$f(z) = \frac{\cot z}{(1-i)} \times \frac{(1+i)}{(1+i)} + \frac{c}{(1-i)}$$

$$= \frac{(1+i)(\cot z)}{2} + c$$

$$f(z) = \frac{\cot z + i\cot z}{2} + c$$

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$f(z) = \ldots$

$i f(z) = iu - v \longrightarrow (2)$

$(1) \times 2 \Rightarrow \quad 2f(z) = 2u + i2v$

$(2) \times -3 \quad -3if(z) = -3iu + 3v$

$2f(z) - 3if(z) = (2u+3v) + i(2v-3u)$

$f(z)(2-3i) = (2u+3v) + i(2v-3u)$

$F(z) = U + iV.$

$f_1 = (U_x)_{(z,0)} = -\operatorname{cosec}^2 z$

$f_2 = (U_y)(z,0) = 0.$

$F(z) = \cot z + c.$

$f(z)(2-3i) = \cot z + c.$

$$f(z) = \frac{\cot z}{2-3i} + \frac{c}{2-3i}$$

$$f(z) = \frac{(2+3i)\cot z}{4-(-1)9} + c$$

$$= \frac{(2+3i)\cot z}{13} + c.$$

Pr. Construct the analytic function $u+v = \frac{x}{x^2+y^2}$ also $f(1)=1$

$f(z) = u + iv \longrightarrow (1)$

$i f(z) = iu - v \longrightarrow (2)$

$(1) \times 1 \rightarrow \quad f(z) = u + iv$

$(2) \times -1 \rightarrow \quad -if(z) = -iu + v$

$f(z)(1-i) = (u+v) + i(v-u)$

$F(z) = U + iV$

$$(U_x)_{(2,0)} = \left[\frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2}\right]_{(2,0)}$$

$$= \left[\frac{(z^2)-(2z^2)}{(z^2)^2}\right] = \left[\frac{-z^2}{z^4}\right] = -\frac{1}{z^2}$$

$$(U_y)_{(2,0)} = \left[\frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2}\right]_{(2,0)}$$

$$= 0.$$

$$F(z) = \int f_1 - i\int f_2 + c.$$

$$= \int -\frac{1}{z^2}dz + c.$$

$$F(z) = \int -z^{-2}dz + c = \frac{1}{z} + c$$

$$f(z)\,(1-i) = \frac{1}{z} + c$$

$$f(z) = \frac{1}{z(1-i)} + \frac{c}{(1-i)}$$

$$= \frac{(1+i)}{z(2)} + c$$

$$f(z) = \frac{1+i}{2z} + c$$

$$f(1) = \frac{1+i}{2} + c = 1$$

$$c = 1 - \frac{1+i}{2} = \frac{2-1-i}{2} = \frac{1-i}{2}$$

$$f(z) = \cancel{\frac{1+i}{2z} + \frac{1-i}{2}} \left[\frac{1+i}{2z} + \frac{1-i}{2}\right]$$

$$\cancel{= \frac{1+i}{2}\left[\frac{1}{z} + \dots\right]} = \frac{1}{2}\left[\frac{1+i}{z} - (1-i)\right]$$

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construct the analytic function $(u-v) = \dfrac{\cos x + \sin x + e^{-y}}{2(\cos x - \cosh y)}$

where $f(\frac{\pi}{2}) = 0$.

$f(z) = u + iv \rightarrow (1)$

$i f(z) = iu - v \rightarrow (2)$

$f(z) + i f(z) = (u-v) + i(u+v)$.

$f(z)(1+i) = (u-v) + i(u+v)$.

$F[z] = U + iV$.

$$(U_x)_{(z,0)} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)[-\sin x + \cos x + 0] - [\cos x + \sin x + e^{-y}][-\sin x]}{(\cos x - \cosh y)^2}\right]_{(z,0)}$$

$$= \frac{1}{2}\left[\frac{(\cos z - \cos 0)[-\sin z + \cos z + 0] - [\cos z + \sin z + 1][-\sin z]}{(\cos z - 1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(\cos z - 1)(\cos z - \sin z) - (\cos z + \sin z - 1)(-\sin z)}{(\cos z - 1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(\cos^2 z - \sin z \times \cos z - \cos z + \sin z) - (-\sin z\cos z - \sin^2 z + \sin z)}{(\cos z - 1)^2}\right]$$

$$= \frac{1}{2}\left(\frac{\cos^2 z - \sin z \cancel{\cos z} - \cos z + \cancel{\sin z} + \sin z \cancel{\cos z} - \sin^2 z \cancel{+ \sin z}}{(\cos z - 1)^2}\right)$$

$$= \frac{1}{2}\left[\frac{1 - \cos z}{(\cos z - 1)^2}\right] = \frac{1}{2}\left[\frac{(1-\cos z)}{(1-\cos z)^2}\right] = \frac{1}{2 \times (1 - \cos z)}$$

$$= \frac{1}{2 \times 2\sin^2 \frac{z}{2}} = \frac{1}{4} \cdot \operatorname{cosec}^2\left(\frac{z}{2}\right).$$

$$(U_y)_{(z,0)} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)(+e^{-y}) - (\cos x + \sin x - e^{-y})(-\sinh y)}{(\cos x - \cosh y)^2}\right]_{(z,0)}$$

$\sin i(y)$

$\sin(0)$

$$= \frac{1}{2}\left[\frac{(\cos z - 1)(1) - (\cos z + \sin z - 1)(0)}{(\cos z - 1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{(\cos z - 1)}}{(\cos z - 1)^{\cancel{2}}}\right] = \frac{1}{2}\,\frac{1}{(\cos z - 1)}$$

$\cos 2A = 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$
$\cos 2A - 1 = -2\sin^2 A$

$$= \frac{1}{2(\cos z - 1)} = \frac{1}{2\left(-2\sin^2\left(\frac{z}{2}\right)\right)}$$

$$= -\frac{1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

$$F(z) = \int -f_1 - i\int f_2 + c$$

$$= \int \frac{1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right) + i\int \frac{1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right) + c$$

$$= \frac{1}{4}\,\frac{\left(-\cot\left(\frac{z}{2}\right)\right)}{1/2} + \frac{1}{4}\cdot i\,\frac{\left(-\cot\left(\frac{z}{2}\right)\right)}{1/2} + c.$$

$$f(z)(1+i) = -\frac{1}{4}\,\frac{\cot\left(\frac{z}{2}\right)}{1/2} - i\frac{1}{4}\,\frac{\cot\left(\frac{z}{2}\right)}{1/2} + c.$$

$$f(z) = -\frac{2}{4}\,\frac{\cot\left(\frac{z}{2}\right)}{(1+i)} - i\,\frac{2}{4}\,\frac{\cot\left(\frac{z}{2}\right)}{(1+i)} + c.$$

$$= -\frac{2}{4_2}\,\frac{(1-i)\cot\left(\frac{z}{2}\right)}{2} - i\,\frac{2}{4_2}\times\frac{(1-i)\cot(z/2)}{2} + c.$$

$$= \frac{(i-1)\cot(z/2)}{4} + \frac{(i-1)}{4}\cot[z/2] + c.$$

$$f(\pi/2) = \frac{(i-1)}{4}\cot(\pi/4) - \frac{(i-1)}{4}\cot(\pi/4) + c = 0$$

$$= \frac{(i-1)}{4} - \frac{(i-1)}{4} + c = 0 \quad \boxed{c=0}$$

$$\cancel{f(z) = \frac{(i-1)}{4}\left[\cot\left(\frac{z}{2}\right) - \cot\left(\frac{z}{2}\right)\right]}.$$

$$\cancel{(1+i)}\, f(z) = -\frac{1}{2}\cot\left(\frac{z}{2}\right)\left[\cancel{1+i}\right] + c$$

$$f(z) = -\frac{1}{2}\cot\left(\frac{z}{2}\right) + c.$$

$$f\left(\frac{\pi}{2}\right) = 0$$

$$-\frac{1}{2}\cot\left(\frac{\pi}{4}\right) + c = 0$$

$$c = 1/2$$

$$\therefore f(z) = -\frac{1}{2}\cot\left(\frac{z}{2}\right) + \frac{1}{2}.$$

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1) $2u+v = e^{2x}[(2x+y)\cos 2y + (x-2y)\sin 2y]$.

$f(z) = u + iv \rightarrow (1)$

$if(z) = iu - v \rightarrow (2)$

$(1)\times 2 \Rightarrow 2f(z) = 2u + i2v$

$(2)\times -1 \Rightarrow -if(z) = -iu + v$

$2f(z) - if(z) = (2u+v) + i(2v-u)$

$f(z)(2-i) = (2u+v) + i(2v-u)$

$F(z) = U + iV$.

$U_x(z,0) = e^{2x}[(2\cos 2y) + 0 + \sin 2y - 0] + [(2x+y)\cos 2y + (x-2y)\sin 2y](2e^{2x})\Big|_{(z,0)}$

$= e^{2z}[(2\cos 0) + 0 + 0 - 0] + [(2z+0)\cos 0 + (z-0)\sin 0](2e^{2z})$

$= e^{2z}[2] + [2z][2e^{2z}] = 2e^{2z}[1+2z]$.

$U_y(z,0) = e^{2x}[-(2x\sin 2y) + 1(-\sin 2y) + \cos 2y + x\cos 2y(2) - (2y\cos 2y + \sin 2y \cdot 2)]\Big|_{(z,0)}$

$= e^{2z}[-0 + 0 + \cos 2(0) + 2z\cos 0 - (0+0)]$.

$= e^{2z}[1+2z]$.

$F(z) = \int f_1 - i\int f_2 + c$

$= 2\int e^{2z}[1+2z] - i \int e^{2z}[1+2z] + c$

$u = 1+2z \quad v = e^{2z}$

$u' = 2 \quad v_1 = e^{2z}/2$

$u'' = 0 \quad v_2 = e^{2z}/4$

$\int uv = uv_1 - u'v_2 + u''v_3 + \dots$

$= (1+2z)\left(\frac{e^{2z}}{2}\right) - 2\frac{e^{2z}}{4} = (1+2z)\frac{e^{2z}}{2} - \frac{e^{2z}}{2}$

$= 2\left[\frac{(1+2z)e^{2z} - e^{2z}}{2}\right] - i\left[\frac{(1+2z)e^{2z} - e^{2z}}{2}\right] + c$

$f(z) = \left((1+2z)e^{2z} - e^{2z}\right)\left(1 - \frac{i}{2}\right) + c$

$f(z)(2-i) = \left(\frac{e^{2z} + 2z e^{2z} - e^{2z}}{2}\right)(2-i) + c = \frac{2ze^{2z}}{2}(2-i) + c$

$f(z) = z \cdot e^{2z} + c$.

(9) $2u+v = \overset{u}{e^x}\overset{v}{(\cos y - \sin y)}$.

ⓐ

$u+iv = f(z) = u+iv$

$i f(z) = iu - v$

① ×2 ⇒ $2f(z) = 2u + i2v$

② ×−1 ⇒ $-if(z) = -iu + v$

$f(z)(2-i) = (2u+v) + i(2v-u)$

$U_x = e^x(\cos y) - e^x \sin y$

$= (e^x(\cos y) - e^x \sin y)_{(z,0)}$

$= (e^z - 0) = e^z$

$U_y = e^x \cos y - e^x \sin y$

$= e^x(-\sin y) - e^x(+\cos y)$

$= e^z(0) - e^z(1)$

$= -e^z$.

$F(z) = \int f_1 - i\int f_2 + c$

$= \int e^z - i\int e^z + c$

$= e^z + ie^z + c$.

$f(z)(2-i) = e^z(1+i) + c$

$f(z) = \frac{e^z(1+i)}{(2-i)} \times \frac{(2+i)}{(2+i)} + c$

$= \frac{e^z(2+2i+i+(-1))}{5} + c$, $= \frac{e^z(3i+1)}{5} + c$.

ⓓ $f(z) = u+iv$

$if(z) = iu - v$

$f(z) + if(z) = (u-v) + i(u+v)$

$f(z)(1+i) = (u-v) + i(u+v)$.

$U_x = (e^x \cos y - e^x \sin y)_{(z,0)}$

$= (e^x(\cos y) - e^x \sin y)_{(z,0)}$

$= (e^z(1) - 0)$

$= e^z$

$U_y = (e^x \cos y - e^x \sin y)_{(z,0)}$

$= (e^x(-\sin y) - (e^x)(\cos y))$

$= (e^z(-\sin 0) - (e^z)(\cos 0))$

$= 0 - e^z = -e^z$.

$F(z) = \int e^z + i\int e^z + c$

$f(z)(1+i) = e^z(1+i) + c$

$f(z) = e^z \frac{\cancel{(1+i)}}{\cancel{(1+i)}} + c \quad = e^z + c$

(1) $u - 2v$

$f(z) = u + iv$

$if(z) = iu - v$

(1) $\times 1 \Rightarrow f(z) = u + iv$

(2) $\times 2 \Rightarrow 2if(z) = i2u - 2v$

$f(z)(1+2i) = (u-2v) + i(v+2u)$

$U_x = (e^x\cos y - e^x\sin y)_{(z,0)}$ $\quad$ $U_y = (e^x\cos y - e^x\sin y)_{(z,0)}$

$= (e^z\cos 0 - 0)$ $\quad$ $= (e^x(-\sin y) - e^x(\cos y))$

$= e^z$ $\quad$ $= -e^z(1)$

$= -e^z$

$F(z) = \int e^z + i\int e^z + c$

$f(z)(1+2i) = e^z(1+i) + c$

$$f(z) = \frac{e^z(1+i)(1-2i)}{(1+2i)(1-2i)} + c$$

$$= \frac{e^z(1-2i+i-2(-1))}{(1-4(-1))} + c$$

$$= \frac{e^z(3-i)}{5} + c$$

(c) $2u + 3v$

$f(z) = u + iv$

$if(z) = iu - v$

(1) $\times 2 \rightarrow 2f(z) = 2u + i2v$

(2) $\times 3 \Rightarrow -i3f(z) = -i3u + 3v$

$f(z)(2-i3) = (2u+3v) + i(2v-3u)$

$F(z) = e^z(1+i) + c$

$f(z)(2-i3) = e^z(1+i) + c$

$$f(z) = \frac{e^z(1+i)(2+3i)}{(2-3i)(2+3i)} + c$$

$$= \frac{e^z(2+3i+2i-3)}{(4-9(-1))} + c$$

$$= \frac{e^z(5i-1)}{13} + c. \quad //$$

Conformal Mapping — simple transformation; standard transformation

standard transformation: $w = \sin z$, $w = \cos z$, $w = \sinh z$, $w = e^z$, $w = z +$

Pr: $w = \sin z$

$$u + iv = \sin z$$
$$= \sin(x+iy)$$
$$= \sin x \cos iy + \cos x \sin iy$$
$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

on comparing,

$$u = \sin x \cdot \cosh y \rightarrow ①$$
$$v = \cos x \cdot \sinh y \rightarrow ②$$

Elimination of y:-

$$\cosh y = \frac{u}{\sin x} \quad ; \quad \sinh y = \frac{v}{\cos x}$$

$$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

Take $x = c$,

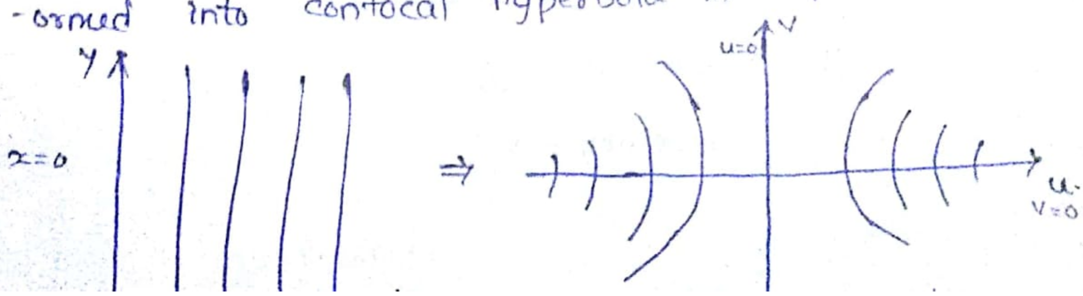
$$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$$

Take, $a^2 = \sin^2 c$, $b^2 = \cos^2 c$.

$$\left(\frac{u^2}{a^2} - \frac{v^2}{b^2}\right) = 1$$

$\therefore$ Equation of hyperbola.

The set of straight lines $(x = c)$ in z-plane is transformed into confocal hyperbola in w-plane.



Elimination of x:-

$$\sin x = \frac{u}{\cosh y}$$

$$\cos x = \frac{v}{\sinh y}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

Take $y = k$ (straight line)

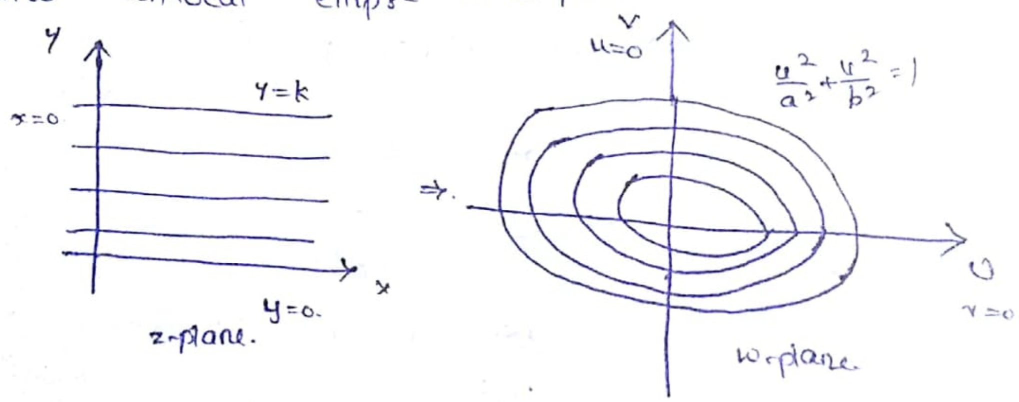
$$\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$$

Take $\cosh^2 c = a^2$, $\sinh^2 c = b^2$

$$\boxed{\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1}$$

$\therefore$ equation of ellipse.

$\therefore$ set of straight lines $(y=k)$ in z-plane is transformed into confocal ellipse in w-plane.



(8) $w = \cos z$

$u + iv = \cos(x + iy)$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos x \cosh y - i \sin x \sinh y$.

$u = \cos x \cosh y \rightarrow (1)$

$v = -\sin x \sinh y \rightarrow (2)$.

Elimination of y:

$$\cosh y = \frac{u}{\cos x} \; ; \quad \sinh y = \frac{-v}{\sin x}$$

$$\cosh^2 y - \sinh^2 y = 1$$

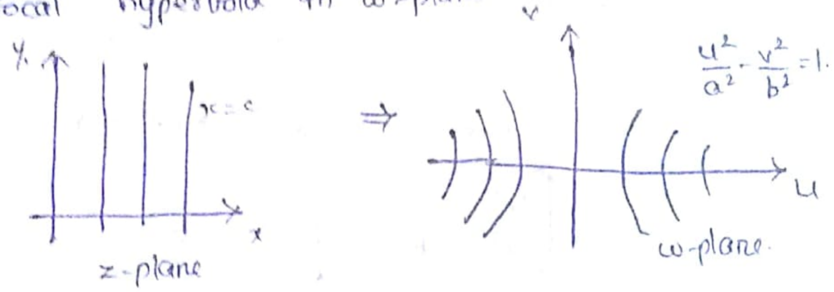
$$\Rightarrow \left(\frac{u}{\cos x}\right)^2 - \left(\frac{-v}{\sin x}\right)^2 = 1 \Rightarrow \frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1$$

Take $x = c$ (straight line)

$$a^2 = \cos^2 c \; ; \; b^2 = \sin^2 c$$

$$\boxed{\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1} \rightarrow \text{hyperbola}$$

$\therefore$ set of straight lines ($x = c$) in z-plane transformed into confocal hyperbola in w-plane.



Elimination of x:

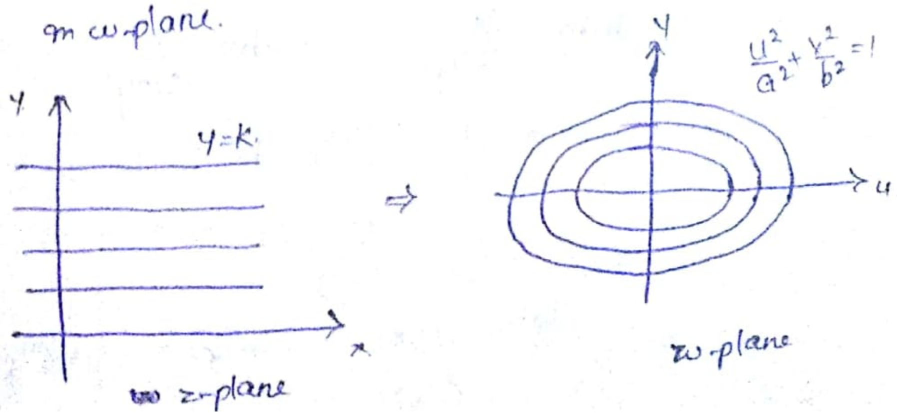
$$\cos x = \frac{u}{\cosh y} \; ; \quad \sin x = \frac{-v}{\sinh y}$$

$$\cos^2 x + \sin^2 x = 1 \Rightarrow \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1.$$

Tak $y = k$ and $\cosh^2 k = a^2$, $\sinh^2 k = b^2$

$$\boxed{\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1} \rightarrow \text{ellipse.}$$

$\therefore$ set of st. Lines $y = k$ in z-plane transformed into confocal ellipse in w-plane.



$\sin(ix) = i\sinh x$

$\sin(hx) = \frac{1}{i}\sin(ix)$

$= -i\sin(ix)$

$= -i\sin(iz)$

$= -i\sin(i(x+iy))$

$= -i\sin(ix-y)$

$= -i[\sin ix\cos y - \cos ix\sin y]$

$= -i[i\sinh x\cos y - \cosh x\sin y]$

$= \sinh x\cos y + i\cosh x\sin y$

$u = \sinh x\cos y \qquad v = \cosh x\sin y$

Elimination of y:-

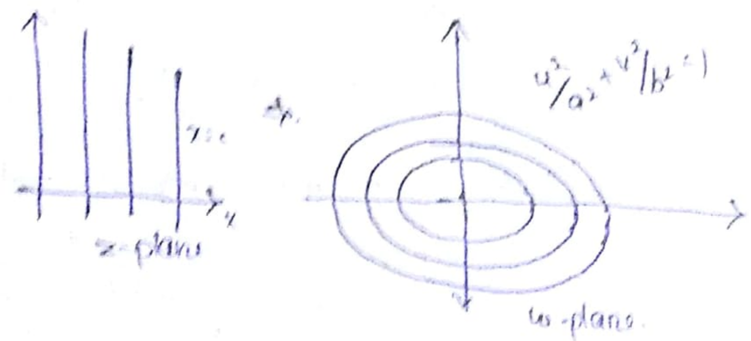
$$\cos y = \frac{u}{\sinh x} \; ; \; \sin y = \frac{v}{\cosh x}$$

$$\cos^2 y + \sin^2 y = 1 \Rightarrow \frac{u^2}{\sinh^2 x} + \frac{v^2}{\cosh^2 x} = 1$$

take $x = c$ and $\sinh^2 c = a^2$ & $\cosh^2 c = b^2$

$$\boxed{\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1} \rightarrow \text{ellipse}$$

$\therefore$ set of st. line $(x=c)$ in z-plane transformed into confocal ellipse in w-plane.



Elimination of x:-

$$\sinh x = \frac{u}{\cos y} \; ; \; \cosh x = \frac{v}{\sin y}$$

$$\cosh^2 x - \sinh^2 x = 1$$

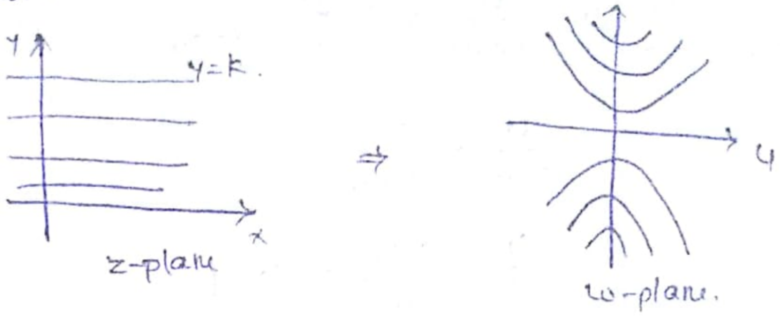
$$\frac{v^2}{\sin^2 y} - \frac{u^2}{\cos^2 y} = 1$$

take $y = k$, $\sin^2 k = a^2$; $\cos^2 k = b^2$

$$\left(\frac{v^2}{a^2} - \frac{u^2}{b^2}\right) = 1 \rightarrow \text{hyperbola}$$

Scanned by CamScanner

∴ set of st. lines $(y=k)$ in z-plane converted into confocal hyperbola in w-plane.



(d) $w = e^z$

$u+iv = e^x . e^{iy}$

$= e^x(\cos y + i\sin y)$

$= e^x\cos y + i e^x \sin y.$

$u = e^x \cos y$

$v = e^x \sin y.$

Eliminating y;

$\cos y = \frac{u}{e^x}$; $\sin y = \frac{v}{e^x}$.

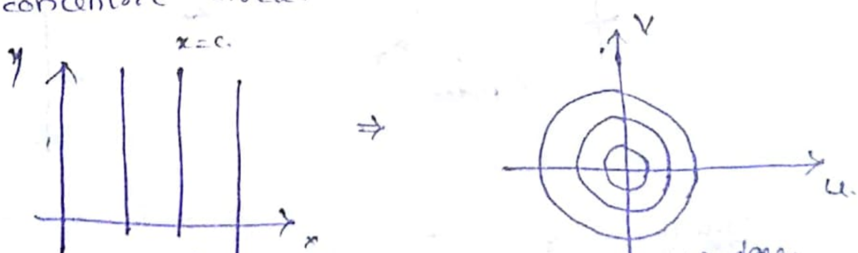
$\cos^2 y + \sin^2 y = 1 \Rightarrow \frac{u^2}{e^{2x}} + \frac{v^2}{e^{2x}} = 1$

$u^2+v^2 = e^{2x} \rightarrow$ circle

case(i):- $x = c$, $u^2+v^2 = e^{2c} \rightarrow$ circle.

$u^2+v^2 = k^2 \rightarrow$ circle.

∴ A set of st. lines $(x=c)$ is in z-plane is transformed into concentric circles in w-plane.



case(ii):- $x=0$; $u^2+v^2 = e^{2(0)} = 1.$

(y-axis) $u^2+v^2 = 1 \rightarrow$ eq. of circle with unit radius.

∴ $x=0$ line in z-plane is transformed into a single circle in w-plane

Elimination of x:-

$$e^x = \frac{u}{\cos y} \; ; \quad e^x = \frac{v}{\sin y}$$

$$\frac{u}{\cos y} = \frac{v}{\sin y}$$

$$\frac{u}{v} = \frac{\cos y}{\sin y} = \cot y$$

$$\boxed{\frac{v}{u} = \tan y.} \Rightarrow v = u \tan y$$

$$\boxed{y = \tan^{-1}\left(\frac{v}{u}\right)}$$

$y = k$ straight line.

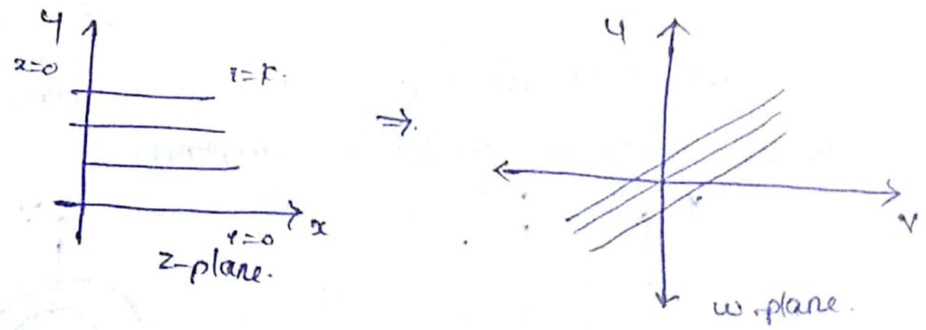
case (iii):-

$$\frac{v}{u} = \tan k.$$

$$v = u(\tan k) = u(m).$$

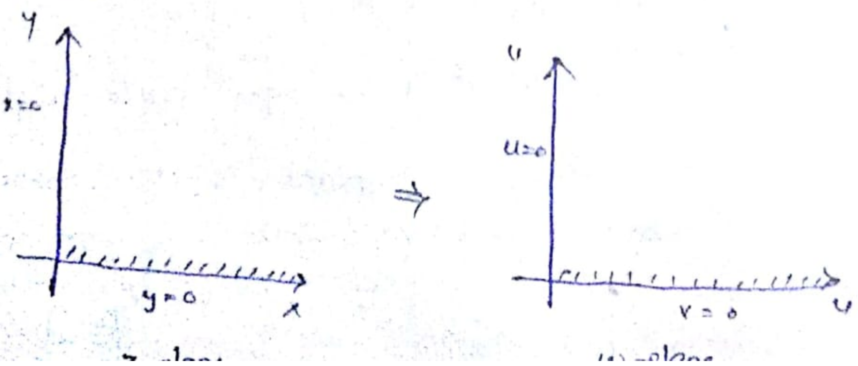
$$\boxed{v = m \times u}$$

$\therefore$ The st-line $y = k$ in z-plane is transformed into set of lines in w-plane.



case (iv):- $y = 0$, $\boxed{\frac{v}{u} = 0}$

$\boxed{v = 0}$



case (v):- $y = \pi/2$

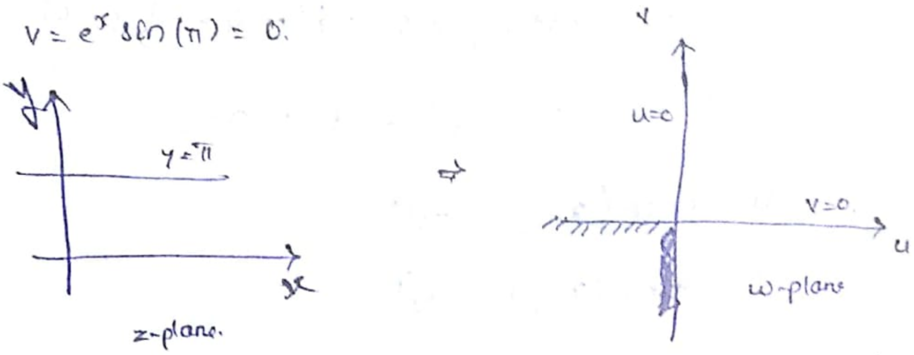
$u = e^x \cdot \cos(\pi/2) = 0$

$v = e^x \cdot \sin(\pi/2) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots (>0)$

case (vi):- $y = \pi$

$u = e^x \cos(\pi) = -e^x$

$v = e^x \sin(\pi) = 0.$

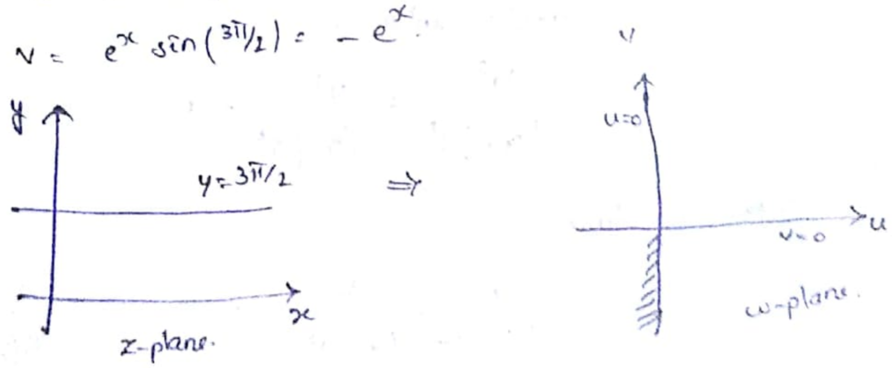


The line $y = \pi/2$ is transformed into the -ve u-axis.

case (vii) :- $y = 3\pi/2$.

$u = e^x \cos(3\pi/2) = 0$

$v = e^x \sin(3\pi/2) = -e^x$.



case (viii): $y = 2\pi$

$u = e^x \cos(2\pi) = e^x$

$v = e^x \sin(2\pi) = 0$

(5) Discuss the transform of $w = z + \frac{1}{z}$ (in polar form) under the transfor $z+\frac{1}{z}$, what will be the image of $|z| = r$ if $r = c$ and $(c \neq 1)$, $(0 < (c=1))$

$$z = r \cdot e^{i\theta} = r[\cos\theta + i\sin\theta]$$

$$= r\cos\theta + ir\sin\theta$$

$$\frac{1}{z} = \frac{1}{r \cdot e^{i\theta}} = \frac{1}{r} \cdot e^{-i\theta}$$

$$= \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

$$u + iv = r\cos\theta + ir\sin\theta + \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

$$= \left(r\cos\theta + \frac{1}{r}\cos\theta\right) + i\left(r\sin\theta - \frac{1}{r}\sin\theta\right)$$

$$u + iv = \cos\theta\left(r + \frac{1}{r}\right) + i\sin\theta\left(r - \frac{1}{r}\right)$$

$$u = \cos\theta\left(r + \frac{1}{r}\right) \quad -(1)$$

$$v = \sin\theta\left(r - \frac{1}{r}\right) \quad -(2)$$

Elimination of θ:-

$$\cos\theta = \frac{u}{\left[r + \frac{1}{r}\right]} \quad ; \quad \sin\theta = \frac{v}{\left(r - \frac{1}{r}\right)}$$

$$\cos^2\theta + \sin^2\theta = 1$$

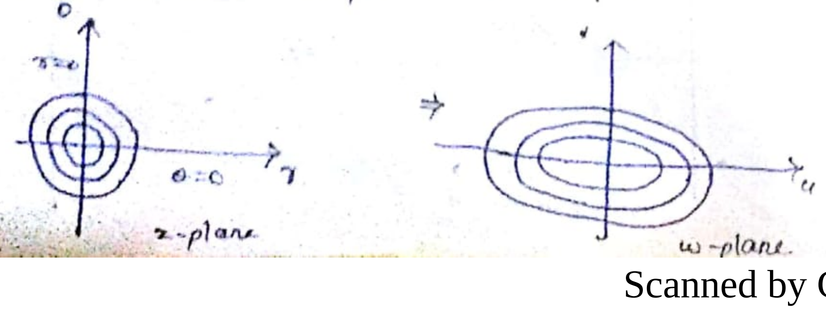
$$\frac{u^2}{\left[r + \frac{1}{r}\right]^2} + \frac{v^2}{\left[r - \frac{1}{r}\right]^2} = 1$$

Case(i):- Take $r = c$,

$$\frac{u^2}{\underbrace{\left(c + \frac{1}{c}\right)^2}_{a^2}} + \frac{v^2}{\underbrace{\left(c - \frac{1}{c}\right)^2}_{b^2}} = 1$$

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \Rightarrow \text{ellipse.}$$

∴ A set of concentric circles in z-plane is transformed into confocal ellipse in the w-plane



case(ii):- Take $r=1$, is a unit radius circle.

$$\frac{u^2}{(r+\frac{1}{r})^2}+\frac{v^2}{(r-\frac{1}{r})^2}=1$$

$$\frac{u^2}{(r+\frac{1}{r})^2}+\infty=1$$

$$\infty \neq 1$$

∴ This case can not be explained.

Elimination of r:-

$$r+\frac{1}{r}=\frac{u}{\cos\theta} \; ; \quad r-\frac{1}{r}=\frac{v}{\sin\theta}$$

$$\left(r+\frac{1}{r}\right)^2-\left(r-\frac{1}{r}\right)^2=\frac{u^2}{\cos^2\theta}-\frac{v^2}{\sin^2\theta}$$

$$2\cancel{r}\times\frac{1}{\cancel{r}}+2\cdot r\cdot\frac{1}{r}=\frac{u^2}{\cos^2\theta}-\frac{v^2}{\sin^2\theta}=4$$

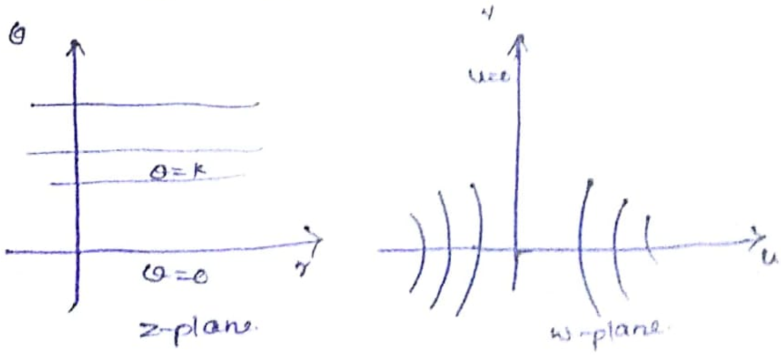
$$\frac{u^2}{4\cos^2\theta}-\frac{v^2}{4\sin^2\theta}=1$$

$\theta=k$,

$$\underbrace{\frac{u^2}{(2\cos\theta)^2}}_{a^2}-\underbrace{\frac{v^2}{(2\sin\theta)^2}}_{b^2}=1$$

$\underbrace{2\cos k}_{a}$, $\underbrace{2\sin k}_{b}$.

$$\frac{u^2}{a^2}-\frac{v^2}{b^2}=1 \Rightarrow \text{hyperbola.}$$



Simple transformations:-

(a) Magnification

(b) translation.

(c) rotation.

(d) Inversion.

(2) To find $|z-2i|=2$

$|z-(0+2i)|=2$

$\therefore$ centre $=(0,2)$

radius $=2$

given $w=\frac{1}{z} \Rightarrow z=\frac{1}{w} \Rightarrow \left|\frac{1}{w}-2i\right|=2 = \left|\frac{1}{u+iv}-2i\right|=2$

$\left|\frac{u-iv}{u^2+v^2}-2i\right|=2 \Rightarrow \left|\frac{(u-iv)-2i(u^2+v^2)}{u^2+v^2}\right|=2$

$\left|\frac{u-iv-2iu^2-2iv^2}{u^2+v^2}\right|=2$

$\left|\frac{u}{u^2+v^2}+i\left[\frac{-v-2u^2-2v^2}{u^2+v^2}\right]\right|=2$

$\sqrt{u^2+(-v-2u^2-2v^2)^2} = [2(u^2+v^2)]^2$

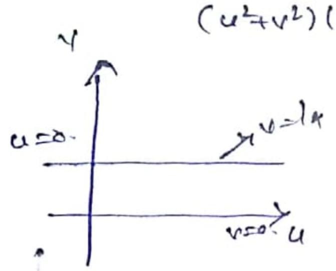
$u^2+(v+2u^2+2v^2)^2 = 4(u^2+v^2)^2$

$u^2+v^2-4v(u^2+v^2)=0.$

$u^2+v^2-4vu^2-4v^3=0.$

$u^2(1-4v)+v^2(1-4v)=0.$

$(u^2+v^2)(1-4v)=0 \Rightarrow 1=4v \Rightarrow \boxed{v=1/4}$



* find the image of the following.

(i) $x=1$ (ii) $\frac{1}{4}<y<\frac{1}{2e}$ (iii) $\frac{1}{4}<y<\frac{2}{4}$

(i) $w=\frac{1}{z}$

$z=\frac{1}{w} \Rightarrow x+iy=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$ $\quad x=\frac{u}{u^2+v^2}$; $y=\frac{-v}{u^2+v^2}$

$1=\frac{u}{u^2+v^2} \Rightarrow u^2+v^2=u.$

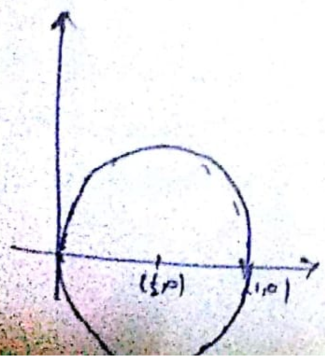
$u^2+v^2-u=0$ $\quad (u^2-u)+v^2=0.$

$(u-\frac{1}{2})^2-(\frac{1}{2})^2+(v-0)^2=0.$

$(u-\frac{1}{2})^2+(v-0)^2=1/4.$

centre $=(1/2,0)$

radius $=1/2.$



(ii) $\frac{1}{4} < y < \frac{1}{2e}$

$\frac{1}{4} = \frac{-v}{u^2+v^2}$

$u^2+v^2 = -4v$

$u^2 + (v^2+4v) = 0.$

$(u-0)^2 + (v+2)^2 - (2)^2 = 0$

$(u-0)^2 + (v+2)^2 = 4$

centre $= (0,-2)$

Radius $= 2.$

$A^2 = v^2$

$A = V.$

$\frac{\cancel{2}4V}{2} = \cancel{2}A B.$

$\boxed{2 = B}$

$\frac{1}{2e} = \frac{-v}{u^2+v^2}$

$u^2+v^2 = -2ve$

$u^2+(v^2+2ve) = 0$

$u^2 + (v^2+2ve) = 0.$

$(u-0)^2 + (v+e)^2 = 0.$

centre $= (0,-e)$

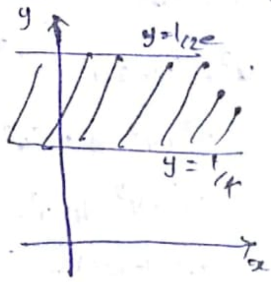
radius $= e.$

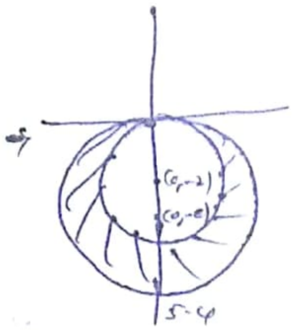
$v^2 = A^2 ..$

$A = V$

$\cancel{2}AB = \cancel{2}ye$

$\boxed{B = e}$





(iii) $\frac{1}{4}$ same. centre $= (0,-2).$

$y = \frac{2}{4} = \frac{1}{2}$

$w = \frac{1}{2} \Rightarrow z = \frac{1}{w}.$

$x = \frac{u}{u^2+v^2}, \; y = \frac{-v}{u^2+v^2}$

$y = \frac{1}{2},$

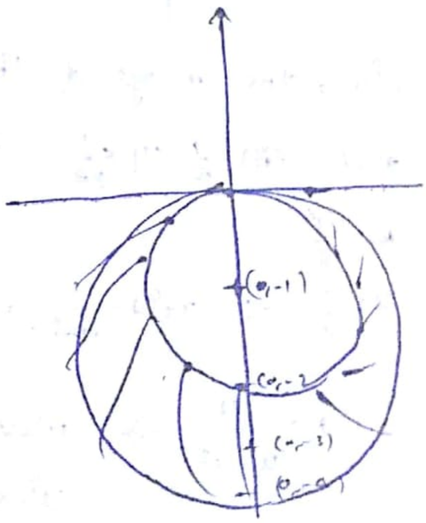
$u^2+v^2 = -2v$

$u^2+v^2+2v = 0.$

$(u-0)^2 + (v+1)^2 = (1)^2$

$c = (0,-1)$

$r = 1$



Pr: Show that the transformation $w = \frac{1}{z}$ transforms all the circles and straight lines in the z-plane into circles or straight lines in the w-plane.

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$x + iy = \frac{1}{u+iv}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

general equation of circle is $a(x^2+y^2) + 2fx + 2gy + c = 0$.

$$\text{centre} = \left(-\frac{f}{a}, -\frac{g}{a}\right) \quad \text{radius} = \sqrt{\frac{g^2}{a^2} + \frac{f^2}{a^2} + c}$$

$$a\left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right] + 2f\left[\frac{u}{u^2+v^2}\right] + 2g\left[\frac{-v}{u^2+v^2}\right] + c = 0$$

$$a\left[\frac{(u^2+v^2)}{(u^2+v^2)^2}\right] + \frac{2fu}{u^2+v^2} - \frac{2gv}{u^2+v^2} + c = 0$$

$$a + 2fu - 2gv + c(u^2+v^2) = 0.$$

case (i):- $a \neq 0, c \neq 0$, circle not passing through the origin in z-plane is transformed into circles not passing through origin in w-plane.

case (ii):- $a \neq 0, c = 0$, In this case, circles passing through the origin in z-plane is transformed into straight line not passing through the origin.

case (iii):- $a = 0, c \neq 0$, in this case, the straight lines not passing through origin in z-plane is transformed into circles passing through origin in w-plane.

case (iv):- $a = 0, c = 0$, In this case, the straight lines passing through the origin in the z-plane is transformed into straight lines passing through origin in w-plane.

Pr: Show that transformation $w = \frac{iz+1}{z+i}$ transform exterior and interior regions of unit circle ($|z| = 1$) into upper and lower half of w-plane.

$$w = \frac{iz+1}{z+i}$$

$$w(z+i) = iz + 1$$

$$wz - iz = 1 - wi$$

$$z(\omega - i) = 1 - \omega i$$

$$z = \frac{1-\omega i}{\omega - i}$$

$$\left|\frac{1-\omega i}{\omega - i}\right| \geq 1$$

$$|1-\omega i| \geq |\omega - i|$$

$$|1-(u+iv)i| \geq |(u+iv-i)|$$

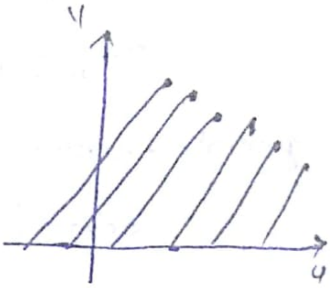
$$|(1+v) - iu| \geq |u+i(v-1)|$$

$$\sqrt{(1+v)^2 + (-u)^2} \geq \sqrt{u^2+(v-1)^2}$$

$$\cancel{u^2}+\cancel{v^2}+\cancel{1}+2v \geq \cancel{u^2}+\cancel{v^2}+\cancel{1}-2v$$

$$4v \geq 0$$

$$v \geq 0.$$



Homework:-

(1) show that $\omega = \frac{z-i}{z+i}$ transforms real axis in z-plane on to $|\omega| = 1$ in ω-plane. show that upper half of z-plane, $Im(z) \geq 0$, goes on to circular $|\omega| \leq 1$

(i) $|\omega| = 1$

$$\left|\frac{z-i}{z+i}\right| = 1$$

$$|z-i| = |z+i|$$

$$|x+iy-i| = |x+iy+i| \Rightarrow |x+i(y-1)| = |x+i(y+1)|$$

$$\cancel{x^2}+(y-1)^2 = \cancel{x^2}+(y+1)^2$$

$$4y = 0 \Rightarrow y = 0 \text{ (x-axis)}$$

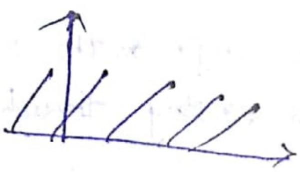
(ii) $|w| \leq 1$

$$\left|\frac{z-i}{z+i}\right| \leq 1$$

$$|z-i| \leq |z+i|$$

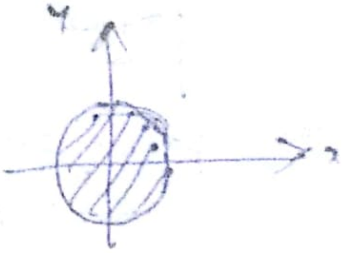
$$|x+i(y-1)| \leq |x+i(y+1)|$$

$$y \geq 0$$



(2) S.T, $\omega = \frac{z-i}{1-iz}$ maps the interior of unit circle on to lower half of ω-plane & also upper half of z-plane on to interior of circle $|\omega| = 1$

(i) interior of unit circle $|z| \le 1$



$$w = \frac{z-i}{1-iz}$$

$$w - iwz = z - i$$

$$z(1+iw) = w + i$$

$$z = \frac{w+i}{1+iw}$$

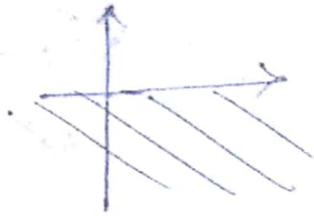
$$\left|\frac{w+i}{1+iw}\right| \le 1 \Rightarrow |w+i| \le |1+iw|$$

$$|(u+iv)+i| \le |1+i(u+iv)|$$

$$|u+i(v+1)| \le |(1-v)+i(u)|$$

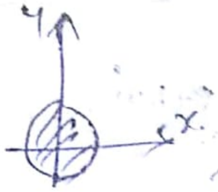
$$u^2+(v+1)^2 \le (1-v)^2+u^2$$

$$v \le 0.$$



(ii) $|w| = 1$

Interior region $|w| \le 1$

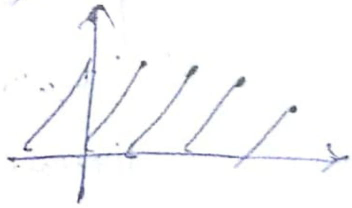


$$\left|\frac{z-i}{z+i}\right| \le 1$$

$$|z-i| \le |z+i|$$

$$|x+i(y-1)| \le |x+i(y+1)|$$

$$y \ge 0.$$



Pr: Prove that $w = \dfrac{2z+3}{z-4}$ maps a circle $x^2+y^2-4x=0$ on to the straight line $4u+3=0$. Also find the image of $|z|=1$ under the above transformation.

$$w = \frac{2z+3}{z-4}$$

$$w(z-4) = 2z+3.$$

$$wz - 4w = 2z+3.$$

$$z(w-2) = 3+4w.$$

$$\boxed{z = \frac{4w+3}{w-2}}$$

In z-plane, $x^2+y^2-4x=0$.

centre $= (-f, -g) = (2, 0) = (a, b)$

radius $= \sqrt{f^2+g^2-c} = \sqrt{2^2+0^2+0} = 2 = r$

$|z - (a+ib)| = r \rightarrow |z-2| = 2$

$$\left|\frac{3+4\omega}{\omega-2}-2\right|=2$$

$$\left|\frac{3+4\omega-2\omega+4}{\omega-2}\right|=2$$

$$|2\omega+7|=2|\omega-2|$$

$$|2u+2iv+7|=2|(u-2)+iv|$$

$$|(2u+7)+i(2v)|=2|(u-2)+iv|$$

$$\sqrt{(2u+7)^2+(2v)^2}=\sqrt{2^2}\times\sqrt{(u-2)^2+v^2}$$

$$\cancel{4u^2}+49+28u+\cancel{4v^2}=\cancel{4u^2}+16-16u+\cancel{4v^2}$$

$$49-16+28u+16u=0$$

$$33+44u=0$$

$$\boxed{4u+3=0}$$

$\Rightarrow |z|=1$

$$\left|\frac{3+4\omega}{\omega-2}\right|=1$$

$$|3+4(u+iv)|=|u+iv-2|$$

$$|(4u+3)+iv|=|(u-2)+iv|$$

$$\sqrt{(4u+3)^2+v^2}=\sqrt{(u-2)^2+v^2}$$

$$16u^2+9+24u+\cancel{v^2}=u^2+4-4u+\cancel{v^2}$$

$$15u^2+28u+5=0.$$

Bilinear:

The transformation $w = \frac{az+b}{cz+d}$, where a, b, c, d are constants and $ad-bc \neq 0$ is called as bilinear transformation.

The above equation $\frac{az+b}{cz+d}$ is both linear in both numerator & denominator in terms of z, So, it is called as bilinear transformation. It is also called as Linear functional (or) Mobious transformation.

* To find the fixed (or) invariant point, solve for z for about two points.

* The bilinear transformation of two points $(z_1, z_2, z_3) \to (w_1, w_2, w_3)$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w_2-w_1)(w-w_3)} = \frac{(z-z_1)(z_2-z_3)}{(z_2-z_1)(z-z_3)}$$

Pr: find the B.T which maps the points $(1, i, -1) \to (2, i, -2)$
$(z_1 z_2 z_3) \quad (w_1 w_2 w_3)$

$$\Rightarrow \frac{(w-2)(i+2)}{(i-2)(w+2)} = \frac{(z-1)(i+1)}{(i-1)(z+1)}$$

$$\Rightarrow \frac{(w-2)}{w+2} \times \frac{(i+2)}{(i-2)} \times \frac{(-i-2)}{(-i-2)} = \frac{(z-1)}{(z+1)} \times \frac{(i+1)}{(i-1)} \times \frac{(-i-1)}{(-i-1)}$$

$$\Rightarrow \frac{(w-2)}{(w+2)} \times \frac{(-i^2-2i-2i-4)}{(1^2+2^2)} = \frac{(z-1)}{(z+1)} \times \frac{(\cancel{-i^2}-i-i-\cancel{1})}{(1^2+1^2)}$$

$$\Rightarrow \frac{(w-2)}{(w+2)} \times \frac{(-3-4i)}{5} = \frac{(z-1)}{(z+1)} \times \frac{(-\cancel{2}i)}{\cancel{2}}$$

$$\Rightarrow \frac{(w-2)}{(w+2)} \times \frac{\cancel{-}(3+4i)}{5(\cancel{-}i)} = \frac{(z-1)}{(z+1)}$$

$$\Rightarrow \frac{(w-2)}{(w+2)} \times \frac{3+4i}{5i} = \frac{z-1}{z+1}$$

$$\Rightarrow \frac{w-2}{w+2} \times \frac{(-3i+4)}{5} = \frac{z-1}{z+1} \Rightarrow \frac{(-3iw+4w+6i-8)}{(5w+10)} = \frac{z-1}{z+1}$$

$\Rightarrow -3i\omega z + 4\omega z + 6iz - 8z - 3i\omega + 4\omega + 6i - 8$

$=$

$5\omega z - 5\omega + 10z - 10.$

$\Rightarrow -3i\omega z + 6iz - 5\omega z - 10z = 8z + 3i\omega - 4\omega - 6i + 8 - 5\omega - 10$

$\omega(-3iz - 5z + 9) = -6iz + 10z + 8z - 2 - 6i$

$$\omega = \frac{z(-6i + 10 + 8) - 2 - 6i}{(-3iz - 5z + 9)}$$

Pr: find the B.T which maps $(-2, 0, 2) \to (0, +i, -i)$
$z_1\ z_2\ z_3 \quad \omega_1\ \omega_2\ \omega_3.$

$$\frac{(\omega - 0)(i + i)}{(i - 0)(\omega + i)} = \frac{(z+2)(-2)}{(+2)(z-2)} \quad (a$$

$$\frac{(\omega)(2i)}{i(\omega + i)} = \frac{-(z+2)}{(z-2)}$$

$$\frac{2\omega}{\omega + i} = \frac{-(z+2)}{(z-2)}$$

$2\omega z - 2(2\omega) = -z\omega - 2\omega - iz - 2i$

$3\omega z - 2\omega = -i(z+2)$

$\omega(3z - 2) = -i(z+2).$

$\omega(2 - 3z) = i(z+2)$

$$\omega = \frac{i(z+2)}{(2-3z)} \quad (b$$

Note: If any one of the point happens to be infinity omit the corresponding factor.

Ex. B·7 of $(0, 1, \infty) \to (-5, -1, 3)$
$(z_1, z_2, z_3) \quad (w_1, w_2, w_3)$

$$\frac{(w+5)(-1-3)}{(-1+5)(w-3)} = \frac{(z-0)}{(1-0)}$$

$$\frac{(w+5)(-\cancel{4})}{(\cancel{4})(w-3)} = \frac{z}{1}$$

$$\frac{(w+5)}{3-w} = \frac{z}{1}$$

$$w+5 = 3z - wz$$

$$w(1+z) = 3z-5$$

$$\boxed{w = \frac{3z-5}{1+z}}$$

Homework:

(a) $(2,1,0) \to (1,0,i)$
$(z_1, z_2, z_3) \quad (w_1, w_2, w_3)$

$$\frac{(w-1)(0-i)}{(0-1)(w-i)} = \frac{(z-2)(1-0)}{(1-2)(z-0)}$$

$$\cancel{-}\frac{(w-1)(-i)}{(w-i)} = \cancel{-}\frac{(z-2)}{z}$$

$$\frac{(-iw+i)}{(w-i)} = \frac{z-2}{z}$$

$$-ziw + zi = zw - 2w - zi + 2i$$

$$w(zi - z - 2) = i(-2z+2)$$

$$w = \frac{i(2-2z)}{(zi-z-2)}$$

(b) $(-1,0,1) \to (0, i, 3)$
$(z_1, z_2, z_3) \quad (w_1, w_2, w_3)$

$$\frac{(w-0)(i-3)}{(i-0)(w-3)} = \frac{(z+1)(0-1)}{(1)(z-1)}$$

$$\frac{w(i-3)}{i(w-3)} = \frac{z+1}{(1-z)} \Rightarrow \cancel{-}\frac{i(i-3)w}{(w-3)} = \cancel{-}\frac{(z+1)}{(z-1)}$$

$$\frac{(-1-3i)w}{w-3} = \frac{z+1}{z-1} \Rightarrow \frac{-w-3iw}{w-3} = \frac{z+1}{z-1}$$

$$\frac{-\omega-3i\omega}{\omega-3} = \frac{z+1}{z-1}$$

$$-\omega z-3i\omega z = z\omega+\omega-3z-3.$$

$$\omega(-z-3iz-z+1) = -3z-3$$

$$(0,-i,2i) \longrightarrow (5i,\infty,1/3)$$

$$\frac{(\omega-5i)}{(\omega-1/3)} = \frac{(z-0)(-i-2i)}{(-i-0)(z-2i)}$$

$$\frac{\cancel{3}(\omega-5i)}{3\omega-1} = \frac{\cancel{3}z}{z-2i}$$

$$(\omega-5i)(z-2i) = z(3\omega-1)$$

$$\omega z-2i\omega-3\omega z = 5iz+10-z.$$

$$-2\omega[z+i] = z[5i-1]+10$$

$$\omega = \frac{z[5i-1]+10}{-2[z+i]}.$$

Pb: find the bilinear transformation which maps the points $z = (-2i, i, \infty) \rightarrow w = (0, -3, 1/3)$ respectively. Also find image of $|z| < 1$.

$$\frac{(w-0)(-3-1/3)}{(-3-0)(w-1/3)} = \frac{(z+2i)}{(3i)}$$

$$\frac{(w)(-10)}{3(-3)(w-1/3)} = \frac{z+2i}{3i}$$

$$\frac{(w)(+10)\cancel{3}}{\cancel{3}(3)(3w-1)} = \frac{z+2i}{\cancel{3}i} \rightarrow 10wi = (3w-1)(z+2i)$$

$$10wi = 3wz + 6wi - z - 2i$$

$$4wi - 3wz = -z - 2i$$

$$w(4i - 3z) = -z - 2i$$

$$w(1) = \frac{-(z+2i)}{4i-3z} = \frac{z+2i}{3z-4i}$$

$$w(3z - 4i) = z + 2i$$

$$3wz - 4iw = z + 2i$$

$$3wz - z = 2i + 4iw$$

$$z(3w-1) = 2i(1+2w)$$

$$z = \frac{2i(2w+1)}{(3w-1)}$$

given $|z| < 1 \Rightarrow \left| \frac{2i[1+2w]}{(3w-1)} \right| < 1$

$$|2i(1+2w)| < |3w-1|$$

$$|2i(1+2u+i2v)| < |3u+3iv-1|$$

$$|2i + 4iu - 4v| < |(3u-1) + i(3v)|$$

$$|-4v + i(2+4u)| < |(3u-1) + i(3v)|$$

$$\sqrt{(-4v)^2 + (2+4u)^2} < \sqrt{(3u-1)^2 + (3v)^2}$$

$$16v^2 + 4 + 16u^2 + 16u < 9u^2 + 1 - 6u + 9v^2$$

$$7v^2 + 7u^2 + 22u + 3 < 0$$

$$u^2 + v^2 + \frac{22}{7}u + \frac{3}{7} < 0 \quad \text{circle}$$

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Q: Find the B.T which map $z=0$ onto $w=-i$ and has -1 and $+1$ as invariant point. Also show that this transformation maps the upper half of z-plane are interior of the unit circle.

$$\frac{(w+i)(-1-1)}{(-1+i)(w-1)} = \frac{(z-0)(-1-1)}{(-1-0)(z-1)}$$

$$\frac{(w+i)(\cancel{-2})}{(-1+i)(w-1)} = \frac{(z-0)(\cancel{-2})}{(-1)(z-1)}$$

$$\frac{(w+i)}{(i-1)(w-1)} = \frac{z}{(1-z)}$$

$$w+i-wz-iz = z(iw-i-w+1)$$

$$w+i-\cancel{w}z-\cancel{i}z = ziw-\cancel{zi}-z\cancel{w}+z$$

$$w-ziw = z-i$$

$$w(1-iz) = z-i$$

$$w = \frac{z-i}{1-iz}$$

$$(w+i) = z(1+iw)$$

$$z = \frac{w+i}{1+iw}$$

z-plane:- upper half of z-plane,

$$y \geq 0$$

$$z = \frac{w+i}{1+iw}$$

$$x+iy = \frac{u+iv+i}{1+iw}$$

$$x+iy = \frac{[u+i(v+1)]}{1+iw} \times \frac{(1-iw)}{(1-iw)}$$

$$= \frac{[u+i(v+1)][1-iw]}{(1+w^2)}$$

$$= \frac{[u-uiw+iv+vw+i+w]}{(1+w^2)}$$

$$= \frac{(u+vw+w)+i(v+1-uw)}{(1+w^2)}$$

$$x+iy = \frac{u+iv+i}{1+i(u+iv)} = \frac{u+i(v+1)}{(1-v)+ui}$$

$$= \frac{u+i(v+1)}{(v-1)+iu} \times \frac{((v-1)-iu)}{((v-1)-iu)}$$

$$= \frac{(u+iv+i)(v-1-iu)}{(v-1)^2+u^2}$$

$$= (uv - \cancel{u} - iu^2 + iv^2 - \cancel{iv} + uv + \cancel{iv} - i + \cancel{u}$$

$$x+iy. = \frac{2uv + i(-u^2+v^2-1)}{(v-1)^2+u^2}$$

$$y = \frac{-u^2+v^2-1}{(v-1)^2+u^2}$$

$$-u^2+v^2-1 \geq 0.$$

$$1 \geq -u^2+v^2$$

Standard results:-

(1) Prove that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \cdot \frac{\partial^2}{\partial z \partial \bar{z}}$

(2)

Let f be a function of x and y

$f \rightarrow (x, y)$

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

$x = \frac{z + \bar{z}}{2} \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2}$; $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$

x is a function of (z and $\bar{z}$).

(3)

$z - \bar{z} = 2iy.$ $\frac{\partial y}{\partial z} = \frac{1}{2i}$; $\frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$

$y = \frac{z - \bar{z}}{2i}$ y is a function of z and $\bar{z}$.

L.H.S: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

R.H.S: $4 \frac{\partial^2 f}{\partial z \partial \bar{z}}$

$\frac{\partial f}{\partial z} = \boxed{\frac{\partial f}{\partial x} \times \frac{\partial x}{\partial z}} + \boxed{\frac{\partial f}{\partial y} \times \frac{\partial y}{\partial z}}$

$2 \frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z}$

Note: whenever we are having ∇^2 term we have to prove about the lemma (∇^2)

(2) $\nabla^2 |f(z)|^2$ $\qquad |z|^2 = z\cdot\bar{z}$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4\partial^2}{\partial z\,\partial\bar{z}}$$

$$|f(z)|^2 = f(z).f(\bar{z})$$

L.H.S: $4\dfrac{\partial^2}{\partial\bar{z}\,\partial z}\left[f(z).f(\bar{z})\right]$

$= 4\dfrac{\partial}{\partial z}\left[f(z).f'(\bar{z})\right]$

$= 4\left[f'(z).f'(\bar{z})\right]$

$= 4\times |f'(z)|^2$

$=$ R.H.S

(3) $\nabla^2 |f(z)|^P$

$\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z\,\partial\bar{z}}$; $\quad |f(z)|^P = \left(\sqrt{f(z).f(\bar{z})}\right)^P$

$= \left(f(z).f(\bar{z})\right)^{P/2}$

$\bar{z}$.

L.H.S $= 4\dfrac{\partial^2}{\partial z\,\partial\bar{z}}\left[f(z).f(\bar{z})\right]^{P/2}$

$= 4\dfrac{\partial}{\partial z}\left[\dfrac{P}{2}\times\left[f(z).f(\bar{z})\right]^{\frac{P}{2}-1}\right]\times f(z).f(\bar{z})\Big]$

$= 4\times\dfrac{\partial}{\partial z}\left[\dfrac{P}{2}\times\left[f(z)\right]^{\frac{P-2}{2}}\left[f(\bar{z})\right]^{\frac{P-2}{2}}.\ f(z).f'(\bar{z})\right]$

$= 4\left[f(\bar{z})\right]^{\frac{P-2}{2}}\times f'(\bar{z})\cdot\dfrac{P}{2}\Big]\cdot\dfrac{\partial}{\partial z}\left[f(z)\right]^{\frac{P-2}{2}+1}$

$= \cancel{4}\left[\dfrac{P}{\cancel{2}}\cdot\left[f(\bar{z})\right]^{\frac{P-2}{2}}.\ f'(\bar{z})\right]\times\dfrac{P}{\cancel{2}}\left[f(z)\right]^{\frac{P}{2}-1}.\ f'(z).$

to prove

$= P^2.f'(z).f'(\bar{z}).\left[f(\bar{z})\right]^{\frac{P-2}{2}}.\left[f(z)\right]^{\frac{P-2}{2}}$

$= P^2.\ |f'(z)|^2\times\left[f(z).f(\bar{z})\right]^{\frac{P-2}{2}}$

$= P^2.\ |f'(z)|^2\times\left[\left(f(z).f'(\bar{z})\right)^{1/2}\right]^{P-2}$

$= P^2\ |f'(z)|^2.\ |f(z)|^{P-2}.$

$=$ RHS

(4) $\log|f(z)| = \log(f(z)\cdot f(\bar z))$

$= \frac{1}{2}\log(f(z)\cdot f(\bar z))$

$= \frac{1}{2}\left[\log(f(z) + \log(f(\bar z))\right]$

$= 4\frac{\partial^2}{\partial z\partial \bar z} \times \frac{1}{2}\left[\log(f(z) + \log(f(\bar z)\right].$

$= 2\frac{\partial}{\partial z}\left[0 + \frac{1}{f(\bar z)}\cdot f'(\bar z)\right].$

$= 2\frac{\partial}{\partial z}\left[\frac{1}{f(\bar z)}\cdot f'(\bar z)\right].$

$= 2\left[\,0\,\right]$

$= 0.$

$=$ R.H.S.

(5) $f(z) = u + iv$

$f(\bar z) = u - iv$

$f(z) + f(\bar z) = 2u \Rightarrow u = \frac{1}{2}\left[f(z) + f(\bar z)\right]$

$f(z) - f(\bar z) = 2iv \Rightarrow v = \frac{1}{2i}\left[f(z) - f(\bar z)\right]$

L.H.S $= \nabla^2 \cdot |Re(f(z)|^2$

$= 4\frac{\partial^2}{\partial z\partial \bar z}\left[\frac{1}{2}[f(z) + f(\bar z)]\right]^2$

$= 4 \times \frac{1}{4}\frac{\partial^2}{\partial z\partial \bar z}\left[[f(z)]^2 + [f(\bar z)]^2 + 2(f(z)\cdot f(\bar z)]\right]$

$= \frac{\partial^2}{\partial z\partial \bar z}\left[[f(z)]^2 + [f(\bar z)]^2 + 2[f(z)\cdot f(\bar z)]\right]$

$= \frac{\partial}{\partial z}\left[2\cdot[f(\bar z)] \times f'(\bar z) + 2f(z)\cdot f'(\bar z)\right].$

$= 0 + 2\cdot f'(z)\cdot f'(\bar z)$

$= 2\cdot|f'(z)|^2$

$=$ R.H.S

⑥ $L.H.S = \nabla^2 |Im f(z)|^2$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left[\frac{1}{2i} \left[f(z) - f(\bar{z}) \right] \right]^2$$

$$= \frac{-\cancel{4}}{\cancel{4}} \frac{\partial^2}{\partial z \partial \bar{z}} \left[\cancel{i^2} \left[f(z) - f(\bar{z}) \right]^2 \right]$$

$$= \frac{-\partial^2}{\partial z \partial \bar{z}} \left[[f(z)]^2 + [f(\bar{z})]^2 - 2 f(z) . f(\bar{z}) \right].$$

$$= -\frac{\partial}{\partial z} \left[0 + 2 f'(\bar{z}) . f(\bar{z}) - 2 . f(z) . f'(\bar{z}) \right].$$

$$= -\left[0 + 0 - 2 . f'(z) . f'(\bar{z}) \right].$$

$$= 2 . \left[f'(z) . f'(\bar{z}) \right]$$

$$= 2 . \; | f'(z) |^2.$$

⑦ $L.H.S = \nabla^2 . |u|^p$

$$= 4 \times \frac{\partial^2}{\partial z \partial \bar{z}} \left[\frac{1}{2} \times [f(z) + f(\bar{z})] \right]^p$$

$$= \frac{4}{2^p} \times \frac{\partial^2}{\partial z \partial \bar{z}} \left[f(z) + f(\bar{z}) \right]^p$$

$$= \frac{4}{2^p} . \frac{\partial}{\partial z} \left[p [f(z) + f(\bar{z})]^{p-1} \times [0 + f'(\bar{z})] \right]$$

$$= \frac{4p}{2^p} . \frac{\partial}{\partial z} \left[[f(z) + f(\bar{z})]^{p-1} \times f'(\bar{z}) \right]$$

$$= \frac{4p}{2^p} \times f'(\bar{z}) \left[(p-1) \left[f(z) + f(\bar{z}) \right]^{p-2} \times [f'(z) + 0] \right]$$

$$= \frac{(4p)(p-1)}{2^p} \times f'(\bar{z}) . f'(z) \left[f(z) + f(\bar{z}) \right]^{p-2}$$

$$= \frac{4p(p-1)}{2^p} . |f'(z)|^2 \left[2u \right]^{p-2}$$

$$= \frac{4p \times p-1}{\cancel{2^p}} \times |f'(z)|^2 \times \frac{\cancel{2^p} \times \cancel{2}}{2^2} \times u^{p-2}$$

$$= p(p-1) \times u^{p-2} . |f'(z)|^2$$

$= R.H.S$

(8) Prove that an Analytic function, with constant modulus is constant.

$f(z) = u + iv$ is analytic function

By C-R equation,

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \rightarrow (1)$$

$|f(z)| = \sqrt{u^2+v^2}$

$= c$ (constant)

$|f(z)| = (u^2+v^2) = c^2 \quad -(2)$

P.d. w.r.t "x"

$(2u)u_x + (2v)(v_x) = 0$

PD w.r.t 'y',

$(2u)(u_y) + (2v)(v_y) = 0$

$u u_x + v v_x = 0 \quad -(3)$

$u u_y + v v_y = 0 \quad -(4)$

$(3) \times u \Rightarrow u^2 u_x + uv v_x = 0.$

$(4) \times v \rightarrow uv u_y + v^2 v_y = 0$

$u^2 u_x + uv[\underbrace{v_x + u_y}_{\cancel{v_x} - \cancel{v_x}}] + v^2 v_y = 0$

$u^2 u_x + v^2 v_y = 0 \qquad [\because v_y = u_x]$

$u_x(u^2+v^2) = 0$

$\Downarrow$

$c^2 \neq 0.$

$\boxed{u_x = 0}$

Illy $u_y = 0 \Rightarrow v_x = 0.$

$\therefore f'(z) = u_x + i v_x$

$= 0 + i0 = 0$

$\therefore f(z)$ is constant

Note: orthogonality , $m_1 m_2 = -1$ $m = \text{slope} = \frac{dy}{dx}$

$\gamma m_1 m_2 = -1$.

(9) $w = u(x,y) + i v(x,y)$ is analytic then family of curves $u(x,y) = c_1$, $v(x,y) = c_2$ cuts orthogonally where c_1 and c_2 are constants.

$$w = u(x,y) + i v(x,y) \qquad \left.\begin{array}{l} u(x,y) = c_1 \\ v(x,y) = c_2 \end{array}\right\} \rightarrow (1)$$

$\downarrow$

By CR equation,

$$\left.\begin{array}{l} u_x = v_y \\ -v_x = u_y \end{array}\right\} \rightarrow (2)$$

$u(x,y) = c_1$

By total diff,

$$u_x \cdot \frac{dx}{dx} + u_y \cdot \frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{u_x}{u_y}.$$

$$\Rightarrow m_1 = \frac{-u_x}{u_y}.$$

$v(x,y) = c_2$.

By total diff w.r.t x,

$$v_x \cdot \frac{dx}{dx} + v_y \cdot \frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = \frac{-v_x}{v_y}.$$

$$\Rightarrow m_2 = \frac{-v_x}{v_y}$$

orthogonal condition,

$$m_1 \times m_2 = -1$$

$$\Rightarrow m_1 \times m_2 = -\frac{u_x}{u_y} \times \frac{-v_x}{v_y}$$

$$= -\frac{\cancel{v_y}}{\cancel{-v_x}} \times \frac{\cancel{v_x}}{\cancel{v_y}}$$

$$\boxed{m_1 m_2 = -1}$$

$\therefore$ curves are orthagonally cutting.

h. verify that following function real and imaginary part cut orthogonally each other.

(a) $f(z) = e^z$ (b) $f(z) = 1/z$ (c) $f(z) = z^2$ (d) $f(z) = z^3$.

(a) $f(z) = e^{z}$

$u+iv = e^{x+iy}$

$= e^{x}.e^{iy} = e^{x}[\cos y + i\sin y]$

$u = e^{x}\cos y$ $\qquad$ $v = e^{x}\sin y.$

$u_x = e^{x}\cos y.$ $\qquad$ $v_x = e^{x}\sin y$

$u_y = -e^{x}\sin y.$ $\qquad$ $v_y = e^{x}\cos y.$

$u_x = v_y$ and $u_y = -v_x.$

$\therefore$ CR equation is satisfied.

$$m_1 = \frac{-u_x}{u_y} = \frac{-e^{x}\cos y}{-e^{x}\sin y}$$

$$= \cot y.$$

$$m_2 = \frac{-v_x}{v_y} = -\frac{\cancel{e}^{x}\sin y}{\cancel{e}^{x}\cos y}$$

$$= -\tan y.$$

$$m_1 \times m_2 = \cot y \times -\tan y$$

$$= \frac{1}{\cancel{\tan y}} \times -\cancel{\tan y}$$

$\boxed{m_1 m_2 = -1}$ $\qquad$ $f(z)$ with u & v cut orthogonally.

(10). C

(b) $f(z) = \frac{1}{z}.$

$$u+iv = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$u = x/_{x^2+y^2}$, $\qquad v = -y/_{x^2+y^2}$

$$u_x = \frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2} \qquad v_x = -\left[\frac{(x^2+y^2)(0)-(y)(2x)}{(x^2+y^2)^2}\right]$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} \qquad = 2xy/_{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2} \qquad v_y = -\left[\frac{(x^2+y^2)(1)-(y)(2y)}{(x^2+y^2)^2}\right]$$

$$= \frac{-2xy}{(x^2+y^2)^2} \qquad = \frac{-[x^2-y^2]}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$u_x = v_y$ & $u_y = -v_x$.

$$m_1 = \frac{-u_x}{u_y} = \frac{y^2-x^2}{-2xy}.$$

$$m_2 = \frac{-v_x}{v_y} = \frac{2xy}{y^2-x^2}$$

$$m_1 \times m_2 = \frac{\cancel{y^2-x^2}}{-2xy} \times \frac{\cancel{2xy}}{\cancel{y^2-x^2}} = -1.$$

$\boxed{m_1 m_2 = -1}$ ∴ $f(z)$ with u & v are orthogonal.

Harmonic functions:-

A function which satisfies the laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \text{ (cartesian) and } \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \text{ (polar)}$$

10. Conjugate Harmonic:-

Any two functions u and v which are harmonic and $f(z) = u+iv$ be analytic. In this case we can say. v is a conjugate harmonic of u and u is a conjugate harmonic v.

The real and imaginary part of analytic function are harmonic functions. The real and imaginary part of analytic function $f(z) = u+iv$ satisfy the laplace equation.

$f(z) = w = u(x,y) + iv(x,y)$ is analytic function.

By CR equation,
$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}\right\} \to (1)$$

P.D w.r.t "x",

$$u_{xx} = \frac{\partial}{\partial x}(v_y) = v_{xy} \qquad u_{xy} = -v_{xx}$$

P.D w.r.t 'y':

$$u_{yx} = v_{yy} \qquad u_{yy} = -v_{yx}.$$

$$u_{xx}+u_{yy} = \cancel{v_{xy}} - \cancel{v_{yx}}$$
$$= 0. \qquad u \text{ is harmonic}$$

$$v_{xx}+v_{yy} = -\cancel{u_{xy}} + \cancel{u_{yx}}$$
$$= 0. \qquad v \text{ is harmonic.}$$

Pr: Prove that the function

$v = \sin x \cosh y + 2\cos x \sinh y + (x^2-y^2+4xy)$ is harmonic. Also find the analytic function $f(z)$ and conjugate harmonic.

$$v = \sin x \cosh y + 2\cos x \sinh y + (x^2-y^2+4xy)$$

$$v_x = (\cos x \cosh y) + 2\sinh y(-\sin x) + 2x - 0 + 4y(1)$$

$$v_{xx} = (-\sin x)\cosh y + 2\sinh y(-\cos x) + 2 - \cancel{0} + \cancel{0}$$

$$v_y = \sin x(\sinh y) + 2\cos x(\cosh y) + (0-2y+4x)$$

$$v_{yy} = \sin x(\cosh y) + 2\cos x(\sinh y) + \cancel{0} - 2 + \cancel{0}$$

$$v_{xx}+v_{yy} = 0+0+0$$
$$= 0.$$

$\therefore v$ is harmonic.

(b) to find $f(z)$:-

$v \to$ imaginary given.

$$f_1 = (v_y)_{(z,0)} = \sin z(\sinh 0) + 2\cos z(\cos 0) - 0 + 4z.$$
$$= 2\cos z + 4z.$$

$$f_2 = (v_x)_{(z,0)} = \cos z(\cosh(0)) + 2\sin 0(-\sin z) + 2z.$$
$$f_2 = \cos z + 2z.$$

$$f(z) = \int f_1 + i\int f_2 + c.$$
$$= \int(2\cos z + 4z)dz + i\int(\cos z + 2z) + c$$
$$= 2\int \cos z\, dz + 4\int z\, dz + i\left[\int \cos z\, dz + 2\int z\, dz\right] + c$$

= 2sinx

= 2sinx

$\therefore$ u =

v =

Pr: Show th

u_x =

u_{xx} =

u_y =

u_{yy} =

u_{xx} +

$= 2[\sin z] + 2z^2 + i[\sin z + z^2] + c$

$= 2[\sin(x+iy)] + 2(x+iy)^2 + i[\sin(x+iy) + (x+iy)^2] + c$

$= 2[\sin x \cos iy + \cos x \sin iy] + 2(x^2 - y^2 + 2ixy)$
$\quad + i[\sin x \cos iy + \cos x \sin iy + x^2 - y^2 + 2ixy] + c.$

$= 2[\sin x \cosh y + i \cos x \sinh y] + 2[x^2 - y^2 + 2ixy]$
$\quad + i[\sin x \cosh y + i \cos x \sinh y + x^2 - y^2 + 2ixy] + c$

$= 2\sin x \cosh y + 2x^2 - 2y^2 - \cos x \sinh y - 2xy$

$= 2\sin x \cosh y - \cos x \sinh y + 2(x^2 - y^2 - xy) + i[2\cos x \sinh y + \sin x \cosh y + x^2 - y^2 + 4ixy].$

$\therefore \quad u = 2\sin x \cosh y - \cos x \sinh y + 2(x^2 - y^2 - xy)$

$V = 2\cos x \sinh y + \sin x \cosh y + x^2 - y^2 + 4xy.$

Pr: Show that $x = \frac{1}{2}\log(x^2+y^2)$, show that harmonic and find conjugate harmonic.

$x = \frac{1}{2}\log(x^2+y^2).$

$u_x = \frac{1}{\cancel{2}} \frac{1}{(x^2+y^2)}(2x) = \frac{x}{x^2+y^2}.$

$u_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}.$

$u_y = \frac{1}{\cancel{2}} \cdot \frac{1}{(x^2+y^2)}(2y)$

$u_{yy} = \frac{(x^2+y^2)(1) - (y)(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$u_{xx} + u_{yy} = \frac{\cancel{(y^2-x^2)}}{(x^2+y^2)^2} + \frac{\cancel{(x^2-y^2)}}{(x^2+y^2)^2} = 0.$

$\therefore$ u is harmonic.

$f'(z) = u_x + iv_x$

$= u_x - iu_y$

$= \frac{x}{x^2+y^2} - i\left[\frac{y}{x^2+y^2}\right]$

$= \frac{(x-iy)}{(x^2+y^2)^2}$

$= \frac{\bar{z}}{(z\bar{z})} = \frac{1}{z}$

Homework:

(1) $u = e^x \cos y$

$u_x = e^x \cos y$

$u_{xx} = e^x \cos y$

$u_y = e^x \cdot (-\sin y)$

$u_{yy} = e^x(-\cos y)$

$u_{xx} + u_{yy} = e^x \cancel{\cos y} - e^x \cancel{\cos y}$

$= 0.$

$\therefore$ u is harmonic

~~$f'(z)$:~~

To find $f(z)$:-

$u \rightarrow$ real part given

$f_1 = (u_x)_{(z,0)} = (e^z \cdot \cos(0)) = e^z$

$f_2 = u_y(z,0) = (e^z(-\sin(0))) = 0.$

$f(z) = \int f_1 - i\int f_2 + c$

$= \int e^z + c$

$f(z) = e^z + c$

$u + iv = e^{x+iy} = e^x[\cos y + i\sin y]$

$u + iv = e^x \cos y + ie^x \sin y$; $\quad u = e^x \cos y$

$v = e^x \sin y$

Scanned by CamScanner

$u_x = -\cos x \sinh y$

$u_{xx} = \sin x \sinh y$

$u_y = -\sin x \cosh y$

$u_{yy} = -\sin x \sinh y.$

$u_{xx} + u_{yy} = -\cancel{\sin x \sinh y} + \cancel{\sin x \sinh y}.$

$= 0.$

$f_1 = (u_x)_{(z,0)} = -\cos z \sinh(0)$

$= 0.$

$f_2 = (u_y)_{(z,0)} = -\sin z \cosh(0)$

$= -\sin z.$

$f(z) = \int f_1 - i \int f_2 + c = 0 - i \int -\sin z + c$

$= i \int \sin z + c$

$= i[\cos z] + c.$

$u+iv. = i[-\cos[x+iy]] + c$

$u+iv = i[-[\cos x \cos iy - \sin x \sin iy]]$

$= i[-[\cos x \cosh y - i \sin x \sinh y]]$

$= -\sin x \sinh y - i \cos x \cosh y.$

$u = -\sin x \sinh y$

$v = -\cos x \cosh y.$

(3) $u = xy(x^2-y^2) = x^3y - xy^3$

$u_x = 3x^2y - y^3$

$u_{xx} = 6xy$

$u_y = x^3 - x3y^2$

$u_{yy} = -6xy$

$u_{xx} + u_{yy} = \cancel{6xy} - \cancel{6xy} = 0.$

$f_1 = (u_x)_{(z,0)} = 3(z^2)(0) - 0 = 0.$

$f_2 = (u_y)_{(z,0)} = (z^3) = z^3$

$f(z) = \int f_1 - i \int f_2 + c = \int 0 - i \int z^3 + c$

$= -i \int z^3 + c = -i\left[\frac{z^4}{4}\right]$

$u+iv = -i\left[\frac{(x+iy)^4}{4}\right]$

$= \frac{1}{4}\left[-i\right.$

Polar form of result (9):-

$w = u(r,\theta) + i\,v(r,\theta)$ be an analytic function then the family of curves $u(r,\theta) = c_1$ and $v(r,\theta) = c_2$ cuts orthogonally.

By CR equation in polar form.

$$\left.\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ v_r &= -\frac{1}{r} u_\theta \end{aligned}\right\} \rightarrow (1)$$

$u(r,\theta) = c_1$

By total diff w.r.t r,

$$u_r \cdot \cancel{\frac{dr}{dr}} + u_\theta \cdot \frac{d\theta}{dr} = 0.$$

$$\boxed{m_1 = \frac{d\theta}{dr} = -\frac{u_r}{u_\theta}}$$

$v(r,\theta) = c_2$

By total diff w.r.t r,

$$v_r \cdot \cancel{\frac{dr}{dr}} + v_\theta \cdot \frac{d\theta}{dr} = 0$$

$$\boxed{m_2 = \frac{d\theta}{dr} = \frac{-v_r}{v_\theta}}$$

$$\therefore\ r^2 m_1 \times m_2 = r^2 \times \frac{-u_r}{u_\theta} \times \frac{v_r}{v_\theta}$$

$$= r^2 \times \frac{v_\theta / r}{u_\theta} \times \frac{u_\theta / r}{v_\theta}$$

$$= \cancel{r^2} \times \frac{\cancel{v_\theta}}{\cancel{r}\cancel{u_\theta}} \times \frac{-\cancel{u_\theta}}{\cancel{r}\cancel{v_\theta}}$$

$$\boxed{r^2 m_1 m_2 = -1}$$

Polar form of result (10):-

$w = u(r,\theta) + i v(r,\theta)$ satisfies the Laplace equation in polar form.

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \Rightarrow f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0 \quad \to (1)$$

By CR equation,

$$\left.\begin{aligned} u_r &= \frac{1}{r} v_\theta \Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ v_r &= -\frac{1}{r} u_\theta \Rightarrow \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}\right\} \to (2)$$

P.d. w.r.t "r"

$$u_{rr} = \frac{1}{r} \times \frac{\partial}{\partial r}(v_\theta) + v_\theta\left(\frac{-1}{r^2}\right)$$

$$\boxed{u_{rr} = \frac{1}{r} \cdot v_{r\theta} - \frac{v_\theta}{r^2}} \to (3)$$

P.d. w.r.t "θ"

$$\frac{\partial}{\partial \theta}(v_r) = -\frac{1}{r} \cdot \frac{\partial}{\partial \theta}(u_\theta)$$

$$\boxed{v_{\theta r} = -\frac{1}{r} \cdot u_{\theta\theta}} \to (4)$$

$$\therefore\ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \left[\frac{v_{r\theta}}{r} - \cancel{\frac{v_\theta}{r^2}}\right] + \cancel{\frac{1}{r}\left[\frac{1}{r} v_\theta\right]} + \frac{1}{r^2}\left[-\cancel{r} v_{\theta r}\right]$$

$$= \left[\cancel{\frac{v_{r\theta}}{r}} - \cancel{\frac{v_{\theta r}}{r}}\right]$$

$$= 0.$$

1. * An analytic function whose real part is constant must be constant itself.

$$f(z) = u(x,y) + i\,v(x,y)$$

$$u(x,y) = c_1$$

$$u_x = 0$$

$$u_y = 0 \Rightarrow -v_x = 0$$

$$v_x = 0.$$

$$f'(z) = u_x + i\,v_x$$

$$= 0 + f(0)$$

$$= 0$$

$$f'(z) = 0 \Rightarrow f(z) = \text{constant}$$

$$\boxed{\therefore f(z) = c}$$

2. * An analytic function whose imaginary part is constant must be constant itself

$$f(z) = u(x,y) + i\,v(x,y)$$

$$v(x,y) = c_2$$

$$v_x = 0 \qquad\qquad u_x = v_y$$

$$v_y = 0 \Rightarrow u_x = 0.$$

$$f'(z) = u_x + i\,v_y$$

$$= 0 + i(0)$$

$$= 0$$

$$f'(z) = 0 \Rightarrow f(z) = \text{constant}$$

$$\boxed{\therefore f(z) = c}$$

3. * If $f(z)$ and $\overline{f(\bar{z})}$ are analytic in a region D then show that $f(z)$ is constant in the same region D.

$$f(z) = u(x,y) + i\,v(x,y) \rightarrow \text{analytic in region D}$$

$$\overline{f(z)} = u(x,y) - i\,v(x,y) \rightarrow \quad " \quad " \quad " \quad "$$

By CR equation, $u_x = v_y$ and $u_y = -v_x$ → (1)

By CR equation,

$u_x = -v_y$ and $u_y = v_x$ → (2)

$2u_x = 0$; $2u_y = 0$.

$u_x = 0$; $u_y = 0$.

$v_x = 0$.

$\therefore f'(z) = u_x + i v_x$

$= 0 + i(0)$

$f'(z) = 0$

$$\boxed{\therefore f(z) = \text{constant}}$$

Applications:-

Fluids → (ability to have flow)

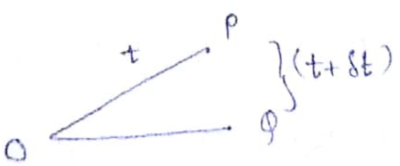
compressible (change in volume when pressure is applied)

Ex: ~~water~~ gases

incompressible (no change in volume when pressure is applied).

Ex: ~~gases~~ water, petrol

Pr: Consider a fluid particle at time t, in motion. Let the position be P at the small interval of time δt, position of the particle be Q.



$\vec{OP} = \vec{r}$

$= x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{V} = \frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t}$$

$$\vec{V} = \frac{d}{dt}(x\vec{i} + y\vec{j} + z\vec{k}) \quad \boxed{\vec{V} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}}$$

* Consider a two dimensional irrotational motion of a incompressible fluid.

1. I

Let $f(z) = u+iv$ be a analytic function

consider motion parallel to xy plane

By total differentiation,

$$\vec{V} = V_x \vec{i} + V_y \vec{j} \quad -(1)$$

irrotational $\rightarrow$ $\text{curl } \vec{V} = \vec{0}$

$$\nabla \times \vec{V} = \vec{0}$$

$$\boxed{\vec{V} = \nabla\phi} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} \quad -(2)$$

(1) & (2)

$$V_x = \frac{\partial\phi}{\partial x} \; ; \quad V_y = \frac{\partial\phi}{\partial y}$$

Incompressible, $\text{div } \vec{V} = 0$

$$\nabla \cdot \vec{V} = 0$$

$$\left(\vec{i}\cdot\frac{\partial}{\partial x} + \vec{j}\cdot\frac{\partial}{\partial y}\right)\left(V_x \cdot \vec{i} + V_y \vec{j}\right) = 0$$

$$\frac{\partial}{\partial x}(V_x) + \frac{\partial}{\partial y}(V_y) = 0$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$\therefore \phi$ is a harmonic function

2.

$\therefore \phi$ is the vector potential.

There exist a conjugate harmonic ψ such that

$f(z) = \phi + i\psi$ is analytic

here ϕ is called as potential function / vector potential

ψ is called as scalar function / scalar "
stream function

$f(z) \rightarrow$ complex potential function.

1. In a two-dimensional flow the stream function is given by $\tan^{-1}\left(\frac{y}{x}\right)$ find the velocity potential, also complex velocity potential

given $\psi = \tan^{-1}\left[\frac{y}{x}\right]$

To find: velocity potential ϕ

$$f(z) = \underset{Re}{\phi} + i\underset{Im}{\psi}$$

Take $\phi = u$, $\psi = v$

$$f_1 = (v_y)_{(z,0)} = \frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)} \times \frac{1}{x} = \frac{x^2}{x^2+y^2} \times \cancel{\frac{1}{x}}$$

$$\cancel{\frac{1}{x}+y^2} = \frac{x}{x^2+y^2}$$

$$= \frac{z}{z^2} = \frac{1}{z}.$$

$$f_2 = (v_x)_{(z,0)} = \frac{1}{1+\left[\frac{y}{x}\right]^2} \times \frac{-y}{x^2} = \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2} = -\frac{y}{x^2+y^2}$$

$$= 0.$$

By, M.T method,

$$f(z) = \int f_1 + i\int f_2 + c$$

$$= \int \frac{1}{z}\,dz + c. \quad = \log z + c.$$

$$f(z) = \log z + c.$$

$$u+iv = \log(x+iy) + c.$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) + c.$$

$\boxed{u = \phi = \frac{1}{2}\log(x^2+y^2)}$ → velocity potential.

2. An incompressible fluid flowing over the xy plane have the velocity potential $x^2-y^2+\frac{x}{x^2+y^2}$. Examine if this is possible and hence find the stream function

$$\phi = x^2-y^2+\frac{x}{x^2+y^2}\,; \quad f(z) = \phi + i\psi$$
$$= u + iv.$$

$$f_1 = (u_x)_{(z,0)} = 2x - 0 + \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}$$

$$= 2x + \frac{y^2-x^2}{(x^2+y^2)^2} \cdot \quad 2z + \frac{0-z^2}{z^4}$$

$$= 2z - 1/z^2.$$

$f_2 = (U_y)_{(z,0)} = 0 - 2y + \frac{(x^2+y^2)(0) - (x)(2y)}{(x^2+y^2)^2}$

$= -2y + (0)$

$f_2 = 0$

$f(z) = \int f_1 - i \int f_2 + c$

$= \int \left(2z - \frac{1}{z^2}\right) dz + c$

$= \int 2z\,dz - \int \frac{1}{z^2}\,dz + c$

$\boxed{f(z) = z^2 + \frac{1}{z} + c}$ → complex velocity potential

$u + iv = (x+iy)^2 + \frac{1}{x+iy} + c.$

$= x^2 - y^2 + 2xiy + \frac{x-iy}{x^2+y^2} + c.$

$u + iv = x^2 - y^2 + \frac{x}{x^2+y^2} + i\left[2xy - \frac{y}{x^2+y^2}\right] + c.$

$\boxed{\therefore \psi = v = 2xy - \frac{y}{x^2+y^2}}$ → stream function

Stream lines and equipotentials:-

* If $u = \phi(x,y) = c_1$ -① and $v = \psi(x,y) = c_2$ -② where c_1 and c_2 are constants then $f(z) = \phi(x,y) + i\psi(x,y)$ be an analytic function and there exist a orthogonal trajectories between ① and ② cuts orthogonally. In this case ① is called as equipotential and ② is called as stream lines (or) lines of forces.

Pr. Find the orthogonal trajectories of $x^4 + y^4 - 6x^2y^2 = c$

$u(x,y) = x^4 + y^4 - 6x^2y^2 = \text{constant}$

$f_1 = (u_x)_{(z,0)} = 4x^3 + 0 - 12xy^2 \Big|_{(z,0)}$

$= 4z^3.$

$f_2 = (u_y)_{(z,0)} = 0 + 4y^3 - 12x^2y \Big|_{(z,0)}$

$= 0 + 0 - 0 = 0.$

$\therefore f(z) = \int f_1 + -i \int f_2 + c$

$= \int 4 z^3 dz + c \quad = \cancel{4} \times \frac{z^4}{\cancel{4}} = z^4 + c.$

$u + iv = (x+iy)^4 + c.$

$(x+a)^n = x^n + {}^nC_1 \cdot x^{n-1} \cdot a^1 + {}^nC_2 \cdot x^{n-2} \cdot a^2 + \dots + a^n$

$= x^4 + 4C_1 \cdot x^3 \cdot (iy) + 4C_2 \cdot x^2 (iy)^2 + 4C_3 \cdot x (iy)^3 + (iy)^4 + c$

$= x^4 + 4 \cdot (x^3)(iy) + 6x^2(-y^2) + 4 \cdot x(-iy^3) + y^4$

$= (x^4 + y^4 - 6x^2y^2) + i\,(4x^3y - 4xy^3)$

$\therefore V(x,y) = 4x^3y - 4xy^3.$

$V_x = 12x^2y - 4y^3.$

$u_y = -V_x = 12x^2y - 4y^3.$

$12x^2\cancel{y} - 4\cancel{y^3} = 12x^2\cancel{y} - 4\cancel{y^3}.$

$V_x = 0.$

$lly \quad V_y = 0.$

$V(x,y)$ must be a constant function.

Pb. Show that the family of concentric circles $x^2+y^2=c$ and family of straigh lines $y = k.x$ cuts orthogonally.

orthogonal $u(x,y) = c \qquad V(x,y) = \frac{y}{x} = k. \Rightarrow y = kx.$

$x^2 + y^2 = c \quad -①$ $\qquad \frac{y}{x} = k \rightarrow ②$

$2x + 2y \frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = k.$

$\cancel{2}y \frac{dy}{dx} = -\cancel{2}x \qquad \boxed{m_2 = k}$

$m_1 \quad \frac{dy}{dx} = -\frac{x}{y}$

$m_1 m_2 = -1$

$= -\frac{x}{y} \times k$

$= -\cancel{y}/\cancel{y}$

$\boxed{m_1 m_2 = -1}$

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Pr show that $r^n = a\sec n\theta$, $r^n = b\operatorname{cosec} n\theta$ cuts orthog[onally]

$$a = \frac{r^n}{\sec n\theta} \qquad b = \frac{r^n}{\operatorname{cosec} n\theta}$$

$$u(r,\theta) = a = r^n \cos n\theta \qquad b = r^n \sin n\theta = V(r,\theta)$$

$$f(z) = u + iV$$

$$= r^n \cos n\theta + i r^n \sin n\theta$$

$$= r^n [\cos n\theta + i \sin n\theta]$$

$$= r^n [\cos\theta + i\sin\theta]^n$$

$$= r^n [e^{i\theta}]^n$$

$$= [r \cdot e^{i\theta}]^n$$

$$\boxed{f(z) = z^n}$$

is always a analytic function and curves cut orthogonally.

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