FOURIER TRANSFORMS

1. State Fourier Integral Theorem

If $f(x)$ is piecewise continuously differentiable and absolutely integrable in $(-\infty, \infty)$ then

then $f(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$

(or) $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{is(x-t)}\,dt\,d\lambda$ **Fourier Integral Formula**

2. Write The Fourier transform pair

Fourier Transform (Complex Form) $F\left[f(x)\right] = F(s) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$

Inverse theorem for complex Fourier Transform / Inverse Fourier Transform:

$$F^{-1}\left[F(f(x))\right] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds$$

are both called Fourier Transform pair.

3. Write The Fourier cosine transform pair

Fourier cosineTransform : $F_c\left[f(x)\right] = F_c(s) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\cos sx\,dx$

Inverse Fourier cosine Transform: $F_c^{-1}\left[F_c\{f(x)\}\right] = f(x) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} F_c(s)\cos sx\,ds$

are both called Fourier cosine Transform pair

4. Write The Fourier sine transform pair

Fourier sineTransform : $F_s\left[f(x)\right] = F_s(s) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\sin sx\,dx$

Inverse Fourier sine Transform: $F_s^{-1}\left[F_s\{f(x)\}\right] = f(x) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} F_s(s)\sin sx\,ds$

are both called Fourier sine Transform pair

5. Find the Fourier transform of $e^{-\alpha|x|}, \alpha \geq 0$

$$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$F\left[e^{-\alpha|x|}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}2\int_0^{\infty} e^{-\alpha x}(\cos sx)\,dx \qquad \left[\because e^{-\alpha|x|}\cos sx \text{ is an even function}\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\alpha x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\alpha}{\alpha^2 + s^2} \right]$$

6. Find the Fourier cosine transform of $e^{-ax}, x \geq 0$

w.k.t., $$F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \qquad \because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$$

7. Find the Fourier sine transform of $e^{-ax}, a > 0$

w.k.t., $$F_s\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \qquad \because \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$$

8. Find Fourier cosine transform of xe^{-ax}

By using property,

$$F_c\left[x\, f(x)\right] = \frac{d}{ds}\left\{F_s\left[f(x)\right]\right\}$$

$$F_c\left[xe^{-ax}\right] = \frac{d}{ds}\left\{F_s\left[e^{-ax}\right]\right\}$$

w.k.t., $$F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]$$ **[by using Q.no 7]**

$$F_c\left[xe^{-ax}\right] = \frac{d}{ds}\left\{ \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\left(a^2 + s^2\right) - s.\, 2s}{\left(a^2 + s^2\right)^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a^2 - s^2}{\left(a^2 + s^2\right)^2} \right]$$

9. Find the Fourier sine transform of $\frac{1}{x}$

w.k.t., $$F_s\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\ dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}\ dx$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}. \qquad \because \int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}, \quad a>0$$

10. Find the Fourier transform of $f(x) = \begin{cases} e^{ikx} & ,a<x<b \\ 0 & ,x<a \text{ and } x>b \end{cases}$

$$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b e^{ikx}\ e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b e^{i(k+s)x}\ dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(k+s)x}}{i(k+s)}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{i(k+s)}\left[e^{i(k+s)b} - e^{i(k+s)a}\right]$$

$$= \frac{i}{\sqrt{2\pi}(k+s)}\left[e^{i(k+s)a} - e^{i(k+s)b}\right]$$

11. Define Self- reciprocal with respect to Fourier Transform

If a transformation of a function *f(x)* is equal to *f(s)* then the function *f(x)* is called self reciprocal.

ie., *F[f(x)] = f(s)* => *F[s] = f(s)*

Ex: $f(x) = e^{\frac{x^2}{2}}$

12. State Parseval's Identity or Plancherel's theorem or Rayleigh's theorem of Fourier Transform

If F(s) is the Fourier transform of *f(x)* , then

$$\int_{-\infty}^{\infty} \left|f(x)\right|^2 dx = \int_{-\infty}^{\infty} \left|F(s)\right|^2 ds$$

13. State convolution Theorem or Faltung theorem in Fourier transform

The Fourier Transform of the convolution of *f(x)* and *g(x)* is the product of their Fourier transforms

i.e., $F\left[f(x)*g(x)\right] = F\left[f(x)\right]F\left[g(x)\right]$

14. Change of scale property

Statement: If F{ *f(x)*} = F(s), For any non zero real 'a' then $F\left[f(ax)\right]=\frac{1}{a}F\left[\frac{s}{a}\right],\ a>0$

Proof: $$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F\left[f(ax)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

Put $ax = y$ $a\,dx = dy$ $dx = dy/a$	$x=-\infty \Rightarrow y=-\infty,$ $x=\infty \Rightarrow y=\infty,$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\, e^{is\frac{y}{a}}\, \frac{dy}{a}$$

$$= \frac{1}{a}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\, e^{i\left(\frac{s}{a}\right)y}\, dy$$

$$F\left[f(ax)\right]=\frac{1}{a}\ F\left(\frac{s}{a}\right)\ ,\ a>0$$

15. Shifting Property:

(i) If F(s) is the Fourier transform of *f(x)*, show that F[*f(x-a)*] = e^{ias} F(s)

Proof: $$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F\left[f(x-a)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

Put x – a = y dx = dy	$x=-\infty \Rightarrow y=-\infty,$ $x=\infty \Rightarrow y=\infty,$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\, e^{is\,(y+a)}\, dy$$

$$= \frac{e^{ias}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\, e^{isy}\, dy$$

$$= e^{ias}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\, e^{isy}\, dy$$

$$F\left[f(x-a)\right] = e^{ias}\ F(s)$$

(ii) If F(s) is the Fourier transform of *f(x)*, show that F[e^{iax} *f(x)*] = F(s+*a*)

Proof:

$$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F\left[e^{iax} f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}\, f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$$

$$F\left[e^{iax} f(x)\right] = F(s+a)$$

16. If F_c (s) is the Fourier cosine transform of f(x), prove that

$$F_c\left[f(x)\cos ax\right] = \frac{1}{2}\left[F_c\left[s+a\right] + F_c\left[s-a\right]\right]$$

Proof:

$$F_c\left[f(ax)\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(ax)\cos sx\, dx$$

$$F_c\left[f(x)\cos ax\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\cos sx\, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left\{\cos(s+a)x + \cos(s-a)x\right\} dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(s+a)x\, dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(s-a)x\, dx$$

$$F_c\left[f(x)\cos ax\right] = \frac{1}{2}\left[F_c\left[s+a\right] + F_c\left[s-a\right]\right]$$

17. If F_c (s) is the Fourier cosine transform of f(x), prove that the Fourier cosine transform of *f(ax)* is $\frac{1}{a} F_c\left[\frac{s}{a}\right]$

$$F_c\left[f(ax)\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(ax)\cos sx\, dx$$

Put $ax = y$ $a\,dx = dy$ $dx = dy/a$	$x=0 \Rightarrow y=0,$ $x=\infty \Rightarrow y=\infty$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(y)\cos\left(\frac{sy}{a}\right)\frac{dy}{a}$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(y)\cos\left(\frac{s}{a}\right)y\, dy$$

$$F_c\left[f(ax)\right] = \frac{1}{a}F_c\left(\frac{s}{a}\right)$$

18. Show that f(x) = 1 , $0<x<\infty$ cannot be represented by a fourier integral.

$$\int_0^\infty |f(x)|\,dx=\int_0^\infty 1\,dx=[x]_0^\infty=\infty$$

and this value tends to ∞ as x tends to ∞. i.e., $\int_0^\infty 1\, f(x)\,dx$ is not convergent.

19. Find the Fourier cosine transform of $f(x)=\begin{cases}\cos x, 0<x<a\\ 0\ ,\ x\ge a\end{cases}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c[f(x)\cos x] = \sqrt{\frac{2}{\pi}}\int_0^a f(x)\cos x\cos sx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^a f(x)\{\cos(s+1)x+\cos(s-1)x\}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1}+\frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1}+\frac{\sin(s-1)a}{s-1}\right]$$, provided $s\neq 1, s\neq -1$

20. Given that $e^{-x^2/2}$ is Self-reciprocal under Fourier cosine transform

(i) find Fourier sine transform of $xe^{-x^2/2}$

(ii) find Fourier cosine transform of $x^2e^{-x^2/2}$.

$$F_c\left[e^{-x^2/2}\right]=e^{-s^2/2}$$

$$F_s\left[xe^{-x^2/2}\right]=-\frac{d}{ds}F_c\left[xe^{-x^2/2}\right]$$

$$=-\frac{d}{ds}\left[e^{-s^2/2}\right]=-e^{-s^2/2}[-s]=se^{-s^2/2}$$

Given, $F_c\left[x^2e^{-x^2/2}\right]=\frac{d}{ds}F_s\left[xe^{-x^2/2}\right]$

$$=\frac{d}{ds}\left[se^{-s^2/2}\right]=\left[se^{-s^2/2}(-s)+e^{-s^2/2}\right]$$

$$=-s^2e^{-s^2/2}+e^{-s^2/2}=(1-s^2)e^{-s^2/2}.$$