K.RAMAKRISHNAN COLLEGE OF ENGINEERING , SAMAYAPURAM
MATHEMATICS - II

UNIT-4- Analytic Functions **Two marks**

1. **Verify $f(z)=z^3$ is analytic or not.**

$$Let \quad f(z) = z^3$$
$$= (x+iy)^3$$
$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$u = x^3 - 3xy^2$	$v = 3x^2y - y^3$
$u_x = 3x^2 - 3y^2$ $u_y = -6xy$	$v_x = 6xy$ $v_y = 3x^2 - 3y^2$

Here $u_x = v_y \quad and \quad u_y = -v_x$

$\therefore f(z)$ satisfies $C-R$ equations.

$\Rightarrow f(z)$ is analytic

2. **Verify whether $f(z)=\bar{z}$ is analytic function or not.**
   (or) Show that the function $f(z)=\bar{z}$ is nowhere differentiable.

$$Let \quad f(z) = \bar{z}$$
$$= x - iy$$
$$= u + iv$$

$u = x$	$v = -y$
$u_x = 1$ $u_y = 0$	$v_x = 0$ $v_y = -1$

Here $u_x \neq v_y$

$\therefore f(z)$ is not satisfied $C-R$ equations.

$\Rightarrow f(z)$ is not analytic

$\Rightarrow \bar{z}$ is nowhere differentiable.

3. **Test the analyticity of the function $f(z)=z\bar{z}$**

$$Let \quad f(z) = z\bar{z}$$
$$= (x+iy)(x-iy)$$
$$= x^2 + y^2$$
$$= (x^2 + y^2) + i0$$
$$= u + iv$$

$u = x^2 + y^2$	$v = 0$
$u_x = 2x$ $u_y = 2y$	$v_x = 0$ $v_y = 0$

Here $u_x \neq v_y \quad and \quad u_y \neq -v_x$

$\therefore f(z)$ is not satisfied $C-R$ equations.

$\Rightarrow z\bar{z}$ is not analytic except $(0,0)$

4. **Show that an analytic function with constant imaginary part is constant.**

Let $w = f(z) = u + iv$ be an analytic function.

Given Imaginary part is constant. $\Rightarrow v = c$ $\quad \Rightarrow v_x = 0$

$v_y = 0$

Since $f(z)$ be an analytic function it satisfies C-R equations

$$\Rightarrow u_x = v_y \ \ and \ \ u_y = -v_x$$

$$u_x = 0 \quad and \quad u_y = 0$$

v is an independent of x and y

$\Rightarrow f(z)$ is constant

5. Are $|z|$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ analytic? Give reason.

No. $|z|$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ are not analytic.

Case (i)

$f(z) = |z|$

$= \sqrt{x^2 + y^2}$

$Here\ u = \sqrt{x^2 + y^2}\ ,\ v = 0$

$\Rightarrow u_x \neq v_y \ \ and \ \ u_y \neq -v_x$

Case (ii)

$f(z) = \mathrm{Re}(z)$

$= x$

$Here\ u = x\ ,\ v = 0$

$\Rightarrow u_x \neq v_y \ \ and \ \ u_y = -v_x$

Case (iii)

$f(z) = \mathrm{Im}(z)$

$= y$

$Here\ u = y\ ,\ v = 0$

$\Rightarrow u_x = v_y \ \ and \ \ u_y \neq -v_x$

$\Rightarrow$ In all cases $v = 0$

C-R equations are not satisfied in all these three cases.

$\Rightarrow$ Given functions are not analytic

6. Show that $u = 2x - x^3 + 3xy^2$ is harmonic

$Given\ \ u = 2x - x^3 + 3xy^2$

$u_x = 2 - 3x^2 + 3y^2$	$u_y = 6xy$
$u_{xx} = -6x$	$u_{yy} = 6x$

$$u_{xx} + u_{yy} = -6x + 6x = 0$$

$\Rightarrow u$ satisfies Laplace equation.

$\Rightarrow u$ is harmonic.

7. Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

$Given \quad u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$	$u_y = -6xy - 6y$
$u_{xx} = 6x + 6$	$u_{yy} = -6x - 6$

$$u_{xx} + u_{yy} = 6x + 6 - 6x - 6$$
$$= 0$$

$\Rightarrow u$ satisfies Laplace equation.

$\Rightarrow u$ is harmonic.

8. Find the map of the circle $|z| = 3$ under the transformation $w = 2z$

$Given \quad w = 2z$

$= 2(x + iy)$

$= u + iv$

$\Rightarrow u = 2x \qquad v = 2y$

$x = \frac{u}{2} \qquad y = \frac{v}{2}$

$Given \quad |z| = 3$

$\Rightarrow |x + iy| = 3$

$\sqrt{x^2 + y^2} = 3$

$x^2 + y^2 = 9$

$\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = 9$

$u^2 + v^2 = 36$

The image of $|z| = 3$ in the z- plane is transformed into $u^2 + v^2 = 36$ in the w-plane.

9. Find the image of the line *x = k* under the transformation $w = \frac{1}{z}$

$Given \quad z = \frac{1}{w}$

$\Rightarrow \quad = \frac{1}{u + iv}$

$= \frac{1}{u + iv} \times \frac{u - iv}{u - iv}$

$= \frac{u - iv}{u^2 + v^2}$

$= \frac{u}{u^2 + v^2} + i\frac{-v}{u^2 + v^2}$

$= x + iy$

$\Rightarrow x = \frac{u}{u^2 + v^2}, \qquad y = \frac{-v}{u^2 + v^2}$

$Given \quad x = k$

$\frac{u}{u^2 + v^2} = k$

$u = k(u^2 + v^2)$

$u^2 + v^2 - \frac{u}{k} = 0$

$Comparing\ with\ eqn\ of\ circle,$

Which is a circle in w-plane whose centre $\left(\frac{1}{2k}, 0\right)$ and radius

$\frac{1}{2k}$

10. State the basic difference between the limit of a function of a real variable and that of a complex variable.

Real Variable	Complex Variable
Limit takes along x axis and y axis or parallel to both axis	Limit takes along any path (straight or curved)

11. State the Cauchy-Riemann equation in polar coordinates satisfied by an analytic function.

$$Given \quad z = re^{i\theta}$$

$$w = f(z) = f(re^{i\theta}) \text{ to be analytic are,}$$

$$\Rightarrow \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

12. Define Conformal.

A transformation that preserves angles between every pair of curves through a point, both in magnitude and direction is called ***conformal*** at that point

ie., A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if $f'(z_0) \neq 0$

A mapping $w = f(z)$ is not conformal at $z = z_0$ if $f'(z_0) = 0$ is called ***Critical point*** of mapping

13. Define Fixed point or invariant point .

Under the transformation $w = f(z)$ is the image of z is itself, then the point is called a ***fixed point*** of the transformation.

ie., The ***fixed point or invariant points*** of the bilinear transformation $w = f(z) = \frac{az + b}{cz + d}$ is obtained from $w = z = f(z)$

Ex: 1. The identity mapping $w = z$ has every point as a fixed point.

2. The mapping $w = \bar{z}$ has infinitely many fixed points.

3. The mapping $w = \frac{1}{z}$ has two fixed points.

14. Find the fixed points of mapping $w = \frac{6z-9}{z}$

The invariant (or) fixed point are given by $w = z$

$$z = \frac{6z-9}{z}$$

$$\Rightarrow \quad z^2 = 6z - 9$$

$$\Rightarrow z^2 - 6z + 9 = 0$$

$$\Rightarrow (z-3)(z-3) = 0$$

$\therefore z = 3, 3$ are fixed points.

15. Find the invariant points of the transformation $w = \frac{2z+6}{z+7}$

The invariant (or) fixed point are given by $w = z$

$$w = \frac{2z+6}{z+7}$$

$$z = \frac{2z+6}{z+7}$$

$$\Rightarrow \quad z^2 + 7z = 2z + 6$$

$$\Rightarrow z^2 + 5z - 6 = 0$$

$$\Rightarrow (z+6)(z-1) = 0$$

$\therefore z = 1, -6$ are fixed points.

16. Find the invariant points of $f(z) = z^2$

The invariant (or) fixed point are given by $w = z$

$$w = z^2$$

$$\Rightarrow \quad z = z^2$$

$$\Rightarrow \quad z^2 - z = 0$$

$$\Rightarrow z(z-1) = 0$$

$\therefore z = 0, 1$ are invariant points.

17. Find the critical points of the transformation $w = 1 + \frac{2}{z}$

$$Given \quad w = 1 + \frac{2}{z} = 1 + 2z^{-1} \qquad ----- (1)$$

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

To Find the critical points

$$\frac{dw}{dz} = \frac{-2}{z^2}$$

$$Put \ \frac{dw}{dz} = 0$$

$$\Rightarrow \frac{-2}{z^2} = 0$$

$$\Rightarrow -2 = 0$$

$\Rightarrow$ *It is absurd*

$$So \ Take \ \frac{dz}{dw} = \frac{z^2}{-2}$$

$$Put \ \frac{dz}{dw} = 0$$

$$\Rightarrow \frac{z^2}{-2} = 0$$

$$\Rightarrow z = 0$$

$\Rightarrow$ *which is the critcial point.*

18. Find the critical points of the transformation $w^2 = (z-\alpha)(z-\beta)$

$$Given \quad w^2 = (z-\alpha)(z-\beta)$$

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

To Find the critical points

Diff (1) w.r.to z,

$$2w \frac{dw}{dz} = (z-\alpha)(1) + (z-\beta)(1)$$

$$= 2z - (\alpha+\beta)$$

$$\frac{dw}{dz} = \frac{2z-(\alpha+\beta)}{2w}$$

$$Put \quad \frac{dw}{dz} = 0$$

$$\Rightarrow \frac{2z-(\alpha+\beta)}{2w} = 0$$

$$\Rightarrow 2z-(\alpha+\beta) = 0$$

$$\Rightarrow \quad z = \frac{\alpha+\beta}{2}$$

Diff (1) w.r.to w,

$$So \ Take \ \frac{dz}{dw} = \frac{2w}{2z-(\alpha+\beta)}$$

$$Put \ \frac{dz}{dw} = 0$$

$$\Rightarrow \frac{2w}{2z-(\alpha+\beta)} = 0$$

$$\Rightarrow \quad w = 0$$

$$(z-\alpha)(z-\beta) = 0$$

$$\Rightarrow \quad z = \alpha, \beta$$

The Critical points are $z = \alpha, \beta, \frac{\alpha+\beta}{2}$

19. Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ **is analytic**

$$Let \quad f(z) = (x+ay) + i(bx+cy)$$

Since $f(z)$ *is analytic*

$\Rightarrow f(z)$ *is satisfied* $C-R$ *equations.*

$u = x + ay$	$v = bx + cy$
$u_x = 1$ $u_y = a$	$v_x = b$ $v_y = c$

Here $\therefore u_x = v_y$ and $v_x = -u_y$

$\Rightarrow 1 = c$ $\quad b = -a$

20. Find the constants a, b, c if $f(z) = x + 2ay + i(3x + by)$ is analytic

$Let \quad f(z) = x + 2ay + i(3x + by)$

$u = x + 2ay$	$v = 3x + by$
$u_x = 1$ $u_y = 2a$	$v_x = 3$ $v_y = b$

$Since\ f(z)\ is\ analytic$

$\Rightarrow f(z)\ is\ satisfied\ \ C - R\ equations.$

$\therefore u_x = v_y$ and $v_x = -u_y$

$\Rightarrow 1 = b$ $\quad 3 = -2a$

$\therefore b = 1$ & $a = -3/2$

21. Show that 2x(1-y) can be the imaginary part of an analytic function

Let v = 2x(1-y)

Since the real and imaginary parts of an analytic function are harmonic functions , they satisfy the Laplace equation.

$v_x = 2(1 - y)$	$v_y = -2x$
$v_{xx} = 0$	$v_{yy} = 0$

$v_{xx} + v_{yy} = 0 + 0$

$= 0$

$\Rightarrow v$ satisfies Laplace equation.

Hence , v can be the imaginary part of an analytic function.

22. Prove that a bilinear transformation has at most two fixed points.

Let the bilinear transformation be $w = \dfrac{az+b}{cz+d}$ ------ (1) if $ad - bc \neq 0$

The fixed point of the transformation are given by $w = z$

$$(1) \Rightarrow \quad z = \frac{az+b}{cz+d}$$

$$\Rightarrow \quad cz^2 + dz = az + b$$

$$\Rightarrow cz^2 + (d-a)z - b = 0$$

If $c \neq 0$, it is quadratic(bilinear) in z, giving two roots and so there are two fixed points.

If $c = 0,\ d \neq a$, there is one fixed point. In this case, It is a linear transformation. So a bilinear transformation has at most two fixed points in the extended plane.

23. Determine the analytic function where real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\varphi_1(x,y) = u_x = 3x^2 - 3y^2 + 6x$	$\varphi_2(x,y) = u_y = -6xy - 6y$
$\varphi_1(z,0) = 3z^2 + 6z$	$\varphi_2(z,0) = 0$

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

By Milne's Thomson method,

$$f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz$$

$$= \int (3z^2 + 6z)\,dz - 0 \qquad (\because u \text{ is given})$$

$$= 3\frac{z^3}{3} + 6\frac{z^2}{2} + c$$

$$= z^3 + 3z^2 + c$$

24. Define Complex Potential Function

In two dimensional steady flow problems in thermodynamics, hydrodynamics and electronics we represent the **complex potential function** as $F(z) = \phi(x,y) + i\,\psi(x,y)$ where

$\phi(x,y)$ = **velocity potential function**

$\psi(x,y)$ = **stream function (or) lines of force**