ENGINEERING MATHEMATICS -III
Laplace Transform
Two marks Questions

1. State the conditions under which Laplace transform of *f(t)* exists.

i. *f* (t) should be **continuous** or **piecewise continuous** in the given closed interval $[a, b]$ where $a > 0$
ii. *f* (t) should be of exponential order.

2. Find the Laplace transform of t sin2t

$$L\left[t\sin 2t\right] = (-1)\frac{d}{ds}\, L\left[\sin 3t\right]$$
$$= (-1)\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$
$$= (-1)\left[\frac{\left(s^2+9\right)(0) - 3\,(2s)}{\left(s^2+9\right)^2}\right]$$
$$= (-1)\left[\frac{-6s}{\left(s^2+9\right)^2}\right]$$
$$= \frac{6s}{\left(s^2+9\right)^2}$$

3. Find the Laplace transform of t cos *a*t

$$L\left[t\ \cos at\right] = (-1)\frac{d}{ds}\, L\left[\cos at\right]$$
$$= (-1)\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$
$$= (-1)\left[\frac{\left(s^2+a^2\right)(1) - s\,(2s)}{\left(s^2+a^2\right)^2}\right]$$
$$= (-1)\left[\frac{s^2+a^2-2s^2}{\left(s^2+a^2\right)^2}\right]$$
$$= \frac{s^2-a^2}{\left(s^2+a^2\right)^2}$$

4. Find the Laplace transform of unit step function.

The unit step function is given by

$$U(t-a) = \begin{cases} 0, & 0 < t < a \\ 1, & a < t < \infty \end{cases}$$

The L.T.of the unit step function is given by

$$L\left[U(t-a)\right] = \int_0^\infty e^{-st}\, U(t-a)\ dt$$
$$= \int_0^a e^{-st}\ (0)\ dt + \int_a^\infty e^{-st}\ (1)\ dt$$
$$= \int_a^\infty e^{-st}\ dt$$
$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$
$$= 0 - \frac{e^{-as}}{-s}$$
$$= \frac{e^{-as}}{s}$$

5. Find the Laplace transform of $\int_0^\infty t\, e^{-3t}\sin 2t\, dt$

$$w.k.t., \int_0^\infty t\ e^{-at}\sin bt\ dt = L\left[t\sin bt\right]_{s=a}$$
$$\int_0^\infty t\ e^{-3t}\sin 2t\ dt = L\left[t\sin 2t\right]_{s=3}$$
$$= \left[(-1)\frac{d}{ds}\ L\left(\sin 2t\right)\right]_{s=3}$$
$$= \left[(-1)\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right]_{s=3}$$
$$= (-1)\left[\frac{\left(s^2+4\right)(0) - 2\,(2s)}{\left(s^2+4\right)^2}\right]_{s=3}$$
$$= \left[\frac{4s}{\left(s^2+4\right)^2}\right]_{s=3}$$
$$= \frac{12}{(9+4)^2}$$
$$= \frac{12}{169}$$

6. Find $L\left(e^{-3t}\sin t\cos t\right)$

$$L\left[e^{-3t}\sin t\cos t\right] = L\left[e^{-3t}\frac{\sin 2t}{2}\right]$$
$$= \frac{1}{2}L\left[e^{-3t}\sin 2t\right]$$
$$= \frac{1}{2}L\left[\sin 2t\right]_{s\to s+3}$$
$$= \frac{1}{2}\left[\frac{2}{s^2+2^2}\right]_{s\to s+3}$$
$$= \frac{1}{(s+3)^2+4}$$
$$= \frac{1}{s^2+6s+13}$$

7. Find the Laplace transform of $f(t) = \begin{cases} 0 & ,t<\frac{2\pi}{3} \\ \cos\left(t-\frac{2\pi}{3}\right) & ,t>\frac{2\pi}{3}\end{cases}$

$$L\left[f(t)\right] = \int_0^\infty e^{-st}\ f(t)\ dt$$
$$= \int_0^{\frac{2\pi}{3}} e^{-st}\ (0)\ dt + \int_{\frac{2\pi}{3}}^\infty e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt$$
$$= \int_0^\infty e^{-s\left(x+\frac{2\pi}{3}\right)}\cos x\ dx$$
$$= e^{\frac{-2\pi s}{3}}\int_0^\infty e^{-sx}\cos x\ dx$$
$$= e^{\frac{-2\pi s}{3}}\left[\frac{e^{-sx}}{s^2+1}\left(\sin x - s\cos x\right)\right]_0^\infty$$
$$= \frac{e^{\frac{-2\pi s}{3}}}{s^2+1}(0+s) \quad = \frac{s\,e^{\frac{-2\pi s}{3}}}{s^2+1}$$

8. Is the linearity property applicable to $L\left\{\frac{1-\cos t}{t}\right\}$ **? Reason out.**

$$L\left[\frac{1-\cos t}{t}\right] = L\left[\frac{1}{t}\right] - L\left[\frac{\cos t}{t}\right] \qquad (\textit{Linearity Property})$$

$L\left[\frac{1}{t}\right]$ *does not exist.* *Since* $\lim\limits_{t\to 0}\frac{1}{t} = \frac{1}{0} = \infty$

$L\left[\frac{\cos t}{t}\right]$ *does not exist.* *Since* $\lim\limits_{t\to 0}\left[\frac{\cos t}{t}\right] = \frac{1}{0} = \infty$

$\therefore$ Linearity property not applicable to $L\left[\frac{1-\cos t}{t}\right]$

9. Find the Laplace transform of $\frac{t}{e^t}$

$$L\left[t\,e^{-t}\right] = (-1)\frac{d}{ds}L\left[e^{-t}\right]$$
$$= (-1)\frac{d}{ds}\left[\frac{1}{s+1}\right]$$
$$= (-1)\frac{d}{ds}\left[(s+1)^{-1}\right]$$
$$= (-1)(-1)(s+1)^{-2}$$
$$= \frac{1}{(s+1)^2}$$

10. Find $L\left[\frac{\sin t}{t}\right]$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L\left[\sin t\right]ds$$
$$= \int_s^\infty\left[\frac{1}{s^2+1}\right]ds$$
$$= \left[\tan^{-1}(s)\right]_s^\infty$$
$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$
$$= \frac{\pi}{2} - \tan^{-1}(s)$$
$$= \cot^{-1}(s)$$

11. Find the Laplace transform of $L\left\{\frac{1-\cos t}{t}\right\}$

$$L\left[\frac{1-\cos t}{t}\right]=\int_s^\infty L\left[1-\cos t\right]\,ds$$
$$=\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+1}\right]ds$$
$$=\log s-\frac{1}{2}\log\left(s^2+1\right)$$
$$=\log s-\log\sqrt{s^2+1}$$
$$=\log\left[\frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$
$$=0-\log\left[\frac{s}{\sqrt{s^2+1}}\right]$$
$$=\log\left[\frac{\sqrt{s^2+1}}{s}\right]$$

12. Find the Laplace transform of $f(t)=\frac{1-e^{-t}}{t}$

$$L\left[\frac{1-e^{-t}}{t}\right]=\int_s^\infty L\left[1-e^{-t}\right]\,ds$$
$$=\int_s^\infty\left[\frac{1}{s}-\frac{1}{s-1}\right]ds$$
$$=\left[\log s-\log(s-1)\right]_s^\infty$$
$$=\log\left[\frac{s}{s-1}\right]_s^\infty$$
$$=\left[\log\frac{s}{s\left(1-\frac{1}{s}\right)}\right]_s^\infty=\left[\log\frac{1}{\left(1-\frac{1}{s}\right)}\right]_s^\infty$$
$$=0-\log\left[\frac{s}{s-1}\right]$$
$$=\log\left[\frac{s-1}{s}\right]$$

13.Verify the Initial value theorem for $f(t)=ae^{-bt}$

$$f(t)=ae^{-bt}$$
$$L\left[f(t)\right]=L\left[ae^{-bt}\right]$$
$$=\frac{a}{s+b}$$

Initial value theorem :

$$\lim_{t\to 0}f(t)=\lim_{s\to\infty}s\,F(s)$$

$$L.H.S.=\lim_{t\to 0}f(t)=\lim_{t\to 0}ae^{-bt}$$
$$=a\qquad ---(1)$$

$$R.H.S.=\lim_{s\to\infty}sF(s)=\lim_{s\to\infty}s\left(\frac{a}{s+b}\right)$$
$$=\lim_{s\to\infty}\left(\frac{as}{s+b}\right)$$
$$=\lim_{s\to\infty}\frac{as}{s\left(1+\frac{b}{s}\right)}$$
$$=\lim_{s\to\infty}\frac{a}{\left(1+\frac{b}{s}\right)}$$
$$=a\qquad ---(2)$$

From (1) & (2),

$LHS=RHS$

I.V.T. is satisfied.

14. Verify the final value theorem for $f(t)=3e^{-2t}$

$$f(t)=3e^{-2t}$$
$$L\left[f(t)\right]=L\left[3e^{-2t}\right]$$
$$=\frac{3}{s+2}$$

Final value theorem :

$$\lim_{t\to\infty}f(t)=\lim_{s\to 0}s\,F(s)$$

$$L.H.S.=\lim_{t\to\infty}f(t)=\lim_{t\to\infty}3e^{-2t}$$
$$=0\qquad ---(1)$$

$$R.H.S.=\lim_{s\to 0}sF(s)=\lim_{s\to 0}s\left(\frac{3}{s+2}\right)$$
$$=\lim_{s\to 0}\left(\frac{3s}{s+2}\right)$$
$$=0\qquad ---(2)$$

From (1) & (2),

$LHS=RHS$

F.V.T. is satisfied.

15. Verify the final value theorem for $f(t) = 1+e^{-t}(\sin t+\cos t)$

$$f(t) = 1+e^{-t}(\sin t+\cos t)$$

$$L[f(t)] = L\left[1+e^{-t}(\sin t+\cos t)\right]$$
$$= L[1]+L\left[e^{-t}(\sin t+\cos t)\right]$$
$$= \frac{1}{s}+L\left[\sin t+\cos t\right]_{s\to s+1}$$
$$= \frac{1}{s}+\left[\frac{1}{s^2+1}+\frac{s}{s^2+1}\right]_{s\to s+1}$$
$$= \frac{1}{s}+\left[\frac{(s+1)}{(s+1)^2+1}+\frac{(s+1)}{(s+1)^2+1}\right]$$
$$= \frac{1}{s}+\left[\frac{(s+2)}{(s+1)^2+1}\right]$$

Final Value Theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

$$L.H.S. = \lim_{t\to\infty} f(t) = \lim_{t\to\infty} 1+e^{-t}(\sin t+\cos t)$$
$$= 1+0 = 1 \qquad ---(1)$$

$$R.H.S. = \lim_{s\to 0} s\,F(s) = \lim_{s\to 0} 1+\left[\frac{s(s+2)}{(s+1)^2+1}\right]$$
$$= 1+0 = 1 \qquad ---(2)$$

From (1) & (2),

LHS = *RHS*

F.V.T. is satisfied.

16. Find $L^{-1}\left[\frac{2s+3}{s^2-4s+13}\right]$

$$L^{-1}\left[\frac{2s+3}{s^2-4s+13}\right] = L^{-1}\left[\frac{2s+3}{\left(s^2-4s+2^2\right)-2^2+13}\right]$$
$$= L^{-1}\left[\frac{2s+3}{(s-2)^2+9}\right] = L^{-1}\left[\frac{2s-4+7}{(s-2)^2+9}\right]$$
$$= L^{-1}\left[\frac{2(s-2)+7}{(s-2)^2+9}\right]$$
$$= L^{-1}\left[\frac{2(s-2)}{(s-2)^2+9}\right]+L^{-1}\left[\frac{7}{(s-2)^2+9}\right]$$
$$= 2\,L^{-1}\left[\frac{(s-2)}{(s-2)^2+3^2}\right]+\frac{7}{3}\,L^{-1}\left[\frac{3}{(s-2)^2+3^2}\right]$$
$$= 2\,e^{2t}L^{-1}\left[\frac{s}{s^2+3^2}\right]+\frac{7}{3}e^{2t}\,L^{-1}\left[\frac{3}{s^2+3^2}\right]$$
$$= 2\,e^{2t}\cos 3t+\frac{7}{3}e^{2t}\sin 3t$$

17. Find $L^{-1}\left[\frac{1}{s^2+4s+5}\right]$

$$L^{-1}\left[\frac{1}{s^2+4s+5}\right] = L^{-1}\left[\frac{1}{\left(s^2+4s+2^2\right)-2^2+5}\right]$$
$$= L^{-1}\left[\frac{1}{(s+2)^2-4+5}\right]$$
$$= L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$
$$= e^{-2t}\,L^{-1}\left[\frac{1}{s^2+1}\right]$$
$$= e^{-2t}\,\sin t$$

18. State the first shifting theorem on Laplace transform

If $L[f(t)] = F(s)$ then $L\left[e^{-at}\,f(t)\right] = [F(s)]_{s\to s+a}$

ie., $L\left[e^{-at}\,f(t)\right] = F(s+a)$

19. Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s+2)}$

$$\frac{1}{(s+1)(s+2)}=\frac{A}{(s+1)}+\frac{B}{(s+2)}$$
$$1=A(s+2)+B(s+1)$$

$Put\ \ s=-1,$ $\quad 1=A(-1+2)$ $\quad A=1$

$Put\ \ s=-2,$ $\quad 1=B(-2+1)$ $\quad B=-1$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=L^{-1}\left[\frac{1}{(s+1)}-\frac{1}{(s+2)}\right]$$
$$=L^{-1}\left[\frac{1}{(s+1)}\right]-L^{-1}\left[\frac{1}{(s+2)}\right]$$
$$=e^{-t}-e^{-2t}$$

20. Find Laplace transform of $\dfrac{s}{(s+1)^2}$

$$L^{-1}\left[\frac{s}{(s+1)^2}\right]=\frac{d}{dt}\ L^{-1}\left[\frac{1}{(s+1)^2}\right]$$
$$=\frac{d}{dt}\left\{e^{-t}L^{-1}\left[\frac{1}{s^2}\right]\right\}$$
$$=\frac{d}{dt}\left\{e^{-t}\,(t)\right\}$$
$$=e^{-t}\,(1)-t(-e^{-t})$$
$$=e^{-t}\,[1+t]$$

21. Find $L^{-1}\left[\cot^{-1}(s)\right]$

$$w.k.t.,\ \ L^{-1}\left[F(s)\right]=-\frac{1}{t}\ L^{-1}\left[F'(s)\right]$$

$$L^{-1}\left[\cot^{-1}(s)\right]=-\frac{1}{t}\ L^{-1}\left[\frac{d}{ds}\cot^{-1}(s)\right]$$
$$=-\frac{1}{t}L^{-1}\left[\frac{-1}{1+s^2}\right]$$
$$=\frac{1}{t}L^{-1}\left[\frac{1}{s^2+1}\right]$$
$$=\frac{1}{t}\sin t$$

22. Find the inverse Laplace transform of $\cot^{-1}\left(\dfrac{k}{s}\right)$

$$w.k.t.,\ \ L^{-1}\left[F(s)\right]=-\frac{1}{t}\ L^{-1}\left[F'(s)\right]$$

$$L^{-1}\left[\cot^{-1}\left(\frac{k}{s}\right)\right]=-\left(\frac{1}{t}\right)\ L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{k}{s}\right)\right]$$
$$=-\left(\frac{1}{t}\right)\ L^{-1}\left[\frac{-1}{\left(1+\frac{k^2}{s^2}\right)}\left(\frac{-k}{s^2}\right)\right]$$
$$=-\left(\frac{1}{t}\right)\ L^{-1}\left[\frac{k}{s^2+k^2}\right]$$
$$=\frac{-1}{t}\sin kt$$

23. Write the value of $t*e^{t}$

$$t*e^t=\int_0^t u.\,e^{t-u}\ du$$
$$=e^t\int_0^t u.\,e^{-u}\ du$$
$$=e^t\left[u\left(\frac{e^{-u}}{-1}\right)-\frac{e^{-u}}{(-1)^2}\right]_0^t$$
$$=e^t\left[-ue^{-u}-e^{-u}\right]_0^t$$
$$=e^t\left[\left(-te^{-t}-e^{-t}\right)-(0-1)\right]$$
$$=e^t-t-1$$

24. Find the inverse Laplace transform of $\dfrac{e^{-as}}{s}$

$$w.k.t.,\ \ L^{-1}\left[\frac{e^{-as}}{s}\right]=u(t)=\begin{cases}0, & t<a\\ 1, & t>a\end{cases}$$