MATHEMATICS - III
UNIT-3- Complex Integration
Two marks

1. Define Singular point

A point $z = a$ at which a function *f(z)* fails to be analytic is called a ***singular point*** or ***singularity*** of *f(z)*

> ***Ex:*** *Consider* $f(z) = \dfrac{1}{z-3}$
> *Here* $z = 3$ *is a singular point of f(z)*

2. Define Poles and Simple poles

- An analytic function f(z) with a singularity at $z=a$ if $\lim_{z\to 2} f(z) = \infty$ then $z=a$ is a ***pole*** of *f(z)*.
- Pole of order one is called ***a simple pole***

> ***Ex:*** *Consider* $f(z) = \dfrac{1}{(z-4)(z-3)^2(z-5)^3}$
> Here $z = 4$ is a simple pole
> $z = 3$ is a pole of order 2
> $z = 5$ is a pole of order 3

3. Define Removable singularity

A point $z = a$ is called ***removable singularity*** of *f(z)* if,

i. $z=a$ is a singular point

ii. $\lim_{z\to a} f(z)$ exists

> ***Ex:*** *Consider* $f(z) = \dfrac{\tan z}{z}$
> *Here* $z = 0$ *is a singular point*
> ⇨ $\lim_{z\to a} \dfrac{\tan z}{z} = 1$
> $\therefore z = 0$ *is an removable singularity.*

4. Define Essential singularity

A point $z = a$ is called ***essential singularity*** of *f(z)* if,

i. $z=a$ is a singular point

ii. *z=a should not be a pole or removable singularity*

> ***Ex:*** *Consider* $f(z) = e^{\frac{1}{z-2}}$
> *At z=2 ,* $f(z) = e^{\frac{1}{0}} = e^{\infty} = \infty$
> $\therefore z = 0$ *is not a pole or removable singularity.*
> $\therefore z = 0$ *is an essential singularity.*

5. Define Isolated Singularity

A point $z = a$ is called ***isolated singularity*** of *f(z)* if,

i. *f(z)* is not analytic at $z=a$

ii. There exists a neighbourhood of $z=a$ containing no other singularity.

> ***Ex:*** *Consider* $f(z) = \dfrac{1}{z}$
> *The function is analytic everywhere except at z=0*
> $\therefore z = 0$ *is an isolated singularity.*

6. Expand $f(z) = \sin z$ in a Taylor series about origin

Taylor's series about z = 0 is,

$$f(z) = f(0) + \frac{(z)}{1!} f'(0) + \frac{(z)^2}{2!} f''(0) + \frac{(z)^3}{3!} f'''(0) + \dots$$

$$= 0 + \frac{(z)}{1!}(1) + \frac{(z)^2}{2!}(0) + \frac{(z)^3}{3!}(-1) + \dots$$

$$\left[\sin z\right]_{z=0} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

Function	Value @ z = 0
$f(z) = \sin z$	$f(0) = \sin 0 = 0$
$f'(z) = \cos z$	$f'(0) = \cos 0 = 1$
$f''(z) = -\sin z$	$f''(0) = -\sin 0 = 0$
$f'''(z) = -\cos z$	$f'''(0) = -\cos 0 = -1$

7. Find the Taylor series for $f(z) = \sin z$ about $z = \frac{\pi}{4}$

Taylor's series about $z = \frac{\pi}{4}$ is

$$f(z) = f\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)}{1!} f'\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \dots$$

$$= \frac{1}{\sqrt{2}} + \frac{\left(z - \frac{\pi}{4}\right)}{1!}\frac{1}{\sqrt{2}} - \frac{\left(z - \frac{\pi}{4}\right)^2}{2!}\frac{1}{\sqrt{2}} - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!}\frac{1}{\sqrt{2}} + \dots$$

$$\left[\sin z\right]_{z=\pi/4} = \frac{1}{\sqrt{2}}\left[1 + \frac{\left(z - \frac{\pi}{4}\right)}{1!} - \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \dots\right]$$

Function	Value @ z = 0
$f(z) = \sin z$	$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$f'(z) = \cos z$	$f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$f''(z) = -\sin z$	$f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f'''(z) = -\cos z$	$f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

8. Evaluate $\oint_c \frac{e^z}{z-1} dz$ if C is $|z| = 2$

Points	Order	Lies
Z = 1	1	inside

Let $f(z) = e^z$

By Cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i \times f(a)$$

$$\therefore \int_c \frac{e^z}{z-1} dz = 2\pi i \times f(1)$$

$$= 2\pi i \times e$$

$$= 2\pi i(e)$$

$$f(z) = e^z$$
$$f(1) = e^1 = e$$

9. Evaluate $\int_c \frac{z\,dz}{(z-1)(z-2)}$ where C is the circle $|z| = \frac{1}{2}$

Points	Order	Lies
Z= 1	1	outside
Z= 2	1	outside

Let $f(z) = \frac{z}{(z-1)(z-2)}$

Here the points both lies outside the given circle $|z| = \frac{1}{2}$

By Cauchy's integral theorem,

$$\int_c f(z) dz = 0$$

$$\therefore \int_c \frac{z}{(z-1)(z-2)} dz = 0$$

10. Evaluate $\int_c \frac{z+4}{z^2+2z} dz$ **where C is the circle** $\left|z-\frac{1}{2}\right| = \frac{1}{3}$

Points	Order	Lies
Z= 0	1	outside
Z= -2	1	outside

Let $f(z) = \frac{z+4}{z(z+2)}$

Here the points both lies outside the given circle $\left|z-\frac{1}{2}\right| = \frac{1}{3}$

By Cauchy's integral theorem,

$$\int_c f(z)\,dz = 0$$

$$\therefore \int_c \frac{z+4}{z^2+2z}\,dz = 0$$

11. Evaluate $\int_c \left[\frac{3z^2+7z+1}{z+1}\right] dz$ **where C is** $|z| = \frac{1}{2}$

Points	Order	Lies
Z= -1	1	outside

Let $f(z) = \frac{3z^2+7z+1}{z+1}$

Here z=-1 lies outside the given circle $|z| = \frac{1}{2}$

By Cauchy's integral theorem,

$$\int_c f(z)\,dz = 0$$

$$\int_c \left[\frac{3z^2+7z+1}{z+1}\right] dz = 0$$

12. Using Cauchy's Integral formula, evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$ **where C is** $|z| = \frac{1}{2}$

Let $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)}$

Points	Order	Lies
Z= -1	1	outside
Z= -2	1	outside

Here the points both lies outside the given circle $|z| = \frac{1}{2}$

By Cauchy's integral theorem,

$$\int_c f(z)\,dz = 0$$

$$\therefore \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)}\,dz = 0$$

13. Identify the type of singularities of the following function: $f(z) = e^{\frac{1}{z-1}}$

$$f(z) = e^{\frac{1}{z-1}}$$

Here z = 1 is a singular point.

At z = 1 we get, $f(z) = e^{\frac{1}{0}} = \infty$ which is not defined.

Also z =1 is not a pole or removable singularity.

⇨ z = 1 is an essential singularity.

14. State Cauchy's Integral formula for derivative

If a function *f(z)* is analytic within and on a simple closed curve C and *a* is any point lying in it,

then $$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} \times f^{(n)}(a)$$

15. If $f(z) = \frac{-1}{z-1} - 2\left[1+(z-1)+(z-1)^2+...\right]$, find the residue of $f(z)$ at $z=1$

Given $$f(z) = (-1)\frac{1}{z-1} - 2\left[1+(z-1)+(z-1)^2+...\right]$$

Res $[f(z): z=1] =$ Coefficient of $\frac{1}{z-1}$ in Laurents series of f(z)

$= -1$

16. Find the residue of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ at its simple pole

By Cauchy's Residue Theorem,

Points	Order
Z = 1	2
Z = -2 Simple pole	1

$$\text{Res}\left[f(z):\ z=-2\right] = \lim_{z\to -2} (z+2)\ f(z)$$

$$= \lim_{z\to -2} (z+2) \frac{z^2}{(z-1)^2(z+2)}$$

$$= \lim_{z\to -2} \frac{z^2}{(z-1)^2}$$

$$= \frac{4}{9}$$

17. Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole

By Cauchy's ResidueTheorem,

Points	Order
Z = 0	3
Z = 2 Simple pole	1

$$\text{Res}\left[f(z):\ z=2\right] = \lim_{z\to 2} (z-2)\ f(z)$$

$$= \lim_{z\to 2} (z-2)\frac{4}{z^3(z-2)}$$

$$= \lim_{z\to 2} \frac{4}{z^3}$$

$$= \frac{4}{8} = \frac{1}{2}$$

18. Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its poles.

Points	Order
Z = -1	2

By Cauchy's ResidueTheorem,

$$\text{Res}\left[f(z):\ z=-1\right] = \lim_{z\to -1} \frac{d}{dz}(z+1)^2\ f(z)$$

$$= \lim_{z\to -1} \frac{d}{dz}(z+1)^2 \frac{e^{2z}}{(z+1)^2}$$

$$= \lim_{z\to -1} \frac{d}{dz} e^{2z}$$

$$= \lim_{z\to -1} 2e^{2z}$$

$$= 2e^{-2}$$

$$= \frac{2}{e^2}$$

19. Find the residue of $f(z)=\dfrac{1-e^{-z}}{z^2}$ **at z = 0**

By Cauchy's ResidueTheorem,

Points	Order
Z = 0	2

$$\operatorname{Res}\left[f(z): z=0\right]=\frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}(z-0)^2 f(z)$$
$$=\lim_{z\to 0}\frac{d}{dz}(z)^2\frac{1-e^{-z}}{z^2}$$
$$=\lim_{z\to 0}\frac{d}{dz}\left(1-e^{-z}\right)$$
$$=\lim_{z\to 0}\left(e^{-z}\right)$$
$$=e^0$$
$$=1$$

20. Find the residue of $f(z)=\dfrac{1-e^{2z}}{z^3}$ **at z = 0**

By Cauchy's ResidueTheorem,

Points	Order
Z = 0	3

$$\operatorname{Res}\left[f(z): z=0\right]=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}(z-0)^3 f(z)$$
$$=\frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}(z)^3\frac{1-e^{2z}}{z^3}$$
$$=\frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left(1-e^{2z}\right)$$
$$=\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left(-2e^{2z}\right)$$
$$=\frac{1}{2}\lim_{z\to 0}\left(-4e^{2z}\right)$$
$$=-2$$

21. Evaluate $\int_c \tan z\, dz$ **where C is** $|z|=2$

$$\int_c \tan z\, dz=\int_c\frac{\sin z}{\cos z}dz=\int_c\frac{P(z)}{Q(z)}dz \qquad \text{where, } P(z)=\sin z,\ Q(z)=\cos z$$

Poles: cos z = 0

⇨ $z=(2n+1)\frac{\pi}{2},\quad n=0,\pm 1,\pm 2,\ldots$

$z=\frac{\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\ldots$

⇨ Out of these poles only $z=\frac{\pi}{2}$ lies inside $|z|=2$

$$\operatorname{Res}\left[f(z): z=\frac{\pi}{2}\right]=\frac{P(a)}{Q'(a)}$$
$$=\frac{P\left(\frac{\pi}{2}\right)}{Q'\left(\frac{\pi}{2}\right)}=\frac{1}{-1}$$
$$=-1$$

$P(z)=\sin z \qquad Q(z)=\cos z$

$P\left(\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1 \qquad Q'(z)=-\sin z$

$Q'\left(\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}=-1$

By CRT,

$$\int_c f(z)dz=2\pi i\times[S.O.R]$$
$$=2\pi i\times[-1]$$
$$\int_c \tan z\, dz=-2\pi i$$