# COMPLEX INTEGRATION

- Cauchy's Integral theorem (Fundamental theorem)
- Cauchy's Integral formula
- Singularities (2m)
- Taylor's Laurant's series
- Cauchy's Residue theorem → type 1, type 2
- Line & Surface Integral (2m)

**Simple curve:-** A curve is said to be simple which does not crosses it self. otherwise the curve is said to be multi curve.

Simple curve      multi

region
- Simple connected region 
- multiple connected region 

**Simple connected region:-** A closed curve is said to be simply connected if the line joining any two points must lie entirely in itself. In other words a simply connected region does not have any holes other wise it is said to be multiple connected region i.e, a region with holes.

**Complex Integration:-**

Let $w = f(z)$ be a analytic function which is also continuous on some region R. Then the integration over the curve C is denoted by $\int_C f(z)dz$ is called as complex integration or contour integration.

**Note:-**

$z = x+iy$

$w = u+iv$

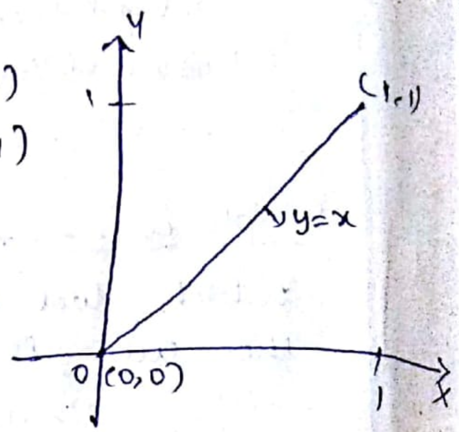
$dz = dx+idy$.

* Evaluate ~~integral 0 to 1+i x-y~~

$$\int_0^{1+i} (x-y+ix^2)\,dz \rightarrow (1)$$ along line from $z=0$ to $z=1+i$

Sol:- Given points $0=(0,0)$

$1+i=(1,1)$




Eqn of st line ~~y=y~~

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{1-0}=\frac{x-0}{1-0}$$

$$y=x.$$

~~Here~~ $z=x+iy$

$dz=dx+idy$

Here the curve $y=x$

$dy=dx$

Substitute all values in I

$$I=\int_{(0,0)}^{(1,1)} (x-y+ix^2)\,[dx+idy]$$

$$=\int_{(0,0)}^{(1,1)} (x-\cancel{y}^{\,x}+ix^2)\,[dx+idx]$$

$$=\int_{x=0}^{x=1} (ix^2)(1+i)\,dx.$$

$$=1+i\int_{x=0}^{x=1} ix^2\,dx$$

$$=i(1+i)\int_0^1 x^2\,dx$$

$$=i(1+i)\left[\frac{x^3}{3}\right]_0^1$$

$$=i(1+i)\left(\frac{1}{3}\right)=(i-1)\frac{1}{3}$$

* Evaluate $\int_{(0,0)}^{(1,1)} [(x^2+y^2)dx - 2xy\,dy]$ — (1)

(i) $y=x$

(ii) $y=x^2$

(iii) $x=y^2$

Sol:

(i) $y=x \Rightarrow dy=dx$   Limits
$x$ varies $0 \to 1$
$y$ varies $0 \to 1$

Substitute in 1

$$I = \int_0^1 (x^2+x^2)\,dx - 2x^2\,dx$$

$$= \int_0^1 2x^2\,dx - \int_0^1 2x^2\,dx = 0$$

$$= \cancel{2\int_0^1 x^2\,dx - 2}$$

(ii) $y=x^2$
$dy=2x\,dx$   Limits $x$ varies $0$ to $1$
$y$ varies $0$ to $1$.   $y: 0 \to 1$

Substitute in (1)

$$I = \int_0^1 (x^2+x^4)\,dx - 2x^3(2x)\,dx$$

$$= \int_0^1 (x^2+x^4)\,dx - 4x^4\,dx.$$

$$= \left[\frac{x^3}{3}+\frac{x^5}{5}\right]_0^1 - 4\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{3}+\frac{1}{5}-\frac{4}{5}$$

$$= \frac{1}{3}-\frac{3}{5} = \frac{5-9}{15} = \frac{-4}{15}$$

(iii) $x = y^2$

$dx = 2y\,dy$

| $y$ | 0 | 1 |
|---|---|---|
| $x = y^2$ | 0 | 1 |

$$I = \int_0^1 (x^2+x)\,dx - 2x\frac{dx}{2}$$

$$= \int_0^1 (x^2+x)\,dx - x\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}.$$

* $$I = \int_{(0,2)}^{(4,0)} (x^2+y^2)\,dx$$

along the curve $y^2 = 4-x$.

H.W. $x^2 = 4-y$

$(x^2+y^2)\,dy$

Sol'n:

$$I = \int_{0,2}^{(4,0)} x^2+y^2\,dx$$

$y^2 = 4-x$

$2y\,dy = -dx.$

| $y$ | 2 | 0 |
|---|---|---|
| $y^2=4-x$ | 0 | 4 |

$4 = 4-x$

$x = 0$

$$I = \int_0^4 (x^2+4-x)\,dx$$

$$= \left[\frac{x^3}{3} + 4x - \frac{x^2}{2}\right]_0^4$$

$$= \left[\frac{64}{3} + 16 - \frac{16}{2}\right]$$

$$= \frac{64}{3} + 8$$

$$= \frac{88}{3}.$$

$\frac{64}{\frac{24}{88}}$

$\cancel{I = y^2 - 4}$

$x^2 = 4 - y$

$2x\,dx = -dy$

$$I = \int_0^4 (x^2 - 4 - x)(-2x\,dx)$$

$$= -2\int_0^4 (x^3 - 4x - x^2)\,dx$$

$$= -2\left[\frac{x^4}{4} - 4\frac{x^2}{2} - \frac{x^3}{3}\right]_0^4$$

$$= -2\left[64 - 2(16) - \frac{64}{3}\right]$$

$$= -2\left[64 - 32 - \frac{64}{3}\right] = -2\left[32 - \frac{64}{3}\right]$$

$$= -2\left[\frac{32}{3}\right] = \frac{-64}{3}.$$

$64 \times 4$

$16 \times 2 = 32$

$96 - 64 = 32$

* Show that $\oint (z+1)\,dz$ where $C$ is the boundary of the square whose vertices are the points $z=0,\ z=1,\ z=1+i,\ z=i$

Sol$^n$

$$\oint (z+1)\,dz$$

$(0,0)$, $(1,0)$

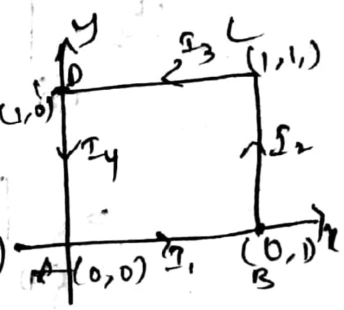
$z=0,\ z=1$

$z=(1+i) = (1,1)$

$z=i = (0,1)$

$z = x+iy$

$dz = dx + i\,dy$




$$I = I_1 + I_2 + I_3 + I_4.$$

x varies from 0 to 0

$y=0$ $dy=0$.

$$I_1 = \int_{AB} (x+1+iy)(dx + i\,dy)$$

$$= \int_{AB} (x+1)\,dx$$

$$= \int_0^1 (x+1)\,dx = \left(\frac{x^2}{2} + x\right)_0^1$$

$$= \frac{1}{2} + 1 = \frac{3}{2}$$

$I_2 =$ BC $\quad$ ~~y~~ x varies from ~~1~~ 1 to 1 $\quad dx = 0$

y varies from 0 to 1

$$I_2 = \int_{BC} (x + (1+i)y)[dx + i\,dy]$$

$$= \int_{BC} (2 + iy)(i\,dy)$$

$$= \int_0^1 (2i - y)\,dy$$

$$= \left[2iy - \frac{y^2}{2}\right]_0^1$$

$$I_2 = 2i - \frac{1}{2}$$

$I_3 =$ CD $\quad$ x varies 1 to 0

y varies 1 to 1

$$I_3 = \int_{CD} (x + (1+i)y)(dx + i\,dy) \qquad y = 1,\ dy = 0$$

$$= \int_{CD} (x + 1 + i)(dx)$$

$$= \int_1^0 (x + 1 + i)\,dx$$

$$= \left[\frac{x^2}{2} + x + ix\right]_1^0$$

$$= -\left[\frac{1}{2} + 1 + i\right]$$

$$= -\left[\frac{3}{2} + i\right]$$

$$= -\frac{3}{2} - i$$

$I_4 =$ D A $\quad$ $x$ varies from $0, 0$ $\quad x=0 \quad dx=0$

$= \int_{DA} (x+1+iy)(dx+idy)$ $\quad y$ varies from 1 to 0

$= \int_{DA} (1+iy)(idy)$

$= \int_1^0 (i-y)\,dy$

$= \left[iy - \frac{y^2}{2}\right]_1^0$

$= -\left[i - \frac{1}{2}\right]$

$= -i + \frac{1}{2}$

$I = I_1 + I_2 + I_3 + I_4$

$= \cancel{\frac{3}{2}} + 2i - \cancel{\frac{1}{2}} - \cancel{\frac{3}{2}} - i - i + \cancel{\frac{1}{2}}$

$= 0.$

* Evaluate $\int_{(1,1)}^{(2,4)} (2x^2+4xy)\,dx + (2x^2-y^2)\,dy$ along the curve $y = x^2$

* P.T $\int_{-2}^{-2+i} (2+z)^2\,dz = \frac{-i}{3}$

# Cauchy's Theorem (or) Cauchy's Fundamental theorem

(or) Integral theorem

~~**Statement:-** If $f(z)$ is analytic inside and on closed curve $C$ of a simply connected region $R$ and if $a$ is any point with in $C$ then $f(a) = \frac{1}{2\pi i}\int \frac{f(z)dz}{z-a}$.~~

**Statement:-** If $f(z)$ is analytic inside and on closed curve $C$ and first derivative $f'(z)$ exist and continuous on the closed curve $C$ then

$$\int_C f(z)\,dz = 0.$$

**Proof:-**

$f(z)$ is analytic on closed curve $C$

$f'(z)$ is exist and continuous on $C$

$\therefore$ By CR eqn. $\therefore u_x, v_x, u_y, v_y$ exists and continuous

$$u_x = v_y, \quad u_y = -v_x.$$

$$z = (x+iy) \Rightarrow dz = dx + i\,dy.$$

$$w = f(z) = u(x,y) + iv(x,y)$$

$$I = \int_C u(x,y) + iv(x,y)(dx + i\,dy)$$

$$= \int_C [u(x,y)dx + i\,u(x,y)dy + i\,v(x,y)dx - v(x,y)dy$$

$$= \underbrace{\int_C [u(x,y)dx - v(x,y)dy]}_{I_1} + i\underbrace{\int_C [u(x,y)dy + v(x,y)dx}_{I_2}$$

By Green's theorem on $XY$ plane.

$$\int_C P\,dx + Q\,dy = \iint_R \left(\underbrace{\frac{\partial Q}{\partial x}}_{Q_x} - \underbrace{\frac{\partial P}{\partial y}}_{P_y}\right) dx\,dy$$

In $I_1$, $P = u(x,y)$, $Q = -v(x,y)$
In $I_2$ $P = v(x,y)$, $Q = u(x,y)$

$$I = \iint_R [-v_x - u_y]\,dx\,dy + i\iint_R (u_x - v_y)\,dx\,dy$$

$$= 0 + i0$$

$$= 0.$$

$$= RHS.$$

**Cauchy's theorem for doubly connected region**

**Statement:** If $f(z)$ is analytic in doubly connected region are bounded by the simply closed curves $C_1$ & $C_2$ then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

**Cauchy's theorem for multi curves**

**Statement:** If $f(z)$ is analytic on the finite no. of closed curves $C_1, C_2 - - - C_n$ then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots \int_{C_n} f(z)dz$$

**Cauchy's Integral formula**

**Stl** If $f(z)$ is analytic with in (inside) and on the closed curve $C$ in a simply connected region and if $a$ is any point with in $C$ then $f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}dz$. (Define index of a point a)
where integration around the curve $C$ being taken in the the direction.

**Proof:** $f(z)$ is analytic with in on closed curve $C$ in a simple connected region

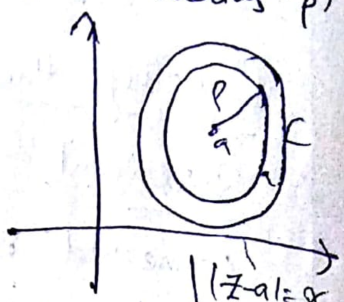
Also 'a' is any point inside C.

~~$f(a) = \frac{1}{2\pi i}\int$~~

$f(z)$ is analytic on C

$\therefore \frac{f(z)}{z-a}$ is also analytic on C. Except at $z=a$

Consider a circle whose centre at 'a' and radius 'ρ'




$\Rightarrow |z-a| = \rho$

$z-a = \rho e^{i\theta}$

$z = a+\rho e^{i\theta}$

$dz = \rho i e^{i\theta} d\theta$

$|z-a| = r$

$|z-a| = \rho e^{i\theta}$

By Cauchy's theorem for doubly connected region C & $C_1$

$$\int_{C} \frac{f(z)dz}{z-a} = \int_{C_1} \frac{f(z)}{z-a} dz.$$

$$= \int_{0}^{2\pi} \frac{f(a+\rho e^{i\theta})}{\cancel{\rho e^{i\theta}}} \cdot \cancel{\rho} i \times \cancel{e^{i\theta}} d\theta$$

$\{C_1$ is a circle $\theta \to 0$ to $2\pi$)

$= \oint \quad \rho \to 0 \quad C_1 \to 0$

$$= \int_{0}^{2\pi} f(a)\, i\, d\theta$$

$$= i f(a) \int_{0}^{2\pi} d\theta$$

$$= i 2\pi f(a)$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-a} dz = f(a)$$

$$\boxed{f(a) = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-a} dz.}$$

Cauchy's integral formula for derivative.

Note: ~~coa~~ Cauchy's integral formula is only valid for inside points, for outside points the index value is simply '0'.

Stm: If $f(z)$ is analytic with in and on a simple closed curve C and a is any point lying inside it then $f'(a) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}dz$.

Proof: $f(z)$ is analytic on closed curve C on simply connected region.

Also 'a' is any point inside C.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz.$$

diff w.r.t. a

$$f'(a) = \frac{d}{da}[f(a)] = \frac{1}{2\pi i}\frac{d}{da}\left[\int_C \frac{f(z)}{(z-a)}dz\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\partial}{\partial a}\left[\frac{f(z)}{(z-a)}\right]dz$$

$$= \frac{1}{2\pi i}\int_C f(z)dz\left[\frac{-1(-1)}{(z-a)^2}\right]$$

$$\boxed{f'(a) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^2}}$$

$$f''(a) = \frac{2}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz.$$

For $n^{th}$ derivative

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

* $$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

$$\int \frac{f(z)}{(z-a)^2}dz = \frac{2\pi i}{1!}f'(a)$$

$$\int_C \frac{f(z)}{(z-a)^3}dz = \frac{2\pi i}{2!}f''(a)$$

* Working rule:-

Case (i): For Outside points use cauchy's Integral theorem, the integration value is '0'.

Case (ii): For Inside points use cauchy's integral formula for the case of two or more inside points use partial fractions.

Case (iii):-

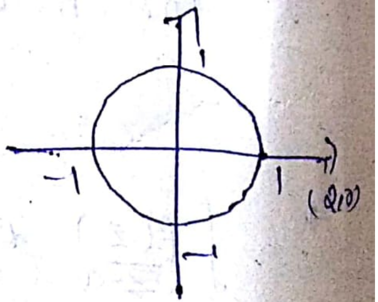
Combination of inside and outside points use cauchy's integral formula but not partial fractions. The integration value can be obtained by taking the outside point to the Numerator

* Evaluate $I_1 = \int_c \frac{z\,dz}{(z-2)}$ c is $|z|=1$

$I_2 = \int_c \frac{dz}{(2z-3)}$ c is $|z|=1$

Sol:- ① By Comparing $I_1$ with

$\int_c \frac{f(z)\,dz}{(z-a)}$




$f(z) = z$, $a=2$.

$|z|=1$

radius $=1$

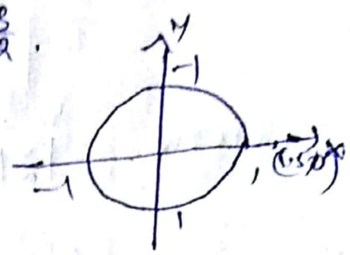
Centre $= (0,0)$

$\therefore a=2$ is a out side point

By cauchy's theorem

$f(z)$ is analytic
$f(z)$ is const
$\therefore \int f(z) \cdot dz = 0.$

(ii) $f(z) = 1, \quad a = \frac{3}{2}.$

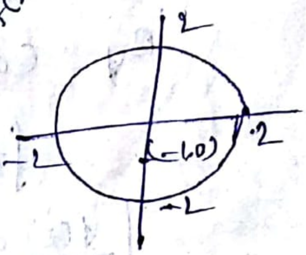



$|z| = 1$
radius $= 1$
Centre $= (0,0)$

$a = \frac{3}{2}$ is a outside point.
By cauchy's theorem.
$f(z)$ is analytic
$f(z)$ is const
$\therefore \int f(z) dz = 0.$

$I_3 = \int_C \frac{z^2 dz}{(z+1)} \quad |z| = 2.$

$f(z) = z^2$




Here $f(z) = z^2$
$a = -1$
$|z| = 2$ radius $= 2$
centre $= (0,0)$
Since $a$ lies inside the region,
By cauchy's integral formula

$\int_C \frac{f(z) dz}{(z-a)} = \frac{2\pi i}{0!} f(a).$ $\qquad z^2 = (-1)^2 = 1$

$\int_C \frac{z^2 dz}{(z+1)} = 2\pi i f(-1)$
$= 2\pi i (1)$

$\int \frac{z^2 dz}{(z+1)} = 2\pi i$

* $I_4$ = with out using Cauchy's residue theorem

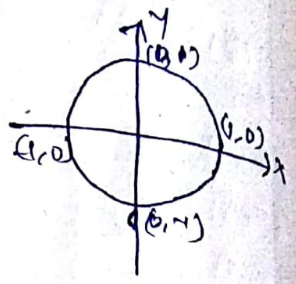
Evaluate $\int_C \frac{dz}{e^z z^2}$

② $\int_C z^2 e^{+1/z}\, dz$ where over the same curve $|z|=1$.

Soln.

① $I_1 = \int \frac{dz}{e^z z^2} = \int \frac{e^{-z}\, dz}{z^2}$.

$f(z) = \frac{1}{\cancel{e^z z^2}} \quad e^{-z}$



Dr = 0

$z^2 = 0$

$z = 0, 0$

$z = 0$ inside 'C'

~~By Cauchy's theorem,~~

$\cancel{\int f(z)\, dz = 0}$

$\int \frac{f(z)}{(z-a)^2} dz = \frac{2\pi i}{1!} f'(a)$

$f(z) = e^{-z}$

$f'(z) = -e^{-z}$

$f'(0) = -e^{-0}$

$= -1$

$\int \frac{e^{-z}}{z^2} dz = \frac{2\pi i}{1!}(-1)$

$= -2\pi i$

$e^{\frac{1}{z}} = 0$

$\log \frac{1}{z} = 0$

$\frac{1}{z} = 0$

$z = 0$

$\frac{1}{z}$

② $I_1 = \int \frac{z^2}{e^{-\frac{1}{z}}} dz$

$f(z) = z^2$

Dr = 0

$e^{-\frac{1}{z}} = 0$ if $z = 0$

$e^{-\infty} = 0$

$\therefore a = 0$ inside C.

$$I_2 = 2\pi i \int_C \frac{f(z)\,dz}{(z-a)} = 2\pi i\, f(a) \qquad f(z) = z^2, \quad f = 0$$

$$\int_C \frac{z^2\,dz}{(z-0)} = 2\pi i\, 0$$

$$= 0.$$

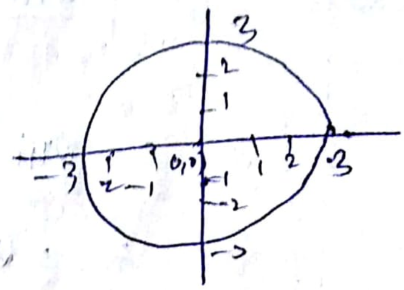
* Without using cauchy's residue theorem evaluate ~~integral over~~ C

$$I_1 = \int_C \frac{z^2\,dz}{(z+1)^2(z+2)} \quad \text{where C is simple closed}$$

curve given by $|z| = 3$

Sol:

$$I = \int_C \frac{z^2\,dz}{(z+1)^2(z+2)}$$



$f(z) = z^2$

$a = 2r = 0$

$(z+1)^2(z+2) = 0$

$(z+1)^2 = 0, \quad z+2 = 0$

$z = -1, -1, \quad z = -2$

$$\frac{1}{(z+1)^2(z+2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z+2}$$

~~put $z = -1$~~

$1 = A(z+1)(z+2) + B(z+2) + C(z+1)^2$

$A(z^2+2z+z+2) \qquad z^2+1+2z$

$z^2+3z+2$

$z = -1 \qquad\qquad$ put $z = -2$

$1 = B(1) \qquad\qquad 1 = C(-1)^2$

$B = 1 \qquad\qquad C = 1$

$Az^2 + Cz^2 = 0$

$A + C = 0$

$A = -1$

$$\frac{1}{(z+1)^2(z+2)} = \frac{-1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)}$$

$f(z) = z^2$

$f(-1)$

$$I = \int_C \frac{f(z)}{(z-a)} dz.$$

$$= \int_C -\frac{f(z)\,dz}{z+1} + \int_C \frac{f(z)}{(z+1)^2} dz + \int_C \frac{f(z)}{z+2} dz$$

$$= \int_C \frac{-z^2}{z+1} dz + \int_C \frac{z^2}{(z+1)^2} dz + \int_C \frac{z^2}{(z+2)} dz$$

$$= -2\pi i\, f(-1) + \frac{2\pi i}{1!} f'(-1) + \frac{2\pi i}{1!} f(-2)$$

$$= -2\pi i + 2\pi i\,(-2) + 2\pi i\; 4$$

$f(z) = z^2$

$f'(z) = 2z$

$f(-1) = 1$

$$= -2\pi i - 4\pi i + 8\pi i$$

$f(-2) = 4$

$$\underline{I} = 2\pi i$$

* Use cauchy's integral formula, evaluate $\int_C \frac{(\sin \pi z^2 + \cos \pi z^2)\, dz}{(z-1)(z-2)}$ where $C$ is $|z| = 3$.

Sol:-

$$I = \int \frac{(\sin \pi z^2 + \cos \pi z^2)\, dz}{(z-1)(z-2)}$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$

$Dr = 0$

$(z-1)(z-2) = 0$

$z = 1, \; z = 2.$

$\therefore$ points lie inside the curve

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

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$1 = A(z-2) + B(z-1)$

Put $z = 2$     put $z = 1$

$B = 1$     $A = -1$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$I = \int_C \frac{-f(z)\,dz}{z-1} + \int_C \frac{+f(z)\,dz}{(z-2)}$$

$f(z) = \sin\pi z^2 + \cos\pi z^2$

$f(1) = \sin\pi + \cos\pi$

$= -1$

$f(2) = \sin 4\pi + \cos 4\pi$

$= 0 + 1$

$$= -2\pi i\, f(+1) + 2\pi i\, f(+2)$$

$$= -2\pi i(-1) + 2\pi i(1)$$

$$= +2\pi i + 2\pi i$$

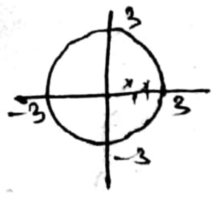
$$I = 4\pi i$$

Hw: (1) $\int_C \frac{e^{2z}\,dz}{(z-1)(z-2)}$    C: $|z| = 3$.

(2) $\int_C \frac{e^{z}\,dz}{(z+2)(z+1)^2}$    $|z| = 3$

(3) $\int_C \frac{e^{-2z}\,dz}{(z+1)^4}$    $|z| = 2$

1A) $I = \int_C \frac{e^{2z}\,dz}{(z-1)(z-2)}$    $|z| = 3$



$f(z) = e^{2z}$    $Dr = 0$

$z = 1, 2$

∴ points lies inside the curve

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$    $A = -1,\ B = +1$

$A, B$

$$\frac{-1}{z-1} + \frac{1}{z-2}$$

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$$I = \int \frac{-f(z)\,dz}{z-1} + \int \frac{f(z)\,dz}{z-2}$$

$$= \int \frac{-e^{2z}}{z-1}\,dz + \int \frac{e^{2z}}{z-2}\,dz \qquad f(z) = e^{2z}$$

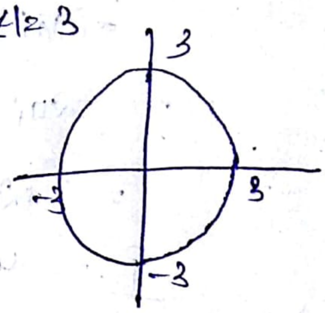
$$f(1) = e^2$$

$$= -2\pi i\, f(1) + 2\pi i\, f(2)$$

$$= -2\pi i\, e^2 + 2\pi i\, e^4$$

$$= 2\pi i\, e^2(-1 + e^2)$$

2A) $I = \int_C \frac{e^z\,dz}{(z+2)(z+1)^2} \quad |z| = 3$




$(z+2)(z+1)^2 = 0$

$z = -2, -1, -1$

$$I = \cancel{\int \frac{f(z)\,dz}{(z+2)} + \int \frac{f(z)\,dz}{(z+1)^2}}$$

$$\frac{1}{(z+2)(z+1)^2} = \frac{A}{z+2} + \frac{B}{(z+1)} + \frac{C}{(z+1)^2}$$

$$1 = A(z+1)^2 + B(z+2)(z+1) + C(z+1)$$

$$Az^2 + B + 2Az + Bz^2 + 3z + 2$$

Put $z = -2$ $\qquad$ Put $z = -1$

$A = 1$ $\qquad$ $C = 1$

$A + B = 0$ $\qquad$ $f(z) = e^z$

$A = -B$ $\qquad$ $f'(z) = e^z$

$B = -1$

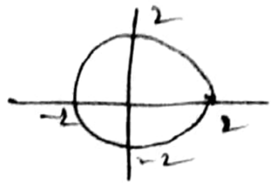
$$I = \int \frac{f(z)\,dz}{z+2} - \int \frac{f(z)\,dz}{z+1} + \int \frac{f(z)\,dz}{(z+1)^2}$$

$$= 2\pi i\, f(-2) - 2\pi i\, f(-1) + 2\pi i\, f'(-1)$$

$= 2\pi i\, e^{-2} - 2\pi i\, e^{-1} + 2\pi i\, e^{-1}$

$= 2\pi i\, e^{-2}$

3A) $I = \int \frac{e^{-2z}\,dz}{(z+1)^4}$ $\quad |z| = 2$.




$z = -1, -1, -1, -1$

$\cancel{I = \int e^{-2z}\,dz}$

$$\frac{1}{(z+1)^4} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{(z+1)^3} + \frac{D}{(z+4)^4}$$

$$1 = A(z+1)^3 + B(z+1)^2 + C(z+1) + D$$

$$= A(z^3+1+3z^2+3z) + B(z^2+2z+1) + C(z+1) + D$$

$A = 0, \quad 3A + B = 0$

$B = 0$

$$= \frac{2\pi i}{3!} f'''(-1)$$

$f(z) = e^{-2z}$

$f'(z) = -2e^{-2z}$

$f''(z) = 4e^{-2z}$

$f'''(z) = -8e^{-2z}$

$f'''(-1) = -8e^{-2(-1)}$

$= -8e^{2}$

$$= \frac{2\pi i}{3!}(-8)e^{2}$$

$$= -\frac{2\pi i(8)e^{2}}{6_{3}}$$

$$= -\frac{8}{3}\pi i e^{2}$$

* $\int_{(1,1)}^{(2,4)} (2x^2+4xy)\,dx + (2x^2-y^2)\,dy$. $\quad y = x^2 \quad x = \sqrt{y}$

$dy = 2x \quad x = \sqrt{y}$

$\cancel{\int_1^4 2x} \quad \int_1^4 (2x^2+4x^3)\,dx + (2x^2-x^4)\,dy\,2x$

$\cancel{\int_1^4 2y} + \int_1^4 2x^2\,dx + \int_1^4 4x^3\,dx + \int_1^4 4x^3 - \int_1^4 2x^5$.

$$2\left[\frac{x^3}{3}\right]_1^4 + 4\left[\frac{x^4}{4}\right]_1^4 + 4\left[\frac{x^4}{4}\right]_1^4 - 2\left[\frac{x^6}{6}\right]_1^4$$

$$= 2\left[\frac{64}{3} - \frac{1}{3}\right] + 4\left[\frac{256}{4} - \frac{1}{4}\right] + 4\left[\frac{256}{4} - \frac{1}{4}\right] - 2\left[\frac{4^6}{6} - \frac{1}{6}\right]$$

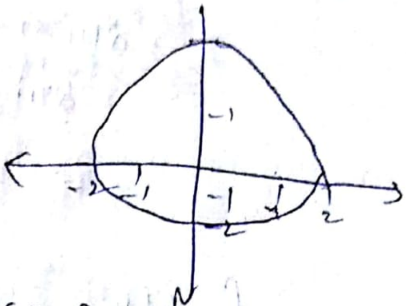
$$= 2\left[\frac{63}{3}\right] + 4\left[\frac{255}{\cancel{4}}\right] - \cancel{4}\left[\frac{255}{\cancel{4}}\right] - 9\left[\frac{3}{6/3}\right]$$

$$= 551$$

* $\int \frac{1}{z^2-1}$ , $|z|=2$

Sol: Centre $(0,0)$

$r = 2$



$(z-1)(z+1)=0$

$z = 1, -1$

$\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1} = A(z-1)+B(z+1)$

If $z=1$, $1 = B(2) = B=\frac{1}{2}$

If $z=-1$, $1 = A(-1-1) = A = \frac{-1}{2}$

$\int \frac{-1}{2(z-1)} dz + \int \frac{1}{2(z+1)} dz$

$= \frac{-1}{2} \int \frac{1 dz}{z-1} + \frac{1}{2} \int \frac{1 dz}{z+1}$

$= \frac{-1}{2} \times 2\pi i \times f(1) + \frac{1}{2} \times 2\pi i \times f(-1)$

$= -\pi i \times 1 + \frac{1}{2} \times 2\pi i \times -1$

$= -2\pi i$

* $\int \frac{z^2+1}{z^2-1} dz$ (=) $|z-1|=1$

Sol: $|z-(0+0i)|=1$

$f(z)=z^2$

$Dr=0$

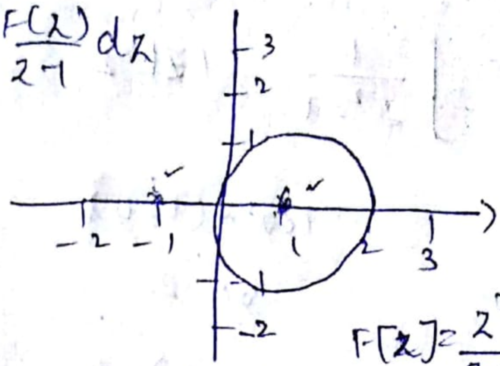
$(z+1)(z-1)=0$

$z=1,-1$

centre $=(1,0)$

radius $= 1$

$$\int \frac{\left[\frac{z^2+1}{z+1}\right]}{z-1} dz = \int \frac{F(z)}{z-1} dz$$



$= 2\pi i \times F[1]$

$= 2\pi i \times 1$

$= 2\pi i$

$F[z] = \frac{z^2+1}{z+1}$

$F[1] = \frac{1+1}{1+1} = 1$

* $\int \frac{z+4}{z^2+2z+5}$ $|z+1+i| = 2$

Sol:

$|z-(-1-i)| = 2$

$f(z) = z+4$ Centre = $(-1,-1)$

radius = 2

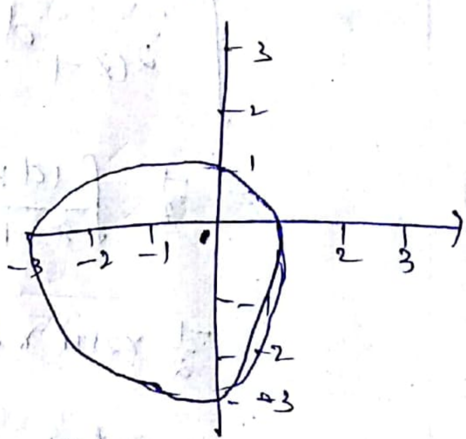
$D = 0$

$z^2+2z+5 = 0$

$z = \frac{-2 \pm \sqrt{4-20}}{2}$

$= \frac{-2 \pm \sqrt{-16}}{2}$

$= \frac{-2 \pm 4i}{2}$



$= -1 \pm 2i$

$= -1+2i, -1-2i$

$(-1,2)$ $(-1,-2)$

$P_1 = -1+2i$

$P_2 = -1-2i$

$$\int \frac{\left[\frac{z+4}{z-P_1}\right]}{z-P_2} dz = \int \frac{F(z)}{[z-P_2]} dz$$

$F[P_2] = \frac{-1-2i+4}{-1-2i+1-2i}$

$= \frac{3-2i}{-4i}$

$= 2\pi i \times F(P_2)$

$= 2\pi i \left[\frac{3-2i}{-4i}\right]$

# Taylor's series and Laurent's series

A function $f(z)$ is analytic about the point $z=a$ in the circle $C$, then $f(z)$

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots \quad (1)$$

Put $z=0 \Rightarrow (0,0)$

$$f(z) = f(0) + \frac{f'(0)}{1!}(z) + \frac{f''(0)}{2!} \times z^2 + \cdots \frac{f^n(0)}{n!} \times z^n + \cdots \infty \quad (2)$$

(Taylor series about origin → Maclaurin series)

**Note:** In eqn (1) there is no negative powers of $(z-a)$, If the series contains positive and negative powers of $(z-a)$ then the corresponding series is called Laurent's series

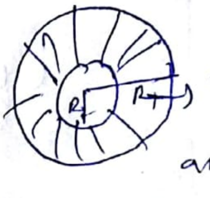
$C_1$ and $C_2$ be two concentric circles,

$|z-a| = R_1$ , $R_1 > R_2$

$|z-a| = R_2$

then A function $f(z)$ is analytic b/w the regions of concentric circles

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{A} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{B}$$




annular region or annulus

is the Laurent's series where A is called regular part / analytic part of Laurent series

B is called principle part of Laurent's series

where the values of $a_n$ and $b_n$ are given by

$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)\,dz}{(z-a)^{n+1}} \;;\; b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)\,dz}{(z-a)^{1-n}}$$

* find the Laurent series expansion of f $f(z) = \frac{z}{(z+1)(z+2)}$ about the region $z=-2$.

$$f(z) = \frac{z}{(z+1)(z+2)}$$

By partial fraction

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$z = A(z+2) + B(z+1)$$

If $z=-1$; $-1 = A(-1+2) \Rightarrow -1 = (A)$
$A = -1$

If $z=-2$; $-2 = B(-2+1) \Rightarrow -2 = -B \Rightarrow B = 2$

$$f(z) = \frac{-1}{z+1} + \frac{2}{z+2} \qquad z=-2$$
$$z+2=0$$

$$= \frac{-1}{(z+1+1)-1} + \frac{2}{z+2}$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= \frac{1}{[1-(z+2)]} + \frac{2}{z+2} = [1-(z+2)]^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$$

$$1+(z+2)+(z+2)^2+(z+2)^3+\cdots$$

$$\sum_{n=0}^{\infty} (z+2)^n + \frac{2}{z+2}$$

* $f(z) = \frac{z}{z^2-3z+2}$ (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $|z-1|<1$

Soln

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$z = A(z-2) + B(z-1)$$

If $z = 2$, $2 = B(2-1) \Rightarrow B = 2$

$z = 1$, $1 = A(1-2) \Rightarrow A = -1$

$$f(z) = \frac{-1}{z-1} + \frac{2}{z-2}$$

$1 < |z| < 2$ — $1 < |z| \Rightarrow \left|\frac{1}{z}\right| < 1$

— $|z| < 2$ $\left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{2}{z-2} - \frac{1}{z-1} = \frac{2}{-2\left[1-\left(\frac{z}{2}\right)\right]} - \frac{1}{z\left[1-\left(\frac{1}{z}\right)\right]}$$

$$= -1\left[1-\left(\frac{z}{2}\right)\right]^{-1} - \frac{1}{z}\times\left[1-\frac{1}{z}\right]^{-1}$$

$$= -1\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$= -1\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -1\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$$

(i) $|z| < 1$

$$f(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

$$= \frac{2}{-2\left[1-\frac{z}{2}\right]} + \frac{1}{1-z}$$

$$= -\left[1-\frac{z}{2}\right]^{-1} + \left[1-z\right]^{-1}$$

$$= -\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right] + \left[1+z+z^2+\cdots\right]$$

$$= -\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty}(z)^n$$

$|z| > 2$

$2 > |z| \Rightarrow \left|\frac{2}{z}\right| < 1$

Note- take the denominator term outside for the factors.

$$f(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

$$= \frac{2}{z\left[1-\frac{2}{z}\right)} - \frac{1}{z\left[1-\frac{1}{z}\right]}$$

$$= \frac{2}{z}\left[1-\frac{2}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{2}$$

$$= \frac{2}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$= \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \sum_{n=0}^{\infty} z^{-n}$$

(iv) $|z-1| < 1$

$u = z-1$

$$f(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

$$= \frac{2}{u-2} - \frac{1}{u} \quad (|z-1| < 1)$$

$$|u| < 1$$

$$= \frac{-2}{1-u} - \frac{1}{u}$$

$$= -2[1-u]^{-1} - \frac{1}{u}$$

$$= -2[1+u+u^2+\cdots] - \frac{1}{u}$$

$$= -2\left[1+(z-1)+(z-1)^2+\cdots\right] - \frac{1}{u}$$

$$= -2\sum_{n=0}^{\infty}(z-1)^n - \frac{1}{u}$$

* Expand Laurent's series for the func

$$f(z) = \frac{(z^2-1)}{(z^2+5z+6)}$$

(i) between 2 & 3 $(2<|z|<3)$

(ii) within 2 $[\,|z|<2\,]$

(iii) outside 3 $[\,|z|>3\,]$

Sol: deg Nr = deg Dr.

$$z^2+5z+6 \overline{)\; z^2-1 \;}\; 1$$
$$\underline{z^2+5z+6}$$
$$-5z-7$$

$$f(z) = Q + \frac{R}{D}$$

$$f(z) = 1 + \frac{(-5z-7)}{z^2+5z+6}$$

$$= 1 - \frac{(5z+7)}{z^2+5z+6}$$

$$\frac{5z+7}{z^2+5z+6} = \frac{A}{z+2} + \frac{B}{z+3}$$

$z = -2$

$5(-2)+7 = A(-2+3)+0$

$-10+7 = A$

$A = -3.$

$z = -3$

$5(-3)+7 = B(-3+2)$

$-15+7 = -B$

$-B = -8$

$B = 8.$

$$f(z) = 1 - \left[\frac{-3}{z+2} + \frac{8}{z+3}\right]$$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3}.$$

(i) $2<|z|<3$ — $2<|z| \Rightarrow \left|\frac{2}{z}\right|<1$ ; $|z|<3 \;\; \left|\frac{z}{3}\right|<1$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

By binomial theorem

$$f(z) = 1 - \left(\frac{3}{z}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \left(\frac{8}{3}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(ii) $|z|<2$

$\left|\frac{z}{2}\right|<1$

$$f(z) = 1 + \frac{3}{2\left[\frac{z}{2}+1\right]} - \frac{8}{3\left[1+\frac{z}{3}\right]}$$

$$= 1 + \frac{3}{2}\left[\frac{z}{2}+1\right]^{-1} - \frac{8}{3}\left[1+\frac{z}{3}\right]^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \dots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \dots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(iii) $|z|>3$

$\left|\frac{3}{z}\right| < 1$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(\frac{3}{z}+1\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(\frac{3}{z}+1\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \dots\right] - \frac{8}{z}\left(1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \dots\right)$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n.$$

(1) $f(z) = \dfrac{2z-5}{z(z^2-z-2)}$ $\quad |z|>2$.

(2) $f(z) = \dfrac{z^2}{(z+2)(z-3)}$ $\quad$ (i) $|z|<2$ $\quad$ (ii) $2<|z|<3$

HW

(3) $f(z) = \dfrac{4z+4}{z(z-3)(z+2)}$ (i) $|z|<1$

(ii) $|z|=2$ & $|z|=3$.

(iii) $|z|>3$.

(1) $f(z) = \dfrac{2z-3}{z(z^2-z-2)}$

$z^2-z-2$

$\dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(-2)}}{2}$

$\dfrac{1\pm\sqrt{1+8}}{2}$

$\dfrac{1+3}{2}, \dfrac{1-3}{2}$

$\dfrac{4}{2}, \dfrac{-2}{2}$

$2, -1$

$$\frac{2z-3}{z(z^2-z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}$$

$$2z-3 = A(z+1)(z-2) + B(z)(z-2) + C(z+1)(z)$$

Put $z=0$ | Put $z=-1$ | $z=2$

$-3 = -2A$, $A = \dfrac{3}{2}$

$-2-3 = -B(-3)$, $-5 = 3B$, $B = \dfrac{-5}{3}$

$4-3 = 2C(3)$, $1 = 6C$, $C = \dfrac{1}{6}$

$$f(z) = \frac{3}{2(z)} - \frac{5}{3(z+1)} + \frac{1}{6(z-2)}$$

$|z|<2$, $\left|\dfrac{z}{2}\right|<1$

$$= \frac{3}{2z} - \frac{5}{3}(1+z)^{-1} + \frac{1}{6(-2)\left(1-\frac{z}{2}\right)}$$

$$= \frac{3}{2z} - \frac{5}{3}(1+z)^{-1} - \frac{1}{12}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{3}{2z} - \frac{5}{3}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{12}$$

$$= \frac{3}{2z} - \frac{5}{3}\left[1-z+z^2-z^3\cdots\right] - \frac{1}{12}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$= \frac{3}{2z} - \frac{5}{3}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{12}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

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(8) $f(z) = \frac{z^2}{(z+2)(z-3)}$

$z^2 - 3z + 2z - 6$

$z^2 - z - 6$

deg of Nu = deg of Dr

$$z^2 - z - 6 \overline{)\, z^2 \,}\ 1$$

$$\frac{-z^2 - z - 6}{\quad z + 6}$$

$f(z) = Q + \frac{R}{D}$

$f(z) = 1 + \frac{+z+6}{z^2}$

$f(z) = 1 + \frac{(z+6)}{z^2}$

$\frac{z+6}{z^2} = \frac{A}{z} + \frac{B}{z^2}$

$z+6 = Az^2 + Bz$

Put $z = 0$ $A =$

$A = 0$ $B = 1$

$\frac{z+6}{z^2} = \frac{1}{z^2}$

$f(z) = 1 + \frac{z+6}{z^2 - z - 6}$

$\frac{z+6}{z^2 - z - 6} = \frac{A}{z+2} + \frac{B}{z-3}$

$z+6 = A(z-3) + B(z+2)$

Put $z = -2$

$-5A = 4$

$A = \frac{-4}{5}$

Put $z = 3$

$5B = 9$

$B = \frac{9}{5}$

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$$f(z) = 1 - \frac{4}{5(z+2)} + \frac{9}{5(z-3)}$$

$|z| < 2$

$\left|\frac{z}{2}\right| < 2$

$$= 1 - \frac{4}{5(2)\left(1+\frac{z}{2}\right)} + \frac{9}{(-3)5\left(1-\frac{z}{3}\right)}$$

$$= 1 - \frac{4}{10}\left(1+\frac{z}{2}\right)^{-1} - \frac{9}{15}\left(1-\frac{z}{3}\right)^{-1}$$

$$= 1 - \frac{4}{10}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right) - \frac{9}{15}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right)$$

$$= 1 - \frac{4}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{9}{15}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$2 < |z| < 3$ — $|z| > 2$, $\left|\frac{2}{z}\right| < 1$; $3 > |z|$; $|z| < 3$, $\left|\frac{z}{3}\right| < 1$

$$f(z) = 1 - \frac{4}{5z\left(1+\frac{2}{z}\right)} + \frac{9}{-5\cdot 3\left(1-\frac{z}{3}\right)}$$

$$= 1 - \frac{4}{5z}\left(1+\frac{2}{z}\right)^{-1} - \frac{9}{15}\left(1-\frac{z}{3}\right)^{-1}$$

$$= 1 - \frac{4}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{9}{5z}\left(\frac{z}{3}\right)^n$$

* Find the Laurent's series expansion for $f(z) = \frac{1}{z(z-1)}$ in a series powers of $z$ and also state the region of validity.

Sol:- $f(z) = \frac{1}{z(z-1)}$

(i) $|z| > 1$

(ii) $|z| < 1$

(iii) $|z| = 0$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$$= A(z-1) + B(z)$$

Put $z=0$ $\quad z=1$

$B=1$

$-A=1$

$A=-1$

$$f(z) = \frac{-1}{z} + \frac{1}{z-1}$$

(i) $|z|<1$

$$= \frac{-1}{z} - \frac{1}{(1-z)}$$

$$= \frac{-1}{z} - (1-z)^{-1}$$

$$= \frac{-1}{z} - \left[1+z+z^2+z^3 \ldots\right]$$

$$= \frac{-1}{z} - \sum_{n=0}^{\infty} z^n$$

(ii) $|z|>1$ $\quad 1>\frac{1}{z}$

$\left|\frac{1}{z}\right|<1$

$$f(z) = \frac{-1}{z} + \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{-1}{z} + \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{-1}{z} + \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+ \ldots\right]$$

$$= \frac{-1}{z} + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

(iii) $|z|=1$

$|z|<1$

$|z|<1$

$\frac{1}{z}\left[1+z+z^2+z^3\right]$.

$\frac{1}{z}=\sum_{n=0}^{\infty} z^n$.

Expand.

* $f(z)=\cos z$ about the point $z=\frac{\pi}{3}$.

* Taylor series at $z=a$.

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots$$

$f(z)=\cos z$ value at $\frac{\pi}{3}$

$f(z)$ $\cos\frac{\pi}{3}=\frac{1}{2}$.

$f'(z)=-\sin z$ $-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$.

$f''(z)=-\cos z$ $-\cos\frac{\pi}{3}=-\frac{1}{2}$.

$f'''(z)=\sin z$ $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$.

$$f(z)=\frac{1}{2}+\left(-\frac{\sqrt{3}}{2}\right)\left(z-\frac{\pi}{3}\right)+\frac{\left(-\frac{1}{2}\right)}{2!}\left(z-\frac{\pi}{3}\right)^2+\frac{\frac{\sqrt{3}}{2}}{3!}\left(z-\frac{\pi}{3}\right)^3+\cdots$$

# Singularities

f(z) is always analytic

Singularities

Zeros I — of f(z) — (Nr = 0)
- zero of order 1 (or) simple zero
- zero of order > 1

Singularities II of f(z) — series expansion — Singular point (poles)
- Regular singular points (RSP) ($Dr = 0$)
- Essential singular point (ESP)
- Isolated singular point (ISP)

evaluation of poles — III — residue of f(z). — Cauchy's residue theorem (1) (2)
- polar order 1
- polar order $n > 1$

(1) **Zeros:** A function $f(z)$ is said to be analytic and a point $z = a$ is called as simple zero if $f(a) = 0$, $f'(a) \neq 0$.

For higher order of the point $z = a$ satisfies $f(a) = 0$ but $f'(a) = 0$.

$f''(a) \neq 0$ zero of order 2

$f'''(a) \neq 0$ zero of order 3.

Consider: $f(z) = (z-1)(z-2)$

The zeros of $f(z)$ is given by Nr = 0

$(z-1)(z-2) = 0$

$z = 1, 2$

$f'(1) = 3 \neq 0$

$f'(2) = 8 \neq 0$.

zero of order 1

$z^2 - 2z - z + 2$

$z^2 - 3z + 2$

$f'(z) = 3z^2 - 3$

$f'(z) = (-1)(z-2) + (z-1)(\cdot)$

$= -z + 2 - 2z + 2$

$= -3z + 4$.

~~If~~ nth derivative is not zero but all the previous (n-1)th derivative is zero.

(II). Singularities of $f(z)$:- singular points (poles)

Rule:- $D_r = 0$.

A function $f(z)$ fails to analytic at the point $z=a$. is called as singula point. It can be classified in to 3 types.

(i) RSP

(ii) ESP

(iii) ISP

⇒ RSP:- A singular point $z=a$ is said to be ~~removable~~ regular removable singular point if the Lorent's series expansion contains infinite no. of positive (power) term (principal part = 0, $b_n = 0$) (Regular part or analytic part)

Eg: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ (infinite no. of powers +ve powers)

$\therefore f(z) = \sin z$

$\frac{\sin z}{1} = 0$

$\sin z = 0$

$\sin z = \sin n\pi$

$z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

RSP for $f(z) = \sin z$

⇒ ESP:- A singular point $z=a$ is said to be essential singular point if the Lorent's series expansion contains infinite no. of negative powers (Principal part) ($\therefore$ Regular part = 0, $a_n = 0$).

Eg: $f(z) = e^{1/z}$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ (infinite no. of negative powers)

$e^{1/z} = 1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \cdots = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \cdots$

$\therefore f(z) = e^{1/z}$

$\cancel{Dr=0}$ $Dr=0$.

$e^{1/z} = 0$.

$e^{1/z} = e^{-\infty}$

$\frac{1}{z} = -\infty$

$\frac{1}{-\infty} = z$

$\underline{z=0}$

$\sin(\frac{1}{z})$ is also an example of ESP.

$z = \pm\frac{1}{n\pi}$.

$\Rightarrow$ **ISP:-**

A singular point $z=a$ is said to be isolated singular point if

(i) $f(z)$ is not analytic at $z=a$

(ii) but $f(z)$ is analytic at some neighbourhoods of 'a'.

Eg: $f(z) = \frac{1}{(z-1)(z-2)}$.

Singular points are $z = 1, 2$

$f(z)$ is not analytic at $z=1,2$.

Because $f(z) = \frac{1}{0}$

but neighbouring points $z = 0.5, 0.6, 0.7$.

$z = 2.1, 2.2, 2.3$.

$f(z)$ is analytic

$z=1$ is ISP.

Note:-

ISP ⟹ is said to be

(i) RSP (or) → if it has no infinite no. of $-ve$ powers.

(ii) ESP (or) → if it has infinite no. of $-ve$ powers.

(iii) poles (or) (Don't) → if it has finite no. of $-ve$ power

Note:-

Poles of order 1 ⟹ singular point of order 1.

(order → power) & In other words In this case $f(z)$ has a R.S.P

$f(z)$ has no zeroes.

Note:-

If $f(z)$ has poles of order (power) > 1

then find $\lim_{z \to 0} f(z) \neq 0$. (or) $\infty$

Note:-

Evaluation of poles are known as residue.

Problems on Zeroes

* Find the zeroes of $f(z) = \frac{z^2+1}{z^2-1}$

Sol: To find zeros

$Nr = 0$

$z^2+1=0$

$z^2=-1$

$z=\pm i$

$$f'(z) = \frac{(z^2-1)(2z)-(z^2+1)(2z)}{(z^2-1)^2}$$

$$= \frac{2z^3-\cancel{2z}-2z^3-\cancel{2z}}{(z^2-1)^2} = \frac{-4z}{(z^2-1)^2}$$

$$f'(i) = \frac{-4i}{(i^2-1)^2} = \frac{-4i}{(-1-1)^2} = \frac{-4i}{4} = -i$$

$\therefore$ $z=i$ is a zero of order 1 (or) simple zero

$$f'(-i) = \frac{-4(-i)}{((-i)^2-1)^2} = \frac{4i}{4} = i$$

$\therefore$ $z=-i$ is a zero of order 1 (or) simple zero

* Find the zeroes of $f(z) = \frac{z^3+1}{z^3-1}$

Sol:
$Nr = 0$

$z^3+1=0$

$z^3=-1$

$z=(-1)^{1/3}$

$=(\cos\pi + i\sin\pi)^{1/3}$

Take $2k\pi$ to arg.

$= \left(\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right)^{1/3}$

$k=0,1,2$

$= \cos\left(\frac{(2k+1)\pi}{3}\right) + i\sin\left((2k+1)\frac{\pi}{3}\right)$

De moivre's the

$k=0 \Rightarrow \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2}+i\frac{\sqrt{3}}{2}$

$k=1 \Rightarrow \cos\pi + i\sin\pi = -1$

$k=2 \Rightarrow \cos\frac{5\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$f'(z) = \frac{(3z^2)(z^3-1)-(z^3+1)(3z^2)}{(z^3-1)^2}$$

$$= \frac{3z^5-3z^2-3z^5-3z^2}{(z^3-1)^2}$$

$$= \frac{-6z^2}{(z^3-1)^2}$$

$f'(-1) = \dfrac{-6(-1)^2}{((-1)^3-1)^2}$

$= \dfrac{-6}{(-1-1)^2} = \dfrac{-6}{4} = \dfrac{-3}{2}$

$\therefore z=-1$ is a zero of order 1 (or) simple zero,

$f'\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \dfrac{-6\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]}{\left[\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3-1\right]^2}$

$= \dfrac{-3[1+i\sqrt{3}]}{\left[[1+i\sqrt{3}]^3-2\right]^2} \times 2$

$= \dfrac{12[1+i\sqrt{3}]}{\left((1+i\sqrt{3})^3-2\right]^2}$

$f'\left[\frac{1}{2}-i\frac{\sqrt{3}}{2}\right] = \dfrac{-6\left[\frac{1}{2}-i\frac{\sqrt{3}}{2}\right]}{\left[\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]^3-1\right]^2}$

$= \dfrac{-6\left[\frac{2}{2}1-i\sqrt{3}\right) \times 2}{\left[[1+i\sqrt{3}]^3-1\right]^2}$

Hw $f(z) = \dfrac{z^3-1}{z^3+1}$

* $\sin\left(\dfrac{1}{z-a}\right)$

Sol:- Nr = 0

$\sin\left(\dfrac{1}{z-a}\right) = \sin n\pi$

$\dfrac{1}{z-a} = n\pi$, $n = \pm1, \pm2, \pm3 \ldots$

$\dfrac{1}{n\pi} = z-a$

$z = \dfrac{1}{n\pi}+a$, $n = \pm1, \pm2 \ldots\ldots$ (infinite no. of zeros)

$\therefore$ All the zeros are simple zeros.

* Find the zeroes of $\dfrac{\sin z - z}{z^3}$

Sol: $\dfrac{\sin z - z}{z^3} = \dfrac{\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots - z}{z^3}$

$$= \frac{\left[-\frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}{z^3}$$

$$= \frac{z^3\left[-\frac{1}{3!} + \frac{z^2}{5!} - \cdots\right]}{z^3}$$

$$= \left[-\frac{1}{3!} + \frac{z^2}{5!} \cdots\right]$$

$\therefore f(z) = \dfrac{\sin z - z}{z^3}$ has infinite zeroes.

* Find the zeroes of $f(z) = \dfrac{1-e^{2z}}{z^4}$

Sol: $Nr = 0$

$1 - e^{2z} = 0$

$1 = e^{2z}$

$= e^0, \cos 2n\pi + i\sin 2n\pi$

$e^{i2n\pi}$

$n = 0, \pm 1, \pm 2, \cdots$

$e^{2z} = e^{i2n\pi}$

Taking log,

$2z = i2n\pi$

$z = \dfrac{i2n\pi}{2}$

$z = in\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

$\therefore$ All are simple zeroes

Problems on singularities (poles)

* Find the nature of singularities of $f(z) = \sin\left(\frac{1}{z+1}\right)$

Sol: $z = -1$ $\quad \sin(\infty) =$ not defined (finite)

$f(z)$ fails to analytic

$z = -1$ I.S.P

$$\sin x = \frac{x}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots$$

$$= \frac{\left(\frac{1}{z+1}\right)}{1!} - \frac{\left(\frac{1}{z+1}\right)^3}{3!} + \frac{\left(\frac{1}{z+1}\right)^5}{5!} - \cdots$$

$$= \frac{1}{(z+1)1!} - \frac{1}{(z+1)^3 3!} + \cdots$$

(infinite no. of the negative powers)

$\therefore z = -1$ is also E.S.P.

* $f(z) = \sin\left(\frac{1}{z-a}\right)$

$z = a$ $\quad (\sin\infty) =$ not defined (finite)

$f(z)$ fails to analytic.

$z = a$ I.S.P.

$$\sin x = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$= \frac{\left(\frac{1}{z-a}\right)}{1!} - \frac{\left(\frac{1}{z-a}\right)^3}{3!} + \frac{\left(\frac{1}{z-a}\right)^5}{5!} - \cdots$$

$$\frac{1}{(z-a)1!} - \frac{1}{(z-a)^3 3!} + \frac{1}{(z-a)^5 5!} - \cdots$$

$\therefore z = a$ is also E.S.P.

* $\sin\left(\frac{1}{1-z}\right)$ Find the nature of singularity.

$z = 1$ $\quad (\sin\infty) =$ not defined (finite)

$f(z)$ fails to analytic

$\therefore z = 1$ is I.S.P.

$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}$

$= \frac{\left(\frac{1}{1-z}\right)}{1!} - \frac{\left(\frac{1}{1-z}\right)^3}{3!} + \frac{\left(\frac{1}{1-z}\right)^5}{5!}$

$= \frac{1}{(1-z)\,1!} - \frac{1}{(1-z)^3 3!} + \cdots$

$= \frac{(1-z)^{-1}}{1!} + \frac{(1-z)^{-3}}{3!} + \cdots$

$z=1$ is R.S.P (infinite no. of the powers)

* $f(z) = \frac{\tan z}{z}$

To find singularities

$Dr = 0$

$z = 0$

At $z=0$, $\frac{\tan z}{z}$ fails to analytic

$\therefore z=0$ is I.S.P.

$\frac{\tan z}{z} = \frac{z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots}{z}$

$= \frac{\cancel{z}\left[1 + \frac{z^2}{3} + \frac{2z^4}{15} + \cdots\right]}{\cancel{z}}$

$= 1 + \frac{z^2}{3} + \frac{z^4}{15} + \cdots$ (infinite no. of +ve pows.

$\therefore z=0$ is R.S.P.

HW

$\tan z$

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* $f(z) = \frac{\sin z}{z}$

Sol: Dr = 0

$z = 0$

At $z=0$ is $\frac{\sin z}{z}$ fails to analytic

$z=0$ is S.P

$$\frac{\sin z}{z} = \frac{\left[\frac{z}{1} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}{z}$$

$$= \frac{z\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]}{z}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$ (infinite no. of +ve powers)

$\therefore$ $z=0$ is R.S.P.

* $f(z) = \frac{z - \sin z}{z}$

Dr = 0

$z = 0$

At $z=0$ is $\frac{z-\sin z}{z}$ fails to analytic.

$$\frac{z - \sin z}{z} = \frac{z - \left[\frac{z}{1} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}{z}$$

$$= \frac{z\left[1 - \left[1 - \frac{z^2}{3!} + \frac{z^4}{5!}\right] - \cdots\right]}{z}$$

$$= \cancel{1} - \cancel{1} + \frac{z^2}{3!} - \frac{z^4}{5!} + \cdots$$

$$= \frac{z^2}{3!} - \frac{z^4}{5!} + \cdots$$ (infinite no. of +ve powers)

$\therefore$ $z=0$ is R.S.P.

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* $f(z) = \frac{\sin z - z}{z^3}$.

Sol:- Dr = 0

$z^3 = 0$

$z = 0, 0, 0$. (pole of order 3)

$z = 0$ is I.S.P.

Since function is analytic at $z = 0$.

$$\frac{\sin z - z}{z^3} = \frac{\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots - \cancel{z}}{z^3}$$

$$= \frac{\cancel{z^3}\left[-\frac{1}{3!} + \frac{z^2}{5!} - \dots\right]}{\cancel{z^3}}$$

$$= -\frac{1}{3!} + \frac{z^2}{5!} + \dots$$ [infinite no. of +ve powers]

$\therefore z = 0$ is R.S.P.

$$\lim_{z \to 0} \frac{\sin z - z}{z^3}$$ (indeterminant form)

L'opitals rule

$$\lim_{z \to 0} \frac{\cos z - 1}{3z^2}$$ (indeterminant form)

L'opitals rule

$$\lim_{z \to 0} \frac{-\sin z}{6z}$$ (indeterminat form)

$$\lim_{z \to 0} \frac{-\cos z}{6} = \frac{-1}{6} \neq 0.$$

$\therefore f(z)$ has R.S.P.

* Note

Limit point of poles is a non isolated essential singularities where as ~~any po~~ Limit point of zeroes is a isolated essential singularity.

* Find the kind of singularity for the function

$f(z) = \frac{\cot \pi z}{(z-a)^2} \rightarrow \frac{\cos \pi z}{\sin \pi z (z-a)^2}$

Sol: $Dr = 0$

$(z-a)^2 = 0$

$z = a, a$ (pole of order 2) (finite)

and also

$\sin \pi z = 0$

$= \sin n\pi$

$n\cancel{\pi} = \cancel{\pi} z$

$n = z$ $\quad n = 0, \pm 1, \pm 2 \ldots$ (infinite)

$z = n$

$\therefore$ limit point of poles of infinite poles non-isolated essential singularity.

* ~~Find~~ classify the kind of singularities for the function

$f(z) = \frac{e^{1/z}}{(z-a)^2}$

Sol:- $Dr = 0$

$(z-a)^2 = 0$

$z = a, a$ (pole of order 2) (finite)

$\therefore \lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{e^{1/z}}{(z-a)^2}$

$\Rightarrow \frac{e^{\infty}}{z} = \infty \neq 0$

$\therefore f(z)$ has R.S.P

$\therefore f(z)$ has no zeroes.

* classify the singularities of $f(z) = \frac{z}{e^z - 1}$

Sol Dr=0

$e^z - 1 = 0$

$e^z = 1$

$z = 0 \longrightarrow$ I S P

$= e^0, z = n\pi i, n = 0, \pm 1, \pm 2 \ldots$

$$\frac{z}{\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots \cancel{-1}\right)}$$

$$\frac{\cancel{z}}{\cancel{z}\left(1 + \frac{z}{2!} + \ldots\right)} = \left(1 + \frac{z}{2} + \frac{z^2}{3!} + \ldots\right)^{-1}$$

$(1+x)^{-1} = 1 - x + x^2 - \ldots$

$1 - \left(\frac{z}{2!} + \ldots\right) + \left(\frac{z}{2!} + \ldots\right)^2 - \ldots$ (Infinite no of +ve power)

$\underset{z \to 0}{Lt} \frac{z}{e^z - 1} \quad \frac{0}{0}$

$\underset{z \to 0}{Lt} \frac{1}{e^z} = 1 \neq 0$

$\therefore f(z)$ has R S P.

$\therefore f(z)$ has no zeroes.

* Find the similar points of $f(z) = \frac{1}{\left(\sin\left(\frac{1}{z-a}\right)\right)}$

Sol:- Dr=0

$\sin\left(\frac{1}{z-a}\right) = 0$

$= \sin n\pi$

$n\pi = \frac{1}{z-a}$

$z = \frac{1}{n\pi}$ $n = 0, \pm 1, \pm 2 \ldots$ (infinite)

$$\frac{1}{\left(\frac{1}{z-a}\right)-\frac{\left(\frac{1}{z-a}\right)^3}{3!}+\cdots}$$

$$\frac{1}{\left(\frac{1}{z-a}\right)\left[1-\frac{\left(\frac{1}{z-a}\right)^2}{3!}+\cdots\right]}$$

$$= (z-a)\frac{1}{\left(1-\frac{1}{3!(z-a)^2}+\cdots\right)}$$

$$= (z-a)\left[1-\frac{1}{3!(z-a)^2}+\cdots\right]^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+\cdots$$

$$= z-a\left[1+\frac{1}{3!(z-a)^2}+\cdots\right]$$

because one term has +ve and another have -ve infinite terms

$\therefore$ Infinite no. of ~~the~~ +ve terms

$\therefore$ R.S.P nor E.S.P.

(or) $f(z)$ has non-essential isolated singularity and non-removable isolated singularity

## Entire function (or) Integral function:-

If a function $f(z)$ which is analytic everywhere in a finite plane except at infinity. is called as an entire function or integral function.

For example $f(z) = e^z$, $\sin^2 z$, $\cos z$, $\cosh z$, $\sinh z$ are integral functions.

## Meromorphic function:-

A function $f(z)$ which is analytic ~~every~~ everywhere except at the poles is called a meromorphic function

exp $f(z) = \frac{1}{(z-1)(z-2)}$ is analytic everywhere except at $z=1,2$

$\therefore$ It is a meromorphic func.

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Residues: (Evaluation of poles) (Dr = 0)

Poles

→ poles of order 1 → $\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z \to a} (z-a) f(z)$

→ poles of order $m > 1$ $\left[\operatorname{Res} f(z)\right]_{z=a} = \frac{1}{(m-1)!} \cancel{\lim_{z \to a}}$

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

→ $f(z) = \frac{P(z)}{Q(z)}$ $\left[\operatorname{Res} f(z)\right]_{(z=a)} = \cancel{\frac{P(z)}{Q'(z)}} \; \cancel{\frac{P(a)}{Q'(a)}} = \cancel{\frac{P(a)}{Q'(a)}}$

$= \frac{P(a)}{Q'(a)}$

→ If $f(z)$ can be expressed as a Lorentz series

then $\left[\operatorname{Res} f(z)\right]_{z=a}$ = Coeff of $\frac{1}{(z-a)}$ term

if $\left[\operatorname{Res} f(z)\right]_{z=a \to \text{outside}} = 0.$

* Find the residues of $f(z) = \frac{4z-3}{(z)(z-1)(z-2)}$ along the curve $|z| = 3$.

Sol:-

$$f(z) = \frac{4z-3}{z(z-1)(z-2)}$$

Dr = 0

$z = 0, 1, 2$

$z = 0$ is pole of order 1.

$z = 1$ is pole of order 1

$z = 2$ is pole of order 1

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z \to a} (z-a) f(z)$$

$$\left[\operatorname{Res} f(z)\right]_{z=0} = \lim_{z \to 0} (\cancel{z}-0) \frac{4z-3}{(\cancel{z})(z-1)(z-2)}$$

$$= \frac{-3}{(-1)(-2)} = -\frac{3}{2}.$$

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \lim_{z\to 1} (z-1) \frac{4z-3}{(z)(z-1)(z-2)}$$

$$= \frac{4-3}{(1-2)} = \frac{4-3}{(-1)}$$

$$= \frac{1}{-1} = -1$$

$$\left[\operatorname{Res} f(z)\right]_{z=2} = \lim_{z\to 2} (z-2) \frac{4z-3}{z(z-1)(z-2)}$$

$$= \frac{8-3}{2(2-1)} = \frac{5}{2}$$

* Find the residue of $f(z) = \frac{z}{(z^2+1)}$ along the curve. $|z| = 2$

Soln $f(z) = \frac{z}{z^2+1}$

$D^r = 0$

$z^2+1 = 0$

$z = \sqrt{-1}$

$z = i, -i$ pole of order 1

$z = 0, 1$

$z = 0, -1$ pole of order 1.

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z\to a} (z-a) f(z)$$

$$\left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z\to i} (z-i) \frac{z}{(z^2+1)}$$

$$= \lim_{z\to i} \cancel{(z-i)} \frac{z}{(z+i)\cancel{(z-i)}}$$

$$= \frac{i}{i+i} = \frac{i}{2i} = \frac{1}{2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-i} = \lim_{z\to -i} \cancel{(z+i)} \frac{z}{\cancel{(z+i)}(z-i)}$$

$$= \frac{-i}{-i-i} = \frac{-i}{-2i} = \frac{1}{2}$$

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* Find the residues of $f(z) = \frac{z+1}{(z-1)(z+2)}$ $|z|=1$

Sol:- $Dr = 0$

$z = 1, -2$

$z = -2$ lies outside

$z = 1$ is pole of order 1

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \lim_{z \to 1} \cancel{(z-1)} \frac{z+1}{\cancel{(z-1)}(z+2)}$$

$$= \frac{1+1}{1+2} = \frac{2}{3}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-2} = 0$$

* Find the residues of $f(z) = \frac{z}{(z-1)^2}$ at $|z|=1$

Sol:- $Dr = 0$

$(z-1)^2 = 0$

$z = 1, 1$ pole of order 2 $\therefore m = 2$

$z = 1, 1$ are inside points $(m-1) = 1$

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d^{2-1}}{dz^{2-1}} \left(\cancel{(z-1)^2} \cdot \frac{z}{\cancel{(z-1)^2}}\right)$$

$$= \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} z$$

$$\left[\operatorname{Res} f(z)\right]_{z=1} = 1$$

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* Find the residues of $f(z) = \frac{z \sin z}{(z-\pi)^3}$ at $z = \pi$

Sol:- Dr = 0

$(z-\pi)^3 = 0$

$z = \pi, \pi, \pi$ pole of order 3 $\quad m = 3$

and inside point

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

$$\left[\operatorname{Res} f(z)\right]_{z=\pi} = \frac{1}{(3-1)!} \lim_{z \to \pi} \frac{d^{3-1}}{dz^{3-1}} \left[\cancel{(z-\pi)^3} \frac{z \sin z}{\cancel{(z-\pi)^3}}\right]$$

$$= \frac{1}{2!} \lim_{z \to \pi} \frac{d^2}{dz^2} z \sin z$$

$$= \frac{1}{2} \lim_{z \to \pi} \frac{d}{dz} (\sin z + z \cos z)$$

$$= \frac{1}{2} \lim_{z \to \pi} \cos z + \cos z - z \sin z$$

$$= \frac{1}{2} \lim_{z \to \pi} 2\cos z - z \sin z.$$

$$= \frac{1}{2} \left[2 \cos \pi - \pi \sin \pi\right]$$

$$= \frac{1}{\cancel{2}} (\cancel{2}(-1))$$

$$\left[\operatorname{Res} f(z)\right]_{z=\pi} = -1 \;//.$$

H.W.

1) $f(z) = \frac{4}{z^3(z-2)}$ (i) at a simple pole (ii) $|z| = 3$.

2) $f(z) = \frac{z^2}{(z-2)(z+1)^2}$ $\quad c \to |z| = 2$

(A) $f(z) = \frac{4}{z^3(z-2)}$

Dr = 0

$z = 0, 0, 0 \to$ pole of order 3

$\to$ pole of order 1 or simple 1.

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(i) at simple pole

$$[\text{Res} f(z)]_{z=2} = \lim_{z \to 2} (z-2) f(z)$$

$$= \lim_{z \to 2} \cancel{(z-2)} \left[ \frac{4}{z^3 \cancel{(z-2)}} \right]$$

$$= \frac{4}{2^3}$$

$$= \frac{4}{8} = \frac{1}{2}$$

(ii) at $|z| = 3$

$z = 0, 0, 0 \rightarrow$ inside $(m = 3)$

$z = 2 \rightarrow$ inside

$$[\text{Res} f(z)]_{z=0} = \frac{1}{(3-1)!} \lim_{z \to 0} \frac{d^{3-1}}{dz^{3-1}} \left[ \cancel{(z-0)^3} \left\{ \frac{4}{\cancel{z^3}(z-2)} \right\} \right]$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[ \frac{4}{z-2} \right]$$

$$= \frac{4}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[ \frac{1}{z-2} \right]$$

$$= \frac{4}{2} \lim_{z \to 0} \frac{d}{dz} \left[ \frac{-1}{(z-2)^2} \right]$$

$$= 2 \lim_{z \to 0} \frac{-1 \times -2}{(z-2)^3}$$

$$= 2 \frac{2}{(0-2)^3}$$

$$= 2 \left[ \frac{2}{-\cancel{8}} \right]$$

$$= \frac{-2}{4} = \frac{-1}{2}$$

2a) $f(z) = \dfrac{z^2}{(z-2)(z+1)^2}$ $|z| = 2$

$D_r = 0$

$z = 2, -1, -1,$ → singular points

$z = 2$ pole of order 1

$z = -1, -1$ pole of order 2 $m = 2$

$$\left[\operatorname{Res} f(z)\right]_{z=2} = \lim_{z\to 2} \cancel{(z-2)} \frac{z^2}{\cancel{(z-2)}(z+1)^2}$$

$$= \frac{2^2}{(2+1)^2}$$

$$= \frac{4}{3^2} = \frac{4}{9}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \frac{1}{(2-1)!} \lim_{z\to(-1)} \frac{d^{2-1}}{dz^{2-1}} \left[(z-(-1))^2 \frac{z^2}{(z-2)(z+1)^2}\right]$$

$$= \frac{1}{1!} \lim_{z\to(-1)} \frac{d}{dz} \cancel{(z+1)^2} \frac{z^2}{(z-2)\cancel{(z+1)^2}}$$

$$= 1 \lim_{z\to -1} \frac{d}{dz} \frac{z^2}{z-2}$$

$$= \lim_{z\to -1} \frac{2z(z-2) - z^2(1)}{(z-2)^2}$$

$$= \frac{2(-1)(-1-2) - (-1)^2}{(-1-2)^2}$$

$$= \frac{-2(-3)-1}{(-3)^2} = \frac{6-1}{9} = \frac{5}{9}$$

* Find the residues of $f(z) = \tan z$ at $z = \frac{\pi}{2}$

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \frac{P(a)}{Q'(a)}$$

$$= \frac{\cancel{[\cos z]_{z=\pi/2}}}{\cancel{(\cos z)'_{z=\pi/2}}}$$

$$= \frac{[\sin z]_{z=\pi/2}}{[-\sin z]_{z=\pi/2}} = -1$$

* $f(z) = \cot z$

$$f(z) = \frac{\cos z}{\sin z}$$

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \frac{P(a)}{Q'(a)}$$

$$= \left[\frac{\cancel{-\sin z}}{\cancel{\sin z}}\right]_{z=\pi/2} = \left[\frac{\cos z}{\cos z}\right]_{z=\pi/2}$$

$$= \cancel{0} = 1$$

* Find the residues of $f(z) = e^{1/z}$.

Sol:

$$f(z) = e^{1/z}$$

$$= \frac{1}{z} + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots$$

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \text{coeff of } \frac{1}{z-a}$$

$$\left[\operatorname{Res} f(z)\right]_{z=0} = \text{coeff of } \frac{1}{z-0}$$

$$= 1$$

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Hw.

(1) $f(z) = \frac{1-e^{-z}}{z^3}$ at $z=0$.

(2) $f(z) = z\cos\left(\frac{1}{z}\right)$ $z=0$.

(A) Soln

$Dr=0$

$z=0,0,0 =$ pole of order 3.

$$\left[\text{Res} f(z)\right]_{z=0} = \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^{3-1}}{dz^{3-1}} \cancel{(z-0)^3} \frac{1-e^{-z}}{\cancel{z^3}}$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} 1-e^{-z}$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz} e^{-z}$$

$$= \frac{1}{2} \lim_{z\to 0} -e^{-z}$$

$$= \frac{1}{2}(-1) = \frac{-1}{2}$$

2A) $f(z) = z\cos\left(\frac{1}{z}\right)$

$$= z\left[1 - \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^4}{4!} - \cdots\right]$$

$$= \left[z - \frac{\frac{z}{z^2}}{2!} + \frac{\frac{z}{z^4}}{4!} - \cdots\right]$$

$$= z - \frac{\frac{1}{z}}{2!} + \frac{\frac{1}{z^3}}{4!}$$

$$\left[\text{Res} f(z)\right]_{z=0} = \text{Coeff of } \frac{1}{z-0}$$

$$= \frac{-1}{2!} = \frac{-1}{2}.$$

# Cauchy's Residue theorem (CRT)

**Statement:** If $f(z)$ is analytic at all points inside and on a simple closed curve 'C' (except for finite no. of poles $z_1, z_2, \ldots z_n$) and on inside we have $\int_C f(z)\,dz = 2\pi i \times$ [sum of residues]

$$\int_C f(z)\,dz = 2\pi i \times [R_1 + R_2 + \cdots R_n]$$

$R_1, R_2 \ldots R_n$ represent residues

**Proof:**

For each point of the poles $[z_1 z_2 \ldots z_n]$ draw a small non-intersecting circle with centre as these poles. Let the circles be $C_1, C_2 \ldots C_n$

Cauchy's Extended theorem

$$\int_C f(z) = \int_{C_1} f(z_1)dz_1 + \int_{C_2} f(z_2)\,dz_2 + \int_{C_3} f(z_3)dz_3 \cdots \int_{C_n} f(z_n)\,dz_n \quad \text{(1)}$$

we know that the residue of $f(z)$ at $z=a$ is given by $\left[\text{Res} f(z)\right]_{z=a} = \frac{1}{2\pi i}\int_C f(z)dz$

(or)

$$2\pi i\,[\text{Res} f(z)] = \int_C f(z)\,dz$$

from (1)

$$\int_C f(z)\,dz = 2\pi i\,[\text{Res} f(z)]_{z=z_1} + 2\pi i\,[\text{Res} f(z)]_{z=z_2} + \cdots + 2\pi i\,[\text{Res} f(z)]_{z=z_n}$$

$$= 2\pi i\,[R_1 + R_2 + \cdots R_n] \text{ (or) sum of residues}$$

* Using Cauchy's residue theorem find the value of the integral $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz$ where $|z| = 3$

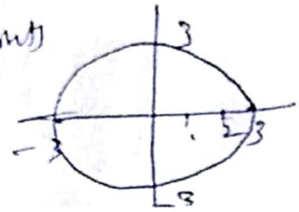
Sol: CRT $\int_C f(z)dz = 2\pi i$ [sum of residues]

Dr = 0

$(z-1)(z-2)$ . 1, 2 inside points

$z=1$ pole of order 1

$z=2$ pole of order 1 .




$R_1 = \left(\operatorname{Res} f(z)\right]_{z=1} = \lim_{z\to 1} (z-1) \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$

$= \frac{\sin\pi + \cos\pi}{1-2} = \frac{-1}{(-1)} = 1$

$R_2 = \left(\operatorname{Res} f(z)\right]_{z=2} = \lim_{z\to 2} (z-2) \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$

$= \frac{\sin 4\pi + \cos 4\pi}{2-1}$

$= \frac{1}{1} = 1$

$\therefore \int_C f(z)dz = \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} = 2\pi i [1+1]$

$= 4\pi i$

* Using CRT find the value of $\int_C \frac{4-3z}{z(z-1)(z-2)}$ where

$C = |z| = \frac{3}{2} = 1.5$

Sol:

Dr = 0

$z = 0, 1, 2$

$z = 0, 1$ inside, and pole of order 1

$z = 2$ outside point $[\operatorname{Res} f(z)]_{z=2} = 0$

$R_1 = [\operatorname{Res} f(z)]_{z=0} = \lim_{z\to 0} (z-0) \frac{4-3z}{z(z-1)(z-2)} = \frac{4}{(-1)(-2)}$

$= \frac{4}{2} = 2$

$R_2 = \left[\text{Res} f(z)\right]_{z=1} = \lim_{z\to 1} \cancel{(z-1)} \dfrac{4-3z}{(z)\cancel{(z-1)}(z-2)}$

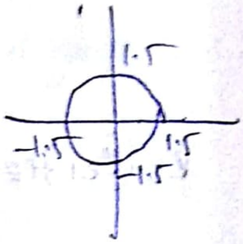
$= \dfrac{4-3}{1-2}$

$= \dfrac{1}{-1} = -1$

$\displaystyle\int_C \frac{4-3z}{z(z-1)(z-2)} = 2\pi i(2-1)$

$= 2\pi i$

* $\displaystyle\int_C \frac{dz}{(z^2+1)(z^2-4)}$ $\quad |z| = \frac{3}{2}$




$z = i, -i$ $(0,1)$ $(0,-1)$ pole of order 1 → inside

$z = -2, 2$ pole of order 1 → outside

$R_1 = \left[\text{Res} f(z)\right]_{z=-2} = 0$

$R_2 = \left[\text{Res} f(z)\right]_{z=2} = 0$

$R_3 = \left[\text{Res} f(z)\right]_{z=i} = \lim_{z\to i} \cancel{(z-i)} \dfrac{1}{(z+i)\cancel{(z-i)}(z^2-4)}$

$= \dfrac{1}{(i+i)(-1-4)}$

$= \dfrac{1}{2i(-5)} = \dfrac{-1}{10i}$

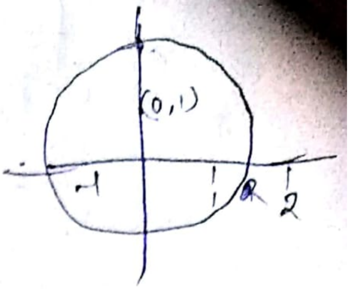
$R_4 = \left[\text{Res} f(z)\right]_{z=-i} = \lim_{z\to -i} \cancel{(z+i)} \dfrac{1}{\cancel{(z+i)}(z-i)(z^2-4)}$

$= \dfrac{1}{(-i-i)(-1-5}$

$= \dfrac{1}{10i}$

$$\int_C \frac{dz}{(z^2+1)(z^2-4)} = 2\pi i \left[ \frac{-10}{101} + \frac{1}{101} \right]$$

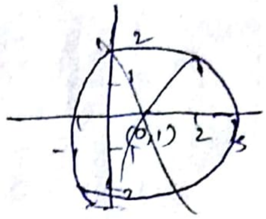
$$= 0$$



$$* \quad \int_C \frac{(z-1)\,dz}{(z+1)^2(z-2)} \qquad (z-1) = 2$$

$\cdot(0,1)$

Sol/ $Dr = 0$



$z = 2, -1, -1$

$z = 2$ ~~inside~~ outside and pole of order 1

$z = -1, -1$, inside and pole of order 2. $m = 2$

$$\left[\operatorname{Res} f(z)\right]_{z=2} = \lim_{z\to 2} \cancel{(z-2)} \frac{(z-1)}{(z+1)^2\cancel{(z-2)}}$$

$$= \frac{2-1}{(2+1)^2} = \frac{1}{9}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \frac{1}{(2-1)!} \lim_{z\to -1} \frac{d^{2-1}}{dz^{2-1}} \left[ \cancel{(z+1)^2} \frac{z-1}{\cancel{(z+1)^2}(z-2)} \right]$$

$$= (1) \lim_{z\to -1} \frac{d}{dz} \frac{z-1}{(z+1)(z-2)}$$

$$= \lim_{z\to -1} \frac{d}{dz} \frac{z-1}{z^2-2z+z-2}$$

$$= \lim_{z\to -1} \frac{d}{dz} \frac{z-1}{z^2-z-2}$$

$$= \lim_{z\to -1} \frac{(z^2-z-2)-(z-1)(2z-1)}{(z^2-z-2)}$$

$$= \lim_{z\to -1} \frac{z^2-z-\cancel{2}-2z^2+z+2z-1}{z^2-z-2}$$

$$= \frac{(-1)^2-(-1)-2-(-1-1)(2(-1)-1)}{((-1)^2-(-1)-2)}$$

$$= \frac{1+1-2-(-2)(-3)}{1+1-2} = \frac{2-2-6}{1}$$

$$= -6$$

$$= \lim_{z \to (-1)} \frac{(z-1)}{(z-2)}$$

$$= \lim_{z \to -1} \frac{(z-2)-(z-1)}{(z-2)^2}$$

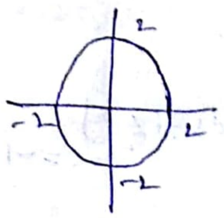
$$= \frac{(-1-2)-(-1-1)}{(-1-2)^2}$$

$$= \frac{-3-(-2)}{(-3)^2} = \frac{-3+2}{9} = \frac{-1}{9}$$

$$\int_C \frac{(z-1)\,dz}{(z+1)^2(z-2)} = 2\pi i\left[-\frac{1}{9}+0\right]$$

$$= \frac{-2\pi i}{9}$$

* $$\int_C \frac{dz}{z^3(z-1)} \qquad |z|=2$$




$Dr = 0$

$z = 0, 0, 0, 1$

$z=0$ pole of order 3
$z=1$ pole of order 1 } inside

$$\left[\mathrm{Res}\, f(z)\right]_{z=1} = \lim_{z \to 1} \cancel{(z-1)} \frac{1}{z^3\cancel{(z-1)}}$$

$$= 1$$

$$\left[\mathrm{Res}\, f(z)\right]_{z=0} = \frac{1}{(3-1)!} \lim_{z \to 0} \frac{d^{3-1}}{dz^{3-1}} \left[\cancel{(z-0)^3} \frac{1}{\cancel{z^3}(z-1)}\right]$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{1}{z-1}$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d}{dz} \frac{-1}{(z-1)^2}$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{(-1)(-2)}{(z-1)^3}$$

$= \frac{1}{2} \cdot \frac{2}{(-1)}$

$= -1$

$$\int_C \frac{dz}{z^3(z-1)} = 2\pi i[1-1]$$

$= 0$

* $\int_C \frac{(2z^2+z)}{(z^2-1)} dz$

(i) $|z-1|=1$,

$|z|=2$.

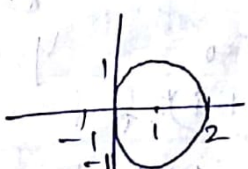
Sol: $Dr = 0$

$z^2 - 1 = 0$

$\therefore z = 1, -1$

(i) $|z-1|=1$

$(1,0)$



$z=1$ inside

$z=-1$ outside.

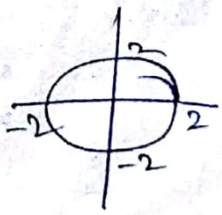
$$\left[\operatorname{Res} f(z,1)\right]_{z=1} = \lim_{z\to 1} \cancel{(z-1)} \frac{2z^2+z}{(z+1)\cancel{(z-1)}}$$

$$= \frac{2+1}{1+1} = \frac{3}{2}.$$

$$\int_C \frac{2z^2+z}{(z^2-1)} dz = 2\pi i\left[\frac{3}{2}+0\right]$$

$= 3\pi i$

(ii) $|z|=2$



$z=-1, 1$ inside

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \lim_{z\to -1} \cancel{(z+1)} \frac{2z^2+z}{\cancel{(z+1)}(z-1)} = \frac{+2-1}{-1-1} = \frac{1}{-2}$$

$$\int_c \frac{2z^2+2}{z^2+1} = 2\pi i\left[\frac{3}{2}-\frac{1}{2}\right]$$

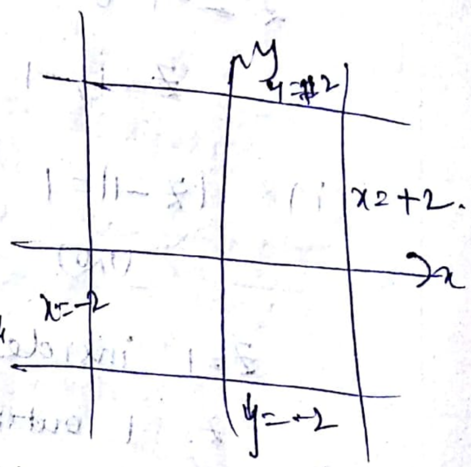
$$= 2\pi i\left[\frac{2}{2}\right]$$

$$= 2\pi i$$

* If $C$ is the boundary of the square whose sides along the straight lines $x=\pm 2$, $y=\pm 2$ and describe in the positive sets. Find the value of

$$\int_c \frac{\tan\left(\frac{z}{2}\right)dz}{(z-1-i)^2}$$

Sol:




$x = \pm 2$

$y = \pm 2$

Or=0

$(z-1-i)^2 = 0$

$z = (1+i)$ in pole of order 2

$$= \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)$$

$$= \frac{1}{(2-1)!}\lim_{z\to(1+i)}\frac{d^{2-1}}{dz^{2-1}}\left[\cancel{(z-(1+i))^2}\frac{\tan\frac{z}{2}}{\cancel{(z-1-i)^2}}\right]$$

$$= 1\lim_{z\to(1+i)}\frac{d}{dz}\tan\left(\frac{z}{2}\right)$$

$$= \lim_{z\to(1+i)}\sec^2\left(\frac{z}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\cdot\sec^2\left(\frac{1+i}{2}\right)$$

HW $\int_C \frac{\tan(\frac{z}{2})\,dz}{(z-a)^2}$ where C is the boundary of square whose sides lie along $x=\pm 2$, $y=\pm 2$

also $-2 < a < +2$

HW $\int_C \frac{z+4\;dz}{(z^2+2z+5)}$ where C is given (i) $|z+1-i|=2$.

(ii) $|z+1+i|=2$.

Sol

$Dr = 0$

$z^2+2z+5=0$

$z = \frac{-2\pm\sqrt{(2)^2-4(5)}}{2} = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm i4}{2}$

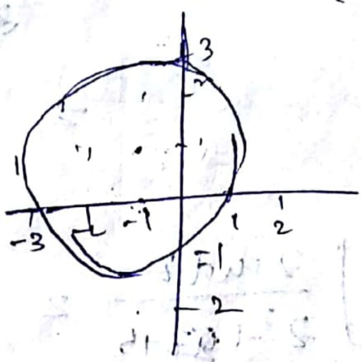
$= -1\pm 2i$

$= (-1,2)$, $(-1,-2)$ order 1.

(i) $|z+1-i|=2$

$-(-1+i)$

Center $(-1,1)$ rad $=2$.




$z = -1+2i$ = inside

$(-1,2)$

$= -1-2i$ = outside

$(-1,-2)$ $\quad [\text{Res} f(z)]_{z=-1-2i} = 0$

$[\text{Res} f(z)]_{z=-1+2i} = \lim_{z\to -1+2i} (z-(-1+2i)) \frac{z+4}{(z-(-1+2i))[z-(-1-2i)]}$

$= \lim_{z\to -1+2i} \cancel{[z+1-2i]} \cdot \left[\frac{z+4}{\cancel{[z+1-2i]}[z+1+2i]}\right]$

$= \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$

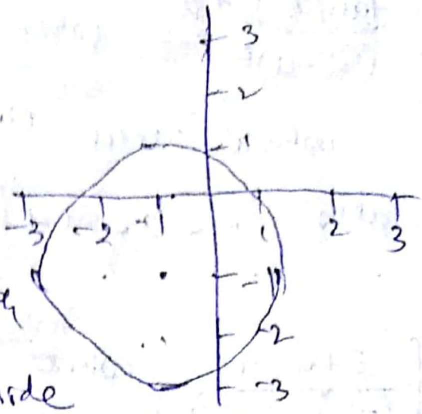
$\int_C \frac{z+4}{(z^2+2z+5)}dz = 2\pi i\,[R_1+R_2]$

$= 2\pi i\left[\frac{3+2i}{4i}\right] = \frac{\pi}{2}\,[3+2i]$

(ii) $|z+1+i| = 2$

Centre $(-1,-1)$

rad $= 2$



$z = (-1,+2) = -1+2i$ outside

$z = (-1,-2) = -1-2i$ inside

$$\left[\operatorname{Res} f(z)\right]_{z=-1+2i} = 0$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1-2i} = \lim_{z\to -1-2i} [z-(-1-2i)] \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}$$

$$= \lim_{z\to -1-2i} \frac{z+4}{[z+1-2i]}$$

$$= \frac{-1-2i+4}{\cancel{-1}-2i\cancel{+1}-2i} = \frac{3-2i}{-4i}$$

$$\int_C \frac{z+4\,dz}{z^2+2z+5} = 2\pi i\left[0+\frac{3-2i}{-\cancel{4}_2 \cancel{i}}\right]$$

$$= -\frac{\pi}{2}(3-2i)$$

## Countour Integration

→ Type I

$$I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

→ Type II

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx.$$

Working rule and common step for type I.

Here $C: |z|=1$ and take $z = e^{i\theta}$
$= \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$.

and integration limits should be checked

$\int_0^{2\pi}$ if $\int_0^{\pi}$ is given $\frac{1}{2}\int_0^{2\pi}$ should be used.

$z + \frac{1}{z} = 2\cos\theta$

$$\boxed{\frac{z^2+1}{2z} = \cos\theta} \quad (1)$$

$z - \frac{1}{z} = 2i\sin\theta$

$$\boxed{\frac{z^2-1}{2iz} = \sin\theta} \quad (2)$$

$z = e^{i\theta}$

$dz = e^{i\theta}\, i\, d\theta$

$= iz\, d\theta$

$$\boxed{d\theta = \frac{dz}{iz}} \quad (3)$$

Substitute (1), (2), (3) in I. and apply CRT we have the required integration value.

Using contour integration evaluate

$$\int_0^{2\pi} \frac{d\theta}{13+5\cos\theta} \qquad |z|=1$$

Soln:

Type I. $\quad z = e^{i\theta}$

$\cos\theta = \frac{z^2+1}{2z}$

$$I = \int_C \frac{dz}{iz \times \left[13 + 5\left[\frac{z^2+1}{2z}\right]\right]}$$

$$= \int_C \frac{dz}{iz\left[\frac{26z + 5z^2 + 5}{2z}\right]} = \frac{2}{i}\int_C \frac{dz}{5z^2+26z+5}$$

$$= \frac{2}{5i}\int_C \frac{dz}{z^2 + \frac{26}{5}z + 1}$$

$$z = \frac{-\left(\frac{26}{5}\right) \pm \sqrt{\left(\frac{26}{5}\right)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{\frac{-26}{5} \pm \frac{24}{5}}{2}$$

$$= \frac{\left(\frac{-26+24}{5}\right)}{2}, \quad \frac{\left(\frac{-26-24}{5}\right)}{2}$$

$$= \frac{-2}{10}, \quad -\frac{50}{10}$$

$$z = -\frac{1}{5}, -5.$$

$z = -5$ outside

$$\left[\operatorname{Res} f(z)\right]_{z=-5} = 0$$

$z = -\frac{1}{5}$ = inside

$$\left[\operatorname{Res} f(z)\right]_{z=-\frac{1}{5}} = \lim_{z \to -\frac{1}{5}} \cancel{\left(z+\frac{1}{5}\right)} \frac{1}{(z+5)\cancel{\left(z+\frac{1}{5}\right)}}$$

$$= \frac{1}{\left(-\frac{1}{5}+5\right)}$$

$$= \frac{1}{\frac{-1+25}{5}} = \frac{5}{24}$$

$$I = \frac{2}{5i} \oint \frac{dz}{z^2 + \frac{26}{5}z + 1} = 2\pi i \left[\frac{5}{24}\right] \times \frac{2}{5i}$$

$$= \frac{5\pi i}{12} \times \frac{2}{5i}$$

$$= \frac{\pi}{6}$$

* $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$

Sol:

$$\cos\theta = \frac{z^2+1}{2z}$$

$$d\theta = \frac{dz}{iz}$$

$$I = \int_C \frac{dz}{iz\left[5+4\left(\frac{z^2+1}{2z}\right)\right]}$$

$$I = \int_C \frac{dz}{i\cancel{z}\left[\frac{10z+4z^2+4}{2\cancel{z}}\right]} = \int_C \frac{dz\,2}{i(10z+4z^2+4)}$$

$$= \frac{2}{i}\int_C \frac{1}{4z^2+10z+4}$$

$$I = \frac{2}{i4}\int_C \frac{1}{z^2+\frac{5}{2}z+1}$$

$$z = \frac{-\frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2-4(1)(1)}}{2(1)} = \cancel{\frac{-5}{2}\pm}$$

$$= -\frac{1}{2}, -2.$$

$$\left[\text{Res}\, f(z)\right]_{z=-2} = 0$$

$$\left[\text{Res}\, f(z)\right]_{z=-\frac{1}{2}} = \lim_{z\to-\frac{1}{2}} \cancel{\left[z+\frac{1}{2}\right]} \left[\frac{1}{\cancel{\left(z+\frac{1}{2}\right)}(z+2)}\right]$$

$$= \frac{1}{\left(-\frac{1}{2}-2\right)} = \frac{1}{\frac{-1+4}{2}}$$

$$= \frac{2}{-5} = +\frac{2}{3}$$

$$I = 2\pi i\left(\frac{1}{2i}\right)\left[+\frac{2}{3}\right]$$

$$= \frac{+2\pi}{3}.$$

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HW-* $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$ } also in $\sin\theta$'

* $\displaystyle\int_0^{\pi} \frac{d\theta}{2+\cos\theta}$

Standard Problem I.

* $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a>b$.

Type I.

$$I = \int_C \frac{dz}{iz\left[a+b\left(\frac{z^2+1}{2z}\right)\right]}$$

$$= \int_C \frac{dz}{iz\left[\frac{2az+bz^2+b}{2z}\right]}$$

$$= \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b}$$

$$= \frac{2}{ib}\int_C \frac{dz}{z^2+\frac{2a}{b}z+1}$$

$$f(z) = \frac{1}{z^2+\frac{2a}{b}z+1} = \frac{1}{(z-\alpha)(z-\beta)}$$

$\cancel{\alpha=}$ Dr=0

$$z^2+\frac{2a}{b}z+1=0$$

$$z = \frac{-\left(\frac{2a}{b}\right)\pm\sqrt{\left(\frac{2a}{b}\right)^2-4(1)(1)}}{2(1)} = \frac{-\frac{2a}{b}\pm\sqrt{\frac{4a^2}{b^2}-4}}{2}$$

$$= \frac{-\frac{2a}{b}\pm 2\sqrt{\frac{a^2}{b^2}-1}}{2} = \frac{-\frac{2a}{b}\pm\frac{2}{b}\sqrt{a^2-b^2}}{2}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b} = \frac{-a \pm \sqrt{a^2-b}}{b}$$

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}, \quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Take $a=2, \quad b=1$

$\alpha = -0.3, \quad \beta = -2.3$

$$\left[\text{Res } f(z)\right]_{z = \alpha = \frac{-a+\sqrt{a^2-b^2}}{b}} = \lim_{z\to\alpha} \cancel{(z-\alpha)} \frac{1}{\cancel{(z-\alpha)}(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta}$$

$$= \frac{1}{\alpha-\beta} =$$

$$= \frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b} + \frac{a+\sqrt{a^2-b^2}}{b}} = \frac{b}{2\sqrt{a^2-b^2}}$$

$$C.R.T = I = \frac{\cancel{2}}{\cancel{b}i} 2\pi i \frac{\cancel{b}}{\cancel{2}\sqrt{a^2-b^2}}$$

$$\boxed{I = \frac{2\pi}{\sqrt{a^2-b^2}}, \ a>b}$$

Standard Pb 2

HW * P.T $\displaystyle\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$ $\quad a=5, \ b=4$

* $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

$$\sin\theta = \frac{z^2-1}{2iz}$$

$$d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \int_C \frac{dz}{iz\left[13+5\left(\frac{z^2-1}{2iz}\right)\right]}$$

$$= \int_C \frac{dz}{\cancel{iz}\left[\frac{26iz+5z^2-5}{2\cancel{iz}}\right]}$$

$$= \frac{2}{5}\int \frac{dz}{z^2 + \frac{26}{5}iz - 1}$$

$$z = \frac{-\frac{26i}{5} \pm \sqrt{\left(\frac{26i}{5}\right)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-\frac{26i}{5} \pm \sqrt{\frac{-676}{25} + 4}}{2}$$

$$= \frac{-\frac{26i}{5} \pm \sqrt{\frac{-676+100}{25}}}{2}$$

$$= \frac{-\frac{26i}{5} \pm \sqrt{\frac{-576}{25}}}{2}$$

$$= \frac{-\frac{26i}{5} \pm \frac{24i}{5}}{2}$$

$$= \frac{-26i+24i}{10}, \frac{-26i-24i}{10}$$

$$= -\frac{2i}{10}, -\frac{50i}{10}$$

$$= -\frac{i}{5}, -5i.$$

$$z = \left(0, \frac{-1}{5}\right) \quad z = (0, -5)$$

$$\left[\text{Res } f(z)\right]_{z=-5} = 0$$

$$\left[\text{Res } f(z)\right]_{z=-\frac{i}{5}} = \lim_{z \to -\frac{i}{5}} \cancel{\left(z+\frac{i}{5}\right)} \left\{\frac{1}{(z+5i)\cancel{\left(z+\frac{i}{5}\right)}}\right.$$

$$= \frac{1}{\left(-\frac{i}{5}+5i\right)}$$

$$= \frac{1}{\frac{-i+25i}{5}}$$

$$= \frac{5}{24i}$$

$$I = \frac{2}{\cancel{8}} \; 2\pi i \; \frac{\cancel{5}}{\cancel{24i}_{12}}$$

$$= \frac{\pi}{6}$$

→ $\displaystyle\int_0^{2\pi} \frac{1\,d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$

**Sol.**

$$\sin\theta = \frac{z^2-1}{2zi}$$

$$d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{dz}{zi\left[5+4\left(\frac{z^2-1}{2zi}\right)\right]} = \int_C \frac{dz}{\cancel{z}i\left[\frac{10zi+4z^2-4}{2\cancel{z}i}\right]}$$

$$= 2\int \frac{dz}{4z^2+10zi-4}$$

$$= \frac{2}{4}\int \frac{dz}{z^2+\frac{5}{2}zi-1}$$

$$= \frac{1}{2}\int \frac{dz}{z^2+\frac{5}{2}zi-1}$$

$$z = \frac{-\frac{5i}{2} \pm \sqrt{\left(\frac{5i}{2}\right)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-\frac{5i}{2} \pm \sqrt{\frac{-25}{4}+4}}{2} = \frac{-\frac{5i}{2} \pm \sqrt{\frac{-25+16}{4}}}{2}$$

$$= \frac{-\frac{5i}{2} \pm \sqrt{\frac{-9}{4}}}{2} = \frac{-\frac{5i}{2} \pm i\frac{3}{2}}{2}$$

$$= \frac{-5i+3i}{4}, \frac{-5i-3i}{4} = \frac{-2i}{4}, \frac{-8i}{4} = \frac{-1}{2}i, -2i$$

$z = -\frac{1}{2}i$ inside

$z = -2i$ outside

$$[\text{Res} f(z)]_{z=-2i} = 0$$

$$[\text{Res} f(z)]_{z=-\frac{1}{2}i} = \lim_{z \to -\frac{1}{2}i} \cancel{\left(z + \frac{1}{2}i\right)} \frac{1}{\cancel{(z+\frac{1}{2}i)}(z+2i)}$$

$$= \frac{1}{\left(-\frac{1}{2}i + 2i\right)} = \frac{2}{(-i+4i)} = \frac{2}{3i}$$

$$I = \frac{1}{2}\int_C \frac{dz}{z^2 + \frac{5}{2}zi - 1} = \frac{1}{\cancel{2}} \cancel{2}\pi \cancel{i} \frac{2}{3\cancel{i}}$$

$$= \frac{2\pi}{3}$$

Standard problem II

$$* \int_0^{\pi} \frac{d\theta}{a + b\cos\theta}$$

Given Integral is $\int_0^{\pi} \Rightarrow \frac{1}{2}\int_0^{2\pi}$

From standard problem I

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

$$\therefore \int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$$

$$= \frac{1}{2} \frac{2\pi}{\sqrt{a^2 - b^2}}$$

$$\boxed{\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}}$$

$$\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$$

Sol: $$\int_C \frac{dz}{iz\left(13+12\left(\frac{z^2+1}{2z}\right)\right)}$$

$$\int_C \frac{dz}{i\left(\frac{26z+12z^2+12}{2}\right)}$$

$$\frac{2}{i}\int_C \frac{dz}{12z^2+26z+12}$$

$$\frac{2}{12i}\int_C \frac{dz}{z^2+\frac{26}{12}z+1}$$

$$\frac{1}{6i}\int_C \frac{dz}{z^2+\frac{13}{6}z+1}$$

$$z = \frac{-\frac{13}{6} \pm \sqrt{\left(\frac{13}{6}\right)^2-4(1)(1)}}{2(1)}$$

$$= \frac{-\frac{13}{6} \pm \sqrt{\frac{169}{36}-4}}{2} = \frac{-\frac{13}{6} \pm \sqrt{\frac{169-144}{36}}}{2}$$

$$= \frac{-\frac{13}{6} \pm \frac{5}{6}}{2} = \frac{-13+5}{12}, \frac{-13-5}{12} = \frac{-8}{12}, \frac{-18}{12}$$

$$= -\frac{2}{3}, \frac{-3}{2}$$

$z = \frac{-2}{3}$ inside

$z = \frac{-3}{2}$ outside

$$\left[\operatorname{Res} f(z)\right]_{z=-\frac{3}{2}} = 0$$

$$\left[\operatorname{Res} f(z)\right]_{z=-\frac{2}{3}} = \lim_{z\to -\frac{2}{3}} \cancel{\left(z+\frac{2}{3}\right)} \frac{1}{\left(z+\frac{3}{2}\right)\cancel{\left(z+\frac{2}{3}\right)}}$$

$$= \frac{1}{\left(-\frac{2}{3}+\frac{3}{2}\right)} = \frac{1}{\left(\frac{-4+9}{6}\right)} = \frac{6}{5}$$

$$\oint I = \frac{1}{6i}\int_C \frac{dz}{z^2+\frac{13}{6}z+1} = \frac{1}{\cancel{6}\cancel{i}} 2\pi \cancel{i}\left[\frac{\cancel{6}}{5}\right] = \frac{2\pi}{5}.$$

* $\int_0^{2\pi} \frac{d\theta}{13+12\sin\theta}$

Sol: $\int_C \frac{dz}{zi\left(13+12\frac{(z^2-1)}{2zi}\right)}$

$\int_C \frac{dz}{zi\left[\frac{26zi+12z^2-12}{2zi}\right]}$

$2\int_C \frac{dz}{12z^2+26zi-12}$

$\frac{1}{6}\int_C \frac{dz}{z^2+\frac{13}{6}zi-1}$

$z = \dfrac{-\frac{13i}{6} \pm \sqrt{\left(\frac{+13i}{6}\right)^2-4(1)(-1)}}{2(1)}$

$= \dfrac{-\frac{13i}{6} \pm \sqrt{\frac{-169}{36}+4}}{2} = \dfrac{-\frac{13i}{6} \pm \sqrt{\frac{-169+144}{36}}}{2}$

$= \dfrac{-\frac{13i}{6} \pm \sqrt{\frac{-25}{36}}}{2} = \frac{-13i+5i}{12}, \frac{-13i-5i}{12}$

$= \frac{-8i}{12}, \frac{-18i}{12} = -\frac{2}{3}i, -\frac{3}{2}i$

$z = -\frac{2}{3}i = \left(0,-\frac{2}{3}\right) =$ inside

$z = -\frac{3}{2}i = \left(0,-\frac{3}{2}\right) =$ outside

$\left[\operatorname{Res} f(z)\right]_{z=-\frac{3}{2}i} = 0$

$\left[\operatorname{Res} f(z)\right]_{z=-\frac{2}{3}i} = \lim_{z\to -\frac{2}{3}i} \cancel{\left(z+\frac{2}{3}i\right)} \frac{1}{\cancel{\left(z+\frac{2}{3}i\right)}\left(z+\frac{3}{2}i\right)}$

$= \frac{1}{\left(-\frac{2}{3}i+\frac{3}{2}i\right)} = \frac{1}{\left(\frac{-4i+9i}{6}\right)}$

$= \frac{6}{5i}$

$$I = \frac{1}{6}\int_C \frac{dz}{z^2+\frac{13}{6}zi+1} = \frac{1}{6}\,2\pi i\,\frac{6}{5i}$$

$$= \frac{2\pi}{5}$$

* $\displaystyle\int_0^{\pi} \frac{d\theta}{2+\cos\theta}$

Sol:- $\displaystyle\int_0^{\pi} \frac{d\theta}{2+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{1}{2}\int_C \frac{dz}{zi\left(2+\frac{z^2+1}{2z}\right)}$

$$= \frac{1}{2}\int_C \frac{dz}{\cancel{z}i\left(\frac{4z+z^2+1}{2\cancel{z}}\right)} = \frac{1}{i}\int_C \frac{dz}{z^2+4z+1}$$

$$z = \frac{-4\pm\sqrt{4^2-4(1)(1)}}{2} = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm\sqrt{12}}{2}$$

$$= \frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3}$$

$z = -2+\sqrt{3},\ -2-\sqrt{3}$

$= -0.26,\ -3.73$

$z = -2+\sqrt{3} =$ inside

$z = -2-\sqrt{3} =$ outside

$\left[\operatorname{Res} f(z)\right]_{z=-2-\sqrt{3}} = 0$

$\left[\operatorname{Res} f(z)\right]_{z=-2+\sqrt{3}} = \lim_{z\to -2+\sqrt{3}} \cancel{(z+2-\sqrt{3})}\dfrac{1}{\cancel{(z+2-\sqrt{3})}(z+2+\sqrt{3})}$

$$= \frac{1}{-\cancel{2}+\sqrt{3}+\cancel{2}+\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

$$I = \frac{1}{i}\int_C \frac{dz}{z^2+4z+1} = \frac{1}{\cancel{i}}\,\cancel{2}\pi \cancel{i}\,\frac{1}{\cancel{2}\sqrt{3}} = \frac{\pi}{\sqrt{3}}$$

* $\displaystyle\int_0^{\pi} \frac{d\theta}{2+\sin\theta}$

* $\displaystyle\int_0^{\pi} \frac{d\theta}{2+\sin\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{1}{2}\int_C \frac{dz}{zi\left(2+\frac{z^2-1}{2zi}\right)}$

$$= \frac{2}{2}\int_C \frac{dz}{4zi+z^2-1} = \int_C \frac{dz}{z^2+4zi-1}$$

$$z = \frac{-4i \pm \sqrt{(4i)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-4i \pm \sqrt{-16+4}}{2} \qquad \frac{-4i \pm \sqrt{-12}}{2}$$

$$= \frac{-4i \pm i2\sqrt{3}}{2} = -2i \pm i\sqrt{3}$$

$z = -2i + i\sqrt{3},\ -2i - i\sqrt{3}$.

$z = -2i + i\sqrt{3}$ = inside

$z = -2i - i\sqrt{3}$ = outside

$$\left[\operatorname{Res} f(z)\right]_{z=-2i-i\sqrt{3}} = 0$$

$$\left[\operatorname{Res} f(z)\right]_{z=-2i+i\sqrt{3}} = \lim_{z\to -2i+i\sqrt{3}} \cancel{(z+2i-i\sqrt{3})} \frac{1}{\cancel{(z+2i-i\sqrt{3})}(z+2i+i\sqrt{3})}$$

$$= \frac{1}{\cancel{-2i}+i\sqrt{3}\cancel{+2i}+i\sqrt{3}}$$

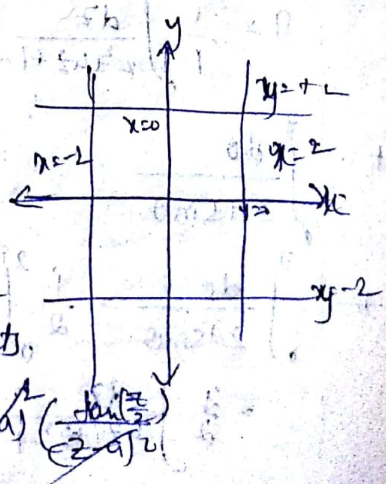
$$= \frac{1}{2i\sqrt{3}}$$

$$I = \int_C \frac{dz}{z^2+4zi-1} = 2\pi i \frac{1}{2i\sqrt{3}}$$

$$= \frac{\pi}{\sqrt{3}} //$$

HW

$\int_C \frac{\tan(\frac{z}{2})\, dz}{(z-a)^2}$ where C is the boundary of square whose sides lie along $x = \pm 2$, $y = \pm 2$ also $-2 < a < +2$.




Sol

Dr = 0

$(z-a)^2 = 0$

$z = a, a$ pole of order 2 and inside points.

$$= \frac{1}{(2-1)!} \lim_{z\to a} \frac{d^{2-1}}{dz^{2-1}} \cancel{(z-a)^2} \left(\frac{\tan(\frac{z}{2})}{\cancel{(z-a)^2}}\right)$$

$= \frac{1}{1} \lim_{z \to a} \frac{d}{dz} \tan\left(\frac{z}{2}\right)$

$$\boxed{I = \frac{1}{2} \sec^2\left(\frac{a}{2}\right)}$$

**Standard problem-**

$$I = \int_0^{2\pi} \frac{1}{a+b\sin\theta}\, d\theta, \quad a > |b|$$

Soln- $C: |z| = 1$

$z = e^{i\theta} = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$, $dz = ie^{i\theta} d\theta$

$d\theta = \frac{dz}{iz}$.

$z + \frac{1}{z} = 2\cos\theta \Rightarrow \frac{z^2+1}{2z} = \cos\theta$

$z - \frac{1}{z} = 2i\sin\theta \Rightarrow \frac{z^2-1}{2iz} = \sin\theta$

$$\therefore I = \int_C \frac{dz}{iz\left[a + b\left(\frac{z^2+1}{\cos\theta}\right)\right]}$$

$$= \int_C \frac{dz}{\cancel{iz}\left[\frac{2aiz + bz^2 - b}{2i\cancel{z}}\right]} = \int_C \frac{2\,dz}{bz^2 + 2aiz - b}$$

$$= \frac{2}{b} \int_C \frac{dz}{z^2 + \frac{2aiz}{b} - 1} \qquad f(z) = \frac{1}{(z-\alpha)(z-\beta)}$$

$D_r = 0$

$$z = \frac{-2ai \pm \sqrt{\left(\frac{2ai}{b}\right)^2 + 4}}{2} = \frac{-2ai \pm \sqrt{-4a^2 + 4b^2}}{2b} \quad (-4(a^2-b^2))$$

$$= \frac{-2ai \pm 2i\sqrt{a^2-b^2}}{2b} = \frac{-ai \pm i\sqrt{a^2-b^2}}{b}$$

$a > b, \therefore a = 2, b = 1$

$$\alpha = \frac{-ai + i\sqrt{a^2-b^2}}{b} = \frac{-2i + i\sqrt{4-1}}{1} = -2i + \sqrt{3}i = -2+1.26i\, ?$$

$$\beta = \frac{-ai - i\sqrt{a^2-b^2}}{b} = -2i - i\sqrt{3} = -3.72i$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=\beta} = 0$$

$$\left[\operatorname{Res} f(z)\right]_{z=\alpha} = \lim_{z \to \alpha} \cancel{(z-\alpha)} \frac{1}{\cancel{(z-\alpha)}(z-\beta)}$$

$$= \frac{1}{\left(\frac{-ai + i\sqrt{a^2-b^2}}{b}\right) - \left(\frac{-ai - i\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{1}{\left(\frac{2i\sqrt{a^2-b^2}}{b}\right)} = \frac{b}{2i\sqrt{a^2-b^2}}$$

$\therefore$ By CRT

$$I = \cancel{2}\,2\pi i\,\frac{\cancel{b}}{\cancel{b}}\,\frac{1}{\cancel{i}\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

$\therefore$ Note! $\int_0^{\pi} \frac{1}{a+b\sin\theta} = \frac{1}{2}\int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta$

$$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}\pi}{\sqrt{a^2-b^2}} = \frac{\pi}{\sqrt{a^2-b^2}}$$

Standard problem

$$I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a} \quad 0<a<1$$

Soln:- $z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$.

$\cos\theta = \frac{z^2+1}{2z}$ $\sin\theta = \frac{z^2-1}{2iz}$. $d\theta = \frac{dz}{iz}$

$$I = \int_C \frac{dz}{iz\left[1-2a\left(\frac{z^2+1}{2z}\right)+a^2\right]}$$

$$= \int_C \frac{dz}{iz\left[\frac{z-a(z^2+1)+a^2z}{z}\right]}$$

$$= \int_C \frac{z\,dz}{iz(z-az^2-a+a^2z)}$$

$$= \frac{1}{i}\int_C \frac{dz}{\underbrace{z+a^2z}_{z(1+a^2)}-az^2-a}$$

$$= \frac{1}{i}\int_C \frac{dz}{-az^2+(1+a^2)z-a}$$

$$= \frac{1}{i}\,\frac{-1}{a}\int_C \frac{dz}{z^2-\left(\frac{1+a^2}{a}\right)z+1}$$

$$f(z) = \frac{1}{z^2 - \left(\frac{1+a^2}{a}\right)z + 1}$$

$Dr = 0$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{\left(\frac{1+a^2}{a}\right) \pm \sqrt{\left(\frac{1+a^2}{a}\right)^2 - 4}}{2}$$

$$= \left(\frac{1+a^2}{a}\right) \pm \sqrt{\frac{(1+a)^2 - 4a^2}{a^2}}$$

$$= \frac{\left(\frac{1+a^2}{a}\right)^2 \pm \frac{1}{a}\sqrt{1+(a^2)^2+2a^2-4a^2}}{2}$$

$$= \frac{\left(\frac{1+a^2}{a}\right) \pm \frac{1}{a}\sqrt{1+a^4-2a^2}}{2}$$

$$= \frac{\left(\frac{1+a^2}{a}\right) \pm \frac{1}{a}\sqrt{(1-a^2)^2}}{2} = \frac{(1+a^2) \pm (1-a^2)}{2a}$$

$$\alpha = \frac{1+a^2+(1-a^2)}{2a} = \frac{2}{2a} = \frac{1}{a}$$

$$\beta = \frac{(1+a^2)-(1-a^2)}{2a} = \frac{2a^2}{2a} = a$$

Take $a = \frac{1}{2}$,

$$\alpha = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$\beta = \frac{1}{2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=\alpha} = 0$$

$$\left[\operatorname{Res} f(z)\right]_{z=\beta} = \lim_{z \to \beta} (z-\beta) \frac{1}{(z-\beta)(z-\alpha)}$$

$$= \lim_{z \to \beta} \frac{z-\beta}{(z-\beta)(z-\beta)}$$

$$= \frac{1}{\beta - \alpha} = \frac{1}{\left(a - \frac{1}{a}\right)} = \frac{a}{a^2 - 1}$$

$\int f(z)\,dz = 2\pi i$ ( sum of res )

$= 2\pi i \left(\frac{a}{a^2-1}\right)$

$= \frac{-1}{ai} \times \frac{2\pi i a}{a^2-1}$

$= \frac{2\pi}{-(a^2-1)} = \frac{2\pi}{1-a^2}$

HW $\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5-4\cos\theta} = \frac{\pi}{6}.$

Soln $C = |z| = 1$

$z = e^{i\theta} = \cos\theta + i\sin\theta =$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$z + \frac{1}{z} = 2\cos\theta \Rightarrow \frac{z^2+1}{2z} = \cos\theta$ ; $z^n = \cos n\theta + i\sin n\theta$

$\frac{z^2-1}{2iz} = \sin\theta$

In Nr $\cos 2\theta = R.P$ of $z^2$

$I = R.P \int_C \frac{z^2\,dz}{zi\left[5-4\left(\frac{z^2+1}{2z}\right)\right]}$

$= R.P \int_C \frac{z^2\,dz}{\cancel{z}i\left[\frac{5z-2z^2-2}{\cancel{z}}\right]}$

$= \frac{R.P}{-2i} \int_C \frac{z^2\,dz}{z^2-\frac{5}{2}z+1}$

$= R.P \frac{1}{-2i} \int_C f(z)\,dz$ $\quad f(z) = \frac{z^2}{z^2-\frac{5}{2}z+1} = \frac{z^2}{(z-\alpha)(z-\beta)}$

$\alpha = 2 \quad \beta = \frac{1}{2}$

$\left[\text{Res } f(z)\right]_{z=2} = 0$

$\left[\text{Res} f(z)\right]_{z=\frac{1}{2}} = \lim_{z\to\frac{1}{2}} \cancel{\left(z-\frac{1}{2}\right)} \frac{z^2}{\cancel{\left(z-\frac{1}{2}\right)}(z-2)} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}-2\right)}$

$= \frac{\frac{1}{4}}{\frac{1-4}{2}} = \frac{1}{4(-3)}^{2} = -\frac{1}{6}$

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BY CRT

$\Omega = R.P \text{ of } \frac{1}{\cancel{2i}} \cancel{2}\pi i \left[ \frac{1}{6} \right]$

$= R.P \text{ of } \frac{\pi}{6} = \frac{\pi}{6}$

* Evaluate $I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$

Sol: $C: |z| = 1$

$z = e^{i\theta} = \cos\theta + i\sin\theta$, $\frac{1}{z} = \cos\theta - i\sin\theta$

$z + \frac{1}{z} = 2\cos\theta$

$\cos\theta = \frac{z^2+1}{2z}$, $z^m = \cos m\theta + i\sin m\theta$

R.P I.P.

$\sin\theta = \frac{z^2-1}{2zi}$

In Nr $\cos 3\theta = $ R.P of $z^3$

$$I = R.P \int_C \frac{z^3\, dz}{zi\left[5 - \cancel{4}^{2}\left(\frac{z^2+1}{\cancel{2}z}\right)\right]}$$

$$= R.P \int_C \frac{z^3 dz}{\cancel{z}i\left[\frac{5z - 2z^2 - 2}{\cancel{z}}\right]}$$

$$= R.P \frac{1}{i} \int_C \frac{z^3\, dz}{[5z - 2z^2 - 2]}$$

$$= R.P \frac{1}{-2i} \int_C \frac{z^3 dz}{\left(z^2 - \frac{5}{2}z + 1\right)}$$

$$= R.P \frac{1}{-2i} \int_C f(z) dz \quad f(z) = \frac{z^3}{z^2 - \frac{5}{2}z + 1} = \frac{z^3}{(z-\alpha)(z-\beta)}$$

$\alpha = 2$, $\beta = \frac{1}{2}$

outside inside

$$\left[\text{Res}\, f(z)\right]_{z=2} = 0$$

$$\left[\text{Res}\, f(z)\right]_{z=\frac{1}{2}} = \lim_{z\to\frac{1}{2}} \cancel{\left(z - \frac{1}{2}\right)} \frac{z^3}{\cancel{\left(z-\frac{1}{2}\right)}(z-2)}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{\frac{1}{2} - 2} = \frac{\frac{1}{8}}{\frac{1-4}{2}} = \frac{2}{24 \cdot -3} = \frac{1}{-12}$$

$$= \frac{-1}{12}$$

By C.R.T.

$\therefore I = R.P \frac{1}{2i} 2\pi i \left[\frac{-1}{12}\right]$

$= R.P \frac{\pi}{12} = \frac{\pi}{12}.$

HW $\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5-4\cos\theta} : \frac{\pi}{6}$

* Evaluate $I = \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta$

Soln $C : |z| = 1$

$z = e^{i\theta} = \cos\theta + i\sin\theta$ $\qquad \cos^2\theta = \frac{1 - \cos 2\theta}{2}$

$z + \frac{1}{z} = 2\cos\theta$ $\qquad = \frac{1 - R.P\text{ of } z^2}{2}$

$\cos\theta = \frac{z^2+1}{2z}$ $\qquad = R.P\left[\frac{1-z^2}{2}\right]$

$I = \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta = \frac{1}{2}\int_C \frac{1-z^2 \quad dz}{2\left(5+4\left(\frac{z^2+1}{2z}\right)\right) iz}$

$= \frac{1}{2} R.P \int_C \frac{1-z^2 \quad dz}{2\left(\frac{5z+2z^2+2}{z}\right) iz} = \frac{1}{4i} R.P \int_C \frac{(1-z^2)\, dz}{2z^2+5z+2}$

$= \frac{1}{8i} R.P \int_C \frac{1-z^2 \, dz}{z^2 + \frac{5z}{2} + 1}$

$= \frac{1}{8i}$ { $Dr = 0$

$\alpha = -\frac{1}{2}, \; -2 \rightarrow$ outside

$-\frac{1}{2} \downarrow$ inside

$[Res f(z)]_{z=-2} = 0$

$[Res f(z)]_{z=-\frac{1}{2}} = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \left[\frac{1-z^2}{\left(z+\frac{1}{2}\right)(z+2)}\right]$

$= \frac{1-\left(\frac{-1}{2}\right)^2}{\left(-\frac{1}{2}+2\right)} = \frac{1-\frac{1}{4}}{\left(\frac{-1+4}{2}\right)} = \frac{\left(\frac{4-1}{4}\right)}{\left(\frac{3}{2}\right)} = \frac{3}{6} = \frac{1}{2}$

$$I = \frac{1}{8i} R.P \int_c \frac{(1-z^2)\,dz}{z^2 - \frac{5}{2}z + 1}$$

$$= \frac{1}{8i} R.P\ (\pi i)\ \frac{1}{2}$$

$$= R.P\ \frac{\pi}{8} = \frac{\pi}{8}$$

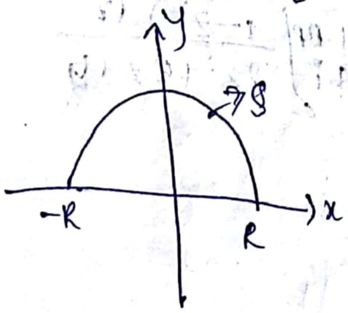
Type II. $\displaystyle\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\,dx$

$$\int_0^{\infty} \Rightarrow \frac{1}{2}\int_{-\infty}^{+\infty}$$

By C.R.T extended form, $x \to z$

$$\int_c f(z)\,dz = \int_{-\infty}^{+\infty} \frac{p(z)}{q(z)}\,dz$$

$$= 2\pi i \left[\text{sum of residues}\right] + \pi i \left[\begin{matrix}\text{sum of residues}\\ \text{on real axis}\end{matrix}\right] \quad \text{--- (1)}$$




From diagram,

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_S f(z)\,dz \quad \text{--- (2)}$$

If $R \to \infty$ $\displaystyle\int_S f(z)\,dz = 0$

SP I :- $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$

HW. SP II :- $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$

SP-III :- $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{a \times a \times 2a} = \frac{\pi}{2a^3}$

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Q. 1:- $\displaystyle\int_{-\infty}^{\infty} \frac{z^2\,dz}{(z^2+a^2)(z^2+b^2)}$

Sol: $D r = 0$

$z = -ai, +ai, \quad z = -bi, +bi$

$(0,a)\ (0,-a) \quad (0,b)\ (0,-b)$

outside points

$$\left[\operatorname{Res} f(z)\right]_{z=ai} = \lim_{z\to ai} \cancel{(z-ai)} \frac{z^2}{(z+ai)\cancel{(z-ai)}(z+bi)(z-bi)}$$

$$= \frac{(ai)^2}{(ai+ai)(ai+bi)(ai-bi)}$$

$$= \frac{-a^2}{(2ai)((ai)^2-(bi)^2)}$$

$$= \frac{-a^2}{2ai[-a^2+b^2]}$$

$$= \frac{\cancel{-}a^2}{\cancel{-}2ai(a^2-b^2)}$$

$$= \frac{a}{2i[a^2-b^2]}$$

$$\left[\operatorname{Res} f(z)\right]_{z=bi} = \lim_{z\to bi} \cancel{z-bi}\left[\frac{z^2}{(z+ai)(z-ai)(z+bi)\cancel{(z-bi)}}\right]$$

$$= \frac{(bi)^2}{(bi+ai)(bi-ai)(bi+bi)}$$

$$= \frac{-b^2}{2bi((bi)^2-(ai)^2)}$$

$$= \frac{-b^2}{2bi(-b^2+a^2)}$$

$$= \frac{-\cancel{b^2}}{2\cancel{b}i(a^2-b^2)} = \frac{-b}{2i(a^2-b^2)}$$

From (1)

$$I = 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right] + \pi i(0) = \cancel{2}\pi \cancel{i}\left[\frac{a-b}{\cancel{2i}(a^2-b^2)}\right] + \pi i(0)$$

$$= \frac{\pi\,\cancel{(a-b)}}{(a+b)\cancel{(a-b)}} = \boxed{\frac{\pi}{a+b}}$$

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from (2) D.ing

$$\int f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}.$$

Eg: $\dfrac{x^2\,dx}{(x^2+25)(x^2+36)}$

$a^2 = 25 \Rightarrow a = 5$

$b^2 = 36 \Rightarrow b = 6$

$$I = \frac{\pi}{a+b} = \frac{\pi}{5+6} = \frac{\pi}{11}$$

* $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-6x+2}{x^4+10x^2+9}\,dx$ → same with 'x' as thw

Sol: $\displaystyle\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} \frac{z^2-6z+2}{z^4+10z^2+9}\,dz.$

Poles Dr = 0,

$z^4+10z^2+9=0$

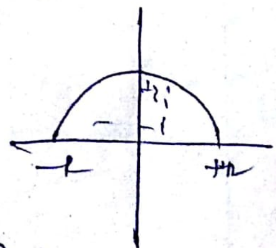
$z^2 = t$

$t^2+10t+9=0$

$t=-1,-9$

$z^2=-1 \qquad z^2=-9$

$z=+i,-i \qquad z=-3i,+3i$




$z=-i,-3i \Rightarrow$ outside.

$$\left[\operatorname{Res} f(z)\right]_{z=-i} = \left[\operatorname{Res} f(z)\right]_{z=-3i} = 0$$

$$\left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z\to i} \cancel{(z-i)}\,\frac{z^2-6z+2}{(z+i)\cancel{(z-i)}(z+3i)(z-3i)}$$

$$= \frac{i^2+6i+2}{z(i+i)(i+3i)(i-3i)}$$

$$\frac{6i+1}{(2i)(4i)(-2i)}$$

$$= \frac{6i+1}{8i^2(2i)}$$

$$= \frac{6i+1}{-16i} = \frac{1-6i}{16i}$$

$$[\text{Res} f(z)]_{z=3i} = \lim_{z\to 3i} \cancel{(z-3i)}\left[\frac{z^2-6z+2}{(z+3i)\cancel{(z-3i)}(z+i)(z-i)}\right]$$

$$= \frac{(3i)^2+6(3i)+2}{(3i+3i)(3i+i)(3i-i)}$$

$$= \frac{-9+18i+2}{(6i)(4i)(2i)}$$

$$= \frac{18i-7}{-48}$$

$$= \frac{7-18i}{48i}$$

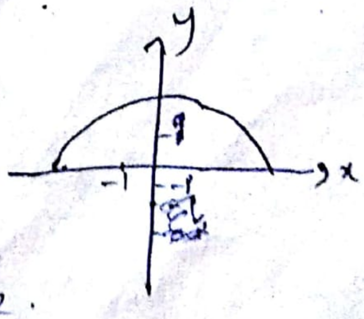
① $= \int_C f(z)\,dz = 2\pi i\left[\frac{10}{48i}\right] + \pi i(0)$

$$= \frac{5\pi}{12}.$$

H.W. $\displaystyle\int_{-\infty}^{\infty} \frac{(x^2-x+2)\,dx}{x^4+10x^2+9}$

* Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x+1)(x^2+1)}$

Sol:- $\displaystyle\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} \frac{z\,dz}{(z+1)(z^2+1)}$




$D = 0 \quad z = -1, i, -i$

outside.

$$[\text{Res} f(z)]_{z=-i} = 0$$

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$$[\operatorname{Res} f(z)]_{z=-1} = \lim_{z \to -1} \frac{\cancel{(z+1)}\, z}{\cancel{(z+1)}(z^2+1)}$$

$$= \frac{(-1)}{(-1)^2+1}$$

$$= \frac{-1}{1+1} = \frac{-1}{2}.$$

$$[\operatorname{Res} f(z)]_{z=i} = \lim_{z \to i} \cancel{(z-i)} \frac{z}{(z+1)(z+i)\cancel{(z-i)}}$$

$$= \frac{i}{(i+1)(i+i)}$$

$$= \frac{\cancel{i}}{2\cancel{i}(1+i)} = \frac{1}{2(i+i)}$$

$$① = \cancel{2}\pi i \left[\frac{1}{\cancel{2}(1+i)}\right] + \pi i \left[\frac{-1}{2}\right]$$

$$\frac{\pi i}{1+i} - \frac{\pi i}{2}$$

$$= \pi i \left[\frac{1}{1+i} - \frac{1}{2}\right]$$

$$= \pi i \left[\frac{2-1-i}{2(1+i)}\right]$$

$$= \pi i \left[\frac{1-i}{2(1+i)} \times \frac{(1-i)}{(1-i)}\right]$$

$$= \pi i \left[\frac{(1-i)^2}{2(1+i)^2}\right]$$

$$= \pi i \left[\frac{(1-i)^2}{8}\right]$$

$$= \cancel{\pi i (1+} \ \frac{\pi i (1-i)^2}{8}$$

SP-II

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$$

Dr = 0

$z = -ai, +ai$ $\quad z = -bi, +bi$

$(0,-a)$ $(0,a)$ $\quad (0,-b)$ $0,b$

outside points

$$[\mathrm{Res}\, f(z)]_{z=ai} = \lim_{z \to ai} \cancel{(z-ai)} \frac{1}{\cancel{(z-ai)}(z+ai)(z-bi)(z+bi)}$$

$$= \frac{1}{(ai+ai)(ai+bi)(ai-bi)}$$

$$= \frac{1}{2ai((ai)^2-(bi)^2)}$$

$$= \frac{1}{2ai(-a^2+b^2)}$$

$$= \frac{1}{2ai(b^2-a^2)}$$

$$[\mathrm{Res}\, f(z)]_{z=bi} = \lim_{z \to bi} \cancel{(z-bi)} \frac{1}{(z+ai)(z-ai)(z+bi)\cancel{(z-bi)}}$$

$$= \frac{1}{(bi+ai)(bi-ai)(bi+bi)}$$

$$= \frac{1}{2bi((bi)^2-(ai)^2)}$$

$$= \frac{1}{2bi(-b^2+a^2)}$$

$$= \frac{-1}{2bi(b^2-a^2)}$$

By CRT

$$I = 2\pi i \left[\frac{1}{2ai(b^2-a^2)} - \frac{1}{2bi(b^2-a^2)}\right] + \pi i(0)$$

$$= \frac{2\pi \cancel{i}}{2\cancel{i}} \left[\frac{b-a}{ab(b-a)(b+a)}\right]$$

$$\boxed{I = \frac{\pi}{ab(a+b)}}$$

Sp. III $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$

Dr = 0

$(x^2+a^2)^2 = 0$

$x^2+a^2=0 \quad x^2+a^2=0$

$x = +ai, -ai$

$x = +ai$, pole of order 2

$= (0, ai)(0, +ai)$

$x = -ai$ pole of order 2 outside

$(0,-ai)(0,-a)$

$$\left[\operatorname{Res} f(z)\right]_{z=ai} = \frac{1}{(2-1)!} \lim_{z\to ai} \frac{d}{dz}\left\{ \frac{\cancel{(z-ai)^2}}{(z+ai)(z+ai)\cancel{(z-ai)^2}} \right\}$$

$$= \frac{1}{1} \lim_{z\to ai} \frac{d}{dz} \frac{1}{(z+ai)^2}$$

$$= \lim_{z\to ai} \frac{-2}{(z+ai)^3}$$

$$= \frac{-2}{(ai+ai)^3}$$

$$= \frac{-2}{(2ai)^3} = \frac{2}{8a^3 i}$$

By CRT

$$I = 2\pi i\left[\frac{2}{8a^3 i}\right] + \pi i(0)$$

$$\boxed{I = \frac{\pi}{2a^3}}$$

HW 2 $\int_{-\infty}^{\infty} \frac{(x^2-x+2)dx}{x^4+10x^2+9}$

Sol $\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} \frac{z^2-z+2}{z^4+10z^2+9}\,dz$

Poles Dr = 0

$$z^4+10z^2+9=0$$

$$z^2=t$$

$$t^2+10t+9=0$$

$$t=-1,-9$$

$$z=+i,-i\ ,\ z=-3i,+3i$$

$$z=-i,-3i=\text{outside} \qquad z=i,+3i=\text{Inside}$$

$$[\operatorname{Res} f(z)]_{z=i}=\lim_{z\to i}\cancel{(z-i)}\frac{z^2-z+2}{\cancel{(z-i)}(z+i)(z-3i)(z+3i)}$$

$$=\frac{i^2-i+2}{(i+i)(i-3i)(i+3i)}$$

$$=\frac{i^2-i+2}{(2i)(-2i)(4i)}=\frac{-i+1}{-(4i^2)(4i)}$$

$$=\frac{\cancel{-i+1}}{\cancel{(-4)(4i)}}\quad\frac{-i+1}{-(16i^3)}$$

$$=\frac{1-i}{16i}$$

$$[\operatorname{Res} f(z)]_{z=3i}=\lim_{z\to 3i}\cancel{(z-3i)}\frac{z^2-z+2}{(z-i)(z+i)(z+3i)\cancel{(z-3i)}}$$

$$=\frac{(3i)^2-(3i)+2}{(3i-i)(3i+i)(3i+3i)}$$

$$=\frac{-9-3i+2}{(2i)(4i)(6i)}$$

$$=\frac{-3i-7}{-48i}$$

$$=\frac{3i+7}{48i}$$

$$I=2\pi i\left[\frac{1-i}{16i}+\frac{3i+7}{48i}\right]$$

$$=2\pi i\left[\frac{3-\cancel{3i}+\cancel{3i}+7}{48i}\right]$$

$$=\cancel{2}\pi\cancel{i}\left[\frac{\cancel{10}^{5}}{\cancel{48i}_{24}}\right]=\frac{5\pi}{12}$$