$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

$f(x) = e^{-ax}$; $g(x) = e^{-bx}$

Nr = 1 , FCT, Nr = 1, FCT

$F_c[f(x)] = F_c[e^{-ax}]$ $\quad F_c[g(x)] = F_c[e^{-bx}]$

$$= \sqrt{\frac{2}{\pi}} \times \frac{a}{s^2+a^2} \qquad = \sqrt{\frac{2}{\pi}} \times \frac{b}{s^2+b^2}$$

R.H.S : $\int_0^\infty F_c[f(x)] \cdot F_c[g(x)]\, ds = \frac{2}{\pi} ab \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}$

L.H.S : $\int_0^\infty f(x) \cdot g(x) \cdot dx = \int_0^\infty e^{-ax} . e^{-bx} . dx$

$$= \int_0^\infty e^{-(a+b)x} . dx = \left[\frac{e^{-(a+b)}}{-(a+b)} \right]_0^\infty = \frac{1}{a+b}$$

∴ L.H.S = R.H.S

$$\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

$$\int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$$

Take $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

Nr = x^2 ; FST Nr = x^2, FST.

$F_s[s] = F_s[e^{-ax}]$ $\quad F_s[e^{-bx}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx . dx.$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx dx \qquad = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+b^2} \right)$$

R.H.S : $\int_0^\infty F_s[f(x)]\, F_s[g(x)] ds = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds.$

L.H.S = $\int_0^\infty f(x) . g(x) . dx = \int_0^\infty e^{-ax} . e^{-bx} . dx$

$$= \int_0^\infty e^{-(a+b)x} \cdot dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty = \frac{1}{a+b}.$$

$$\therefore \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2(a+b)}$$

Take $f(x) = e^{-ax}$

Nr = 1, FCT $F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \times \int_0^\infty e^{-ax} \cos sx \, dx.$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{(s^2+a^2)}$$

Passeval's identity,

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$$

$$\text{R.H.S} = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2} \right) \right]^2 ds.$$

$$= \frac{2}{\pi} \int_0^\infty \frac{a^2}{s^2+a^2} \times ds = \frac{2a^2}{\pi} \int_0^\infty \frac{1}{s^2+a^2} ds.$$

$$\text{L.H.S.} = \int_0^\infty [f(x)]^2 dx.$$

$$= \int_0^\infty (e^{-ax})^2 dx = \int_0^\infty e^{-2ax} dx = \left[\frac{e^{-2ax}}{-2a} \right]_0^\infty = \frac{1}{2a}$$

$$\frac{1}{2a} = \frac{2a^2}{\pi} \int_0^\infty \frac{1}{(s^2+a^2)^2} ds.$$

$$\therefore \int_0^\infty \frac{1}{(s^2+a^2)^2} ds = \frac{\pi}{4a^3}.$$

Take $f(x) = e^{-ax}$

$Nr = 1$; FST

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right).$$

Parseval's identity, $\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(s)]^2 ds.$

$$R.H.S = \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right)^2 ds = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds.$$

$$L.H.S = \int_0^\infty [f(x)]^2 = \int_0^\infty (e^{-ax})^2 dx = \int_0^\infty e^{-2ax} . dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a}.$$

$$\therefore \frac{1}{2a} = \frac{2}{\pi} \int_0^\infty \frac{s^2}{s^2+a^2} ds.$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds = \frac{\pi}{4a}.$$

$F(x) = x^{m-1}$, $\frac{1}{\sqrt{x}}$ is SR under FCT.

$$\Gamma n = \int_0^\infty e^{y} . y^{n-1} dy \; ; \; F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} . \cos sx . dx$$

take $y = isx$.

y	0	∞
x	0	∞

$dy = (is) dx$

$$\Gamma n = \int_0^\infty e^{-isx} (isx)^{n-1} (is)^1 . dx$$

$$\frac{\Gamma n}{(is)^n} = \int_0^\infty x^{n-1} (\cos sx - i \sin sx) dx$$

$$\frac{\Gamma n}{s^n} (i)^{-n} = \int_0^\infty x^{m-1} . \cos sx . dx - i \int_0^\infty x^{n-1} \sin sx \, dx$$

$$\frac{\Gamma n}{s^n} \left[\cos \frac{n\pi}{2} - i \sin \frac{2\pi}{2}\right] = \int_0^\infty x^{n-1} \cos sx . dx - i \int_0^\infty x^{m-1} \sin sx . dx$$

Real part: $\frac{\Gamma n}{s^n}\left[\cos\frac{n\pi}{2}\right] = \int_0^\infty x^{n-1}\cos sx \cdot dx$ —①

: J-T :

img part: $-\frac{\Gamma n}{s^n}\sin\left(\frac{n\pi}{2}\right) = -\int_0^\infty x^{n-1}\sin sx \cdot dx$ —②

from ① and ②

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \times \frac{\Gamma n}{s^n}\cos\left(\frac{n\pi}{2}\right) \rightarrow *$$

$$F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}} \times \frac{\sqrt{n}}{s^n} \times \sin\left(\frac{n\pi}{2}\right) \rightarrow **$$

b) $F_c\left(\frac{1}{\sqrt{x}}\right)$

$= F_c(x^{-1/2})$

$= F_c(x^{-1/2+1-1}) = F_c(x^{1/2-1}) \quad \therefore n = 1/2$

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}} \times \frac{\Gamma 1/2}{3^{1/2}} \times \cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{s}}$$

$$f(x) = \begin{cases} a-|x|, & |x|<a \\ 0, & |x|>a \end{cases}$$

(1) F-T, $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) \cdot e^{isx} \cdot dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)(\cos sx + i\underset{\to 0}{\sin sx})dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a (a-x) \cdot \cos sx \cdot dx$$

$u = a-x \qquad v = \cos sx$

$u' = -1 \qquad v_1 = \sin sx/s$

$u'' = 0 \qquad v_2 = -\cos sx/s^2$

$$= \sqrt{\frac{2}{\pi}}\left[(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[(a-a\times 0) - \left(\frac{\cos sa}{s^2}\right)\right] - \left[a(0) - \frac{1}{s^2}\right]\Big] = \sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2(\frac{as}{2}}{s^2}\right.$$

F-I-T :

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} . ds.$$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \times \sqrt{\frac{2}{\pi}} \cdot \int_{-\infty}^{\infty} \frac{\sin^2(\frac{as}{2})}{s^2} [\cos x + i \sin x_0] dx$$

$$= \frac{2}{\pi} \times 2 \times \int_0^{\infty} \frac{\sin^2(as/2)}{s^2} (\cos sx) ds$$

take $x=0 ; a=2$

$$f(0) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2(as/2)}{s^2} ds$$

$$a = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2(as/2)}{s^2} ds.$$

$$2 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2(s)}{s^2} ds. \quad t=s.$$

$$\boxed{\therefore \int_0^{\infty} \left[\frac{\sin t}{t}\right]^2 dt = \pi/2}$$

By Peresaral's identity,

$$\text{L.H.S} = \int_{-\infty}^{\infty} (a-|x|))^2 dx$$

$$= \int_{-a}^{a} (a-x)^2 dx = 2\int_0^a (a-x)^2 = 2\int_0^a (a^2+x^2-2ax) dx$$

$$= 2 \left[a^2x + \frac{x^3}{3} - \frac{2ax^2}{2}\right]_0^a = 2\left[\left(a^3 + \frac{a^3}{3} - a^3\right) - (0+0+0)\right]$$

$$= 2a^3/3 //$$

R.H.S :

$$= \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$= \int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \frac{\sin^2(as/2)}{s^2}\right] ds$$

$$= \frac{4\times2}{\pi} \times \int_{-\infty}^{\infty} \frac{\sin^2(as/2)}{s^4} ds.$$

$$\frac{2\times8}{\pi} = \frac{16}{\pi} \int_0^{\infty} \frac{\sin^4 t}{t^4} ds. \quad a=2 ; s=t.$$

$$\therefore \int_0^{\infty} \frac{\sin^4 t}{t^4} dt = \pi/3$$

$$f(x) = \begin{cases} a^2-x^2, & |x|<a \\ 0, & |x|>a \end{cases}$$

F.T ; $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) . e^{isx} . dx.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (a^2-x^2) (\cos sx + i \underset{\downarrow}{\sin} sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \int_0^a (a^2-x^2) (\cos sx) dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a (a^2-x^2) (\cos sx) \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a (a^2-x^2) (\cos sx) dx.$$

$u = a^2-x^2$

$u' = -2x$

$u'' = -2$

$v = \cos sx$

$v_1 = \sin sx / s$

$v_2 = -\cos sx / s^2$

$v_3 = -\sin sx / s^3.$

$$= \sqrt{\frac{2}{\pi}} \left[(a^2-x^2) \left(\frac{\sin sx}{s}\right) - (2x) \left(\frac{\cos sx}{s^2}\right) + (2) \left(\frac{\sin sx}{s^3}\right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 (\sin as - as \cos as)}{s^3} \right]$$

$$= 2 \sqrt{\frac{2}{\pi}} \left[\frac{\sin as - as \cos as}{s^3} \right]$$

F.I.T, $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) . e^{-isx} . dx$

$$= \frac{1}{\sqrt{2\pi}} \times 2 \times \int_{-\infty}^{\infty} \left[\frac{\sin as - as \cos as}{s^3} \right] [\cos sx - i \underset{\downarrow 0}{\sin sx}] \, ds$$

take $x = 0$; $a = 1$; $s = t$; $ds = dt$.

$$1 = \frac{2}{\pi} \times 2 \int_0^{\infty} \left[\frac{\sin s - s \cos s}{s^3} \right] ds.$$

$$\therefore \int_0^{\infty} \frac{\sin t - t \cos t}{t^3} dt = \frac{\pi}{4}$$

By persavel's identity, $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$

L.H.S : $= \int_{-\infty}^{\infty}(a^4+x^4-2a^2x^2)\,dx \;=2\int_0^a (a^4+x^4-2a^2x^2)\,dx$

$= 2\left[\frac{8a^5}{15}\right] = \frac{16a^5}{15}.$

R.H.S : $\frac{16a^5}{15} = \int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]\right]^2 ds$

$\frac{16a^5}{15} = 4\times\frac{2}{\pi}\times 2\int_0^{\infty}\left[\frac{\sin as - as\cos as}{s^3}\right]^2 ds$

Put $s=t, a=1$.

$\therefore \int_0^{\infty}\left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt = \pi/15$

Convolution:

Given, $f(x) = g(x) = e^{-x^2}$

R.H.S : To prove : $F[f(x)*g(x)] = F[f(x)]\,F[g(x)]$.

W.K.T. $F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-isx}\,dx$

$F[e^{-x^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-isx}\,dx.$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2+isx)}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-[(x+is/2)^2+\frac{s^2}{4}]}\,dx.$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x+is/2)^2}\cdot e^{-s^2/4}\,dx$

$= e^{-s^2/4}\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{-(x+is/2)^2}\,dx$

Put $t = x+\frac{is}{2}$ | $x\to-\infty$ | $t\to-\infty$

$dt = dx$ | $x\to\infty$ | $t\to\infty$

$= e^{-s^2/4}\left[\frac{1}{\sqrt{2t}}\right]\int_{-\infty}^{\infty} e^{-t^2}\,dt.$

$= e^{-s^2/4}\left[\frac{1}{\sqrt{2\pi}}\right]\sqrt{\pi}$

$= \frac{1}{\sqrt{2}} \times e^{-s^2/4}$

$F[f(x)] = \frac{1}{\sqrt{2}} e^{-s^2/4}$

$F[g(x)] = F[f(x)] = \frac{1}{\sqrt{2}} \cdot e^{-s^2/4}$

$\therefore F[g(x)] \cdot F[g(x)] = \left[\frac{1}{\sqrt{2}} e^{-s^2/4}\right]\left[\frac{1}{\sqrt{2}} \cdot e^{-s^2/4}\right]$

$= \frac{1}{2} \times e^{-s^2/2}$ — ①

L.H.S :

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \cdot g(x-u) \cdot du.$$

$$e^{-x^2} \cdot e^{-x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{-(x-u)^2} du.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-u^2)-u^2} \cdot du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2(u-x/2)^2} e^{-x^2/2} du.$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2(u-x/2)^2} du.$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2(u-x/2)^2} du.$$

$t = u - x/2$ | $u \to -\infty \Rightarrow t \to -\infty$

$dt = du$ | $u \to \infty \Rightarrow t \to \infty.$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \times \int_{-\infty}^{\infty} e^{-2t^2} dt$$

Put $y = \sqrt{2}\, t$

$dy = \sqrt{2} \cdot dt$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} \cdot \frac{dy}{\sqrt{2}}$$

$$= \frac{e^{-x^2/2}}{2\sqrt{\pi}} \times \int_{-\infty}^{\infty} e^{-y^2} dy.$$

$$= \frac{1}{2} . e^{-x^2/2}$$

$$F[f(x) * g(x)] = F\left[\frac{1}{2} e^{-x^2/2}\right]$$

$$= \frac{1}{2} F[e^{-x^2/2}].$$

$$F[e^{-x^2/2}] = e^{-s^2/2}$$

$$\therefore F[f(x) * g(x)] = \frac{1}{2} \times e^{-s^2/2} \longrightarrow ②$$

① = ② //