Unit-4

Fourier Transforms.

* Infinite form of fourier series is fourier transform

Overview - Fourier transforms

- Self reciprocal F.T (aq)
- parsevals identity F.T (ds)
- Convolution (dt)
- F.I. Formula
- Relation b/w L-T, F.T

Derivations

- F. cosine T
- F. I. C T
- Finite Fourier T.
- properties (2m)
- F. sin T
- F. I. S. T.
- Sol. of B.V.P
- Integral eqns

Def of Fourier Transform (F.T) [Infinite fourier transform or complex fourier transform].

A function $f(x)$ which is define and continuous on $(-\infty$ to $+\infty)$ then the F.T of $f(x)$ is given by

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx. \quad \text{--- (1)}$$

F.I.T is given by

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds. \quad \text{--- (2)}$$

Note Eq (1) & (2) are called as fourier transform pairs.

Note The other forms of F.T and F.I.T is given in the following table.

F.T $F(s)$	F.I.T $f(x)$
$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}ds$
$\int_{-\infty}^{\infty} f(x)\, e^{isx}dx$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}ds$
$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{isx}dx$	$\int_{-\infty}^{\infty} F(s)\, e^{-isx}dx.$

Fourier integral theorem for $f(x)$

Fourier integral theorem for $f(x)$ which is integrable and is piece wise continuous and ~~double~~ in $(-\infty, \infty)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-is(t-x)}\, dt\, ds$$

Derivation of Fourier inversion Formula or Theorem

Statement:- Let $f(x)$ be a function which satisfying Dirichlet's condition in every finite interval $(-l, +l)$. Denote the fourier ~~of~~ transform of $f(x)$ by $F(s)$. Then at every point of continuity the value of $f(x)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx} ds$$

Proof:-

By fourier integral theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-isx}\, e^{-ist}\, dt\, ds$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}dt \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}ds$$

t→dummy

$$= F(s)\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} ds$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds}$$

Derivation of parseval's identity / theorem (or) Rayleigh Theorem / Plancherel's Identity

Statement:- If $F(s)$ is the fourier transform of $f(x)$ then the relationship between F.T and F.I.T is given by Parseval's identity.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

Proof: W.K.T the convolution of two func is given by

$$f(x)*g(x) \text{ (or) } f*g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

By convolution theorem

$$F[f(x)*g(x)] = F(s) \cdot G(s)$$

where $F(s) = F[f(x)]$

$G(s) = F[g(x)]$

$$\Rightarrow f(x)*g(x) = F^{-1}[F(s)\, G(s)]$$

$$\cancel{\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} f(t) g(x-t) dt = \cancel{\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} F(s) G(s) e^{-isx} ds.$$

$$g(t) = \overline{f(t)}$$

$$G(s) = \overline{F(s)}$$

$$\int_{-\infty}^{\infty} f(t)\, \overline{f(t)}\, dt = \int_{-\infty}^{\infty} F(s)\, \overline{F(s)}\, ds$$

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} (F(s))^2 ds.$$

($t \to$ dummy)

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

H.

1. Show that $f(x) = e^{-x^2/2}$ is self reciprocal.

Sol:

$$F.T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{+isx} dx.$$

$$e^{-x^2/2} e^{isx} = e^{\frac{-x^2}{2} + isx}$$

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

$$= e^{\frac{-x^2+2isx}{2}}$$

$$\int_{0}^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$

$$= e^{-\left[\frac{x^2-2isx}{2}\right]}$$

$$x^2 - 2isx = x^2 - 2isx + (is)^2 - (is)^2.$$

$A^2 = x^2$

$A = x$

$$= e^{-\frac{[(x-is)^2 + s^2]}{2}}$$

$$= e^{-\left[\frac{(x-is)^2}{(\sqrt{2})^2}\right]} e^{-s^2/2}$$

$$= e^{-\left[\frac{x-is}{\sqrt{2}}\right]^2} \times e^{-s^2/2}$$

$$F.T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{2}\right)^2} e^{-s^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{2}\right)^2} dx.$$

put $y = \frac{x-is}{\sqrt{2}}$

Limits		
x	$-\infty$	∞
y	$-\infty$	∞

$$dy = \frac{1}{\sqrt{2}} dx$$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \sqrt{2}$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy.$$

$$= \frac{e^{-s^2/2} \sqrt{\pi}}{\sqrt{\pi}} = e^{-s^2/2}. \text{ (In terms of 's')}$$

$\therefore f(x)$ is self reciprocal under F.T.

* Under the fourier transform ~~show that~~ ~~f(x) =~~ $f(x) = e^{-a^2x^2}$ where a is +ve ~~is self reciprocal~~. and hence show that $e^{-x^2/2}$ is self reciprocal. where as e^{-x^2} is not self reciprocal.

Sol:-

$$F.T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{+isx} dx.$$

$$e^{-a^2x^2} e^{+isx} = e^{-a^2x^2+isx}$$

$$= e^{-(a^2x^2 - isx)}$$

$A^2 = a^2x^2$ $\quad B = \frac{is}{2a}$

$2AB = \frac{2a(x)(is)}{2a}$

$$= e^{-\left[a^2x^2 - \frac{2a}{2a} isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]}$$

$$= e^{-\left(\left[ax - \frac{is}{2a}\right]^2 + \frac{s^2}{4a^2}\right]}$$

$$F.T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} e^{-\frac{s^2}{4a^2}} dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx.$$

Put $y = ax - \frac{is}{2a}$ $\quad \left| \begin{array}{l} x \; -\infty \; \infty \\ y \; -\infty \; \infty \end{array} \right.$

$dy = a dx$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} \frac{dy}{a}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{e^{-s^2/4a^2}\sqrt{\pi}}{a\sqrt{2\pi}}$$

$$F.T = F(s) = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

$$F(s) = F[f(x)] = \frac{e^{-s^2/4a^2}}{\sqrt{2}\,a}$$

$$F[e^{-a^2x^2}] = \frac{e^{-s^2/4a^2}}{\sqrt{2}\,a} \quad \text{--- (1)}$$

$F[e^{-x^2/2}] =$ put $a^2 = \frac{1}{2}$ in (1)

$$a = \frac{1}{\sqrt{2}}$$

$$= \frac{e^{-s^2/(4(1/2))}}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)}$$

$$= e^{-s^2/2}$$

$\therefore f(x)$ is self reciprocal under F.T.

$F[e^{-x^2}] \Rightarrow$ put $a^2 = 1$ in (1)

$$a = 1$$

$$F(s) = \frac{e^{-s^2/4(1)}}{\sqrt{2}(1)}$$

$$= \frac{e^{-s^2/4}}{\sqrt{2}}$$

$\therefore f(x)$ is not self reciprocal under F.T.

* Find F.T of $f(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$

(a) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ (F.I.T)

Sol:- (b) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$ (P.I)

Sol:-

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$\therefore |x| < 1 \Rightarrow -1 < x < 1.$

$|x| = \begin{cases} +x & x \text{ is +ve} \\ -x & x \text{ is -ve} \end{cases}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} [1-|x|] e^{isx}\, dx$$

$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a & \text{even} \\ 0 & \text{odd} \end{cases}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} [1-|x|] [\cos sx + i s \sin(x)]\, dx,$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} [1-|x|] \cos sx\, dx + i \int_{-1}^{1} [1-|x|] \sin sx\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{1} [1-x] \cos sx\, dx$$

$u = 1-x \qquad v = \cos x$

$u' = -1 \qquad v_1 = +\frac{\sin sx}{s}$

$v_2 = -\frac{\cos sx}{s^2}$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[1-x \left[\frac{\sin sx}{s}\right) - \left(\frac{\cos sx}{s^2}\right) \right]_{x=0}^{x=1}$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ 0 - \frac{\cos s}{s^2} \right\} - 0 \left\{ 0 - \frac{\cos 0}{s^2} \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \times \left[\frac{1-\cos s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{2 \sin^2(s/2)}{s^2}$$

$$F(s) = \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{\sin^2(s/2)}{s^2} \qquad s \longrightarrow -\infty \text{ to } \infty$$

(ii) F.I.T =

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds.$$

$$= \frac{1}{\sqrt{2\pi}} 2\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin^2(s/2)}{s^2} e^{-isx}\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(s/2)}{s^2} [\cos sx - i \sin sx] \, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(s/2)}{s^2} (\cos sx) \, ds.$$

$$= \frac{2}{\pi} 2 \times \int_{0}^{\infty} \frac{\sin^2(s/2)}{s^2} \cos sx \, ds$$

Take $x=0$

$$f(0) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin^2(s/2)}{s^2} \, ds.$$

$$\Rightarrow \frac{\pi}{4} = \int_{0}^{\infty} \frac{\sin^2(s/2)}{s^2} \, ds.$$

Take $t = s/2$ $\quad s = 2t$ $\quad s: 0 \; \infty$

$dt = \frac{ds}{2}$ $\quad t = \frac{s}{2}: 0 \; \infty.$

$ds = 2dt$

$$\frac{\pi}{4} = \int_{0}^{\infty} \frac{\sin^2 t \; 2dt}{(2t)^2}$$

$$\frac{\pi}{\cancel{4}_2} = \frac{1}{\cancel{2}} \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$$

$$\frac{\pi}{2} = \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2} //.$$

(ii) Parsevals formula

$$\int_{-\infty}^{\infty} |f(x)|^2 dx \quad \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

$$L.H.S = \int_{-1}^{+1} |1-|x||^2 dx = 2\int_{0}^{1} [1-x]^2 dx$$

Scanned by CamScanner

$$= 2\left\{\int_0^1 1\,dx + \int_0^1 x^2 dx\right\} + 2\int_0^1 x\,dx$$

$$= 2\left\{[x]_0^1 + \left[\frac{x^3}{3}\right]_0^1\right\} + 2\left[\frac{x^2}{2}\right]_0^1$$

$$= 2\left\{1 + \frac{1}{3}\right\} - 2\left[\frac{1}{2}\right]$$

$$= 2\left\{\cancel{\frac{3+1}{3}}\right\} = \frac{4}{3} \quad +\frac{2}{3}.$$

$$R.H.S = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$|F(s)|^2 = 4\,\frac{2}{\pi} \times \frac{\sin^4(s/2)}{s^4}$$

$$= \frac{8}{\pi}\int_{-\infty}^{\infty} \frac{\sin^4(s/2)}{s^4}\,ds. \qquad t = \frac{s}{2} \quad dt = \frac{ds}{2}.$$

$$= \frac{8\times 2}{\pi}\int_0^{\infty} \frac{\sin^4(s/2)}{s^4}\,ds.$$

$$= \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 t \; 2dt}{(2t)^4}$$

By parseval

$$\frac{2}{3} = \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 t}{t^4}\,dt$$

$$\int_0^{\infty} \frac{\sin^4 t}{t^4}\,dt = \frac{\pi}{3}.$$

$*$ $f(x) = \begin{cases} a-|x| & , |x|<a \\ 0 & , |x|>a \end{cases}$

Sol: (a) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

(b) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Sol: F.T $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{+isx}\, dx$,

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|]\, e^{+isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|]\, [\cos sx - i \sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|] \cos sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_{0}^{a} (a-x) \cos x\, dx.$$

$u = a-x$, $u' = -1$

$v = \cos x$

$v_1 = \frac{+\sin sx}{s}$

$v_2 = \frac{-\cos sx}{s^2}$

$$= \sqrt{\frac{2}{\pi}} \left[(a-x)\left[\frac{+\sin sx}{s}\right] - (-1)\left[-\frac{\cos sx}{s^2}\right] \right]_{x=0}^{x=a}$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos as}{s^2} + \frac{1}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos as}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}}\; \frac{2 \sin^2\left(\frac{as}{2}\right)}{s^2}$$

(i) From Fourier Inverse transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\, e^{-isx}\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \left[\cos sx - i \sin sx\right] ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} (\cos sx)\, ds$$

put $x = 0$, $a = 2$

$$f(0) = \frac{2 \times 2}{\pi} \int_{0}^{\infty} \frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\, ds.$$

$a(2)$

$$\frac{(2)\pi}{4 \cdot 2} = \int_{0}^{\infty} \frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\, ds$$

$s = 0, \infty$

$t = \frac{s}{2}$, $t = 0, \infty$

$\boxed{s = t}$

$dt = \frac{ds}{2}$

$ds = 2\, dt$

$$\boxed{\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}}$$

$$\frac{a\pi}{2} = \frac{1}{4} \int_{0}^{\infty} \frac{\sin^2 t}{(2t)^2}\, 2\, dt$$

$$\frac{a\pi}{2} = \frac{1}{2} \int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt$$

$$\frac{a\pi}{2} = \int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt$$

(ii) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$

LHS $= \int_{-a}^{a} (a-|x|)^2 dx = 2\int_0^a (a-x)^2 dx$

$= 2\left\{\int_0^a a^2 dx + \int_0^a x^2 dx - \int_0^a 2ax\, dx\right\}$

$= 2\left\{a^2 (x)_0^a + \left(\frac{x^3}{3}\right)_0^a - 2a\left(\frac{x^2}{2}\right)_0^a\right\}$

$= 2\left\{a^3 + \frac{a^3}{3} - \frac{2a^3}{2}\right\} = 2\left\{\frac{2a^3}{3}\right\}$

RHS $= \int_{-\infty}^{\infty} (F(s))^2 ds$. $(F(s))^2 = \frac{4}{\cancel{\ }} \frac{2}{\pi} \frac{\sin^4\left(\frac{as}{2}\right)}{s^4}$

$\frac{8}{\pi}\int_{-\infty}^{\infty} \frac{\sin^4\left(\frac{as}{2}\right)}{s^4} ds.$

$\frac{2a^3}{3} = \frac{8}{\pi}\int_{-\infty}^{\infty} \frac{\sin^4\left(\frac{as}{2}\right)}{s^4} ds$

$a = 2 \qquad s \to t$

$\frac{2(8)}{3} = \frac{8}{\pi} \times 2 \int_0^{\infty} \frac{\sin^4 t}{t^4} dt.$

$\frac{\pi}{3} = \int_0^{\infty} \frac{\sin^4 t}{t^4} dt$

$$\boxed{\int_0^{\infty} \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3}}$$

//..

* Show that the F.T of

$$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| < a \end{cases} = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$$

(a) $$\int_0^\infty \frac{\sin t - t\cos t}{t^3} = \frac{\pi}{4}$$

(b) $$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 = \frac{\pi}{15}.$$

Sol) F.T

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{+isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2 - x^2)\left[\cos sx - i\sin sx\right] dx.$$

$$= \frac{(1)2}{\sqrt{2\pi}}\int_0^a (a^2 - x^2)\cos sx\, dx$$

$u = a^2 - x^2$ $\qquad$ $v = \cos sx$

$u' = -2x$ $\qquad$ $v_1 = \cancel{\cos}\ \frac{\sin sx}{s}$

$u'' = -2$ $\qquad$ $v_2 = -\frac{\cos sx}{s^2}$

$\qquad\qquad$ $v_3 = -\frac{\sin sx}{s^3}$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (2x)\left[\frac{\cos sx}{s^2}\right] + 2\,\frac{\sin sx}{s^3}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[-2a\left[\frac{\cos sa}{s^2}\right] + \frac{2\sin sa}{s^3}\right] - 0\left[\cancel{\frac{2x}{s^2}}\right]\right\}$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{-a\cos sa}{s^2} + \frac{\sin sa}{s^3}\right]$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - sa\cos sa}{s^3}\right]$$

(a) F.I.T

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} \frac{\sin sa - sa\cos sa}{s^3}\left[\cos sx - i\sin sx\right] ds$$

$$= \frac{2}{\pi}\times 2\int_{0}^{\infty} \frac{\sin sa - sa\cos sa}{s^3}\cos sx\, ds$$

$x = 0,\ a = 1$

$$f(0) = \frac{2}{\pi}\times 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds$$

$f(0)\ a^2 - x^2$
$= (1)^2 - 0$
$= 4$

$s \to t$

$$1 = \frac{2\times 2}{\pi}\int_{0}^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt$$

$$\boxed{\int_{0}^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}}$$

(b) Parseval's formula

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds.$$

$$L.H.S. = \int_{-a}^{a} (a^2 - x^2)^2\, dx =$$

$$= 2\int_{0}^{a} (a^2 - x^2)^2\, dx = 2\int_{0}^{a} (a^4 + x^4 - 2a^2x^2)\, dx$$

$$= 2\left[\int_{0}^{a} a^4\, dx + \int_{0}^{a} x^4\, dx - \int_{0}^{a} 2a^2x^2\, dx\right]$$

$$= 2\left[a^4[x]_0^a + \left[\frac{x^5}{5}\right]_0^a - 2a^2\left[\frac{x^3}{3}\right]_0^a\right]$$

Scanned by CamScanner

$$= 2\left[a^5 + \frac{a^5}{5} - \frac{2a^5}{3}\right]$$

$$= 2\left[\frac{15a^5 + 3a^5 - 10a^5}{15}\right]$$

$$= 2\left(\frac{8a^5}{15}\right)$$

$$= \frac{16a^5}{15}$$

R.H.S

$$|F(s)|^2 = 4 \cdot \frac{2}{\pi}\left[\frac{\sin sa - sa\cos a}{s^3}\right]^2$$

$$= \frac{8}{\pi}\left[\frac{\sin sa - sa\cos a}{s^3}\right]^2$$

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa - sa\cos a}{s^3}\right)^2 ds$$

$$16a = \frac{8}{\pi} \times 2 \int_{0}^{\infty}\left(\frac{\sin sa - sa\cos sa}{s^3}\right)^2 ds$$

as $LHS = RHS$

$$\frac{16a^5}{15} = \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin sa - sa\cos sa}{s^3}\right)^2 ds$$

Put $a = 1$ $\quad s \to dt$

$ds = dt$

$$\frac{\pi}{15} = \int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt$$

$$\boxed{\int_0^{\infty}\left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt = \frac{\pi}{15}}$$

HW

Show that the F.T of

$$f(x) = \begin{cases} 4-x^2, & |x|<2 \\ 0, & |x|<2 \end{cases}$$

(a) $\int_0^\infty \frac{\sin t - t\cos t}{t^2}$ $\frac{\pi}{4}$

(b) $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2$ $\frac{\pi}{15}$

* Find the F.T of $f(x) = \begin{cases} 0 & |x|<a \\ 1 & |x|>a \end{cases}$

where a the integer and hence deduce the

(a) $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$. (b) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Sol:

F.T

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{+isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} 1.(\cos sx + i\sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^a \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[+\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left(+\frac{\sin sa}{s}\right] + 0$$

$$\boxed{F(s) = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}}$$

(ii) F.I.T

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin sa}{s} (\cos sx - i\sin sx) ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin sa}{s}\cos sx\, ds$$

put $x=0$ $a=1$ $s=t$ $ds=dt$

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin t}{t}\,dt$$

$$\boxed{\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}}$$

(ii) By parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

LHS $\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-a}^{a} (1^2)\,dx$

$$= [x]_{-a}^{a} = a-(-a) = 2a$$

RHS $= \int_{-\infty}^{\infty} |F(s)|^2\,ds =$

$$|F(s)|^2 = \frac{2}{\pi}\,\frac{\sin^2 sa}{s^2}$$

$$\int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin^2 sa}{s^2}\right) ds = \frac{2(2)}{\pi}\int_0^\infty \frac{\sin^2 sa}{s^2}\,ds$$

$s \to t$ $a=1$
$ds=dt$

$$2 = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$$

$$\boxed{\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}}$$

Hw

Scanned by CamScanner

* Find the F.T of $f(x)=\begin{cases}1-x^2, & |x|<1\\ 0, & |x|>1\end{cases}$ Hence S.T.

(a) $\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \pi/16$

(b) $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)^2 dx = \pi/15.$

* Find the F.T of $f(x)=\{ e^{-a|x|}, \; a>0$

and hence deduce the (a) $\int_0^\infty \frac{\cos xt}{a^2+t^2}\,dt = \frac{\pi}{2a}e^{-a|x|}$

(b) $\int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a^3}$

(c) $F\left[x\, e^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}}\cdot 2a$

Sol:

$$F.T\,F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\left[\cos sx + i\sin sx\right] dx$$

$$= \frac{1}{\sqrt{2\pi}}\times 2\int_0^\infty e^{-ax}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \quad \because \int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}$$

$$\int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}$$

$$= \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$$

$$\boxed{F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]}$$

(a) F.I.T

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,dx$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}}\sqrt{\frac{2}{\pi}}\,a\int_{-\infty}^{\infty}\frac{1}{a^2+s^2}(\cos sx - i\sin sx)\,dx$$

$$f(x) = \frac{a}{\pi}\times 2\int_{0}^{\infty}\frac{1}{a^2+s^2}\cos sx\,dx$$

$s = t \quad ds = dt$

$$\frac{\pi}{2a}f(x) = \int_0^{\infty}\frac{\cos xt}{a^2+t^2}\,dt$$

$$\int_0^{\infty}\frac{\cos xt}{a^2+t^2}\,dt = \frac{\pi}{2a}e^{-a|x|}$$

(b) $$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|F(s)|^2\,ds.$$

$$LHS = \int_{-\infty}^{\infty}\left(e^{-a|x|}\right)^2 dx$$

$$= 2\int_0^{\infty}\left(e^{-ax}\right)^2 dx$$

$$= 2\int_0^{\infty} e^{-2ax}\,dx.$$

$$= 2\left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty}$$

$$= \frac{-1}{a}\left[0 - 1\right]$$

$$= \frac{-1}{a}(-1) = \frac{1}{a}$$

R.H.S.-

$$|F(s)|^2 = \frac{2}{\pi}\left[\frac{a}{a^2+s^2}\right]^2$$

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{a}{a^2+s^2}\right]^2 ds$$

$$= \frac{2\times 2}{\pi}\int_0^{\infty}\left[\frac{a}{a^2+s^2}\right]^2 ds$$

L.H.S = R.H.S.

$$\frac{1}{a} = \frac{4a^2}{\pi}\int_0^{\infty}\frac{1}{(a^2+s^2)^2}\,ds$$

$s \rightarrow x$

$ds = dx$

$$\frac{\pi}{4a^3} = \int_0^{\infty}\frac{1}{x^2+a^2}\,dx.$$

$$\boxed{\int_0^{\infty}\frac{1}{x^2+a^2}\,dx = \frac{\pi}{4a^3}.}$$

(c) Derivative ~~Modulation~~ property.

$$F\left[x^n f(x)\right] = (-i)^n \frac{d^n}{ds^n}\left[F(s)\right]$$

$$F\left[x\, e^{-a|x|}\right] = (-i)^1 \frac{d^1}{ds^1}\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+s^2}$$

$$= -i\sqrt{\frac{2}{\pi}}\,a\,\frac{-2s}{(s^2+a^2)^2}$$

$$\boxed{F\left[x\, e^{-a|x|}\right] = +i\sqrt{\frac{2}{\pi}}\,\frac{2as}{(s^2+a^2)^2}}$$

H.W Find F.T of $f(x) = \begin{cases} 1 & |x|<2 \\ 0 & |x|>2 \end{cases}$
(1)

(a) $\int_0^\infty \frac{\sin \alpha}{\alpha} d\alpha$ (b) $\int_0^\infty \left(\frac{\sin \beta}{\beta}\right)^2 d\beta$

Find F.T
(1) $f(x) = e^{-|x|}$ S.T $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$.

* Find F.T of $f(x) = \frac{1}{\sqrt{|x|}}$; $x = \begin{cases} x & , x>0 \\ -x & , x<0 \end{cases}$

Sol:- $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{1}{\sqrt{-x}} e^{isx} dx + \int_0^\infty \frac{1}{\sqrt{x}} e^{isx} dx.$

$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{\cos sx}{\sqrt{x}} dx$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty R.P \text{ of } \frac{(e^{-isx})}{\sqrt{x}} dx$

$= R.P \text{ of } \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-isx}}{\sqrt{x}} dx \times \frac{is}{\sqrt{is}\sqrt{is}}$

$= R.P \text{ of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-isx} \frac{d(isx)}{\sqrt{isx}} \times \frac{1}{\sqrt{is}}$

$\left[\begin{array}{lll} t = isx & x \to -\infty & \infty \\ & t \to -\infty & \infty \end{array} \right]$

$= R.P \text{ of } \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{is}} \int_0^\infty \frac{e^{-t} dt}{\sqrt{t}}$

$= R.P \text{ of } \sqrt{\frac{2}{\pi}} \times \frac{1}{\sqrt{is}} \int_0^\infty e^{-t} t^{-1/2} dt$

$= R.P \text{ of } \sqrt{\frac{2}{\pi}} \times \frac{1}{\sqrt{is}} \sqrt{\pi}$ $\quad \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx.$

$F(s) = R.P \text{ of } \frac{\sqrt{2}}{\sqrt{s}} i^{-1/2}$

$r\cos\theta + ir\sin\theta = 0 + i$ —— 0

$\boxed{r=1}$ $\cos\theta = 0$ $\quad r\cos\theta = 0$
$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ $\quad r\sin\theta = 1$

Sub $r=1$ & $\theta=\frac{\pi}{2}$ in (1)

$(i) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$(i)^{-1/2} = (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^{-1/2}$

$= \cos\left[\frac{\pi}{2} \times \frac{-1}{2}\right] + i\sin\frac{\pi}{2} \times \frac{-1}{2})$

$i^{-1/2} = \cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)$

$\therefore F(s) = \frac{\sqrt{2}}{\sqrt{s}} \cos\left(\frac{\pi}{4}\right)$

$= \frac{\sqrt{2}}{\sqrt{s}} \times \frac{1}{\sqrt{2}}$

$f(s) = \frac{1}{\sqrt{s}}.$

HW: $f(x) = \begin{cases} 4-x^2, & |x|<2 \\ 0, & |x|<2 \end{cases}$

F.T

$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} 4-x^2\, \cancel{dx}\, e^{isx}$

$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} 4-x^2 (\cos sx + i\sin sx)\, dx$

$= \frac{1 \times 2}{\sqrt{2\pi}} \int_{0}^{2} (4-x^2) \cos sx\, dx.$

$= \sqrt{\frac{2}{\pi}} \cancel{\int_0^2 4\cos sx\, dx} - \int_0^2 x^2 \cos sx\, dx$

$= \sqrt{\frac{2}{\pi}} \left\{ \cancel{4\left[\sin sx\right]_0^2} - \right.$

$u = 4-x^2$ $\quad v = \cos sx$

$u' = -2x$ $\quad v_1 = \frac{\sin sx}{s}$

$u'' = -2$ $\quad v_2 = -\frac{\cos sx}{s^2}$

$v_3 = -\frac{\sin sx}{s^3}$

$$= \sqrt{\frac{2}{\pi}} \left[(4-x^2)\frac{\sin sx}{s} - (2x)\frac{\cos sx}{s^2} + (2)\frac{\sin sx}{s^3} \right]_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[-4\frac{\cos 2s}{s^2} + 2\frac{\sin 2s}{s^3} \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin 2s}{s^3} - \frac{2\cos 2s}{s^2} \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin 2s - 2s\cos 2s}{s^3} \right]$$

(a) F.I.T

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= 2\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{\sin 2s - 2s\cos 2s}{s^3}\right] e^{-isx}\, ds$$

$$= \frac{2\times 2}{\pi} \int_0^{\infty} \frac{\sin 2s - 2s\cos 2s}{s^3} \cos sx\, ds.$$

put $x=0$ $\quad$ $2s = t$

$2ds = dt$

$ds = \frac{dt}{2}$

$$4 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin t - t\cos t}{(t/2)^3} \frac{dt}{2}.$$

$$1 = \frac{1}{2\pi} \int_0^{\infty} \frac{\sin t - t\cos t}{t^3} dt \cdot 2^{3/2}$$

$\frac{4}{2\pi}$

$$\boxed{\int_0^{\infty} \frac{\sin t - t\cos t}{t^3} = \frac{\pi}{4}.}$$

(b) Parseval's identity.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

$$LHS = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$= \int_{-2}^{2} (4-x^2)^2 dx. = 2\int_0^2 (4-x^2)^2 dx.$$

$$= 2\left[\int_0^2 16\,dx + \int_0^2 x^4\,dx - \int_0^2 8x^2\,dx\right]$$

$$= 2\left[16[x]_0^2 + \left[\frac{x^5}{5}\right]_0^2 - 8\left[\frac{x^3}{3}\right]_0^2\right]$$

$$= 2\left[16(2) + \frac{32}{5} - \frac{64}{3}\right]$$

$$= 2\left[\frac{480 + 96 - 320}{15}\right] = 2\left[\frac{160+96}{15}\right] = \frac{512}{15}$$

R.H.S

$$(F(s))^2 = \frac{4 \cdot 2}{\pi}\left[\frac{\sin 2s - 2s\cos 2s}{s^3}\right]$$

$$\int_0^\infty [F(s)]^2 ds = \frac{8}{\pi}\int_0^\infty \left[\frac{\sin 2s - 2s\cos 2s}{s^3}\right]^2 ds.$$

$$= \frac{8 \times 4}{\pi}\int_0^\infty \left(\frac{\sin 2s - 2s\cos 2s}{s^3}\right)^2 ds$$

$2s = t$

$2ds = dt$

$ds = \frac{dt}{2}$

$$\frac{512}{15} = \frac{16}{\pi}\int_0^\infty \left[\frac{\sin t - t\cos t}{(t/2)^3}\right]^2 \frac{dt}{2}.$$

$$\frac{512}{15} = \frac{8}{\pi}\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right]^2 2^6\,dt$$

$$\frac{512}{15} = \frac{8 \times 64}{\pi}\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt$$

$$\boxed{\frac{\pi}{15} = \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt}$$

HW

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cancel{(1-x^2)}\, f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \frac{1 \times 2}{\sqrt{2\pi}} \int_{0}^{1} (1-x^2) \cos sx \, dx .$$

$(1-x^2) = u$ $\quad$ $v = \cos sx$

$-2x = u'$ $\quad$ $v_1 = \frac{\sin sx}{s}$

$u'' = -2$ $\quad$ $v_2 = \frac{-\cos sx}{s^2}$

$v_3 = -\frac{\sin sx}{s^3}$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2) \frac{\sin sx}{s} - (2x)\left(\frac{-\cos sx}{s^2}\right) + \left(2 \frac{\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[-2\frac{\cos s}{s^2} + 2\frac{\sin s}{s^3} \right]$$

$$= \sqrt{\frac{2}{\pi}}\, 2 \left[\frac{\sin s - s\cos s}{s^3} \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$$

(a) F.I.T

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds.$$

$$= 2\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} [\cos sx - i \sin sx]\, ds$$

$$= \frac{2 \times 2}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos sx \, ds$$

put $x = \frac{1}{2}$

$$\frac{3}{4} = \frac{4}{\pi}\int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right) ds$$

$$f\left(\tfrac{1}{2}\right) = 1 - \left(\tfrac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4}$$

$$\int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}.$$

(b) Parsevals Identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

LHS:

$$\int_{-1}^{1} (1-x^2)^2 dx = 2\int_0^1 (1-x^2)^2 dx$$

$$= 2\left[\int_0^1 1\, dx + \int_0^1 x^4 dx - \int_0^1 2x^2 dx\right]$$

$$= 2\left[[x]_0^1 + \left[\frac{x^5}{5}\right]_0^1 - 2\left[\frac{x^3}{3}\right]_0^1\right] = 2\left[1 + \frac{1}{5} - \frac{2}{3}\right]$$

$$= 2\left[\frac{15+3-10}{15}\right] = 2\left[\frac{8}{15}\right] = \frac{16}{15}$$

R.H.S

$$|F(s)|^2 = \frac{8}{\pi}\left[\frac{\sin s - s\cos s}{s^3}\right]^2$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \frac{16}{\pi}\int_0^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds.$$

$$\frac{16}{\pi}\int_0^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \frac{16}{15}$$

$s \to x$

$$\boxed{\int_0^{\infty}\left[\frac{x\cos x - \sin x}{x^3}\right]^2 dx = \frac{\pi}{15}}$$

Hw

$$f(x) = \begin{cases} 1 & |x| < 2 \\ 0 & |x| > 2 \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} (\cos sx + i \sin sx)\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{2} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin 2s}{s}\right]$$

(i)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin 2s}{s} [\cos sx - i s \sin sx]\, ds$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin 2s}{s} \cos sx\, ds$$

$x = 0$

$2s = \alpha$

$2\, ds = d\alpha$

$ds = \frac{d\alpha}{2}$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \alpha}{(\alpha/2)} \frac{d\alpha}{2}$$

$$\Rightarrow 1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \alpha}{\alpha} \frac{d\alpha \cdot 2}{2}$$

$$\int_{0}^{\infty} \frac{\sin \alpha}{\alpha}\, d\alpha = \frac{\pi}{2}$$

(ii) Parsevals identity

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{-\infty}^{\infty} |F(s)|\, ds$$

Scanned by CamScanner

LHS $= \int_{-2}^{2} 1^2\, dx$

$= [x]_{-2}^{2} = 2-(-2) = 4$

RHS $=$

$(F(s))^2 = \frac{2}{\pi}\left[\frac{\sin 2s}{s}\right]^2$

$\frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin 2s}{s}\right]^2 ds = \frac{2}{\pi}\times 2\int_0^{\infty}\left[\frac{\sin 2s}{s}\right]^2 ds$

$2s = \beta$

$2\,ds = d\beta$

$ds = \frac{d\beta}{2}$

$= \frac{4}{\pi}\int_0^{\infty}\frac{(\sin\beta)^2}{(\beta/2)^2}\frac{d\beta}{2}$

$= \frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin\beta}{\beta}\right)^2 4\, d\beta$

$= \frac{8}{\pi}\int_0^{\infty}\left(\frac{\sin\beta}{\beta}\right)^2 d\beta$

$\cancel{4} = \frac{8}{\pi}\int_0^{\infty}\left(\frac{\sin\beta}{\beta}\right)^2 d\beta$

$\int_0^{\infty}\left(\frac{\sin\beta}{\beta}\right)^2 d\beta = \frac{\pi}{2}$ //

HW: $f(x) = e^{-|x|}$

F.T $= F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{+isx}\, dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|} e^{+isx}\, dx$

$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos sx\, dx$

$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{1+s^2}\right]$

I.F.T $\cancel{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}} F(s)\, e^{-isx}\, dx$

Scanned by CamScanner

$$\cancel{\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1^2+s^2}\,ds}$$

$$= \cancel{\frac{2}{\pi}\int_0^\infty}$$

By Parsevals identity

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$$

L.H.S

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} \left(e^{-|x|}\right)^2 dx$$

$$= 2\int_0^\infty e^{-2x}\,dx$$

$$= \cancel{2}\left[\frac{e^{-2x}}{-\cancel{2}}\right]_0^\infty$$

$$= -\left[e^{-2x}\right]_0^\infty = -(0-1) = +1$$

RHS

$$|F(s)|^2 = \frac{2}{\pi}\left[\frac{1}{1+s^2}\right]^2$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{1+s^2}\right]^2 ds = \frac{4}{\pi}\int_0^\infty \frac{1}{(s^2+1)^2}\,ds$$

$s \to x$

$$1 = \frac{4}{\pi}\int_0^\infty \left(\frac{1}{x^2+1}\right)^2 dx$$

$$\boxed{\int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{4}}$$

Scanned by CamScanner

formulae:-

Fourier Cosine Tranform = F.C.T $\Rightarrow F_c(s) = F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$

Fourier Inverse cosine transform FICT

$$f_c(x) = F_c^{-1}[F_c(s)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\, ds$$

Fourier Sine Tranform

$$F.S.T \Rightarrow F_s(s) = F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

F.I.S.T

$$f(x) = F_s^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\, ds.$$

Relation b/w FCT & FST.

$$F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$$

$$F_c[x f(x)] = \frac{d}{ds}[F_s(s)]$$

$$\int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}$$

$$\int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}.$$

Scanned by CamScanner

* Find the F.C.T of $f(x) = \frac{e^{-ax}}{x}$ and hence evaluate $F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$. $a, b > 0$

Soln-

F.C.T

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx.$$

Differentiate on both sides w.r.t LHS variable

$$\frac{d}{ds}\left[F_c(s)\right] = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial s} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} -\sin sx (\cancel{x}) \cdot dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} (+\sin sx) \, dx. \qquad \boxed{\because \int_0^\infty e^{-ax} \sin bx = \frac{b}{a^2+b^2}}$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2+s^2}\right]$$

Taking $\int$ w.r.t 's'

$$\int \frac{d}{ds} F_c(s) = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s \times 2}{(a^2+s^2)2} ds.$$

$$F_c(s) = -\frac{1}{2}\sqrt{\frac{2}{\pi}} \log\left[s^2+a^2\right]$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = F_c(s) = -\frac{1}{\sqrt{2\pi}} \log\left[s^2+a^2\right] \text{ --- } ①$$

Replace $a \to b$.

$$F_c\left[\frac{e^{-bx}}{x}\right] = F_c(s) = \frac{-1}{\sqrt{2\pi}} \log\left[s^2-b^2\right] \text{ --- } ②$$

① - ②

$$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}} \left[\log(s^2+a^2) - \log(s^2-b^2)\right]$$

$$= \frac{-1}{\sqrt{2\pi}} \left[\log\left(\frac{s^2+a^2}{s^2-b^2}\right)\right]$$

* Find the F.S.T of $f(x) = \frac{e^{-ax}}{x}$ and hence evaluate $F_s\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$ $a, b > 0$

Sol// F.S.T

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

Differentiate on both sides w.r.t LHS variable

$$\frac{d}{ds}[F_s(s)] = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial s} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos sx (\cancel{x}) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2 + a^2}\right]$$

~~Tak~~ Taking $\int$ w.r.t s

$$\int \frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{a}{s^2 + a^2} \, ds$$

$$F_s(s) = \cancel{a} \sqrt{\frac{2}{\pi}} \frac{1}{\cancel{a}} \tan^{-1}\left(\frac{s}{a}\right)$$

$$F_s\left[\frac{e^{-ax}}{x}\right] = F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) \quad \text{--- (1)}$$

replace $a \to b$

$$F_s\left[\frac{e^{-bx}}{x}\right] = F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{b}\right) \quad \text{--- (2)}$$

(2) - (3)

$$F_s\left[\frac{e^{-ax} - e^{-bx}}{x}\right] = \sqrt{\frac{2}{\pi}} \left[\tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)\right] //.$$

Imp 15m Find F.C.T
*

$f(x) = e^{-ax}$, $a>0$ and find F.C.T an applying inversion formula. Find the fourier sine transform of $\frac{x}{x^2+a^2}$ and also F.C.T of $\frac{1}{x^2+a^2}$ and also evaluate $F_s[xe^{-ax}]$

Sol:-

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

$$F_s(xe^{-ax}] = -\frac{d}{ds}\{F_c[e^{-ax}]\}$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}\right]$$

$$= -a\sqrt{\frac{2}{\pi}} \frac{d}{ds} \frac{1}{s^2+a^2}$$

$$= -a\sqrt{\frac{2}{\pi}} \frac{-1}{(s^2+a^2)^2}(2s)$$

$$= \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$$

FIST:-

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx \, dx$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2+a^2} dx \quad x = \frac{n\pi}{2}, n \in Z.$$

$$\Rightarrow \frac{\pi}{2} \times f(x) = \int_0^\infty \frac{s}{s^2+a^2} ds$$

$$s \to x$$

$$\Rightarrow \frac{\pi}{2} e^{-a\pi/2} = \int_0^\infty \frac{x}{x^2+a^2} dx.$$

Find F.S.T of $f(x) = e^{-ax}$ $a>0$, by applying inversion formula. Find the F.S.T of $\frac{x}{x^2+a^2}$ and F.C.T of $\frac{1}{x^2+a^2}$ and evaluate $F_c[xe^{-ax}]$.

Soln

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

$$F_c[xe^{-ax}] = \frac{d}{ds}\left[F_s(e^{-ax})\right]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{s^2+a^2 - s(2s)}{(s^2+a^2)^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a^2-s^2}{(s^2+a^2)^2}$$

$$= \cancel{\sqrt{\frac{2}{\pi}} \frac{1}{(s^2+a^2)^2}} = \sqrt{\frac{2}{\pi}} \frac{a^2-s^2}{(s^2+a^2)^2}$$

FICTL

$$f(x) = \sqrt{\frac{2}{\pi}} \int F_c(s)[f(x)] \cos sx \, ds.$$

$$= \frac{2}{\pi} \int_0^\infty \cancel{\frac{s \cos sx}{s^2+a^2}} \, dx$$

$$= \frac{2}{\pi} \int_0^\infty \frac{a \cos sx}{a^2+s^2} \, ds \qquad x=0$$

$$\frac{\pi}{2a}\cancel{e^{-ax}}(1) = \int_0^\infty \frac{1}{x^2+a^2} \, dx$$

* Find F.C.T of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

Sol: $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}}$ $\quad \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^a [\cos(x(1+s)) + \cos(x(1-s))] dx.$

$= \frac{1}{\sqrt{2\pi}} \left[\int_0^a \cos(x(1+s)) dx + \int_0^a \cos(x(1-s)) dx \right]$

$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{\sin(x(1+s))}{1+s} \right)_0^a + \left(\frac{\sin(x(1-s))}{1-s} \right)_0^a \right]$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(a(1+s))}{1+s} + \frac{\sin(a(1-s))}{1-s} \right]$

Function will be valid for all conditions except at $s = 1, -1$

* Show that $f(x) = e^{-x^2/2}$ is self reciprocal F.C.T.

Sol: $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos sx \, dx$

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_{-\infty}^\infty e^{-x^2/2} \cos sx \, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} \cos sx \, dx$ $\quad \cos sx = R.P \text{ of } e^{isx}.$

$= R.P \text{ of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} \cdot e^{isx} dx$

(refer pb. 1)

$= R.P [e^{-s^2/2}]$

$= e^{-s^2/2}$

$\therefore$ self reciprocal

Find F.S.T of $f(x) = \begin{cases} \sin x, & 0<x<a \\ 0, & x>a. \end{cases}$

Sol: $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^a \sin x \sin sx \, dx.$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^a \cos[x(1-s)] - \cos[x(1+s)] \, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\int_0^a \cos(x(1-s)) dx - \int_0^a \cos(x(1+s)) dx \right]$

$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{\sin(x(1-s))}{(1-s)} \right)_0^a - \left(\frac{\sin(x(1+s))}{1+s} \right)_0^a \right]$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(a(1-s))}{1-s} - \frac{\sin(a(1+s))}{1+s} \right]$

Function will be valid for all conditions except at $s = 1, -1$

Under F.S.T we have ~~I.P of~~ $f(e^{...})$

I.P of $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx} \, dx.$

(refer Pb 1)

$= I.P \left[e^{-s^2/2} \right]$

$= 0$

$\therefore$ It is not self reciprocal.

* Show that $f(x) = e^{-x^2/2}$ is self reciprocal under F.C.T of $e^{-a^2x^2}$ and hence deduce $F_s[x e^{-a^2x^2}]$

$\boxed{\text{Derive F.T}}$

Sol:- $F_c[e^{-a^2x^2}] = F_c(s) = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$

Take $a^2 = \frac{1}{2}$, $a = \frac{1}{\sqrt{2}}$

$= \frac{1}{(\frac{1}{\sqrt{2}})\sqrt{2}} e^{-s^2/4(\frac{1}{2})} = e^{-s^2/2}$

$F_s[x e^{-a^2x^2}] = \frac{-d}{ds} \cdot [F_c[e^{-a^2x^2}]]$

$= \frac{-1}{a\sqrt{2}} \frac{d}{ds} e^{-a^2/4a^2}$

$= \frac{-1}{a\sqrt{2}} e^{(-s^2/4a^2)} \times \left(\frac{-2s}{4a^2}\right)$

$= \frac{1}{2\sqrt{2}a^3} s e^{-s^2/4a^2} e^{-s^2/4a}$.

* $F_c[x^{n-1}]$ & $F_s[x^{n-1}]$, $0<x<1$. P.T $\frac{1}{\sqrt{x}}$ is self reciprocal under F.C.T. Evaluate F.C.T and F.S.T for $f(x) = x^{n-1}$ where $0<x<1$ by using gamma integration

Sol:- $F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \cos sx \, dx$

w.k.T $\Gamma n = \int_0^\infty e^{-y} y^{n-1} dy$.

Take $y = isx$
$dy = is\,dx$

y	0	∞
x	0	∞

$\Gamma n = \int_0^\infty e^{-isx} (isx)^{n-1} (is) \, ds$

$$= \int_0^\infty e^{-isx} (x^{n-1}) (is)^{n-1} (is)^{n} dx$$

$$= (is)^n \int_0^\infty x^{n-1} [\cos sx - i \sin sx] dx$$

$$= (is)^n \left[\int_0^\infty x^{n-1} \cos sx \, dx - i \int_0^\infty x^{n-1} \sin sx \, dx \right]$$

$$\frac{\Gamma n}{(is)^n} = \int_0^\infty x^{n-1} \cos sx \, dx - i \int_0^\infty x^{n-1} \sin sx \, dx$$

$$\frac{\Gamma n}{s^n} (i)^{-n} = \int_0^\infty x^{n-1} \cos sx \, dx - i \int_0^\infty x^{n-1} \sin sx \, dx .$$

$$\frac{\Gamma n}{s^n} \left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right]^{-n} = \int_0^\infty x^{n-1} \cos sx \, dx - i \int_0^\infty x^{n-1} \sin sx \, dx .$$

R.P. $\frac{\Gamma n}{s^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right) = \int_0^\infty x^{n-1} \cos sx \, dx - i \int_0^\infty x^{n-1} \sin sx \, dx$

$$\frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} = \int_0^\infty x^{n-1} \cos sx \, dx . \quad \text{--- (1)}$$

I.P.

$$\frac{\Gamma n}{s^n} \sin \frac{n\pi}{2} = \int_0^\infty x^{n-1} \sin sx \, dx . \quad \text{--- (2)}$$

From (1) & (2)

$$F_c [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos\left(\frac{n\pi}{2}\right) \quad \text{--- (A)}$$

$$F_s (x^{n-1}) = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \times \sin\left(\frac{n\pi}{2}\right). \quad \text{--- (B)}$$

(ii) $F_c \left[\frac{1}{\sqrt{x}}\right]$ is S.R.

$$= F_c \left[x^{(-\frac{1}{2}+1)-1} \right]$$

$$= F_c \left[x^{\frac{1}{2}-1} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{\Gamma \frac{1}{2}}{s^{1/2}} \cos\left(\frac{\frac{1}{2}\pi}{2}\right)$$

$$= \sqrt{2} \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{s}}$$

* Problems on F.C.T & F.S.I.T.

* Find the F.C.T of e^{-ax}, $a>0$ and hence deduce integral $\int_0^{\infty} \frac{\cos sx \, ds}{a^2+s^2} = \frac{\pi}{2a} e^{-ax}$.

Sol:

~~Derive I.T.~~

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx.$$

Here $a=a$ $b=s$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right)$$

F.I.C.T

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx \, ds.$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} a \int_0^{\infty} \frac{\cos sx}{s^2+a^2} ds.$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos sx}{s^2+a^2} ds$$

$$\boxed{\frac{\pi}{2a} e^{-ax} = \int_0^{\infty} \frac{\cos sx}{s^2+a^2} ds}$$

Take $x=0$

$$f(0) \ \frac{\pi}{2a} = \int_0^{\infty} \frac{1}{s^2+a^2} ds.$$

* Find the F.C.T of $f(x) = e^{-|x|}$ and hence deduce the $\int_0^{\infty} \frac{\cos x t \, dt}{1+t^2} = \frac{\pi}{2} e^{-|x|}$

Sol:

F.C.T $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-|x|} \cos sx \, dx.$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos sx \, dx$$

$a=1$, $b=s$

$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{s^2+1}\right)$

F.I.C.T

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos sx}{s^2+1} \, ds$$

$s \to t$
$ds = dt$.

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{s^2+1} \, ds$$

$$\frac{\pi}{2} e^{-|x|} = \int_0^\infty \frac{\cos xt}{1+t^2} \, dt$$

* Find the F.S.T of e^{-ax}, $a>0$ and hence deduce $\int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds = \frac{\pi}{2} e^{-ax}$.

Sol:

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

Here $a = a$, $b = s$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

F.I.S.T

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds$$

$$\frac{\pi}{2} e^{-ax} = \int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds.$$

* Find F.S.T of e^{-x} and hence deduce the $\int_0^\infty \frac{x \sin mx}{x^2+1} = \frac{\pi}{2} e^{-m}$.

Sol:

$$F_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$a = 1$, $b = s$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+1}\right]$$

FIST

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s \sin sx}{s^2+1} \, ds$$

$s \to x \quad x \to m$

$$f(m) = \frac{2}{\pi} \int_0^\infty \frac{x \sin xm}{x^2+1} \, dx$$

$$\frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x \sin xm}{x^2+1} \, dx$$

Evaluation of definite integrals using Transformation method

or Fourier transform technique

Parseval indentity for cosine $= \int_0^\infty f(x) g(x) \, dx = \int_0^\infty F_c(f(x)) F_c(g(x))$

Parseval identity for sine $= \int_0^\infty f(x) g(x) \, dx = \int_0^\infty F_s(f(x)) F_s(g(x)) \, dy.$

I $\Rightarrow \int_0^\infty \frac{1 \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \frac{\pi}{ab(a+b)}$

II $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$

III $= \int_0^\infty \frac{x^2 \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \frac{\pi}{a+b}$

HW IV $= \int_0^\infty \frac{x^2 \, dx}{(x^2+a^2)^2} = \frac{\pi}{4a}$

If x^2 is in numerator used F.S.T

If 1 " used F.C.T

I Take $f(x) = e^{-ax}$

Here is 1, F.C.T

$$F_c[f(x)] = F_c[e^{-ax}]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

Scanned by CamScanner

$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$, $g(x) = e^{-bx}$, $a \to b$

$F_c(g(x)) = \frac{b}{s^2+b^2}$

$R.H.S = \int_0^\infty F_c[f(x)]\, F_c[g(x)]\, ds = \frac{2}{\pi} ab \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}$

__LHS__ $\int_0^\infty f(x)\, g(x)\, dx$.

$= \int_0^\infty e^{-ax} e^{-bx} dx = \int_0^\infty e^{-(a+b)} dx$

$= \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty = \frac{1}{a+b}$.

LHS = RHS.

$\frac{1}{(a+b)} \frac{\pi}{2ab} = \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}$ $\Rightarrow s \to x$

II

Take $f(x) = e^{-ax}$

Nos. 1 F.C.T

$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$

__RHS__ $= \int_0^\infty F_c(f(x))\, F_c(g(x))\, ds = \frac{2}{\pi} a^2 \int_0^\infty \frac{1}{(s^2+a^2)^2} ds$.

__LHS__ $= \int_0^\infty e^{-ax} e^{-ax} dx = \int_0^\infty e^{-2ax} dx$

$= \left[\frac{e^{-2ax}}{-2a} \right]_0^\infty = -\frac{0}{2a} + \frac{1}{2a} = \frac{+1}{2a}$

LHS = RHS

$\frac{2}{\pi} a^2 \int_0^\infty \frac{1}{s^2+a^2} ds = \frac{+1}{2a}$

$\int_0^\infty \frac{1}{s^2+a^2} ds = \frac{1}{2a} \times \frac{\pi}{2a^2}$

$\int_0^\infty \frac{1}{s^2+a^2} ds = \frac{\pi}{4a^3}$ $s \to x$

III. Take $f(x) = e^{-ax}$

N.B x^2

$$F.ST = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right)$$

$$g(x) = e^{-bx}$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+b^2}\right)$$

$$R.H.S = \int_0^\infty F_s\{f(x)\} F_s(g(x))$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds.$$

LHS = $\int_0^\infty f(x)\, g(x)\, dx$

$$\int_0^\infty e^{-ax} e^{-bx} dx = \int_0^\infty e^{-(a+b)x} dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{1}{a+b}$$

LHS = R.H.S

$$\frac{1}{a+b} = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds,$$

$$\int_0^\infty \frac{s^2 \, ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2(a+b)}$$

Hw S.T $f(x) = \frac{1}{\sqrt{x}}$ is reciprocal under F.ST & FCT.

IV

Take $f(x) = e^{-ax}$

N_r is x^2

$F_s(s)$ F.S.T $= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$

$F_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2} \right)$

R.H.S $= \int_0^\infty (F_s f(x))^2 = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$

$LHS = \int_0^\infty (f(x))^2 dx = \int_0^\infty (e^{-ax})^2 dx = \int_0^\infty e^{-2ax} dx$

$= \left[\frac{e^{-2ax}}{-2a} \right]_0^\infty = \frac{1}{2a}$

LHS = RHS

$\frac{1}{2a} = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$

$\int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds = \frac{\pi}{4a}$

$s \to x$

$ds = dx$

$\int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx = \frac{\pi}{4a}$

~~Hw~~ ~~F.C.T~~ $\cancel{F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{\sqrt{x}} \sin sx \, dx}$

~~F.T =~~

~~$f(s)$ =~~

* Find the F.C.T $e^{-a^2x^2}$ & hence deduce that $e^{-x^2/2}$ is self reciprocal w.r.to cosine transform

Sol:- $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx \, dx$

Scanned by CamScanner

$$= \sqrt{\frac{2}{\pi}}\, R.P\, \frac{1}{2}\int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$-a^2x^2+isx = -\left[(a^2x^2 - isx\times\frac{2a}{2a} + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]$$

$$\text{i.e.} = -\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]$$

$$= -\left(ax-\frac{is}{2a}\right)^2 - \frac{s^2}{4a^2}$$

$$= \sqrt{\frac{2}{\pi}}\, R.p \times \frac{1}{2}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} . e^{-s^2/4a^2}\, dx.$$

$$= \sqrt{\frac{2}{\pi}}\, e^{-s^2/4a^2}\, R.P \times \frac{1}{2}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx.$$

put $y = ax - \frac{is}{2a}$

$$dy = a\,dx \Rightarrow dx = \frac{dy}{a}$$

x $-\infty$ ∞

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\, e^{-s^2/4a^2} \times \frac{1}{2} R.P \int_{-\infty}^{\infty} e^{-y^2} \frac{dy}{a}$$

$y = ax - \frac{is}{2a}$ $-\infty$ ∞

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\, e^{-s^2/4a^2}\, \frac{1}{2} R.P \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\, e^{-s^2/4a^2} \times \frac{1}{2} R.P\, 2\int_{0}^{\infty} e^{-y^2} dy$$

$y^2 = t$ $\qquad y$ 0 ∞

$2y\,dy = dt$ $\qquad t = y^2$ 0 ∞

$$dy = \frac{dt}{2y} = \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\, e^{-s^2/4a^2} . \cancel{2} R.P \int_0^{\infty} \frac{e^{-t}}{\cancel{2}\sqrt{t}} dt$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\, e^{-s^2/4a^2} \times \frac{1}{2} R.P \int_0^{\infty} e^{-t}\, t^{\left(-\frac{1}{2}+1\right)-1} dt$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\, e^{-s^2/4a^2} \times \frac{1}{2}\int_0^{\infty} e^{-t}\, t^{1/2-1} dt$$

$$= \frac{1}{a\sqrt{2\pi}}\, e^{-s^2/4a^2} \int_0^{\infty} e^{-t}\, t^{1/2-1} dt$$

$$= \frac{1}{a\sqrt{2\pi}}\, e^{-s^2/4a^2} \quad \therefore \Gamma n = \int_0^{\infty} e^{-t} t^{n-1} dt,\ n = \frac{1}{2}$$

$F_c[f(x)] = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$ $\Gamma\frac{1}{2} \to \sqrt{\pi}$

$F_c[f(x)] = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2} = F_c[e^{-a^2x^2}]$

Put $a^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}}$

$F_c[e^{-x^2/2}] = \frac{1}{\frac{1}{\sqrt{2}} \times \sqrt{2}} e^{-s^2/4(1/2)}$

$= e^{-s^2/2} = f(s)$

$\therefore f(x)$ is self reciprocal w.r.t to cosine transform

* Find F.C.T of $f(x) = \begin{cases} x & , (0,1) \\ 2-x & ; (1,2) \\ 0 & , x>2 \end{cases}$

Sol: $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx \right]$

$= \sqrt{\frac{2}{\pi}} \left\{ \left[(x)\left(\frac{\sin sx}{s}\right) - (1)\left(-\frac{\cos sx}{s^2}\right) \right]_0^1 + \left[(2-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(-\frac{\cos sx}{s^2}\right) \right]_1^2 \right\}$

$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right] + \left[-\frac{\cos 2s}{s^2} - \left(\frac{\sin s}{1} - \frac{\cos s}{s^2} \right) \right] \right\}$

$= \sqrt{\frac{2}{\pi}} \left[\frac{-1}{s^2} - \frac{\cos 2s}{s^2} \right] = -\sqrt{\frac{2}{\pi}} \times \frac{1}{s^2} (1 + \cos 2s)$

* Find the F.S.T of e^{-ax} $a>0$ & P.T $\int_0^\infty \frac{s}{s^2+a^2} \sin sx \, dx = \frac{\pi}{2} e^{-ax}$

Sol: $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$

$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2} \right)$

Deduction:

Using F.I.S.T

~~$F_s^{-1}\left[\frac{e^{-sa}}{s}\right]$ $f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}(x/a)$~~

~~Dedc~~

~~Take $a=0$~~

~~$F_s^{-1}\left[\frac{e^{0}}{s}\right] = F_s^{-1}\left[\frac{1}{s}\right]$~~

~~$= \sqrt{\frac{2}{\pi}} \tan^{-1}(\infty)$~~

~~$= \sqrt{\frac{2}{\pi}} \times \pi/2$~~

~~$= \sqrt{\frac{\pi}{2}}$.~~

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$

$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$

$= \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$

$\int_0^\infty \frac{s}{s^2+a^2} \sin sx \, dx = \frac{\pi}{2} e^{-ax}$

* Find $f(x)$ if its sine transformation is e^{-as}/s & Hence find $F_s^{-1}\left[\frac{1}{s}\right]$

Sol:

$F_s[f(x)] = \frac{e^{-as}}{s}$

By FIST

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(f(x)) \sin sx \, ds$

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-sa}}{s} \sin sx \, ds$

Diff w.r.t to x and $\int$ w.r.t s

$\frac{d f(x)}{dx} = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty \frac{e^{-sa}}{s} \sin sx (ds)$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-sa}}{\cancel{s}} \cancel{s} \cos sx \, ds$

$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+x^2}\right)$

Taking $\int$ on both side

$f(x) = a\sqrt{\frac{2}{\pi}} \int \frac{dx}{a^2+x^2}$

$= a\sqrt{\frac{2}{\pi}} \times \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

$F_s^{-1}\left[\frac{e^{-sa}}{s}\right] = f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)$

Take $a=0$

$F_s^{-1}\left[\frac{e^{0}}{s}\right] = F_s^{-1}\left[\frac{1}{s}\right]$

$= \sqrt{\frac{2}{\pi}} \tan^{-1}(\infty)$

$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}$

$F_c^{-1}\left[\frac{1}{s}\right] = \sqrt{\frac{\pi}{2}}$

Solving integral equations using F.C.T & F.S.T

Working rule:

S1: change ~~s = x~~ $\alpha = s$ and mul both sides by $\sqrt{\frac{2}{\pi}}$ so that to get $F_c(s)$ or $F_s(s)$ —①

S2:- Use Inversion formula for getting $f(x)$ F.I.C.T / F.I.S.T.

S3:- Substitute $f(x)$ obtained in 'S2' in to $F_c(s)$ or $F_s(s)$. —②

S4:- Compare ① & ②.

* Solve the integral equation $\int_0^\infty f(x)\cos\alpha x\,dx = e^{-\alpha}$ and hence show that $\int_0^\infty \frac{\cos\alpha x\,dx}{1+x^2} = \frac{\pi}{2}e^{-\alpha}$

Sol: S1. $\alpha = s$ multiply by $\sqrt{\frac{2}{\pi}}$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = e^{-s}\sqrt{\frac{2}{\pi}}$$

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\,e^{-s} \quad \text{—①}$$

S2:- $f(x) \to$ F.I.C.T

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\,ds.$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-s}\cos sx\,ds$$

$a = 1 \quad b = x$

$$\boxed{f(x) = \frac{2}{\pi}\left[\frac{1}{x^2+1}\right]} \qquad \boxed{\int_0^\infty e^{-ax}\cos bx = \frac{a}{a^2+b^2}}$$

S3/- Sub $f(x)$ in F.C.T.

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \frac{2}{\pi} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos sx}{1+x^2} \, dx.$$

$$F_c(s) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_0^\infty \frac{\cos sx}{1+x^2} \, dx \quad \text{--- (2)}$$

S4/ Compare (1) & (2)

$$\sqrt{\frac{2}{\pi}} e^{-s} = \frac{2}{\pi} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos sx}{1+x^2} \, dx$$

$$\int_0^\infty \frac{\cos sx}{1+x^2} \, dx = e^{-s} \frac{\pi}{2}.$$

$s \to \lambda$

$$\boxed{\int_0^\infty \frac{\cos \lambda x}{1+x^2} \, dx = e^{-\lambda} \frac{\pi}{2}.}$$

* Solve the integral equation

$$\int_0^\infty f(x) \sin sx \, dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$$

Sol:- S1:- Mul by $\sqrt{\frac{2}{\pi}}$ on both sides

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx = \sqrt{\frac{2}{\pi}} \begin{cases} 1, & 0 \le s \le 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$$

$$F_s(s) = F_s\{f(x)\} = \frac{2}{\pi} \left[\int_0^1 1 \sin sx \, dx + \int_1^2 2 \sin sx \, dx + 0 \right]$$

$$= \frac{2}{\pi} \left[\left[\frac{-\cos sx}{s} \right]_0^1 + 2 \left[\frac{-\cos sx}{s} \right]_1^2 \right]$$

Scanned by CamScanner

$= \frac{2}{\pi}\left[\left(-\frac{\cos s}{s}+\frac{1}{s}\right)+2\left(-\frac{\cos 2s}{s}+\frac{\cos s}{s}\right)\right]$

$= \frac{2}{\pi}\left[-\frac{\cos s}{s}+\frac{1}{s}-\frac{2\cos 2s}{s}+\frac{2\cos s}{s}\right]$

$= \frac{2}{\pi}\left[\frac{1}{s}-\frac{2\cos 2s}{s}+\frac{\cos s}{s}\right]$

$= \frac{2}{\pi s}\left[1-2\cos 2s+\cos s\right]$

* Solve the integral eqn $\int_0^\infty f(x)\cos \alpha x\, dx = \begin{cases} 1-\alpha, & 0\le\alpha\le 1 \\ 0, & \alpha>1 \end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$.

Sol:- take $\alpha = s$ and multi on both sides by $\sqrt{\frac{2}{\pi}}$

S1 $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\begin{cases} 1-s, & 0\le s\le 1 \\ 0, & s>1 \end{cases}$ —(1)

$F_c[s] = \sqrt{\frac{2}{\pi}}\begin{cases} 1-s, & 0\le s\le 1 \\ 0, & s>1 \end{cases}$

S2:- $f(x) \rightarrow$ F.I.C.T.

$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\, ds$.

$= \frac{2}{\pi}\left[\int_0^1 (1-s)\cos sx\, ds\right] +$

$= \frac{2}{\pi}$ $u = 1-s$ $\quad v = \cos sx$

$u' = -1$ $\quad v_1 = \frac{\sin sx}{x}$

$v_2 = -\frac{\cos sx}{x^2}$

$f(x) = \frac{2}{\pi}\left[(1-s)\left(\frac{\sin sx}{x}\right) - \frac{\cos sx}{x^2}\right]_0^1$

$= \frac{2}{\pi}\left[-\frac{\cos x}{x^2}+\frac{1}{x^2}\right] = \frac{2}{\pi}\left[\frac{1-\cos x}{x^2}\right]$ —(2)

$$f(x) = \frac{2}{\pi}\left[\frac{2\sin^2(\frac{x}{2})}{x^2}\right]$$

$$\therefore f(x) = \frac{4}{\pi}\frac{\sin^2(x/2)}{x^2}$$

S3: sub $f(x)$ in F.C.T

$$F_c(S) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{4}{\pi}\int_0^\infty \frac{\sin^2(x/2)}{x^2}\cos sx\,dx.$$

Take $S=0$

$$= \cancel{\frac{4}{\pi}\sqrt{\frac{2}{\pi}}}$$

$$= \frac{4}{\pi}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin^2(x/2)}{x^2}\,dx \quad \text{——— (2)}$$

S4 Compare (1) & (2)

$$\cancel{\sqrt{\frac{2}{\pi}}}\begin{cases} 1-s & ; 0 \le s \le 1 \\ 0, & s>1 \end{cases} = \frac{4}{\pi}\cancel{\sqrt{\frac{2}{\pi}}}\int_0^\infty \frac{\sin^2(x/2)}{x^2}\,dx$$

$\because S=0$

R.H.S = 1

$$1 \times \frac{\pi}{4} = \int_0^\infty \frac{\sin^2(x/2)}{x^2}\,dx$$

$x/2 = t$

$\frac{dx}{2} = dt$

$dx = 2dt$

$x = 2t$

$$\frac{\pi}{4} = \int_0^\infty \frac{\sin^2 t}{(2t)^2}\,2dt$$

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}.$$

HW: Use parseval's identity ~~find the~~ of F.C.T evaluate $\int_0^\infty \frac{\sin ax\,dx}{a^2+x^2}$ if $f(x) = \begin{cases} 1, & |x|<a \\ 0, & |x|>a>0 \end{cases}$

$g(x) = e^{-a|x|}, \quad a>0.$

Imp **Relation between Laplace transform and Fourier Transform**

Consider $f(x) = \begin{cases} e^{-xt} g(t), & t>0 \\ 0, & t<0. \end{cases}$ where

fourier transform $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt.$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-xt} g(t) e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty g(t) e^{-xt+ist} dt.$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty g(t) e^{-t(x-is)} dt$$

$p = x - is.$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty g(t) e^{-pt} dt.$$

$$F[f(t)] = F(s) = \frac{1}{\sqrt{2\pi}} L[g(t)]$$

The fourier transform, $F(s)$ is obtained by multiplying $\frac{1}{\sqrt{2\pi}}$ with the Laplace transform of corresponding function.

Finite fourier transform

Let $f(x)$ be sectionally continuous over the range $(0, l)$. It can be classified into cosine transform, sine transform.

Formulae:-

$$F.S.T = F_s(P) = \bar{f}_s(P) = \int_0^l f(x) \sin\left(\frac{P\pi x}{l}\right) dx \quad \text{--- (1)}$$

$$F.C.T = F_c(P) = \bar{f}_c(P) = \int_0^l f(x) \cos\left(\frac{P\pi x}{l}\right) dx \quad \text{--- (2)}$$

$$F.I.S.T = f(x) = \frac{2}{l}\sum_{P=1}^{\infty} F_s(P) \sin\left(\frac{P\pi x}{l}\right) \quad \text{--- (3)}$$

$$\cancel{F.I.C.T = f(x) = \frac{1}{l}\sum_{P=1}^{\infty} F_c(P) \cos\left(\frac{P\pi x}{l}\right) dx}$$

$$F.I.C.T = f(x) = \frac{1}{l} F_c(0) + \frac{2}{l}\sum_{P=1}^{\infty} F_c(P) \cos\left(\frac{P\pi x}{l}\right) \quad \text{--- (4)}$$

Derivation of inversion formula for F.I.S.T.

St:- Let $f(x)$ denotes a function which is sectionally continuous in the interval $(0, l)$. Then the inversion formula:

$$F.I.S.T,\ f(x) = \frac{2}{l}\sum_{P=1}^{\infty} F_s(P) \sin\left(\frac{P\pi x}{l}\right) dx.$$

Proof:- The fourier series of $f(x)$ in full range is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

For half range sine series a_0 and $a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$n = p$

$$b_p = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{p\pi x}{l}\right) dx.$$

$$b_p = \frac{2}{l} F_s(p) \quad \text{from (1)}$$

$$\therefore f(x) = \sum_{p=1}^{\infty} \frac{2}{l} F_s(p) \sin\left(\frac{p\pi x}{l}\right) dx$$

$$\boxed{f(x) = \frac{2}{l}\sum_{p=1}^{\infty} F_s(p) \sin\left(\frac{p\pi x}{l}\right) dx}$$

Inversion formula cosine transform

St:- Let $f(x)$ denotes a function which is sectionally continuous in the interval $(0, l)$ Where the finite fourier cosine transform of $f(x)$ over $(0, l)$ is given by.

$$F.I.C.T = f(x) = \frac{1}{l} F_c(0) + \frac{2}{l}\sum_{p=1}^{\infty} F_c(p) \cos\left(\frac{p\pi x}{l}\right).$$

Proof:- W.K.T

The fourier series of $f(x)$ in full range is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

Then the half range cosine series is given by.

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\, dx$, $a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

$(n=p)$

$$= \frac{2}{l}\int_0 f(x)\cos\left(\frac{p\pi x}{l}\right)dx$$

$$a_p = \frac{2}{l}\ F_c(p)$$

$$a_0 = \frac{2}{l}\times F_c(0)$$

$$f(x) = \frac{2}{l}\times\frac{1}{2}F_c(0) + \frac{2}{l}\sum_{p=1}^{\infty} F_c(p)\cos\left(\frac{p\pi x}{l}\right)$$

$$\boxed{f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{p=1}^{\infty} F_c(p)\cos\left(\frac{p\pi x}{l}\right)}$$

* Find (i) $f(x)=1$, $(0,\pi)$ HW (iv) $f(x) = \begin{cases} 1, & 0<x<\pi/2 \\ -1, & \pi/2<x<\pi \end{cases}$

(ii) $f(x)=x$, $(0,l)$ HW (v) $f(x)=x^3$ $(0,l)$

HW (iii) $f(x)=x^2$ $(0,l)$ (vi) $f(x)=e^{ax}$ $(0,l)$

F.S.T and F.C.T, F.S.I.T, F.C.I.T

Sol:- $F_s(p) = F.S.T = \int_0^l f(x)\sin\left(\frac{p\pi x}{l}\right)dx$.

Here $l=\pi$

$$F_s(p) = \int_0^{\pi} 1\cdot\sin\left(\frac{p\cancel{\pi}x}{\cancel{\pi}}\right)dx$$

$$= \int_0^{\pi}\sin px\,dx$$

$$= \left[\frac{-\cos px}{p}\right]_0^{\pi} = \frac{1}{p}$$

$$= \frac{-1}{p}\left[(-1)^p - 1\right]$$

if $p=$ even

$F_s(p)=0$

if $p=$ odd

$= \frac{2}{p}$.

Scanned by CamScanner

$F_c(p) = \int_0 f(x) \cos\left(\frac{p\pi x}{l}\right) dx$

$l = \pi$

$F_c(p) = \int_0^{\pi} 1 \cdot \cos\left(\frac{p\pi x}{\pi}\right) dx$

$= \left[\frac{\sin px}{p}\right]_0^{\pi} = \frac{1}{p}\left[\sin px\right]_0^{\pi}$

$= \frac{1}{p}[0-0] = 0.$

(ii) $F_s(p) = \int_0^{l} x \sin\left(\frac{p\pi x}{l}\right) dx$

$u = x \qquad v = \sin\left(\frac{p\pi x}{l}\right)$

$u' = 1 \qquad v_1 = \frac{-\cos\left(\frac{p\pi x}{l}\right)}{\left(\frac{p\pi}{l}\right)}$

$v_2 = \frac{-\sin\left(\frac{p\pi x}{l}\right)}{\left(\frac{p\pi}{l}\right)^2}$

$= \left[x\left[\frac{-\cos\left(\frac{p\pi x}{l}\right)}{\left(\frac{p\pi}{l}\right)}\right] + \frac{\sin\left(\frac{p\pi x}{l}\right)}{\left(\frac{p\pi}{l}\right)^2}\right]_0^{l}$

$= \left[-l\left(\frac{l}{p\pi}\cos(p\pi)\right) + \left(\frac{l}{p\pi}\right)^2 \sin(p\pi)\right]$

$= \frac{l^2}{p\pi}\left[\frac{\sin p\pi}{p\pi} - \cos p\pi\right].$

$= -\frac{l^2}{p\pi}\cos p\pi = \frac{-l^2}{p\pi}(-1)^p$ //.

$F_c(p) = \int_0^{l} x \cos\left(\frac{p\pi x}{l}\right) dx$

$u = x \qquad v = \cos\left(\frac{p\pi x}{l}\right)$

$u' = 1 \qquad v_1 = \frac{l}{p\pi}\sin\left(\frac{p\pi x}{l}\right)$

$v_2 = \left(\frac{l}{p\pi}\right)^2\left(-\cos\left(\frac{p\pi x}{l}\right)\right)$

$$= \left[x \left(\frac{l}{p\pi} \sin\left(\frac{p\pi x}{l}\right) \right) + \left(\frac{l}{p\pi}\right)^2 \left(\cos\left(\frac{p\pi x}{l}\right) \right) \right]_0^l$$

$$= \left[\frac{l^2}{p\pi} \sin p\pi + \left(\frac{l}{p\pi}\right)^2 \cos p\pi - \frac{l^2}{(p\pi)^2} \right]$$

$$= \cancel{\left[\left(\frac{l}{p\pi}\right)^2 (-1)^p - \frac{l^2}{p\pi} \right]} \quad \left(\frac{l}{p\pi}\right)^2 \left[(-1)^p - 1 \right]$$

$$= \frac{l^2}{p\pi} \left[\cancel{\frac{(-1)^p}{p\pi}} - 1 \right] = \begin{cases} 0 & p = \text{Even} \\ -\frac{2l^2}{p\pi^2} & p = \text{odd} \end{cases}$$

(vi) $F_s(p) = \int_0^l f(x) \sin\left(\frac{p\pi x}{l}\right) dx.$

$$= \int_0^l e^{ax} \sin\left(\frac{p\pi x}{l}\right) dx.$$

$a = a$, $b = \frac{p\pi}{l}$

$$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$\int e^{ax} \cos bx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx].$$

$$= \left\{ \frac{e^{ax}}{a^2 + \left(\frac{p\pi}{l}\right)^2} \left[a \sin\left(\frac{p\pi x}{l}\right) - \frac{p\pi}{l} \cos \frac{p\pi x}{l} \right] \right\}_0^l$$

$$= \left\{ \frac{e^{al}}{a^2 + \left(\frac{p\pi}{l}\right)^2} \left[a \sin p\pi - \frac{p\pi}{l} \cos p\pi \right] - \frac{1}{a^2 + \left(\frac{p\pi}{l}\right)^2} \left[-\frac{p\pi}{l} \right] \right\}$$

$$= \frac{e^{al}}{a^2 + (p\pi)^2} \left[-\frac{p\pi}{l} \cos p\pi \right] + \frac{p\pi}{a^2 + \left(\frac{p\pi}{l}\right)^2}$$

$$= \frac{l^2}{a^2l^2+p^2\pi^2}\left[e^{-al} \times \frac{p\pi}{l}(-1)^p + \frac{p\pi}{l}\right]$$

$$= \frac{l}{a^2l^2+p^2\pi^2} \times \cancel{p\pi}\left[e^{-al}(-1)^p p\pi + p\pi\right].$$

$$= \frac{l}{a^2l^2+p^2\pi^2} \, \frac{p\pi}{l}\left[1-e^{al}(-1)^p\right] \text{ ans.}$$

(iii) $F_c(p) = \int_0^l f(x) \cos\left(\frac{p\pi x}{l}\right) dx.$

$$= \int_0^l e^{ax} \sin\left(\frac{p\pi x}{l}\right) dx$$

$$= \left\{\frac{e^{ax}}{a^2+\left(\frac{p\pi}{l}\right)^2}\left[a\cos\left(\frac{p\pi x}{l}\right) + \frac{p\pi}{l}\sin\frac{p\pi x}{l}\right]\right\}_0^l$$

$$= \left\{\frac{e^{al}}{a^2+\left(\frac{p\pi}{l}\right)^2}\left[a\cos p\pi + \frac{p\pi}{l}\sin p\pi\right] - \frac{1}{a^2+\left(\frac{p\pi}{l}\right)^2}(a)\right\}$$

$$= \frac{1}{a^2+\left(\frac{p\pi}{l}\right)^2}\left[a(-1)^p\right] - \frac{a}{a^2+\left(\frac{p\pi}{l}\right)^2}$$

$$= \frac{a}{a^2+\left(\frac{p\pi}{l}\right)^2}\left[(-1)^p - 1\right].$$

* Find $f(x)$ if its finite F.S.T is given by $\frac{2\pi}{p^3}(-1)^p$.

sol:

F.S.T

$$F_s(p) = \frac{2\pi}{p^3}(-1)^p$$

$$F.I.S.T = \frac{2}{l}\sum_{p=1}^{\infty} F_s(p)\sin\left(\frac{p\pi x}{l}\right)$$

$$= \frac{2}{l}\sum_{p=1}^{\infty} \frac{2\pi}{p^3}(-1)^p \sin\left(\frac{p\pi x}{l}\right)$$

$$= \frac{4\pi}{l}\sum_{p=1}^{\infty} \frac{(-1)^p}{p^3}\sin\left(\frac{p\pi x}{l}\right)$$

* Find $f(x)$ if its finite F.S.T is given by

(i) $F_s(p) = \frac{1-\cos p\pi}{p^2\pi^2}$, $p = 1,2,3,\ldots$, $0<x<\pi$.

(ii) $F_s(p) = \frac{16(-1)^{p-1}}{p^3}$, $p=1,2,3\ldots$ $0<x<8$.

(iii) $F_s(p) = \frac{\cos\left(\frac{2\pi p}{3}\right)}{(2p+1)^2}$, $p=1,3,3\ldots$ $0<x<1$

* Find the inversion formula using finite fourier cosine transform,

(i) $F_c(p) = \frac{1}{2p}\sin\left(\frac{p\pi}{2}\right)$, $p=1,2,3\ldots$

$= \frac{\pi}{4}$, $p=0$, $0<x<2\pi$.

(ii) $F_c(p) = \begin{cases} \frac{6\sin\left(\frac{p\pi}{2}\right) - \cos p\pi}{(2p+1)\pi}, & p=1,2,3\ldots \\ \frac{2}{\pi}, \; p=0, & 0<x<4 \end{cases}$

(iii) $F_c(p) = \begin{cases} \cos\left(\frac{p\pi}{3}\right), & p = 1,2,3 \ldots \\ 1, & p = 0 \end{cases}$ $0 < x < l$

* Imp

Find the finite fourier sine transform of $f(x) = 1$, $(0, \pi)$ use inversion theorem and find the fourier sine series from the func and hence proved

(a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = \frac{\pi}{4}$

(b) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$ (parseval's identity)

Sol/ $l = \pi$

F.S.T $\quad F_s(p) = \int_0^l 1 \cdot \sin\left(\frac{p\pi x}{\pi}\right) dx$

$$= \int_0^{\pi} \sin(px)\, dx$$

$$= \left[\frac{-\cos px}{p}\right]_0^{\pi} = \frac{-1}{p}[\cos p\pi - 1]$$

$$= \frac{-1}{p}\left[(-1)^p - 1\right]$$

$$\therefore F_s(p) = \begin{cases} \frac{2}{p} & p \to \text{odd} \\ 0, & p \to \text{even} \end{cases}$$

$$\text{F.I.S} = \frac{2}{l} \sum_{p=1}^{\infty} F_s(p) \sin\left(\frac{p\pi x}{l}\right) \qquad l = \pi$$

$$1 = \frac{2}{\pi} \sum_{p=\text{odd}}^{\infty} \frac{2}{p} \sin\left(\frac{p\pi x}{\pi}\right)$$

$$= \frac{2}{\pi} \sum_{p=\text{odd}}^{\infty} \frac{2}{p} \sin px$$

$$\frac{\pi}{4} f(x) = \sum_{p=\text{odd}}^{\infty} \frac{\sin px}{p}$$

Take $x = \frac{\pi}{2}$

$$\frac{\pi}{4} f\left(\frac{\pi}{2}\right) = \sum_{p=\text{odd}}^{\infty} \frac{\sin p\left(\frac{\pi}{2}\right)}{p}$$

$$\boxed{\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} \cdots}$$

W.K.T the parsevals identity F.S

$$\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{UL-LL}\int_{LL}^{UL}[f(x)]^2\,dx.$$

$a_0 = a_n = 0$

$$\Rightarrow \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 = \frac{1}{\pi-0}\int_0^{\pi} 1^2\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\frac{2}{\pi}\int_0^{\pi} 1 \sin\left(\frac{p\pi x}{\pi}\right)dx$$

$$\text{cl. } b_p = \frac{2}{\pi}\int_0^{\pi}\sin px\,dx.$$

$$= \frac{2}{\pi}\left[\frac{-\cos px}{p}\right]_0^{\pi}$$

$$= \frac{-2}{\pi p}\left[(-1)^p - 1\right]$$

$$b_p = \begin{cases} \frac{4}{\pi p} & p \to \text{odd} \\ 0 & p \to \text{even} \end{cases}$$

$n = p$

$$\therefore \frac{1}{2}\sum_{p=odd} b_p^2 = 1$$

$$\frac{1}{2}\sum_{p=odd}\frac{16}{\pi^2 p^2} = 1$$

$$\sum_{p=odd}^{\infty}\frac{1}{n} = \frac{\pi^2}{8}$$

$$\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}}$$

(iii) $F_s(p) = \int_0^l x^2 \sin\left(\frac{p\pi x}{l}\right) dx$.

$= \int_0^l$ $u = x^2$ $\quad v = \sin\left(\frac{p\pi x}{l}\right)$

~~$v_1 = \cos p\pi d$~~

$u' = 2x$ $\quad v_1 = \frac{-l}{p\pi}\cos\left(\frac{p\pi x}{l}\right)$

$u'' = 2$ $\quad v_2 = \frac{-l^2}{(p\pi)^2}\sin\left(\frac{p\pi x}{l}\right)$

$v_3 = \frac{l^3}{(p\pi)^3}\cos\left(\frac{p\pi x}{l}\right)$

$$= \left[-(x^2)\left(\frac{l}{p\pi}\right)\cos\left(\frac{p\pi x}{l}\right) + (2x)\frac{l^2}{(p\pi)^2}\sin\left(\frac{p\pi x}{l}\right) - \frac{2l^3}{(p\pi)^3}\cos\left(\frac{p\pi x}{l}\right)\right]_0^l$$

$$= \left[\frac{-l^3}{p\pi}\cos p\pi + \frac{2l^3}{(p\pi)^2}\sin p\pi - \frac{2l^3}{(p\pi)^3}\cos p\pi\right] + \frac{2l^3}{(p\pi)^3}$$

$$= \left[\frac{-l^3}{p\pi}(-1)^p - \frac{2l^3}{(p\pi)^3}(-1)^p + \frac{2l^3}{(p\pi)^3}\right]$$

$$= \frac{-l^3}{p\pi}\left[(-1)^p - \frac{2}{(p\pi)^2}(-1)^p + \frac{2}{(p\pi)^2}\right]$$

$$F_c(p) = \int_0^l x^2 \cos\left(\frac{p\pi x}{l}\right) dx$$

$u = x^2$ $\quad v = \cos\left(\frac{p\pi x}{l}\right)$

$u' = 2x$ $\quad v_1 = \frac{+l}{p\pi}\sin\left(\frac{p\pi x}{l}\right)$

$u'' = 2$ $\quad v_2 = \frac{-l^2}{(p\pi)^2}\cos\left(\frac{p\pi x}{l}\right)$

$v_3 = \frac{-l^3}{(p\pi)^2}\sin\left(\frac{p\pi x}{l}\right)$

Scanned by CamScanner

$$= \left[(x l^2) \frac{l}{p\pi} \sin\left(\frac{p\pi x}{l}\right) + (2x) \frac{l^2}{(p\pi)^2} \cos\left(\frac{p\pi x}{l}\right) - 2\frac{l^3}{(p\pi)^3} \sin\left(\frac{p\pi x}{l}\right) \right]_0^l$$

$$= \left[\frac{l^3}{p\pi} \sin p\pi + \frac{2 l^3}{(p\pi)^2} \cos p\pi - \frac{2l^3}{(p\pi)^3} \sin p\pi \right]$$

$$= \cancel{\frac{l^3}{p\pi}} \; \frac{2l^3}{(p\pi)^2} (-1)^p.$$

Scanned by CamScanner

Formulae for B.V.P

* $F_s\left[\frac{\partial u}{\partial x}\right] = \frac{-p\pi}{l} F_c(u)$

* $F_c\left[\frac{\partial u}{\partial x}\right] = \frac{p\pi}{l} F_s(u) - u(0,t) + (-1)^p u(l,t)$

* $F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{-p^2\pi^2}{l^2} F_s(p) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(l,t)\right]$

* $F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{p^2\pi^2}{l^2} F_c(p) + \left[\frac{\partial u}{\partial x}(l,t)\cos p\pi - \frac{\partial u}{\partial x}(0,t)\right]$

Problems Solving Boundary value problems Using finiite F.S.T etc

1. Using finite F.T solve $\frac{du}{dt} = \frac{\partial^2 u}{\partial x^2}$ given $u(0,t)=0$ & $u(4,t)=0$ $u(x,0)=2x$, where $0<x<4, t>0$.

Sol: Given that $u(0,t) \rightarrow$ Take finite F.S.T,

x by $\sin\left(\frac{p\pi x}{l}\right)$, $l=4$ we have

$$\int_0^4 \frac{\partial u}{\partial t} \sin\left(\frac{p\pi x}{4}\right) dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin\left(\frac{p\pi x}{4}\right) dx$$

$$\Rightarrow F_s\left[\frac{du}{dt}\right] = F_s\left[\frac{d^2u}{dx^2}\right]$$

$$\Rightarrow \frac{d}{dt}\left[F_s(u)\right] = \frac{-p^2\pi^2}{l^2} F_s(u) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(l,t)\right]$$

$$= \frac{-p^2\pi^2}{16}\left[F_s(u)\right] + 0$$

$$\Rightarrow \frac{d\left[F_s(u)\right]}{\left[F_s(u)\right]} = \frac{-p^2\pi^2}{16} dt.$$

Taking $\int$ w.r.t 't'.

$$\log\left[F_s(u)\right] = \frac{-p^2\pi^2}{16}t + C$$

$$\Rightarrow F_s(u) = e^{-p^2\pi^2 t/16} e^{c} \sim A.$$

$$\int_0^{l} u(x,t)\sin\left(\frac{p\pi}{l}\right)dx = A e^{-p^2\pi^2 t/16} \quad \text{—(1)}$$

Given $u(x,0) = 2x$

$$\Rightarrow \int_0^4 u(x,0)\sin\left(\frac{p\pi x}{4}\right)dx = A$$

$$\Rightarrow \int_0^4 2x\sin\left(\frac{p\pi x}{4}\right)dx = A.$$

$u = x$ $\quad$ $v = \sin\left(\frac{p\pi x}{4}\right)$

$u' = 1$ $\quad$ $v_1 = -\cos\left(\frac{p\pi x}{4}\right)\times\frac{4}{p\pi}$

$v_2 = -\sin\left(\frac{p\pi x}{4}\right)\times\frac{16}{p^2\pi^2}$

$$A = 2\left[-x\cos\left(\frac{p\pi x}{4}\right)\times\frac{4}{p\pi} + \frac{16}{p^2\pi^2}\sin\left(\frac{p\pi x}{4}\right)\right]_0^4$$

$$A = 2\left[-4\cos\left(\frac{p\pi 4}{4}\right)\cdot\frac{4}{p\pi} + \frac{16}{p^2\pi^2}\sin\left(\frac{4p\pi}{4}\right)\right] - [0+0]$$

$$A = \frac{-32}{p\pi}\cos p\pi \Rightarrow (-1)^p$$

Sub in (1)

$$\therefore F_s(u) = \frac{-32}{p\pi}(-1)^p \times e^{-p^2\pi^2 t/16}$$

(or)

$F_s\left[u(x,t)\right]$

By finite Fourier inversion for sine.

$$f(x) = \sum_{p=1}^{\infty} F_s(p) \sin\left(\frac{p\pi x}{l}\right)$$

Here

$$u(x,t) = \frac{2}{l} \sum_{p=1}^{\infty} F_s[u(x,t)] \sin\left(\frac{p\pi x}{4}\right)$$

$$= \frac{-32}{\pi} \times \frac{2}{4} \times \sum_{p=1}^{\infty} (-1)^p \times e^{-p\pi^2 t/16} \times \sin\left(\frac{p\pi x}{4}\right)$$

$$= \frac{-16}{\pi} \sum_{n=1}^{\infty} (-1)^p \times e^{-p^2\pi^2/16} \sin\left(\frac{p\pi x}{4}\right)$$

2. Solve : $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 6$, $t > 0$ $\left(\frac{\partial u}{\partial x}\right)(0,t) = 0$,

$\left(\frac{\partial u}{\partial x}\right)(6,t) = 0$ & $u(x,0) = 2x$.

Sol:- $\frac{\partial u}{\partial x}$ is given at $(0,t)$ & $(6,t)$ take finite F.C.T

$\times$ by $\cos\left(\frac{p\pi x}{l}\right)$, $l = 6$, ~~we~~ have

$$\int_0^6 \left(\frac{\partial u}{\partial t}\right) \cos\left(\frac{p\pi x}{6}\right) dx = \int_0^6 \left(\frac{\partial^2 u}{\partial x^2}\right) \cos\left(\frac{p\pi x}{6}\right) dx.$$

$$F_c\left[\frac{\partial u}{\partial t}\right] = F_c\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{d}{dt} F_c(u) = \frac{-p^2\pi^2}{l^2} F_c(u) + \frac{\partial u}{\partial t}(l,t)\cos p\pi - \frac{\partial u}{\partial x}(0,t)$$

$$= \frac{-p^2\pi^2}{36} F_c(u) + \underbrace{\frac{\partial u}{\partial t}(6,t)}_{0} \cos p\pi - \underbrace{\frac{\partial u}{\partial u}\Big|(0,t)}_{0}$$

$$\frac{d[F_c(u)]}{[F_c(u)]} = \frac{-p^2\pi^2}{36} dt$$

Taking ~~int~~

$$\log F_c[u] = \frac{-p^2\pi^2 t}{36} + C$$

Taking exp,

$$F_c[u] = e^C e^{-p^2\pi^2 t/36}$$

$$= A e^{-p^2\pi^2 t/36} \quad \text{---} (1)$$

$$\Rightarrow \int_0^l u(x,t)\cos\left(\frac{p\pi x}{l}\right)dx = Ae^{-p^2\pi^2 t/36}$$

Using $t=0$, $u(x,0)=2x$, $l=6$

$$\Rightarrow \int_0^6 u(x,0)\cos\left(\frac{p\pi x}{6}\right)dx = A$$

$$\int_0^6 2x\cos\left(\frac{p\pi x}{6}\right)dx = A$$

$u = x$ $\quad v = \cos\left(\frac{p\pi x}{6}\right)$

$u' = 1$ $\quad v_1 = \sin\left(\frac{p\pi x}{6}\right)\times\frac{6}{p\pi}$

$v_2 = -\frac{36}{p^2\pi^2}\cos\left(\frac{p\pi x}{6}\right)$

$$A = 2\left[\frac{6}{p\pi}\times x\times\sin\left(\frac{p\pi x}{6}\right)+\frac{36}{p^2\pi^2}\cos\left(\frac{p\pi x}{6}\right)\right]$$

$$A = 2\left\{\left[0+\frac{36(-1)^p}{p^2\pi^2}\right]-\left[0+\frac{36}{p^2\pi^2}\right]\right\}$$

$$\boxed{A = \frac{72}{p^2\pi^2}\left[(-1)^p-1\right]}$$ sub in (1)

$$F_c(u) = \frac{72}{p^2\pi^2}\left[(-1)^p-1\right]e^{-p^2\pi^2 t/36}$$

By inversion formula for cosine

$$f(x) = \frac{1}{l}F_c(0)+\frac{2}{l}\sum_{p=1}^{\infty}F_c(s)\cos\left(\frac{p\pi x}{l}\right)dx$$

Here

$$u(x,t) = \frac{1}{6}\times F_c(0)+\frac{2}{6}\sum_{p=1}^{\infty}\frac{72}{p^2\pi^2}\left[(-1)^p-1\right]e^{-p^2\pi^2 t/36}$$

$$= \frac{1}{6}\int_0^6 (2x)\,dx + 24\sum_{p=1}^{\infty}\frac{[(-1)^p - 1]}{p^2\pi^2} e^{-p^2\pi^2 t/36}$$

$$= \frac{2}{6}\left[\frac{x^2}{2}\right]_0^6 + 24\sum_{p=1}^{\infty}\frac{[(-1)^p - 1]}{p^2\pi^2} e^{-p^2\pi^2 t/36}$$

$$= 6 + \frac{24}{\pi^2}\sum_{p=1}^{\infty}\frac{[(-1)^p - 1]}{p^2} e^{-p^2\pi^2 t/36}.$$

Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $0<x<4$, $t>0$ given $u(0,t)=0$
$u(4,t)=0$ $u(x,0)=3\sin \pi x + 2\sin 5x$

$u(0,t)$ is given, take finite Fourier sine transformation -

x by $\sin\left(\frac{p\pi x}{l}\right)$ $l=4$, we have

$$\int_0^4 \left(\frac{\partial u}{\partial t}\right)\sin\left(\frac{p\pi x}{4}\right)dx = 2\int_0^4 \left(\frac{\partial^2 u}{\partial x^2}\right)\sin\left(\frac{p\pi x}{4}\right)dx.$$

$$\Rightarrow F_s\left[\frac{\partial u}{\partial t}\right] = 2F_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial}{\partial t}[F_s(u)] = 2\left[\frac{-p^2\pi^2}{l^2}F_s(u) + \frac{p\pi}{l}[u(0,t) - (-1)^p u(4,t)]\right]$$

$$= -\frac{2p^2\pi^2}{16}F_s(u)$$

$$\Rightarrow \frac{d[F_s(u)]}{F_s(u)} = \frac{-p^2\pi^2}{8}dt$$

Taking $\int$.

$$\Rightarrow \log F_s(u) = \frac{-p^2\pi^2}{8}t + C$$

Taking exp

$$\Rightarrow F_s(u) = e^{-p^2\pi^2 t}e^{c} = Ae^{-p^2\pi^2 t/8} \quad \text{——} (1)$$

Scanned by CamScanner

Using $t=0$, $l=4$ $\quad u(x,0) = 3\sin\pi x - 2\sin 5\pi x$

$$\Rightarrow \int_0^4 u(x,t)\,\sin\left(\frac{p\pi x}{4}\right) dx = A e^{-p^2\pi^2 t/8}$$

$$\Rightarrow A = \int_0^4 (3\sin\pi x - 2\sin 5\pi x)\,\sin\left(\frac{p\pi x}{4}\right) dx$$

$$= 0, \; p \neq 4, \; p \neq 5\times 4 = 20$$

when $p=4$ $\quad A = \int_0^4 [3\sin\pi x - 2\sin 5\pi x]\,\sin\pi x\, dx$

$$= 3\int_0^4 \sin^2\pi x\, dx - 2\int_0^4 \sin 5\pi x\,\sin(\pi x)\, dx$$

$$= 3\int_0^4 \left[\frac{1-\cos 2\pi x}{2}\right] dx - \frac{2}{2}\int_0^4 \cos(5\pi x - \pi x) - \cos(5\pi x + \pi x)$$

$$= \frac{3}{2}\left[x - \frac{\sin 2\pi x}{2\pi}\right]_0^4 - \left[\frac{\sin 4\pi x}{4\pi} - \frac{\sin 6\pi x}{6\pi}\right]_0^4$$

$$= \frac{3}{2}\times 4$$

$$= 6.$$

when $p=20$

$$A = \int_0^4 [3\sin\pi x - 2\sin 5\pi x]\,\sin\left(\frac{20\pi x}{4}\right) dx$$

$$= 3\int_0^4 \sin\pi x\,\sin 5\pi x\, dx - 2\int_0^4 \sin^2 5\pi x\, dx$$

$$= -2\int_0^4 \frac{[1-\cos 10\pi x]}{2}\, dx$$

$$= -\frac{2}{2}\left[x - \frac{\sin 10\pi x}{10\pi}\right]_0^4$$

$$= -4 \qquad \text{Sub in } (1)$$

$F_s(u) = 6e^{\cdots/8} - 4e^{\cdots/8}$
$(p=4) \qquad (p=20)$

By inversion formula for sine.

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty} F_s(u)\sin\left(\frac{p\pi x}{l}\right)$$

$$= \frac{2\cdot 1}{4\cdot 2}\left[6e^{-p^2\pi^2 t/8}\sin\left(\frac{4\pi x}{4}\right) - 4e^{-p^2\pi^2 t/8}\sin\left(\frac{20\pi x}{4}\right)\right]$$

$$= \frac{1}{2}\left[6e^{-16\pi^2 t/8}\sin(\pi x) - 4e^{-50\pi^2 t/8}\sin(5\pi x)\right]$$

$$= 3e^{-2\pi^2 t}\sin(\pi x) - 4e^{-50\pi^2 t}\sin(5\pi x).$$

4. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi$, $t > 0$, given $u(0,t) = u(\pi,t) = t$, $t > 0$, $u(x,0) = \sin^3 x$.

Sol: Given that $u(0,t)$ & $u(\pi,t)$, take Finite F.S.T $\times$ by $\sin\left(\frac{p\pi x}{l}\right)$, $l = \pi$, we have,

$$\Rightarrow \int_0^{\pi} \frac{\partial u}{\partial t}\sin\left(\frac{p\pi x}{\pi}\right)dx = \int_0^{\pi}\frac{\partial^2 u}{\partial x^2}\sin\left(\frac{p\pi x}{\pi}\right)dx$$

$$\Rightarrow F_s\left[\frac{\partial u}{\partial t}\right] = F_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{d}{dt}\left[F_s(u)\right] = \frac{-p^2\pi^2}{l^2}F_s(u) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(l,t)\right]$$

$$= \frac{-p^2\pi^2}{\pi^2}F_s(u) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(\pi,t)\right]$$
$\qquad\qquad 0 \qquad\qquad 0$

$$= -p^2\left(F_s(u)\right).$$

$$\Rightarrow \frac{d[F_s(u)]}{F_s[u]} = -p^2 dt$$

Taking int

$$\log F_s(u) = -p^2 t + c$$

Taking Exp.;

$F_s(u) = e^{-p^2t} e^{c} = A \cdot e^{-p^2t}$ —①

using $t=0$, $u(x,0) = \sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$

$\Rightarrow \int_0^l u(x,t)_0 \sin\left(\frac{p\pi x}{l}\right) dx = A e^{-p^2t}_{t=0}$

i.e. $A \Rightarrow \frac{1}{4}\int_0^\pi (3\sin x - \sin 3x) \sin\left(\frac{p\pi x}{\pi}\right) = 0 \quad p \neq 1, \; p \neq 3$

when $p=1$, $A = \frac{1}{4}\int_0^\pi (3\sin x - \sin 3x)\sin x \, dx$

$= \frac{1}{4} \times \int_0^\pi 3\sin^2 x \, dx - \int_0^\pi \sin 3x \sin x \, dx$

$= \frac{1}{4}\left[\frac{3}{2}\int_0^\pi (1-\cos 2x)\, dx - \frac{1}{2}\int_0^\pi (\cos 2x - \cos 4x)\, dx\right.$

$= \frac{1}{4}\left[\frac{3}{2}\left(x - \frac{\sin 2x}{2}\right)\right]_0^\pi - \frac{1}{2}(\sin$

$= \frac{3\pi}{8}$

when $p=3$, $A = \frac{1}{4}\int_0^\pi (3\sin x - \sin 3x)\sin 3x \, dx$

$= \frac{1}{4}\int_0^\pi 3\sin x \sin 3x \, dx - \int_0^\pi \sin^2 3x \, dx$

$= \frac{-1}{4} \times \frac{1}{2}\int_0^\pi [1 - \cos 6x]\, dx$

$= -\frac{1}{8}[x - \frac{\sin 6x}{6}]_0^\pi$

$= \frac{-\pi}{8}$ sub in ①

$F_s(u) = \frac{3\pi}{8} e^{-p^2t} - \frac{\pi}{8} e^{-p^2t}$

$(p=3)$

By inversion formula for sine

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty} F_s(u)\sin\left(\frac{p\pi x}{l}\right)$$

$$= \frac{2}{\pi}\left[\frac{3\pi}{8}e^{-p^2t}\sin\left(\frac{\pi x}{\pi}\right) - \frac{\pi}{8}e^{-p^2t}\sin\left(\frac{3\pi x}{\pi}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{3\pi}{8}e^{-t}\sin x - \frac{\pi}{8}e^{-9t}\sin 3x\right]$$

$$= \frac{2}{\pi}\times\frac{3\pi}{8_4}e^{-t}\sin x - \frac{2}{\pi}\times\frac{\pi}{8_4}e^{-9t}\sin 3x.$$

$$= \frac{3}{4}\times e^{-t}\sin x - \frac{e^{-9t}}{4}\sin 3x.$$

5) ~~Stove~~ solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 10$, given $u(0,t) = u(10,t)$

for $t > 0$ & $u(x,0) = 10x - x^2$, $0 < x < 10$.

~~$u(0,t)$ & $u(x,0) = 10x - x^2$~~,

Sol $u(0,t)$ & $u(10,t)$ is given take finite F.S.T

i.e by $\sin\left(\frac{p\pi x}{l}\right)$, $l = 10$, we have.

$$\int_0^{10}\frac{du}{dt}\sin\left(\frac{p\pi x}{10}\right)dx = \int_0^{10}\frac{\partial^2 u}{\partial x^2}\sin\left(\frac{p\pi x}{10}\right)dx$$

$$F_s\left(\frac{du}{dt}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \frac{d}{dt}[F_s(u)] = \frac{-p^2\pi^2}{l^2}F_s(u) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(l,t)\right]$$

$$= \frac{-p^2\pi^2}{100}F_s(u) + \frac{p\pi}{10}\left[u(0,t) - (-1)^p u(l,t)\right]$$

$$\Rightarrow \frac{d[F_s(u)]}{F_s(u)} = \frac{-p^2\pi^2}{100}dt$$

Taking $\int$

$$\Rightarrow \log F_s(u) = \frac{-p^2\pi^2 t}{100} + C$$

Taking ep

$\Rightarrow F_s(u) = e^{-p^2\pi^2 k t/100}$ $e^{c} = e^{-p^2\pi^2 t/100} A$ —— (1)

Using $t=0$, $l=10$, $u(x,0) = 10x - x^2$, $0 < x < 10$, we have

$\Rightarrow \int_0^{10} u(x,t) \sin\left(\frac{p\pi x}{10}\right) = e^{-p^2\pi^2 t/100} A$

$A = \int_0^{10} (10x - x^2) \sin\left(\frac{p\pi x}{10}\right) dx$

$u = 10x - x^2$ $\quad v = \sin\left(\frac{p\pi x}{10}\right)$

$u' = 10 - 2x$ $\quad v_1 = \frac{-10}{p\pi} \cos\left(\frac{p\pi x}{10}\right)$

$u'' = -2$ $\quad v_2 = \frac{-100}{p^2\pi^2} \sin\left(\frac{p\pi x}{10}\right)$

$v_3 = \frac{1000}{p^3\pi^3} \cos\left(\frac{p\pi x}{10}\right)$

$$A = \left[(10x - x^2)\left(\frac{-10}{p\pi}\right)\cos\left(\frac{p\pi x}{10}\right) + \left(\frac{10-2x}{p^2\pi^2}\right) 100 \sin\left(\frac{p\pi x}{10}\right) - \frac{2000}{p^3\pi^3} \cos\left(\frac{p\pi x}{10}\right)\right]_0^{10}$$

$$= \left[0 - 0 - \frac{2000}{p^3\pi^3} \cos\left(\frac{p\pi \times 10}{10}\right)\right] - \left[0 + 0 - \frac{2000}{p^3\pi^3}(1)\right]$$

$$= -\frac{2000}{p^3\pi^3}\left[(-1)^p - 1\right]$$

$$= \begin{cases} 0 & p \to \text{even} \\ \frac{4000}{p^3\pi^3} & \text{odd} \end{cases}$$ sub in (1)

$\therefore F_s(u) = \frac{4000}{p^3\pi^3} e^{-p^2\pi^2 t/100}$ $p \to$ odd

Using inversion formula for sine, we have

$$f(x) = \frac{2}{l} \sum_{p=1}^{\infty} F_s(p) \sin\left(\frac{p\pi x}{l}\right)$$

$$\Rightarrow u(x,t) = \frac{1}{10} \sum_{p=odd}^{\infty} \frac{4000}{p^3 \pi^3} e^{-p^2\pi^2 t/100} \sin\left(\frac{p\pi x}{l}\right)$$

$$= \frac{800}{\pi^2} \sum_{p=odd}^{\infty} \frac{1}{p^3} e^{-p^2\pi^2 t/100} \sin\left(\frac{p\pi x}{10}\right).$$

Scanned by CamScanner

Applications of finite fourier sine and cosine transforms

The one dimensional heat eqn in 2-d is given by $\frac{\partial u}{\partial t} = \alpha^2 \cdot \frac{\partial^2 u}{\partial x^2}$

$$\alpha = \frac{\rho}{k \times c}$$

$\alpha^2 \rightarrow$ diffusivity.

$k \rightarrow$ specific heat

$c \rightarrow$ thermal conductivity

$\rho \rightarrow$ density.

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{-p^2\pi^2}{l^2} F_s(p) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(l,t)\right]$$