Periodic functions

1) Find the laplace transform of the rectangular wave given by $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

$$f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$$

$p = 2b$

$$= \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st}\,dt + \int_b^{2b} e^{-st}(-1)\,dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^b - \left[\frac{e^{-st}}{-s} \right]_b^{2b} \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left[-\frac{1}{s} \left[e^{-st} \right]_0^b + \frac{1}{s} \left[e^{-st} \right]_b^{2b} \right]$$

$$= \frac{1}{s(1-e^{-2bs})} \left[-(e^{-bs}-1) + [e^{-2bs} - e^{-sb}] \right]$$

$$= \frac{1}{s(1-e^{-2bs})} \left[-e^{-bs} + 1 + (e^{-bs})^2 - e^{-bs} \right]$$

$$(1-e^{-bs})^2$$

1

Find the laplace transform of the half wave rectifier function

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

$$P = \frac{2\pi}{\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\, dt + 0 \right]$$

(sin wt)

(cos wt)

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2} \left[-s\sin\omega t - \omega\cos\omega t\right] \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega}\omega + \omega}{s^2+\omega^2} \right]$$

$$= \frac{\omega\left[1+e^{-s\pi/\omega}\right]}{\left[1-e^{-s\pi/\omega}\right]\left[1+e^{-s\pi/\omega}\right]\left[s^2+\omega^2\right]}$$

$$= \frac{\omega}{(s^2+\omega^2)(1-e^{-s\pi/\omega})}$$

Find the laplace transform of $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t & a < t < 2a \end{cases}$, $f(t+2a) = f(t)$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st} t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right.$$

$$= \frac{1}{1-e^{-2as}} \left[\left[-t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) + \frac{e^{-st}}{s^2} \right]_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left[-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right] - \left(-\frac{1}{s^2}\right) \right] + \left[\left(\frac{e^{-2as}}{s^2}\right) - \left(-\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right) \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right]$$

$$= \frac{[1-e^{-as}]^2}{s^2(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s^2(1+e^{-as})}.$$

$$= \frac{1}{s^2} \tanh\left[\frac{as}{2}\right]$$

Find the laplace transform of $|\sin t|$

$p = \pi$

$$= \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st} \sin t\, dt$$ [∵ sint value is +ve (0,π

$$= \frac{1}{1-e^{-\pi s}} \left[\frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-\pi s}} \left[\frac{1+e^{-\pi s}}{s^2+1} \right]$$

$$= \frac{1+e^{-\pi s}}{1-e^{-\pi s}} \left[\frac{1}{s^2+1} \right] = \coth\left[\frac{\pi s}{2}\right] \left(\frac{1}{s^2+1}\right)$$

$$= \left[\frac{1}{s^2+1}\right] \coth\left[\frac{\pi s}{2}\right]$$

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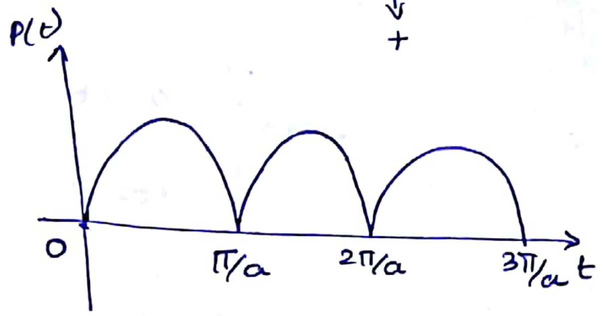
Full wave sine $|\sin at|$ $(0, \pi/a)$ $f(t) = \begin{cases} +\sin at & (0, \pi/2a) \\ -\sin at & (\pi/2a, \pi/a) \end{cases}$

$\downarrow$ $+$

$P = \pi/a$



$$= \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

$$= \left(\frac{1}{1-e^{-\pi s/a}}\right) \int_0^{\frac{\pi}{a}} e^{-st} \sin at \; dt$$

$$\int e^{at} \sin bt\,dt = \frac{e^{at}}{a^2+b^2} [a \sin bt - b \cos bt]$$

$a = -s$

$b = a$

$$= \frac{1}{1-e^{-\pi s/a}} \left[\frac{e^{-st}}{(s^2+a^2)} [-s \sin at - a \cos at] \right]_{t=0}^{t=\pi/a}$$

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$$= \frac{1}{1-e^{-\pi s/a}} \left[\frac{e^{-\frac{\pi s}{a}}}{s^2+a^2} \left[-s \sin\left(\frac{\cancel{a}\pi}{\cancel{a}}\right)_0 - a \cos\left(\frac{a\pi}{a}\right)_{-1} \right] \right.$$

L.L

$$\left. - \left[\frac{1}{s^2+a^2} [\cancel{s \sin 0}_0 - a \cos 0_1] \right] \right]$$

$$= \frac{1}{(1-e^{-\pi s/a})(s^2+a^2)} [a e^{-\pi s/a} + a]$$

$$= \frac{a}{s^2+a^2} \times \frac{1+e^{-\pi s/a}}{1-e^{-\pi s/a}} \Rightarrow \frac{a}{s^2+a^2} \times \coth\left(\frac{\pi s}{2a}\right) //$$

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Full cosine $f(t) = |\cos at|$ $\quad (0, \pi/a)$

$$f(t) = \begin{cases} +\cos at & , \ (0, \pi/2a) \\ -\cos at & , \ (\pi/2a, \pi/a) \end{cases}$$

$$= \frac{1}{1-e^{-\pi s/a}} \int_0^{\pi/a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\pi s/a}} \left[\int_0^{\pi/2a} e^{-st} \cos at\,dt - \int_{\pi/a}^{\pi/2a} e^{-st} \cos at\,dt \right]$$

$$\int e^{at} \cos bt\,dt = \frac{e^{at}}{a^2+b^2}\left[a\cos bt + b\sin bt\right]$$

Here, $\quad a = -s \quad b = a$

$$\Rightarrow \frac{1}{1-e^{-\pi s/a}} \left[\left\{ \frac{e^{-st}}{s^2+a^2}\left[-s\cos at + a\sin at\right] \right\}_{u.L\ (t=\pi/2a)} - \left\{ \frac{e^{-st}}{s^2+a^2}\left[-s\,\cancel{\cos at} + a\,\cancel{\sin at}\right] \right\}_{L.L\ (t=0)} \right]$$

$$- \left[\left\{ \frac{e^{-st}}{s^2+a^2}\left[-s\cos at + a\sin at\right] \right\}_{u.L\ (t=\pi/a)} - \left\{ \frac{e^{-st}}{s^2+a^2}\left[-s\cos at + a\sin at\right] \right\}_{L.L.\ (t=\pi/2a)} \right]$$

$$\Rightarrow \frac{1}{1-e^{-\pi s/a}(s^2+a^2)} \left[e^{-\pi s/2a}\left[-s\overset{0}{\cos}\left(\frac{a\pi}{2\cancel{a}}\right) + a\overset{1}{\sin}\left(\frac{a\pi}{2\cancel{a}}\right)\right] + s \right]$$

$$- \left[-e^{-\pi s/a}\left[-s\overset{-1}{\cos}\left(\frac{a\pi}{\cancel{a}}\right) + a\overset{0}{\sin}\left(\frac{a\pi}{a}\right)\right]\right]$$

$$- \left[-e^{-\pi s/2a}\left[-s\overset{0}{\cos}\left(\frac{a\pi}{2\cancel{a}}\right) + a\overset{1}{\sin}\left(\frac{a\pi}{2a}\right)\right]\right]$$

$$= \frac{\left[a e^{-\pi s/2a} + s - s e^{-\pi s/a} + a e^{-\pi s/2a}\right]}{(s^2+a^2)\,(1-e^{-\pi s/a})}$$

$$= \frac{2a\; e^{-\pi s/a}}{(s^2+a^2)(1-e^{-\pi s/a})} + \frac{s\,\cancel{(1-e^{-\pi s/a})}}{(s^2+a^2)\;\cancel{(1-e^{-\pi s/a})}}$$

$$= \frac{a}{(s^2+a^2)} \times \frac{1}{\dfrac{(1-e^{-\pi s/a})}{(2e^{-\pi s/a})}} + \frac{s}{s^2+a^2} \qquad \because \sinh\theta = \frac{e^{+\theta} - e^{-\theta}}{2}$$

$$\Rightarrow \frac{a}{s^2+a^2}\; \frac{1}{\dfrac{e^{\pi s/2a} - e^{-\pi s/a + \pi s/2a}}{2}} + \frac{s}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2}\; \frac{1}{\dfrac{e^{+\pi s/2a} - e^{-\pi s/2a}}{2}} + \frac{s}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2} \times \frac{1}{\sinh\left(\frac{\pi s}{2a}\right)} + \frac{s}{s^2+a^2}$$

or

$$\Rightarrow \frac{a}{s^2+a^2} \times \cosh\left(\frac{\pi s}{2a}\right) + \frac{s}{s^2+a^2} \,//.$$

Convulsion :

$$L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b^2-a^2} L^{-1}\left[\frac{b^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b^2-a^2} L^{-1}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2}\frac{1}{a} L^{-1}\left[\frac{a}{s^2+a^2}\right] - \frac{1}{b^2-a^2}\frac{1}{b} L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$= \frac{1}{a(b^2-a^2)} \sin at - \frac{1}{b(b^2-a^2)} \sin bt$$

$$= \frac{1}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s^2+a^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+b^2} - \frac{a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+b^2}\right] - a^2 L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right] - \frac{a^2}{b^2-a^2}L^{-1}\left[\frac{b^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2}L^{-1}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+\omega^2)^2}\right]$$

$$= L^{-1}\left[\frac{(s^2+\omega^2)+(s^2-\omega^2)}{2(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s^2+\omega^2} + \frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega}\sin\omega t + t\cos\omega t\right]$$

$$L^{-1}\left[\frac{1}{(s^2+\omega^2)^2}\right]$$

$$= L^{-1}\left[\frac{(s^2+\omega^2)-(s^2-\omega^2)}{2\omega^2(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2\omega^2} L^{-1}\left[\frac{1}{s^2+\omega^2} - \frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2\omega^2}\left[\frac{1}{\omega}\sin\omega t - t\cos\omega t\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{\cos at - \cos bt}{b^2-a^2}$$

$$LHS = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$= \cos at * \frac{1}{a}\sin bt$$

$t = u$

$t = t-u$

$$= \cos au * \frac{1}{a}\sin b(t-u)$$

$$= \frac{1}{a}\int_0^t \sin(bt-bu)(\cos au)$$

$$\sin A\cos B = \frac{1}{2}\sin(A+B) + \sin(A-B)$$

$$= \frac{1}{2a}\int_0^t \sin(bt-bu+au) + \sin(bt-bu-au)\,du$$

$$= \frac{1}{2a}\int_0^t \sin(bt\ (a-b)u + \sin bt\ (a+b)u\ du$$

$$= \frac{1}{2a}\left[-\frac{\cos bt(a-b)u}{a-b} - \frac{\cos bt(a+b)u}{a+b}\right]$$

$$= \frac{1}{2a}\left[\frac{1}{a+b} - \frac{1}{a-b}\right][\cos at - \cos b]$$

$$= \frac{1}{2a}\left(\frac{a-b-a+b}{a^2-b^2}\right)[\cos at - \cos bt]$$

$$= \frac{\cos at - \cos bt}{b^2-a^2}$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] \cdot * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \cos at + \frac{1}{a}\sin at$$

$$= \frac{1}{a}\int_0^t \cos au + \frac{1}{b}\sin a(t-u)$$

$$= \frac{1}{a}\int_0^t \cos au + \sin at - au$$

$$= \frac{1}{a}\int_0^t \sin at - au + \cos au \; du$$

$$= \frac{1}{a}\int_0^t \sin(at - au + au) + \sin(at - au - au)\,du$$

$$= \frac{1}{a}\int_0^t \sin at + \sin at - 2au \; du$$

$$= \frac{1}{a}\int_0^t u(\sin at) + a(\sin t - u)\,du$$

$$\cancel{= \frac{1}{2a}\int_0^t t\sin at + \frac{\cos at}{2a} - \cos}$$

$$= \frac{1}{2a}\, t\sin at$$