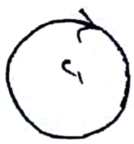
-: Complex Integration :-

* Cauchy's Integral Theorm / fundamental theorm.
* Cauchy's Integral formula
* Cauchy's Residual Theorm < type - I, type - II
* Taylor's and Lawrantz series.
* singularities (2M) * Line and surface integral (2M)

Curve < simple curve, multi curve.

Simple curve: A curve is said to be simple if it does not crosses itself otherwise the curve is said to be multicurve.

Simple curve:

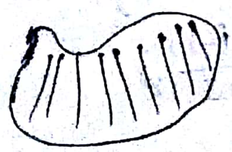


multicurve:

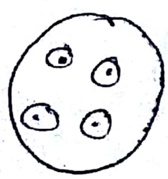
Region: < simply connected, multiple region.

A closed curve is said to be simply connected if the line joining any two points must lie entirely in itself. In other words, a simply connected region does not have any holes. otherwise it is said to be multiply connected region i.e, a region with holes.

simply connected.



multiple region:



Complex Integration: Let $w = f(z)$ be a analytic function which is also continous on one region then integration over the region C is denoted by $\int_C f(z)dz$ is called complex integration (or) contour integration.

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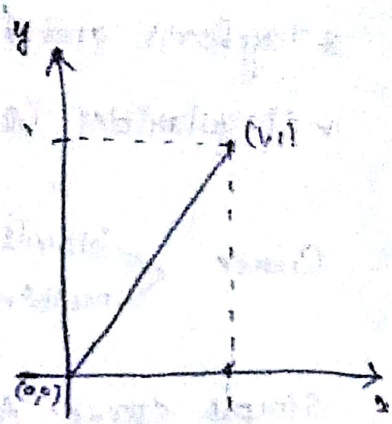
$w = f(z) = u + iv$.

$dz = dx + i\,dy$.

Ex: Evaluate $\int_0^{1+i} (x - y + ix^2)\,dz$ along the line from $z=0$ to $z = 1+i$



Given points: $O = (0,0)$

$1+i = (1,1)$

Equation: $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$\frac{y-0}{1-0} = \frac{x-0}{1-0}$

$\boxed{y = x}$

$z = x + iy$.

$dz = dx + i\,dy$.

here the curve $y = x$

$dy = dx$

$$I = \int_{(0,0)}^{(1,1)} (\cancel{x} - \cancel{y} + ix^2)\,(dx + i\underbrace{dy}_{dx})$$

$$= \int_{(0,0)}^{(1,1)} (ix^2)(1+i)\,dx.$$

$$= \int_{x=0}^{x=1} (ix^2)(1+i)\,dx = i(1+i)\int_0^1 x^2\,dx$$

$$= i(1+i)\left[\frac{x^3}{3}\right]_0^1$$

$$= i(1+i)\left(\frac{1}{3}\right)$$

$$= \frac{(i-1)}{3}$$

Q1: Evaluate $\int_{(0,0)}^{(1,1)} [x^2+y^2]\,dx - 2xy\,dy$ along the curve (i) $y=x$
(ii) $y=x^2$
(iii) $x=y^2$.

(i) $y=x$.

$y=x$ — x varies from 0 to 1

$dy=dx$ — y varies from 0 to 1.

$z=x+iy$.

$dz=dx+idy$

$$I = \int [(2x^2)\,dx - 2x^2\,dx]$$

$$= 0.$$

(ii) $y=x^2$

$dy = 2dx\,x$

$$I = \int [x^2+x^4]\,dx - 2x^3(2x\,dx).$$

$$= \int_0^1 (x^2+x^4)\,dx - 4x^4\,dx.$$

$$= \int_0^1 [x^2+x^4-4x^4]\,dx = \int_0^1 [x^2-3x^4]\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 - 3\cdot\left[\frac{x^5}{5}\right]_0^1$$

$$= \left[\frac{1}{3}\right] - 3\left[\frac{1}{5}\right]$$

$$= \frac{1}{3} - \frac{3}{5} = \frac{5-9}{15} = -4/15.$$

(iii) $x=y^2$

$dx = 2y\,dy$

$$I = \int [y^4+y^2]\,2y\,dy - 2(y^2)(y)\,dy.$$

$$= \int_0^1 [2y^5 + \cancel{2y^3}]\,dy - \cancel{2y^3}\,dy.$$

x	0	1
$x=y^2$	0	1

$$= 2\left[\frac{y^6}{6}\right]_0^1 = \frac{2}{6}\left[y^6\right]_0^1$$

$$= \frac{1}{3}(1-0)$$

$$= 1/3.$$

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Ex. $I = \int_{(0,2)}^{(4,0)} (x^2+y^2)\,dx$ along the curve $y^2=4-x$.

$$y^2 = 4-x.$$

$$= \int_{(0,2)}^{(4,0)} (x^2+4-x)\,dx.$$

$$= \int_0^4 (x^2-x+4)\,dx = \left[\frac{x^3}{3}\right]_0^4 - \left[\frac{x^2}{2}\right]_0^4 + \left[4x\right]_0^4$$

$$= \frac{64}{3} - \frac{16}{2} + 16.$$

$$= \frac{64}{3} + 16 - \frac{16}{2}$$

$$= \frac{64}{3} + \frac{16}{2}$$

$$= \frac{128+48}{6} = \frac{\cancel{176}^{88}}{\cancel{6}_3} = 88/3.$$

$(4,0)$

*

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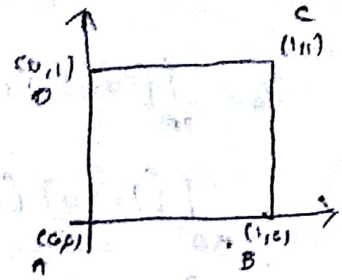
Q: Show that $\int_c (z+1)\,dz$ where c is the boundary of square with vertices $z=0, z=1, z=1+i, z=i$.

$$(0,0) \quad (1,0) \quad (1,1) \quad (0,1).$$

$z = x+iy.$

$dz = dx + i\,dy.$



$$I = \int (z+1)\,dz$$

$$I = I_1 + I_2 + I_3 + I_4$$

$I_1 = ?$ $\quad$ A (0,0) $\quad$ B (1,0)

x varies from 0 to 1

$y = c \Rightarrow dy = 0.$

$$I_1 = \int_{(0,0)}^{(1,0)} (x+1) + i(y)\,[dx + i\,dy]$$

$$= \int_0^1 (x+1)\,dx = \left[\frac{x^2}{2}\right]_0^1 + [x]_0^1$$

$$= \frac{1}{2} + 1 = 3/2$$

$I_2 =$ $\quad$ B (1,0) $\quad$ C (1,1).

y varies from 0 to 1

$x = 1$, $dx = 0.$

$$I_2 = \int_{BC} (x+1) + (iy)\,[dx + i\,dy]$$

$$= \int_{BC} [(1+1) + iy][0 + i\,dy]. = \int_{BC} [2+iy]\,i\,dy. = \int_0^1 [2+iy]\,i\,dy.$$

$$= \int_0^1 2i\,dy - y\,dy.$$

$$= 2i[y]_0^1 - \left[\frac{y^2}{2}\right]_0^1$$

$$= 2i(1) - \frac{1}{2}. \quad = 2i - \frac{1}{2}.$$

$I_3 =$ $\quad$ C (1,1) $\quad$ D (0,1)

x varies from 1 to 0.

$y = 1$, $dy = 0.$

$$I_3 = \int_{CD} [(x+1) + iy][dx + i\,dy]$$

$$= \int_{CD} [x+1] + i]\,[dx]. = \int_1^0 x\,dx + 1\,dx + i\,dx$$

$$= \left[\frac{x^2}{2}\right]_1^0 + 1[x]_1^0 + i(x)_1^0$$

$$= -\frac{1}{2} + (-1) - i = -\frac{1}{2} - 1 - i = -\frac{3}{2} - i$$

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$I_4 = \underset{(0,1)}{D} \quad \underset{(0,0)}{A}$

$x = 0 \quad dx = 0.$

$y = 1$ to $0.$

$$I_4 = \int_{DA} [(x+1) + iy][dx + idy].$$

$$= \int_{DA} (1+iy)(i\,dy).$$

$$= \int_1^0 i\,dy - y\,dy.$$

$$= i[y]_1^0 - \left[\frac{y^2}{2}\right]_1^0$$

$$= i[-1] - \left[\frac{-1}{2}\right]$$

$$= -i + \frac{1}{2}.$$

$$I = \cancel{\frac{3}{2}} + \cancel{2i} - \cancel{\frac{1}{2}} - \cancel{\frac{3}{2}} - \cancel{i - i} + \cancel{\frac{1}{2}}$$

$$= 0.$$

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Cauchy's fundamental theorm / Cauchy's Integral Theorm:-

~~Statement:- If $f(z)$ is analytic inside and on closed curved c in a simply connected region R and a is any point within C then~~

~~$f(a) =$~~

Statement:- If $f(z)$ is analytic inside and on closed curve C and first derivative $f'(z)$ exist and continous on the closed curve C then

$$I = \int_C f(z)\, dz = 0$$

Proof:- given — $f(z)$ is analytic on closed curve C

$f'(z)$ exist and continous onc

$\to$ By CR equation, $u_x = v_y$ & $u_y = -v_x$.

u_x, u_y, v_x, v_y exist & continous

$$z = x + iy \Rightarrow dz = dx + i\,dy.$$

$$\omega = f(z) = u(x,y) + i\,v(x,y)$$

$$I = \int_C [u(x,y) + i\,v(x,y)]\,[dx + i\,dy]$$

$$= \int_C [u(x,y)\,dx + i\,u(x,y)\,dy + i\,v(x,y)\,dx - v(x,y)\,dy]$$

$$= \int_C [u(x,y)\,dx - v(x,y)\,dy] + i\,[u(x,y)\,dy + v(x,y)\,dx]$$

$$= \underbrace{\int_C u(x,y)\,dx - v(x,y)\,dy}_{I_1} + i \underbrace{\int_C u(x,y)\,dy + v(x,y)\,dx}_{I_2}.$$

By Green's Theorm in XY plane

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

$$(Q_x - P_y.)$$

In I_1; $P = u(x,y)$; $Q = -v(x,y)$

In I_2; $P = v(x,y)$; $Q = u(x,y)$

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$$I = \iint_R [-V_x - U_y]\,dx\,dy + i \iint_R [V_x - U_y]\,dx\,dy$$

$$= \iint_R [\cancel{U_y} - \cancel{U_y}]\,dx\,dy + i \iint_R [\cancel{V_y} - \cancel{V_y}]\,dx\,dy \quad \text{(By CR equation)}$$

$= 0 + i(0)$

$= 0.$

$= R.H.S$

Cauchy's theorm for doubly connected region:-

Statement:- If $f(z)$ is analytic in the doubly connected region (or) bounded by the simple closed curves c_1 and c_2 $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$

Cauchy's theorm for multicurve:

Statement:- If $f(z)$ is analytic on the finite no. of closed curves $c_1, c_2, \ldots c_n$ then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz.$$

Cauchy's Integral formula:-

Statement:- If $f(z)$ is analytic within (inside) and on closed curve c in a simply connected region R and if a is any point within C, (inside c) then $f(a) = \dfrac{1}{2\pi i} \int_c \dfrac{f(z)dz}{(z-a)}$

also called as index value of point a.

where the integration around the curve C being taken in the positive direction.

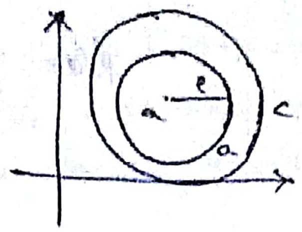
Proof:- given $f(z)$ is analytic within and on closed curve C in a simply closed region R.

Also a is any point inside C.

$\Rightarrow f(z)$ is analytic on c

$\Rightarrow \dfrac{f(z)}{z-a}$ is also analytic on c except at $z=a$

Consider a circle with centre "a" and radius "ρ"



$\Rightarrow |z-a| = \rho$

$\Rightarrow z-a = \rho.e^{i\theta}$

$z = a + \rho.e^{i\theta}$

$dz = \rho.i\ e^{i\theta} d\theta$

By cauchy's theorem for doubly region c and c_1

$$\Rightarrow \int_c \frac{f(z)}{(z-a)} dz = \int_{c_1} \frac{f(z)dz}{(z-a)}$$

$$= \int_0^{2\pi} \frac{f(a+\rho.e^{i\theta})}{\cancel{\rho}.\cancel{e^{i\theta}}} \times \cancel{\rho}.i.\cancel{e^{i\theta}} d\theta. \quad \left| \begin{array}{l} c_1 \to \text{circle} \\ \theta \to 0 \text{ to } 2\pi \end{array} \right.$$

$\rho \to 0, \quad c_1 \to 0$

$$= \int_0^{2\pi} f(a).\, i\, d\theta$$

$$= i \times f(a) \left(\int_0^{2\pi} d\theta \right)$$

$$\int \frac{f(z)}{(z-a)} dz = 2\pi i. f(a)$$

$\rightarrow |z-a| = r.$

$\rightarrow z-a = r.e^{i\theta}$

$|z| = r.$

$z = re^{i\theta}$

$$\boxed{\therefore f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)} dz.}$$

Cauchy's Integral formula for derivative :-

Note: CIT is valid only for inside points. for outside point the index value is simply zero.

Statement: If $f(z)$ is analytic within and on a simple closed curve C and a is any point lying inside it then $f'(a) = \dfrac{1}{2\pi i} \displaystyle\int_c \frac{f(z)dz}{(z-a)^2}$.

Proof:- step-I: State and prove cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)} \rightarrow (1)$$

Taking diffe w.r.t "a"

$$f'(a) = \frac{d}{da}[f(a)] = \frac{1}{2\pi i} \times \frac{d}{da}\left[\int_C \frac{f(z)\,dz}{(z-a)}\right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\partial}{\partial a}\left[\frac{f(z)}{(z-a)}\right] dz$$

$$= \frac{1}{2\pi i} \int_C f(z)\,dz \cdot \frac{-1}{(z-a)^2} \times (-1)$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^2}.$$

Hly

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^3}$$

$$f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^4}$$

$$\vdots$$

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\boxed{\int_C \frac{f(z)\,dz}{(z-a)} = 2\pi i \times f(a)}$$

$$\int_C \frac{f(z)\,dz}{(z-a)^2} = 2\pi i \times f'(a)$$

$$\int_C \frac{f(z)\,dz}{(z-a)^3} = \frac{2\pi i}{2!} \times f''(a).$$

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Working rule:-

case (I):- Outside points:- Use cauchy's integral theorm, the integration value is 0.

case (II):- Inside points:- use cauchy's integral formula, for the case of a (or) more inside points use partial fraction.

case (III):- Combination of inside and outside:

use cauchy's integral formula but not partial fraction, the integration values can be obtained by taking the outside point to the numerator.

Pr: Evaluate $I_1 = \int_c \frac{z\,dz}{(z-2)}$, where c is $|z| = 1$

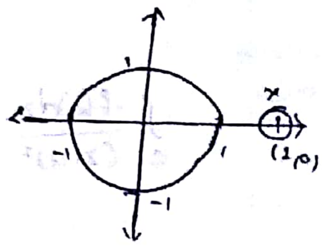
$I_2 = \int_c \frac{dz}{(2z-3)}$, where c is $|z| = 1$

* $I_1 = \int_c \frac{z\,dz}{(z-2)} \Rightarrow \int_c \frac{f(z)dz}{(z-a)}$

$\Rightarrow f(z) = z.$

$a = 2.$

c is $|z| = 1$



$a = 2$ outside point over c

By cauchy's theorm, f(z) is analytic & f(z) is continous.

$\therefore \int_c f(z)dz = 0.$

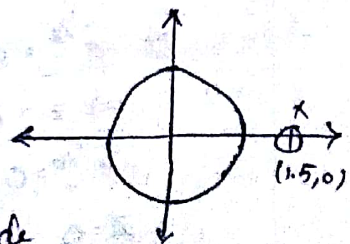
* $I_2 = \int_c \frac{dz}{2z-3} \Rightarrow \int \frac{f(z)dz}{(z-a)}$

$\Rightarrow f(z) = 1$

$2\left[z-\frac{3}{2}\right] = 0$

$z = 3/2$

$a = 1.5.$ is outside



By cauchy's theorm, f(z) is analytic & continous $\therefore \int f(z)dz = 0.$

* $I_3 = \int_C \frac{z^2 dz}{z+1}$ $\quad |z| = 2$

$f(z) = z^2$; $a = -1$ $\qquad f(z) = z^2$

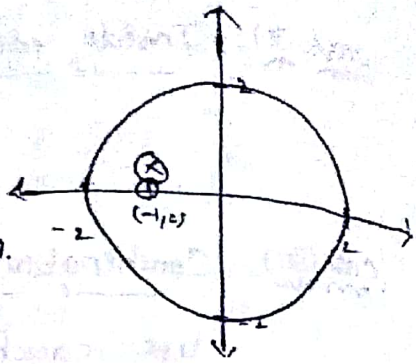
$f(-1) = (-1)^2$

$= 1$

$\int_C \frac{f(z)dz}{z-a} = 2\pi i\, f(a)$

$= 2\pi i\ (1)$

$= 2\pi i$



a is point inside the Region.

R. without using cauchy's Residue theorm, evaluate $\int_C \frac{dz}{z^2 e^z}$ $\quad \int_C z^2 . e^{1/z}$

over the same curve. $C \rightarrow |z| = 1$

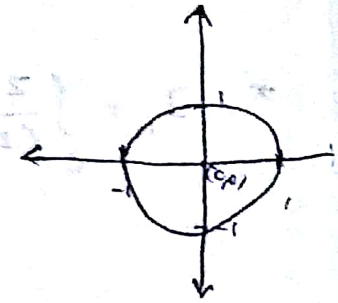
* $I_1 = \int_C \frac{dz}{z^2 . e^z}$ $\Rightarrow \int_C \frac{f(z)dz}{(z-a)} =$

$f(z) = e^{-z}$

$Dr = 0$

$z^2 = 0$

$z = (0,0)$.



$\int_C \frac{f(z)dz}{(z-a)^2} = \frac{2\pi i}{1!} . f'(a)$ $\qquad f(z) = e^{-z}$

$f'(z) = e^{-z}(-1) = -e^{-z}$

$= \frac{2\pi i}{1!}(-1)$ $\qquad f'(a) = f'(0) = -e^{-0} = -1$.

$= -2\pi i$

* $J_2 = \int_C \frac{z^2 dz}{e^{-1/z}}$

$f(z) = z^2$

$D_r = 0$. $\qquad I_2 = 2\pi i$

$\Rightarrow e^{-1/z} = 0$.

$\Rightarrow z = 0$

$e^{-1/0} = 0$.

$z = 0$ lies inside c.

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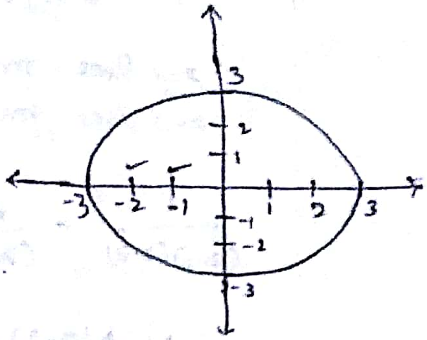
$$\int_c \frac{f(z)dz}{(z-a)} = 2\pi i \times f(a)$$ $f(z) = z^2$

$= 2\pi i \times 0$ $f(a) = a^2$

$= 0.$ $f(0) = 0$

Q. without using cauchy's residue theorm evaluate $\int_c \frac{z^2}{(z+1)^2(z+2)} dz$ where c is the simple closed curved given by $|z| = 3$

$$I = \int_c \frac{z^2 dz}{(z+1)^2(z+2)}$$



$f(z) = z^2$

$Dr = 0$

$\Rightarrow z = -1, -1, -2.$

$\therefore z = -1$ lies inside c

$z = -2$ lies inside c

$$\frac{1}{(z+1)^2(z+2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z+2)}$$

$$1 = A(z+1)(z+2) + B(z+2) + C(z+1)^2$$

Put $z = -1$,

$1 = A(0) + B(1) + C(0)$

$\boxed{B = 1}$

Put $z = -2$.

$1 = A(0) + B(0) + C(-1)^2$

$\boxed{C = 1}$ and also $\boxed{A = -1}$ by comparing coeffs.

$$\therefore \frac{1}{(z+1)^2(z+2)} = \frac{-1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)}$$

$f(z) = z^2$

$f(-1) = 1$

$f(2) = 4$

$f'(z) = 2z$

$f'(-1) = -2$

$$I = \int_c \frac{-f(z)dz}{(z+1)} + \int_c \frac{f(z)dz}{(z+1)^2} + \int_c \frac{f(z)dz}{(z+2)}$$

$$= \int_c \frac{-z^2 dz}{(z+1)} + \int_c \frac{z^2 dz}{(z+1)^2} + \int_c \frac{z^2 dz}{(z+2)}$$

$= -2\pi i \times f(-1) + 2\pi i \times f'(-1) + 2\pi i \times f(2)$

$= -2\pi i \times 1 + 2\pi i(-2) + 2\pi i(4)$

$= -2\pi i - 4\pi i + 8\pi i = 2\pi i$

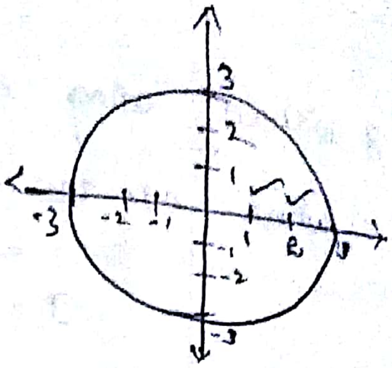
Pr: Use Cauchy's integral formula, evaluate $\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$ where c is $|z| = 3$.

$f(z) = \sin\pi z^2 + \cos\pi z^2$

Dr = 0.

$(z-1)(z-2) = 0$.

$z = 1, 2$.



$z=1$ lies inside c

$z=2$ lies inside C

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

$z = 2$,

$1 = B(1) \Rightarrow \boxed{B=1}$

$z = 1$,

$1 = A(-1) \Rightarrow \boxed{A=-1}$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$I = \int_c \frac{-f(z)dz}{(z-1)} + \int_c \frac{f(z)dz}{(z-2)}$$

$$\int_c = (-1) \times 2\pi i \times f(1) + 2\pi i \times f(2)$$

$$= (-1) \times 2\pi i (-1) + 2\pi i (+1)$$

$$= 2\pi i + 2\pi i = 4\pi i$$

$f(z) = \sin\pi z^2 + \cos\pi z^2$.

$f(1) = \sin\pi + \cos\pi$

$= -1$

$f(2) = \sin\pi(4) + \cos\pi(4)$

$= +1$

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14. Evaluate $\int_c \frac{z\,dz}{(z-1)(z-2)^2}$, $c \to |z-2| = \frac{1}{2}$ centre $= (2,0)$

radius $= 1/2 = 0.5$

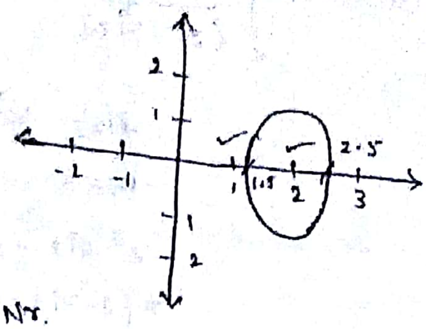
hence, $f(z) = z$.

$\therefore$ Dr $= 0$

$(z-1)(z-2)^2 = 0$

$z = 1, \underbrace{2, 2}_{\text{inside}}$.

$1 \to$ outside



take outside point to Nr.

$$= \int_c \frac{\left(\frac{z}{z-1}\right) dz}{(z-2)^2} = \int \frac{F(z)\,dz}{(z-2)^2} = \frac{2\pi i}{1!} \times F'(a)$$

$$F(z) = \frac{z}{z-1}$$

$$F'(z) = \frac{(z-1)(1) - (z)(1)}{(z-1)^2}$$

$$= \frac{\cancel{z}-1-\cancel{z}}{(z-1)^2} = \frac{-1}{(z-1)^2}$$

$$F'(2) = \frac{-1}{(2-1)^2} = \frac{-1}{(1)^2} = -1$$

$a = 2$;

$= 2\pi i \times F'(2)$

$= 2\pi i \times (-1)$

$= -2\pi i$.

15. Evaluate $\int_c \frac{(z+4)\,dz}{(z^2+2z+5)}$ $c \to |z+1-i| = 2$

$|z-(-1+i)| = 2 \Rightarrow$ centre $= (-1,1)$

radius $= 2$.

$f(z) = z+4$

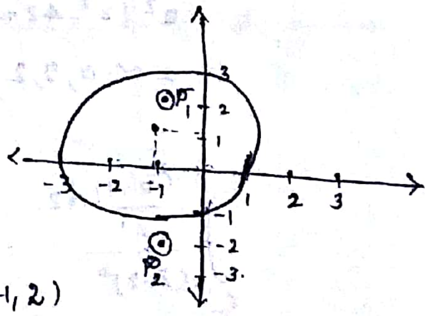
Dr $= 0$.

$z^2 + 2z + 5 = 0$

$$z = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$= -1 \pm 2i$ $(-1, 2)$, $(-1,-2)$



$$= \int_c \frac{\frac{z+4}{(z-P_2)}}{(z-P_1)} dz = \int \frac{F(z)\,dz}{(z-P_1)} = 2\pi i \times F(P_1)$$

$= 2\pi i \times \frac{(3+2i)}{4i}$

$= \frac{3}{2}\pi + \pi i$

$$F(z) = \frac{z+4}{z-(-1-2i)} \qquad F(P_1) = F(-1+2i)$$

$$= \frac{-1+2i+4}{\cancel{-1}+2i\cancel{+1}+2i} = \frac{3+2i}{4i}$$

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without using cauchy's Residue theorem, evaluate

$\int_c \frac{(z^3+1)dz}{(z^3-3iz)}$ $c \rightarrow |z|=1$ centre $=(0,0)$ $r=1$

$f(z) = z^3+1$

$Dr = 0$

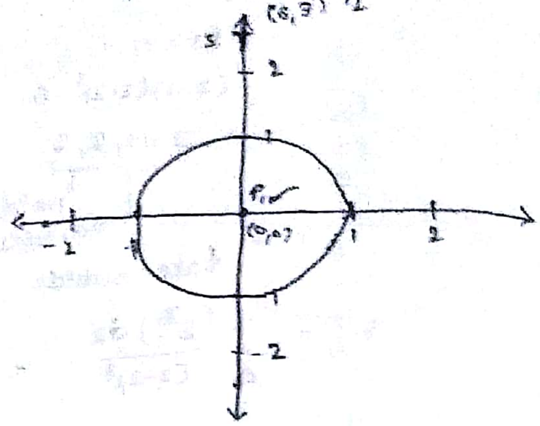
$z^2 - 3iz = 0$

$z[z-3i] = 0$

$z=0, \quad z-3i=0$

$\downarrow \qquad \downarrow$

$(0,0) \qquad (0,3)$



$$\int \frac{z^3+1}{z(z-3i)} dz = \int \frac{\frac{(z^3+1)}{(z-3i)}}{z} dz = \int \frac{F[z]}{(z-0)} dz$$

$F[z] = \frac{z^3+1}{z-3i} = \frac{1}{-3i}$

$= 2\pi i \times F[0]$

$= 2\pi i \times \frac{1}{-3i}$

$= -\frac{2\pi}{3}$

$\int \frac{z+1}{z^4-4z^3+4z^2}$, $c \rightarrow |z-2-i| = 2$

centre $= (2,1)$

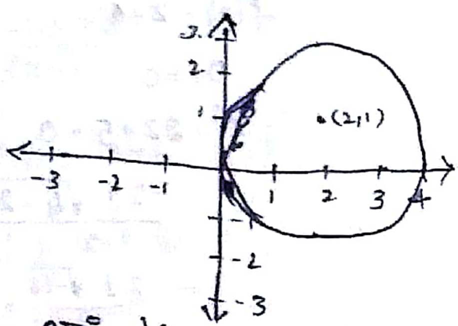
$r = 2$.

$f(z) = z+1$

$Dr = 0$.

$z^2(z^2-4z+4) = 0$

$z = 0, 0, 2, 2$.



$$\int_C \frac{\frac{(z+1)}{z^2} dz}{(z-2)^2} = \int_C \frac{F[z] dz}{(z-2)^2} = \frac{2\pi i}{1!} \times F'(2)$$

$F[z] = \frac{z+1}{z^2}$

$= \frac{2\pi i}{1!} \times \frac{-1}{2}$

$= -\pi i$

$F'[z] = \frac{z^2(1) - (z+1)(2z)}{z^4}$

$= \frac{z^2 - (2z^2+2z)}{z^4}$

$= \frac{-z^2-2z}{z^4}$

$= -\frac{z+2}{z^3}$

$F'(2) = -\frac{2+2}{8} = \frac{-4}{8} = \frac{-1}{2}$.

Pr: Evaluate $\int_C \frac{7z-1}{z^2-3z-4}$, c is given by $x^2+4y^2=4$

$\frac{x^2}{4}+\frac{y^2}{1}=1$ (ellipse)

$f(z)=7z-1$

$Dr=0$

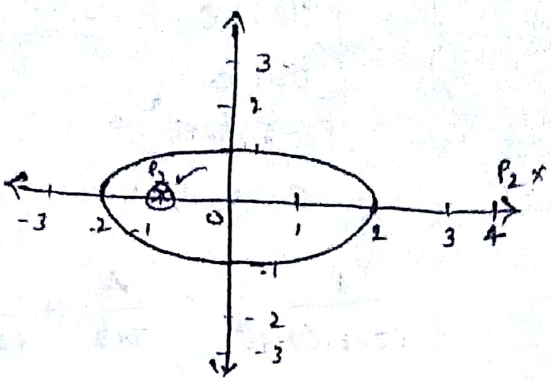
$z^2-3z-4=0$

$(z+1)(z-4)=0$

$z=-1, 4$

$\downarrow \quad \downarrow$

$(-1,0) \quad (4,0)$



$$= \int \frac{\frac{7z-1}{z-4}}{(z+1)} dz = \int \frac{F[z]}{(z+1)} dz = 2\pi i \times F[-1].$$

$$= 2\pi i \times \frac{8}{5}$$

$$= \frac{16\pi i}{5}$$

$F[z] = \frac{7z-1}{z-4}$

$F[-1] = \frac{-7-1}{-1-4}$

$= \frac{-8}{-5} = \frac{8}{5}$

Homework:- (04-09-18)

(1) $\int_C \frac{e^{2z}dz}{(z-1)(z-2)}$ $\Rightarrow$ $c \to |z|=3.$

centre $=(0,0)$
radius $=3$

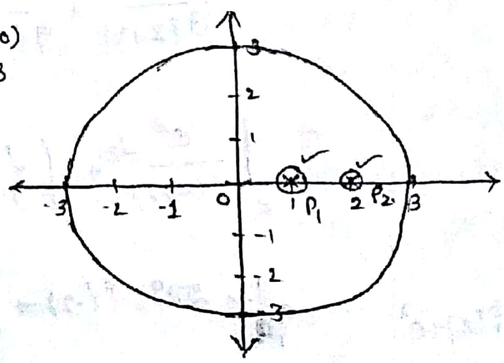
$f(z)=e^{2z}$

$Dr=0$

$(z-1)(z-2)=0$

$z=1, 2$

$=(1,0); (2,0)$



$z=1$ lies inside

$z=2$ lies inside

$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1}+\frac{B}{z-2}$ $= \frac{-1}{z-1}+\frac{1}{z-2}$

$1 = A(z-2)+B(z-1)$

If $(z=1)$, $1=A(-1)$

$\boxed{A=-1}$

If $z=2$, $1=B(2-1)$

$\boxed{B=1}$

$$= \int \frac{-e^{z}f(z)}{z-1} + \int \frac{e^{z}dz}{z-2}$$

$$= -2\pi i \times f(1) + 2\pi i \cdot f(2)$$

$$= -2\pi i \times e + 2\pi i \times e^2$$

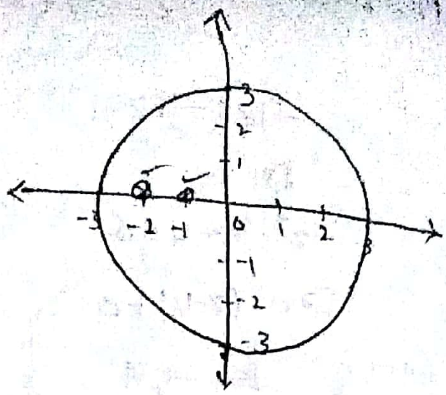
$$= 2\pi i\,(e^4 - e^2)$$

$f(z)=e^{2z}$

$f(1)=e^2$

$f(2)=e^4$

(2) $\int_C \frac{e^z dz}{(z+2)(z+1)^2}$ $\quad C = |z| = 3$



$f(z) = e^z$

$Dr = 0$

$(z+2)(z+1)^2 = 0$

$z = -2, -1, -1.$

$$\frac{1}{(z+2)(z+1)^2} = \frac{A}{z+2} + \frac{B}{(z+1)^2} + \frac{C}{(z+1)}$$

$$1 = A(z+1)^2 + B(z+2) + C(z+2)(z+1)$$

Put $z = -1$,

$1 = B(1+2) \quad \Rightarrow B = 1/3.$

Put $z = 2$,

$1 = A(3)^2 \Rightarrow A = 1/9.$

compare coeff.

z^2, $\quad 0 = A + C$

$A = -C \quad \boxed{C = -1/9}$

$$= \frac{1}{3(z+2)} + \frac{1}{9(z+1)^2} - \frac{1}{9(z+1)}$$

$$= \int_C \frac{e^z}{3(z+2)} dz + \int \frac{1}{9} \frac{e^z dz}{(z+1)^2} - \int \frac{1}{9} \frac{e^z dz}{(z+1)}$$

$f(z) = e^z$

$f(-1) = e^{-1} = 1/e$

$f(-2) = e^{-2} = 1/e^2$

$f'(z) = e^z$

$= e^{-1}$

$= 1/e$

$$= \frac{1}{3} \times 2\pi i \times f(-2) + \frac{1}{3} \times \frac{2\pi i}{1!} \times f'(-1) - \frac{1}{9} \times 2\pi i \times f(-1)$$

$$= \frac{1}{3} \times 2\pi i \times \frac{1}{e^2} + \frac{1}{3} \times 2\pi i \left(\frac{+1}{e}\right) - \frac{1}{9} \times 2\pi i \times \frac{1}{e}$$

$$\cancel{= \frac{2\pi i}{3e}\left[\frac{1}{e} + \frac{1}{3} - \frac{1}{3}\right]} = \frac{2\pi i}{3e}\left[\frac{1}{3e} + 1 - \frac{1}{3}\right].$$

$$\cancel{= \frac{2\pi i}{3e}\left[\frac{1}{e} - \frac{2}{3}\right]} = \frac{2\pi i}{3e}\left[\frac{1}{3e} + \frac{2}{3}\right].$$

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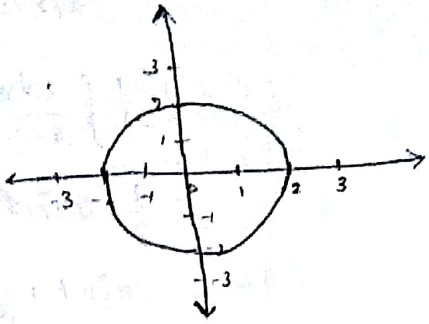
(3) $\int \frac{e^{2z}dz}{(z+1)^4}$ $c = |z| = 2$.   centre $= (0,0)$
$r = 2$.

$f(z) = e^{2z}$

$Dr = 0$,

$(z+1)^4 = 0$

$z = -1, -1, -1, -1$



$$\frac{1}{(z+1)^4} = \frac{A}{(z+1)^4} + \frac{B}{(z+1)^3} + \frac{C}{(z+1)^2} + \frac{D}{(z+1)}$$

$1 = A + B(z+1) + C(z+1)^2 + D(z+1)^3$.

$1 = A + Bz + B + Cz^2 + 2Cz + C + Dz^3 + 3Dz^2 + 3Dz + D$.

coeff. of z^3, $0 = D$

z^2, $0 = C + 3D$

$C = 0$.

z, $0 = B + 2C + 3D$.

$B = 0$.

G $1 = A + C + D$ $\boxed{A = 1}$

$= \frac{1}{(z+1)^4}$

$I = \int \frac{e^{2z}dz}{(z+1)^4} = \frac{2\pi i}{3!} \times f'''(-1)$

$= \frac{2\pi i}{6 3} \times 8 \times e^{-2}$

$= \frac{8}{3}\pi \times i \times \frac{1}{e^2}$

$= \frac{8}{3e^2}\pi i$.

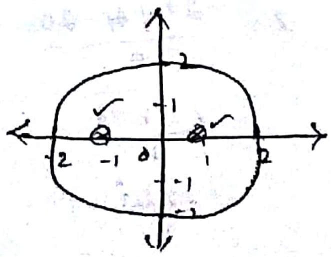
$f(z) = e^{2z}$
$f'(z) = e^{2z}(2)$
$f''(z) = e^{2z}(4)$
$f'''(z) = 8 \times e^{2z}$.
$f'''(-1) = e^{2(-1)} \times 8$.

Homework:- (05-09-18).

(1) $\int \frac{1}{z^2-1}$, $|z| = 2$.   centre $= (0,0)$
$r = 2$.

$(z-1)(z+1) = 0$. $f(z) = 1$

$z = 1, -1$.



$\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$ $1 = A(z-1) + B(z+1)$

If $z = 1$, $1 = B(2) \Rightarrow B = \frac{1}{2}$

If $z = -1$, $1 = A(-1-1) \Rightarrow A = -\frac{1}{2}$.

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$$= \int \frac{-1}{2(z-1)} dz + \int \frac{1}{2(z+1)} dz$$

$$= -\frac{1}{2}\int \frac{1 dz}{(z-1)} + \frac{1}{2}\int \frac{1 dz}{(z+1)}$$

$$= -\frac{1}{2} \times 2\pi i \times f(1) + \frac{1}{2} \times 2\pi i \times f(-1)$$

$$= -\pi i \times 1 + \frac{1}{2} \times 2\pi i \times -1$$

$$= -\pi i - \pi i$$

$$= -2\pi i.$$

(2) $\int \frac{z^2+1}{z^2-1} dz$, $C \Rightarrow$ $|z-1| = 1$

$|z-(1+0i)| = 1$

$f(z) = z^2$

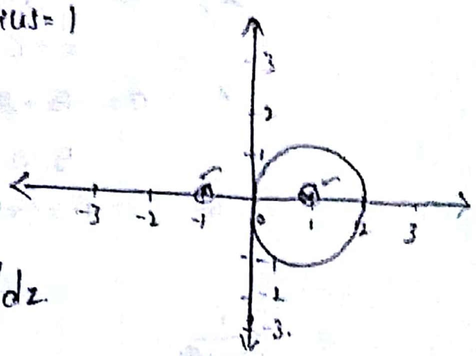
centre $= (1,0)$

radius $= 1$

$Dr = 0$

$(z+1)(z-1) = 0$

$z = 1, -1$



$$= \int \frac{\left[\frac{z^2+1}{z+1}\right]}{z-1} dz = \int \frac{F[z]}{z-1} dz.$$

$= 2\pi i \times F[1]$ $\qquad F[z] = \frac{z^2+1}{z+1}$

$= 2\pi i \times 1$ $\qquad F[1] = \frac{1+1}{1+1} = 1$

$= 2\pi i.$

(3) $\int \frac{z+4}{z^2+2z+5}$ $\qquad |z+1+i| = 2.$

$|z-(-1-i)| = 2.$

$f(z) = z+4.$

centre $= (-1,-1)$

$Dr = 0.$

$r = 2.$

$z^2+2z+5 = 0.$

$$z = \frac{-2 \pm \sqrt{4-20}}{2}$$

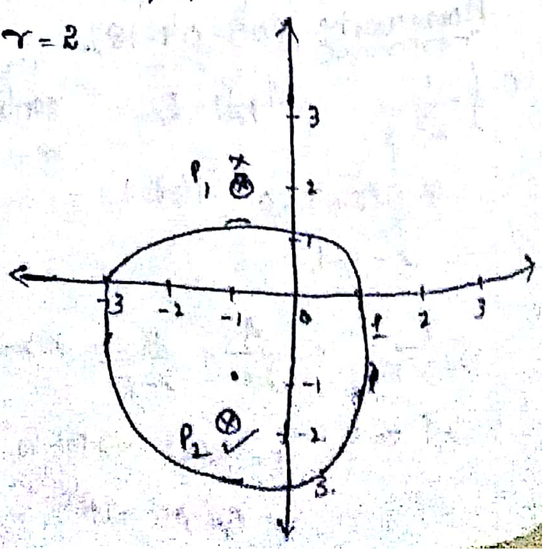
$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i.$$

$= -1+2i, -1-2i.$

$(-1,2)$ $\quad (-1,-2).$



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$P_1 = -1+2i$
$P_2 = -1-2i$

$$= \int_c \frac{\left[\frac{z+4}{z-P_1}\right]}{[z-P_2]}\, dz = \int \frac{F[z]}{[z-P_1]}\, dz.$$

$$= \oint 2\pi i \times F[P_2] \qquad F(P_2) = \frac{-1-2i+4}{\cancel{+1}-2i\cancel{-1}-2i}$$

$$= 2\pi i \times \left[\frac{3-2i}{-4i}\right] \qquad = \frac{3-2i}{-4i}$$

(4) $\int \frac{z+1}{z^2+2z+4}$ $\quad C \rightarrow |z+1+i| = 2.$

centre $= (-1,-1)$

$f(z) = z+1$ $\quad |z-(-1-i)| = 2$

$Dr = 0$ $\quad$ radius $= 2.$

$z^2+2z+4 = 0.$

$$z = \frac{-2 \pm \sqrt{4-16}}{2}$$

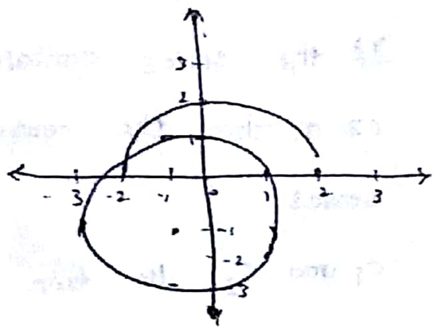
$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$= -1 \pm \sqrt{3}i$$

$(-1, \sqrt{3})$ $\quad (-1, -\sqrt{3})$

$(-1, 1.732)$ $\quad (-1, -1.732).$



Taylor's series and Laurent's series:-

A function $f(z)$ is analytic about the point $z=a$ in the circle C, then
$$f(z) = f(a) + \frac{f'(a)}{1!}\times(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}\times(z-a)^n + \cdots \infty \quad \text{—①}$$

Put $z=0 \Rightarrow (0,0)$

$$f(z) = f(0) + \frac{f'(0)}{1!}\times(z) + \frac{f''(0)}{2!}\times z^2 + \cdots + \frac{f^n(0)}{n!}\times z^n + \cdots \infty \quad \text{—②}$$

(Taylor series about origin → Maclaurin series)

Note: In equation① there is no negative powers of $(z-a)$

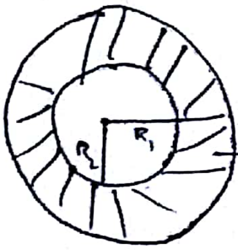
If the series contains positive and negative powers of $(z-a)$ then the corresponding series is called Laurents series.

c_1 and c_2 be two concentric circles,

$$|z-a| = R_1$$
$$|z-a| = R_2 \qquad R_1 > R_2.$$

then A function $f(z)$ is analytic between the regions of concentric circles.

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n(z-a)^n}_{A} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{B}$$



is the Laurent's series where

A is called regular part / analytic part of Laurent series.

B is called principle part of Laurent series.

where the values of a_n and b_n are given by

$$a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)dz}{(z-a)^{n+1}} \quad ; \quad b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(z)dz}{(z-a)^{1-n}}$$

Problems on Laurent series:

Pr. Find the Laurent series expansion of $f(z) = \frac{z}{(z+1)(z+2)}$ about the region $z = -2$.

$$f(z) = \frac{z}{(z+1)(z+2)}$$

By partial fractions,

$$\frac{z}{(z+1)(z+2)} = \frac{A}{(z+1)} + \frac{B}{(z+2)}$$

$$z = A(z+2) + B(z+1)$$

If $z=-1$, $-1 = A(-1+2) \Rightarrow -1 = A(1) \Rightarrow A = -1$

$z=-2$, $-2 = B(-2+1) \Rightarrow -2 = -B \Rightarrow B = 2$

$$f(z) = \frac{-1}{(z+1)} + \frac{2}{(z+2)} \qquad z=-2$$
$$z+2=0.$$

$$= \frac{-1}{(z+1+1)-1} + \frac{2}{(z+2)}$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= \frac{1}{[1-(z+2)]} + \frac{2}{z+2} = [1-(z+2)]^{-1} + \frac{2}{z+2}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$= 1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots + \frac{2}{z+2}$$

$$= \sum_{n=0}^{\infty} (z+2)^n + \frac{2}{z+2}.$$

Pr. $f(z) = \frac{z}{(z^2-3z+2)}$, (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $|z-1|<1$

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$z = A(z-2) + B(z-1)$$

if $z=2$, $2 = B(2-1) \Rightarrow B=2.$

$z=1$, $1 = A(1-2) \Rightarrow A=-1.$

$$f(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)} = \frac{2}{z-2} - \frac{1}{(z-1)}$$

(ii) $1 < |z| < 2$ $\begin{cases} 1 < |z| \Rightarrow \left|\frac{1}{z}\right| < 1 \\ |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1 \end{cases}$

$$\therefore f(z) = \frac{2}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{2}{-2\left[1-\left(\frac{z}{2}\right)\right]} - \frac{1}{z\left[1-\left(\frac{1}{z}\right)\right]}$$

$$= -1\left[1-\left(\frac{z}{2}\right)\right]^{-1} - \frac{1}{z} \times \left(1-\frac{1}{z}\right]^{-1}$$

$$= -1\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+ \cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right]$$

$$= -1\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -1\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}.$$

(i) $|z| < 1$

$$f(z) = \frac{2}{z-2} - \frac{1}{(z-1)}$$

$$= \frac{\cancel{2}}{-\cancel{2}\left[1-\frac{z}{2}\right]} + \frac{1}{(1-z)}$$

$$= -\left[1-\frac{z}{2}\right]^{-1} + [1-z]^{-1}$$

$$= -\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right] + \left[1+z+z^2+\cdots\right]$$

$$= -\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty}(z)^n.$$

(iii) $|z| > 2$.

$2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

Note:- take the denominator term outside for the both factors.

$$= \frac{2}{z\left[1-\frac{2}{z}\right]} - \frac{1}{z\left[1-\frac{1}{z}\right]}$$

$$= \frac{2}{z}\left[1-\frac{2}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{2}$$

$$= \frac{2}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right] = \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)$$

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(iv) $|z-1|<1$

$u = z-1$

$$f(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

$$= \frac{2}{u-2} - \frac{1}{u} \quad ; \quad |z-1|<1$$

$$|u|<1$$

$$= \frac{-2}{[1-u]} - \frac{1}{u}$$

$$= -2[1-u]^{-1} - \frac{1}{u}$$

$$= -2\left[1+u+u^2+\cdots\right] - \frac{1}{u}$$

$$= -2\left[1+(z-1)+(z-1)^2+\cdots\right] - \frac{1}{u}$$

$$= -2\sum_{n=0}^{\infty}(z-1)^n - \frac{1}{u}.$$

Homework:-

(1) $f(z) = \dfrac{1}{1-z^2}$, $\quad z=1$

$$f(z) = \frac{1}{(1-z)(1+z)} = \frac{A}{1-z} + \frac{B}{1+z}$$

$$1 = A(1+z) + B(1-z)$$

If $z=1$, $\quad 1 = A(2) \Rightarrow A = 1/2$

If $z=-1$, $\quad 1 = B(2) \Rightarrow B = 1/2$.

$$f(z) = \frac{1}{2(1-z)} + \frac{1}{2(1+z)} \qquad z=1 \Rightarrow 1-z=0.$$

$z-1=0$.

$$= \frac{1}{2(1-z)} + \frac{1}{-2(-1-z)} = \frac{1}{2(1-z)} - \frac{1}{2(-1+1-1-z)}$$

$$= \frac{1}{2(1-z)} - \frac{1}{2(-2+1-z)} = \frac{1}{2(1-z)} - \frac{1}{-4+2(1-z)}$$

$$= \frac{1}{2(1-z)} - \left[-4+2(1-z)\right]^{-1}$$

$$= \frac{1}{2(1-z)} - 4\left[-1+\left[\frac{1-z}{2}\right]\right]^{-1}$$

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Q. Expand Laurent series for the function $f(z) = \frac{z^2-1}{(z^2+5z+6)}$

(a) between 2 and 3 $(2<|z|<3)$.

(b) within 2 $(|z|<2)$

(c) outside 3 $(|z|>3)$

$$\begin{array}{r|l} & 1 \quad \text{Quotient} \\ z^2+5z+6 & z^2-1 \\ \text{Divisor} & z^2+5z+6 \\ \hline & -5z-7 \quad \text{Remainder} \end{array}$$

Dividend = Divisor × Quotient + Remainder

$$= Q + \frac{R}{D}$$

$$f(z) = 1 - \left[\frac{(5z+7)}{(z^2+5z+6)}\right]$$

$$\frac{5z+7}{(z^2+5z+6)} = \frac{A}{(z+2)} + \frac{B}{(z+3)}$$

$$5z+7 = A(z+3) + B(z+2)$$

If $z=-2$, $-3 = A(+1)$ $\boxed{A=-3}$

If $z=-3$ $-8 = B(-1)$ $\boxed{B=8}$

$$f(z) = 1 + \left[\frac{3}{z+2} - \frac{8}{z+3}\right]$$

(i) $2<|z|<3$ $\quad 2<|z| \Rightarrow \left|\frac{2}{z}\right|<1$

$|z|<3 \Rightarrow \left|\frac{z}{3}\right|<1$

$$\therefore f(z) = 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(\frac{z}{3}+1)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\dots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\dots\right)$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n.$$

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(ii) $|z| < 2$

$\left|\frac{z}{2}\right| < 1$

$$f(z) = 1 - \left[\frac{(-3)}{(z+2)} + \frac{8}{(z+3)}\right]$$

$$= 1 + \frac{3}{2\left[\frac{z}{2}+1\right]} - \frac{8}{3\left[\frac{z}{3}+1\right]}$$

$$= 1 + \frac{3}{2}\left[1+\frac{z}{2}\right]^{-1} - \frac{8}{3}\left[1+\frac{z}{3}\right]^{-1}$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(iii) $|z| > 3$

$3 < |z|$

$\left|\frac{3}{z}\right| < 1$

$$f(z) = 1 - \left[\frac{(-3)}{(z+2)} + \frac{8}{(z+3)}\right]$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

(3) $f(z) = \dfrac{2z-5}{z(z^2-z-2)}$ $\quad |z| > 2 \Rightarrow 2 < |z|$

$\left|\frac{2}{z}\right| < 1$

$$\frac{2z-5}{z(z^2-z-2)} = \frac{A}{z} + \frac{Bz+C}{(z^2-z-2)}$$

$$2z-5 = A(z^2-z-2) + (Bz+C)(z)$$

$$= Az^2 - Az - 2A + Bz^2 + Cz = z^2(A+B) + z(-A+C) - 2A$$

compare coeff ^{n}, z^2, $0 = A+B$.

z, $2 = -A+C$

Con, $-5 = -2A$ $\quad A = \frac{5}{2}$, $\quad C = 2 + \frac{5}{2} = $ [illegible]

$B = -\frac{12}{5}$

$$f(z) = \frac{\frac{2}{5}}{z} + \frac{-\frac{12}{5}z + \frac{12}{5}}{(z^2-z-2)} = \frac{2}{5z} - \frac{12z}{5(z^2-z-2)} + \frac{12}{5(z^2-z-2)}$$

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$$\cancel{f(z)} = \frac{2}{5z} - \frac{12z}{5z\left[z-1-\frac{2}{z}\right]}$$

$$f(z) = \frac{2z-5}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}.$$

$$2z-5 = A(z+1)(z-2) + B(z)(z-2) + C(z)(z+1)$$

If $z=0$, $-5 = A(1)(-2) \Rightarrow A = 5/2$

If $z=2$, $-1 = C(2)(3) \Rightarrow C = -1/6$.

If $z=-1$, $-7 = B(-1)(-3) \Rightarrow B = -7/3$.

$$\therefore f(z) = \frac{5}{2z} - \frac{7}{3(z+1)} - \frac{1}{6(z-2)}$$

$|z| > 2$ $\quad 2 < |z|$ $\quad \left|\frac{2}{z}\right| < 1$

$$= \frac{5}{2z} - \frac{7}{3z(1+\frac{1}{z})} - \frac{1}{6z(1-\frac{2}{z})}$$

$$= \frac{5}{2z} - \frac{7}{3z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{5}{2z} - \frac{7}{3z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{6z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n.$$

Pr: $f(z) = \frac{z^2}{(z+2)(z-3)}$ (i) $|z| < 2$ (ii) $2 < |z| < 3$.

$$(z^2-z-6)\overline{\big)\, z^2}\quad 1$$

$$\frac{z^2-z-6}{\;\;z+6}$$

$$= Q + \frac{R}{D}.$$

$$= 1 + \frac{z+6}{(z+2)(z-3)}$$

$$\frac{z+6}{(z+2)(z-3)} = \frac{A}{z+2} + \frac{B}{z-3} \Rightarrow A(z-3) + B(z+2) = z+6$$

If $z=3$, $9 = B(5) \Rightarrow B = 9/5$

If $z=-2$, $4 = A(-5) \Rightarrow A = -4/5$.

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$$f(z) = 1 + \left[\frac{-4/5}{(z+2)} + \frac{9/5}{(z-3)}\right].$$

(i) $|z|<2 \qquad \left|\frac{z}{2}\right|<1$

$$= 1 - \frac{4}{5(z+2)} + \frac{9}{5(z-3)}$$

$$= 1 - \frac{4}{5\times 2\left(\frac{z}{2}+1\right)} + \frac{9}{5\times 3\left(\frac{z}{3}-1\right)}$$

$$= 1 - \frac{2}{5\left(1+\frac{z}{2}\right)} - \frac{3}{5\left(1-\frac{z}{3}\right)}$$

$$= 1 - \frac{2}{5}\left(1+\frac{z}{2}\right)^{-1} - \frac{3}{5}\left(1-\frac{z}{3}\right)^{-1}$$

$$= 1 - \frac{2}{5}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{3}{5}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

(ii) $2<|z|<3$ $\quad 2<|z| \Rightarrow \left|\frac{2}{z}\right|<1$; $\quad |z|<3 \Rightarrow \left|\frac{z}{3}\right|<1$

$$= 1 - \frac{4}{5(z+2)} + \frac{9}{5(z-3)}$$

$$= 1 - \frac{4}{5z\left(1+\frac{2}{z}\right)} + \frac{9}{15\left(\frac{z}{3}-1\right)}$$

$$= 1 - \frac{4}{5z}\left(1+\frac{2}{z}\right)^{-1} - \frac{3}{5\left(1-\frac{z}{3}\right)}$$

$$= 1 - \frac{4}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{3}{5}\left(1-\frac{z}{3}\right)^{-1}$$

$$= 1 - \frac{4}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{3}{5}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n.$$

8. Find the Laurent series expansion for the function $f(z) = \frac{1}{z(z-1)}$ and also state the region of validity.

$$f(z) = \frac{1}{z(z-1)}$$

(i) $|z|<1 \Rightarrow \left|\frac{z}{1}\right|<1$

$1 = A(z-1) + B(z)$

$z=0, \quad A=-1$

$z=1, \quad B=1$

$$= \frac{-1}{z} + \frac{1}{z-1}$$

$$= \frac{-1}{z} + \frac{1}{-(1-z)}$$

$$= \frac{-1}{z} - \frac{1}{(1-z)}$$

$$= \frac{-1}{z} - \sum_{n=0}^{\infty} (z)^n$$

case (ii):-

$$= \frac{-1}{z} + \frac{1}{z-1} \qquad |z| > 1$$

$$1 < |z|$$

$$\left|\frac{1}{z}\right| < 1$$

$$= \frac{-1}{z} + \frac{1}{z(1-\frac{1}{z})}$$

$$= \frac{-1}{z} + \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{-1}{z} + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

case (iii):-

$$|z| = 1$$

$$f(z) = -\frac{1}{z} + \frac{1}{z-1}$$

$$= \frac{-1}{z} - \frac{1}{(1-z)}$$

$$= \frac{-1}{z} - (1-z)^{-1}$$

$$= \frac{-1}{z} - \sum_{n=0}^{\infty} (z)^n$$

P2. Expand $f(z) = \cos z$ about the point $z = \pi/3$.

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

$f(z) = \cos z$	$f'(z) = -\sin z$	value
$f(a) = \cos a$	at $a = \pi/3$	$= \cos(\frac{\pi}{3}) = \frac{1}{2}$

$f(z) = \cos z$

$f'(z) = -\sin z$

$$f'(\tfrac{\pi}{3}) = -\sqrt{3}/2$$

$$f''(z) = -\cos z$$

$$= -\cos(\pi/3) = -1/2$$

$$f'''(z) = \sin z = \sin(\pi/3) = +\sqrt{3}/2$$

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$$f(z) = \frac{1}{2} - \frac{\sqrt{3}}{2} \times \left(z - \frac{\pi}{3}\right) - \frac{1/2}{2!}(z-a)^2 + \cdots$$

$$\therefore f(z) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(z - \frac{\pi}{3}\right) - \frac{1}{4}\left(z - \pi/3\right)^2 + \cdots$$

Singularities:-

$z=a$, $f(z)$ is analytic function.

Singularities (singular point).

- zeroes of $f(z)$
  - Rule: $Nr = 0$.
    - simple zero (or) zero of order 1, $f(a)=0$.
    - zero of order >1.
- singularities of $f(z)$
  - * series expansion
  - (poles) singular point ($Dr = 0$)
    - Regular singular point (RSP)
    - Essential singular point (ESP)
    - Isolated singular point (ISP)
- Residues of $f(z)$ — Evaluation of poles
  - pole of order 1
  - pole of order >1

Def: A function $f(z)$ is said to be analytic and a point $z=a$ is called as simple zero if $f(a)=0$, $f'(a)\neq 0$

for higher order

if the point $z=a$ satisfies $f(a)=0$ but $f'(a)=0$,

$f''(a) \neq 0 \rightarrow$ zero of order 2.

$f'''(a) \neq 0 \rightarrow$ zero of order 3.

* nth derivative $\neq 0$ zero of order n, but derivatives of previous are zero.

Consider $f(z) = (z-1)(z-2)$

$Nr = 0$.

$(z-1)(z-2) = 0$.

$z = 1, 2$.

$f'(z) = (z-1)(1) + (z-2)(1)$

$= 2z - 3$.

$f'(1) = 2(1) - 3$

$= -1 \neq 0$

$f'(2) = 2(2) - 3$

$= 1 \neq 0$

$\therefore$ $z=1$ is the zero of order 1.

$z=2$ is the zero of order 1.

Singular point (poles): Rule: Dr = 0.

A function f(z) fails to analytic at the point z = a is called as singular point. It can be classified into three types (a) RSP (b) ESP (c) ISP

* Regular singular point:-

* A singular point z = a is said to be regular sp if the Laurent expansion contains infinite number of positive terms. (Regular part (or) Analytic part) and also principple part = 0 ($b_n = 0$).

Ex: $f(z) = \sin z = 1 + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ (infinite +ve terms)

$\frac{\sin z}{1} = 0.$

$\sin z = 0$

$\sin z = \sin(n\pi)$

$z = n\pi,\quad n = 0, \pm1, \pm2, \cdots$

* Essential singular point:

* A singular point z = a is said to be essential singular point if Laurent series expansion contains infinite number of negative terms (Principle part) and also regular part = 0 ($a_n = 0$).

Ex: $f(z) = e^{1/z}$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$\therefore e^{1/z} = 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \cdots$

$= 1 + \frac{1}{z \times 1!} + \frac{1}{z^2} \times \frac{1}{2!} + \cdots$

$= 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \cdots$ [infinite no. of terms in -ve powers]

$f(z) = e^{1/z}$

$Dr = 0.$

$e^{1/z} = 0.$ $\quad \frac{1}{z} = -\infty$

$= e^{-\infty}$ $\quad \frac{1}{-} = z \Rightarrow \boxed{\quad}$

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Ex: $f(z) = \sin\left(\frac{1}{z}\right)$

$= 1 + \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} + \cdots$

$= 1 + \frac{1}{z^3 \cdot 3!} + \frac{1}{z^5 \cdot 5!} + \cdots$

$= 1 + \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} + \cdots$ (infinite no. of -ve terms).

$f(z) = \sin\left(\frac{1}{z}\right)$

$Dr = 0$

$\frac{1}{z} = n\pi$

$\boxed{z = \pm\frac{1}{n\pi}}$

* Isolated Singular Point:-

* A singular point $z=a$ is said to be isolated singular point if

(a) $f(z)$ is not analytic at $z=a$.

(b) but $f(z)$ is analytic at some neighbourhoods of a.

Ex: $f(z) = \frac{1}{(z-1)(z-2)}$

$Dr = 0$

$(z-1)(z-2) = 0$

$z = 1, 2$ are the singular points

At $z=1$, $f(z) = \infty$

$f(z)$ is not analytic at $z=1$

At $z=2$, $f(z) = \infty$.

$f(z)$ is not analytic at $z=2$.

But for neighbourhoods of 1 and 2 $f(z)$ is analytic.

Note:- * I.S.P $\Rightarrow$ is said to be

(i) R.S.P $\rightarrow$ if it has no -ve powers. (all are +ve) infinite no. of powers

(ii) E.S.P $\rightarrow$ if it has infinite -ve powers

(iii) poles $\rightarrow$ if it has finite no. of -ve powers.

* poles of order 1 $\Rightarrow$ singular point of order 1.
(order → power)

In this case $f(z)$ has a RSP. in other words $f(z)$ has no zeroes.

* If $f(z)$ has poles of order (power) greater than 1,

then find $\lim_{z \to 0} f(z) \neq 0$

* Evaluation of poles are known as residues.

* Residues < simple residue. / residue of order "n".

Problems on zeroes:-

① $f(z) = \dfrac{z^2+1}{z^2-1}$ → Nr.

To find zeroes, Nr = 0.

$z^2+1 = 0.$

$z^2 = -1$

$z = \pm i$ < $+i$, $-i$

$$f'(z) = \frac{(z^2-1)(2z) - (z^2+1)(2z)}{(z^2-1)^2}$$

$$= \frac{\cancel{2z^3} - 2z - \cancel{2z^3} - 2z.}{(z^2-1)^2} = \frac{-4z}{(z^2-1)^2}$$

$$f'(i) = \frac{-4i}{(-1-1)^2} = \frac{-4i}{(-2)^2} = \frac{-\cancel{4}i}{4} = -i \neq 0$$

$\therefore z = i$ is a zero of order of 1. / simple zero.

$$f'(-i) = \frac{4i}{((-i)^2-1)^2} = \frac{4i}{(-1-1)^2} = \frac{4i}{4} = i \neq 0.$$

$\therefore z = -i$ is a zero of order of 1 / simple zero

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② $f(z) = \frac{z^3+1}{z^3-1}$

to find zeroes, Nr = 0.

$z^3+1=0.$

$z^3=-1$

$z=(-1)^{1/3}$

$=[\cos\pi + i\sin\pi]^{1/3}$

$=[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{1/3}$

$=\left[\cos\frac{(2k+1)\pi}{3}+i\sin\frac{(2k+1)\pi}{3}\right]$ Demoivre's theorm,

$k=0,1,2.$

$k=0;\quad \cos\frac{\pi}{3}+i\sin\frac{\pi}{3} = \frac{1}{2}+i\frac{\sqrt{3}}{2} = \frac{1+i\sqrt{3}}{2}.$

$k=1;\quad \cos\pi+i\sin\pi = -1$

$k=2;\quad \cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3} = \frac{1}{2}-\frac{i\sqrt{3}}{2} = \frac{1-i\sqrt{3}}{2}.$

∴ All the points are simple zeroes.

③ $f(z) = \frac{z^3-1}{z^3+1}$

to find zeroes, Nr = 0.

$z^3-1=0.$

$z^3=1$

$z=(1)^{1/3}$

$=[\cos 0+i\sin 0]^{1/3}$

$=[\cos(2k\pi+0)+i\sin(2k\pi+0)]^{1/3}$

$=\cos\left[\frac{2k\pi}{3}\right]+i\sin\frac{2k\pi}{3}.$

$k=0,1,2.$

$k=0;\quad \cos 0+i\sin 0 = 1$

$k=1;\quad \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} = \frac{-1+\sqrt{3}i}{2}$ $\qquad -\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$k=2;\quad \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3} = \frac{-1-i\sqrt{3}}{2}.$

∴ All the points are simple zeroes.

④ $\sin\left[\frac{1}{z-a}\right]$

$N_r = 0.$

$\sin\left[\frac{1}{z-a}\right] = \sin n\pi$

$\frac{1}{z-a} = n\pi \quad ; n = \pm 1, \pm 2, ---$

$z-a = \frac{1}{n\pi}$

$z = \frac{1}{n\pi} + a \; ; \; n = \pm 1, \pm 2, ---$ (infinite no. of zeroes)

$\therefore$ All the zeroes are simple zeroes.

limiting case specail problems: ⑤ $\frac{(\sin z) - (z)}{z^3}$

$= \frac{\cancel{\frac{z}{1!}} - \frac{z^3}{3!} + \frac{z^5}{5!} - --- \cancel{-z}}{z^3}$

$= \frac{-\frac{z^3}{3!} + \frac{z^5}{5!} - ---}{z^3}$

taking z^3 outside

$= \frac{\cancel{z^3}\left(-\frac{1}{3!} + \frac{z^2}{5!} - ---\right)}{\cancel{z^3}}$

$N_r = 0$

$\therefore f(z) = \frac{\sin z - z}{z^3}$ has infinite no. of zeroes.

⑥ $f(z) = \frac{1 - e^{2z}}{z^4}$

$N_r = 0 ; \quad 1 - e^{2z} = 0$

$e^{2z} = 1$

$= e^0 , \quad \cos 2n\pi + i\sin 2n\pi$

(or)

$e^{i2n\pi}$

$n = 0, \pm 1, \pm 2, ---$

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$e^{2z} = e^{i2n\pi}$

taking log,

$2z = i2n\pi$

$z = in\pi$, $n = 0, \pm 1, \pm 2,$ (infinite)

$\therefore$ All are simple zeroes).

Problems on singularities (Poles):-

1) find the nature of singularity for $f(z) = \sin\left[\frac{1}{z+1}\right]$

$\sin\left[\frac{1}{z+1}\right]$

$z = -1$, $\sin(\infty)$ x finite

$f(z)$ fails to be analytic

$z = -1$ is ISP.

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \frac{\left(\frac{1}{z+1}\right)}{1!} - \frac{\left(\frac{1}{z+1}\right)^3}{3!} + \frac{\left(\frac{1}{z+1}\right)^5}{5!} - \cdots$$

$$= \frac{(z+1)^{-1}}{1!} - \frac{(z+1)^{-3}}{3!} + \frac{(z+1)^{-5}}{5!} - \cdots \quad \text{(-ve powers)}$$

$\therefore z = -1$ is also ESP.

2) find the nature of singularity for $f(z) = \sin\left(\frac{1}{z-a}\right)$.

$z = a$, $\sin(\infty)$ x finite

$\therefore f(z)$ fails to be analytic.

$z = a$ is ISP.

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \frac{\left(\frac{1}{z-a}\right)}{1!} - \frac{\left(\frac{1}{z-a}\right)^3}{3!} + \frac{\left(\frac{1}{z-a}\right)^5}{5!} - \cdots$$

$$= \frac{(z-a)^{-1}}{1!} - \frac{(z-a)^{-3}}{3!} + \frac{(z-a)^{-5}}{5!} - \cdots \quad \text{(-ve powers of } z\text{)}.$$

$\therefore z = a$ is also ESP

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③ find the nature of singularity $f(z) = \sin\left(\frac{1}{1-z}\right)$

$$f(z) = \sin\left(\frac{1}{1-z}\right)$$

if $z=1$, $f(z)$ fails to be analytic.

$z=1$ is ISP

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \cdots$$

$$= \frac{\frac{1}{(1-z)}}{1!} - \frac{\frac{1}{(1-z)^3}}{3!} + \cdots$$

$$= \frac{(1-z)^{-1}}{1!} - \frac{(1-z)^{-3}}{3!} + \cdots$$

$\therefore$ $z=$ is also R.S.P.

* find the singularity of following:

(a) $\frac{\tan z}{z}$

for singularity, $Dr = 0$

$z = 0$.

At $z=0$, $\frac{\tan z}{z}$ fails to be analytic.

$\therefore$ $z=0$ is an ISP.

$$\frac{\tan z}{z} = \frac{z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots}{z}$$

$$= \frac{\cancel{z}\left[1 + \frac{z^2}{3} + \frac{2z^4}{15} + \cdots\right]}{\cancel{z}}$$

$$= 1 + \frac{z^2}{3} + \frac{2z^4}{15} + \cdots$$ (infinite +ve terms)

$\therefore$ $z=0$ is a R.S.P

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(b) $f(z) = \tan z$.

$z = \frac{\pi}{2}$

$\therefore f(z)$ fails to be infinity.

$\therefore z = \frac{\pi}{2}$ is ISP.

$\tan z = z + \frac{z^2}{3} + \frac{2z^5}{15} + \dots$ (infinite +ve terms)

$\therefore z = \frac{\pi}{2}$ is also RSP

(c) $f(z) = \frac{\sin z}{z}$.

Dr $= 0$;

$z = 0$.

At $z = 0$, $\frac{\sin z}{z}$ fails to be analytic

$\therefore z = 0$ is an ISP.

$$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots}{z}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

$\therefore z = 0$ is also R.S.P.

(d) ~~$f(z) = \frac{\sin z}{z^3}$~~ $f(z) = \frac{z - \sin z}{z^2}$

~~Dr $= 0$. $z^3 = 0$~~

Dr $= 0$

$z = 0$.

At $z = 0$ $\frac{z - \sin z}{z}$ fails to be analytic.

$\therefore z = 0$ is ISP.

$$\frac{z - \sin z}{z} = \frac{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right]}{z} = \frac{\cancel{z} - \cancel{z} + \frac{z^3}{3!} - \frac{z^5}{5!} - \dots}{z}$$

$$= \frac{z\left[\frac{z^2}{3!} - \frac{z^4}{5!} + \dots\right]}{z} = \frac{z^2}{3!} - \frac{z^4}{5!} + \dots$$

$\therefore z = 0$ is also an RSP.

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(e) $f(z) = \frac{\sin z - z}{z^3}$.

$Dr = 0$

$z^3 = 0$.

$z = 0, 0, 0$ (Pole of order 3)
(power)

$z = 0 \rightarrow$ ISP.

$$f(z) = \frac{\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots - \cancel{z}}{z^3}$$

$$= \frac{\cancel{z^3}\left[\frac{-1}{3!} + \frac{z^2}{5!} - \dots\right]}{\cancel{z^3}}$$

$$= \left[\frac{-1}{3!} + \frac{z^2}{5!} - \dots\right]$$ [infinite positive terms]

$\therefore z = 0$ is a R.S.P.

$\therefore \lim_{z \to 0} \frac{\sin z - z}{z^3} \left[\frac{0}{0}\right]$

L'hospital

$\lim_{z \to 0} \frac{\cos z - 1}{3z^2} \left[\frac{0}{0}\right]$.

L'hospital.

$\lim_{z \to 0} \frac{-\sin z}{6z} \left[\frac{0}{0}\right]$

L'hospital.

$\lim_{z \to 0} \frac{-\cos z}{6}$

$= -1/6 \neq 0$.

$\therefore f(z)$ has RSP.

$\therefore f(z)$ has no zeroes.

Note: Limit point of poles is a non-isolated essential singularity where as any point of zeroes is an isolated essential singularity.

* $f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin \pi z \, (z-a)^2}$

Dr = 0; $(z-a)^2 = 0$.

$z = a, a$. (pole of order 2) (finite)

Also; $\sin \pi z = 0$

$\sin \pi z = \sin n\pi$

$\cancel{\pi} z = n\cancel{\pi}$

$z = n$; $n = 0, \pm 1, \pm 2, \ldots$ (infinite).

∴ Limit pt of infinite poles, non-isolated ESP.

* Classify the singularity of $f(z) = \frac{e^{1/z}}{(z-a)^2}$.

Dr = 0; $(z-a)^2 = 0$.

$z = a, a$ (pole of order 2)

$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{e^{1/z}}{(z-a)^2}$

$= \frac{e^{\infty}}{a^2} = \infty$

$\neq 0$

∴ $f(z)$ has R.S.P

∴ $f(z)$ has no zeroes.

* Classify the singularity of $f(z) = \frac{z}{e^z - 1}$.

Dr = 0, $e^z - 1 = 0 \rightarrow e^z = 1$

$e^z = e^0$

$\boxed{z = 0}$ is an ISP.

$\frac{z}{e^z - 1} = \frac{z}{\cancel{1} + \frac{z}{1!} + \frac{z^2}{2!} + \cdots - \cancel{1}} = \frac{\cancel{z}}{\cancel{z}\left[1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right]} = \frac{1}{1 + \left[\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right]}$

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$$= \left[1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)\right]^{-1}$$

$$\cancel{= 1 - \frac{1}{2}\left[\frac{z}{2!}\right] + \left[\right.}$$

$$= 1 - \left[\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right] + \left[\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right]^2 - \left[\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right]^3 + \cdots$$

(infinite +ve terms).

$= RSP.$

$$\lim_{z \to 0} \frac{z}{e^z - 1} \quad \left[\frac{0}{0}\right].$$

$$= \lim_{z \to 0} \frac{1}{e^z}.$$

$= 1 \neq 0.$

$\because f(z)$ is R.S.P

$\therefore f(z)$ has no zeroes.

* Find the singular points of $f(z) = \dfrac{1}{\sin\left(\frac{1}{z-a}\right)}$

Also state their nature.

Dr $= 0$: $\quad \sin\left(\frac{1}{z-a}\right) = 0$

$$\sin\left(\frac{1}{z-a}\right) = \sin n\pi$$

$$\frac{1}{z-a} = n\pi$$

$$z - a = \frac{1}{n\pi} \Rightarrow z = a + \frac{1}{n\pi},$$

$n = 0, \pm 1, \pm 2, \cdots$ (infinite)

$\therefore$ it is not an ISP.

$$f(z) = \frac{1}{\frac{\left(\frac{1}{z-a}\right)}{1!} - \frac{\left(\frac{1}{z-a}\right)^3}{3!} + \cdots} = \frac{1}{\left(\frac{1}{z-a}\right)\left[1 - \frac{\left(\frac{1}{z-a}\right)^2}{3!} + \cdots\right]}$$

$$= (z-a)\left[1 - \left(\frac{1}{3!(z-a)^2} - \cdots\right)\right]^{-1}$$

$$= (z-a)\left[1 + \left(\frac{1}{3!(z-a)^2} - \cdots\right) + \cdots\right].$$

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∴ infinite no. of -ve terms and one +ve term

∴ neither RSP nor ESP

* f(z) has non isolated essential singularity and non-removable isolated singularity.

Entire function (or) Integral function:-

A function f(z) which is analytic everywhere in the finite plane except at infinity is called an entire function (or) integral function

For example $f(z) = e^z$, $\sin z$, $\sinh z$, $\cos z$, $\cosh z$.

Meromorphic function:-

A function f(z) analytic ~~except~~ every where except at the poles is called as meromorphic functions.

For example, $f(z) = \frac{1}{(z-1)(z-2)}$ is analytic everywhere except at z=1 and z=2.

∴ It is meromorphic function.

Residues:- (Evaluation of Poles) (Dr = 0).

Poles

- poles of order 1 → $[\text{Res}(f(z))]_{z=a} = \lim_{z\to a} (z-a) f(z)$
- poles of order m>1 → $[\text{Res} f(z)]_{z=a} = \frac{1}{m-1} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$
- $f(z) = \frac{P(z)}{\varphi(z)}$ = $[\text{Res}(f(z))]_{z=a} = \frac{P(a)}{\varphi'(a)}$
- If f(z) can be expressed as a laurentz series then.

$$[\text{Res} f(z)]_{z=a} = \text{coeff of } \frac{1}{(z-a)} \text{ term}$$

if $[\text{Res} f(z)]_{z=a} = 0$ outside.

* Find the residues of $f(z) = \frac{4z-3}{(z)(z-1)(z-2)}$ along the curve $|z| = 3$.

$$f(z) = \frac{4z-3}{z(z-1)(z-2)}$$

$Dr = 0$

$z = 0, 1, 2$

$z = 0$ is pole of order 1

$z = 1$ is pole of order 1

$z = 2$ is pole of order 1

$$[\text{Res } f(z)]_{z=a} = \lim_{z \to a} (z-a) f(z)$$

$$[\text{Res } f(z)]_{z=0} = \lim_{z \to 0} \cancel{(z-0)} \frac{4z-3}{\cancel{(z)}(z-1)(z-2)} = \frac{-3}{(-1)(-2)} = -3/2$$

$$[\text{Res } f(z)]_{z=1} \Rightarrow \lim_{z \to 1} \cancel{(z-1)} \frac{4z-3}{z\cancel{(z-1)}(z-2)} = \frac{1}{1(-1)} = -1$$

$$[\text{Res } f(z)]_{z=2} \Rightarrow \lim_{z \to 2} \cancel{(z-2)} \frac{4z-3}{z(z-1)\cancel{(z-2)}} = \frac{5}{2} = 5/2$$

* find the residue of $f(z) = \frac{z}{z^2+1}$ along the curve $|z| = 1$

$$f(z) = \frac{z}{z^2+1}$$

$Dr = 0$

$z^2 + 1 = 0$

$z^2 = -1$

$z = i, -i$

$z = 0, 1$ pole of order 1

$z = 0, -1$ pole of order 1

$$[\text{Res } f(z)]_{(z=a)} = \lim_{z \to a} (z-a) f(z)$$

$$[\text{Res } f(z)]_{z=i} = \lim_{z \to i} (z-i) \frac{z}{z^2+1}$$

$$= \lim_{z \to i} \cancel{(z-i)} \frac{z}{(z+i)\cancel{(z-i)}} = \frac{\cancel{i}}{2\cancel{i}} = 1/2$$

$$[\text{Res} f(z)]_{z=-i} = \underset{z\to -i}{Lt} (z+i) \frac{z}{z^2+1}$$

$$= \underset{z\to -i}{Lt} (z+i) \frac{z}{(z+i)(z-i)}$$

$$= \frac{-\cancel{i}}{-2\cancel{i}} = \frac{1}{2}.$$

* find the residues of $f(z) = \frac{z+1}{(z-1)(z+2)}$ along $z=1$

$Dr = 0.$

$(z-1)(z+2) = 0$

$\therefore z = 1, -2.$

$z = -2$ lies outside.

$z = 1$ pole of order 1.

$$[\text{Res} f(z)]_{z=1} = \underset{z\to 1}{Lt} \cancel{(z-1)} \frac{z+1}{\cancel{(z-1)}(z+2)} = \frac{2}{3} = \frac{2}{3}.$$

$$[\text{Res} f(z)]_{z=-2} = 0$$

* find the residues of $f(z) = \frac{z}{(z-1)^2}$ at $|z|=1$

$Dr = 0$

$(z-1)^2 = 0.$

$z = 1, 1$ poles of order 2, $m=2$

$$[\text{Res} f(z)]_{z=a} = \frac{1}{(m-1)!} \underset{z\to a}{Lt} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

$$[\text{Res} f(z)]_{z=1} = \frac{1}{(2-1)!} \underset{z\to 1}{Lt} \frac{d^{2-1}}{dz^{2-1}} \left[\cancel{(z-1)^2} \cdot \frac{z}{\cancel{(z-1)^2}}\right]$$

$$= \frac{1}{1!} \underset{z\to 1}{Lt} \frac{d}{dz}(z)$$

$$[\text{Res} f(z)]_{z=1} = 1$$

* find the residues of $f(z) = \frac{z\sin z}{(z-\pi)^3}$ at $z=\pi$

$Dr = 0.$

$(z-\pi)^3 = 0.$

$z = \pi, \pi, \pi$ poles of order 3, $m = 3$

$$[\text{Res } f(z)]_{z=a} = \frac{1}{(m-1)!} \underset{z \to a}{Lt} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^{m-1} f(z) \right]$$

$$[\text{Res } f(z)]_{z=\pi} = \frac{1}{2!} \underset{z\to\pi}{Lt} \frac{d^2}{dz^2} \left[\cancel{(z-\pi)^3} \times \frac{z\sin z}{\cancel{(z-\pi)^3}} \right]$$

$$= \frac{1}{2!} \underset{z\to\pi}{Lt} \frac{d^2}{dz^2} (z\sin z)$$

$$= \frac{1}{2} \underset{z\to\pi}{Lt} \frac{d}{dz} (\sin z + z\cos z)$$

$$= \frac{1}{2} \underset{z\to\pi}{Lt} \left(\cos z - z\sin z + \cos z \right)$$

$$= \frac{1}{2} \underset{z\to\pi}{Lt} \; 2\cos z - z\sin z$$

$$= \frac{1}{2} \left[2\cos\pi - \pi \sin\pi \right]$$

$$= \frac{1}{\cancel{2}} (\cancel{2}(-1)) = -1.$$

Homework: (1) $f(z) = \frac{4}{z^3(z-2)}$ (i) at a simple pole (ii) $|z| = 3$

(2) $f(z) = \frac{z^2}{(z-2)(z+1)^2}$, along $|z| = 2$

(1) $f(z) = \frac{4}{z^3(z-2)}$

$Dr = 0.$

$z = 0, 0, 0 \rightarrow$ pole of order 3.

$z = 2 \rightarrow$ pole of order 1 (or) simple 1

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(i) at simple pole,

$$\left[\operatorname{Res} f(z)\right]_{z=2} = \lim_{z\to 2} (z-2) f(z)$$

$$= \lim_{z\to 2} (\cancel{z-2}) \left[\frac{4}{z^3(\cancel{z-2})}\right]$$

$$= \frac{4}{8} = \frac{1}{2}.$$

(ii) at $|z| = 3$

$z = 0,0,0 \rightarrow$ inside $(m=3)$

$z = 2 \rightarrow$ inside.

$$\left[\operatorname{Res} f(z)\right]_{z=0} = \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[\cancel{(z-0)^3} \cdot \frac{4}{\cancel{z^3}(z-2)}\right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} \frac{4}{(z-2)}$$

$$= 2 \lim_{z\to 0} \frac{d^2}{dz^2}\left(\frac{1}{z-2}\right).$$

$$= 2 \cdot \lim_{z\to 0} \frac{d}{dz}\left(\frac{-1}{(z-2)^2}\right)$$

$$= 2 \times \frac{-1 \times -2}{(z-2)^3} = \lim_{z\to 0} \frac{4}{(z-2)^3} = \frac{4}{(0-2)^3} = \frac{4}{-8} = -\frac{1}{2}$$

(2) $f(z) = \dfrac{z^2}{(z-2)(z+1)^2}$ $\quad |z| = 2.$

Dr $= 0$,

$z = 2, -1, -1 \rightarrow$ inside points

$z = 2$ pole of order 1

$z = -1$ pole of order 2 $(m=2)$

(i) $$\left[\operatorname{Res} f(z)\right]_{z=2} = \lim_{z\to 2} (\cancel{z-2}) \frac{z^2}{(\cancel{z-2})(z+1)^2}$$

$$= \frac{4}{9}$$

$$= 4/9.$$

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(ii) $[\text{Res } f(z)]_{z=-1} = \frac{1}{(2-1)!} \lim_{z\to -1} \frac{d^{2-1}}{dz^{2-1}}\left[\cancel{(z-(-1))^2} \frac{z^2}{(z-2)\cancel{(z+1)^2}}\right]$

$$= \frac{1}{1!} \cdot \lim_{z\to -1} \frac{d^1}{dz}\left[\frac{z^2}{z-2}\right]$$

$$= \lim_{z\to -1} \frac{(z-2)(2z) - (z^2)(1)}{(z-2)^2}$$

$$= \frac{2(-1)(-1-2) - (-1)^2(1)}{(-1-2)^2}$$

$$= \frac{6-1}{9} = 5/9.$$

* find the residues of $f(z) = \tan z$ at $z = \pi/2$

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

$$[\text{Res } f(z)]_{z=a} = \frac{P(a)}{Q'(a)}$$

$$= \frac{[\sin z]_{z=\pi/2}}{[-\sin z]_{z=\pi/2}} = -1$$

* $f(z) = \cot z$

$$f(z) = \frac{\cos z}{\sin z}$$

$$[\text{Res } f(z)]_{z=a} = \frac{P(a)}{Q'(a)}$$

$$= \left[\frac{\cos z}{\cos z}\right]_{z=\pi/2}$$

$$= 1.$$

* find residue of $f(z) = e^{1/z}$

$$f(z) = e^{1/z}$$

$$= 1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots$$

$[\text{Res } f(z)]_{z=a} = \text{coeff of } \frac{1}{z-a}$

$[\text{Res } f(z)]_{z=a} = \text{coeff of } \frac{1}{z-a} = 1.$

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Hw

1. $f(z) = \dfrac{1-e^{-z}}{z^3}$ at $z=0$.

2. $f(z) = z \cdot \cos\left(\frac{1}{z}\right)$, $z=0$.

Cauchy's Residue Theorm (CRT) :-

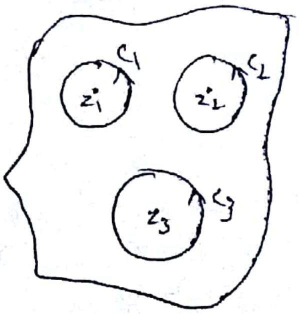
If $f(z)$ is analytic at all points inside and on a simple closed curve C (except for finite number of poles $z_1, z_2, \ldots z_n$) and on inside C, we have

$$\int_C f(z)\,dz = 2\pi i \times [\text{sum of residues}]$$

$$\int_C f(z)\,dz = 2\pi i \times [R_1 + R_2 + \cdots + R_n]$$

$R_1, R_2, \ldots R_n$ represent n residues.

Proof:- For each point of the poles $z_1, z_2, \ldots z_n$ draw a small non-intersecting circle with centre as $z_1, z_2, \ldots z_n$. Let the circles be $C_1, C_2, \ldots C_n$.



Cauchy's extended theorm,

$$\int_C f(z)\,dz = \int_{C_1} f(z_1)\,dz_1 + \int_{C_2} f(z_2)\,dz_2 + \cdots + \int_{C_n} f(z_n)\,dz_n \quad \text{--- (1)}$$

we know that, the residue of $f(z)$ at $z=a$ is given by $[\operatorname{Res} f(z)]_{z=a} = \dfrac{1}{2\pi i}\int_C f(z)\,dz$

(or)

$$\int_C f(z)\,dz = 2\pi i\,[\operatorname{Res} f(z)]_{z=a}$$

$$= 2\pi i \times [\operatorname{Res} f(z)]_{z=z_1} + 2\pi i \times [\operatorname{Res} f(z)]_{z=z_2} + \cdots + 2\pi i\,[\operatorname{Res} f(z)]_{z=z_n}$$

$$= 2\pi i\,[R_1 + R_2 + \cdots + R_n]$$

$$= 2\pi i\,[\text{sum of residues}]$$

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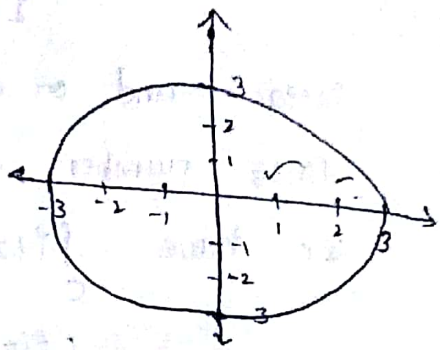
Pr: Using cauchy residue theorm find the value of

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz \qquad c : |z| = 3.$$

$$\int_c f(z) dz = 2\pi i \times [\text{sum of Residues}]$$

Poles, $D_r = 0$; $\qquad f(z) = \sin \pi z^2 + \cos \pi z^2$



$(z-1)(z-2) = 0.$

$z = 1$ pole of order 1.

$z = 2$ pole of order 1.

$$R_1 = [\text{Res} f(z)]_{z=1} = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} \frac{\cancel{(z-1)} \sin \pi z^2 + \cos \pi z^2}{\cancel{(z-1)}(z-2)}$$

$$= \frac{\sin \pi + \cos \pi}{(1-2)} = \frac{0 + (-1)}{-1} = \frac{-1}{-1} = 1.$$

$$R_2 = [\text{Res} f(z)]_{z=2} = \lim_{z \to 2} (z-2) f(z)$$

$$= \lim_{z \to 2} \frac{\cancel{(z-2)} \sin \pi z^2 + \cos \pi z^2}{(z-1)\cancel{(z-2)}}$$

$$= \frac{\sin \pi(4) + \cos \pi(4)}{(2-1)} = \frac{1}{1} = 1.$$

$$\int_c f(z) dz = 2\pi i \times [R_1 + R_2]$$

$$= 2\pi i [1+1]$$

$$= 4\pi i$$

Pr: Using CRT $\displaystyle\int_c \frac{4-3z}{z(z-1)(z-2)}$ where $c : |z| = \frac{3}{2}$.

$D_r = 0$

$z(z-1)(z-2) = 0$

$z = 0$, pole of order 1

$z = 1$, Pole of order 1.

$$R_1 = [\text{Res} f(z)]_{z=0} = \lim_{z\to 0} (z-0) \frac{4-3z}{(z)(z-1)(z-2)}$$

$$= \lim_{z\to 0} \frac{4-3z}{(z-1)(z-2)}$$

$$= \frac{4}{(-1)(-2)} = \frac{4}{2} = 2.$$

$$R_2 = [\text{Res} f(z)]_{z=1} = \lim_{z\to 1} \cancel{(z-1)} \frac{4-3z}{z\cancel{(z-1)}(z-2)}$$

$$= \frac{4-3(1)}{1\,(1-2)} = \frac{4-3}{-1} = \frac{1}{-1} = -1.$$

$$\int_C f(z)dz = 2\pi i[R_1+R_2] = 2\pi i[2-1]$$

$$= 2\pi i$$

Pr: $\int_C \frac{dz}{(z^2+1)(z^2-4)}$; $|z| = \frac{3}{2}$.

Poles, Dr = 0

$(z^2+1)(z^2-4)=0.$

$z^2 = -1$

$z = +i$ pole of order 1

$z = -i$ " " " 1.

$\underbrace{(0,1) \quad (0,-1)}_{\text{inside}}$

$z^2-4=0.$

$z=+2$ pole of order 1

$z=-2$ pole of order 1.

$\underbrace{(2,0) \quad (-2,0)}_{\text{outside}}$

$$R_1 = [\text{Res} f(z)]_{z=i} = \lim_{z\to i} \cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}(z^2-4)}$$

$$= \frac{1}{(i+i)(-5)} = \frac{1}{(2i)(-5)} = -\frac{1}{10i}$$

$$R_2 = [\text{Res} f(z)]_{z=-i} = \lim_{z\to -i} \cancel{(z+i)} \frac{1}{\cancel{(z+i)}(z-i)(z^2-4)}$$

$$= \frac{1}{(-i-i)(-1-4)} = \frac{1}{(-2i)(-5)} = \frac{1}{10i}$$

$$\int_C f(z)dz = 2\pi i\left[\cancel{-\frac{1}{10i}} + \cancel{\frac{1}{10i}}\right]$$

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* Evaluate $\int_c \frac{(z-1)dz}{(z+1)^2(z-2)}$ $|z-i|=2$

$|z-(0+1i)|=2$

centre $=(0,1)$

$Dr=0$

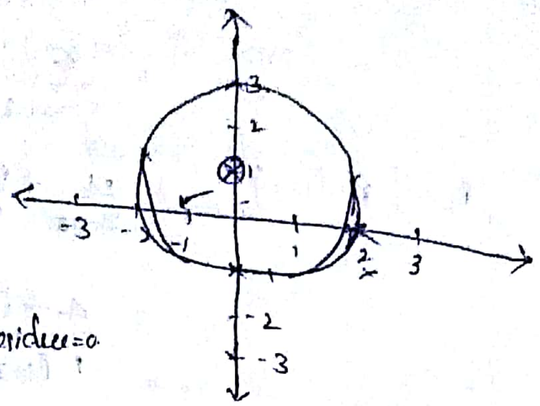
$(z+1)^2(z-2)=0$

$(z+1)(z+1)(z-2)=0$

$z=-1,-1,2$

$z=-1$ is order 2

$z=2$ is order 1. $\to$ residue $=0$



$R=\left[Res\,f(z)\right]_{z=-1}=\frac{1}{(m-1)!}\,\underset{z\to-1}{Lt}\,\frac{d^{m-1}}{dz^{m-1}}\left[\cancel{(z+1)^2}\frac{(z-1)}{\cancel{(z+1)^2}(z-2)}\right]dz$

$=\frac{1}{(2-1)!}\,\underset{z\to-1}{Lt}\,\frac{d^{2-1}}{dz^{2-1}}\left[\frac{z-1}{\cancel{(z+1)}(z-2)}\right]dz$

$=\underset{z\to-1}{Lt}\,\frac{d}{dz}\left[\frac{z-1}{\cancel{(z+1)}(z-2)}\right]$

$=\underset{z\to-1}{Lt}\left[\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right]$

$=\underset{z\to-1}{Lt}\left[\frac{\cancel{z}-2\cancel{-z}+1}{(z-2)^2}\right]$

$=\frac{-1}{(-1-2)^2}=\frac{-1}{(-3)^2}=\frac{-1}{9}$ //.

$\therefore \int_c f(z)dz=2\pi i\left[R_1+R_2\right]$

$=2\pi i\left[\frac{-1}{9}\right]=-\frac{2\pi i}{9}$.

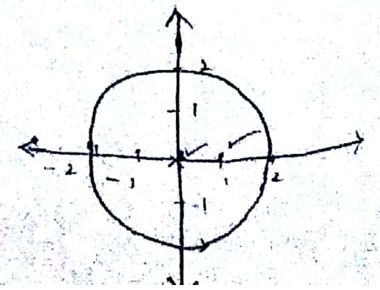
Pr: $\int_c \frac{dz}{z^3(z-1)}$ C: $|z|=2$.

$Dr=0$;

$z^3(z-1)=0$.

$z=0,0,0$ Pole of order 3

$z=1$ Pole of order 1.



R_1: $[\text{Res } f(z)]_{z=1} = \lim_{z\to 1} \cancel{(z-1)} \dfrac{1}{z^3\cancel{(z-1)}}$

$= \lim_{z\to 1} \dfrac{1}{z^3} = \dfrac{1}{1} = 1$

R_2: $[\text{Res } f(z)]_{z=0} = \dfrac{1}{(m-1)!} \lim_{z\to 0} \dfrac{d^{m-1}}{dz^{m-1}} \left[\cancel{(z-0)^3} \cdot \dfrac{1}{\cancel{z^3}(z-1)} \right]$

$= \dfrac{1}{2!} \lim_{z\to 0} \dfrac{d^2}{dz^2} \left[\dfrac{1}{z-1} \right]$

$= \dfrac{1}{2} \times \lim_{z\to 0} \dfrac{d}{dz} \left(\dfrac{-1}{(z-1)^2} \right)$

$= \dfrac{1}{2} \times -1 \times \lim_{z\to 0} \dfrac{d}{dz} \left(\dfrac{1}{(z-1)^2} \right)$

$= \dfrac{1}{2} \times (-1)(-2) \lim_{z\to 0} \left[\dfrac{1}{(z-1)^3} \right]$

$= \dfrac{\cancel{2}}{\cancel{2}} \times \lim_{z\to 0} \dfrac{1}{(z-1)^3} = \dfrac{1}{(0-1)^3} = -1$

$\therefore \int_c f(z)\,dz = 2\pi i (R_1 + R_2)$

$= 2\pi i [1 - 1]$

$= 0.$

* $\int \dfrac{(2z^2+z)\,dz}{(z^2-1)}$ (i) $|z-1| = 1$ (ii) $|z| = 2$

(i) $|z - (1+0i)| = 1$

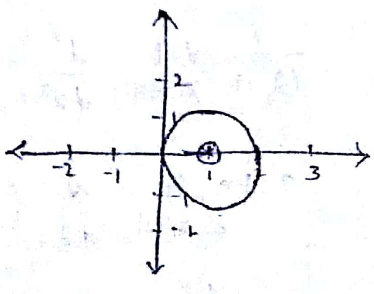
centre $= (1,0)$ $r = 1$.

Dr $= 0$ $z^2 - 1 = 0$.

$(z+1)(z-1) = 0$

$z = 1$ inside

$z = -1$ outside $R_2 = 0$.



$R_1 = [\text{Res } f(z)]_{z=1} = \lim_{z\to 1} \cancel{(z-1)} \dfrac{2z^2+z}{\cancel{(z-1)}(z+1)}$

$= \dfrac{2(1)+1}{(1+1)} = 3/2.$

$\int_c f(z)\,dz = \cancel{2}\pi i \left[\frac{3}{\cancel{2}} + 0\right] = 3\pi i.$

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(ii) $|z| = 2$

$$R_1 = [\text{Res} f(z)]_{z=1} = \underset{z \to 1}{lt} \cancel{(z-1)} \frac{2z^2+z}{\cancel{(z-1)}(z+1)}$$

$$= \frac{2(1)+1}{(1+1)} = 3/2$$

$$R_2 = [\text{Res} f(z)]_{z=-1} = \underset{z \to -1}{lt} \cancel{(z+1)} \frac{2z^2+z}{(z-1)\cancel{(z+1)}}$$

$$= \frac{2(-1)^2+(-1)}{(-1-1)} = \frac{2(1)-1}{-2} = \frac{1}{-2}$$

$$\int_C f(z)\,dz = 2\pi i \left[\frac{3}{2} - \frac{1}{2}\right]$$

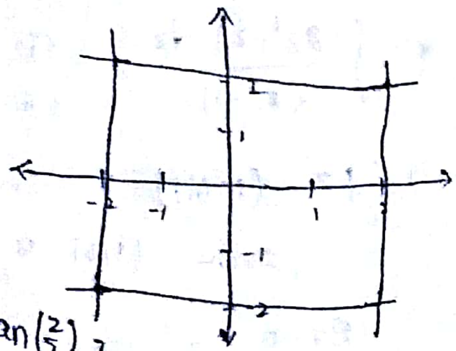
$$= 2\pi i.$$

* If c is boundary of square whose sides along straight lines $x = \pm 2$, $y = \pm 2$ and describing the +ve sense. Find the value of $\int_C \frac{\tan(\frac{z}{2})dz}{(z-1-i)^2}$

Dr = 0.

$= (z-1-i)^2 = 0.$

$z = (1+i)$ pole of order 2.



$$R_1 = (\text{Res}\, f(z))_{z=1+i}$$

$$= \frac{1}{(2-1)!} \underset{z \to (1+i)}{lt} \frac{d^{2-1}}{dz^{2-1}} \left[\frac{\cancel{(z-(1+i))^2} \tan(\frac{z}{2})}{\cancel{(z-(1+i))^2}} \right]$$

$$= 1 \times \underset{z \to (1+i)}{lt} \frac{d}{dz} \left[\tan\left(\frac{z}{2}\right)\right].$$

$$= \underset{z \to (1+i)}{lt} \sec^2\left(\frac{z}{2}\right) \times \frac{1}{2}.$$

$$= \frac{1}{2} \times \sec^2\left(\frac{1+i}{2}\right)$$

$$\int_C f(z)\,dz = 2\pi i \times \frac{1}{2}\left[\sec^2\left(\frac{1+i}{2}\right)\right]$$

$$= \pi i \sec^2\left(\frac{1+i}{2}\right)$$

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* $\displaystyle\int_C \frac{\tan\left(\frac{z}{2}\right)}{(z-a)^2}\,dz$ C is boundary of square whose sides lies along $x = \pm 2,\ y = \pm 2$. Also, $-2 < a < 2$.

* $\displaystyle\int \frac{(z+4)}{z^2+2z+5}\,dz$ C : (i) $|z+1-i| = 2$.
(ii) $|z+1+i| = 2$.

(i) $|z+1-i| = 2$

$|z-(-1+i)| = 2$

centre = $(-1, 1)$; radius = 2.

$Dr = 0$:

$z^2 + 2z + 5 = 0$

$z = \dfrac{-2 \pm \sqrt{4-20}}{2}$

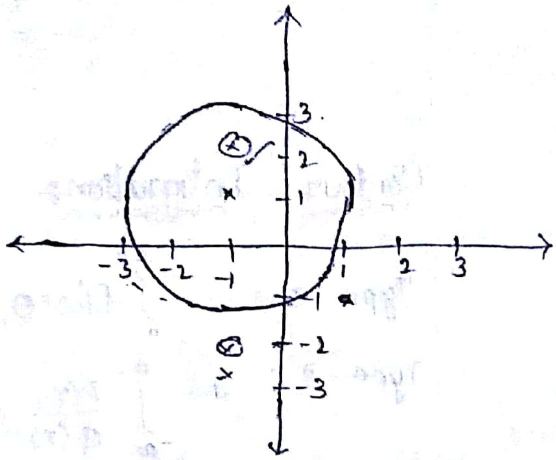
$= \dfrac{-2 \pm 4i}{2}$

$= -1 \pm 2i$

$-1+2i$ $\quad$ $-1-2i$

$(-1, 2)$ $\quad$ $(-1,-2)$. $\rightarrow R_2 = 0$.

$\downarrow R_1$



$R_1 = \left[\operatorname{Res} f(z)\right]_{z=-1+2i} = \lim\limits_{z\to(-1+2i)} \cancel{(z-(-1+2i))}\dfrac{z+4}{\cancel{(z-(-1+2i))}\,(z-(-1-2i))}$

$= \lim\limits_{z\to(-1+2i)} \dfrac{z+4}{(z+1+2i)}$

$= \dfrac{-1+2i+4}{(\cancel{-1}+2i\cancel{+1}+2i)} = \dfrac{3+2i}{4i}$

$= \dfrac{3}{4i} + \dfrac{2\cancel{i}}{4\cancel{i}}$

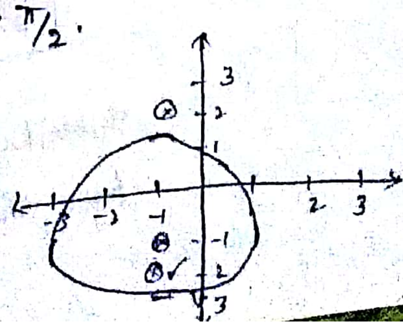
$= \dfrac{1}{2} - \dfrac{3}{4}i$.

$\therefore \displaystyle\int_C f(z)\,dz = (2\pi i)\left(\dfrac{3+2i}{\cancel{4i}_{\,2}}\right)$

$= (3+2i) \times \pi/2$.

* $|z-(-1-i)| = 2$.

centre = $(-1,-1)$ radius = 2.



$z = (-1+2i)\ (-1-2i)$.

$$R_1 = \left[\operatorname{Res} f(z)\right]_{z=-1-2i} = \lim_{z \to (-1-2i)} \cancel{(z-(-1-2i))} \frac{z+4}{(z-(-1+2i))\cancel{(z-(-1-2i))}}$$

$$= \frac{-1-2i+4}{-1-2i+1-2i}$$

$$= \frac{3-2i}{-4i}$$

$$\int_C f(z)\,dz = \cancel{2}\pi \cancel{i} \left(\frac{3-2i}{-\cancel{4}\cancel{i}_{2}}\right)$$

$$= \frac{(2i-3)}{1} \times \frac{\pi}{2}$$

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