Standard Derivations:- (type- I)

* $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} \quad (a>b)$

$z = e^{i\theta} = \cos\theta + i\sin\theta$ *

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{z^2+1}{2z}$; $\sin\theta = \frac{z^2-1}{2iz}$; $d\theta = \frac{dz}{iz}$

$I = \int_C \frac{dz}{iz\left[a+b\left[\frac{z^2+1}{2z}\right]\right]}$

$= \int_C \frac{dz}{i\cancel{z}\left[\frac{2az+bz^2+b}{2\cancel{z}}\right]} = \frac{2}{i}\int \frac{dz}{bz^2+2az+b} = \frac{2}{bi}\int \frac{dz}{z^2+\frac{2a}{b}z+1}$ Poles, $Dr = 0$.

$z^2 + \frac{2a}{b}z + 1 = 0 \Rightarrow z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-\frac{2a}{b} \pm \frac{2(\sqrt{a^2-b^2})}{b}}{2} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$

$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$; $\beta = -\frac{a-\sqrt{a^2-b^2}}{b}$.

$\left[\text{Res}(f(z))\right]_{z=\frac{-a-\sqrt{a^2-b^2}}{b}} = 0$ (outside)

$\left[\text{Res}\, f(z)\right]_{z=\frac{-a+\sqrt{a^2-b^2}}{b}} = \cancel{\left(z-\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)\right)} \times \frac{1}{\cancel{\left(z-\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)\right)}\left(z-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)\right)}$

$= \frac{1}{\left(\frac{\cancel{-a}+\sqrt{a^2-b^2}+\cancel{a}+\sqrt{a^2-b^2}}{b}\right)} = \frac{b}{2\sqrt{a^2-b^2}}$

$\therefore I = 2\pi \cancel{i} \times \frac{\cancel{b}}{\cancel{2}\sqrt{a^2-b^2}} \times \frac{\cancel{2}}{\cancel{b}\cancel{i}} = \frac{2\pi}{\sqrt{a^2-b^2}}$

* $I = \int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\left[\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}\right]$

$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}\pi}{\sqrt{a^2-b^2}}$

$= \pi / \sqrt{a^2-b^2}$.

* $I = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$, $a>|b|$

$z = e^{i\theta} = \cos\theta + i\sin\theta$ $\quad \cos\theta = \frac{z^2+1}{2z}$; $\sin\theta = \frac{z^2-1}{2iz}$; $d\theta = \frac{dz}{iz}$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore I = \int_C \frac{dz}{iz\left[a+b\left[\frac{z^2-1}{2iz}\right]\right]} = \int_C \frac{dz}{\cancel{iz}\left[\frac{2aiz+bz^2-b}{2i\cancel{z}}\right]} = \frac{2}{b}\int_C \frac{dz}{z^2+\frac{2ai}{b}z-1}$$; Poles, Dr = 0.

$$z = \frac{-\frac{2ai}{b} \pm \sqrt{-\frac{4a^2}{b^2}-4}}{2} = \frac{-\frac{2ai}{b} \pm \frac{2i}{b}\sqrt{a^2-b^2}}{2} = \frac{-ai \pm i\sqrt{a^2-b^2}}{b}$$

$$\therefore \alpha = \frac{-ai+i\sqrt{a^2-b^2}}{b} \; ; \; \beta = \frac{-ai-i\sqrt{a^2-b^2}}{b}$$

$$\left[\operatorname{Res} f(z)\right]_{z=\frac{-ai-i\sqrt{a^2-b^2}}{b}} = 0 \text{ (outside)}$$

$$\left[\operatorname{Res} f(z)\right]_{z=\frac{-ai+i\sqrt{a^2-b^2}}{b}} = \lim_{z\to\frac{-ai+i\sqrt{a^2-b^2}}{b}} \cancel{\left(z-\left(\frac{-ai+i\sqrt{a^2-b^2}}{b}\right)\right)} \frac{1}{\cancel{\left(z-\left(\frac{-ai+i\sqrt{a^2-b^2}}{b}\right)\right)}\left(z-\left(\frac{-ai-i\sqrt{a^2-b^2}}{b}\right)\right)}$$

$$= \frac{1}{\frac{-\cancel{ai}+i\sqrt{a^2-b^2}+\cancel{ai}+i\sqrt{a^2-b^2}}{b}} = \frac{b}{2i\sqrt{a^2-b^2}}$$

$$\therefore I = \cancel{2}\pi \cancel{i} \times \frac{2}{\cancel{b}} \times \frac{\cancel{b}}{\cancel{2i}\sqrt{a^2-b^2})} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

* Note:-

$$\int_0^{\pi} \frac{d\theta}{a+b\sin\theta} = \frac{1}{2}\left[\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}\right] = \frac{1}{\cancel{2}} \times \frac{\cancel{2}\pi}{\sqrt{a^2-b^2}}$$

$$= \pi/\sqrt{a^2-b^2}.$$

* $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2}$, $0<a<1$

$z = e^{i\theta} = \cos\theta + i\sin\theta$ $\quad \cos\theta = \frac{z^2+1}{2z}$; $\sin\theta = \frac{z^2-1}{2iz}$; $d\theta = \frac{dz}{iz}$.

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$$I = \int_c \frac{dz}{iz\left[1-2a\left(\frac{z^2+1}{2z}\right)+a^2\right]} = \int_c \frac{dz}{i\cancel{z}\left[\frac{z-az^2-a+a^2z}{\cancel{z}}\right]} = \frac{1}{i}\int_c \frac{dz}{-az^2+(1+a^2)z-a}$$

$$= \frac{1}{-ia}\int_c \frac{dz}{z^2-\left(\frac{1+a^2}{a}\right)z+1}$$ Poles, $Dr=0$;

$$z^2-\left(\frac{1+a^2}{a}\right)z+1=0 \Rightarrow z = \frac{\left(\frac{1+a^2}{a}\right)\pm\sqrt{\left(\frac{1+a^2}{a}\right)^2-4}}{2} = \frac{\left(\frac{1+a^2}{a}\right)\pm\sqrt{\frac{(1+a^2)^2-4a^2}{a^2}}}{2}$$

$$= \frac{\left(\frac{1+a^2}{a}\right)\pm\frac{1}{a}\sqrt{(1-a^2)^2}}{2} = \frac{(1+a^2)\pm(1-a^2)}{2a}$$

$$\alpha = \frac{2}{2a} = \frac{1}{a}\ ; \quad \beta = \frac{2a^2}{2a} = a.$$

$$\left[\text{Res}\, f(z)\right]_{z=\frac{1}{a}} = 0 \quad \text{(outside)}.$$

$$\left[\text{Res}\, f(z)\right]_{z=a} = \underset{z\to a}{Lt}\, (\cancel{z-a})\frac{1}{(\cancel{z-a})(z-\frac{1}{a})}$$

$$= \frac{1}{(a-\frac{1}{a})} = \frac{a}{a^2-1}$$

$$\therefore I = 2\pi\cancel{i}\times\frac{\cancel{a}}{a^2-1}\times\frac{-1}{\cancel{i}\cancel{a}} = \frac{-2\pi}{a^2-1} = \frac{2\pi}{1-a^2}.$$

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Contour Integration:-

Type-I : $I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \qquad c: |z| = 1$

Type-II : $I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$

Type-I problems:-

* $\int_0^{2\pi} \frac{d\theta}{3+5\cos\theta} = I$

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$z + \frac{1}{z} = 2\cos\theta$

$\boxed{\cos\theta = \frac{z^2+1}{2z}}$

$z - \frac{1}{z} = 2i\sin\theta$

$\boxed{\sin\theta = \frac{z^2-1}{2z}}$

$z = e^{i\theta}$

$dz = e^{i\theta} \cdot i\, d\theta$

$\boxed{d\theta = \frac{dz}{iz}}$

$I = \int_c \frac{dz}{iz\left[13+5\left[\frac{z^2+1}{2z}\right]\right]}$

$= \int_c \frac{dz}{i\cancel{z}\left[\frac{26z+5z^2+5}{2\cancel{z}}\right]}$

$= \frac{2}{i} \int \frac{dz}{5z^2+26z+5}$

$= \frac{2}{5i} \int \frac{dz}{z^2+\frac{26}{5}z+1}$

Poles, $Dr = 0 \Rightarrow z^2 + \frac{26}{5}z + 1 = 0$

$\Rightarrow z = \frac{-\frac{26}{5} \pm \sqrt{\left(\frac{676}{25}\right)-4}}{2} = \frac{-\frac{26}{5} \pm \sqrt{\frac{576}{25}}}{2} = \frac{-\frac{26}{5} \pm \frac{24}{5}}{2} = \begin{cases} \alpha = \frac{-\frac{26}{5}+\frac{24}{5}}{2} = -1/5 \\ \beta = \frac{-\frac{26}{5}-\frac{24}{5}}{2} = -5 \end{cases}$

(outside)

$[\text{Res } f(z)]_{z=-5} = 0$ (outside)

$[\text{Res } f(z)]_{z=-\frac{1}{5}} = \lim_{z \to -\frac{1}{5}} \cancel{\left(z+\frac{1}{5}\right)} \cdot \frac{1}{\cancel{\left(z+\frac{1}{5}\right)}(z+5)}$

$= \frac{1}{-\frac{1}{5}+5} = \frac{5}{24}$

$\therefore I = \cancel{2}\pi\cancel{i} \times \frac{\cancel{2}}{\cancel{5}\cancel{i}} \times \frac{\cancel{5}}{\cancel{24}_{6}} = \pi/6$

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$$* \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$$

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$$\cos\theta = \frac{z^2+1}{2z} \quad ; \quad \sin\theta = \frac{z^2-1}{2iz} \quad ; \quad d\theta = \frac{dz}{iz}$$

$$I = \int_c \frac{dz}{iz\left[5+4\left[\frac{z^2+1}{2z}\right]\right]}$$

$$= \int_c \frac{dz}{i\cancel{z}\left[\frac{10z+4z^2+4}{2\cancel{z}}\right]}$$

$$= \frac{2}{i}\int_c \frac{dz}{4z^2+10z+4} = \frac{2}{4i}\int_c \frac{dz}{z^2+\frac{10}{4}z+1} \quad ; \text{ Poles, } Dr = 0$$

$$z^2 + \frac{10}{4}z + 1 = 0 \Rightarrow z = \frac{-\frac{10}{4} \pm \sqrt{\frac{100}{16}-4}}{2} = \frac{-\frac{10}{4} \pm \frac{6}{4}}{2} = \begin{cases} \alpha = \frac{-\frac{10}{4}+\frac{6}{4}}{2} = -\frac{1}{2} \\ \beta = \frac{-\frac{10}{4}-\frac{6}{4}}{2} = -2 \quad \times \text{(outside)} \end{cases}$$

$$[\text{Res } f(z)]_{z=-2} = 0 \quad \text{(outside)}$$

$$[\text{Res } f(z)]_{z=-\frac{1}{2}} = \lim_{z \to -\frac{1}{2}} \cancel{\left(z+\frac{1}{2}\right)} \times \frac{1}{\cancel{\left(z+\frac{1}{2}\right)}(z+2)} = \frac{1}{-\frac{1}{2}+2}$$

$$= \frac{1}{\frac{3}{2}} = 2/3.$$

$$\therefore I = \cancel{2}\pi\cancel{i} \times \frac{2}{3} \times \frac{1}{\cancel{2i}} = 2\pi/3.$$

$$* \int_0^{\pi} \frac{d\theta}{2+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_c \frac{dz}{iz\left[2+1\left[\frac{z^2+1}{2z}\right]\right]} = \int_c \frac{dz}{i\cancel{z}\left[\frac{4z+z^2+1}{2\cancel{z}}\right]} = \frac{2}{i}\int_c \frac{dz}{z^2+4z+1}$$

Poles, $Dr = 0$.

$$z^2+4z+1 = 0 \Rightarrow z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

$\alpha = -2+\sqrt{3}$ ✓

$\beta = -2-\sqrt{3}$ × (outside)

$\left[\text{Res}\, f(z)\right]_{z=(-2-\sqrt{3})} = 0$ (outside) ②

$\left[\text{Res}\, f(z)\right]_{z=(-2+\sqrt{3})} = \lim_{z\to(-2+\sqrt{3})} \cancel{(z-(-2+\sqrt{3}))} \dfrac{1}{\cancel{(z-(-2+\sqrt{3}))}(z-(-2-\sqrt{3}))}$

$= \dfrac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})} = \dfrac{1}{-\cancel{2}+\sqrt{3}+\cancel{2}+\sqrt{3}} = \dfrac{1}{2\sqrt{3}}$

$\therefore I = 2\pi i \times \dfrac{2}{i} \times \dfrac{1}{2\sqrt{3}} \times \dfrac{1}{2}$

$= \pi/\sqrt{3}$

* $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \dfrac{z^2+1}{2z}$; $\sin\theta = \dfrac{z^2-1}{2iz}$; $d\theta = \dfrac{dz}{iz}$

$= \displaystyle\int_c \frac{dz}{iz\left[13+12\left[\frac{z^2+1}{2z}\right]\right]} = \int_c \frac{dz}{i\cancel{z}\left[\frac{26z+12z^2+12}{2\cancel{z}}\right]} = \frac{2}{i}\int_c \frac{dz}{12z^2+26z+12} = \frac{2}{12i}\int_c \frac{dz}{z^2+\frac{26}{12}z+1}$

Poles, $Dr = 0$;

$z = \dfrac{-\frac{13}{6} \pm \sqrt{\frac{169}{36}-4}}{2} = \dfrac{-\frac{13}{6} \pm \frac{5}{6}}{2} = \alpha = \dfrac{-\frac{13}{6}+\frac{5}{6}}{2} = -\dfrac{2}{3}$ ✓

$\beta = \dfrac{-\frac{13}{6}-\frac{5}{6}}{2} = -\dfrac{3}{2}$ × (outside)

$\left[\text{Res}\, f(z)\right]_{z=-\frac{3}{2}} = 0$ (outside)

$\left[\text{Res}\, f(z)\right]_{z=-\frac{2}{3}} = \lim_{z\to -\frac{2}{3}} \cancel{\left(z+\frac{2}{3}\right)} \dfrac{1}{\cancel{\left(z+\frac{2}{3}\right)}\left(z+\frac{3}{2}\right)}$

$= \dfrac{1}{-\frac{2}{3}+\frac{3}{2}} = \dfrac{1}{\frac{-4+9}{6}} = \dfrac{6}{5}$

$\therefore I = 2\pi i \times \dfrac{1}{\cancel{6}i} \times \dfrac{\cancel{6}}{5} = 2\pi/5$

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* $\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta}$

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{z^2+1}{2z}$; $\sin\theta = \frac{z^2-1}{2z}$; $d\theta = \frac{dz}{iz}$

$= \int_0^{2\pi} \frac{dz}{iz\left[5-4\left[\frac{z^2-1}{2iz}\right]\right]} = \int_c \frac{dz}{\cancel{iz}\left[\frac{10iz-4z^2+4}{2i\cancel{z}}\right]} = 2\int \frac{dz}{-4z^2+10iz+4}$

$= \frac{1}{-2}\int \frac{dz}{+z^2 \underset{-}{+} \frac{5}{2}iz \underset{-}{+} 1}$. Poles, $Dr = 0$.

$+z^2 - \frac{5}{2}iz - 1 = 0 \Rightarrow z = \frac{\frac{5}{2} \pm \sqrt{\frac{-25}{4}+4}}{2} = \frac{\frac{5}{2} \pm \sqrt{\frac{-9}{4}}}{2} = \alpha = \frac{\frac{5}{2}+\frac{3}{2}i}{2} = i/2$

$\beta = \frac{\frac{5}{2}-\frac{3}{2}i}{2} = 2i$

$[Res\, f(z)]_{z=2i} = 0$ (outside)

$[Res\, f(z)]_{z=i/2} = \lim_{z \to i/2} \cancel{\left(z-\frac{i}{2}\right)} \frac{1}{\cancel{\left(z-\frac{i}{2}\right)}(z-2i)}$

$= \frac{1}{\left(\frac{i}{2}-2i\right)} = \frac{1\times 2}{i-4i} = \frac{2}{-3i}$

$\therefore I = 2\pi \cancel{i} \times \frac{\cancel{2}}{\cancel{-3i}} \times \frac{-1}{\cancel{2}}$

$= 2\pi/3.$

* $\int_0^{\pi} \frac{\sin^2\theta\, d\theta}{5+4\cos\theta}$; $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ $= R.P\left[\frac{1-\cos 2\theta}{2}\right]$

$I = \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{5+4\cos\theta} = \frac{1}{2}.R.P\int_c \frac{(1-z^2)\,dz}{2iz\left[5+4\left[\frac{z^2+1}{2z}\right]\right]} = \frac{1}{2}RP\int_c \frac{(1-z^2)dz}{\cancel{2iz}\left[\frac{10z+4z^2+4}{\cancel{2z}}\right.}$

$= \frac{1}{2i} R.P \int_c \frac{(1-z^2)dz}{4z^2+10z+4} = \frac{1}{8i} RP \int_c \frac{(1-z^2)dz}{z^2+\frac{5}{2}z+1}$, Poles, $Dr=0$

$z^2+\frac{5}{2}z+1=0 \Rightarrow z = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4}-4}}{2} = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2}$; $\alpha = -1/2$

$\beta = -2$

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③

$\left[Res\, f(z)\right]_{z=-2} = 0$ (outside)

$\left[Res\, f(z)\right]_{z=\frac{-1}{2}} = \lim_{z\to\frac{-1}{2}} \cancel{(z+\frac{1}{2})} \frac{(1-z^2)}{\cancel{(z+\frac{1}{2})}(z+2)} = \frac{1-1/4}{\frac{-1}{2}+2} = \frac{3/4}{3/2} = \frac{2}{4} = 1/2$

$\therefore I = \cancel{2}\pi \cancel{i} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{8i}} \times \frac{1}{2}$

$= \frac{\pi}{16}.$ (R.P)

* $I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta.$

$z = e^{i\theta} = \cos\theta + i\sin\theta$ $\quad \cos\theta = \frac{z^2+1}{2z}$; $\sin\theta = \frac{z^2-1}{2iz}$; $d\theta = \frac{dz}{iz}$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$I = R.P \int_c \frac{z^3\, dz}{iz\left[5-4\left[\frac{z^2+1}{2z}\right]\right]} = R.P \int_c \frac{z^3 dz}{i\cancel{z}\left[\frac{10z-4z^2-4}{2\cancel{z}}\right]} = \frac{2}{i} R.P \int_c \frac{z^3 dz}{-4z^2+10z-4}$

$= \frac{-2}{4i} R.P \int_c \frac{z^3 dz}{z^2-\frac{10}{4}z+1}$ $\quad$ Poles, $Dr = 0.$

$z^2 - \frac{10}{4}z + 1 = 0. \Rightarrow z = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4}-4}}{2} = \alpha = \frac{\frac{5}{2}+\frac{3}{2}}{2} = 2$ ✗

$\beta = \frac{\frac{5}{2}-\frac{3}{2}}{2} = \frac{1}{2}.$ ✓

$\left[Res\, f(z)\right]_{z=2} = 0$ (outside)

$\left[Res\, f(z)\right]_{z=\frac{1}{2}} = \lim_{z\to\frac{1}{2}} \cancel{(z-\frac{1}{2})} \frac{z^3}{\cancel{(z-\frac{1}{2})}(z-2)} = \frac{1/8}{(\frac{1}{2}-2)} = \frac{1}{\cancel{8}4} \times \frac{-\cancel{2}}{3} = -1/12$

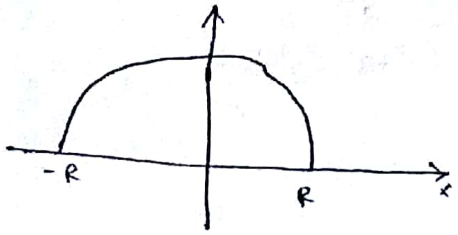
$\therefore I = \cancel{2}\pi\cancel{i} \times \frac{-1}{12} \times \frac{-1}{\cancel{2i}} = R.P\, \frac{\pi}{12} = \pi/12.$

Type-II : ⑤

$$\int_{-\infty}^{\infty} \frac{P(x)}{\varphi(x)} dx \iff \int_{0}^{\infty} \frac{P(x)}{\varphi(x)} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{P(x)}{\varphi(x)} dx$$

By CRT equation, $(x \to z)$ $\int_C f(z) dz = \int_{-\infty}^{\infty} \frac{P(z)}{\varphi(z)} dz$

$$= 2\pi i \left[\text{sum of residues}\right] + \pi i \left[\begin{array}{c}\text{sum of residues} \\ \text{on real axis}\end{array}\right] \quad -①$$



From dia,

$$\int_C f(z) dz = \int_{-R}^{R} f(x) dx + \int_S f(z) dz \quad -②$$

Standard Derivations:

$$* \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \int_C \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)} . \quad \text{Poles, Dr} = 0.$$

$$(z^2+a^2)(z^2+b^2) = 0$$

$z^2 = -a^2$ $\quad$ $z^2 = -b^2$

$z = \pm ai$ $\quad$ $z = \pm bi$

ai $\quad$ $-ai$ $\quad$ bi $\quad$ $-bi$

$(0,a)$ $\quad$ $(0,-a)$ outside $\quad$ $(0,b)$ $\quad$ $(0,-b)$ outside

$$\left[\text{Res } f(z)\right]_{z=ai} = \lim_{z \to ai} \cancel{(z-ai)} \times \frac{z^2}{\cancel{(z-ai)}(z+ai)(z-bi)(z+bi)}$$

$$= \frac{(ai)^2}{(ai+ai)(ai-bi)(ai+bi)} = \frac{\cancel{-}a^2}{(2ai)\cancel{i^2}(a^2-b^2)} = \frac{a}{2i(a^2-b^2)}$$

$$\left[\text{Res } f(z)\right]_{z=bi} = \lim_{z \to bi} \cancel{(z-bi)} \frac{z^2}{(z-ai)(z+ai)\cancel{(z-bi)}(z+bi)} = \frac{(bi)^2}{(bi-ai)(bi+ai)(2bi)}$$

$$= \frac{\cancel{-}b^2}{(2bi)\cancel{i^2}(b^2-a^2)} = \frac{b}{2i(b^2-a^2)} = \frac{-b}{2i(a^2-b^2)}$$

$$\therefore I = 2\pi i \left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right] = \frac{\cancel{2}\pi \cancel{i}\cancel{(a-b)}}{\cancel{2i}\cancel{(a-b)}(a+b)} = \frac{\pi}{(a+b)}$$

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{(a+b)}$$ *

* $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $a>0, b>0$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} \Rightarrow \int_C f(z)dz = \frac{1}{2}\int_C \frac{dz}{(z^2+a^2)(z^2+b^2)}$$ *

Now, $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_S f(z)dz$

$R \to \infty$, $\int_S f(z)dz \to 0$

$z^2 = -a^2$ | $z^2 = -b^2$
$z = +ai$ | $z = +bi$
$z = -ai$ (outside) | $z = -bi$ (outside)

$$[\text{Res } f(z)]_{z=ai} = \lim_{z\to ai} \cancel{(z-ai)} \times \frac{1}{\cancel{(z-ai)}(z+ai)(z+bi)(z-bi)}$$

$$= \frac{1}{(2ai)\,i(a+b)\,i(a-b)} = \frac{-1}{2ai(a^2-b^2)}$$

$$[\text{Res } f(z)]_{z=bi} = \lim_{z\to bi} \cancel{(z-bi)} \frac{1}{(z-ai)(z+ai)(z+bi)\cancel{(z-bi)}}$$

$$= \frac{1}{(bi-ai)(bi+ai)(bi+bi)} = \frac{-1}{(b^2-a^2)(2bi)} = \frac{1}{2bi(a^2-b^2)}$$

$$\therefore I = 2\pi i\left[\frac{1}{(a^2-b^2)2bi} - \frac{1}{2ai(a^2-b^2)}\right] = \cancel{2}\pi \cancel{i} \times \frac{1}{(a^2-b^2)\cancel{2i}}\left[\frac{1}{b}-\frac{1}{a}\right].$$

$$= \frac{\pi}{\cancel{(a-b)}(a+b)}\left[\frac{\cancel{a-b}}{ab}\right]\frac{1}{2} = \frac{\pi}{2ab(a+b)}$$

$$* \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

if $a=b$.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+a^2)} = \frac{\pi}{2a(a)(a+a)} = \frac{\pi}{2a^2(2a)} = \frac{\pi}{4a^3}.$$

$$* \int_{-\infty}^{\infty} \frac{x^2-6x+2}{x^4+10x^2+9}\,dx$$

$$\int_c f(z)\,dz = \int_{-\infty}^{\infty} \frac{z^2-6z+2}{z^4+10z^2+9}\,dz.$$

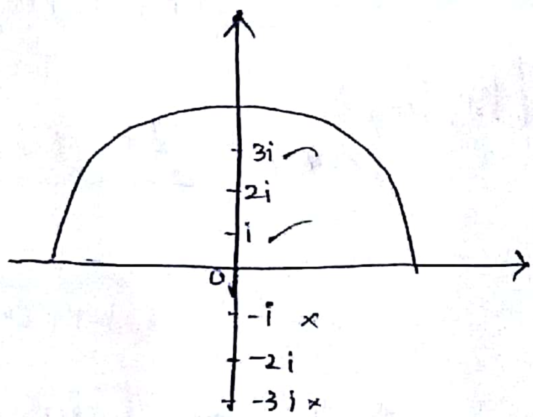
Poles, Dr $=0$, $z^2=t \Rightarrow t^2+10t+9=0.$

$(t+9)(t+1)=0.$

$t=-9 \quad t=-1$

$z^2=-9 \quad z^2=-1$

$z=\pm 3i \quad z=\pm i$



$$\left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z\to i} \cancel{(z-i)} \frac{z^2-6z+2}{\cancel{(z-i)}(z+i)(z-3i)(z+3i)}$$

$$= \frac{i^2-6i+2}{(2i)(-2i)(4i)} = \frac{1-6i}{-16(-i)} = \frac{1-6i}{16i}$$

$$\left[\operatorname{Res} f(z)\right]_{z=3i} = \lim_{z\to 3i} \cancel{(z-3i)} \frac{9i^2-6(3i)+2}{\cancel{(z-3i)}(z+3i)(z-i)(z+i)}$$

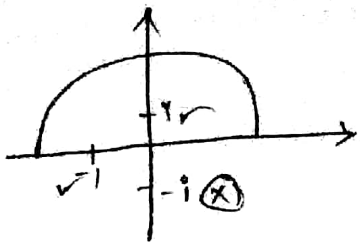
$$= \frac{-14-6i+2}{(6i)(2i)(4i)} = \frac{18i+7}{48i}$$

$$\therefore I = 2\pi i\left[\frac{18i+7}{48i} + \frac{1-6i}{16i}\right] + \pi i[0]$$

$$= 2\pi i\left[\frac{18i+7}{48i} + \frac{3-18i}{48i}\right]$$

$$= \cancel{2\pi i}\left[\frac{10}{\cancel{48i}_{\cancel{24}\,12}}\right] = 5\pi/12.$$

* $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x+1)(x^2+1)}$



$$= \int_C f(z)\,dz = \int_{-\infty}^{\infty} \frac{z\,dz}{(z+1)(z^2+1)}$$

Poles $\cancel{Dr}=0$, $(z+1)(z^2+1)=0$

$z=-1,\ z=+i,\ z=-i$ (outside).

$$\left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z\to i} \cancel{(z-i)} \frac{z}{(z+1)\cancel{(z-i)}(z+i)}$$

$$= \frac{\cancel{i}}{(1+i)\,(2\cancel{i})} = \frac{1}{2+2i}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \lim_{z\to -1} \cancel{(z+1)} \frac{z}{\cancel{(z+1)}(z-i)(z+i)}$$

$$= \frac{-1}{(-1-i)\,(-1+i)} = \frac{-1}{((-1)^2-(i)^2)} = -\frac{1}{2}.$$

$$\therefore\ I = \cancel{2}\pi i \left[\frac{+1}{\cancel{2}(1+i)}\right] + \pi i\left[\frac{-1}{2}\right].$$

$$= \frac{\pi i}{(1+1)} - \frac{\pi i}{2} = \pi i\left[\frac{2-1-i}{2(1+i)}\right]$$

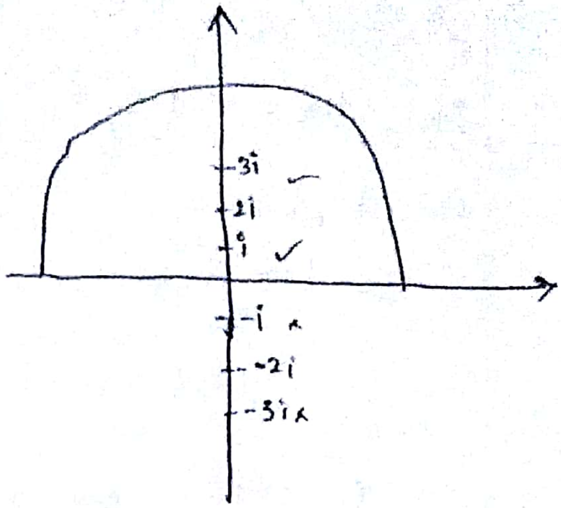
$$= \pi i\left[\frac{1-i}{2(1+i)}\right] = \frac{\pi i\,[1-i]^2}{(1^2-(i)^2)} = \frac{\pi i\,(1-i)^2}{2(2)}$$

$$= \frac{\pi i}{4}(1-i)^2$$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} \frac{x\,dx}{(x+1)(x^2+1)} = \frac{\pi i}{4}(1-i)^2.$$

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx$$

$$\int_C f(z)\, dz = \int_{-\infty}^{\infty} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\, dz$$



Put $z^2 = t$, Poles, $Dr = 0$

$$t^2 + 10t + 9 = 0$$

$$(t+9)(t+1) = 0$$

$$z^2 = -9 \; ; \; z^2 = -1$$

$$z = \pm 3i \; ; \; z = \pm i$$

$$f(z) = \frac{z^2 - z + 2}{(z+3i)(z-3i)(z+i)(z-i)}$$

$$\left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z \to i} \cancel{(z-i)} \frac{(z^2 - z + 2)}{(z+3i)(z-3i)(z+i)\cancel{(z-i)}}$$

$$= \frac{-1 - i + 2}{(4i)(-2i)(2i)} = \frac{1-i}{-16(-i)} = \frac{1-i}{16i}$$

$$\left[\operatorname{Res} f(z)\right]_{z=3i} = \lim_{z \to 3i} \cancel{(z-3i)} \frac{(z^2 - z + 2)}{\cancel{(z-3i)}(z+3i)(z-i)(z+i)}$$

$$= \frac{-9 - 3i + 2}{(6i)(2i)(4i)} = \frac{-7-3i}{48(-i)} = \frac{7+3i}{48i}$$

$$\therefore I = 2\pi i \left[\frac{7+3i}{48i} + \frac{3-3i}{48i}\right]$$

$$= \cancel{2}\pi \cancel{i} \left[\frac{\cancel{10}\,5}{\cancel{48}\,\cancel{i}}\right] \quad \cancel{24}\;12$$

$$= 5\pi/12.$$

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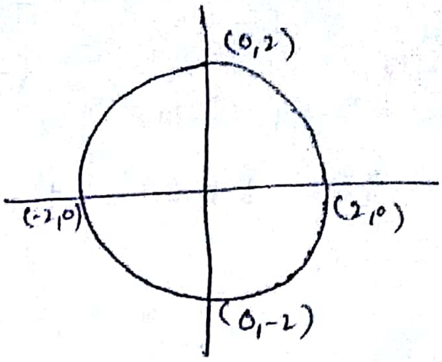
* $\int_C \frac{4z^3+2z}{z^4+z^2+1}$; $|z|=2$

Poles, Dr=0, Let $z^2=t$



$\Rightarrow \quad t^2+t+1=0.$

$t = \frac{-1\pm\sqrt{1-4}}{2}$

$= \frac{-1\pm i\sqrt{3}}{2}$

$\therefore \quad \alpha = \frac{-1+i\sqrt{3}}{2}$; $\beta = \frac{-1-i\sqrt{3}}{2}$

$\therefore \quad f(z) = \frac{4z^3+2z}{(z^2-\alpha)(z^2-\beta)}$

$\left[\text{Res } f(z)\right]_{z^2=\alpha} = \frac{1}{(2-1)!} \underset{z^2\to\alpha}{Lt} \frac{d}{dz}\left[(z^2-\alpha)f(z)\right]$

$= \underset{z^2\to\alpha}{Lt} \frac{d}{dz}\left[\frac{4z^3+2z}{z^2-\beta}\right]$

$= \underset{z^2\to\alpha}{Lt} \left\{\frac{(z^2-\beta)(12z^2+2)-(4z^3+2z)(2z)}{(z^2-\beta)^2}\right\}$

$= \underset{z^2\to\alpha}{Lt} \frac{2(z^2-\beta)(6z^2+1)}{(z^2-\beta)^2} - 4z^2(2z^2+1)$

$= \frac{2(\alpha-\beta)(6\alpha+1)-4\alpha(2\alpha+1)}{(\alpha-\beta)^2}$

$\alpha-\beta = i\sqrt{3}$; $6\alpha+1 = -3+i3\sqrt{3}+1 = -2+i3\sqrt{3}$

$\therefore \left[\text{Res } f(z)\right]_{z^2=\alpha} = \frac{(2i\sqrt{3})(-2+i3\sqrt{3}) - (i\sqrt{3})(-1+i\sqrt{3})}{(i\sqrt{3})^2}$

$= \frac{2(i\cancel{\sqrt{3}})}{(i\sqrt{3})^2}\left[-2+3i\sqrt{3}+1-i\sqrt{3}\right]$

$= \frac{2}{i\sqrt{3}}\left(2i\sqrt{3}+-1\right).$

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$$\left[\operatorname{Res} f(z)\right]_{z^2=\beta} = \frac{1}{(2-1)!} \lim_{z^2\to\beta} \frac{d}{dz}\left[\frac{4z^3+2z}{(z^2-\alpha)}\right]$$

$$= \lim_{z^2\to\beta} \frac{(z^2-\alpha)(12z^2+2)-(4z^3+2z)(2z)}{(z^2-\alpha)^2}$$

$$= \frac{2(\beta-\alpha)(6\beta+1)-4\beta(2\beta+1)}{(\beta-\alpha)^2}$$

$\beta-\alpha = -i\sqrt{3}$, $1+2\beta = -i\sqrt{3}$, $1+6\beta = -2-3i\sqrt{3}$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z^2=\beta} = \frac{2(-i\sqrt{3})(-2-3i\sqrt{3})-4\left[\frac{-1-i\sqrt{3}}{2}\right](-i\sqrt{3})}{(-i\sqrt{3})^2}$$

$$= \frac{2(\cancel{-i\sqrt{3}})}{(-i\sqrt{3})^{\cancel{2}}}\left[-2-3i\sqrt{3}-(-1-i\sqrt{3})\right]$$

$$= \frac{2}{-i\sqrt{3}}(-1-2i\sqrt{3})$$

$$= \frac{2(2i\sqrt{3}+1)}{i\sqrt{3}}$$

$\therefore$ By CRT, $\int_C f(z)dz = 2\pi i\left[\frac{2}{\sqrt{3}i}(2i\sqrt{3}-1)+\frac{2}{\sqrt{3}i}(2i\sqrt{3}+1)\right]$

$$= 2\pi i\left[2\times\frac{2\cancel{i\sqrt{3}}}{\cancel{i\sqrt{3}}}\right]$$

$$= 8\pi i$$

$$\therefore \int_C \frac{4z^3+2z}{(z^4+z^2+1)}dz = 8\pi i.$$

$$* \int_0^{2\pi} \frac{d\theta}{\cos\theta+\sin\theta+4}$$

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{z^2+1}{2z}$; $\sin\theta = \frac{z^2-1}{2iz}$; $d\theta = \frac{dz}{iz}$

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$$\int_0^{2\pi} \frac{d\theta}{\cos\theta + 2\sin\theta + 4} = \int_C \frac{dz}{iz\left[\frac{z^2+1}{2z} + 2\left[\frac{z^2-1}{2iz}\right] + 4\right]}$$

$$= \int_C \frac{dz}{iz\left[\frac{i(z^2+1) + 2(z^2-1) + 8iz}{2iz}\right]}$$

$$= 2\int_C \frac{dz}{z^2 i + i + 2z^2 - 2 + 8zi}$$

$$= 2\int_C \frac{dz}{z^2(2+i) + 8iz + (i-2)}$$

$$= \frac{2}{(2+i)}\int_C \frac{dz}{z^2 + \frac{8i}{(2+i)}z + \frac{(i-2)}{(2+i)}}$$

Poles, Dr = 0; $Z = \dfrac{-\frac{8i}{(2+i)} \pm \sqrt{\frac{-64}{(2+i)^2} - \frac{4(i-2)}{(2+i)}}}{2}$

$$= \frac{-\frac{8i}{(2+i)} \pm \sqrt{\frac{-64-(-1-4)}{(2+i)^2}}}{2}$$

$$= \frac{-4i \pm i\sqrt{11}}{(2+i)}$$

$\therefore \alpha = \dfrac{-4i + i\sqrt{11}}{2+i}$; $\beta = \dfrac{-4i - i\sqrt{11}}{2+i}$

$$[\operatorname{Res} f(z)]_{z=\alpha} = \lim_{z\to\alpha} \cancel{(z-\alpha)} \frac{1}{\cancel{(z-\alpha)}(z-\beta)}$$

$$= \frac{1}{\alpha-\beta} = \frac{2+i}{2i\sqrt{11}}$$

$$\therefore I = \cancel{2}\pi\cancel{i} \times \frac{(2+i)}{\cancel{2i}(\sqrt{11})} = \frac{(2+i)\pi}{\sqrt{11}}$$

$\therefore I = \dfrac{2\pi}{\sqrt{11}}$ (Real part)