Engineering Mathematics II

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Preface

I am deeply gratified with the enthusiastic response shown by students and faculty members alike to all my books on Engineering Mathematics and Mathematics in general throughout the country.

The motivation behind writing this book is to meet the requirements of students of first-year undergraduate course on Engineering Mathematics offered to the students of engineering. The contents have been covered in adequate depth for semester II of various universities/ deemed universities across the country. It offers a balanced coverage of both theory and problems. Lucid writing style supported by step-by-step solutions to all problems enhances understanding of the concepts. The book has ample number of solved and unsolved problems of different variety to help students and teachers learning and teaching this subject.

I hope that the book will be received by both the faculty and the students as enthusiastically as my other books on Engineering Mathematics. Critical evaluation and suggestions for the improvement of the book will be highly appreciated and acknowledged.

T VEERARAJAN

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UNIT-1
MATRICES

UNIT 1

Matrices

1.1 INTRODUCTION

In the lower classes, the students have studied a few topics in Elementary Matrix theory. They are assumed to be familiar with the basic definitions and concepts of matrix theory as well as the elementary operations on and properties of matrices. Though the concept of rank of a matrix has been introduced in the lower classes, we briefly recall the definition of rank and working procedure to find the rank of a matrix, as it will be of frequent use in testing the consistency of a system of linear algebraic equations, that will be discussed in the next section.

1.1.1 Rank of a Matrix

Determinant of any square submatrix of a given matrix A is called a minor of A. If the square submatrix is of order r, then the minor also is said to be of order r.

Let A be an $m \times n$ matrix. The rank of A is said to be 'r', if

(i) there is at least one minor of A of order r which does not vanish and
(ii) every minor of A of order $(r + 1)$ and higher order vanishes.

In other words, the rank of a matrix is the largest of the orders of all the non-vanishing minors of that matrix. Rank of a matrix A is denoted by $R(A)$ or $\rho(A)$.

To find the rank of a matrix A**,** we may use the following procedure:

We first consider the highest order minor (or minors) of A. Let their order be r. If any one of them does not vanish, then $\rho(A) = r$. If all of them vanish, we next consider minors of A of next lower order $(r - 1)$ and so on, until we get a non-zero minor. The order of that non-zero minor is $\rho(A)$.

This method involves a lot of computational work and hence requires more time, as we have to evaluate many determinants. An alternative method to find the rank of a matrix A is given below:

Reduce A to any one of the following forms, (called normal forms) by a series of elementary operations on A and then find the order of the unit matrix contained in the normal form of A:

$$[I_r];\ [I_r \mid O];\ \left[\frac{I_r}{O}\right];\ \left[\frac{I_r \mid O}{O \mid O}\right]$$

Here I_r denotes the unit matrix of order r and O is zero matrix.

By an *elementary operation* on a matrix (denoted as E-operation) we mean any one of the following operations or transformations:

(i) Interchange of any two rows (or columns).
(ii) Multiplication of every element of a row (or column) by any non-zero scalar.
(iii) Addition to the elements of any row (or column), the same scalar multiples of corresponding elements of any other row (or column).

Note ☑ The alternative method for finding the rank of a matrix is based on the property that the rank of a matrix is unaltered by elementary operations.

Finally we observe that we need not necessarily reduce a matrix A to the normal form to find its rank. It is enough we reduce A to an equivalent matrix, whose rank can be easily found, by a sequence of elementary operations on A. The methods are illustrated in the worked examples that follow.

1.2 VECTORS

A set of n numbers $x_1, x_2, \ldots, x_n$ written in a particular order (or an ordered set of n numbers) is called an *n-dimensional vector* or a *vector of order n*. The n numbers are called the components or elements of the vector. A vector is denoted by a single letter X or Y etc. The components of a vector may be written in a row as $X = (x_1, x_2, ..., x_n)$ or in a column as $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. These are called respectively row vector and column vector. We note that a row vector of order n is a $1 \times n$ matrix and a column vector of order n is an $n \times 1$ matrix.

1.2.1 Addition of Vectors

The sum of two vectors of the same dimension is obtained by adding the corresponding components.

i.e., if $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$,
then $X + Y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

1.2.2 Scalar Multiplication of a Vector

If k is a scalar and $X = (x_1, x_2, \ldots, x_n)$ is a vector, then the scalar multiple kX is defined as

$$kX = (kx_1, kx_2, \ldots, kx_n).$$

1.2.3 Linear Combination of Vectors

If a vector X can be expressed as $X = k_1X_1 + k_2X_2 + \cdots + k_rX_r$ then X is said to be a linear combination of the vectors $X_1, X_2, \ldots, X_r$.

1.3 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE OF VECTORS

The vectors $X_1, X_2, \ldots, X_r$ are said to be *linearly dependent* if we can find scalars $k_1, k_2, \ldots k_r$, which are not all zero, such that $k_1X_l + k_2X_2 + \cdots + k_rX_r = 0$.

A set of vectors is said to be *linearly independent* if it is not linearly dependent, i.e. the vectors $X_1, X_2, \ldots, X_r$ are linearly independent, if the relation $k_1X_1 + k_2X_2 + \ldots k_rX_r = 0$ is satisfied only when $k_l = k_2 = \cdots = k_r = 0$.

Note ☑ When the vectors $X_1, X_2, \ldots, X_r$ are linearly dependent, then $k_1X_1 + k_2X_2 + \cdots + k_rX_r = 0$, where at least one of the k's is not zero. Let $k_m \neq 0$.

Thus
$$X_m = -\frac{k_1}{k_m}\cdot X_1 - \frac{k_2}{k_m}X_2 - \cdots - \frac{k_r}{k_m}X_r\,.$$

Thus at least one of the given vectors can be expressed as a linear combination of the others.

1.4 METHODS OF TESTING LINEAR DEPENDENCE OR INDEPENDENCE OF A SET OF VECTORS

Method 1 Using the definition directly.

Method 2 We write the given vectors as row vectors and form a matrix. Using elementary row operations on this matrix, we reduce it to echelon form, i.e. the one in which all the elements in the r^{th} column below the r^{th} element are zero each. If the number of non-zero row vectors in the echelon form equals the number of given vectors, then the vectors are linearly independent. Otherwise they are linearly dependent.

Method 3 If there are n vectors, each of dimension n, then the matrix formed as in method (2) will be a square matrix of order n. If the rank of the matrix equals n, then the vectors are linearly independent. Otherwise they are linearly dependent.

1.5 CONSISTENCY OF A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Consider the following system of m linear algebraic equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system can be represented in the matrix form as $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \hdashline a_{m1} & a_{m2}\cdots & \cdots & a_{mn} \end{bmatrix},\; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\; B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix A is called *the coefficient matrix* of the system, X is the matrix of unknowns and B is the matrix of the constants.

If $B \equiv O$, a zero matrix, the system is called a system of *homogeneous linear equations;* otherwise, the system is called a system of *linear non-homogeneous equations.*

The $m \times (n + 1)$ matrix, obtained by appending the column vector B to the coefficient matrix A as the additional last column, is called the *augmented matrix* of the system and is denoted by $[A, B]$ or $[A \mid B]$.

i.e. $$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \hdashline a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

1.5.1 Definitions

A set of values of $x_1, x_2 \ldots, x_n$. which satisfy all the given m equations simultaneously is called a *solution* of the system.

When the system of equations has a solution, it is said to be *consistent.* Otherwise the system is said to be *inconsistent.*

A consistent system may have either only one or infinitely many solutions.
When the system has only one solution, it is called *the unique solution.*

The necessary and sufficient condition for the consistency of a system of linear non-homogeneous equations is provided by a theorem, called Rouches's theorem, which we state below without proof.

1.5.2 Rouche's Theorem

The system of equations $AX = B$ is consistent, if and only if the coefficient matrix A and the augmented matrix $[A, B]$ are of the same rank.

Thus to discuss the consistency of the equations $AX = B$ (m equations in n unknowns), the following procedure is adopted:

We first find $R(A)$ and $R(A, B)$.

(i) If $R(A) \neq R(A, B)$, the equations are inconsistent

(ii) If $R(A) = R(A, B) =$ the number of unknowns n, the equations are consistent and have a unique solution.
In particular, if A is a non-singular (square) matrix, the system $AX = B$ has a unique solution.

(iii) If $R(A) = R(A, B) <$ the number of unknowns n, the equations are consistent and have an infinite number of solutions.

1.5.3 System of Homogeneous Linear Equations

Consider the system of homogeneous linear equations $AX = O$ (m equations in n unknowns)

i.e
$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \text{-----------------} \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = 0 \end{array}$$

This system is always consistent, as $R(A) = R(A, O)$. If the coefficient matrix A is non-singular, the system has a unique solution, namely, $x_1 = x_2 = \cdots = x_n = 0$. This unique solution is called *the trivial solution,* which is not of any importance.

If the coefficient matrix A is singular, i.e. if $|A| = 0$, the system has an infinite number of non-zero or non-trivial solutions.

The method of finding the non-zero solution of a system of homogeneous linear equations is illustrated in the worked examples that follow.

WORKED EXAMPLE 1(a)

Example 1.1 Show that the vectors $X_1 = (1, 1, 2)$, $X_2 = (1, 2, 5)$ and $X_3 = (5, 3, 4)$ are linearly dependent. Also express each vector as a linear combination of the other two.

Method 1

Let $$k_1X_1 + k_2X_2 + k_3X_3 = 0$$

i.e. $$k_1(1, 1, 2) + k_2(1, 2, 5) + k_3(5, 3, 4) = (0, 0, 0)$$

$\therefore$
$$k_1 + k_2 + 5k_3 = 0 \quad (1)$$
$$k_1 + 2k_2 + 3k_3 = 0 \quad (2)$$
$$2k_1 + 5k_2 + 4k_3 = 0 \quad (3)$$

(2) – (1) gives $k_2 - 2k_3 = 0$ or $k_2 = 2k_3$ (4)

Using (4) in (3), $k_1 = -7k_3$ (5)

Taking $k_3 = 1$, we get $k_1 = -7$ and $k_2 = 2$.

Thus $$-7X_1 + 2X_2 + X_3 = 0 \quad (6)$$

$\therefore$ The vectors X_1, X_2, X_3 are linearly dependent.

From (6), we get $$X_1 = \frac{2}{7}X_2 + \frac{1}{7}X_3,$$

$$X_2 = \frac{7}{2}X_1 - \frac{1}{2}X_3 \quad \text{and} \quad X_3 = 7X_1 - 2X_2$$

Method 2

Writing X_1, X_2, X_3 as row vectors, we get

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \left(R_2' = R_2 - R_1,\ R_3' = R_3 - 5R_1\right)$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \left(R_3'' = R_3' + 2R_2'\right)$$

In the echelon form of the matrix, the number of non-zero vectors = 2 (< the number of given vectors).

$\therefore X_1, X_2, X_3$ are linearly dependent.

Now
$$\begin{aligned}0 = R_3'' &= R_3' + 2R_2' \\ &= (R_3 - 5R_1) + 2(R_2 - R_1) \\ &= -7R_1 + 2R_2 + R_3\end{aligned}$$

i.e. $\quad -7X_1 + 2X_2 + X_3 = 0$

As before, $\quad X_1 = \frac{2}{7}X_2 + \frac{1}{7}X_3,\ X_2 = \frac{7}{2}X_1 - \frac{1}{2}X_3 \quad$ and $\quad X_3 = 7X_1 - 2X_2.$

Method 3

$$|A| = 0 \quad \therefore R(A) \neq 3; \quad R(A) = 2$$

$\therefore$ The vectors X_1, X_2, X_3 are linearly dependent.

Example 1.2 Show that the vectors $X_1 = (1, -1, -2, -4)$, $X_2 = (2, 3, -1, -1)$, $X_3 = (3, 1, 3, -2)$ and $X_4 = (6, 3, 0, -7)$ are linearly dependent. Find also the relationship among them.

$$A = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \left(R_2' = R_2 - 2R_1,\ R_3' = R_3 - 3R_1,\ R_4' = R_4 - 6R_1\right)$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \left(R_2'' = \frac{1}{5}R_2';\ R_3'' = R_3';\ R_4'' = R_4'\right)$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \end{bmatrix} = \left(R_3''' = R_3'' - 4R_2'';\ R_4''' = R_4'' - 9R_2''\right)$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(R_4'''' = R_4''' - R_3'''\right)$$

Number of non-zero vectors in echelon form of the matrix $A = 3$.

$\therefore$ The vectors X_1, X_2, X_3, X_4 are linearly dependent.

Now
$$0 = R_4'''' = R_4''' - R_3'''$$
$$= \left(R_4'' - 9R_2''\right) - \left(R_3'' - 4R_2''\right)$$
$$= R_4' - R_3' - R_2'$$
$$= \left(R_4 - 6R_1\right) - \left(R_3 - 3R_1\right) - \left(R_2 - 2R_1\right)$$
$$= -R_1 - R_2 - R_3 + R_4$$

$\therefore$ The relation among X_l, X_2, X_3, X_4 is
$$-X_1 - X_2 - X_3 + X_4 = 0 \quad \text{or} \quad X_1 + X_2 + X_3 - X_4 = 0.$$

Example 1.3 Show that the vectors $X_1 = (2, -2, 1)$, $X_2 = (1, 4, -1)$ and $X_3 = (4, 6, -3)$ are linearly independent.

Method 1

Let $\quad k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$

i.e. $k_1 (2, -2, 1) + k_2 (1, 4, -1) + k_3 (4, 6, -3) = (0, 0, 0)$

$\therefore \qquad 2k_1 + k_2 + 4k_3 = 0 \qquad (1)$

$\qquad -2k_1 + 4k_2 + 6k_3 = 0 \qquad (2)$

$\qquad k_1 - k_2 - 3k_3 = 0 \qquad (3)$

From (1) and (2), $\qquad k_2 + 2k_3 = 0 \qquad (4)$

From (2) and (3), $\qquad k_2 = 0 \qquad (5)$

$\therefore \qquad k_1 = 0 = k_2 = k_3.$

$\therefore$ The vectors X_1, X_2, X_3 are linearly independent.

Method 2

$$A = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 4 & -1 \\ 4 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 \\ 2 & -2 & 1 \\ 4 & 6 & -3 \end{bmatrix} \left(R_1' = R_2\ ;\ R_2' = R_1\right)$$
$$\sim \begin{bmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & -10 & 1 \end{bmatrix} \left(R_2'' = R_2' - 2R_1'\ ;\ R_3'' = R_3' - 4R_1'\right)$$
$$\sim \begin{bmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & 0 & -2 \end{bmatrix} \left(R_3''' = R_3'' - R_2''\right)$$

Number of non-zero vectors in the echelon form of A = number of given vectors, $\therefore X_1, X_2, X_3$ are linearly independent.

Example 1.4 Show that the vectors $X_1 = (1, -1, -1, 3)$, $X_2 = (2, 1, -2, -1)$ and $X_3 = (7, 2, -7, 4)$ are linearly independent.

$$A = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 3 \\ 2 & 1 & -2 & -1 \\ 7 & 2 & -7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 3 & 0 & -7 \\ 0 & 9 & 0 & -17 \end{bmatrix} \left(\begin{aligned} & R_2' = R_2 - 2R_1\ ; \\ & R_3' = R_3 - 7R_1 \end{aligned}\right)$$

$$\sim\begin{bmatrix}1 & -1 & -1 & 3\\0 & 3 & 0 & -7\\0 & 0 & 0 & 4\end{bmatrix}\left(R_3'' = R_3' - 3R_2'\right)$$

Number of non-zero vectors in the echelon form of A = number of given vectors.
$\therefore X_1, X_2, X_3$ are linearly independent.

Example 1.5 Test for the consistency of the following system of equations:

$$\begin{aligned}x_1 + 2x_2 + 3x_3 + 4x_4 &= 5\\6x_1 + 7x_2 + 8x_3 + 9x_4 &= 10\\11x_1 + 12x_2 + 13x_3 + 14x_4 &= 15\\16x_1 + 17x_2 + 18x_3 + 19x_4 &= 20\\21x_1 + 22x_2 + 23x_3 + 24x_4 &= 25\end{aligned}$$

The given equations can be put as

$$\begin{bmatrix}1 & 2 & 3 & 4\\6 & 7 & 8 & 9\\11 & 12 & 13 & 14\\16 & 17 & 18 & 19\\21 & 22 & 23 & 24\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}5\\10\\15\\20\\25\end{bmatrix}$$

i.e. $$AX = B \text{ (say)}$$

Let us find the rank of the augmented matrix $[A, B]$ by reducing it to the normal form

$$[A, B] = \begin{bmatrix}1 & 2 & 3 & 4 & 5\\6 & 7 & 8 & 9 & 10\\11 & 12 & 13 & 14 & 15\\16 & 17 & 18 & 19 & 20\\21 & 22 & 23 & 24 & 25\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 3 & 4 & 5\\5 & 5 & 5 & 5 & 5\\10 & 10 & 10 & 10 & 10\\15 & 15 & 15 & 15 & 15\\20 & 20 & 20 & 20 & 20\end{bmatrix}\begin{pmatrix}R_2 \to R_2 - R_1\\R_3 \to R_3 - R_1\\R_4 \to R_4 - R_1\\R_5 \to R_5 - R_1\end{pmatrix}$$

Note ☑ If two matrices A and B are equivalent, i.e. are of the same rank, it is denoted as $A \sim B$.

$$\sim\begin{bmatrix}1 & 2 & 3 & 4 & 5\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\end{bmatrix}\left(R_2 \to \frac{1}{5}R_2,\ R_3 \to \frac{1}{10}R_3,\ R_4 \to \frac{1}{15}R_4,\ R_5 \to \frac{1}{20}R_5\right)$$

$$\sim\begin{bmatrix}1 & 2 & 3 & 4 & 5\\0 & -1 & -2 & -3 & -4\\0 & -1 & -2 & -3 & -4\\0 & -1 & -2 & -3 & -4\\0 & -1 & -2 & -3 & -4\end{bmatrix}\left(R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_1,\ R_5 \to R_5 - R_1\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -2 & -3 & -4 \end{bmatrix} \begin{matrix} (C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1, \\ C_4 \to C_4 - 4C_1, C_5 \to C_5 - 5C_1) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} [C_2 \to -C_2, C_3 \to C_3 \div (-2), \\ C_4 \to C_4 \div (-3), C_5 \to C_5 \div (-4)] \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (R_3 \to R_3 - R_2, R_4 \to R_4 - R_2, \\ R_5 \to R_5 - R_2) \end{matrix}$$

$$\sim \left[\begin{array}{cc:ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} (C_3 \to C_3 - C_2, C_4 \to C_4 - C_2, \\ C_5 \to C_5 - C_2) \end{matrix}$$

Now $[A, B]$ has been reduced to the normal form $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$

The order of the unit matrix present in the normal form = 2.
Hence the rank of $[A, B] = 2$.
The rank of the coefficient matrix A can be found as 2, in a similar manner.
Thus $R(A) = R[A, B] = 2$
$\therefore$ The given system of equations is consistent and possesses many solutions.

Example 1.6 Test for the consistency of the following system of equations:

$$x_1 - 2x_2 - 3x_3 = 2;\ 3x_1 - 2x_2 = -1;\ -2x_2 - 3x_3 = 2;\ x_2 + 2x_3 = 1.$$

The system can be put as

$$\begin{bmatrix} 1 & -2 & -3 \\ 3 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

i.e. $AX = B$ (say)

$$[A, B] = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 3 & -2 & 0 & -1 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} (R_2 \to R_2 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 9 & -7 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} (C_2 \to C_2 + 2C_1, C_3 \to C_3 + 3C_1, C_4 \to C_4 - 2C_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \end{bmatrix} (R_2 \to R_4,\ R_4 \to R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & -11 \end{bmatrix} (R_3 \to R_3 + 2R_2, R_4 \to R_4 - 4R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & -11 \end{bmatrix} (C_3 \to C_3 - 2C_2, C_4 \to C_4 - C_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -15 \end{bmatrix} (R_4 \to R_4 - R_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -15 \end{bmatrix} (C_4 \to C_4 - 4C_3)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(R_4 \to -\frac{1}{15} R_4\right)$$

$\therefore\ R[A, B] = 4$

But $R(A) \neq 4$, as A is a (4×3) matrix.

In fact $R(A) = 3$, as the value of the minor

$$\begin{vmatrix} 3 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} \neq 0$$

Thus $R(A) \neq R[A, B]$

$\therefore$ The given system is inconsistent.

Example 1.7 Test for the consistency of the following system of equations and solve them, if consistent, by matrix inversion.

$$x - y + z + 1 = 0;\ x - 3y + 4z + 6 = 0;\ 4x + 3y - 2z + 3 = 0;$$
$$7x - 4y + 7z + 16 = 0.$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 4 \\ 4 & 3 & -2 \\ 7 & -4 & 7 \end{bmatrix}$$

$$[A, B] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -3 & 4 & -6 \\ 4 & 3 & -2 & -3 \\ 7 & -4 & 7 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & -2 & 3 & -5 \\ 0 & 7 & -6 & 1 \\ 0 & 3 & 0 & -9 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - 4R_1, \\ R_4 \to R_4 - 7R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -5 \\ 0 & 7 & -6 & 1 \\ 0 & 3 & 0 & -9 \end{bmatrix} (C_2 \to C_2 + C_1,\ C_3 \to C_3 - C_1,\ C_4 \to C_4 + C_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & 7 \\ 0 & -5 & 3 & -2 \\ 0 & -9 & 0 & 3 \end{bmatrix} (R_2 \leftrightarrow R_3 \text{ and then } C_2 \leftrightarrow C_4)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -27 & 33 \\ 0 & 0 & -54 & 66 \end{bmatrix} \begin{matrix} (R_3 \to R_3 + 5R_2,\ R_4 \to R_4 + 9R_2) \\ \text{and then} \\ (C_3 \to C_3 + 6C_2,\ C_4 \to C_4 - 7C_2) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \left(C_3 \to -\frac{1}{27} C_3, C_4 \to \frac{1}{33} C_4 \right)$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} R_4 \to R_4 - 2R_3 \text{ and then} \\ C_4 \to C_4 - C_3 \end{pmatrix}$$

$\therefore$ $R[A, B] = 3$. Also $R(A) = 3$

$\therefore$ The given system is consistent and has a unique solution.

To solve the system, we take any three, say the first three, of the given equations.

i.e. $$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 4 \\ 4 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ -3 \end{bmatrix}$$

i.e. $$AX = B, \text{ say}$$

$\therefore$ $$X = A^{-1} B \qquad (1)$$

Let $$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 4 \\ 4 & 3 & -2 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now A_{11} = co-factor of a_{11} in $|A| = -6$

$A_{12} = 18$; $A_{13} = 15$; $A_{21} = 1$; $A_{22} = -6$; $A_{23} = -7$;

$A_{31} = -1$; $A_{32} = -3$; $A_{33} = -2$.

$\therefore$ $$\text{Adj}(A) = \begin{bmatrix} -6 & 1 & -1 \\ 18 & -6 & -3 \\ 15 & -7 & -2 \end{bmatrix}$$

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} = -9$$

$\therefore$ $$A^{-1} = \frac{1}{|A|} \text{adj } A = -\frac{1}{9} \begin{bmatrix} -6 & 1 & -1 \\ 18 & -6 & -3 \\ 15 & -7 & -2 \end{bmatrix} \qquad (2)$$

Using (2) in (1),

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} -6 & 1 & -1 \\ 18 & -6 & -3 \\ 15 & -7 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ -6 \\ -3 \end{bmatrix}$$

$$= -\frac{1}{9}\begin{bmatrix} 3 \\ 27 \\ 33 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -3 \\ -\frac{11}{3} \end{bmatrix}$$

$\therefore$ Solution of the system is $x = -\frac{1}{3}, y = -3, z = -\frac{11}{3}$

Example 1.8 Test for the consistency of the following system of equations and solve them, if consistent:

$$3x + y + z = 8; -x + y - 2z = -5; x + y + z = 6; -2x + 2y - 3z = -7.$$

Note ☑ As the solution can be found out by any method, when the system is consistent, we may prefer the triangularisation method (also known as Gaussian elimination method) to reduce the augmented matrix $[A, B]$ to an equivalent matrix. Using the equivalent matrix, we can test the consistency of the system and also find the solution easily when it exists. In this method, we use *only elementary row operations* and convert the elements below the principal diagonal of A as zeros.

$$[A, B] = \begin{bmatrix} 3 & 1 & 1 & 8 \\ -1 & 1 & -2 & -5 \\ 1 & 1 & 1 & 6 \\ -2 & 2 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ -1 & 1 & -2 & -5 \\ 3 & 1 & 1 & 8 \\ -2 & 2 & -3 & -7 \end{bmatrix} (R_1 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & -2 & -10 \\ 0 & 4 & -1 & 5 \end{bmatrix} (R_2 \to R_2 + R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 + 2R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 1 & 3 \end{bmatrix} (R_3 \to R_3 + R_2, R_4 \to R_4 - 2R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(R_4 \to R_4 + \frac{1}{3} R_3\right) \tag{1}$$

Now, Determinant of $[A, B]$ = − Determinant of the equivalent matrix = 0. ($\because$ Two rows interchanged in the first operation)

$\therefore$
$$R[A, B] \leq 3$$

Now $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{vmatrix} = -6 \neq 0$

$\therefore$ $R\,[A, B] = R\,(A) = 3$ = the number of unknowns.

$\therefore$ The system is consistent and has a unique solution.

A system of equations equivalent to the given system is also obtained from the equivalent matrix in (1).

The equivalent equations are

$$x + y + z = 6, \qquad 2y - z = 1 \qquad \text{and} \qquad -3z = -9$$

Solving them backwards, we get

$$x = 1, y = 2, z = 3.$$

Example 1.9 Examine if the following system of equations is consistent and find the solution if it exists.

$$x + y + z = 1, 2x - 2y + 3z = 1; x - y + 2z = 5; 3x + y + z = 2.$$

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & 3 & 1 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & -2 & 1 & 4 \\ 0 & -2 & -2 & -1 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 - R_1, \\ R_4 \to R_4 - 3R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & -\frac{5}{2} & -\frac{1}{2} \end{bmatrix} \left(R_3 \to R_3 - \frac{1}{2} R_2, R_4 \to R_4 - \frac{1}{2} R_2 \right)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 22 \end{bmatrix} (R_4 \to R_4 + 5R_3)$$

It is obvious that det $[A, B] = 4$ and det $[A] = 3$

$\therefore$ $$R\,[A, B] \neq R\,[A].$$

$\therefore$ The system is inconsistent.

Note ☑ The last row of the equivalent matrix corresponds to the equation $0 \cdot x + 0 \cdot y + 0 \cdot z = 22$, which is absurd. From this also, we can conclude that the system is inconsistent.

Example 1.10 Solve the following system of equations, if consistent:

$$x + y + z = 3,\ x + y - z = 1;\ 3x + 3y - 5z = 1.$$

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & -1 & 1 \\ 3 & 3 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -8 & -8 \end{bmatrix} (R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_3 \to R_3 - 4R_2)$$

$\therefore$ All the third order determinants vanish

$\therefore$ $$R\,[A, B] \neq 3$$

Consider $\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix}$, which is a minor of both A and $[A, B]$.

The value of this minor $= -2 \neq 0$

$\therefore$ $R\,(A) = R\,[A, B] <$ the number of unknowns.

$\therefore$ The system is consistent with many solutions.

From the first two rows of the equivalent matrix, we have $x + y + z = 3$ and $-2z = -2$

i.e. $$z = 1 \quad \text{and} \quad x + y = 2.$$

$\therefore$ The system has a one parameter family of solutions, namely $x = k,\ y = 2 - k,\ z = 1$, where k is the parameter.

Giving various values for k, we get infinitely many solutions.

Example 1.11 Solve the following system of equations, if consistent:

$$x_1 + 2x_2 - x_3 - 5x_4 = 4;\ x_1 + 3x_2 - 2x_3 - 7x_4 = 5;\ 2x_1 - x_2 + 3x_3 = 3.$$

$$[A, B] = \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \\ 2 & -1 & 3 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & -5 & 5 & 10 & -5 \end{bmatrix} (R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (R_3 \to R_3 + 5R_2)$$

$\therefore$ $R\,[A, B] \neq 3$ ($\because$ the last row contains only zeros)

Similarly $R\,(A) \neq 3$.

Since $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \neq 0$, $R\,(A) = R\,[A, B] = 2 <$ the number of unknowns.

$\therefore$ The given system is consistent with many solutions.

From the first two rows of the equivalent matrix, we have

$$x_1 + 2x_2 - x_3 - 5x_4 = 4 \qquad (1)$$

and $$x_2 - x_3 - 2x_4 = 1 \qquad (2)$$

As there are only 2 equations, we can solve for only 2 unknowns.

Hence the other 2 unknowns are to be treated as parameters.

Taking $x_3 = k$ and $x_4 = k'$, we get

$$x_2 = 1 + k + 2k' \qquad \text{[from (2)]}$$

and $$x_1 = 4 - 2\,(1 + k + 2k') + k + 5k' \qquad \text{[from (1)]}$$

i.e. $$x_1 = 2 - k + k'$$

$\therefore$ The given system possesses a two parameter family of solutions.

Note ☑ From the Examples (10) and (11), we note that the number of parameters in the solution equals the difference between the number of unknowns and the common rank of A and $[A, B]$.

Example 1.12 Find the values of k, for which the equations $x + y + z = 1$, $x + 2y + 3z = k$ and $x + 5y + 9z = k^2$ have a solution. For these values of k, find the solutions also.

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & k \\ 1 & 5 & 9 & k^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & k-1 \\ 0 & 4 & 8 & k^2-1 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & k-1 \\ 0 & 0 & 0 & k^2 - 4k + 3 \end{bmatrix} (R_3 \to R_3 - 4R_2) \qquad (1)$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \therefore\ R(A) = 2$$

If the system possesses a solution, $R\,[A, B]$ must also be 2.

$\therefore$ The last row of the matrix in (1) must contain only zeros.

$\therefore$ $$k^2 - 4k + 3 = 0 \qquad \text{i.e. } k = 1 \text{ or } 3.$$

For these values of k, $R\,(A) = R\,[A, B] = 2 <$ the number of unknowns.

$\therefore$ The given system has many solutions.

Case (i) $k = 1$

The first two rows of (1) give the equivalent equations as

$$x + y + z = 1 \qquad (2)$$

and $$y + 2z = 0 \qquad (3)$$

Puting $z = \lambda$, the one-parameter family of solutions of the given system is

$$x = \lambda + 1,\ y = -\,2\lambda \qquad \text{and} \qquad z = \lambda$$

Case (ii) $k = 3$

The equivalent equations are

$$x + y + z = 1 \qquad (2)$$

and $$y + 2z = 2 \tag{4}$$

Putting $z = \mu$, the one-parameter family of solutions of the given system is

$$x = \mu - 1, y = 2 - 2\mu, z = \mu.$$

Example 1.13 Find the condition satisfied by a, b, c, so that the following system of equations may have a solution:

$$x + 2y - 3z = a;\ 3x - y + 2z = b;\ x - 5y + 8z = c.$$

$$[A, B] = \begin{bmatrix} 1 & 2 & -3 & a \\ 3 & -1 & 2 & b \\ 1 & -5 & 8 & c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & -7 & 11 & c-a \end{bmatrix} (R_2 \to R_2 - 3R_1, R_3 \to R_3 - R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & 0 & 0 & 2a-b+c \end{bmatrix} (R_3 \to R_3 - R_2) \tag{1}$$

$$A \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 11 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore\ R(A) = 2$$

If the given system possesses a solution, $R[A, B] = 2$.

$\therefore$ The last row of (1) should contain only zeros.

$\therefore$ $2a - b + c = 0$. Only when this condition is satisfied by a, b, c, the system will have a solution.

Example 1.14 Find the value of k such that the following system of equations has (i) a unique solution, (ii) many solutions and (iii) no solution.

$$kx + y + z = 1;\ x + ky + z = 1;\ x + y + kz = 1.$$

$\therefore$
$$A = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$$

$$\begin{aligned} |A| &= k(k^2 - 1) + (1 - k) + (1 - k) \\ &= (k - 1)(k^2 + k - 2) \\ &= (k - 1)^2 (k + 2) \end{aligned}$$

$|A| = 0$, when $k = 1$ or $k = -2$

When $k \neq 1$ and $k \neq -2$, $|A| \neq 0$ $\quad \therefore R(A) = 3$

Then the system will have a unique solution.

When $k = 1$, the system reduces to the single equation $x + y + z = 1$.
In this case, $R(A) = R[A, B] = 1$.
$\therefore$ The system will have many solutions.
(i.e. a two parameter family of solutions)
When $k = -2$,

$$[A, B] = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} (R_1 \leftrightarrow R_2)$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & 0 \end{bmatrix} (R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1)$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} (R_3 \to R_3 + R_2)$$

Now

$$\begin{vmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad \therefore\ R(A) < 3$$

$$\begin{vmatrix} 1 & -2 \\ 0 & -3 \end{vmatrix} \neq 0 \quad \therefore R(A) = 2$$

$$\begin{vmatrix} -2 & 1 & 1 \\ -3 & 3 & 3 \\ 0 & 0 & 3 \end{vmatrix} = \text{a minor of } [A, B] \neq 0$$

$\therefore$ $R[A, B] = 3$. Thus $R(A) \neq R[A, B]$.
$\therefore$ The system has no solution.

Example 1.15 Investigate for what values of λ, μ, the equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions.

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix} (R_3 \to R_3 - R_2)$$

$$\therefore\ A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \text{ and } |A| = \lambda - 3$$

If $\lambda \neq 3$, $|A| \neq 0$ $\quad \therefore R(A) = 3$

$\therefore$ When $\lambda \neq 3$ and μ takes any value, the system has a unique solution.

If $\lambda = 3$, $|A| = 0$ and a second order minor of A, i.e. $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \neq 0$

$\therefore R(A) = 2$.

When $\lambda = 3$,
$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \mu - 10 \end{bmatrix} \tag{1}$$

When $\lambda = 3$ and $\mu = 10$, the last row of (1) contains only zeros.

$\therefore \quad R[A, B] \neq 3$ and clearly $R[A, B] = 2$.

Thus, when $\lambda = 3$ and $\mu = 10$, $R(A) = R[A, B] = 2$.

$\therefore$ The system has an infinite number of solutions.

When $\lambda = 3$ and $\mu \neq 10$, a third order minor of $[A, B]$, i.e.

$$\begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 4 \\ 0 & 0 & \mu - 10 \end{vmatrix} = \mu - 10 \neq 0$$

$\therefore \quad R[A, B] = 3$

Thus, when $\lambda = 3$ and $\mu \neq 10$, $R(A) \neq R[A, B]$.

$\therefore$ The given system has no solution.

Example 1.16 Test whether the following system of equations possess a non-trivial solution.

$$x_1 + x_2 + 2x_3 + 3x_4 = 0; \quad 3x_1 + 4x_2 + 7x_3 + 10x_4 = 0;$$
$$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0; \quad 6x_1 + 8x_2 + 13x_3 + 16x_4 = 0.$$

The given system is a homogeneous linear system of the form $AX = 0$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 5R_1, \\ R_4 \to R_4 - 6R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix} (R_3 \to R_3 - 2R_2, R_4 \to R_4 - 2R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} (R_4 \to R_4 - R_3)$$

$\therefore |A| = 4$ i.e. A is non-singular

$$R(A) = R[A, 0] = 4$$

$\therefore$ The system has a unique solution, namely, the trivial solution.

Example 1.17 Find the non-trivial solution of the equations $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 11z = 0$, if it exists.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -10 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 7R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} (R_3 \to R_3 - 2R_2) \qquad (1)$$

$\therefore |A| = 0$ and $\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} \neq 0$ $\quad \therefore R(A) = 2$

$\therefore$ The system has non-trivial solution. From the first two rows of (1), we see that the given equations are equivalent to

$$x + 2y + 3z = 0 \qquad (2)$$

and

$$-2y - 5z = 0 \qquad (3)$$

Putting $z = k$, we get $y = -\frac{5}{2}k$ from (3) and $x = 2k$.

Thus the non-trivial solution is $x = 4k$, $y = -5k$ and $z = 2k$.

Example 1.18 Find the non-trivial solution of the equations $x - y + 2z - 3w = 0$, $3x + 2y - 4z + w = 0$, $5x - 3y + 2z + 6w = 0$, $x - 9y + 14z - 2w = 0$, if it exists.

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 5 & -3 & 2 & 6 \\ 1 & -9 & 14 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -10 & 10 \\ 0 & 2 & -8 & 21 \\ 0 & -8 & 12 & 1 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 5R_1, \\ R_4 \to R_4 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -8 & 21 \\ 0 & -8 & 12 & 1 \end{bmatrix} \left(R_2 \to \frac{1}{5} R_2\right)$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -4 & 17 \\ 0 & 0 & -4 & 17 \end{bmatrix} (R_3 \to R_3 - 2R_2, R_4 \to R_4 + 8R_2)$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -4 & 17 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_4 \to R_4 - R_3)$$

$\therefore$ $|A| = 0$ i.e. $R(A) < 4$

$\therefore$ The system has a non-trivial solution.

The system is equivalent to

$$x - y + 2z - 3w = 0 \quad (1)$$

$$y - 2z + 2w = 0 \quad (2)$$

$$-4z + 17w = 0 \quad (3)$$

Putting $w = 4k$, we get $z = 17k$ from (3), $y = 26k$ from (2) and $x = 4k$.

Thus the non-trivial solution is $x = 4k$, $y = 26k$, $z = 17k$ and $w = 4k$.

Example 1.19 Find the values of λ for which the equations $x + (\lambda + 4)\, y + (4\lambda + 2)z = 0$, $x + 2(\lambda + 1)\, y + (3\lambda + 4)\, z = 0$, $2x + 3\lambda y + (3\lambda + 4)\, z = 0$ have a non-trivial solution. Also find the solution in each case.

$$A = \begin{bmatrix} 1 & \lambda + 4 & 4\lambda + 2 \\ 1 & 2\lambda + 2 & 3\lambda + 4 \\ 2 & 3\lambda & 3\lambda + 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \lambda + 4 & 4\lambda + 2 \\ 0 & \lambda - 2 & -\lambda + 2 \\ 0 & \lambda - 8 & -5\lambda \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - 2R_1 \end{pmatrix} \quad (1)$$

For non-trivial solution, $|A| = 0$

i.e. $$-5\lambda\,(\lambda - 2) - (\lambda - 8)\,(2 - \lambda) = 0$$

i.e. $$-4\lambda^2 + 16 = 0$$

$\therefore$ $$\lambda = \pm\, 2$$

When $\lambda = 2$, the system is equivalent to

$$x + 6y + 10z = 0$$

$$-6y - 10z = 0, \quad \text{from (1)}$$

Putting $z = 3k$, we get $y = -5k$ and $x = 0$

i.e. the solution is $x = 0$, $y = -5k$ and $z = 3k$.

When $\lambda = -2$, the system is equivalent to

$$x + 2y - 6z = 0$$

$$-4y + 4z = 0, \quad \text{from (1)}$$

Putting $z = k$, we get $y = k$ and $x = 4k$.

i.e. the solution is $x = 4k$, $y = k$ and $z = k$.

EXERCISE 1(a)

Part A

(Short Answer Questions)

1. Define the linear dependence of a set of vectors.
2. Define the linear independence of a set of vectors.
3. If a set of vectors is linearly dependent, show that at least one member of the set can be expressed as a linear combination of the other members.
4. Show that the vectors $X_1 = (1, 2)$, $X_2 = (2, 3)$ and $X_3 = (4, 5)$ are linearly dependent.
5. Show that the vectors $X_1 = (0, 1, 2)$, $X_2 = (0, 3, 5)$ and $X_3 = (0, 2, 5)$ are linearly dependent.
6. Express $X_1 = (1, 2)$ as a linear combination of $X_2 = (2, 3)$ and $X_3 = (4, 5)$.
7. Show that the vectors (1, 1, 1), (1, 2, 3) and (2, 3, 8) are linearly independent.
8. Find the value of a if the vectors $(2, -1, 0)$, $(4, 1, 1)$ and $(a, -1, 1)$ are linearly dependent.
9. What do you mean by consistent and inconsistent systems of equations. Give examples.
10. State Rouche's theorem.
11. State the condition for a system of equations in n unknowns to have (i) one solution, (ii) many solutions and (iii) no solution.
12. Give an example of 2 equations in 2 unknowns that are (i) consistent with only one solution and (ii) inconsistent.
13. Give an example of 2 equations in 2 unknowns that are consistent with many solutions.
14. Find the values of a and b, if the equations $2x - 3y = 5$ and $ax + by = -10$ have many solutions.
15. Test if the equations $x + y + z = a$, $2x + y + 3z = b$, $5x + 2y + z = c$ have a unique solution, where a, b, c are not all zero.
16. Find the value of λ, if the equations $x + y - z = 10$, $x - y + 2z = 20$ and $\lambda x - y + 4z = 30$ have a unique solution.
17. If the augmented matrix of a system of equations is equivalent to $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$, find the value of λ, for which the system has a unique solution.
18. If the augmented matrix of a system of equations is equivalent to $\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & \lambda + 1 & \mu - 3 \end{bmatrix}$, find the values of λ and μ for which the system has only one solution.

19. If the augmented matrix of a system of equations is equivalent to $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 5 & 4 & 2 \\ 0 & 0 & \lambda-2 & \mu-3 \end{bmatrix}$, find the values of λ and μ for which the system has many solutions.
20. If the augmented matrix of a system of equations is equivalent to $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & \lambda-8 & \mu-11 \end{bmatrix}$, find the values of λ and μ for which the system has no solution.
21. Do the equations $x - 3y - 8z = 0$, $3x + y = 0$ and $2x + 5y + 6z = 0$ have a non-trivial solution? Why?
22. If the equations $x + 2y + z = 0$, $5x + y - z = 0$ and $x + 5y + \lambda z = 0$ have a non-trivial solution, find the value of λ.
23. Given that the equations $x + 2y - z = 0$, $3x + y - z = 0$ and $2x - y = 0$ have non-trivial solution, find it.

Part B

Show that the following sets of vectors are linearly dependent. Find their relationship in each case:

24. $X_1 = (1, 2, 1)$, $X_2 = (4, 1, 2)$, $X_3 = (6, 5, 4)$, $X_4 = (-3, 8, 1)$.
25. $X_1 = (3, 1, -4)$, $X_2 = (2, 2, -3)$, $X_3 = (0, -4, 1)$, $X_4 = (-4, -4, 6)$
26. $X_1 = (1, 2, -1, 3)$, $X_2 = (0, -2, 1, -1)$, $X_3 = (2, 2, -1, 5)$
27. $X_1 = (1, 0, 4, 3)$, $X_2 = (2, 1, -1, 1)$, $X_3 = (3, 2, -6, -1)$
28. $X_1 = (1, -2, 4, 1)$, $X_2 = (1, 0, 6, -5)$, $X_3 = (2, -3, 9, -1)$ and $X_4 = (2, -5, 7, 5)$.
29. Determine whether the vector $x_5 = (4, 2, 1, 0)$ is a linear combination of the set of vectors $X_1 = (6, -1, 2, 1)$, $X_2 = (1, 7, -3, -2)$, $X_3 = (3, 1, 0, 0)$ and $X_4 = (3, 3, -2, -1)$.

    Show that each of the following sets of vectors is linearly independent.
30. $X_1 = (1, 1, 1)$; $X_2 = (1, 2, 3)$; $X_3 = (2, -1, 1)$.
31. $X_1 = (1, -1, 2, 3)$; $X_2 = (1, 0, -1, 2)$; $X_3 = (1, 1, -4, 0)$
32. $X_1 = (1, 2, -1, 0)$ $X_2 = (1, 3, 1, 2)$; $X_3 = (4, 2, 1, 0)$; $X_4 = (6, 1, 0, 1)$.
33. $X_1 = (1, -2, -3, -2, 1)$; $X_2 = (3, -2, 0, -1, -7)$; $X_3 = (0, 1, 2, 1, -6)$; $X_4 = (0, 2, 2, 1, -5)$.
34. Test for the consistency of the following system of equations:

$$\begin{bmatrix} 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 \\ 15 & 16 & 17 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 14 \\ 19 \end{bmatrix}$$

Test for the consistency of the following systems of equations and solve, if consistent:

35. $2x - 5y + 2z = -3; -x - 3y + 3z = -1; x + y - z = 0; -x + y = 1.$
36. $3x + 5y - 2z = 1; x - y + 4z = 7; -6x - 2y + 5z = 9; 7x - 3y + z = 4.$
37. $2x + 2y + 4z = 6; 3x + 3y + 7z = 10; 5x + 7y + 11z = 17; 6x + 8y + 13z = 16.$

Test for the consistency of the following systems of equations and solve, if consistent:

38. $x + 2y + z = 3; 2x + 3y + 2z = 5; 3x - 5y + 5z = 2; 3x + 9y - z = 4.$
39. $2x + 6y - 3z = 18; 3x - 4y + 7z = 31; 5x + 3y + 3z = 48; 8x - 3y + 2z = 21.$
40. $x + 2y + 3z = 6; 5x - 3y + 2z = 4; 2x + 4y - z = 5; 3x + 2y + 4z = 9.$
41. $x + 2y = 4; 10y + 3z = -2; 2x - 3y - z = 5; 3x + 3y + 2z = 1.$
42. $2x_1 + x_2 + 2x_3 + x_4 = 6; x_1 - x_2 + x_3 + 2x_4 = 6; 4x_1 + 3x_2 + 3x_3 - 3x_4 = -1; 2x_1 + 2x_2 - x_3 + x_4 = 10$
43. $2x + y + 5z + w = 5; x + y + 3z - 4w = -1; 3x + 6y - 2z + w = 8; 2x + 2y + 2z - 3w = 2.$

Test for the consistency of the following systems of equations and solve, if consistent:

44. $x - 3y - 8z = -10; 3x + y = 4z; 2x + 5y + 6z = 13.$
45. $5x + 3y + 7z = 4; 3x + 26y + 2z = 9; 7x + 2y + 10z = 5.$
46. $x - 4y - 3z + 16 = 0; 2x + 7y + 12z = 48; 4x - y + 6z = 16; 5x - 5y + 3z = 0.$
47. $x - 2y + 3w = 1; 2x - 3y + 2z + 5w = 3; 3x - 7y - 2z + 10w = 2.$
48. $x_1 + 2x_2 + 2x_3 - x_4 = 3; x_1 + 2x_2 + 3x_3 + x_4 = 1; 3x_1 + 6x_2 + 8x_3 + x_4 = 5.$
49. Find the values of k, for which the equations $x + y + z = 1, x + 2y + 4z = k$ and $x + 4y + 10z = k^2$ have a solution. For these values of k, find the solutions also.
50. Find the values of λ, for which the equtions $x + 2y + z = 4, 2x - y - z = 3\lambda$ and $4x - 7y - 5z = \lambda^2$ have a solution. For these values of λ, find the solutions also.
51. Find the condition on a, b, c, so that the equations $x + y + z = a, x + 2y + 3z = b, 3x + 5y + 7z = c$ may have a one-parameter family of solutions.
52. Find the value of k for which the equations $kx - 2y + z = 1, x - 2ky + z = -2$ and $x - 2y + kz = 1$ have (i) no solution, (ii) one solution and (iii) many solutions.
53. Investigate for what values of λ, μ the equations $x + y + 2z = 2, 2x - y + 3z = 2$ and $5x - y + \lambda z = \mu$ have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions.
54. Find the values of a and b for which the equations $x + y + 2z = 3, 2x - y + 3z = 4$ and $5x - y + az = b$ have (i) no solution, (ii) a unique solution, (iii) many solutions.
55. Find the non-trivial solution of the equations $x + 2y + z = 0; 5x + y - z = 0$ and $x + 5y + 3z = 0$, if it exists.
56. Find the non-trivial solution of the equations $x + 2y + z + 2w = 0; x + 3y + 2z + 2w = 0; 2x + 4y + 3z + 6w = 0$ and $3x + 7y + 4z + 6w = 0$, if it exists.
57. Find the values of λ for which the equations $3x + y - \lambda z = 0, 4x - 2y - 3z = 0$ and $2\lambda x + 4y + \lambda z = 0$ possess a non-trivial solution. For these values of λ, find the solution also.
58. Find the values of λ for which the equations $(11 - \lambda) x - 4y - 7z = 0, 7x - (\lambda + 2) y - 5z = 0, 10x - 4y - (6 + \lambda) z = 0$ possess a non-trivial solution. For these values of λ, find the solution also.

1.6 EIGENVALUES AND EIGENVECTORS

1.6.1 Definition

Let $A = [a_{ij}]$ be a square matrix of order n. If there exists a *non-zero* column vector X and a scalar λ, such that

$$AX = \lambda X$$

then λ is called an eigenvalue of the matrix A and X is called the eigenvector corresponding to the eigenvalue λ.

To find the eigenvalues and the corresponding eigenvectors of a square matrix A, we proceed as follows:

Let λ be an eigenvalue of A and X be the corresponding eigenvector. Then, by definition,

$AX = \lambda X = \lambda IX$, where I is the unit matrix of order n.

i.e.
$$(A - \lambda I)\, X = 0 \tag{1}$$

i.e.
$$\left\{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \hdashline 0 & 0 & 0 & \dots & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e.
$$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots (a_{nn} - \lambda)\, x_n &= 0 \end{aligned} \tag{2}$$

Equations (2) are a system of homogeneous linear equations in the unknowns x_1, $x_2, \dots, x_n$.

Since $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is to be a non-zero vector,

$x_1, x_2, \dots, x_n$ should not be all zeros. In other words, the solution of the system (2) should be a non-trivial solution.

The condition for the system (2) to have a non-trivial solution is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \tag{3}$$

i.e.
$$|A - \lambda I| = 0 \tag{4}$$

The determinant $|A - \lambda I|$ is a polynomial of degree n in λ and is called *the characteristic polynomial* of A.

The equation $|A - \lambda I| = 0$ or the equation (3) is called *the characteristic equation* of A.

When we solve the characteristic equation, we get n values for λ. These n roots of the characteristic equation are called the *characteristic roots* or *latent roots* or *eigenvalues* of A.

Corresponding to each value of λ, the equations (2) possess a non-zero (non-trivial) solution X. X is called *the invariant vector* or *latent vector* or *eigenvector* of A corresponding to the eigenvalue λ.

Notes ☑

1. Corresponding to an eigenvalue, the non-trivial solution of the system (2) will be a one-parameter family of solutions. Hence the eigenvector corresponding to an eigenvalue is not unique.
2. If all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix A are distinct, then the corresponding eigenvectors are linearly independent.
3. If two or more eigenvalues are equal, then the eigenvectors may be linearly independent or linearly dependent.

1.6.2 Properties of Eigenvalues

1. A square matrix A and its transpose A^T have the same eigenvalues.
   Let $A = (a_{ij})$; $i, j = 1, 2, \ldots, n$.
   The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} \tag{1}$$

   The characteristic polynomial of A^T is

$$|A^T - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} - \lambda & \ldots & a_{n2} \\ \hdashline a_{1n} & a_{2n} & \ldots & a_{nn} - \lambda \end{vmatrix} \tag{2}$$

   Determinant (2) can be obtained by changing rows into columns of determinant (1).

   $\therefore \qquad |A - \lambda I| = |A^T - \lambda I|$

   $\therefore$ The characteristic equations of A and A^T are identical.

   $\therefore$ The eigenvalues of A and A^T are the same.

2. The sum of the eigenvalues of a matrix A is equal to the sum of the principal diagonal elements of A. (The sum of the principal diagonal elements is called *the Trace* of the matrix.)

   The characteristic equation of an n^{th} order matrix A may be written as

$$\lambda^n - D_1\lambda^{n-1} + D_2\lambda^{n-2} - \cdots + (-1)^n D_n = 0, \tag{1}$$

   where D_r is the sum of all the r^{th} order minors of A whose principal diagonals lie along the principal diagonal of A.

(*Note* ☑ $D_n = |A|$). We shall verify the above result for a third order matrix.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The characteristic equation of A is given by

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} = 0 \qquad (2)$$

Expanding (2), the characteristic equation is

$$(a_{11} - \lambda)\left\{\lambda^2 - (a_{22} + a_{33})\lambda + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\}$$

$$- a_{12}\left\{-a_{21}\lambda + \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right\} + a_{13}\left\{a_{31}\lambda + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right\} = 0$$

i.e. $$-\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2$$

$$-\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\}\lambda + |A| = 0$$

i.e. $\lambda^3 - D_1\,\lambda^2 + D_2\,\lambda - D_3 = 0$, using the notation given above.
This result holds good for a matrix of order n.

Note ☑ This form of the characteristic equation provides an alternative method for getting the characteristic equation of a matrix.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A.
$\therefore$ They are the roots of equation (1).

$$\therefore \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_n = \frac{-(-D_1)}{1} = D_1$$
$$= a_{11} + a_{22} + \cdots + a_{nn}$$
$$= \text{Trace of the matrix } A.$$

3. The product of the eigenvalues of a matrix A is equal to $|A|$.
   If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, they are the roots of

$$\lambda^n - D_1\lambda^{n-1} + D_2\lambda^{n-2} - \cdots + (-1)^n D_n = 0.$$

$$\therefore \text{ Product of the roots } = \frac{(-1)^n \cdot (-1)^n\, D_n}{1}$$

i.e. $\lambda_1, \lambda_2 \ldots \lambda_n = D_n = |A|$.

1.6.3 Aliter

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of $|A - \lambda I| = 0$

$\therefore$ $|A - \lambda I| \equiv (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$, since L.S. is a n^{th} degree polynomial in λ whose leading term is $(-1)^n \lambda^n$.

Putting $\lambda = 0$ in the above identity, we get $|A| = (-1)^n (-\lambda_1)(-\lambda_2) \ldots (-\lambda_n)$ i.e. $\lambda_1 \lambda_2 \ldots \lambda_n = |A|$.

1.6.4 Corollary

If $|A| = 0$, i.e. A is a singular matrix, at least one of the eigenvalues of A is zero and conversely.

4. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix A, then
   (i) $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the eigenvalues of the matrix kA, where k is a non-zero scalar.
   (ii) $\lambda_1^p, \lambda_2^p, \ldots, \lambda_n^p$ are the eigenvalues of the matrix A^p, where p is a positive integer.
   (iii) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigenvalues of the inverse matrix A^{-1}, provided $\lambda_r \neq 0$ i.e. A is non-singular.

(i) Let λ_r be an eigenvalue of A and X_r the corresponding eigenvector.
Then, by definition,

$$AX_r = \lambda_r x_r \tag{1}$$

Multiplying both sides of (1) by k,

$$(kA)X_r = (k\lambda_r) X_r \tag{2}$$

From (2), we see that $k\lambda_r$ is an eigenvalue of kA and the corresponding eigenvector is the same as that of λ_r, namely X_r.

(ii) Premultiplying both sides of (1) by A,

$$\begin{aligned} A^2 X_r &= A(AX_r) \\ &= A(\lambda_r X_r) \\ &= \lambda_r (AX_r) \\ &= \lambda_r^2 X_r \end{aligned}$$

Similarly $A^3 X_r = \lambda_r^3 X_r$ and so on.

In general, $A^p X_r = \lambda_r^p X_r$ (3)

From (3), we see that λ_r^p is an eigenvalue of A^p with the corresponding eigenvector equal to X_r, which is the same for λ_r.

(iii) Premultiplying both sides of (1) by A^{-1},

$$A^{-1}(AX_r) = A^{-1}(\lambda_r X_r)$$

i.e. $$X_r = \lambda_r (A^{-1} X_r)$$

$\therefore$ $$A^{-1} X_r = \frac{1}{\lambda_r} X_r \tag{4}$$

From (4), we see that $\frac{1}{\lambda_r}$ is an eigenvalue of A^{-1} with the corresponding eigenvector equal to X_r, which is the same for λ_r.

5. The eigenvalues of a real symmetric matrix (i.e. a symmetric matrix with real elements) are real.

Let λ be an eigenvalue of the real symmetric matrix and X be the corresponding eigenvector.

Then $$AX = \lambda \tag{1}$$

Premultiplying both sides of (1) by $\bar{X}^T$ (the transpose of the conjugate of X), we get

$$\bar{X}^T AX = \lambda \bar{X}^T X \tag{2}$$

Taking the complex conjugate on both sides of (2),

$$X^T \bar{A} \bar{X} = \bar{\lambda} X^T \bar{X} \text{ (assuming that } \lambda \text{ may be complex)}$$

i.e. $$X^T A \bar{X} = \bar{\lambda} X^T \bar{X} \ (\because \bar{A} = A, \text{ as } A \text{ is real}) \tag{3}$$

Taking transpose on both sides of (3),

$$\bar{X}^T A^T X = \bar{\lambda} \bar{X}^T X \qquad \left[\because (AB)^T = B^T A^T\right]$$

i.e. $$\bar{X}^T A X = \bar{\lambda} \bar{X}^T X \qquad \left[\because (A)^T = A, \text{ as } A \text{ is symmetric}\right] \tag{4}$$

From (2) and (4), we get

$$\lambda \bar{X}^T X = \bar{\lambda} \bar{X}^T X$$

i.e. $$(\lambda - \bar{\lambda}) \bar{X}^T X = 0$$

$\bar{X}^T X$ is an 1×1 matrix, i.e. a single element which is positive

$\therefore$ $$\lambda - \bar{\lambda} = 0$$

i.e. λ is real.

Hence all the eigenvalues are real.

6. The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

Note ☑ Two column vectors $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ are said to be orthogonal, if their inner product $(x_1 y_1 + x_2 y_2 + \cdots x_n y_n) = 0$ i.e. if $X^T Y = 0$.

Let λ_1, λ_2 be any two distinct eigenvalues of the real symmetric matrix A and X_1, X_2 be the corresponding eigenvectors respectively.

Then $$AX_1 = \lambda_1 X_1 \tag{1}$$

and $$AX_2 = \lambda_2 X_2 \tag{2}$$

Premultiplying both sides of (1) by, X_2^T we get

$$X_2^T AX_1 = \lambda_1 X_2^T X_1$$

Taking the transpose on both sides,

$$X_1^T AX_2 = \lambda_1 X_1^T X_2 \qquad (\because A^T = A) \tag{3}$$

Premultiplying both sides of (2) by X_1^T, we get

$$X_1^T AX_2 = \lambda_2 X_1^T X_2 \tag{4}$$

From (3) and (4), we have

$$\lambda_1 X_1^T X_2 = \lambda_2 X_1^T X_2$$

i.e. $$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$$

Since $$\lambda_1 \neq \lambda_2, X_1^T X_2 = 0$$

i.e. the eigenvectors X_1 and X_2 are orthogonal.

WORKED EXAMPLE 1(b)

Example 1.1 Given that $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, verify that the eigenvalues of A^2 are the squares of those of A.

Verify also that the respective eigenvectors are the same.

The characteristic equation of A is $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

i.e. $$(5 - \lambda)(2 - \lambda) - 4 = 0$$

i.e. $$\lambda^2 - 7\lambda + 6 = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 6$.

The eigenvector corresponding to any λ is given by $(A - \lambda I) X = 0$

i.e. $$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

When $\lambda = 1$, the eigenvector is given by the equations

$$4x_1 + 4x_2 = 0 \text{ and}$$

$$x_1 + x_2 = 0, \text{ which are one and the same.}$$

Solving, $x_1 = -x_2$. Taking $x_1 = 1, x_2 = -1$.

$\therefore$ The eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

When $\lambda = 6$, the eigenvector is given by

$$-x_1 + 4x_2 = 0$$

and

$$x_1 - 4x_2 = 0$$

Solving, $x_1 = 4x_2$

Taking $x_2 = 1, x_1 = 4$

$\therefore$ The eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Now
$$A^2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}$$

The characteristic equation of A^2 is $\begin{vmatrix} 29-\lambda & 28 \\ 7 & 8-\lambda \end{vmatrix} = 0$

i.e. $(29 - \lambda)(8 - \lambda) - 196 = 0$

i.e. $\lambda^2 - 37\lambda + 36 = 0$

i.e. $(\lambda - 1)(\lambda - 36) = 0$

$\therefore$ The eigenvalues of A^2 are 1 and 36, that are the squares of the eigenvalues of A, namely 1 and 6. When $\lambda = 1$, the eigenvector of A^2 is given by

$$\begin{bmatrix} 28 & 28 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \quad \text{i.e.} \quad 28x_1 + 28x_2 = 0 \quad \text{and} \quad 7x_1 + 7x_2 = 0$$

Solving, $x_1 = -x_2$. Taking $x_1 = 1, x_2 = -1$.

When $\lambda = 36$, the eigenvector of A^2 is given by

$$\begin{bmatrix} -7 & 28 \\ 7 & -28 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \text{ i.e.} -7x_1 + 28x_2 = 0 \quad \text{and} \quad 7x_1 - 28x_2 = 0.$$

Solving, $x_1 = 4x_2$. Taking $x_2 = 1, x_1 = 4$.

Thus the eigenvectors of A^2 are

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, which are the same as the respective eigenvectors of A.

Example 1.2 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

i.e. $(1 - \lambda)\{\lambda^2 - 6\lambda + 4\} - (1 - \lambda - 3) + 3(1 - 15 + 3\lambda) = 0$

i.e. $-\lambda^3 + 7\lambda^2 - 36 = 0$ or $\lambda^3 - 7\lambda^2 + 36 = 0$ (1)

i.e. $(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$ [$\because \lambda = -2$ satisfies (1)]

i.e. $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$

$\therefore$ The eigenvalues of A are $\lambda = -2, 3, 6$.

Case (i) $\lambda = -2$.

The eigenvector is given by

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (2)$$

i.e.
$$x_1 + 7x_2 + x_3 = 0$$
$$3x_1 + x_2 + 3x_3 = 0$$

Solving these equations by the rule of cross-multiplication, we have

$$\frac{x_1}{21-1} = \frac{x_2}{3-3} = \frac{x_3}{1-21}$$

i.e.
$$\frac{x_1}{20} = \frac{x_2}{0} = \frac{x_3}{-20} \qquad (3)$$

Note ☑ To solve for x_1, x_2, x_3, we have taken the equations corresponding to the second and third rows of the matrix in step (2). The proportional values of x_1, x_2, x_3 obtained in step (3) are the co-factors of the elements of the first row of the determinant of the matrix in step (2). This provides an alternative method for finding the eigenvector.

From step (3), $x_1 = k$, $x_2 = 0$ and $x_3 = -k$.

Usually the eigenvector is expressed in terms of the simplest possible numbers, corresponding to $k = 1$ or -1.

$\therefore$ $x_1 = 1, \qquad x_2 = 0, \qquad x_3 = -1$

Thus the eigenvector corresponding to $\lambda = -2$ is

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

Values of x_1, x_2, x_3 are proportional to the co-factors of –2, 1, 3 (elements of the first row i.e. –5, 5, –5.

i.e.
$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \quad \text{or} \quad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore$
$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\text{or} \qquad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Note ☑ Since the eigenvalues of A are distinct, the eigenvectors X_1, X_2, X_3 are linearly independent, as can be seen from the fact that the equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$ is satisfied only when $k_1 = k_2 = k_3 = 0$.

Example 1.3 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is given by

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0, \text{ where}$$

D_1 = the sum of the first order minors of A that lie along the main diagonal of A
$= 0 + 0 + 0$
$= 0$

D_2 = the sum of the second order minors of A whose principal diagonals lie along the principal diagonal of A.

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= -3$

$D_3 = |A| = 2$

Thus the characteristic equation of A is

$$\lambda^3 - 3\lambda - 2 = 0$$

$$\text{i.e.} \qquad (\lambda + 1)^2(\lambda - 2) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = -1, -1, 2$.

Case (i) $\lambda = -1$.

The eigenvector is given by

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

All the three equations reduce to one and the same equation $x_1 + x_2 + x_3 = 0$. There is one equation in three unknowns.

$\therefore$ Two of the unknowns, say, x_1 and x_2 are to be treated as free variables (parameters). Taking $x_1 = 1$ and $x_2 = 0$, we get $x_3 = -1$ and taking $x_1 = 0$ and $x_2 = 1$, we get $x_3 = -1$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 2.$

The eigenvector is given by

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Values of x_1, x_2, x_3 are proportional to the co-factors of elements in the first row.

i.e. $$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

or $$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ $$x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note ☑ Though two of the eigenvalues are equal, the eigenvectors X_1, X_2, X_3 are found to be linearly independent.

Example 1.4 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(\lambda^2-4) + 2(-1-\lambda-1) + 2(3-1+\lambda) = 0$$

i.e. $$(2-\lambda)(\lambda-2)(\lambda+2) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = -2, 2, 2$.

Case (i) $\lambda = -2$

The eigenvector is given by

$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ $\dfrac{x_1}{-8} = \dfrac{x_2}{-2} = \dfrac{x_3}{14}$ (by taking the co-factors of elements of the third row) i.e. $$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7}$$

$\therefore$ $$X_1 = \begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}$$

Case (ii) $\lambda = 2$.

The eigenvector is given by

$$\begin{vmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ $$\frac{x_1}{0} = \frac{x_2}{4} = \frac{x_3}{4} \quad \text{or} \quad \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ $$X_2 = X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Note ☑ Two eigenvalues are equal and the eigenvectors are linearly dependent.

Example 1.5 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

Can you guess the nature of A from the eigenvalues? Verify your answer.

The characteristic equation of A is

$$\begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{vmatrix} = 0$$

i.e. $$(11 - \lambda)(\lambda^2 + 8\lambda - 8) + 4(8 - 7\lambda) - 7(10\lambda - 8) = 0$$

i.e. $$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 0, 1, 2$.

Case (i) $\lambda = 0$.

The eigenvector is given by $$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ $$\frac{x_1}{-8} = \frac{x_2}{-8} = \frac{x_3}{-8}$$

or $$\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{1}$$

$\therefore$ $$X_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Case (ii) $\lambda = 1.$

The eigenvector is given by $\begin{bmatrix}10 & -4 & -7\\7 & -3 & -5\\10 & -4 & -7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$

$\therefore$ $$\frac{x_1}{-1}=\frac{x_2}{1}=\frac{x_3}{-2}$$

$\therefore$ $$X_2=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

Case (iii) $\lambda = 2.$

The eigenvector is given by $\begin{bmatrix}9 & -4 & -7\\7 & -4 & -5\\10 & -4 & -8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$

$\therefore$ $$\frac{x_1}{12}=\frac{x_2}{6}=\frac{x_3}{12}$$

or $$\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{2}$$

$\therefore$ $$X_3=\begin{bmatrix}2\\1\\2\end{bmatrix}$$

Since one of the eigenvalues of A is zero, product of the eigenvalues = $|A|$ = 0, i.e. A is non-singular. It is verified below:

$$\begin{vmatrix}11 & -4 & -7\\7 & -2 & -5\\10 & -4 & -6\end{vmatrix}=11(12-20)+4(-42+50)-7(-28+20)=0.$$

Example 1.6 Verify that the sum of the eigenvalues of A equals the trace of A and that their product equals $|A|$, for the matrix

$$A=\begin{bmatrix}1 & 0 & 0\\0 & 3 & -1\\0 & -1 & 3\end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

i.e. $$(1-\lambda)(\lambda^2 - 6\lambda + 8) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 2, 4$.
Sum of the eigenvalues = 7.
Trace of the matrix = 1 + 3 + 3 = 7
Product of the eigenvalues = 8.

$$|A| = 1 \times (9 - 1) = 8.$$

Hence the properties verified.

Example 1.7 Verify that the eigenvalues of A^2 and A^{-1} are respectively the squares and reciprocals of the eigenvalues of A, given that

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

i.e. $$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 3, 2, 5$.

Now $$A^2 = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{bmatrix}$$

The characteristic equation of A^2 is

$$\begin{vmatrix} 9-\lambda & 5 & 38 \\ 0 & 4-\lambda & 42 \\ 0 & 0 & 25-\lambda \end{vmatrix} = 0$$

i.e. $$(9-\lambda)(4-\lambda)(25-\lambda) = 0$$

$\therefore$ The eigenvalues of A^2 are 9, 4, 25, which are the squares of the eigenvalues of A.

Let $$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A_{11} = Co-factor of $a_{11} = 10$; $A_{12} = 0$; $A_{13} = 0$;

$A_{21} = -5$; $A_{22} = 15$; $A_{23} = 0$; $A_{31} = -2$; $A_{32} = -18$; $A_{33} = 6$

$|A| = 30.$

$$\therefore \qquad A^{-1} = \frac{1}{30}\begin{bmatrix} 10 & -5 & -2 \\ 0 & 15 & -18 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{15} \\ 0 & \frac{1}{2} & -\frac{3}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

The characteristic equation of A^{-1} is

$$\begin{vmatrix} \frac{1}{3}-\lambda & -\frac{1}{6} & -\frac{1}{15} \\ 0 & \frac{1}{2}-\lambda & -\frac{3}{5} \\ 0 & 0 & \frac{1}{5}-\lambda \end{vmatrix} = 0$$

i.e.
$$\left(\frac{1}{3}-\lambda\right)\left(\frac{1}{2}-\lambda\right)\left(\frac{1}{5}-\lambda\right) = 0$$

$\therefore$ The eigenvalues of A^{-1} are $\frac{1}{3}, \frac{1}{2}, \frac{1}{5}$, which are the reciprocals of the eigenvalues of A.

Hence the properties verified.

Example 1.8 Find the eigenvalues and eigenvectors of (adj A), given that the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

i.e.
$$(2-\lambda)^3 - (2-\lambda) = 0$$

i.e.
$$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 2, 3$.

Case (i) $\quad \lambda = 1.$

The eigenvector is given by $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore \qquad X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 2$.

The eigenvector is given by $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

i.e. $-x_3 = 0$ and $-x_1 = 0$

$\therefore$ $x_1 = 0$, $x_3 = 0$ and x_2 is arbitrary. Let $x_2 = 1$

$$\therefore \qquad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

$$\therefore \qquad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The eigenvalues of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$ with the eigenvectors X_1, X_2, X_3.

Now $$\frac{\text{adj } A}{|A|} = A^{-1}$$

i.e. adj $A = |A| \cdot A^{-1} = 6A^{-1}$ ($\because |A| = 6$ for the given matrix A)

$\therefore$ The eigenvalues of (adj A) are equal to 6 times those of A^{-1}, namely, 6, 3, 2. The corresponding eigenvectors are X_1, X_2, X_3 respectively.

Example 1.9 Verify that the eigenvectors of the real symmetric matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

are orthogonal in pairs.

The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

i.e. $$(3-\lambda)(\lambda^2 - 8\lambda + 14) + (\lambda - 3 + 1) + (1 + \lambda - 5) = 0$$

i.e. $$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

i.e. $$(\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 2, 3, 6$.

Case (i) $\lambda = 2$.

The eigenvector is given by $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{-2} \quad \text{or} \quad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$\therefore$ $$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Now

$$X_1^T\ X_2 = \begin{bmatrix}1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$$

$$X_2^T\ X_3 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix} = 0$$

$$X_3^T\ X_1 = \begin{bmatrix}1 & -2 & 1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix} = 0$$

Hence the eigenvectors are orthogonal in pairs.

Example 1.10 Verify that the matrix

$$A = \frac{1}{3}\begin{bmatrix}2 & 2 & 1\\-2 & 1 & 2\\1 & -2 & 2\end{bmatrix}$$

is an orthogonal matrix. Also verify that $\frac{1}{\lambda}$ is an eigenvalue of A, if λ is an eigenvalue and that the eigenvalues of A are of unit modulus.

Note ☑ A square matrix A is said to be orthogonal if $AA^T = A^TA = I$.

Now

$$AA^T = \frac{1}{3}\begin{bmatrix}2 & 2 & 1\\-2 & 1 & 2\\1 & -2 & 2\end{bmatrix} \times \frac{1}{3}\begin{bmatrix}2 & -2 & 1\\2 & 1 & -2\\1 & 2 & 2\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}9 & 0 & 0\\0 & 9 & 0\\0 & 0 & 9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = I$$

Similarly we can prove that $A^TA = I$.
Hence A is an orthogonal matrix.
The characteristic equation of $3A$ is

$$\begin{vmatrix}2-\lambda & 2 & 1\\-2 & 1-\lambda & 2\\1 & -2 & 2-\lambda\end{vmatrix} = 0$$

i.e. $(2 - \lambda)(\lambda^2 - 3\lambda + 6) - 2(2\lambda - 4 - 2) + (4 - 1 + \lambda) = 0$

i.e. $\lambda^3 - 5\lambda^2 + 15\lambda - 27 = 0$

i.e. $(\lambda - 3)(\lambda^2 - 2\lambda + 9) = 0$

$\therefore$ The eigenvalues of $3A$ are given by

$$\lambda = 3 \qquad \text{and} \qquad \lambda = \frac{2 \pm \sqrt{4-36}}{2} = 1 \pm i\,2\sqrt{2}$$

$\therefore$ The eigenvalues of A are

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1 + i\,2\sqrt{2}}{3}, \quad \lambda_3 = \frac{1 - i\,2\sqrt{2}}{3}$$

Now $$\frac{1}{\lambda_1} = 1 = \lambda_1$$

$$\frac{1}{\lambda_2} = \frac{3}{1 + i2\sqrt{2}} = \frac{3(1 - i2\sqrt{2})}{(1 + i\,2\sqrt{2})(1 - i\,2\sqrt{2})} = \frac{1 - i\,2\sqrt{2}}{3} = \lambda_3$$

and similarly $\dfrac{1}{\lambda_3} = \lambda_2$.

Thus, if λ is an eigenvalue of an orthogonal matrix, $\dfrac{1}{\lambda}$ is also an eigenvalue.

Also $|\lambda_1| = |1| = 1$.

$$|\lambda_2| = \left|\frac{1}{3} + \frac{i\,2\sqrt{2}}{3}\right| = \sqrt{\frac{1}{9} + \frac{8}{9}} = 1$$

Similarly, $|\lambda_3| = 1$.

Thus the eigenvalues of an orthogonal matrix are of unit modulus.

EXERCISE 1(b)

Part A

(Short Answer Questions)

1. Define eigenvalues and eigenvectors of a matrix.
2. Prove that A and A^T have the same eigenvalues.
3. Find the eigenvalues of $2A^2$, if $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.
4. Prove that the eigenvalues of $(-3A^{-1})$ are the same as those of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.
5. Find the sum and product of the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$.
6. Find the sum of the squares of the eigenvalues of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

7. Find the sum of the eigenvalues of $2A$, if $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

8. Two eigenvalues of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the third eigenvalue.

9. If the sum of two eigenvalues and trace of a 3×3 matrix A are equal, find the value of $|A|$.

10. Find the eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

11. Find the sum of the eigenvalues of the inverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$.

12. The product of two eigenvalues of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigenvalue.

Part B

13. Verify that the eigenvalues of A^{-1} are the reciprocals of those of A and that the respective eigenvectors are the same with respect to the matrix

$$A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}.$$

14. Show that the eigenvectors of the matrix $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Find the eigenvalues and eigenvectors of the following matrices:

15. $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

16. $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

18. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

22. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$

25. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

    What can you infer about the matrix A from the eigenvalues? Verify your answer.

26. Given that $A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$, verify that the sum and product of the eigenvalues of A are equal to the trace of A and $|A|$ respectively.

27. Verify that the eigenvalues of A^2 and A^{-1} are respectively the squares and reciprocals of the eigenvalues of A, given that $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$.

28. Find the eigenvalues and eigenvectors of (adj A), when $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

29. Verify that the eigenvectors of the real symmetric matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$.

    are orthogonal in pairs.

30. Verify that the matrix $A = \dfrac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is orthogonal and that its eigenvalues are of unit modulus.

1.7 CAYLEY-HAMILTON THEOREM

This theorem is an interesting one that provides an alternative method for finding the inverse of a matrix A. Also any positive integral power of A can be expressed, using this theorem, as a linear combination of those of lower degree. We give below the statement of the theorem without proof:

1.7.1 Statement of the Theorem

Every square matrix satisfies its own characteristic equation.

This means that, if $c_0 \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n = 0$ is the characteristic equation of a square matrix A of order n, then

$$c_0 A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I = 0 \tag{1}$$

Note: ☑ When λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by $c_n I$ to get the result of Cayley-Hamilton theorem, where I is the unit matrix of order n.

Also 0 in the R.S. of (1) is a null matrix of order n.

1.7.2 Corollary

(1) If A is non-singular, we can get A^{-1}, using the theorem, as follows:
Multiplying both sides of (1) by A^{-1} we have

$$c_0 A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I + c_n A^{-1} = 0$$

$$\therefore \qquad A^{-1} = -\frac{1}{c_n}\left(c_0 A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I\right).$$

(2) If we multiply both sides of (1) by A, $c_0 A^{n+1} + c_1 A^n + \cdots + c_{n-1} A^2 + c_n A = 0$

$$\therefore \qquad A^{n+1} = -\frac{1}{c_0}\left(c_1 A^n + c_2 A^{n-1} + \cdots + c_{n+1} A^2 + c_n A\right)$$

Thus higher positive integral powers of A can be computed, if we know powers of A of lower degree.

1.7.3 Similar Matrices

Two matrices A and B are said to be similar, if there exists a non-singular matrix P such that $B = P^{-1} AP$.

When A and B are connected by the relation $B = P^{-1} AP$, B is said to be obtained from A by a similarity transformation.

When B is obtained from A by a similarity transformation, A is also obtained from B by a similarity transformation as explained below:

$$B = P^{-1} AP$$

Premultiplying both sides by P and postmultiplying by P^{-1}, we get

$$PBP^{-1} = PP^{-1} APP^{-1}$$
$$= A$$

Thus $\qquad A = PBP^{-1}$

Now taking $P^{-1} = Q$, we get $A = Q^{-1} BQ$.

1.8 PROPERTY

Two similar matrices have the same eigenvalues.

Let A and B be two similar matrices.

Then, by definition, $B = P^{-1} AP$

$$\therefore \qquad B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1} AP - P^{-1} \lambda IP$$

$$= P^{-1}(A - \lambda I)P$$

$$\therefore \qquad |B - \lambda I| = |P^{-1}|\,|A - \lambda I|\,|P|$$
$$= |A - \lambda I|\,|P^{-1}P|$$
$$= |A - \lambda I|\,|I|$$
$$= |A - \lambda I|$$

Thus A and B have the same characteristic polynomials and hence the same characteristic equations.

$\therefore$ A and B have the same eigenvalues.

1.8.1 Diagonalisation of a Matrix

The process of finding a matrix M such that $M^{-1}AM = D$, where D is a diagonal matrix, is called diagonalisation of the matrix A. As $M^{-1}AM = D$ is a similarity transformation, the matrices A and D are similar and hence A and D have the same eigenvalues.

The eigenvalues of D are its diagonal elements. Thus, if we can find a matrix M such that $M^{-1}AM = D$, D is not any arbitrary diagonal matrix, but it is a diagonal matrix whose diagonal elements are the eigenvalues of A.

The following theorem provides the method of finding M for a given square matrix whose eigenvectors are distinct and hence whose eigenvectors are linearly independent.

1.8.2 Theorem

If A is a square matrix with distinct eigenvalues and M is the matrix whose columns are the eigenvectors of A, then A can be diagonalised by the similarity transformation $M^{-1}AM = D$, where D is the diagonal matrix whose diagonal elements are the eigenvalues of A.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalues of A and $X_1, X_2, \ldots, X_n$ be the corresponding eigenvectors.

Let $M = [X_1, X_2, \ldots, X_n]$, which is an $n \times n$ matrix, called the Modal matrix.

$\therefore$ $AM = [AX_1, AX_2, \ldots, AX_n]$ [*Note* ☑ Each AX_r is a $(n \times 1)$ column vector]

Since X_r is the eigenvector of A corresponding to the eigenvalue λ_r,

$$AX_r = \lambda X_r \quad (r = 1, 2, \ldots n)$$

$$\therefore \qquad AM = [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n]$$

$$= [X_1, X_2, \ldots, X_n]\begin{bmatrix} \lambda_1 & 0 & 0 & -- & 0 \\ 0 & \lambda_2 & 0 & -- & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & -- & \lambda_n \end{bmatrix}$$

$$= MD \tag{1}$$

As $X_1, X_2, \ldots, X_n$ are linearly independent column vectors, M is a non-singular matrix Premultiplying both sides of (1) by M^{-1}, we get $M^{-1}AM = M^{-1}MD = D$.

Note ☑ For this diagonalisation process, A need not necessarily have distinct eigenvalues. Even if two or more eigenvalues of A are equal, the process holds good, provided the eigenvectors of A are linearly independent.

1.9 CALCULATION OF POWERS OF A MATRIX *A*

Assuming A satisfies the conditions of the previous theorem,

$$D = M^{-1} AM$$

$$\therefore \quad A = M\,D\,M^{-1}$$

$$A^2 = (M\,D\,M^{-1})\,(M\,D\,M^{-1})$$

$$= MD(M^{-1}\,M)DM^{-1}$$

$$= MD^2M^{-1}$$

Similarly, $A^3 = MD^3\,M^{-1}$

Extending, $A^k = M\,D^k\,M^{-1}$

$$= M\begin{bmatrix} \lambda_1^k & 0 & 0 & -- & 0 \\ 0 & \lambda_2^k & 0 & -- & 0 \\ - & - & - & -- & - \\ 0 & 0 & 0 & -- & \lambda_n^k \end{bmatrix} M^{-1}$$

1.10 DIAGONALISATION BY ORTHOGONAL TRANSFORMATION OR ORTHOGONAL REDUCTION

If A is a real symmetric matrix, then the eigenvectors of A will be not only linearly independent but also pairwise orthogonal. If we normalise each eigenvector X_r, i.e. divide each element of X_r by the square-root of the sum of the squares of all the elements of X_r and use the normalised eigenvectors of A to form the normalised modal matrix N, then it can be proved that N is an orthogonal matrix. By a property of orthogonal matrix, $N^{-1} = N^T$.

$\therefore$ The similarity transformation $M^{-1}\,A\,M = D$ takes the form $N^T AN = D$.

Transforming A into D by means of the transformation $N^T AN = D$ is known as orthogonal transformation or orthogonal reduction.

Note: ☑ Diagonalisation by orthogonal transformation is possible only for a real symmetric matrix.

WORKED EXAMPLE 1(c)

Example 1.1 Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and also use it to find A^{-1}.

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

i.e. $(1-\lambda)(\lambda^2-3\lambda-4)-3(4-4\lambda-3)+7(8-2+\lambda)=0$

i.e. $\lambda^3-4\lambda^2-20\lambda-35=0$

Cayley-Hamilton theorem states that

$$A^3-4A^2-20A-35\,I=0 \tag{1}$$

which is to be verified.

Now,
$$A^2=\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}=\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}$$

$$A^3=A\cdot A^2=\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}=\begin{bmatrix}135&152&232\\140&163&208\\60&76&111\end{bmatrix}$$

Substituting these values in (1), we get,

$$\text{L.S.}=\begin{bmatrix}135&152&232\\140&163&208\\60&76&111\end{bmatrix}-\begin{bmatrix}80&92&92\\60&88&148\\40&36&56\end{bmatrix}-\begin{bmatrix}20&60&140\\80&40&60\\20&40&20\end{bmatrix}-\begin{bmatrix}35&0&0\\0&35&0\\0&0&35\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$= \text{R.S.}$

Thus Cayley-Hamilton theorem is verified. Premultiplying (1) by A^{-1},

$$A^2-4A-20I-35A^{-1}=0$$

$\therefore$
$$A^{-1}=\frac{1}{35}\left(A^2-4A-20I\right)$$

$$=\frac{1}{35}\left(\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}-\begin{bmatrix}4&12&28\\16&8&12\\4&8&4\end{bmatrix}-\begin{bmatrix}20&0&0\\0&20&0\\0&0&20\end{bmatrix}\right)$$

$$=\frac{1}{35}\begin{bmatrix}-4&11&-5\\-1&-6&25\\6&1&-10\end{bmatrix}$$

Example 1.2 Verify that the matrix $A=\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}$ satisfies its characteristic equation and hence find A^4.

The characteristic equation of A is

$$\begin{vmatrix}2-\lambda&-1&2\\-1&2-\lambda&-1\\1&-1&2-\lambda\end{vmatrix}=0$$

i.e. $(2 - \lambda)(\lambda^2 - 4\lambda + 3) + (\lambda - 2 + 1) + 2(1 - 2 + \lambda) = 0$

i.e. $$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0 \qquad (1)$$

According to Cayley-Hamilton theorem, A satisfies (1), i.e.

$$A^3 - 6A^2 + 8A - 3I = 0 \qquad (2)$$

which is to be verified.

Now $$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

Substituting these values in (2),

$$\text{L.S.} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{R.S.}$$

Thus A satisfies its characteristic equation.

Multiplying both sides of (2) by A, we have,

$$A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$\therefore \quad A^4 = 6A^3 - 8A^2 + 3A \qquad (3)$$

$$= 6(6A^2 - 8A + 3I) - 8A^2 + 3A, \text{ using (2)}$$

$$= 28A^2 - 45A + 18I \qquad (4)$$

A^4 can be computed by using either (3) or (4).

From (4),

$$A^4 = \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

Example 1.3 Use Cayley-Hamilton theorem to find the value of the matrix given by $(A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I)$, if the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

i.e. $(2-\lambda)(\lambda^2 - 3\lambda + 2) + \lambda - 1 = 0$

i.e. $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$\therefore$ $A^3 - 5A^2 + 7A - 3I = 0$, by Cayley-Hamilton theorem (1)

Now the given polynomial in A

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 8A - 2I) + I$

$= 0 + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$, by (1)

$= A^2 + A + I$, again using (1) (2)

Now $$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

Substituting in (2), the given polynomial

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 1.4 Find the eigenvalues of A and hence find A^n (n is a positive integer), given that $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

i.e. $\lambda^2 - 4\lambda - 5 = 0$

$\therefore$ The eigenvalues of A are $\lambda = -1, 5$

When λ^n is divided by $(\lambda^2 - 4\lambda - 5)$, let the quotient be $Q(\lambda)$ and the remainder be $(a\lambda + b)$.

Then $\lambda^n \equiv (\lambda^2 - 4\lambda - 5)\, Q(\lambda) + (a\lambda + b)$ (1)

Put $\lambda = -1$ in (1). $-a + b = (-1)^n$ (2)

Put $\lambda = 5$ in (l). $5a + b = 5^n$ (3)

Solving (2) and (3), we get

$$a = \frac{5^n - (-1)^n}{6} \quad \text{and} \quad b = \frac{5^n + 5(-1)^n}{6}$$

Replacing λ by the matrix A in (1), we have

$$A^n = (A^2 - 4A - 5I)\,Q(A) + aA + bI$$
$$= 0 \times Q(A) + aA + bI \text{ (by Cayley-Hamilton theorem)}$$
$$= \left\{\frac{5^n - (-1)^n}{6}\right\}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \left\{\frac{5^n + 5(-1)^n}{6}\right\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For example, when $n = 3$,

$$A^3 = \left(\frac{125+1}{6}\right)\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \left(\frac{125-5}{6}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 21 & 42 \\ 84 & 63 \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$
$$= \begin{bmatrix} 41 & 42 \\ 84 & 83 \end{bmatrix}$$

Example 1.5 Diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ by similarity transformation and hence find A^4.

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 2 & -7 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{vmatrix} = 0$$

i.e. $(2 - \lambda)(\lambda^2 + 2\lambda - 5) - 2(-6 - 2\lambda + 7) = 0$

i.e. $\lambda^3 - 13\lambda + 12 = 0$

i.e. $(\lambda - 1)(\lambda - 3)(\lambda + 4) = 0$

∴ Eigenvalues of A are $\lambda = 1, 3, -4$.

Case (i) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 2 & -7 \\ 2 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

∴ $$\frac{x_1}{-2} = \frac{x_2}{8} = \frac{x_3}{2}$$

∴ $$X_1 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -1 & 2 & -7 \\ 2 & -2 & 2 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore \qquad \dfrac{x_1}{10} = \dfrac{x_2}{12} = \dfrac{x_3}{2}$

$\therefore \qquad X_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$

Case (iii) $\quad \lambda = -4$.

The eigenvector is given by $\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore \qquad \dfrac{x_1}{3} = \dfrac{x_2}{-2} = \dfrac{x_3}{2}$

$\therefore \qquad X_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

Hence the modal matrix is $M = \begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$

Let $\qquad M \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then the co-factors are given by

$$A_{11} = 14, \quad A_{12} = 10, \quad A_{13} = 2, \quad A_{21} = -7, \quad A_{22} = 5, \quad A_{23} = -6,$$
$$A_{31} = -28, \quad A_{32} = -10, \quad A_{33} = 26.$$

and $\qquad |M| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 70.$

$\therefore \qquad M^{-1} = \dfrac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}$

The required similarity transformation is

$$M^{-1} A\, M = D(1, 3, -4) \tag{1}$$

which is verified as follows:

$$AM = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 15 & -12 \\ -4 & 18 & 8 \\ -1 & 3 & -8 \end{bmatrix}$$

$$M^{-1}\,A\,M = \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}\begin{bmatrix} 1 & 15 & -12 \\ -4 & 18 & 8 \\ -1 & 3 & -8 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 70 & 0 & 0 \\ 0 & 210 & 0 \\ 0 & 0 & -280 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

A^4 is given by $$A^4 = M\,D^4\,M^{-1} \qquad (2)$$

$$D^4\,M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix} \times \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 810 & 405 & -810 \\ 512 & -1536 & 6656 \end{bmatrix}$$

$$M\,D^4\,M^{-1} = \begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix} \times \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 810 & 405 & -810 \\ 512 & -1536 & 6656 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 5600 & -2590 & 15890 \\ 3780 & 5530 & -18060 \\ 1820 & -2660 & 12530 \end{bmatrix}$$

i.e. $$A^4 = \begin{bmatrix} 80 & -37 & 227 \\ 54 & 79 & -258 \\ 26 & -38 & 179 \end{bmatrix}$$

Example 1.6 Find the matrix M that diagonalises the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ by means of a similarity transformation. Verify your answer. The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(\lambda^2 - 5\lambda + 4) - 2(1-\lambda) + (\lambda - 1) = 0$$

i.e. $$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

i.e. $$(\lambda - 1)^2 (\lambda - 5) = 0$$

$\therefore$ The eigenvalues of A are λ = 5, 1, 1.

Case (i) $\lambda = 5$.

The eigenvector is given by $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

All the three equations are one and the same, namely, $x_1 + 2x_2 + x_3 = 0$

Two independent solutions are obtained as follows:

Putting $x_2 = -1$ and $x_3 = 0$, we get $x_1 = 2$

Putting $x_2 = 0$ and $x_3 = -1$, we get $x_1 = 1$

$\therefore$ $$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Hence the modal matrix is

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the co-factors are given by

$A_{11} = 1, \quad A_{12} = 1, \quad A_{13} = 1, \quad A_{21} = 2, \quad A_{22} = -2, \quad A_{23} = 2$

$A_{31} = 1, \quad A_{32} = 1, \quad A_{33} = -3$ and

$$|M| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 4$$

$$\therefore \qquad M^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

The required similarity transformation is

$$M^{-1}\, A\, M = D(5, 1, 1) \tag{1}$$

We shall now verify (1).

$$AM = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 1 \\ 5 & -2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$M^{-1}\, A\, M = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 2 & 1 \\ 5 & -2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= D(5, 1, 1).$$

Example 1.7 Diagonalise the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation. The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(\lambda^2 - 2\lambda - 3) - (-\lambda - 1) - (-\lambda - 1) = 0$$

i.e. $$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

i.e. $$(\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$$

$\therefore$ The eigenvalues of A are 1 = –1, 1, 4.

Case (i) $\lambda = -1$.

The eigenvector is given by $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\therefore \qquad X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{-4} = \frac{x_2}{2} = \frac{x_3}{-2}$$

$$\therefore \qquad X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 4$.

The eigenvector is given by $\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Hence the modal matrix $M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Normalising each column vector of *M*, i.e. dividing each element of the first column by $\sqrt{2}$, that of the second column by $\sqrt{6}$ and that of the third column by $\sqrt{3}$, we get the normalised modal matrix *N*.

Thus $$N = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

The required orthogonal transformation that diagonalises A is

$$N^T A N = D(-1, 1, 4) \tag{1}$$

which is verified below:

$$A N = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

$$N^T \ A N = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= D(-1, 1, 4).$$

Example 1.8 Diagonalise the matrix $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of an orthogonal transformation.

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(6-\lambda)(2-\lambda) - 16(6-\lambda) = 0$$

i.e. $$(6-\lambda)(\lambda^2 - 4\lambda - 12) = 0$$

i.e. $$(6-\lambda)(\lambda-6)(\lambda+2) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = -2, 6, 6$.

Case (i) $\lambda = -2$.

The eigenvector is given by $\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{32} = \frac{x_2}{0} = \frac{x_3}{-32}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

We get only one equation,

i.e. $$x_1 - x_3 = 0 \qquad (1)$$

From this we get, $x_1 = x_3$ and x_2 is arbitrary.

x_2 must be so chosen that X_2 and X_3 are orthogonal among themselves and also each is orthogonal with X_1.

Let us choose X_2 arbitrarily as $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Note ☑ This assumption of X_2 satisfies (1) and x_2 is taken as 0.

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

X_3 is orthogonal to X_1

$\therefore$ $$a - c = 0 \qquad (2)$$

X_3 is orthogonal to X_2

$\therefore$ $$a + c = 0 \qquad (3)$$

Solving (2) and (3), we get $a = c = 0$ and b is arbitrary.

Taking $b=1, X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Note ☑ Had we assumed X_2 in a different form, we should have got a different X_3.

For example, if $X_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, then $X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

The modal matrix is $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

The normalised model matrix is

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

The required orthogonal transformation that diagonalises A is

$$N^T AN = D(-2, 6, 6) \tag{1}$$

which is verified below:

$$AN = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{bmatrix}$$

$$N^T AN = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= D(-2, 6, 6)$$

Note ☑ From the above problem, it is clear that diagonalisation of a real symmetric matrix is possible by orthogonal transformation, even if two or more eigenvalues are equal.

EXERCISE 1(c)

Part A

(Short Answer Questions)

1. State Cayley-Hamilton theorem.
2. Give two uses of Cayley-Hamilton theorem.
3. When are two matrices said to be similar? Give a property of similar matrices.
4. What do you mean by diagonalising a matrix?
5. Explain how you will find A^k, using the similarity transformation $M^{-1} AM = D$.
6. What is the difference between diagonalisation of a matrix by similarity and orthogonal transformations?
7. What type of matrices can be diagonalised using (i) similarity transformation and (ii) orthogonal transformation?
8. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.
9. Use Cayley-Hamilton theorem to find the inverse of $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$.
10. Use Cayley-Hamilton theorem to find A^3, given that $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$.

11. Use Cayley-Hamilton theorem to find $(A^4 - 4A^3 - 5A^2 + A + 2I)$, when $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

12. Find the modal matrix that will diagonalise the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.

Part B

13. Show that the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies its own characteristic equation and hence find A^{-1}.

14. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ and hence find A^{-1}

15. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ and hence find A^{-1}.

16. Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4.

17. Verify that the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ satisfies its a own characteristic equation and hence find A^4.

18. Find A^n, using Cayley-Hamilton theorem, when $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$. Hence find A^4.

19. Find A^n, using Cayley-Hamilton theorem, when $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$. Hence find A^3.

20. Given that $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, compute the value of $(A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 31A - 36I)$, using Caylay-Hamilton theorem.

Diagonalise the following matrices by similarity transformation:

21. $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$; find also the fourth power of this matrix.

23. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ 24. $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ 25. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

26. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Diagonalise the following matrices by orthogonal transformation:

27. $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$ 28. $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ 29. $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

30. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

1.11 QUADRATIC FORMS

A homogeneous polynomial of the second degree in any number of variables is called a quadratic form.

For example, $x_1^2 + 2x_2^2 - 3x_3^2 + 5x_1x_2 - 6x_1x_3 + 4x_2x_3$ is a quadratic form in three variables.

The general form of a quadratic form, denoted by Q in n variables is

$$\begin{aligned} Q = {} & c_{11}x_1^2 + c_{12}x_1x_2 + \cdots + c_{1n}x_1x_n \\ & + c_{21}x_2x_1 + c_{22}x_2^2 + \cdots + c_{2n}x_2x_n \\ & + c_{31}x_3x_1 + c_{32}x_3x_2 + \cdots + c_{3n}x_3x_n \\ & + (\text{- - - - - - - - - - - - - - - -}) \\ & + c_{n1}x_nx_1 + c_{n2}x_nx_2 + \cdots + c_{nn}x_n^2 \end{aligned}$$

i.e.
$$Q = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_ix_j$$

In general, $c_{ij} \neq c_{ji}$. The coefficient of $x_i x_j = c_{ij} + c_{ji}$.

Now if we define $a_{ij} = \frac{1}{2}\left(c_{ij} + c_{ji}\right)$, for all i and j, then $a_{ii} = c_{ii}$, $a_{ij} = a_{ji}$ and $a_{ij} + a_{ji} = 2a_{ij} = c_{ij} + c_{ji}$.

$\therefore Q = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$, where $a_{ij} = a_{ji}$ and hence the matrix $A = [a_{ij}]$ is a symmetric matrix. In matrix notation, the quadratic form Q can be represented as $Q = X^T AX$, where

$$A=[a_{ij}],\ X=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\quad\text{and}\quad X^T=[x_1, x_2, \ldots, x_n]\cdot$$

The symmetric matrix $A=[a_{ij}]=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}$ is called *the matrix of the quadratic form Q*.

Note ☑ To find the symmetric matrix A of a quadratic form, the coefficient of x_i^2 is placed in the a_{ii} position and $\left(\dfrac{1}{2}\times\text{coefficient } x_i\, x_j\right)$ is placed in each of the a_{ij} and a_{ji} positions.

For example, (i) if $Q=2x_1^2-3x_1x_2+4x_2^2$, then

$$A=\begin{bmatrix}2 & -\dfrac{3}{2}\\ -\dfrac{3}{2} & 4\end{bmatrix}$$

(ii) if $Q=x_1^2+3x_2^2+6x_3^2-2x_1x_2+6x_1x_3+5x_2x_3$,

then $A=\begin{bmatrix}1 & -1 & 3\\ -1 & 3 & \dfrac{5}{2}\\ 3 & \dfrac{5}{2} & 6\end{bmatrix}$

Conversely, the quadratic form whose matrix is

$$\begin{bmatrix}3 & \dfrac{1}{2} & 0\\ \dfrac{1}{2} & 0 & 6\\ 0 & 6 & -7\end{bmatrix}\text{ is } Q=3x_1^2-7x_3^2+x_1x_2+12x_2x_3$$

1.11.1 Definitions

If A is the matrix of a quadratic form Q, $|A|$ is called *the determinant* or *modulus of Q*.

The rank r of the matrix A is called *the rank of the quadratic form.*

If $r < n$ (the order of A) or $|A| = 0$ or A is singular, the quadratic form is called *singular*. Otherwise it is non-singular.

1.11.2 Linear Transformation of a Quadratic Form

Let $Q = X^T AX$ be a quadratic form in the n variables $x_1, x_2, \ldots, x_n$.

Consider the transformation $X = PY$, that transforms the variable set $X = [x_1, x_2, \ldots, x_n]^T$ to a new variable set $Y = [y_1, y_2, \ldots, y_n]^T$, where P is a non-singular matrix.

We can easily verify that the transformation $X = PY$ expresses each of the variables $x_1, x_2, \ldots, x_n$ as homogeneous linear expressions in $y_1, y_2, \ldots, y_n$. Hence $X = PY$ is called a non-singular linear trans formation.

By this transformation, $Q = X^T AX$ is transformed to

$$\begin{aligned} Q &= (PY)^T A(PY) \\ &= Y^T \left(P^T AP\right) Y \\ &= Y^T BY, \text{ where } B = P^T AP \end{aligned}$$

Now
$$\begin{aligned} B^T &= \left(P^T AP\right)^T = P^T A^T P \\ &= P^T AP \qquad (\because A \text{ is sysmmetric}) \\ &= B \end{aligned}$$

$\therefore$ B is also a symmetric matrix.

Hence B is the matrix of the quadratic form $Y^T BY$ in the variables $y_1, y_2, \ldots, y_n$. Thus $Y^T B\ Y$ is the linear transform of the quadratic form $X^T AX$ under the linear transformation $X = PY$, where $B = P^T AP$.

1.11.3 Canonical Form of a Quadratic Form

In the linear transformation $X = PY$, if P is chosen such that $B = P^T\ A\ P$ is a diagonal

matrix of the form $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, then the quadratic form Q gets reduced as

$$\begin{aligned} Q &= Y^T BY \\ &= [y_1, y_2, \ldots, y_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$

This form of Q is called *the sum of the squares form of Q* or *the canonical form of Q*.

1.11.4 Orthogonal Reduction of a Quadratic Form to the Canonical Form

If, in the transformation $X = PY$, P is an orthogonal matrix and if $X = PY$ transforms the quadratic form Q to the canonical form then Q is said to be reduced to the canonical form by an orthogonal transformation.

We recall that if A is a real symmetric matrix and N is the normalised modal matrix of A, then N is an orthogonal matrix such that $N^T AN = D$, where D is a diagonal matrix with the eigenvalues of A as diagonal elements.

Hence, to reduce a quadratic form $Q = X^T AX$ to the canonical form by an orthogonal transformation, we may use the linear transformation $X = NY$, where N is the normalised modal matrix of A. By this orthogonal transformation, Q gets transformed into $Y^T DY$, where D is the diagonal matrix with the eigenvalues of A as diagonal elements.

1.11.5 Nature of Quadratic Forms

When the quadratic form $X^T AX$ is reduced to the canonical form, it will contain only r terms, if the rank of A is r.

The terms in the canonical form may be positive, zero or negative.

The number of positive terms in the canonical form is called *the index* (p) of the quadratic form.

The excess of the number of positive terms over the number of negative terms in the canonical form i.e. $p - (r - p) = 2p - r$ is called the *signature(s)* of the quadratic form i.e. $s = 2p - r$.

The quadratic form $Q = X^T A X$ in n variables is said to be

(i) positive definite, if $r = n$ and $p = n$ or if all the eigenvalues of A are positive.
(ii) negative definite, if $r = n$ and $p = 0$ or if all the eigenvalues of A are negative.
(iii) positive semidefinite, if $r < n$ and $p = r$ or if all the eigenvalues of $A \geq 0$ and at least one eigenvalue is zero.
(iv) negative semidefinite, if $r < n$ and $p = 0$ or if all the eigenvalues of $A \leq 0$ and at least one eigenvalue is zero.
(v) indefinite in all other cases or if A has positive as well as negative eigenvalues.

WORKED EXAMPLE 1(d)

Example 1.1 Reduce the quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form by an orthogonal transformation. Also find the rank, index, signature and nature of the quadratic form.

$$\text{Matrix of the Q.F. is } A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

Refer to the worked example (7) in section 1(c).
The eigenvalues of A are $-1, 1, 4$.
The corresponding eigenvectors are $[0, 1, 1]^T$ $[2, -1, 1]^T$ and $[1, 1, -1]^T$ respectively.

The modal matrix $M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

The normalised modal matrix $N = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$

Hence $N^T AN = D\,(-1, 1, 4)$, where D is a diagonal matrix with $-1, 1, 4$ as the principal diagonal elements.

$\therefore$ The orthogonal transformation $X = NY$ will reduce the Q.F. to the canonical form $-y_1^2 + y_2^2 + 4y_3^2$

Rank of the Q.F. = 3.

Index = 2

Signature = 1

Q.F. is indefinite in nature, as the canonical form contains both positive and negative terms.

Example 1.2 Reduce the quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form by orthogonal reduction. Find also the nature of the quadratic form.

Matrix of the Q.F. is $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$

Refer to worked example (8) in section 1(c).

The eigenvalues of A are $-2, 6, 6$.

The corresponding eigenvectors are $[1, 0, -1]^T$, $[1, 0, 1]^T$ and $[0, 1, 0]^T$ respectively.

Note ☑ Though two of the eigenvalues are equal, the eigenvectors have been so chosen that all the three eigenvectors are pairwise orthogonal.

The modal matrix $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

The normalised modal matrix is given by

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Hence $N^T AN$ = Diag (−2, 6, 6)

∴ The orthogonal transformation $X = NY$

i.e.
$$x_1 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$
$$x_2 = y_2$$
$$x_3 = -\frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$

will reduce the given Q.F. to the canonical form $-2y_1^2 + 6y_2^2 + 6y_3^2$.

The Q.F. is indefinite in nature, as the canonical form contains both positive and negative terms.

Example 1.3 Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semidefinite. Give also a non-zero set of values (x_1, x_2, x_3) which makes this quadratic form zero.

Matrix of the Q.F. is $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$

i.e. $$(1-\lambda)\{(2-\lambda)(1-\lambda)-1\}-(1-\lambda)=0$$

i.e. $$(1-\lambda)(\lambda^2-3\lambda)=0$$

∴ The eigenvalues of A are $\lambda = 0, 1, 3$.

When $\lambda = 0$, the elements of the eigenvector are given by $x_1 - x_2 = 0, -x_1 + 2x_2 + x_3 = 0$ and $x_2 + x_3 = 0$.

Solving these equations, $x_1 = 1, x_2 = 1, x_3 = -1$

∴ The eigenvector corresponding to $\lambda = 0$ is

$$[1, 1, -1]^T$$

When $\lambda = 1$, the elements of the eigenvector are given by $-x_2 = 0, -x_1 + x_2 + x_3 = 0$ and $x_2 = 0$.

Solving these equations, $x_1 = 1, x_2 = 0, x_3 = 1$.

∴ When $\lambda = 1$, the eigenvector is

$$[1, 0, 1]^T$$

When $\lambda = 3$, the elements of the eigenvector are given by $-2x_1 - x_2 = 0, -x_1 - x_2 + x_3 = 0$ and $x_2 - 2x_3 = 0$

Solving these equation, $x_1 = -1, x_2 = 2, x_3 = 1$.

∴ When $\lambda = 3$, the eigenvector is $[-1, 2, 1]^T$.

Now the modal matrix is $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

The normalised modal matrix is

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Hence $N^T AN$ = Diag (0, 1, 3)

∴ The orthogonal transformation $X = NY$.

i.e.

$$x_1 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3$$

$$x_3 = -\frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$$

will reduce the given Q.F. to the canonical form $0 \cdot y_1^2 + y_2^2 + 3y_3^2 = y_2^2 + 3y_3^2$.

As the canonical form contains only two terms, both of which are positive, the Q.F. is positive semi-definite.

The canonical form of the Q.F. is zero, when $y_2 = 0$, $y_3 = 0$ and y_1 is arbitrary.

Taking $y_1 = \sqrt{3}$, $y_2 = 0$ and $y_3 = 0$, we get $x_1 = 1$, $x_2 = 1$ and $x_3 = -1$.

These values of x_1, x_2, x_3 make the Q.F. zero.

Example 1.4 Determine the nature of the following quadratic forms without reducing them to canonical forms:

(i) $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

(ii) $5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3\, x_1$

(iii) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1\, x_2 - 8x_2\, x_3 - 4x_3\, x_1$.

Note ☑ We can find the nature of a Q.F. without reducing it to canonical form. The alternative method uses the principal sub-determinants of the matrix of the Q.F., as explained below:

Let $A = (a_{ij})_{n\times n}$ be the matrix of the Q.F.

Let $D_1 = |a_{11}| = a_{11}$, $\quad D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$,

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ etc. and } D_n = |A|$$

$D_1, D_2, D_3, \ldots D_n$ are called the principal sub-determinants or principal minors of A.

(i) The Q.F. is positive definite, if $D_1, D_2, \ldots, D_n$ are all positive i.e. $D_n > 0$ for all n.
(ii) The Q.F. is negative definite, if $D_1, D_3, D_5, \ldots$ are all negative and $D_2, D_4, D_6, \ldots$ are all positive i.e. $(-1)^n D_n > 0$ for all n.
(iii) The Q.F. is positive semidefinite, if $D_n \geq 0$ and least one $D_i = 0$.
(iv) The Q.F. is negative semidefinite, if $(-1)^n \; D_n \geq 0$ and at least one $D_i = 0$.
(v) The Q.F. is indefinite in all other cases.

(i) $Q = x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

Matrix of the Q.F. is $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

Now $\quad D_1 = |1| = 1; \; D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2 \;$;

$$D_3 = 1 \cdot (18 - 1) - 1 \cdot (6 - 2) + 2(1 - 6) = 3.$$

D_1, D_2, D_3 are all positive.

$\therefore$ The Q.F. is positive definite.

(ii) $Q = 5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3x_1$.

$$A = \begin{bmatrix} 5 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 14 \end{bmatrix}$$

Now $D_1 = 5; \; D_2 = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24;$

$$D_3 = |A| = 5 \cdot (70 - 64) - 1 \cdot (14 - 32) - 4 \cdot (-8 + 20)$$
$$= 30 + 18 - 48 = 0$$

D_1 and D_2 are > 0, but $D_3 = 0$

$\therefore$ The Q.F. is positive semidefinite.

(iii) $Q = 2x_1^2 + x_2^2 - 3x_3^2 + 12x_1\, x_2 - 8x_2\, x_3 - 4x_3\, x_1$

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

Now $D_1 = |2| = 2; \; D_2 = \begin{vmatrix} 2 & 6 \\ 6 & 1 \end{vmatrix} = -34;$

$$D_3 = |A| = 2\cdot(-3-16) - 6\cdot(-18-8) - 2(-24+2)$$
$$= -38 + 156 + 44 = 162$$

$\therefore$ The Q.F. is indefinite.

Example 1.5 Reduce the quadratic forms $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 4x_2x_3 + 18x_3x_1$ and $2x_1^2 + 5x_2^2 + 4x_1x_2 + 2x_3x_1$ simultaneously to canonical forms by a real non-singular transformation.

Note ☑ We can reduce two quadratic forms $X^T AX$ and $X^T BX$ to canonical forms simultaneously by the same linear transformation using the following theorem, (stated without proof):

If A and B are two symmetric matrices such that the roots of $|A - \lambda B| = 0$ are all distinct, then there exists a matrix P such that $P^T AP$ and $P^T BP$ are both diagonal matrices.

The procedure to reduce two quadratic forms simultaneously to canonical forms is given below:

(1) Let A and B be the matrices of the two given quadratic forms.
(2) Form the characteristic equation $|A - \lambda B| = 0$ and solve it. Let the eigenvalues (roots of this equation) be $\lambda_1, \lambda_2, \ldots, \lambda_n$.
(3) Find the eigenvectors X_i ($i = 1, 2, \ldots, n$) corresponding to the eigenvalues λ_i, using the equation $(A - \lambda_i B) X_i = 0$.
(4) Construct the matrix P whose column vectors are $X_1, X_2, \ldots, X_n$. Then $X = PY$ is the required linear transformation.
(5) Find $P^T AP$ and $P^T BP$, which will be diagonal matrices.
(6) The quadratic forms corresponding to these diagonal matrices are the required canonical forms.

The matrix of the first quadratic form is

$$A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$$

The matrix of the second quadratic form is

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation is $|A - \lambda B| = 0$

i.e.
$$\begin{vmatrix} 6-2\lambda & 2-2\lambda & 9-\lambda \\ 2-2\lambda & 3-5\lambda & 2 \\ 9-\lambda & 2 & 14 \end{vmatrix} = 0$$

Simplifying, $5\lambda^3 - \lambda^2 - 5\lambda + 1 = 0$

i.e. $(\lambda - 1)(5\lambda - 1)(\lambda + 1) = 0$

$$\therefore \lambda = -1, \frac{1}{5}, 1$$

When $\lambda = -1$, $(A - \lambda B)X = 0$ gives the equations.
$8x_1 + 4x_2 + 10x_3 = 0$; $4x_1 + 8x_2 + 2x_3 = 0$; $10x_1 + 2x_2 + 14x_3 = 0$.

Solving these equations, $\dfrac{x_1}{-72} = \dfrac{x_2}{24} = \dfrac{x_3}{48}$

$$\therefore \qquad X_1 = [-3, 1, 2]^T$$

When $\lambda = \dfrac{1}{5}, (A - \lambda B)\,X = 0$ gives the equations.

$28x_1 + 8x_2 + 44x_3 = 0$; $8x_1 + 10x_2 + 10x_3 = 0$; $44x_1 + 10x_2 + 70x_3 = 0$.

Solving these equations, $\dfrac{x_1}{-360} = \dfrac{x_2}{72} = \dfrac{x_3}{216}$

$$\therefore \qquad X_2 = [-5, 1, 3]^T$$

When $\lambda = 1, (A - \lambda B)X = 0$ gives the equations

$$4x_1 + 8x_3 = 0; \quad -2x_2 + 2x_3 = 0; \quad 8x_1 + 2x_2 + 14x_3 = 0$$

$$\therefore \qquad X_3 = [2, -1, -1]^T.$$

Now $P = [X_1, X_2, X_3] = \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

Now
$$P^T AP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ -1 & -1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the Q.F. $X^T AX$ is reduced to the canonical form $y_1^2 + y_2^2 + y_3^2$.

Now $P^T B P = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & -1 & -3 \\ -5 & -5 & -5 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the Q.F. $X^T B X$ is reduced to the canonical form $-y_1^2 + 5y_2^2 + y_3^2$.
Thus the transformation $X = PY$ reduces both the Q.F.'s to canonical forms.

Note ☑ $X = PY$ is not an orthogonal transformation, but only a linear non-singular transformation.

EXERCISE 1(d)

Part A

(Short answer questions)

1. Define a quadratic form and give an example for the same in three variables:
2. Write down the matrix of the quadratic form $3x_1^2 + 5x_2^2 + 5x_3^2 - 2x_1x_2 + 2x_2x_3 + 6x_3x_1$.
3. Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$.
4. When is a Q.F. said to be singular? What is its rank then?
5. If the Q.F. $X^T AX$ gets transformed to $Y^T BY$ under the transformation $X = PY$, prove that B is a symmetric matrix.
6. What do you mean by canonical form of a quadratic form? State the condition for $X = PY$ to reduce the Q.F. $X^T AX$ into the canonical form.
7. How will you find an orthogonal transformation to reduce a Q.F. $X^T AX$ to the canonical form?
8. Define index and signature of a quadratic form.
9. Find the index and signature of the Q.F. $x_1^2 + 2x_2^2 - 3x_3^2$.
10. State the conditions for a Q.F. to be positive definite and positive semidefinite.

Part B

11. Reduce the quadratic form $2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2$ to canonical form by an orthogonal transformation. Also find the rank, index and signature of the Q.F.
12. Reduce the Q.F. $3x_1^2 - 3x_2^2 - 5x_3^2 - 2x_1x_2 - 6x_2x_3 - 6x_3x_1$ to canonical form by an orthogonal transformation. Also find the rank, index and signature of the Q.F.
13. Reduce the Q.F. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ to canonical form by an orthogonal transformation. Also state its nature.
14. Obtain an orthogonal transformation which will transform the quadratic form $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ into sum of squares form and find also the reduced form.
15. Find an orthogonal transformation which will reduce the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ into the canonical form and hence find its nature.
16. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to the canonical form through an orthogonal transformation and hence show that it is positive definite. Find also a non-zero set of values for x_1, x_2, x_3 that will make the Q.F. zero.
17. Reduce the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ to a canonical form by orthogonal reduction. Find also a set of non-zero values of x_1, x_2, x_3, which will make the Q.F. zero.
18. Reduce the quadratic form $5x_1^2 + 26x_2^2 + 10x_3^2 + 6x_1x_2 + 4x_2x_3 + 14x_3x_1$ to a canonical form by orthogonal reduction. Find also a set of non-zero values of x_1, x_2, x_3, which will make the Q.F. zero.

19. Determine the nature of the following quadratic forms without reducing them to canonical forms:

 (i) $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_2x_3 + 18x_1x_3 + 4x_1x_2$

 (ii) $x_1^2 - 2x_1x_2 + x_2^2 + x_3^2$

 (iii) $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_3x_1$

20. Find the value of λ so that the quadratic form $\lambda(x_1^2 + x_2^2 + x_3^2) + 2x_1x_2 - 2x_2 x_3 + 2x_3x_1$ may be positive definite.
21. Find real non-singular transformations that reduce the following pairs of quadratic forms simultaneously to the canonical forms.

 (i) $6x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_3x_1$ and $5x_1^2 + x_2^2 + 5x_3^2 - 2x_1x_2 + 8x_3x_1$.

 (ii) $3x_1^2 + 3x_2^2 - 3x_3^2 + 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ and $4x_1x_2 + 2x_2x_3 - 2x_3x_1$.

 (iii) $2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_2x_3 - 4x_3x_1$ and $2x_2x_3 - 2x_1x_2 - x_2^2$.

 (iv) $3x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_2 - 4x_2x_3$ and $5x_1^2 + 5x_2^2 + x_3^2 - 8x_1x_2 - 2x_2x_3$.

ANSWERS

Exercise 1(a)

Part A

(6) $X_1 = -\frac{1}{2}X_2 + \frac{3}{2}X_3$
(8) $a = 8$
(12) $x + 2y = 3$ and $2x - y = 1$; $x + 2y = 3$ and $2x + 4y = 5$
(13) $x + 2y = 3$ and $2x + 4y = 6$ (14) $a = -4$, $b = 6$
(15) Have a unique solution. (16) $\lambda \neq 5$
(17) No unique solution for any value of λ.
(18) $\lambda \neq -1$ and μ = any value (19) $\lambda = 2$ and $\mu = 3$
(20) $\lambda = 8$ and $\mu \neq 11$ (21) No, as $|A| \neq 0$
(22) $\lambda = 3$ (23) $x = k, y = 2k, z = 5k$

Part B

(24) $-7X_1 + X_2 + X_3 + X_4 = 0$
(25) $2X_1 - X_2 - X_3 + X_4 = 0$
(26) $2X_1 + X_2 - X_3 = 0$
(27) $X_1 - 2X_2 + X_3 = 0$
(28) $X_1 - X_2 + X_3 - X_4 = 0$
(29) Yes. $X_5 = 2X_1 + X_2 - 3X_3 + 0.X_4$
(34) $R(A) = R[A, B] = 2$; Consistent with many solutions.

(35) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent
(36) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent
(37) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent
(38) Consistent; $x = -1, y = 1, z = 2$
(39) Consistent; $x = 3, y = 5, z = 6$
(40) Consistent; $x = 1, y = 1, z = 1$
(41) Consistent; $x = 2, y = 1, z = -4$
(42) Consistent; $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$
(43) Consistent; $x = 2, y = \frac{1}{5}, z = 0, w = \frac{4}{5}$
(44) Consistent; $x = 2k - 1, y = 3 - 2k, z = k$
(45) Consistent; $x = \frac{7-16k}{11}, y = \frac{k+3}{11}, z = k$
(46) Consistent; $x = \frac{16}{3} - \frac{9}{5}k, y = \frac{16}{3} - \frac{6}{5}k, z = k$
(47) Consistent; $x = 3 - 4k - k', y = 1 - 2k + k', z = k, w = k'$
(48) Consistent; $x_1 = -2k + 5k' + 7, x_2 = k, x_3 = -2k' - 2, x_4 = k'$
(49) $k = 1, 2$: When $k = 1, x = 2\lambda + 1, y = -3\lambda, z = \lambda$
When $k = 2, x = 2\mu, y = 1 - 3\mu, z = \mu$
(50) $\lambda = 1, 8$: When $\lambda = 1, x = k + 2, y = 1 - 3k, z = 5k$
When $\lambda = 8, x = \frac{1}{5}(k+52), \quad y = -\frac{1}{5}(3k+16), z = k$
(51) a + $2b - c = 0$
(52) No solution, when $k = 1$; one solution, when $k \neq 1$ and -2; many solutions, when $k = -2$.
(53) No solution when $\lambda = 8$; and $\mu \neq 6$; unique solution, when $\lambda \neq 8$ and μ = any value; many solutions when $\lambda = 8$ and $\mu = 6$.
(54) If $a = 8$, $b \neq 11$ no solution; If $a \neq 8$ and b = any value, unique solution; If $a = 8$ and $b = 11$, many solutions
(55) $x = k, y = -2k, z = 3k$
(56) $x = -4k, y = 2k, z = -2k, w = k$
(57) $\lambda = 1, -9$; When $\lambda = 1$, $x = k, y = -k, z = 2k$ and when $\lambda = -9$, $x = 3k, y = 9k, z = -2k$
(58) $\lambda = 0, 1, 2$; When $\lambda = 0$, solution is (k, k, k); When $\lambda = 1$, solution is $(k, -k, 2k)$; When $\lambda = 2$, solution is $(2k, k, 2k)$.

Exercise 1(b)

(3) 2, 50
(5) −2, −1
(6) 38
(7) 36
(8) 5
(9) 0
(10) $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$
(11) $\frac{47}{60}$
(12) 2
(15) 1, 3 −4; $(-2, 1, 4)^T, (2, 1, -2)^T, (1, -3, 13)^T$
(16) $1, \sqrt{5}, -\sqrt{5}; (1, 0, -1)^T, (\sqrt{5}-1, 1, -1)^T, (\sqrt{5}+1, \ -1, 1)^T$
(17) 1, 3, −4; $(-1, 4, 1)^T, (5, 6, 1)^T, (3, -2, 2)^T$

(18) 5, −3, −3; $(1, 2, -1)^T$, $(2, -1, 0)^T$, $(3, 0, 1)^T$
(19) 5, 1, 1, ; $(1, 1, 1)^T$, $(2, -1, 0)^T$, $(1, 0, -1)^T$
(20) 8, 2, 2; $(2, -1, 1)^T$, $(1, 2, 0)^T$, $(1, 0, -2)^T$
(21) 3,.2, 2; $(1, 1, -2)^T$, $(5, 2, -5)^T$
(22) −2, 2, 2 ; $(4, 1, -7)^T$, $(0, 1, 1)^T$
(23) 2, 2, 2; $(1, 0, 0)^T$
(24) 1, 1, 6, 6; $(0, 0, 1, 2)^T$, $(1, -2, 0, 0)^T$, $(0, 0, 2, -1)^T$ and $(2, 1, 0, 0)^T$
(25) 0, 3, 15; $(1, 2, 2)^T$, $(2, 1, -2)^T$, $(2, -2, 1)^T$; A is singular
(26) Eigenvalues are 5, −10, −20; Trace = −25; $|A|$ = 1000
(28) 1, 4, 4; $(1, -1, 1)^T$, $(2, -1, 0)^T$, $(1, 0, -1)^T$
(29) −1, 1, 4; $(0, 1, 1)^T$, $(2, -1, 1)^T$, $(1, 1, -1)^T$

Exercise 1(c)

(9) $\frac{1}{36}\begin{bmatrix} 6 & -3 \\ -2 & 7 \end{bmatrix}$

(10) $\begin{bmatrix} -19 & 57 \\ 38 & 76 \end{bmatrix}$

(11) $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

(12) $M = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$

(13) $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(14) $\frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

(15) $-\frac{1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

(16) $\begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$

(17) $\begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$

(18) $A^n = \left(\frac{6^n - 2^n}{4}\right)\cdot\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} + \left(\frac{3.2^n - 6^n}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$

(19) $A^n = \left(\frac{9^n - 4^n}{5}\right)\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} + \left(\frac{9.4^n - 4.9^n}{5}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 463 & 399 \\ 266 & 330 \end{bmatrix}$

(20) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(21) $D(1, 3, -4);\quad M = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & -3 \\ -4 & -2 & 13 \end{bmatrix}$

(22) $D(1, 2, 3);\quad M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix};\quad A^4 = \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & 160 & 81 \end{bmatrix}$

(23) $D(2, 3, 6); \quad M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

(24) $D(4, -2, -2); \quad M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

(25) $D(8, 2, 2); \quad M = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

(26) $D(2, -1, -1); \quad M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

(27) $D(0, 3, 14); \quad N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$

(28) $D(1, 3, 4); \quad N = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

(29) $D(4, 1, 1); \quad N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

(30) $D(5, -3, -3); \quad N = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$

Exercise 1(d)

(2) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 5 \end{bmatrix}$

(3) $2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_2x_3 - 4x_3x_1$

(4) Singular, when $|A| = 0$; Rank $r < n$

(6) $P^T AP$ must be a diagonal matrix

(9) index = 2 and signature = 1

(11) $N = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}; Q = y_1^2 + 3y_2^2 + 6y_3^2; r = 3;\ p = 3; s = 3$

(12) $N = \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \\ 0 & -\frac{5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & \frac{3}{\sqrt{14}} \end{bmatrix}; Q = 4y_1^2 - y_2^2 - 8y_3^2; r = 3;\ p = 1; s = -1$

(13) $N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}; Q = 8y_1^2 + 2y_2^2 + 2y_3^2;$ Q is positive definite

(14) $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}; Q = 4y_1^2 + y_2^2 + y_3^2$

(15) $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}; Q = 2y_1^2 - y_2^2 - y_3^2;$ Q is indefinite

(16) $N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}; Q = 3y_2^2 + 15y_3^2;\ x_1 = 1,\ x_2 = 2,\ x_3 = 2$

(17) $N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}; Q = 3y_2^2 + 14y_3^2;\ x_1 = 1,\ x_2 = -5,\ x_3 = 4$

(18) $N = \begin{bmatrix} \frac{16}{\sqrt{378}} & -\frac{2}{\sqrt{14}} & \frac{1}{\sqrt{27}} \\ -\frac{1}{\sqrt{378}} & \frac{1}{\sqrt{14}} & \frac{5}{\sqrt{27}} \\ -\frac{11}{\sqrt{378}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{27}} \end{bmatrix}; Q = 14y_2^2 + 27y_3^2;\ x_1 = 16,\ x_2 = -1,\ x_3 = -11$

(19) (i) positive definite; (ii) positive semidefinite; (iii) indefinite

(20) $\lambda > 2$

(21) (i) $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}; Q_1 = -y_1^2 + 4y_2^2 + 2y_3^2; Q_2 = y_1^2 + 4y_2^2 + y_3^2$

(ii) $P = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -2 & 0 & 0 \end{bmatrix}; Q_1 = -16y_1^2 + 4y_2^2 + 8y_3^2; Q_2 = 4y_1^2 - 4y_2^2 + 4y_3^2$

(iii) $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; Q_1 = y_1^2 + y_2^2 + y_3^2; Q_2 = y_2^2 - y_3^2$

(iv) $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}; Q_1 = 2y_1^2 + 4y_2^2 - y_3^2; Q_2 = y_1^2 + 4y_2^2 + y_3^2$

UNIT-2
VECTOR CALCULUS

UNIT **2**

Vector Calculus

2.1 INTRODUCTION

In Vector Algebra we mostly deal with constant vectors, viz. vectors which are constant in magnitude and fixed in direction. In Vector Calculus we deal with variable vectors i.e. vectors which are varying in magnitude or direction or both. Corresponding to each value of scalar variable t, if there exists a value of the vector $\bar{F}$, then $\bar{F}$ is called a *vector function of the scalar variable t* and is denoted as $\bar{F} = \bar{F}(t)$. For example, the position of a particle that moves continuously on a curve in space varies with respect to time t. Hence the position vector $\bar{r}$ of the particle with respect to a fixed point is a function of time t, i.e. $\bar{r} = \bar{r}(t)$.

Also the position vector of the particle varies from point to point. Hence it can also be considered as a function of the point. If the points are specified by their rectangular cartesian co-ordinates (x, y, z), then $\bar{r}$ is a function of the scalar variables x, y, z, i.e. $\bar{r} = \bar{r}\,(x, y, z)$.

A physical quantity, that is a function of the position of a point in space, is called *a scalar point function* or *a vector point function*, according as the quantity is a scalar or vector. Temperature at any point in space and electric potential are examples of scalar point functions. Velocity of a moving particle and gravitational force are examples of vector point functions.

When a point function is defined at every point of a certain region of space, then that region is called a *field*. The field is called a *scalar field* or a *vector field*, according as the point function is a scalar point function or vector point function. In Vector Calculus, though differentiation and integration of vector time functions and vector point functions are dealt with, we will be concerned with the latter only.

2.2 VECTOR DIFFERENTIAL OPERATOR ∇

We now consider an operator ∇ (to be read as 'del') which is useful in defining three quantities known as the gradient, the divergence and the curl, that are useful in engineering applications.

The operator ∇ is defined as

$$\nabla = \bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}$$

∇ is called the vector differential operator, as it behaves like a vector (though not a vector) with $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ as coefficients of $\bar{i}, \bar{j}, \bar{k}$ respectively. When writing $\nabla = \sum \bar{i}\,\frac{\partial}{\partial x}$, it should be noted that $\bar{i}$ and $\frac{\partial}{\partial x}$ are written as the first and second factors respectively.

2.2.1 Gradient of a Scalar Point Function

Let $\phi\,(x, y, z)$ be a scalar point function defined in a certain region of space. Then the vector point function given by

$$\nabla\phi = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right)\phi = \bar{i}\,\frac{\partial\phi}{\partial x} + \bar{j}\,\frac{\partial\phi}{\partial y} + \bar{k}\,\frac{\partial\phi}{\partial z}$$

is defined as *the gradient of* ϕ and shortly denoted as *grad* ϕ.

Note ☑
1. $\nabla\,\phi$ should not be written as $\phi\,\nabla$.
2. When ∇ combines with ϕ, neither $\cdot$ nor $\times$ should be put between ∇ and ϕ.
3. If ϕ is a constant, $\nabla\,\phi = 0$.
4. $\nabla\,(c_1\phi_1 \pm c_2\phi_2) = c_1\,\nabla\phi_1 \pm c_2\,\nabla\,\phi_2$ where c_1 and c_2 are constants and ϕ_1, ϕ_2 are scalar point functions.
5. $\nabla\,(\phi_1\,\phi_2) = \phi_1\,\nabla\,\phi_2 + \phi_2\,\nabla\,\phi_1$.
6. $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\,\nabla\phi_1 - \phi_1\,\nabla\phi_2}{\phi_2^2}$, if $\phi_2 \neq 0$.
7. If $v = f(u)$, then $\nabla\,v = f'(u)\,\nabla\,u$.

2.2.2 Directional Derivative of a Scalar Point Function $\phi\,(x, y, z)$

Let P and Q be two neighbouring points whose position vectors with respect to the origin O be $\bar{r}\,(=\overline{OP})$ and $\bar{r} + \overline{\Delta r}\,(=\overline{OQ})$ respectively, so that $\overline{PQ} = \overline{\Delta r}$ and $PQ = \Delta r$. Let ϕ and $\phi + \Delta\phi$ be the values of a scalar point function ϕ at the points P and Q respectively.

Then $\frac{\mathrm{d}\phi}{\mathrm{d}r} = \lim\limits_{\Delta r \to 0}\left(\frac{\Delta\phi}{\Delta r}\right)$ is called the directional derivative of ϕ in the direction OP.

i.e. $\frac{\mathrm{d}\phi}{\mathrm{d}r}$ gives the rate of change of ϕ with respect to the distance measured in the direction of $\bar{r}$.

In particular, $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ are the directional derivatives of ϕ at $P(x, y, z)$ in the directions of the co-ordinate axes.

2.2.3 Gradient as a Directional Derivative

Let $\phi\,(x, y, z)$ be a scalar point function. Then $\phi\,(x, y, z) = c$ represents, for various values of c, a family of surfaces, called *the level surfaces of the function* ϕ.

Consider two level surfaces of ϕ, namely $\phi = c_1$ and $\phi = c_2$ passing through two neighbouring points $P\,(\bar{r})$ and $Q\,(\bar{r}+\Delta\bar{r})$. [Refer to Fig. 2.1]

Let the normal at P to the level surface $\phi = c_1$ meet $\phi = c_2$ at R.

Clearly the least distance between the two surfaces = $PR = \Delta n$.

If $Q\hat{P}R = \theta$, then, from the figure PQR, which is almost a right angled triangle,

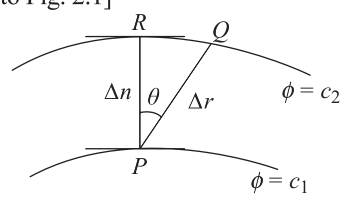


Fig. 2.1

$$\Delta n = \Delta r \cos\theta \qquad (1)$$

If $\hat{n}$ is the unit vector along PR, i.e. in the direction of outward drawn normal at P to the surface $\phi = c_1$, then (1) can be written as

$$\Delta n = \hat{n}\cdot\Delta\bar{r}, \qquad \text{where } \Delta\bar{r} = \overline{PQ}.$$

or
$$dn = \hat{n}\cdot d\bar{r} \qquad (2)$$

$\therefore$
$$d\phi = \frac{d\phi}{dn}\,dn$$

$$= \frac{d\phi}{dn}\,\hat{n}\cdot d\bar{r} \qquad \text{[by using (2)]} \qquad (3)$$

Also
$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{[Refer to Chapter 4 of Part I]}$$

$$= \left(\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}\right)\cdot(dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$= \nabla\phi\cdot d\bar{r} \qquad (4)$$

From (3) and (4),

$$\nabla\phi\cdot d\bar{r} = \frac{d\phi}{dn}\,\hat{n}.d\bar{r}$$

Since $\overline{PQ} = \Delta\bar{r}$ (or $d\,\bar{r}$) is arbitrary,

$$\nabla\phi = \frac{d\phi}{dn}\,\hat{n} \qquad (5)$$

From (1),
$$\frac{d\phi}{dn} = \frac{d\phi}{dr\,(\cos\theta)}$$

i.e. $$\frac{d\phi}{dr} = \cos\theta \frac{d\phi}{dn}$$

i.e. $$\frac{d\phi}{dr} \le \frac{d\phi}{dn} \quad [\because \cos\theta \le 1]$$

i.e. the maximum directional derivative of ϕ is $\frac{d\phi}{dn}$, that is the directional derivative ϕ in the direction of $\hat{n}$.

Thus, from (5), we get the following interpretation of $\nabla\phi$:

$\nabla\phi$ is a vector whose magnitude is the greatest directional derivative of ϕ and whose direction is that of the outward drawn normal to the level surface $\phi = c$.

WORKED EXAMPLE 2(a)

Example 2.1 Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point P (1, −2, −1), (i) that is maximum, (ii) in the direction of PQ, where Q is (3, −3, −2).

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y}\overline{j} + \frac{\partial\phi}{\partial z}\overline{k}$$

$$= (2xyz + 4z^2)\,\overline{i} + x^2z\,\overline{j} + (x^2y + 8xz)\,\overline{k}$$

$$\therefore \quad (\nabla\phi)_{(1,-2,-1)} = 8\overline{i} - \overline{j} - 10\overline{k}$$

The magnitude of $(\nabla\phi)_P$ is the greatest directional derivative of ϕ at P.

Thus the maximum directional derivative of ϕ at $(1, -2, -1) = \sqrt{64+1+100} = \sqrt{165}$ units.

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 2\overline{i} - \overline{j} - \overline{k}$$

Directional derivative of ϕ in the direction of $\overline{PQ}$ = Component (or projection) of $\nabla\phi$ along $\overline{PQ}$.

$$= \frac{\nabla\phi\cdot\overline{PQ}}{|\overline{PQ}|}$$

$$= \frac{(8\overline{i} - \overline{j} - 10\overline{k})\cdot(2\overline{i} - \overline{j} - \overline{k})}{\sqrt{4+1+1}}$$

$$= \frac{27}{\sqrt{6}} \text{ units.}$$

Example 2.2 Find the unit normal to the surface $x^3 - xyz + z^3 = 1$ at the point (1, 1, 1).

Note ☑ Unit normal to a surface $\phi = c$ at a point means the unit vector $\hat{n}$ in the direction of the outward drawn normal to the surface at the given point.

The given equation of the surface $x^3 - xyz + z^3 = 1$ is taken as $\phi(x, y, z) = c$.

$\therefore$ $$\phi = x^3 - xyz + z^3$$

$\nabla\phi$ is a vector acting in the direction of the outward drawn normal to the surface $\phi = c$.

Now $$\nabla\phi = (3x^2 - yz)\bar{i} - xz\bar{j} + (3z^2 - xy)\bar{k}$$

$$(\nabla\phi)_{(1,1,1)} = 2\bar{i} - \bar{j} + 2\bar{k} = \bar{n} \quad \text{(= a vector in the direction of the normal)}$$

$\therefore$ $$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$= \frac{1}{3}(2\bar{i} - \bar{j} + 2\bar{k})$$

Example 2.3 Find the directional derivative of the function $\phi = xy^2 + yz^2$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point $(-1, 2, 1)$.

The equation of the surface $x \log z - y^2 + 4 = 0$ is identified with $\psi(x, y, z) = c$.

$\therefore$ $$\psi(x, y, z) = x \log z - y^2 \text{ and } c = -4.$$

The direction of the normal to this surface is the same as that of $\nabla\psi$.

Now $$\nabla\psi = (\log z)\bar{i} - 2y\bar{j} + \frac{x}{z}\bar{k}$$

$\therefore$ $$(\nabla\psi)_{(-1,2,1)} = -4\bar{j} - \bar{k} = \bar{b} \text{ (say)}$$

$$\phi = xy^2 + yz^3$$

$$\nabla\phi = y^2\bar{i} + (2xy + z^3)\bar{j} + 3yz^2\bar{k}$$

$$(\nabla\phi)_{(2,-1,1)} = \bar{i} - 3\bar{j} - 3\bar{k}$$

Directional derivative of ϕ in the direction of $\bar{b}$

$$= \frac{\nabla\phi \cdot \bar{b}}{|\bar{b}|}$$

$$= \frac{(\bar{i} - 3\bar{j} - 3\bar{k}) \cdot (-4\bar{j} - \bar{k})}{\sqrt{16+1}}$$

$$= \frac{15}{\sqrt{17}} \text{ units.}$$

Example 2.4 Find the angle between the normals to the surface $xy = z^2$ at the points $(-2, -2, 2)$ and $(1, 9, -3)$.

Angle between the two normal lines can be found out as the angle between the vectors acting along the normal lines.

Identifying the equation $xy = z^2$ with $\phi(x, y, z) = c$, we get $\phi = xy - z^2$ and $c = 0$.

$$\nabla\phi = y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$(\nabla\phi)_{(-2,-2,2)} = -2\bar{i} - 2\bar{j} - 4\bar{k} = \bar{n}_1 \text{ (say)}$$

$$(\nabla\phi)_{(1,9,-3)} = 9\bar{i} + \bar{j} + 6\bar{k} = \bar{n}_2 \text{ (say)}$$

$\bar{n}_1$ and $\bar{n}_2$ are vectors acting along the normals to the surface at the given points.

$\therefore$ If θ is the required angle,

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \frac{-44}{\sqrt{24}\cdot\sqrt{118}} = -\frac{11}{\sqrt{177}}$$

$\therefore$ $$\theta = \cos^{-1}\left\{-\frac{11}{\sqrt{177}}\right\}$$

Example 2.5 Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at the point (6, 4, 3).

Angle between two surfaces at a point of intersection is defined as the angle between the respective normals at the point of intersection.

Identifying $x^2 - y^2 - z^2 = 11$ with $\phi = c$,

we have $\phi = x^2 - y^2 - z^2$ and $c = 11$.

$\therefore$ $$\nabla\phi = 2x\bar{i} - 2y\bar{j} - 2z\bar{k}$$

$$(\nabla\phi)_{(6,4,3)} = 12\bar{i} - 8\bar{j} - 6\bar{k} = \bar{n}_1$$

Identifying $xy + yz - zx = 18$ with $\psi = c'$,

we have $\psi = xy + yz - zx$ and $c' = 18$

$$\nabla\psi = (y - z)\,\bar{i} + (z + x)\bar{j} + (y - x)\bar{k}$$

$\therefore$ $$\nabla\psi_{\ (6,4,3)} = \bar{i} + 9\bar{j} - 2\bar{k}$$

If θ is the angle between the surfaces at (6, 4, 3), then

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$= \frac{-48}{\sqrt{244}\sqrt{86}} = -\frac{24}{\sqrt{61\times 86}}$$

$\therefore$ $$\theta = \cos^{-1}\left\{\frac{-24}{\sqrt{5246}}\right\}$$

Example 2.6 Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point (1, −1, 2).

Identifying $2xz^2 - 3\,xy - 4x = 7$ with $\phi = c$,

we have $\phi = 2xz^2 - 3xy - 4x$ and $c = 7$.

$\therefore$ $$\nabla\phi = (2z^2 - 3y - 4)\bar{i} - 3x\bar{j} + 4xz\bar{k}$$

$$(\nabla\phi)_{(1,-1,2)} = 7\bar{i} - 3\bar{j} + 8\bar{k}$$

$(\nabla\phi)_{(1,-1,2)}$ is a vector in the direction of the normal to the surface $\phi = c$.

$\therefore$ D.R.'s of the normal to the surface ($\phi = c$) at the point (1,−1, 2) are (7, −3, 8).

Now the tangent plane is the plane passing through the point (1, −1, 2) and having the line whose D.R.'s are (7, − 3, 8) as a normal.

$\therefore$ Equation of the tangent plane is

$$7(x-1)-3(y+1)+8(z-2)=0$$

i.e. $$7x-3y+8z-26=0.$$

Example 2.7 Find the constants a and b, so that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ may cut orthogonally at the point (1,−1, 2).

Two surfaces are said to cut orthogonally at a point of intersection, if the respective normals at that point are perpendicular.

Identifying $5x^2 - 2yz - 9x = 0$ with $\phi_1 = c$,

we have $$\nabla\phi_1 = (10x-9)\bar{i} - 2z\bar{j} - 2y\bar{k}$$

$\therefore$ $$(\nabla\phi_1)_{(1,-1,2)} = \bar{i} - 4\bar{j} + 2\bar{k} = \bar{n}_1 \text{ (say)}$$

Identifying $ax^2y + bz^3 = 4$ with $\phi_2 = c'$.

we have $$(\nabla\phi_2) = 2axy\bar{i} + ax^2\bar{j} + 3bz^2\bar{k}$$

$\therefore$ $$(\nabla\phi_2)_{(1,-1,2)} = -2a\bar{i} + a\bar{j} + 12b\bar{k} = \bar{n}_2 \text{ (say)}$$

Since the surfaces cut orthogonally, $\bar{n}_1 \perp \bar{n}_2$.

i.e. $$\bar{n}_1 \cdot \bar{n}_2 = 0$$

i.e. $$-6a + 24b = 0$$

i.e. $$-a + 4b = 0 \tag{1}$$

Since (1,−1, 2) is a point of intersection of the two surfaces, it lies on $ax^2y + bz^3 = 4$

$\therefore$ $$-a + 8b = 4 \tag{2}$$

Solving (1) and (2), we get $a = 4$ and $b = 1$.

Example 2.8 If $\bar{r}$ is the position vector of the point (x, y, z), $\bar{a}$ is a constant vector and $\phi = x^2 + y^2 + z^2$, prove that (i) grad $(\bar{r}\cdot\bar{a}) = \bar{a}$ and (ii) $\bar{r}\cdot \text{grad } \phi = 2\phi$.

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

Let $$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

$\therefore$ $$\bar{r}\cdot\bar{a} = a_1x + a_2y + a_3z$$

$\therefore$ $$\text{grad }(\bar{r}\cdot\bar{a}) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

$$\phi = x^2 + y^2 + z^2$$

$\therefore$ $$\text{grad } \phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$\therefore$ $$\bar{r}\cdot \text{grad } \phi = 2(x^2 + y^2 + z^2) = 2\phi.$$

Example 2.9 If $\bar{r}$ is the position vector of the point (x, y, z) with respect to the origin, prove that $\nabla(r^n) = nr^{n-2}\bar{r}$.

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$\therefore$
$$r^2 = |\bar{r}^2| = x^2 + y^2 + z^2 \tag{1}$$

$\therefore$
$$\nabla(r^n) = \frac{\partial}{\partial x}(r^n)\bar{i} + \frac{\partial}{\partial y}(r^n)\bar{j} + \frac{\partial}{\partial z}(r^n)\bar{k}$$

$$= nr^{n-1}\left(\frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}\right) \tag{2}$$

From (l), $\quad 2r\dfrac{\partial r}{\partial x} = 2x$

i.e. $\quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$ (3)

Using (3) in (2), we have

$$\nabla(r^n) = nr^{n-1}\left(\frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}\right)$$

$$= nr^{n-2}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= nr^{n-2}\bar{r}.$$

Example 2.10 Find the function ϕ, if grad ϕ

$$= (y^2 - 2xyz^3)\bar{i} + (3 + 2xy - x^2z^3)\bar{j} + (6z^3 - 3x^2yz^2)\bar{k}.$$

$$\nabla\phi = (y^2 - 2xyz^3)\bar{i} + (3 + 2xy - x^2z^3)\bar{j} + (6z^3 - 3x^2yz^2)\bar{k} \tag{1}$$

By definition,
$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k} \tag{2}$$

Comparing (1) and (2), we get

$$\frac{\partial\phi}{\partial x} = y^2 - 2xyz^3 \tag{3}$$

$$\frac{\partial\phi}{\partial y} = 3 + 2xy - x^2z^3 \tag{4}$$

$$\frac{\partial\phi}{\partial z} = 6z^3 - 3x^2yz^2 \tag{5}$$

Integrating both sides of (3) partially with respect to x (i.e. treating y and z as constants),

$$\phi = xy^2 - x^2yz^3 + a \text{ function not containing } x \tag{6}$$

Note ☑ When we integrate both sides of an equation ordinarily with respect to x, we usually add an arbitrary constant in one side. When we integrate partially, we add an arbitrary function of the other variables y and z, i.e. an arbitrary function independent of x.

Integrating (4) partially with respect to y.

$$\phi = 3y + xy^2 - x^2yz^3 + \text{a function not containing } y \tag{7}$$

Integrating (5) partially with respect to z,

$$\phi = \frac{3}{2}z^4 - x^2yz^3 + \text{ a function not containing } z \tag{8}$$

(6), (7) and (8) give only particular forms of ϕ. The general form of ϕ is obtained as follows:

The terms which do not repeat in the R.S's of (6), (7) and (8) should necessarily be included in the value of ϕ.

The terms which repeat should be included only once in the value of ϕ.

The last terms indicate that there is a term of ϕ which is independent of x, y, z, i.e. a constant.

$$\therefore \quad \phi = 3y + \frac{3}{2}z^4 + xy^2 - x^2yz^3 + c\,.$$

EXERCISE 2(a)

Part A

(Short Answer Questions)

1. Define grad ϕ and give its geometrical meaning.
2. If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla(r) = \frac{1}{r}\bar{r}$.
3. If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla(|\bar{r}|^2) = 2\bar{r}$.
4. If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla f(r) = \frac{1}{r}f'(r)\bar{r}$.
5. Find grad ϕ at the point $(1, -2, -1)$ when $\phi = 3x^2y - y^3z^2$.
6. Find the maximum directional derivative of $\phi = x^3y^2z$ at the point (1, 1, 1).
7. Find the directional derivative of $\phi = xy + yz + zx$ at the point (1, 2, 3) along the x-axis.
8. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum?
9. If the temperature at any point in space is given by $T = xy + yz + zx$, find the direction in which the temperature changes most rapidly with distance from the point (1, 1, 1).
10. The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

Part B

11. If $\phi = xy + yz + zx$ and $\bar{F} = x^2y\bar{i} + y^2z\bar{j} + z^2x\bar{k}$, find $\bar{F}\cdot$ grad ϕ and $\bar{F}\times$ grad ϕ at the point $(3, -1, 2)$.

12. Find the directional derivative of $\phi = 2xy + z^2$ at the point (1, −1, 3) in the direction of $\bar{i} + 2\bar{j} + 2\bar{k}$.
13. Find the directional derivative of $\phi = xy^2 + yz^3$ at the point P (2, −1, 1) in the direction of PQ where Q is the point (3, 1, 3).
14. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point (2, −2, 3).
15. Find the directional derivative of the scalar function $\phi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point (3, 1, 3).
16. Find the angle between the normals to the surface $xy^3z^2 = 4$ at the points (−1, −1, 2) and (4, 1, −1).
17. Find the angle between the normals to the surface $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).
18. Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point (2, −1, 2).
19. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, −2, 1).
20. Find the angle between the tangent planes to the surfaces $x \log z - y^2 = -1$ and $x^2y + z = 2$ at the point (1, 1, 1).
21. Find the equation of the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point (1, −3, 2).
22. Find the values of λ and μ, if the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at the point (1, −1, 2).
23. Find the values of a and b, so that the surfaces $ax^3 - by^2z = (a + 3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at the point (2, −1, −3).
24. Find the scalar point function whose gradient is $(2xy - z^2)\bar{i} + (x^2 + 2yz)\bar{j} + (y^2 - 2zx)\bar{k}$.
25. If $\nabla\phi = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$, find $\phi(x, y, z)$, given that $\phi(1, -2, 2) = 4$.

2.2 THE DIVERGENCE OF A VECTOR

If $\bar{F}(x, y, z)$ is a differentiable vector point function defined at each point (x, y, z) in some region of space, then *the divergence of* $\bar{F}$, denoted as *div* $\bar{F}$, is defined as

$$\begin{aligned}\text{div}\quad \bar{F} &= \nabla\cdot\bar{F}\\ &= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{F}\\ &= \bar{i}\cdot\frac{\partial\bar{F}}{\partial x} + \bar{j}\cdot\frac{\partial\bar{F}}{\partial y} + \bar{k}\cdot\frac{\partial\bar{F}}{\partial z}\end{aligned}$$

Formula for $\nabla\cdot\bar{F}$, ***when*** $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(F_1\bar{i} + F_2\bar{j} + F_3\bar{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note ☑ Since $\bar{F}$ is a vector point function, F_1, F_2 and F_3 are scalar point functions and hence $\nabla \cdot \bar{F}$ is also a scalar point function.

2.3.1 Physical meaning of $\nabla \cdot \bar{F}$

(i) If $\bar{V} = V_x \bar{i} + V_y \bar{j} + V_z \bar{k}$ is a vector point function representing the instantaneous velocity of a moving fluid at the point (x, y, z), then $\nabla \cdot \bar{V}$ represents the rate of loss of the fluid per unit volume at that point.

(ii) If the vector point function $\bar{V}$ represents an electric flux, then $\nabla \cdot \bar{V}$ represents the amount of flux that diverges per unit volume in unit time.

(iii) If the vector point function $\bar{V}$ represents heat flux, then $\nabla \cdot \bar{V}$ represents the rate at which heat is issuing from the concerned point per unit volume.

In general, if $\bar{F}$ represents any physical quantity, then $\nabla \cdot \bar{F}$ gives at each point the rate per unit volume at which the physical quantity is issuing from that point. It is due to this physical interpretation of $\nabla \cdot \bar{F}$, it is called the divergence of $\bar{F}$.

2.3.2 Solenoidal Vector

If $\bar{F}$ is a vector such that $\nabla \cdot \bar{F} = 0$ at all points in a given region, then it is said to be a solenoidal vector in that region.

2.3.3 Curl of a Vector

If $\bar{F}$ (x, y, z) is a differentiable vector point function defined at each point (x, y, z) in some region of space, then *the curl of* $\bar{F}$ or *the rotation of* $\bar{F}$, denoted as *curl* $\bar{F}$ or *rot* $\bar{F}$ is defined as

$$\begin{aligned}
\text{Curl}\,\bar{F} &= \nabla \times \bar{F} \\
&= \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z} \right) \times \bar{F} \\
&= \bar{i} \times \frac{\partial \bar{F}}{\partial x} + \bar{j} \times \frac{\partial \bar{F}}{\partial y} + \bar{k} \times \frac{\partial \bar{F}}{\partial z}
\end{aligned}$$

Note ☑ Curl $\bar{F}$ is also a vector point function.

Formula for $\boldsymbol{\nabla \times \bar{F}}$, ***when*** $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$ (where F_1, F_2 and F_3 are scalar point functions):

$$\begin{aligned}
\nabla \times \bar{F} &= \sum \bar{i} \times \frac{\partial}{\partial x} (F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}) \\
&= \sum \frac{\partial F_1}{\partial x} (\bar{i} \times \bar{i}) + \frac{\partial F_2}{\partial x} (\bar{i} \times \bar{j}) + \frac{\partial F_3}{\partial x} (\bar{i} \times \bar{k})
\end{aligned}$$

$$= \sum \left(\frac{\partial F_2}{\partial x} \bar{k} - \frac{\partial F_3}{\partial x} \bar{j} \right)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \bar{k}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

2.3.4 Physical Meaning of Curl $\bar{F}$

If $\bar{F}$ represents the linear velocity of the point (x, y, z) of a rigid body that rotates about a fixed axis with constant angular velocity $\bar{\omega}$, then curl $\bar{F}$ at that point represents $2\bar{\omega}$.

2.3.5 Irrotational Vector

If $\bar{F}$ is a vector such that $\nabla \times \bar{F} = 0$ at all points in a given region, then it is said to be an irrotational vector in that region.

2.3.6 Scalar Potential of an Irrotational Vector

If $\bar{F}$ is irrotational, then it can be expressed as the gradient of a scalar point function.

Let $\quad \bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

Since $\bar{F}$ is irrotational, Curl $\bar{F} = 0$

i.e.
$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

i.e.
$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \bar{k} = 0$$

$$\therefore \quad \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}; \ \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}; \ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \tag{1}$$

Equations (1) are satisfied when

$$F_1 = \frac{\partial \phi}{\partial x}, F_2 = \frac{\partial \phi}{\partial y} \quad \text{and} \quad F_3 = \frac{\partial \phi}{\partial z}$$

$$\therefore \qquad \bar{F} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$
$$= \nabla\phi$$

If $\bar{F}$ is irrotational and $\bar{F} = \nabla\phi$, then ϕ is called the scalar potential of $\bar{F}$.

2.3.7 Expansion Formulae Involving Operations by ∇

Expansion formulas involving operations by one or two ∇'s are given below: The proofs of some of them are given and those of the rest are left as exercise to the students.

1. If $\bar{u}$ and $\bar{v}$ are vector point functions, then $\nabla\cdot(\bar{u}\pm\bar{v}) = \nabla\cdot\bar{u}\pm\nabla\cdot\bar{v}$.
2. If $\bar{u}$ and $\bar{v}$ are vector point functions, then $\nabla\times(\bar{u}\pm\bar{v}) = \nabla\times\bar{u}\pm\nabla\times\bar{v}$.
3. If ϕ is a scalar point function and $\bar{F}$ is a vector point function, then

$$\nabla\cdot(\phi\bar{F}) = \phi(\nabla\cdot\bar{F}) + (\nabla\phi)\cdot\bar{F}$$

Proof:
$$\nabla\cdot(\phi\bar{F}) = \sum\bar{i}\cdot\frac{\partial}{\partial x}(\phi\bar{F})$$
$$= \sum\bar{i}\cdot\left(\phi\frac{\partial\bar{F}}{\partial x} + \frac{\partial\phi}{\partial x}\bar{F}\right)$$
$$= \phi\sum\bar{i}\cdot\frac{\partial\bar{F}}{\partial x} + \left(\sum\bar{i}\frac{\partial\phi}{\partial x}\right)\cdot\bar{F}$$
$$= \phi\left(\nabla\cdot\bar{F}\right) + (\nabla\phi)\cdot\bar{F}$$

4. If ϕ is a scalar point function and $\bar{F}$ is a vector point function, then

$$\nabla\times(\phi\bar{F}) = \phi(\nabla\times\bar{F}) + (\nabla\phi)\times\bar{F}$$

5. If $\bar{u}$ and $\bar{v}$ are vector point functions, then $\nabla\cdot(\bar{u}\times\bar{v}) = \bar{v}\cdot\text{curl}\,\bar{u} - \bar{u}\cdot\text{curl}\,\bar{v}$.

Proof:
$$\nabla\cdot(\bar{u}\times\bar{v}) = \sum\bar{i}\cdot\frac{\partial}{\partial x}(\bar{u}\times\bar{v})$$
$$= \sum\bar{i}\cdot\left(\frac{\partial\bar{u}}{\partial x}\times\bar{v} + \bar{u}\times\frac{\partial\bar{v}}{\partial x}\right)$$
$$= \sum\bar{i}\cdot\left(\frac{\partial\bar{u}}{\partial x}\times\bar{v}\right) - \sum\bar{i}\cdot\left(\frac{\partial\bar{v}}{\partial x}\times\bar{u}\right)$$
$$= \sum\left(\bar{i}\times\frac{\partial\bar{u}}{\partial x}\right)\cdot\bar{v} - \sum\left(\bar{i}\times\frac{\partial\bar{v}}{\partial x}\right)\cdot\bar{u}$$

[∵ the value of a scalar triple product is unaltered, when dot and cross are interchanged]

$$=\left(\sum \bar{i}\times\frac{\partial\bar{u}}{\partial x}\right)\cdot\bar{v}-\left(\sum\bar{i}\times\frac{\partial\bar{v}}{\partial x}\right)\cdot\bar{u}$$

$$=\bar{v}\cdot\text{Curl}\,\bar{u}-\bar{u}\cdot\text{Curl}\,\bar{v}$$

6. If $\bar{u}$ and $\bar{v}$ are vector point functions, then

$$\nabla\times(\bar{u}\times\bar{v})=(\nabla\cdot\bar{v})\bar{u}+(\bar{v}\cdot\nabla)\bar{u}-(\nabla\cdot\bar{u})\bar{v}-(\bar{u}\cdot\nabla)\bar{v}$$

Note ☑ In this formula, $\nabla\cdot\bar{v}$ and $\bar{v}\cdot\nabla$ are not the same, $\nabla\cdot\bar{v}$ means div $\bar{v}$, but $\bar{v}\cdot\nabla$ represents the operator $v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}$, if $\bar{v}=v_x\bar{i}+v_y\bar{j}+v_z\bar{k}\cdot$ Thus $(\bar{v}\cdot\nabla)\bar{u}$ represents $v_x\frac{\partial\bar{u}}{\partial x}+v_y\frac{\partial\bar{u}}{\partial y}+v_z\frac{\partial\bar{u}}{\partial z}\cdot$

7. If $\bar{u}$ and $\bar{v}$ are vector point functions, then $\nabla(\bar{u}\cdot\bar{v})=\bar{v}\times\text{curl}\,\bar{u}+\bar{u}\times\text{curl}\,\bar{v}$ $+(\bar{v}\cdot\nabla)\bar{u}+(\bar{u}\cdot\nabla)\bar{v}\cdot$

8. If ϕ is a scalar point function, then div (grad ϕ) = $\nabla\cdot(\nabla\phi)=\nabla^2\phi$, where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator* and $\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$ is called *the Laplacian of* ϕ. $\nabla^2\phi=0$ is called the Laplace equation.

Note ☑ ∇^2 can also operate on a vector point function $\bar{F}$ resulting in $\nabla^2\bar{F}=\frac{\partial^2\bar{F}}{\partial x^2}+\frac{\partial^2\bar{F}}{\partial y^2}+\frac{\partial^2\bar{F}}{\partial z^2}\cdot$

9. If ϕ is a scalar point function, then curl (grad ϕ) = $\nabla\times(\nabla\phi)=0$.

Proof grad $\phi=\left(\frac{\partial\phi}{\partial x}\right)\bar{i}+\left(\frac{\partial\phi}{\partial y}\right)\bar{j}+\left(\frac{\partial\phi}{\partial z}\right)\bar{k}$

$$\therefore\ \text{curl (grad }\phi)=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}$$

$$=\left(\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial z\,\partial y}\right)\bar{i}+\left(\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial x\,\partial z}\right)\bar{j}+\left(\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial y\,\partial x}\right),$$

$$=0.$$

Note ☑ This result means that (grad ϕ) is always an irrotational vector.

10. If $\bar{F}$ is a vector point function, then div $(\text{curl}\,\bar{F}) = \nabla \cdot (\nabla \times \bar{F}) = 0$.

Proof: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\therefore \qquad \text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

$$\therefore \text{ div } \left(\text{curl}\,\bar{F}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} + \frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}$$

$$= 0.$$

Note ☑ This result means that (curl $\bar{F}$) is always a solenoidal vector.

11. If $\bar{F}$ is a vector point function, then

$$\text{curl } (\text{curl}\,\bar{F}) = \nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}.$$

Proof: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

Then curl $\bar{F} = \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right)\bar{i} + \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right)\bar{j} + \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right)\bar{k}$

$\therefore$ curl (curl $\bar{F}$)

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right) & \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right) & \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right) \end{vmatrix}$$

$$= \sum\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right]\bar{i}$$

$$= \sum\left[\left(\frac{\partial^2 F_2}{\partial y\,\partial x} + \frac{\partial^2 F_3}{\partial z\,\partial x}\right) - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]\bar{i}$$

$$= \sum\left[\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x\,\partial y} + \frac{\partial^2 F_3}{\partial x\,\partial z}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]\bar{i}$$

$$=\sum\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_1\right]\bar{i}$$

$$=\sum\left[\frac{\partial}{\partial x}(\nabla\cdot\bar{F})-\nabla^2 F_1\right]\bar{i}$$

$$=\left[\bar{i}\,\frac{\partial}{\partial x}(\nabla\cdot\bar{F})+\bar{j}\,\frac{\partial}{\partial y}(\nabla\cdot\bar{F})+\bar{k}\,\frac{\partial}{\partial z}(\nabla\cdot\bar{F})\right]-\nabla^2\left[F_1\bar{i}+F_2\bar{j}+F_3\bar{k}\right]$$

$$=\nabla(\nabla\cdot\bar{F})-\nabla^2\bar{F}.$$

12. If $\bar{F}$ is a vector point function, then

$$\text{grad (div}\,\bar{F})=\nabla(\nabla\cdot\bar{F})=\nabla\times(\nabla\times\bar{F})+\nabla^2\bar{F}\ .$$

Note ☑ Rewriting the formula (11), this result is obtained.

WORKED EXAMPLE 2(b)

Example 2.1 When $\phi = x^3 + y^3 + z^3 - 3xyz$, find $\nabla\phi$, $\nabla\cdot\nabla\phi$ and $\nabla\times\nabla\phi$ at the point (1, 2, 3).

$$\phi = x^3 + y^3 + z^3 - 3x\,yz$$

$$\nabla\phi=\sum\left(\frac{\partial\phi}{\partial x}\right)\bar{i}$$

$$=3(x^2-yz)\bar{i}+3(y^2-zx)\bar{j}+3(z^2-xy)\bar{k}$$

$$\nabla\cdot\nabla\phi=\frac{\partial}{\partial x}[3(x^2-yz)]+\frac{\partial}{\partial y}[3(y^2\ \ zx)]+\frac{\partial}{\partial z}[3(z^2-xy)]$$

$$=6(x+y+z)$$

$$\nabla\times\nabla\phi=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3(x^2-yz) & 3(y^2-zx) & 3(z^2-xy)\end{vmatrix}$$

$$=(-3x+3x)\bar{i}-(-3y+3y)\bar{j}+(-3z+3z)\bar{k}$$

Note ☑ $\nabla\times\nabla\phi = 0$, for any ϕ, as per the expansion formula (9).

$$(\nabla\phi)_{(1,2,3)}=-15\bar{i}+3\bar{j}+21\bar{k}$$

$$(\nabla\cdot\nabla\phi)_{(1,2,3)}=36 \quad\text{and}\quad (\nabla\times\nabla\phi)_{(1,2,3)}=0.$$

Example 2.2 If $\bar{F}=(x^2-y^2+2xz)\bar{i}+(xz-xy+yz)\bar{j}+(z^2+x^2)\bar{k}$, find $\nabla\cdot\bar{F},\nabla(\nabla\cdot\bar{F}),\nabla\times\bar{F},\nabla\cdot(\nabla\times\bar{F})$ and $\nabla\times(\nabla\times\bar{F})$ at the point (1, 1, 1).

$$\bar{F}=(x^2-y^2+2xz)\bar{i}+(xz-xy+yz)\bar{j}+(z^2+x^2)\bar{k}$$

$$\nabla\cdot\bar{F}=\frac{\partial}{\partial x}(x^2-y^2+2xz)+\frac{\partial}{\partial y}(xz-xy+yz)+\frac{\partial}{\partial z}(z^2+x^2)$$

$$=(2x+2z)+(-x+z)+2z$$

$$=x+5z$$

$$\nabla(\nabla\cdot\bar{F})=\frac{\partial}{\partial x}(x+5z)\bar{i}+\frac{\partial}{\partial y}(x+5z)\bar{j}+\frac{\partial}{\partial z}(x+5z)\bar{k}$$

$$=\bar{i}+5\bar{k}$$

$$\nabla\times\bar{F}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2-y^2+2xz & xz-xy+yz & z^2+x^2\end{vmatrix}$$

$$=-(x+y)\bar{i}-(2x-2x)\bar{j}+(y+z)\bar{k}$$

$$=-(x+y)\bar{i}+(y+z)\bar{k}$$

$$\nabla\cdot(\nabla\times\bar{F})=\frac{\partial}{\partial x}[-(x+y)]+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(y+z)$$

$$=-1+0+1=0$$

Note ☑ $\nabla\cdot(\nabla\times\bar{F})=0$, for any $\bar{F}$, as per the expansion formula

$$\nabla\times(\nabla\times\bar{F})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ -(x+y) & 0 & y+z\end{vmatrix}$$

$$=\bar{i}+\bar{k}$$

$\therefore$

$$(\nabla\cdot\bar{F})_{(1,1,1)}=6;\quad [\nabla(\nabla\cdot\bar{F})]_{(1,1,1)}=\bar{i}+5\bar{k};$$

$$(\nabla\times\bar{F})_{(1,1,1)}=-2\bar{i}+2\bar{k};\quad [\nabla\cdot(\nabla\times\bar{F})]_{(1,1,1)}=0;$$

$$[\nabla\times(\nabla\times\bar{F})]_{(1,1,1)}=\bar{i}+\bar{k}$$

Example 2.3 If $\bar{a}$ is a constant vector and $\bar{r}$ is the position vector of the point (x, y, z,) with respect to the origin, prove that (i) grad $(\bar{a}\cdot\bar{r})=\bar{a}$, (ii) div $(\bar{a}\times\bar{r})$ = 0 and (iii) curl $(\bar{a}\times\bar{r})=2\bar{a}$.

$$\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$$

Let $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$, where a_1, a_2, a_3 are constants.

$\therefore$

$$\bar{a}\cdot\bar{r}=a_1x+a_2y+a_3z$$

$$\therefore \quad \text{grad } (\bar{a}\cdot\bar{r}) = \sum \frac{\partial}{\partial x}(a_1x + a_2y + a_3z)\bar{i}$$
$$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$
$$= \bar{a}$$

$$\bar{a}\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$
$$= (a_2z - a_3y)\bar{i} + (a_3x - a_1z)\bar{j} + (a_1y - a_2x)\bar{k}$$

$$\therefore \quad \text{div}(\bar{a}\times\bar{r}) = \frac{\partial}{\partial x}(a_2z - a_3y) + \frac{\partial}{\partial y}(a_3x - a_1z) + \frac{\partial}{\partial z}(a_1y - a_2x)$$
$$= 0$$

$$\text{curl}(\bar{a}\times\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$$
$$= (a_1 + a_1)\bar{i} - (-a_2 - a_2)\bar{j} + (a_3 + a_3)\bar{k}$$
$$= 2(a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$$
$$= 2\bar{a}$$

Example 2.4 Show that $\bar{u} = (2x^2 + 8xy^2z)\bar{i} + (3x^3y - 3xy)\bar{j} - (4y^2z^2 + 2x^3z)\bar{k}$ is not solenoidal, but $\bar{v} = xyz^2\bar{u}$ is solenoidal.

$$\nabla\cdot\bar{u} = \frac{\partial}{\partial x}(2x^2 + 8xy^2z) + \frac{\partial}{\partial y}(3x^3y - 3xy) + \frac{\partial}{\partial z}\{-(4y^2z^2 + 2x^3z)\}$$
$$= (4x + 8y^2z) + (3x^3 - 3x) - (8y^2z + 2x^3)$$
$$= x^3 + x$$
$$\neq 0, \text{ for all points } (x, y, z)$$

$\therefore$ $\bar{u}$ is not solenoidal.

$$\bar{v} = xyz^2\bar{u}$$
$$= (2x^3yz^2 + 8x^2y^3z^3)\bar{i} + (3x^4y^2z^2 - 3x^2y^2z^2)\bar{j} - (4xy^3z^4 + 2x^4yz^3)\bar{k}$$

$$\therefore \quad \nabla\cdot\bar{v} = (6x^2yz^2 + 16xy^3z^3) + (6x^4yz^2 - 6x^2yz^2) - (16xy^3z^3 + 6x^4yz^2)$$
$$= 0, \text{ for all points } (x, y, z)$$

$\therefore$ $\bar{v}$ is solenoidal.

Example 2.5 Show that $\bar{F} = (y^2 - z^2 + 3yz - 2x)\bar{i} + (3xz + 2xy)\bar{j} + (3xy - 2xz + 2z)\bar{k}$ is both solenoidal and irrotational.

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$
$$= -2 + 2x - 2x + 2$$
$$= 0, \text{ for all points } (x, y, z)$$

$\therefore\ \overline{F}$ is a solenoidal vector.

$$\nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$
$$= (3x - 3x)\overline{i} - (3y - 2z + 2z - 3y)\overline{j} + (3z + 2y - 2y - 3z)\overline{k}$$
$$= 0, \text{ for all points } (x, y, z)$$

$\therefore\ \overline{F}$ is an irrotational vector.

Example 2.6 Show that $\overline{F} = (y^2 + 2xz^2)\overline{i} + (2xy - z)\overline{j} + (2x^2z - y + 2z)\overline{k}$ is irrotational and hence find its scalar potential.

$$\nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y^2 + 2xz^2) & (2xy - z) & (2x^2z - y + 2z) \end{vmatrix}$$
$$= (-1 + 1)\overline{i} - (4xz - 4xz)\overline{j} + (2y - 2y)\overline{k}$$
$$= 0, \text{ for all points } (x, y, z)$$

$\therefore\ \overline{F}$ is irrotational.

Let the scalar potential of $\overline{F}$ be ϕ.

$$\therefore \qquad \overline{F} = \nabla\phi$$
$$= \frac{\partial \phi}{\partial x}\overline{i} + \frac{\partial \phi}{\partial y}\overline{j} + \frac{\partial \phi}{\partial z}\overline{k}$$

$$\therefore \qquad \frac{\partial \phi}{\partial x} = y^2 + 2xz^2$$

Integrating partially w.r.t. x;

$$\phi = xy^2 + x^2z^2 + \text{a function independent of } x \tag{1}$$

$$\frac{\partial \phi}{\partial y} = 2xy - z$$

Integrating partially w.r.t. y;

$$\phi = xy^2 - yz + \text{a function independent of } y \tag{2}$$

$$\frac{\partial \phi}{\partial z} = 2x^2z - y + 2z$$

Integrating partially w.r.t. z;

$$\phi = x^2z^2 - yz + z^2 + \text{a function independent of } z \qquad (3)$$

From (1), (2), (3), we get $\phi = xy^2 + x^2z^2 - yz + z^2 + c$.

Example 2.7 Find the values of the constants a, b, c, so that $\bar{F} = (axy + bz^3)\bar{i} + (3x^2 - cz)\bar{j} + (3xz^2 - y)\bar{k}$ may be irrotational. For these values of a, b, c, find also the scalar potential of $\bar{F}$.

$\bar{F}$ is irrotational.

$\therefore \qquad \nabla \times \bar{F} = 0$

i.e.
$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (axy + bz^3) & (3x^2 - cz) & (3xz^2 - y) \end{vmatrix} = 0$$

i.e. $\qquad (-1 + c)\bar{i} - (3z^2 - 3bz^2)\bar{j} + (6x - ax)\bar{k} = 0$

$\therefore \qquad c - 1 = 0, \quad 3z^2(1 - b) = 0, \quad x(6 - a) = 0$

$\therefore \qquad a = 6, \quad b = 1, \quad c = 1.$

Using these values of a, b, c,

$$\bar{F} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$$

Let ϕ be the scalar potential of $\bar{F}$.

$$\therefore \qquad \bar{F} = \nabla\phi = \frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j} + \frac{\partial \phi}{\partial z}\bar{k}$$

$$\frac{\partial \phi}{\partial x} = 6xy + z^3, \quad \frac{\partial \phi}{\partial y} = 3x^2 - z, \quad \frac{\partial \phi}{\partial z} = 3xz^2 - y$$

Integrating partially w.r.t. the concerned variables,

$$\phi = 3x^2y + xz^3 + \text{a function independent of } x \qquad (1)$$

$$\phi = 3x^2y - yz + \text{a function independent of } y \qquad (2)$$

$$\phi = xz^3 - yz + \text{a function independent of } z \qquad (3)$$

From(l), (2) and (3), we get

$$\phi = 3x^2y + xz^3 - yz + c$$

Example 2.8 If ϕ and ψ are scalar point functions, prove that (i) $\phi\,\nabla\phi$ is irrotational and (ii) $\nabla\phi \times \nabla\psi$ is solenoidal.

$$\begin{aligned} \nabla \times \phi\nabla\phi &= \phi(\nabla \times \nabla\phi) + \nabla\phi \times \nabla\phi, \text{ by expansion formula} \\ &= \phi(0) + 0, \text{ by expansion formula} \\ &= 0 \end{aligned}$$

$\therefore\ \phi\nabla\phi$ is irrotational

$$\nabla \cdot (\bar{u} \times \bar{v}) = \bar{v} \cdot \text{curl}\,\bar{u} - \bar{u} \cdot \text{curl}\,\bar{v}, \text{ by expansion formula}$$

$$\therefore \quad \nabla \cdot (\nabla\phi \times \nabla\psi) = \nabla\psi \cdot \text{curl}\,(\nabla\phi) - \nabla\phi \cdot \text{curl}\,(\nabla\psi)$$
$$= \nabla\psi \cdot 0 - \nabla\phi \cdot 0 = 0.$$

$\therefore$ $(\nabla\phi \times \nabla\psi)$ is solenoidal.

Example 2.9 If $r = |\bar{r}|$, where $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla^2 (r^n) = n(n+1)\, r^{n-2}$ and hence deduce that $\frac{1}{r}$ satisfies Laplace equation.

We have already proved, in worked example (9) of the previous section, that $\nabla (r^n) = nr^{n-2}\,\bar{r}$.

Now
$$\nabla^2 (r^n) = \nabla \cdot (\nabla r^n)$$
$$= \nabla \cdot (nr^{n-2}\,\bar{r})$$
$$= n\left[\nabla (r^{n-2}) \cdot \bar{r} + r^{n-2}\,(\nabla \cdot \bar{r})\right], \text{ by expansion formula}$$
$$= n\left[(n-2)\, r^{n-4}\,\bar{r} \cdot \bar{r} + 3r^{n-2}\right]$$

$$\left[\text{since } \nabla \cdot \bar{r} = \nabla \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3\right]$$

$$\therefore \quad \nabla^2(r^n) = n\,[(n-2)\, r^{n-4}\, r^2 + 3r^{n-2}]$$
$$= n\,(n+1)\, r^{n-2}$$

Taking $n = -1$ in the above result,

$$\nabla^2 \left(\frac{1}{r}\right) = (-1)(0)\, r^{-3} = 0$$

i.e. $\frac{1}{r}$ satisfies Laplace equation.

Example 2.10 If u and v are scalar point functions, prove that

$$\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u.$$

$$\nabla \cdot (u\nabla v - v\nabla u) = \nabla \cdot (u\nabla v) - \nabla \cdot (v\nabla u)$$
$$= (\nabla u \cdot \nabla v + u\nabla^2 v) - (\nabla v \cdot \nabla u + v\nabla^2 u)$$

[by the expansion formula]

$$= u\nabla^2 v - v\nabla^2 u.$$

Example 2.11 If u and v are scalar point functions and $\bar{F}$ is a vector point function such that $u\bar{F} = \nabla v$, prove that $\bar{F} \cdot$ curl $\bar{F} = 0$.

Given
$$u\bar{F} = \nabla v$$

$$\therefore \quad \bar{F} = \frac{1}{u}\nabla v$$

$$\therefore \qquad \text{Curl}\,\bar{F} = \nabla \times \left(\frac{1}{u}\nabla v\right)$$

$$= \frac{1}{u}(\nabla \times \nabla v) + \nabla\left(\frac{1}{u}\right) \times \nabla v, \text{ by expansion formula}$$

$$= \nabla\left(\frac{1}{u}\right) \times \nabla v, \text{ since } \nabla \times \nabla v = 0.$$

Now
$$\bar{F} \cdot \text{curl}\,\bar{F} = \frac{1}{u}\left\{\nabla v \cdot \left(\nabla \frac{1}{u} \times \nabla v\right)\right\}$$

$$= \frac{1}{u}(0), \qquad [\because \text{ Two factors are equal in the scalar triple product}]$$

$$= 0.$$

Example 2.12 If $r = |\bar{r}|$, where $\bar{r}$ is the position vector of the point (x, y, z) with respect to the origin, prove that (i) $\nabla f(r) = \dfrac{f'(r)}{r}\bar{r}$ and

(ii) $\nabla^2 f(r) = f''(r) + \dfrac{2}{r} f'(r)$.

$$r^2 = x^2 + y^2 + z^2$$

$$\therefore \qquad 2r\frac{\partial r}{\partial x} = 2x$$

$\therefore \; \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$. Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$.

Now
$$\nabla f(r) = \sum \bar{i}\,\frac{\partial}{\partial x} f(r)$$

$$= f'(r)\sum \frac{\partial r}{\partial x}\bar{i}$$

$$= f'(r)\sum \frac{x}{r}\bar{i}$$

$$= \frac{f'(r)}{r}\bar{r}$$

$$\nabla^2 f(r) = \nabla \cdot \nabla f(r)$$

$$= \nabla \cdot \left\{\frac{f'(r)}{r}\bar{r}\right\}$$

$$= \nabla\left\{\frac{f'(r)}{r}\right\} \cdot \bar{r} + \frac{f'(r)}{r}\nabla \cdot \bar{r}$$

$$= \left\{\frac{r f''(r) - f'(r)}{r^2}\right\}\nabla(r) \cdot \bar{r} + 3\frac{f'(r)}{r} \qquad [\because \nabla \cdot \bar{r} = 3]$$

$$=\left\{\frac{r f''(r)-f'(r)}{r^2}\right\}\frac{1}{r}\bar{r}\cdot\bar{r}+\frac{3f'(r)}{r}$$

$$\left[\because \nabla(r)=\sum\frac{\partial r}{\partial x}\bar{i}=\sum\frac{x}{r}i=\frac{1}{r}\bar{r}\right]$$

$$=\left\{\frac{r f''(r)-f'(r)}{r^2}\right\}\frac{1}{r}(r^2)+\frac{3f'(r)}{r}$$

$$=f''(r)+\frac{2}{r}f'(r)$$

Example 2.13 Find $f(r)$ if the vector $f(r)\ \bar{r}$ is both solenoidal and irrotational.

$f(r)\ \bar{r}$ is solenoidal

$\therefore$ $\quad \nabla\cdot\{f(r)\bar{r}\}=0$

i.e. $\quad \nabla f(r)\cdot\bar{r}+f(r)\nabla\cdot\bar{r}=0$ [by expansion formula]

i.e. $\quad \dfrac{f'(r)}{r}\bar{r}\cdot\bar{r}+3f(r)=0$ [Refer to the previous problem]

i.e. $\quad rf'(r)+3f(r)=0$

i.e. $\quad \dfrac{f'(r)}{f(r)}+\dfrac{3}{r}=0$

Integrating both sides w.r.t. r,

$$\log f(r)+3\log r=\log c$$

i.e. $$\log r^3 f(r)=\log c$$

$\therefore$ $$f(r)=\frac{c}{r^3} \tag{1}$$

$f(r)\ \bar{r}$ is also irrotational

$\therefore$ $\quad \nabla\times\{f(r)\bar{r}\}=0$

i.e. $\quad \nabla f(r)\times\bar{r}+f(r)\nabla\times\bar{r}=0$ [by expansion formula]

i.e. $$\frac{f'(r)}{r}(\bar{r}\times\bar{r})+0=0\left[\because \nabla\times\bar{r}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0\right]$$

i.e. $$\frac{f'(r)}{r}(0)+0=0$$

This is true for all values of $f(r)$ (2)

From (1) and (2), we get that $f(r)\ \bar{r}$ is both solenoidal and irrotational if $f(r)=\dfrac{c}{r^3}$

Example 2.14 If ϕ is a scalar point function, prove that $\nabla\phi$ is both solenoidal and irrotational, provided ϕ is a solution of Laplace equation.

$\nabla\cdot\nabla\phi=0$, only when $\nabla^2\phi=0$.

i.e, $\quad \nabla\phi$ is solenoidal, only when $\nabla^2\phi=0$ (1)

$\nabla \times \nabla\phi = 0$, always [by expansion formula]

i.e. $\nabla\phi$ is irrotational always (2)

From (1) and (2),

$\nabla\phi$ is both solenoidal and irrotational, when $\nabla^2\phi = 0$, i.e. when ϕ is a solution of Laplace equation.

Example 2.15 If $\bar{F}$ is solenoidal, prove that curl curl curl curl $\bar{F} = \nabla^4 \bar{F}$.

Since $\bar{F}$ is solenoidal, $\nabla \cdot \bar{F} = 0$ (1)

By the expansion formula,

$$\text{Curl curl } \bar{F} = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F} \quad (2)$$

$$= -\nabla^2 \bar{F} \text{ [by (1)]}$$

$\therefore$ Curl curl curl curl $\bar{F}$ $= \text{curl curl } (-\nabla^2 \bar{F})$

$$= -\text{ curl curl } (\nabla^2 \bar{F})$$

$$= -[\nabla\{\nabla \cdot \nabla^2 \bar{F}\} - \nabla^2(\nabla^2 \bar{F})], \text{ by using (2)}$$

$$= -[\nabla\{\nabla^2(\nabla \cdot \bar{F})\} - \nabla^4 \bar{F}], \quad \text{by interchanging the operations } \nabla \cdot \text{ and } \nabla^2$$

$$= \nabla^4 \bar{F} \text{ \{by using (1)\}}$$

EXERCISE 2(b)

Part A

(Short Answer Questions)

1. Define divergence and curl of a vector point function.
2. Give the physical meaning of $\nabla \cdot \bar{F}$
3. Give the physical meaning of $\nabla \times \bar{F}$
4. When is a vector said to be (i) solenoidal, (ii) irrotational ?
5. Prove that the curl of any vector point function is solenoidal.
6. Prove that the gradient of any scalar point function is irrotational.
7. If $\bar{r}$ is the position vector of the point (x, y, z) w.r.t. the origin, find div $\bar{r}$ and curl $\bar{r}$.
8. If $\bar{F} = 3xyz^2\,\bar{i} + 2xy^3\,\bar{j} - x^2yz\,\bar{k}$, find $\nabla \cdot \bar{F}$ at the point (1,–1, 1).
9. If $\bar{F} = (x^2 + yz)\,\bar{i} + (y^2 + 2zx)\,\bar{j} + (z^2 + 3xy)\,\bar{k}$, find $\nabla \times \bar{F}$ at the point (2,–1, 2).
10. If $\bar{F} = (x + y + 1)\,\bar{i} + \bar{j} - (x + y)\,\bar{k}$, show that $\bar{F}$ is perpendicular to curl $\bar{F}$.
11. If $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, prove that curl curl $\bar{F} = 0$.
12. Show that $\bar{F} = (x + 2y)\,\bar{i} + (y + 3z)\,\bar{j} + (x - 2z)\bar{k}$ is solenoidal.
13. Show that $\bar{F} = (\sin y + z)\,\bar{i} + (x\cos y - z)\,\bar{j} + (x - y)\bar{k}$ is irrotational.
14. Find the value of λ, so that $\bar{F} = \lambda y^4z^2\bar{i} + 4x^3z^2\bar{j} + 5x^2y^2\bar{k}$ may be solenoidal.
15. Find the value of λ, if $\bar{F} = (2x - 5y)\bar{i} + (x + \lambda y)\bar{j} + (3x - z)\bar{k}$ is solenoidal.

16. Find the value of a, if $\overline{F} = (axy - z^3)\overline{i} + (a-2)x^2\overline{j} + (1-a)xz^2\,\overline{k}$ is irrotational.
17. Find the values of a, b, c, so that the vector $\overline{F} = (x+y+az)\overline{i} + (bx+2y-z)\overline{j} + (-x+cy+2z)\overline{k}$ may be irrotational.
18. If $\bar{u}$ and $\overline{v}$ are irrotational, prove that $(\overline{u}\times\overline{v})$ is solenoidal.
19. If ϕ_1 and ϕ_2 are scalar point functions, prove that $\nabla \times (\phi_1\ \nabla\phi_2) = -\nabla \times (\phi_2\nabla\phi_1)$.
20. If $\nabla\phi$ is a solenoidal vector, prove that ϕ is a solution of Laplace equation.

Part B

21. If $u = x^2yz$ and $v = xy - 3z^2$, find (i) $\nabla \cdot (\nabla u \times \nabla v)$ and $\nabla \times (\nabla u \times \nabla v)$ at the point (1, 1, 0).
22. Find the directional drivative of $\nabla \cdot (\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2 z = 3x + z^2$, where $\phi = x^2 y^2 z^2$
23. If $\overline{F}=3x^2\,\overline{i}+5xy^2\,\overline{j}+xyz^3\,\overline{k}$, find $\nabla\cdot\overline{F}, \nabla(\nabla\cdot\overline{F}), \nabla\times\overline{F}, \nabla\cdot(\nabla\times\overline{F})$ and $\nabla\times(\nabla\times\overline{F})$ at the point (1, 2, 3)
24. If $\bar{a}$ is a constant vector and $\overline{r}$ is the position vector of (x, y, z) w.r.t. the origin, prove that $\nabla\times[(\overline{a}\cdot\overline{r})\overline{r}]=\overline{a}\times\overline{r}$.
25. Prove that $\overline{F}=3yz\overline{i}+2zx\,\overline{j}+4xy\,\overline{k}$ is not irrotational, but $(x^2\,y\,z^3)\,\overline{F}$ is irrotational. Find also its scalar potential.
26. Show that $\overline{F}=(z^2+2x+3y)\,\overline{i}+(3x+2y+z)\,\overline{j}+(y+2zx)\,\overline{k}$ is irrotational, but not solenoidal. Find also its scalar potential.
27. Find the constants a, b, c, so that $\overline{F}=(x+2y+az)\,\overline{i}+(bx-3y-z)\,\overline{j}+(4x+cy+2z)\,\overline{k}$ may be irrotational. For these values of a, b, c, find its scalar potential also.
28. Find the smallest positive integral values of a, b, c, if $\overline{F}=axyz^3\,\overline{i}+bx^2z^3\,\overline{j}+cx^2yz^2\,\overline{k}$ is irrotational. For these values of a, b, c, find its scalar potential also.
29. If $\overline{r}$ is the position vector of the point (x, y, z) w.r.t. the origin, prove that

    (i) $\nabla\cdot\left(\dfrac{1}{r}\overline{r}\right)=\dfrac{2}{r}$ and (ii) $\nabla\left[\nabla\cdot\left(\dfrac{1}{r}\overline{r}\right)\right]=-\dfrac{2}{r^3}$
30. Find the value of n, if $r^n\ \overline{r}$ is both solenoidal and irrotational, when $\overline{r}=x\overline{i}+y\overline{j}+z\overline{k}$.

2.4 LINE INTEGRAL OF VECTOR POINT FUNCTIONS

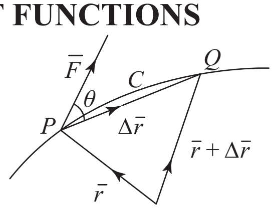


Fig. 2.2

Let $\overline{F}$ (x, y, z) be a vector point function defined at all points in some region of space and let C be a curve in that region (Fig. 2.2).

Let the position vectors of two neighbouring points P and Q on C be $\overline{r}$ and $\overline{r}+\Delta\overline{r}$ .Then $\overline{PQ}=\Delta\overline{r}$ If $\overline{F}$ acts at P in a direction that makes an angle θ with $\overline{PQ}$, then $\overline{F} \cdot \Delta\overline{r} = F(\Delta r)\cos\theta$

In the limit, $\overline{F} \cdot \mathrm{d}\overline{r} = F\,\mathrm{d}r\cos\theta$.

Note ☑ Physically $\overline{F} \cdot \mathrm{d}\overline{r}$ means the elemental work done by the force $\overline{F}$ through the displacement $\mathrm{d}\overline{r}$.

Now the integral $\int_C \overline{F} \cdot \mathrm{d}\overline{r}$ is defined as *the line integral of* $\overline{F}$ *along the curve C*.

Since $\int_C \overline{F} \cdot \mathrm{d}\overline{r} = \int_C F\cos\theta\,\mathrm{d}r$, it is also called the line integral of the tangential component of $\overline{F}$ along C.

Note ☑ (1) $\int_{A\,(C)}^{B} \overline{F} \cdot \mathrm{d}\overline{r}$ depends not only on the curve C but also on the terminal points A and B.

(2) Physically $\int_{A\,(C)}^{B} \overline{F} \cdot \mathrm{d}\overline{r}$ denotes the total work done by the force $\overline{F}$ in displacing a particle form A to B along the curve C.

(3) If the value of $\int_{A}^{B} \overline{F} \cdot \mathrm{d}\overline{r}$ does not depend on the curve C, but only on the terminal points A and B, $\overline{F}$ is called a *Conservative vector*. Similarly, if the work done by a force $\overline{F}$ in displacing a particle from A to B does not depend on the curve along which the particle gets displaced but only on A and B, the force $\overline{F}$ is called a *Conservative force*.

(4) If the path of integration C is a closed curve, the line integral is denoted as $\oint_C \overline{F} \cdot \mathrm{d}\overline{r}$.

(5) When $\overline{F} = F_1\,\overline{i} + F_2\,\overline{j} + F_3\,\overline{k}$,

$$\int_C \overline{F} \cdot \mathrm{d}\overline{r} = \int_C (F_1\,\overline{i} + F_2\,\overline{j} + F_3\,\overline{k}) \cdot (\mathrm{d}x\,\overline{i} + \mathrm{d}y\,\overline{j} + \mathrm{d}z\,\overline{k})$$

$$(\because \overline{r} = x\overline{i} + y\overline{j} + z\overline{k})$$

$$= \int_C (F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z)$$, which is evaluated as in the problems in Chapter 5 of Part I.

(6) $\int_C \phi\,\mathrm{d}r$, where ϕ is a scalar point function and $\int_C \overline{F} \times \mathrm{d}\overline{r}$ are also line integrals.

2.4.1 Condition for $\overline{F}$ to be Conservative

If $\overline{F}$ is an irrotational vector, it is conservative.

Proof: Since $\overline{F}$ is irrotational, it can be expressed as $\nabla\phi$. i.e., $\overline{F} = \nabla\phi$

$$\int_A^B \overline{F} \cdot d\overline{r} = \int_A^B \nabla\phi \cdot d\overline{r}$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}\,\overline{i} + \frac{\partial\phi}{\partial y}\,\overline{j} + \frac{\partial\phi}{\partial z}\,\overline{k} \right) \cdot (dx\,\overline{i} + dy\,\overline{j} + dz\,\overline{k})$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz \right)$$

$$= \int_A^B d\phi$$

$= [\phi]_A^B$, whatever be the path of integration

$= \phi(B) - \phi(A)$

$\therefore$ $\overline{F}$ is conservative.

Note ☑ If $\overline{F}$ is irrotational (and hence conservative) and C is a closed curve, then $\oint_C \overline{F} \cdot d\overline{r} = 0$.

[$\because$ $\phi(A) = \phi(B)$, as A and B coincide]

2.4.2 Surface Integral of Vector Point Function

Let S be a two sided surface, one side of which is considered arbitrarily as the positive side.

Let $\overline{F}$ be a vector point function defined at all points of S.

Let dS be the typical elemental surface area in S surrounding the point (x, y, z).

Let $\hat{n}$ be the unit vector normal to the surface S at (x, y, z) drawn in the positive side (or outward direction)

Let θ be the angle between $\overline{F}$ and $\hat{n}$.

$\therefore$ The normal component of $\overline{F} = \overline{F} \cdot \hat{n} = F \cos\theta$

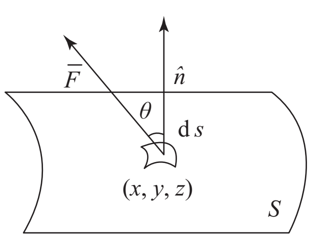


Fig. 2.3

The integral of this normal component through the elemental surface area dS over the surface S is called *the surface integral of* $\overline{F}$ *over S* and denoted as $\int_S F\cos\theta\, dS$ or $\int_S \overline{F} \cdot \hat{n}\, dS$.

If $d\overline{S}$ is a vector whose magnitude is dS and whose direction is that of $\hat{n}$, then $d\overline{S} = \hat{n}\, dS$.

$\therefore$ $\int_S \overline{F} \cdot \hat{n}\, dS$ can also be written as $\int_S \overline{F} \cdot d\overline{S}$.

Note ☑ (1) If S is a closed surface, the outer surface is usually chosen as the positive side.

(2) $\int_S \phi \, \mathrm{d}\bar{S}$ and $\int_S \bar{F} \times \mathrm{d}\bar{S}$, where ϕ is a scalar point function are also surface integrals.

(3) When evaluating $\int_S \bar{F} \cdot \hat{n} \, \mathrm{d}\bar{S}$, the surface integral is first expressed in the scalar form and then evaluated as in problems in Chapter 5 of part I.

To evaluate a surface integral in the scalar form, we convert it into a double integral and then evaluate. Hence the surface integral $\int_S \bar{F} \cdot \mathrm{d}\bar{S}$ is also denoted as $\iint_S \bar{F} \cdot \mathrm{d}\bar{S}$.

WORKED EXAMPLE 2(c)

Example 2.1 Evaluate $\int_C \phi \, \mathrm{d}\bar{r}$, where C is the curve $x = t$, $y = t^2$, $z = (1 - t)$ and $\phi = x^2 y\,(1 + z)$ from $t = 0$ to $t = 1$.

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\therefore \quad \mathrm{d}\bar{r} = \mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k}$$

Hence the given line integral $\mathrm{I} = \int_C x^2 y(1+z)\,(\mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k})$

$$= \bar{i} \int_C x^2 y(1+z)\,\mathrm{d}x + \bar{j} \int_C x^2 y(1+z)\,\mathrm{d}y + \bar{k} \int_C x^2 y\,(1+z)\,\mathrm{d}z$$

$$= \bar{i} \int_0^1 t^4\,(2-t)\,\mathrm{d}t + \bar{j} \int_0^1 t^4\,(2-t)\,2t\,\mathrm{d}t + \bar{k} \int_0^1 t^4\,(2-t)\,(-\mathrm{d}t)$$

$$= \bar{i}\left[2\frac{t^5}{5} - \frac{t^6}{6}\right]_0^1 + \bar{j}\left[4\frac{t^6}{6} - 2\frac{t^7}{7}\right]_0^1 + \bar{k}\left[-2\frac{t^5}{5} + \frac{t^6}{6}\right]_0^1$$

$$= \frac{7}{30}\bar{i} + \frac{8}{21}\bar{j} - \frac{7}{30}\bar{k}.$$

Example 2.2 If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$, evaluate $\int_C \bar{F} \times \mathrm{d}\bar{r}$, where C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from (0, 0, 0) to (1, 2, 1).

$$\bar{F} \times \mathrm{d}\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ xy & -z & x^2 \\ \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \end{vmatrix}$$

$$= -(z\,\mathrm{d}z + x^2\,\mathrm{d}y)\bar{i} - (xy\,\mathrm{d}z - x^2\,\mathrm{d}x)\,\bar{j} + (xy\,\mathrm{d}y + z\mathrm{d}\,x)\bar{k}$$

$\therefore$ The given line integral

$$= \int_C [-(z\mathrm{d}z + x^2\ \mathrm{d}y)\bar{i} - (xy\ \mathrm{d}z - x^2\ \mathrm{d}x)\bar{j} + (xy\,\mathrm{d}y + z\mathrm{d}x)\bar{k}]$$

$$= \int_0^1 [-(t^3 \cdot 3t^2 \cdot + t^4 \cdot 2)\,\mathrm{d}t\ \bar{i} - (2t^3 \cdot 3t^2 - t^4 \cdot 2t)\,\mathrm{d}t\ \bar{j} + (2t^3 \cdot 2 + t^3\ 2\,t)\,\mathrm{d}t\ \bar{k}]$$

[$\because$ (0, 0, 0,) corresponds to $t = 0$ and (1, 2, 1) corresponds to $t = 1$]

$$= -\bar{i}\int_0^1 (3t^5 + 2t^4)\,\mathrm{d}t - \bar{j}\int_0^1 (6t^5 - 2t^5)\,\mathrm{d}t + \bar{k}\int_0^1 (4t^3 + 2t^4)\,\mathrm{d}t$$

$$= -\bar{i}\left(3\frac{t^6}{6} + 2\frac{t^5}{5}\right)_0^1 - \bar{j}\left(4\frac{t^6}{6}\right)_0^1 + \bar{k}\left(t^4 + 2\frac{t^5}{5}\right)_0^1$$

$$= -\frac{9}{10}\bar{i} - \frac{2}{3}\bar{j} + \frac{7}{5}\bar{k}$$

Example 2.3 Find the work done when a force $\bar{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$ displaces a particle in the xy-plane from (0, 0) to (1, 1) along the curve (i) $y = x$, (ii) $y^2 = x$. Comment on the answer.

$$W = \text{Work done by } \bar{F} = \int_C \bar{F} \cdot \mathrm{d}\bar{r}$$

$$= \int_C [(x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}] \cdot (\mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k})$$

$$= \int_C [(x^2 - y^2 + x)\,\mathrm{d}x - (2xy + y)\,\mathrm{d}y]$$

Case (i) C is the line $y = x$.

$$\therefore \quad W_1 = \int_{\substack{y = x \\ (\mathrm{d}y = \mathrm{d}x)}} [(x^2 - y^2 + x)\,\mathrm{d}x - (2xy + y)\,\mathrm{d}y]$$

$$= \int_0^1 (-2x^2)\,\mathrm{d}x$$

$$= -\frac{2}{3}$$

Case (ii) C is the curve $y^2 = x$.

$$\therefore \quad W_2 = \int_{\substack{x = y^2 \\ (\mathrm{d}x = 2y\ \mathrm{d}y)}} [(x^2 - y^2 + x)\,\mathrm{d}x - (2xy + y)\,\mathrm{d}y]$$

$$= \int_0^1 (2y^5 - 2y^3 - y)\,\mathrm{d}y$$

$$= -\frac{2}{3}$$

Comment As the works done by the force, when it moves the particle along two different paths from (0, 0) to (1, 1), are equal, the force may be a conservative force.

In fact, $\bar{F}$ is a conservative force, as $\bar{F}$ is irrotational.

It can be verified that the work done by $\bar{F}$ when it moves the particle from (0, 0) to (1, 1) along any other path (such as $x^2 = y$) is also equal to $-\frac{2}{3}$.

Example 2.4 Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r} = \cos t\bar{i} + \sin t\,\bar{j} + t\bar{k}$ from $t = 0$ to $t = 2\pi$.

From the vector equation of the curve C, we get the parametric equations of the curve as $x = \cos t$, $y = \sin t$, $z = t$.

Work done by $\bar{F} = \int_C \bar{F} \cdot \mathrm{d}\bar{r}$

$$= \int_C (z\bar{i} + x\bar{j} + y\bar{k}) \cdot (\mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k})$$

$$= \int_C (z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z)$$

$$= \int_0^{2\pi} \left[t(-\sin t) + \cos^2 t + \sin t\right]\mathrm{d}t$$

$$= \left[t\cos t - \sin t + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) - \cos t\right]_0^{2\pi}$$

$$= (2\pi + \pi - 1) - (-1)$$

$$= 3\pi$$

Example 2.5 Evaluate $\oint_C \bar{F} \cdot \mathrm{d}\bar{r}$, where $\bar{F} = (\sin y)\bar{i} + x(1 + \cos y)\bar{j} + z\bar{k}$ and C is the circle $x^2 + y^2 = a^2$ in the xy-plane.

Given integral $= \int_C \left[(\sin y)\bar{i} + x(1 + \cos y)\bar{j} + z\bar{k}\right] \cdot (\mathrm{d}x\,\bar{i} + \mathrm{d}y\,\bar{j} + \mathrm{d}z\,\bar{k})$

$$= \int_{\left(\substack{x^2 + y^2 = a^2 \\ z = 0}\right)} \left[\sin y\,\mathrm{d}x + x(1 + \cos y)\,\mathrm{d}y + z\,\mathrm{d}z\right]$$

$$= \int_{x^2 + y^2 = a^2} \left[\sin y\,\mathrm{d}x + x(1 + \cos y)\,\mathrm{d}y\right]$$

Since C is a closed curve, we use the parametric equations of C, namely $x = a\cos\theta$, $y = a\sin\theta$ and the parameter θ as the variable of integration. To move around the circle C once completely, θ varies from 0 to 2 π.

Now, given integral $= \int \left[(\sin y\,\mathrm{d}x + x\cos y\,\mathrm{d}y) + x\,\mathrm{d}y\right]$

$$= \int \left[\mathrm{d}(x\sin y)\,x\,\mathrm{d}y\right]$$

$$= \int_0^{2\pi} \left\{\mathrm{d}\left[a\cos\theta \cdot \sin(a\sin\theta)\right] + a^2\cos\theta\,\mathrm{d}\theta\right\}$$

$$= \left[a\cos\theta \cdot \sin(a\sin\theta) + \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) \right]_0^{2\pi}$$

$$= \pi a^2$$

Example 2.6 Find the work done by the force $\bar{F} = (2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k}$, when it moves a particle from (1, −2, 1) to (3, 1, 4) along any path.

To evaluate the work done by a force, the equation of the path and the terminal points must be given. As the equation of the path is not given in this problem, we guess that the given force $\bar{F}$ is conservative. Let us verify whether $\bar{F}$ is conservative, i.e. irrotational.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= (0 - 0)\bar{i} - (3z^2 - 3z^2)\bar{j} + (2x - 2x)\bar{k}$$

$$= 0$$

∴ $\bar{F}$ is irrotational and hence conservative.

∴ Work done by $\bar{F}$ depends only on the terminal points.

Since $\bar{F}$ is irrotational, let $\bar{F} = \nabla\phi$.

It is easily found that $\phi = x^2 y + z^3 x + c$.

$$\text{Work done by } \bar{F} = \int_{(1,-2,1)}^{(3,1,4)} \bar{F} \cdot d\bar{r}$$

$$= \int_{(1,-2,1)}^{(3,1,4)} \nabla\phi \cdot d\bar{r}$$

$$= \int_{(1,-2,1)}^{(3,1,4)} d\phi$$

$$= [\phi(x, y, z)]_{(1,-2,1)}^{(3,1,4)}$$

$$= \left[x^2 y + z^2 x + c\right]_{(1,-2,1)}^{(3,1,4)}$$

$$= (201 + c) - (-1 + c)$$

$$= 202.$$

Example 2.7 Find the work done by the force $\bar{F} = y(3x^2 y - z^2)\bar{i} + x(2x^2 y - z^2)$ $\bar{j} - 2xyz\bar{k}$, when it moves a particle around a closed curve C.

To evaluate the work done by a force, the equation of the path C and the terminal points must be given.

Since C is a closed curve and the particle moves around this curve once completely, any point (x_0, y_0, z_0) can be taken as the initial as well as the final point.

But the equation of C is not given. Hence we guess that the given force $\bar{F}$ is conservative, i.e. irrotational. Actually it is so, as verified below.

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y^2x^2 - yz^2 & 2x^3y - z^2x & -2xyz \end{vmatrix}$$

$$= (-2xz + 2xz)\bar{i} - (-2yz + 2yz)\bar{j} + (6x^2y - z^2 - 6x^2y + z^2)\bar{k}$$

$$= 0$$

Since $\bar{F}$ is irrotational, let $\bar{F} = \nabla\phi$.

$\therefore$ Work done by $\bar{F} = \oint_C \bar{F}\cdot d\bar{r}$

$$= \oint_C \nabla\phi\cdot d\bar{r}$$

$$= \int_{(x_0, y_0, z_0)}^{(x_0, y_0, z_0)} d\phi$$

$$= \phi(x_0, y_0, z_0) - \phi(x_0, y_0, z_0)$$

$$= 0$$

Example 2.8 Evaluate $\iint_S \bar{A}\cdot d\bar{S}$, where $\bar{A} = 12x^2y\bar{i} - 3yz\bar{j} + 2z\bar{k}$ and S is the portion of the plane $x + y + z = 1$ included in the first octant (Fig. 2.4).

Given integral $I = \iint_S \bar{A}\cdot\hat{n}\, d\bar{S}$, where $\hat{n}$ is the unit normal to the surface S given by

$\phi = c$,

i.e. $x + y + z = 1$

$\therefore$ $\phi = x + y + z$

$$\nabla\phi = \bar{i} + \bar{j} + \bar{k}$$

$$\hat{n} = \frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$$

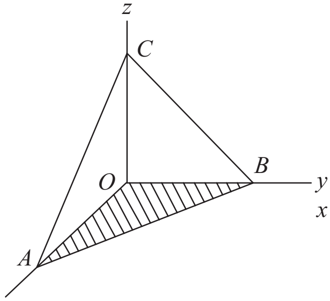


Fig. 2.4

$$\therefore \quad I = \iint_S (12x^2y\bar{i} - 3yz\bar{j} + 2z\bar{k})\cdot\frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})\, dS$$

$$= \frac{1}{\sqrt{3}}\iint_S (12x^2y - 3yz + 2z)\, dS$$

To convert the surface integral as a double integral, we project the surface S on the xoy - plane. Then dS cos γ= dA, where γ is the angle between the surface S and the xoy - plane, i.e. the angle between $\hat{n}$ and $\bar{k}$. $\therefore \cos\gamma = \hat{n}\cdot\bar{k}$

$$\therefore \qquad dS = \frac{dA}{\hat{n}\cdot k} = \frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$\therefore \qquad I = \frac{1}{\sqrt{3}}\iint\limits_{\Delta OAB}(12x^2y - 3yz + 2z)\frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

[$\because$ the projection of S on the xoy plane is ΔOAB]

$$= \iint\limits_{\Delta OAB}\left\{12x^2y - 3y(1-x-y) + 2(1-x-y)\right\}dx\,dy$$

Note ☑ To express the integrand as a function of x and y only, z is expressed as a function of x and y from the equation of S.

$$I = \int_0^1\int_0^{1-y}(12x^2y + 3xy + 3y^2 - 5y - 2x + 2)\,dx\,dy$$

$$= \int_0^1\left[4y(1-y)^3 + \frac{3y}{2}(1-y)^2 + 3y^2(1-y) - 5y(1-y) - (1-y)^2 + 2(1-y)\right]dy$$

$$= \frac{49}{120}$$

Example 2.9 Evaluate $\iint\limits_S \bar{F}\cdot d\bar{S}$, where $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant.

Given integral $I = \iint\limits_S \bar{F}\cdot\hat{n}\,dS$, where $\hat{n}$ is the unit normal to the surface S given by

$\phi = c$ i.e. $\quad x^2 + y^2 + z^2 = 1.$

$\phi = x^2 + y^2 + z^2$

$$\therefore \qquad \nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\therefore \qquad \hat{n} = \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{\sqrt{4(x^2+y^2+z^2)}}$$

$$= \frac{2(x\bar{i} + y\bar{j} + z\bar{k})}{\sqrt{4\times 1}} \qquad [\because \text{the point } (x, y, z) \text{ lies on } S]$$

$$= x\bar{i} + y\bar{j} + z\bar{k}.$$

$$\therefore \qquad I = \iint\limits_S (yz\bar{i} + zx\bar{j} + xy\bar{k})\cdot(x\bar{i} + y\bar{j} + z\bar{k})\,dS$$

$$= \iint_S 3xyz \, dS$$

$$= \iint_R 3xyz \frac{dx\, dy}{\hat{n} \cdot \bar{k}},$$

where R is the region in the xy-plane bounded by the circle $x^2 + y^2 = 1$ and lying in the first quadrant.

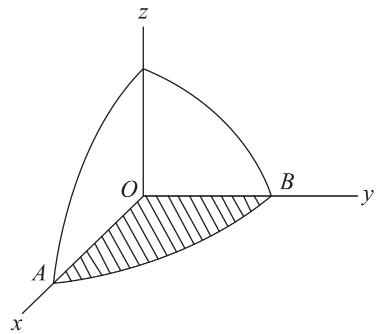


Fig. 2.5

$$I = \iint_R 3xyz \frac{dx\, dy}{z}$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} 3xy \, dx\, dy$$

$$= \int_0^1 3y \left(\frac{x^2}{2}\right)_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2} \int_0^1 y\,(1 - y^2)\, dy$$

$$= \frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)_0^1$$

$$= \frac{3}{8}$$

Example 2.10 Evaluate $\iint_S \bar{F} \cdot dS$ if, $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ contained in the first octant between the planes $z = 0$ and $z = 2$.

Given integral $I = \iint_S \bar{F} \cdot \hat{n}\, dS$, where $\hat{n}$ is the unit normal to the surface S given by $\phi = c$, i.e. $x^2 + y^2 = 9$

$\therefore \qquad \phi = x^2 + y^2$

$$\nabla\phi = 2x\bar{i} + 2y\bar{j}$$

$$\therefore \qquad \hat{n} = \frac{2xi + 2y\bar{j}}{\sqrt{4(x^2 + y^2)}}$$

$$= \frac{2(x\bar{i} + y\bar{j})}{\sqrt{4 \times 9}} \qquad [\because \text{ the point } (x, y, z) \text{ lies on } S]$$

$$= \frac{1}{3}(x\bar{i} + y\bar{j})$$

$$\therefore\ \mathrm{I} = \iint_S (yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k})\cdot\frac{1}{3}(x\bar{i} + y\bar{j})\,\mathrm{d}S$$

$$= \frac{1}{3}\iint_S (xyz + 2y^3)\,\mathrm{d}S$$

$$= \frac{1}{3}\iint_R (xyz + 2y^3)\frac{\mathrm{d}x\,\mathrm{d}z}{\hat{n}\cdot\bar{j}}$$

where R is the rectangular region OABC in the xoz-plane, got by projecting the cylindrical surface S on the xoz-plane (Fig. 2.6).

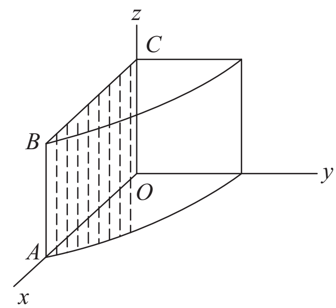


Fig. 2.6

$$\mathrm{I} = \frac{1}{3}\iint_R (xyz + 2y^3)\frac{\mathrm{d}x\,\mathrm{d}z}{\frac{y}{3}}$$

$$= \iint_R (xz + 2y^2)\,\mathrm{d}x\,\mathrm{d}z$$

$$= \int_0^2\int_0^3 [xz + 2(9 - x^2)]\,\mathrm{d}x\,\mathrm{d}z$$

$$= \int_0^2 \left(\frac{9}{2}z + 18\times 3 - 2\times 9\right)\mathrm{d}z$$

$$= \left(\frac{9}{2}\cdot\frac{z^2}{2} + 36\cdot z\right)_0^2 = 81$$

EXERCISE 2(c)

Part A

(Short Answer Questions)

1. Define line integral of a vector point function.
2. When is a force said to be conservative? State also the condition to be satisfied by a conservative force.
3. If $\bar{F}$ is irrotational, prove that it is conservative.
4. Define surface integral of a vector point function.
5. Explain how $\iint_S \bar{F}\cdot\mathrm{d}\bar{S}$ is evaluated.
6. Evaluate $\int_C \bar{r}\cdot\mathrm{d}\bar{r}$, where C is the line $y = x$ in the xy-plane from (1, 1) to (2, 2).
7. Find the work done by the force $\bar{F} = x\bar{i} + 2y\bar{j}$ when it moves a particle on the curve $2y = x^2$ from (0, 0) to (2, 2).
8. Prove that the force $\bar{F} = (2x + yz)\bar{i} + (xz - 3)\bar{j} + xy\bar{k}$ is conservative.

9. Evaluate $\iint_S (yz\bar{i} + zx\bar{j} + xy\bar{k}) \cdot d\bar{S}$, where S is the region bounded by $x = 0$, $x = a$, $y = 0$, $y = b$ and lying in the xoy-plane.
10. Find the work done by a conservative force when it moves a particle around a closed curve.

Part B

11. Evaluate $\int_C \phi \, d\bar{r}$, where $\phi = 2xyz^2$ and C is the curve given by $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$.
12. Evaluate $\int_C \bar{F} \times d\bar{r}$ along the curve $x = \cos t$, $y = 2 \sin t$, $z = \cos t$ from $t = 0$ to $t = \dfrac{\pi}{2}$, given that $\bar{F} = 2x\bar{i} + y\bar{j} + z\bar{k}$.
13. Evaluate $\int_C \bar{F} \times d\bar{r}$ along the curve $x = \cos\theta$, $y = \sin\theta$, $z = 2\cos\theta$ from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$, given that $\bar{F} = 2y\bar{i} - z\bar{j} + x\bar{k}$
14. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ along the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, given that $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$.
15. Find the work done by the force $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, when it moves a particle along the curve $y = 2x^2$ in the xy-plane from (0, 0) to (1, 2).
16. Find the work done by the force $\bar{F} = (y + 3z)\bar{i} + (2z + x)\bar{j} + (3x + 2y)\bar{k}$, when it moves a particle along the curve $x = a \cos t$, $y = a \sin t$, $z = \dfrac{2at}{\pi}$ between the points $(a, 0, 0)$ and $(0, a, a)$.
17. Find the work done by the force $\bar{F} = (2y + 3)\bar{i} + xz\bar{j} + (yz - x)\bar{k}$ when it moves a particle along the line segment joining the origin and the point (2, 1, 1).
18. Evaluate $\oint_C \bar{F} \cdot d\bar{r}$, where C is the circle $x^2 + y^2 = 4$ in the xy-plane, if $\bar{F} = (2x - y - z)\bar{i} + (x + y - z^2)\bar{j} + (3x - 2y + 4z)\bar{k}$.
19. Find the work done by the force $\bar{F} = (x^2 + y^2)\bar{i} + (x^2 + z^2)\bar{j} + y\bar{k}$, when it moves a particle along the upper half of the circle $x^2 + y^2 = 1$ from the point (−1, 0) to the point (1, 0).
20. Find the work done by the force $\bar{F} = (e^x z - 2xy)\bar{i} + (1 - x^2)\bar{j} + (e^x + z)\bar{k}$, when it moves a particle from (0, 1, −1) to (2, 3, 0) along any path.
21. Find the work done by the force $\bar{F} = (y^2 \cos x + z^3)\bar{i} + (2y \sin x - 4)\bar{j} + (3xz^2 + 2)\bar{k}$, when it moves a particle from (0, 1, −1) to $\left(\dfrac{\pi}{2}, -1, 2\right)$ along any path.

22. Find the work done by the force $\overline{F} = y^2\overline{i} + 2(xy+z)\overline{j} + 2y\overline{k}$, when it moves a particle around a closed curve C.
23. Evaluate $\iint_S \overline{F}\cdot \mathrm{d}\overline{S}$, where $\overline{F} = xy\overline{i} - x^2\overline{j} + (x+z)\overline{k}$ and S is the part of the plane $2x + 2y + z = 6$ included in the first octant.
24. Evaluate $\iint_S \overline{F}\cdot \mathrm{d}\overline{S}$, where $\overline{F} = y\overline{i} - x\overline{j} + 4\overline{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.
25. Evaluate $\iint_S \overline{F}\cdot \mathrm{d}\overline{S}$, where $\overline{F} = z\overline{i} + x\overline{j} - 3y^2z\overline{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between the planes $z = 0$ and $z = 5$.

2.5 INTEGRAL THEOREMS

The following three theorems in Vector Calculus are of importance from theoretical and practical considerations:

1. Green's theorem in a plane
2. Stoke's theorem
3. Gauss Divergence theorem

Green's theorem in a plane provides a relationship between a line integral and a double integral.

Stoke's theorem, which is a generalisation of Green's theorem, provides a relationship between a line integral and a surface integral. In fact, Green's theorem can be deduced as a particular case of Stoke's theorem. Gauss Divergence theorem provides a relationship between a surface integral and a volume integral.

We shall give the statements of the above theorems (without proof) below and apply them to solve problems:

2.5.1 Green's Theorem in a Plane

If C is a simple closed curve enclosing a region R in the xy-plane and $P(x, y)$, $Q(x, y)$ and its first order partial derivatives are continuous in R, then

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

where C is described in the anticlockwise direction.

2.5.2 Stoke's Theorem

If S is an open two sided surface bounded by a simple closed curve C and if $\overline{F}$ is a vector point function with continuous first order partial derivatives on S, then

$$\oint_C \overline{F}\cdot \mathrm{d}\overline{r} = \iint_S \text{curl } \overline{F}\cdot \mathrm{d}\overline{S},$$

where C is described in the anticlockwise direction as seen from the positive tip of the outward drawn normal at any point of the surface S.

2.5.3 Gauss Divergence Theorem

If S is a closed surface enclosing a region of space with volume V and if $\bar{F}$ is a vector point function with continuous first order partial derivatives in V, then

$$\iint_S \bar{F}\cdot d\bar{S} = \iiint_V (\text{div}\,\bar{F})\, dv\,.$$

2.5.4 Deduction of Green's Theorem from Stoke's Theorem

Stoke's theorem is $$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot d\bar{S} \tag{1}$$

Take S to be a plane surface (region) R in the xy-plane bounded by a simple closed curve C.

Also take $\bar{F} = P(x, y)\bar{i} + Q(x, y)\bar{j}$

$$\text{Then curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\bar{k}$$ [$\because$ P and Q are functions of x and y]

Inserting all these in (1), we get

$$\oint_C (P\bar{i} + Q\bar{j})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k}) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\bar{k}\cdot d\bar{S}$$

i.e. $$\oint_C (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,\bar{k}\cdot\hat{n}\,dS \tag{2}$$

Now $\hat{n}$ is the unit vector in the outward drawn normal direction to the surface R. Normal at any point of R, that lies in the xy-plane, is parallel to the z-axis.

Taking the positive direction of the z-axis as the positive (outward) direction of $\hat{n}$, we get $\hat{n} = \bar{k}$.

Using this in (2), we get

$$\oint_C (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dx\,dy$$

[$\because$ dS = elemental plane surface area in the xy-plane
= $dx\,dy$]

Note ☑ If the surface S is a plane surface, problems in Stoke's theorem will reduce to problems in Green's theorem in a plane.

2.5.5 Scalar form of Stoke's Theorem

Take $\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$ in Stoke's theorem, where P, Q, R are functions of x, y, z.

Then $$\bar{F}\cdot \mathrm{d}\bar{r} = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z \tag{1}$$

$$\text{Curl } \bar{F} = (R_y - Q_z)\bar{i} + (P_z - R_x)\bar{j} + (Q_x - P_y)\bar{k}$$

$\therefore$ $$\text{Curl } \bar{F}\cdot \mathrm{d}\bar{S} = \text{curl } \bar{F}\cdot \hat{n}\,\mathrm{d}S$$
$$= (R_y - Q_z)(\hat{n}\cdot\bar{i})\mathrm{d}S + (P_z - R_x)(\hat{n}\cdot\bar{j})\mathrm{d}S + (Q_x - P_y)(\hat{n}\cdot\bar{k})\mathrm{d}S$$
$$= (R_y - Q_z)\,\mathrm{d}y\,\mathrm{d}z + (P_z - R_x)\,\mathrm{d}z\,\mathrm{d}x + (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y \tag{2}$$

[$\because \mathrm{d}S\cos\gamma = \mathrm{d}x\,\mathrm{d}y$ i.e. $\mathrm{d}S\,(\hat{n}\cdot\bar{k}) = \mathrm{d}x\,\mathrm{d}y$]

Inserting (1) and (2) in Stoke's theorem, it reduces to the scalar form

$$\int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z) = \iint_S [(R_y - Q_z)\,\mathrm{d}y\,\mathrm{d}z + (P_z - R_x)\,\mathrm{d}z\,\mathrm{d}x + (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y] \tag{3}$$

Note ☑ If we take P and Q as functions of x and y only and $R = 0$ in (3), we get Green's theorem in a plane.

2.5.6 Scalar form of Gauss Divergence Theorem

Take $\bar{F} = P\bar{i} + Q\bar{j} + R\bar{k}$ in Divergence theorem, where P, Q, R are functions of x, y, z.

Then div $$\bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{1}$$

$$\bar{F}\cdot\mathrm{d}\bar{S} = \bar{F}\cdot\hat{n}\,\mathrm{d}S$$
$$= P(\hat{n}\cdot\bar{i})\,\mathrm{d}S + Q\,(\hat{n}\cdot\bar{j})\,\mathrm{d}S + R\,(\hat{n}\cdot\bar{k})\,\mathrm{d}S$$
$$= P\,\mathrm{d}y\,\mathrm{d}z + Q\,\mathrm{d}z\,\mathrm{d}x + R\,\mathrm{d}x\,\mathrm{d}y \tag{2}$$

Inserting (1) and (2) in Divergence theorem, we get

$$\iint_S (P\mathrm{d}y\,\mathrm{d}z + Q\mathrm{d}z\,\mathrm{d}x + R\mathrm{d}x\,\mathrm{d}y) = \iiint_V (P_x + Q_y + R_z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{3}$$

which is the scalar form of Divergence theorem.

Note ☑ (3) is also called *Green's theorem in space*.

2.5.7 Green's Identities

In Divergence theorem, take $\bar{F} = \phi\nabla\psi$, where ϕ and ψ are scalar point functions.
Then div $(\phi\,\nabla\,\psi) = \nabla\cdot(\phi\,\nabla\,\psi)$
$$= \phi\,\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$\therefore$ Divergence theorem becomes

$$\iint_S \phi\nabla\psi\cdot\mathrm{d}\bar{S} = \iiint_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,\mathrm{d}V \tag{1}$$

Interchanging ϕ and ψ,

$$\iint_S \psi \nabla \phi \cdot d\bar{S} = \iiint_V (\psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi) dV \quad (2)$$

(1) $-$ (2) gives,

$$\iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\bar{S} = \iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV \quad (3)$$

Note ☑ (1) or (2) is called *Green's first identity* and (3) is called *Green's second identity*.

WORKED EXAMPLE 2(d)

Example 2.1 Verify Green's theorem in a plane with respect to $\int_C (x^2 - y^2)\, dx + 2xy\, dy$, where C is the boundary of the rectangle in the xoy-plane bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.

(OR)

Verify Stoke's theorem for a vector field defined by $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ in the rectangular region in the xoy-plane bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.

Stoke's theorem is $\int_C \bar{F} \cdot d\bar{r} = \iint_S \text{curl}\, \bar{F} \cdot d\bar{S}$

Now
$$\text{curl}\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$
$$= (2y + 2y)\,\bar{k}$$

$\therefore$ We have to verify that

$$\int_C [(x^2 - y^2)\bar{i} + 2xy\,\bar{j}] \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k}) = \iint_S 4y\bar{k} \cdot \hat{n}\, dS$$

i.e.
$$\int_C [(x^2 - y^2)\, dx + 2xy\, dy] = \iint_R 4y\, dx\, dy \text{ [Fig. 2. 7]} \quad (1)$$
$$\left[\because \hat{n} = \bar{k}\right]$$

(1) is also the result for the given function as per Green's theorem.

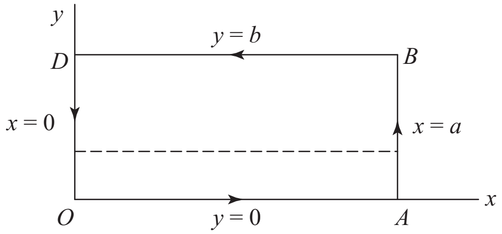


Fig. 2.7

$$\text{L.S. of (1)} = \int\limits_{\substack{OA \\ \begin{pmatrix} y=0 \\ dy=0 \end{pmatrix}}} + \int\limits_{\substack{AB \\ \begin{pmatrix} x=a \\ dx=0 \end{pmatrix}}} + \int\limits_{\substack{BD \\ \begin{pmatrix} y=b \\ dy=0 \end{pmatrix}}} + \int\limits_{\substack{DO \\ \begin{pmatrix} x=0 \\ dx=0 \end{pmatrix}}} [(x^2 - y^2)\,dx + 2xy\,dy]$$

[$\because$ the boundary C consists of 4 lines]

$$= \int_0^a x^2 dx + \int_0^b 2ay\,dy + \int_a^0 (x^2 - b^2)\,dx + 0$$

Note ☑ To simplify the line integral along each line, we make use of the equation of the line and the corresponding value of dx or dy

$$= \left(\frac{x^3}{3}\right)_0^a + a(y^2)_0^b - \left(\frac{x^3}{3} - b^2 x\right)_0^a$$

$$= 2\,ab^2$$

$$\text{R.S. of (1)} = \int_0^b \int_0^a 4y\,dx\,dy$$

$$= \int_0^b 4y\,(x)_0^a\,dy = 2a(y^2)_0^b$$

$$= 2\,a\,b^2.$$

Since L.S. of (1) = R.S. of (1), Stoke's theorem (Green's theorem) is verified.

Example 2.2 Verify Green's theorem in a plane for $\int\limits_C [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$, where C is the boundary of the region defined by the lines $x = 0$, $y = 0$ and $x + y = 1$.

Green's theorem is $\int\limits_C (P\,dx + Q\,dy) = \iint\limits_R (Q_x - P_y)\,dx\,dy$

$\therefore$ For the given integral,

$$\int\limits_C [3x^2 - 8y^2]\,dx + (4y - 6xy)\,dy] = \iint\limits_R 10y\,dx\,dy \text{ [Fig. 2.8]} \qquad (1)$$

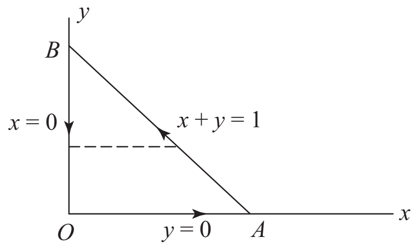


Fig. 2.8

$$\text{L.S. of (1)} = \int\limits_{\substack{OA \\ \begin{pmatrix} y=0 \\ dy=0 \end{pmatrix}}} + \int\limits_{\substack{AB \\ \begin{pmatrix} x+y=1 \\ x=1-y \\ dx=-dy \end{pmatrix}}} + \int\limits_{\substack{BO \\ \begin{pmatrix} x=0 \\ dx=0 \end{pmatrix}}} [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$$

$$= \int_0^1 3x^2 \mathrm{d}x + \int_0^1 [\{3(1-y)^2 - 8y^2\}(-\mathrm{d}y) + \{4y - 6y\,(1-y)\}\,\mathrm{d}y] + \int_1^0 4y\,\mathrm{d}y$$

$$= \int_0^1 3x^2\,\mathrm{d}x + \int_0^1 (11y^2 + 4y - 3)\,\mathrm{d}y - \int_0^1 4y\,\mathrm{d}y = 1 + \left(\frac{11}{3} + 2 - 3\right) - 2$$

$$= \frac{5}{3}$$

$$\text{R.S. of (1)} = \int_0^1 \int_0^{1-y} 10y\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 10y\,(1-y)\,\mathrm{d}y$$

$$= \left(5y^2 - 10\frac{y^3}{3}\right)_0^1 = \frac{5}{3}$$

Since L.S. of (1) = R.S. of (1), Green's theorem is verified.

Example 2.3 Use Stoke's theorem to evaluate $\int_C \bar{F} \cdot \mathrm{d}\bar{r}$, where $\bar{F} = (\sin x - y)\,\bar{i} - \cos x\bar{j}$ and C is the boundary of the triangle whose vertices are (0, 0), $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

Note ☑ Evaluating $\int_C \bar{F} \cdot \mathrm{d}\bar{r}$ by using Stoke's theorem means expressing the line integral in terms of its equivalent surface integral and then evaluating the surface integral.

By Stoke's theorem, $\int_C \bar{F} \cdot \mathrm{d}\bar{r} = \iint_S \operatorname{curl} \bar{F} \cdot \mathrm{d}\bar{S}$, where S is any open two-sided surface bounded by C. [Fig. 2.9]

To simplify the work, we shall choose S as the plane surface R in the xoy-plane bounded by C.

$$\therefore \int_C \bar{F} \cdot \mathrm{d}\bar{r} = \iint_R \operatorname{curl} \bar{F} \cdot \bar{k}\,\mathrm{d}x\,\mathrm{d}y \quad [\because \text{ For the } xoy\text{-plane, } \hat{n} = \bar{k} \text{ and } \mathrm{d}S = \mathrm{d}x\,\mathrm{d}y]$$

For this problem,

$$\operatorname{curl} \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\sin x - y) & -\cos x & 0 \end{vmatrix}$$

$$= (\sin x + 1)\bar{k}$$

$\therefore$ The given line integral $= \iint_R (1+\sin x)\,\mathrm{d}x\,\mathrm{d}y$

$$= \int_0^1 \int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (1+\sin x)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \left(x - \cos x\right)_{\frac{\pi y}{2}}^{\frac{\pi}{2}}\,\mathrm{d}y$$

$$= \int_0^1 \left(\frac{\pi}{2} - \frac{\pi y}{2} + \cos\frac{\pi y}{2}\right)\mathrm{d}y$$

$$= \left(\frac{\pi}{2}y - \frac{\pi y^2}{4} + \frac{2}{\pi}\sin\frac{\pi y}{2}\right)_0^1$$

$$= \frac{\pi}{4} + \frac{2}{\pi}.$$

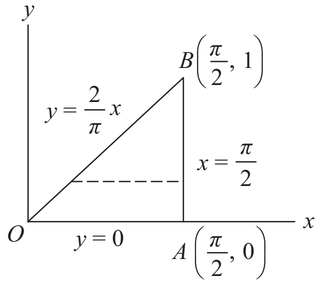


Fig. 2.9

Example 2.4 Use Green's theorem in a plane to evaluate $\oint_C [(2x - y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y]$, where C is the boundary of the circle $x^2 + y^2 = a^2$ in the xoy-plane.

By Green's theorem in a plane,

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\therefore \oint_C [(2x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y] = \iint_R [1-(-1)]\,\mathrm{d}x\,\mathrm{d}y$$

$$= 2\iint_R \mathrm{d}x\,\mathrm{d}y \text{[Fig. 2.10]}$$

$$= 2 \times \text{area of the region } R$$

$$= 2\pi a^2$$

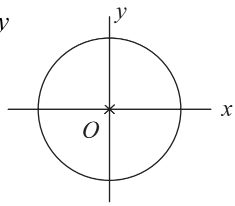


Fig. 2.10

Example 2.5 Verify Stoke's theorem when $\bar{F} = (2xy - x^2)\,\bar{i} - (x^2 - y^2)\,\bar{j}$ and C is the boundary of the region enclosed by the parabolas $y^2 = x$ and $x^2 = y$.

$$\text{Stoke's theorem is } \int_C \bar{F}\cdot \mathrm{d}\bar{r} = \iint_S \text{curl } \bar{F}\cdot\hat{n}\,\mathrm{d}S$$

$$\text{Now curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - x^2 & -x^2 + y^2 & 0 \end{vmatrix}$$

$$= (-2x - 2x)\bar{k}$$

$$= -4x\bar{k}$$

∴ Stoke's theorem becomes

$$\int_C [(2xy - x^2)\,dx - (x^2 - y^2)\,dy] = \iint_R -4x\bar{k}\cdot\bar{k}\,dx\,dy$$

$$= \iint_R -4x\,dx\,dy \text{ [Fig. 2.11]} \qquad (1)$$

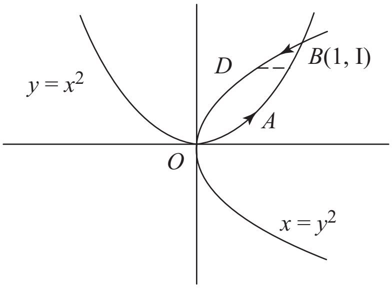


Fig. 2.11

$$\text{L.S. of (1)} = \int_{\substack{OAB \\ \left(\substack{y=x^2 \\ dy=2x\,dx}\right)}} + \int_{\substack{BDO \\ \left(\substack{x=y^2 \\ dx=2y\,dy}\right)}} [(2xy - x^2)\,dx - (x^2 - y^2)\,dy]$$

$$= \int_0^1 (2x^5 - x^2)\,dx + \int_1^0 (3y^4 - 2y^5 + y^2)\,dy$$

[∵ the coordinates of B are found as (1, 1) by solving the equations $y = x^2$ and $x = y^2$]

$$= \frac{-3}{5}$$

$$\text{R.S. of (1)} = \int_0^1 \int_{y^2}^{\sqrt{y}} -4x\,dx\,dy$$

$$= -2\int_0^1 (x^2)_{y^2}^{\sqrt{y}}\,dy$$

$$= -2\int_0^1 (y - y^4)\,dy$$

$$= -\frac{3}{5}$$

Since L.S. of (1) = R.S. of (1), Stoke's theorem is verified.

Example 2.6 Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_C (x\,dy - y\,dx)$ Hence find the area bounded (i) by the parabola $y^2 = 4a\,x$ and its latus rectum and (ii) by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\text{By Green's theorem, } \int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R (Q_x - P_y)\mathrm{d}x\,\mathrm{d}y$$

Taking $P = -\dfrac{y}{2}$ and $Q = \dfrac{x}{2}$, we get

$$\frac{1}{2}\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x) = \iint_R \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right] \mathrm{d}x\,\mathrm{d}y$$

$$= \iint_R \mathrm{d}x\,\mathrm{d}y$$

=Area of the region R enclosed by C.

(i) Area bounded by the parabola and its latus rectum $= \dfrac{1}{2}\displaystyle\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$ [Fig. 2.12]

$$= \frac{1}{2}\left[\int_{LOL'} + \int_{L'SL} (x\,\mathrm{d}y - y\,\mathrm{d}x)\right]$$

$$= \frac{1}{2}\left[\int_{\left(\substack{x=\frac{y^2}{4a} \\ \mathrm{d}x=\frac{y}{2a}\mathrm{d}y}\right)} (x\,\mathrm{d}y - y\,\mathrm{d}x) + \int_{\substack{x=a \\ \mathrm{d}x=0}} (x\,\mathrm{d}y - y\,\mathrm{d}x)\right]$$

$$= \frac{1}{2}\left[\int_{2a}^{-2a} -\frac{y^2}{4a}\mathrm{d}y + \int_{-2a}^{2a} a\,\mathrm{d}y\right]$$

$$= \int_0^{2a} \frac{y^2}{4a}\mathrm{d}y + \int_0^{2a} a\,\mathrm{d}y$$

$$= \frac{8}{3}a^2.$$

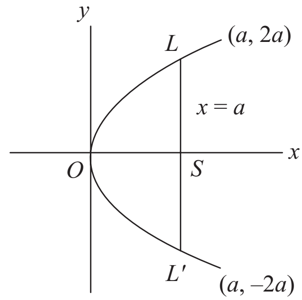


Fig. 2.12

(ii) Area bounded by the ellipse

$$= \frac{1}{2}\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$$

$$= \frac{1}{2}\int_{\left(\substack{x=a\cos\theta \\ y=b\sin\theta}\right)} (x\,\mathrm{d}y - y\,\mathrm{d}x) \quad \text{[Fig. 2.13]}$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)\,\mathrm{d}\theta$$

$$= \pi ab.$$

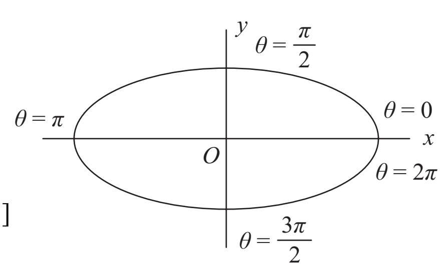


Fig. 2.13

Example 2.7 Use Stoke's theorem to prove that (i) curl (grad ϕ) = 0 and (ii) div (curl $\bar{F}$) = 0.

(i) Stoke's theorem is $\iint_S \text{curl } \bar{F} \cdot d\bar{S} = \oint_C \bar{F} \cdot d\bar{r}$

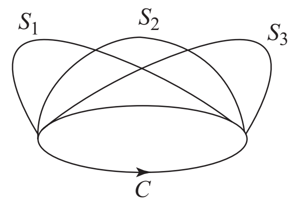


Fig. 2.14

Taking $\bar{F} = \text{grad } \phi$, we have

$$\iint_S \text{curl (grad } \phi) \cdot d\bar{S} = \oint_C \text{grad } \phi \cdot d\bar{r}$$
$$= \oint_C d\phi$$
$$= 0$$

The above result is true for any open two-sided surface S, provided it is bounded by the same simple closed curve C. [Fig. 2.14]

$\therefore$ curl (grad ϕ) $\cdot\, d\bar{S} = 0$, [for any S and hence for any $d\bar{S}$]

$\therefore$ curl (grad ϕ) $= 0$

(ii) Gauss divergence theorem is

$\iiint_V (\text{div } \bar{F})\, dv = \iint_S \bar{F} \cdot d\bar{S}$, where S is a closed surface enclosing a volume V.

Replacing $\bar{F}$ by curl $\bar{F}$, we have

$$\iiint_V \text{div (curl } \bar{F})\, dv = \iint_S \text{curl } \bar{F} \cdot d\bar{S} \tag{1}$$

The surface integral in (1) appears to be the same as that in Stoke's theorem. However Stoke's theorem cannot be used to simplify the R.S. of (1), as S is a closed surface. In order to enable us to use Stoke's theorem, we divide S into two parts S_1 and S_2 by a plane. S_1 and S_2 are open two-sided surfaces, each of which is bounded by the same closed curve C as given in Fig. 2.15.

Now (1) becomes

$$\iiint_V \text{div (curl } \bar{F})\, dv = \iint_{S_1} \text{curl } \bar{F} \cdot d\bar{S} + \iint_{S_2} \text{curl } \bar{F} \cdot d\bar{S}$$
$$= \oint_C \bar{F} \cdot d\bar{r} + \oint_C \bar{F} \cdot d\bar{r}$$

[by Stoke's theorem]

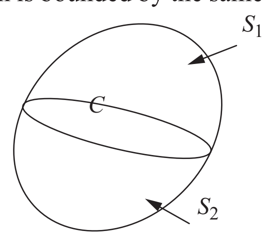


Fig. 2.15

Note ☑ If C is described in the anticlockwise sense as seen from the positive tip of the outer normal to S_1, it will be described in the clockwise sense as seen from the positive tip of the outer normal to S_2.

$$\therefore \quad \iiint_V \text{div (curl } \bar{F})\, dv = \oint_C \bar{F} \cdot d\bar{r} - \oint_C \bar{F} \cdot d\bar{r}$$
$$= 0.$$

Since V is arbitrary, div (curl $\bar{F}$) $= 0$.

Example 2.8 Evaluate $\int_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$, by using Stoke's theorem, where C is the boundary of the rectangle defined by $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$.

The scalar form of Stoke's theorem is $\int_C (P\,dx + Q\,dy + R\,dz) = \iint_S [(R_y - Q_z)$ $dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy]$

Taking $P = \sin z$, $Q = -\cos x$, $R = \sin y$, we get

$$\int_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$$

$$= \iint_S (\cos y\,dy\,dz + \cos z\,dz\,dx + \sin x\,dx\,dy)$$

$$= \iint_S \sin x\,dx\,dy \qquad [\because S \text{ is the rectangle in the } z = 3 \text{ plane and hence } dz = 0]$$

$$= \int_0^1 \int_0^\pi \sin x\,dx\,dy$$

$$= -(\cos x)_0^\pi$$

$$= 2$$

Example 2.9 Evaluate $\iint_S (x\,dy\,dz + 2y\,dz\,dx + 3z\,dx\,dy)$ where S is the closed surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Scalar from of divergence theorem is

$$\iint_S (P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy) = \iiint_V (P_x + Q_y + R_z)\,dV$$

Taking $P = x$, $Q = 2y$, $R = 3z$,

the given surface integral = $\iiint_V 6\,dv = 6V$

$$= 6 \times \frac{4}{3}\pi a^3$$

$$= 8\pi a^3.$$

Example 2.10 Verify Stoke's theorem for $\bar{F} = xy\bar{i} - 2yz\,\bar{j} - zx\bar{k}$ where S is the open surface of the rectangular parallelopiped formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ above the xoy-plane.

$$\text{Stoke's theorem is } \int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl } \bar{F}\cdot d\bar{S}$$

Here
$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$$

$$= 2y\bar{i} + z\bar{j} + x\bar{k}$$

$\therefore$ Stoke's theorem takes form

$$\int_C (xy\,\mathrm{d}x - 2yz\,\mathrm{d}y - zx\,\mathrm{d}z) = \iint_S (2y\bar{i} + z\bar{j} - x\bar{k})\cdot \mathrm{d}\bar{S} \tag{1}$$

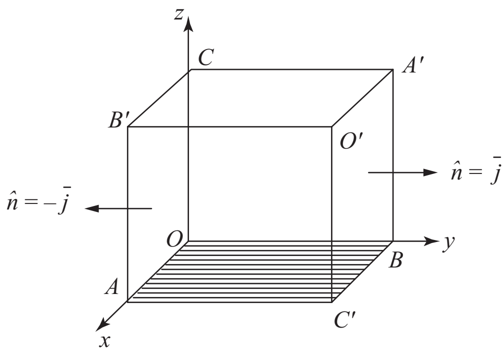


Fig. 2.16

The open cuboid S is made up of the five faces $x = 0$, $x = 1$, $y = 0$, $y = 2$ and $z = 3$ and is bounded by the rectangle $OAC'B$ lying on the xoy-plane. [Fig. 2.16]

$$\text{L.S. of (1)} = \int_{OAC'B} (xy\,\mathrm{d}x - 2yz\,\mathrm{d}y - zx\,\mathrm{d}z)$$

$$= \int_{OAC'B} xy\,\mathrm{d}x \quad \text{[Fig. 2.17]} \qquad (\because \text{ the boundary } C \text{ lies on the plane } z = 0)$$

$$= \int_{\substack{OA\\ \begin{pmatrix} y=0\\ \mathrm{d}y=0\end{pmatrix}}} + \int_{\substack{AC'\\ \begin{pmatrix} x=1\\ \mathrm{d}x=0\end{pmatrix}}} + \int_{\substack{C'B\\ \begin{pmatrix} y=2\\ \mathrm{d}y=0\end{pmatrix}}} + \int_{\substack{BO\\ \begin{pmatrix} x=0\\ \mathrm{d}x=0\end{pmatrix}}} (xy\,\mathrm{d}x)$$

$$= \int_1^0 2x\,\mathrm{d}x$$

$$= -1.$$

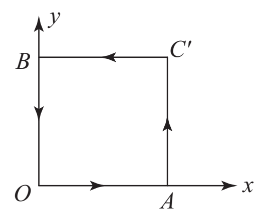


Fig. 2.17

$$\text{R.S. of (1)} = \iint_{\begin{pmatrix} x=0\\ \hat{n}=-\bar{i}\end{pmatrix}} + \iint_{\begin{pmatrix} x=1\\ \hat{n}=\bar{i}\end{pmatrix}} + \iint_{\begin{pmatrix} y=0\\ \hat{n}=-\bar{j}\end{pmatrix}} + \iint_{\begin{pmatrix} y=2\\ \hat{n}=\bar{j}\end{pmatrix}} + \iint_{\begin{pmatrix} z=3\\ \hat{n}=\bar{k}\end{pmatrix}} (2y\bar{i} + z\bar{j} - x\bar{k})\cdot \hat{n}\,\mathrm{d}s$$

Note ☑ $\hat{n}$ is the unit normal at any point of the concerned surface. For example, at any point of the plane surface $y = 0$, the outward drawn normal is parallel to the y-axis, but opposite in direction. $\therefore$ $\hat{n}$ at any point of $y = 0$ is equal to $-\bar{j}$.

Similarly $\hat{n}$ for $y = 2$ is equal to $\bar{j}$ and so on.
Using the relevant value of $\hat{n}$ and simplifying the integrand, we have

$$\text{R. S. of (1)} = -\iint_{x=0} 2y\,\mathrm{d}S + \iint_{x=1} 2y\,\mathrm{d}S - \iint_{y=0} z\,\mathrm{d}S + \iint_{y=2} z\,\mathrm{d}S - \iint_{z=3} x\,\mathrm{d}S$$

$$= -\int_0^3\int_0^2 2y\,dy\,dz + \int_0^3\int_0^2 2y\,dy\,dz - \int_0^1\int_0^3 z\,dz\,dx + \int_0^1\int_0^3 z\,dz\,dx - \int_0^2\int_0^1 x\,dx\,dy$$

[∵ Elemental plane (surface) area dS on $x = 0$ and $x = 1$ are equal, each equal to dy dz etc.]

$$= -\int_0^2\int_0^1 x\,dx\,dy \qquad (\because \text{ the other integrals cancel themselves})$$

$$= -\int_0^2 \left(\frac{x^2}{2}\right)_0^1 dy$$

$$= -1.$$

Thus Stoke's theorem is verified.

Example 2.11 Verify Stoke′s theorem for $\bar{F} = y^2 z\bar{i} + z^2 x\bar{j} + x^2 y\bar{k}$ where S is the open surface of the cube formed by the planes $x = \pm a$, $y = \pm a$ and $z = \pm a$, in which the plane $z = -a$ is cut.

$$\text{Stoke's theorem is } \int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl } \bar{F}\cdot d\bar{S}$$

Here

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= (x^2 - 2zx)\bar{i} + (y^2 - 2xy)\bar{j} + (z^2 - 2yz)\bar{k}$$

∴ Stoke's theorem takes the form

$$\int_C (y^2 z\,dx + z^2 x\,dy + x^2 y\,dz) = \iint_S [(x^2 - 2zx)\bar{i} + (y^2 - 2xy)\bar{j} + (z^2 - 2yz)\bar{k}]\cdot d\bar{S} \qquad (1)$$

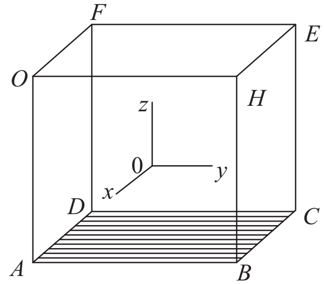


Fig. 2.18

The open cube S is bounded by the square ABCD that lies in the plane $z = -a$ (Fig. 2.18)

$$\text{L.S. of (1)} = \int_{(z=-a)} (y^2 z\,dx + z^2 x\,dy + x^2 y\,dz)$$

$$= \int_{ABCD} (-ay^2 dx + a^2 x\,dy) \qquad [\because dz = 0, \text{ as } z = -a]$$

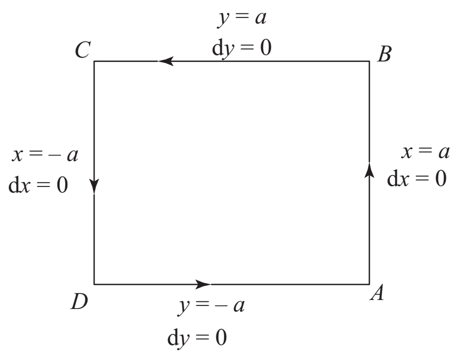


Fig. 2.19

$$\text{L.S. of (1)} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} (-ay^2\,dx + a^2\,x\,dy)\ \big(\text{Fig. 2.19}\big)$$

$$= \int_{-a}^{a} a^3 dy - \int_{a}^{-a} a^3 dx - \int_{a}^{-a} a^3 dy - \int_{-a}^{a} a^3 dx$$

$$= 4\,a^4.$$

$$\text{R.S. of (1)} = \iint_{\left(\substack{x=-a\\ \hat{n}=-\bar{i}}\right)} + \iint_{\left(\substack{x=a\\ \hat{n}=\bar{i}}\right)} + \iint_{\left(\substack{y=-a\\ \hat{n}=-\bar{j}}\right)} + \iint_{\left(\substack{y=a\\ \hat{n}=\bar{j}}\right)}$$

$$+ \iint_{\left(\substack{z=a\\ \hat{n}=\bar{k}}\right)} [(x^2 - 2zx)\,\bar{i} + (y^2 - 2xy)\,\bar{j} + (z^2 - 2yz)\,\bar{k}] \cdot \hat{n} dS$$

$$= \iint_{x=-a} (2zx - x^2)\,dS + \iint_{x=a} (x^2 - 2zx)\,dS + \iint_{y=-a} (2xy - y^2)\,dS$$

$$+ \iint_{y=a} (y^2 - 2xy)\,dS + \iint_{z=a} (z^2 - 2yz)\,dS$$

$$= \iint (-2az - a^2)\,dy\,dz + \iint (a^2 - 2az)\,dy\,dz$$

$$+ \iint (-2ax - a^2)\,dz\,dx + \iint (a^2 - 2ax)\,dz\,dx$$

$$+ \iint (a^2 - 2ay)\,dx\,dy$$ [using the equations of the planes to simplify the integrands]

$$= \int_{-a}^{a}\int_{-a}^{a} -4az\,dy\,dz + \int_{-a}^{a}\int_{-a}^{a} -4ax\,dz\,dx$$

$$+ \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2ay)\,dx\,dy$$

$$= 0 + 0 + 4a^4 = 4a^4.$$

Thus Stokes' theorem is verified.

Example 2.12 Verify Stokes' theorem for $\overline{F} = -y\overline{i} + 2yz\overline{j} + y^2\overline{k}$, where S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ and C is the circular boundary on the xoy-plane.

$$\text{Curl } \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & 2yz & y^2 \end{vmatrix}$$

$$= \overline{k}.$$

Stoke's theorem is $\int\limits_C \overline{F} \cdot \mathrm{d}\overline{r} = \iint\limits_S \overline{F} \cdot \mathrm{d}\overline{S}$

Here $$\oint\limits_C (-y \,\mathrm{d}x + 2yz \,\mathrm{d}y + y^2 \,\mathrm{d}z) = \iint\limits_S \overline{k} \cdot \mathrm{d}\overline{S} \qquad (1)$$

C is the circle in the xoy-plane whose equation is $x^2 + y^2 = a^2$ and whose parametric equations are $x = a \cos\theta$ and $y = a \sin\theta$.

$\therefore$ $$\text{L.S. of (1)} = \int\limits_{x^2+y^2=a^2} -y \,\mathrm{d}x \qquad (\because C \text{ lies on } z = 0)$$

$$= \int_0^{2\pi} a^2 \sin^2\theta \,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$= \pi a^2.$$

$$\text{R.S. of (1)} = \iint\limits_S \overline{k} \cdot \hat{n} \,\mathrm{d}S, \text{ where}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}, \text{ where } \phi = x^2 + y^2 + z^2$$

$$= \frac{2(x\overline{i} + y\overline{j} + z\overline{k})}{\sqrt{4(x^2 + y^2 + z^2)}}$$

$$= \frac{x\overline{i} + y\overline{j} + z\overline{k}}{a} \qquad [\because \text{the point } (x, y, z) \text{ lies on } \phi = a^2]$$

$\therefore$ $$\text{R.S. of (1)} = \iint\limits_S \frac{z}{a} \,\mathrm{d}S$$

$$= \iint\limits_R \frac{z}{a} \frac{\mathrm{d}x \,\mathrm{d}y}{\hat{n} \cdot \overline{k}}, \text{ where } R \text{ is the projection of } S \text{ on the } xoy\text{-plane.}$$

$$= \iint\limits_R \mathrm{d}x \,\mathrm{d}y, \text{ where } R \text{ is the region enclosed by } x^2 + y^2 = a^2.$$

$$= \pi a^2$$

Thus Stoke's theorem is verified.

Example 2.13 If S is the surface of the sphere $x^2 + y^2 + z^2 = 1$, evaluate $\iint_S (x\bar{i} + 2y\bar{j} + 3z\bar{k}) \cdot d\bar{S}$.

By Divergence theorem,

$$\iint_S \bar{F} \cdot d\bar{S} = \iiint_V (\text{div } \bar{F})\, dv$$

$$\therefore \quad \iint_S \bar{F} \cdot d\bar{S} = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)\right] dV$$

$$= 6 \iiint_V dV$$

$= 6V$, where V is the volume enclosed by S.

$$= 6 \times \frac{4}{3}\pi$$

$$= 8\pi.$$

Example 2.14 Verify Gauss divergence theorem for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$, where S is the surface of the cuboid formed by the planes $x = 0$, $x=a$, $y=0$, $y=b$, $z = 0$ and $z = c$.

Divergence theorem is

$$\iint_S \bar{F} \cdot d\bar{S} = \iiint_V (\text{div } \bar{F})\, dV \qquad (1)$$

S is made up of six plane surfaces.

$$\therefore \quad \text{L.S. of (1)} = \iint_{\substack{x=0 \\ \hat{n}=-\bar{i}}} + \iint_{\substack{x=a \\ \hat{n}=\bar{i}}} + \iint_{\substack{y=0 \\ \hat{n}=-\bar{j}}} + \iint_{\substack{y=b \\ \hat{n}=\bar{j}}} + \iint_{\substack{z=0 \\ \hat{n}=-\bar{k}}}$$

$$+ \iint_{\substack{z=c \\ \hat{n}=\bar{k}}} (x^2\bar{i} + y^2\,\bar{j} + z^2\bar{k}) \cdot \hat{n}\, dS$$

$$= \iint_{x=0} -x^2 dS + \iint_{x=a} x^2 dS + \iint_{y=0} -y^2 dS + \iint_{y=b} y^2 dS$$

$$- \iint_{z=0} z^2 dS + \iint_{z=c} z^2 dS$$ (on using the relevant values of $\hat{n}$)

$$= a^2 \int_0^c \int_0^b dy\, dz + b^2 \int_0^a \int_0^c dz\, dx + c^2 \int_0^b \int_0^a dx\, dy$$

$$= abc\,(a + b + c)$$

$$\text{R.S. of (1)} = \iiint_V (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= \int_0^c \int_0^b \int_0^a (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= \int_0^c \int_0^b (x^2 + 2y \cdot x + 2z \cdot x)_0^a \mathrm{d}y \ \mathrm{d}z$$

$$= \int_0^c (a^2 y + ay^2 + 2az \cdot y)_0^b \ \mathrm{d}z$$

$$= \int_0^c (a^2 b + ab^2 + 2ab\, z) \ \mathrm{d}z$$

$$= a^2bc + + ab^2c + abc^2$$

$$= abc\,(a + b + c)$$

Thus divergence theorem is verified.

Example 2.15 Verify divergence theorem for $\bar{F} = x^2\bar{i} + z\,\bar{j} + yz\,\bar{k}$ over the cube formed by

$$x = \pm 1, \quad y = \pm 1, \quad z = \pm 1.$$

Divergence theorem is $\iint_S \bar{F} \cdot \mathrm{d}\bar{S} = \iiint_V (\operatorname{div} \bar{F})\,\mathrm{d}V$ (1)

$$\text{L.S. of (1)} = \iint\limits_{\substack{x=-1\\ \hat{n}=-\bar{i}}} + \iint\limits_{\substack{x=1\\ \hat{n}=\bar{i}}} + \iint\limits_{\substack{y=-1\\ \hat{n}=-\bar{j}}} + \iint\limits_{\substack{y=1\\ \hat{n}=\bar{j}}} + \iint\limits_{\substack{z=-1\\ \hat{n}=-\bar{k}}}$$

$$+ \iint\limits_{\substack{z=1\\ \hat{n}=\bar{k}}} (x^2\,\bar{i} + z\,\bar{j} + yz\,\bar{k}) \cdot \hat{n} \ \mathrm{d}S$$

$$= \iint\limits_{x=-1} -x^2 \mathrm{d}S + \iint\limits_{x=1} x^2 \mathrm{d}S + \iint\limits_{y=-1} -z \ \mathrm{d}S + \iint\limits_{y=1} z \ \mathrm{d}S$$

$$\iint\limits_{z=-1} -yz \ \mathrm{d}S + \iint\limits_{z=1} yz \ \mathrm{d}S \qquad \text{(using the relevant values of } \hat{n})$$

$$= 0$$

$$\text{R.S. of (1)} = \iiint\limits_v (2x + y) \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}z$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x + y) \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}z$$

$$= \int_{-1}^{1}\int_{-1}^{1} (2y \ \mathrm{d}y \ \mathrm{d}z)$$

$$= 0$$

Thus divergence theorem is verified.

EXERCISE 2(d)

Part A

(Short Answer Questions)

1. State Green's theorem in a plane or the connection between a line integral and a double integral.
2. State Stoke's theorem or the connection between a line integral and a surface integral.
3. State Gauss divergence theorem or the connection between a surface integral and a volume integral.
4. Deduce Green's theorem in a plane from Stoke's theorem.
5. State the scalar form of Stoke's theorem.
6. Give the scalar form of divergence theorem.
7. Derive Green's identities from divergence theorem.
8. Use Stoke's theorem to prove that $\nabla \times \nabla \phi = 0$.
9. Use the integral theorems to prove $\nabla \cdot (\nabla \times \bar{F}) = 0$.
10. Evaluate $\oint_C (yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z)$ where C is the circle given by $x^2 + y^2 + z^2 = 1$ and $z = 0$.
11. Evaluate $\iint_S (x\,\mathrm{d}y\,\mathrm{d}z + y\,\mathrm{d}z\,\mathrm{d}x + z\,\mathrm{d}x\,\mathrm{d}y)$ over the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
12. If C is a simple closed curve and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, prove that $\int_C \bar{r} \cdot \mathrm{d}\bar{r} = 0$.
13. If C is a simple closed curve and ϕ is a scalar point function, prove that $\int_C \phi \nabla \phi \cdot \mathrm{d}\bar{r} = 0$.
14. If S is a closed surface and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, find the value $\iint_S \nabla\left(\frac{1}{r}\right) \cdot \mathrm{d}\bar{S}$.
15. If S is a closed surface enclosing a volume V, evaluate $\iint_S \nabla(r^2) \cdot \mathrm{d}\bar{S}$.
16. Evaluate $\int_C [(x - 2y)\mathrm{d}x + (3x - y)\mathrm{d}y]$, where C is the boundary of a unit square.
17. Evaluate $\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x)$, where C is the circle $x^2 + y^2 = a^2$.
18. If $\bar{A} = \text{curl}\,\bar{F}$, prove that $\iint_S \bar{A} \cdot \mathrm{d}\bar{S} = 0$, where S is any closed surface.
19. If S is any closed surface enclosing a volume V and if $\bar{A} = a\,x\bar{i} + by\,\bar{j} + c\,z\bar{k}$, prove that $\iint_S \bar{A} \cdot \mathrm{d}\bar{S} = (a + b + c)V$.
20. If $\bar{r} = x\,\bar{i} + y\bar{j} + z\bar{k}$ and S the surface of a sphere of unit radius, find $\iint_S \bar{r} \cdot \mathrm{d}\bar{S}$.

Part B

21. Verify Green's theorem in a plane with respect to $\int_C (x^2\,\mathrm{d}x - xy\,\mathrm{d}y)$, where C is the boundary of the square formed by $x = 0, y = 0, x = a, y = a$.
22 Verify Stoke's theorem for $\overline{F} = x^2\overline{i} + xy\overline{j}$ in the square region in the xy-plane bounded by the lines $x = 0, y = 0, x = 2, y = 2$.
23. Use Green's theorem in a plane to evaluate $\int_C \left[x^2(1+y)\,\mathrm{d}x + (x^3 + y^3)\,\mathrm{d}y\right]$ where C is the square formed by $x = \pm 1$ and $y = \pm 1$.
24. Use Stoke's theorem to find the value of $\int_c \overline{F}\cdot \mathrm{d}\overline{r}$, when $\overline{F} = (xy - x^2)\,\overline{i} + x^2 y\,\overline{j}$ and C is the boundary of the triangle in the xoy-plane formed by $x = 1$, $y = 0$, and $y = x$.
25. Verify Green's theorem in a plane with respect to $\oint_C \left[(2x^2 - y^2)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y\right]$, where C is the boundary of the region in the xoy-plane enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = 1$.
26. Verify Stoke's theorem for $\overline{F} = (xy + y^2)\overline{i} + x^2\,\overline{j}$ in the region in the xoy-plane bound by $y = x$ and $y = x^2$.
27. Use Green's theorem in a plane to find the finite area enclosed by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.
28. Use Green's theorem in a plane to find the area of the region in the xoy-plane bounded by $y^3 = x^2$ and $y = x$.
29. Verify Stoke's theorem for $\overline{F} = (y - z + 2)\overline{i} + (yz + 4)\overline{j} - xz\overline{k}$, where S is the open surface of the cube formed by $x = 0, x = 2, y = 0, y = 2$ and $z = 2$.
30. Verify Stoke's theorem for $\overline{F} = (x^2 - y^2)\overline{i} + 2xy\overline{j} + xyz\overline{k}$ over the surface of the box bounded by the planes $x = 0, x = a, y = 0, y = b$ and $z = c$.
31. Verify Stoke's theorem for $\overline{F} = (2x - y)\,\overline{i} - yz^2\overline{j} - y^2 z\overline{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is the circular boundary in the xoy-plane.
32. Verify Gauss divergence theorem for $\overline{F} = (x^2 - yz)\overline{i} + (y^2 - zx)\overline{j} + (z^2 - xy)\overline{k}$ and the closed surface of the rectangular parallelopiped formed by $x = 0, x = 1, y = 0, y = 2, z = 0$ and $z = 3$.
33. Verify divergence theorem for (i) $\overline{F} = 4xz\overline{i} - y^2\overline{j} + yz\overline{k}$ and (ii) $\overline{F} = (2x - z)\overline{i} + x^2 y\overline{j} - xz^2\overline{k}$, when S is the closed surface of the cube formed by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
34. Use divergence theorem to evaluate $\iint_S (yz^2\overline{i} + zx^2\overline{j} + 2z^2\overline{k})\cdot \mathrm{d}\overline{S}$, where S is the closed surface bounded by the xoy-plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane.

35. Use divergence theorem to evaluate $\iint_S (4x\bar{i} - 2y^2\bar{j} + z^2\bar{k})\cdot d\bar{S}$, where S is the closed surface bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$.

ANSWERS

Exercise 2(a)

(5) $-12\bar{i} - 9\bar{j} - 16\bar{k}$ (6) $\sqrt{14}$ (7) 5

(8) $\frac{1}{\sqrt{19}}(\bar{i} + 3\bar{j} - 3\bar{k})$ (9) $\frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$ (10) $\frac{1}{3}(2\bar{i} + 2\bar{j} - \bar{k})$

(11) 25; $-56\bar{i} + 30\bar{j} - 47\bar{k}$ (12) $\frac{14}{3}$ (13) $-\frac{11}{3}$

(14) $\pm\frac{1}{3}(-\bar{i} + 2\bar{j} + 2\bar{k})$ (15) $-\frac{27}{\sqrt{11}}$ (16) $\cos^{-1}\left\{\frac{-45}{\sqrt{2299}}\right\}$

(17) $\cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$ (18) $\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$ (19) $\cos^{-1}\left(-\frac{3}{7\sqrt{6}}\right)$

(20) $\cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$ (21) $2x - y - 3z + 1 = 0$ (22) $\lambda = \frac{5}{2}$; $\mu = 1$

(23) $a = -\frac{7}{3}$; $b = \frac{64}{9}$ (24) $x^2y - xz^2 + y^2z + c$ (25) $\phi = x^2yz^3 + 20$

Exercise 2(b)

(7) 3; 0 (8) 4 (9) $2(\bar{i} + \bar{j} + \bar{k})$

(14) λ can take any value (15) $\lambda = -1$ (16) $a = 4$

(17) $a = -1, b = 1, c = -1$ (21) 0; $3\bar{k}$ (22) $\frac{124}{\sqrt{21}}$

(23) 80; $80\bar{i} + 37\bar{j} + 36\bar{k}$; $27\bar{i} - 54\bar{j} + 20\bar{k}$; 0; $74\bar{i} + 27\bar{j}$

(25) $x^3y^2z^4$ (26) $x^2 + y^2 + 3xy + yz + z^2x$

(27) $a = 4$, $b = 2$, $c = -1$; $\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx$

(28) x^2yz^3 (30) -3

Exercise 2(c)

(6) 3 (7) 6 (9) $\frac{a^2b^2}{2}$

(10) 0
(11) $\frac{8}{11}\bar{i} + \frac{4}{5}\bar{j} + \bar{k}$
(12) $-\pi\bar{i} + \frac{1}{2}\bar{j} + \frac{3\pi}{2}\bar{k}$
(13) $\left(2 - \frac{\pi}{4}\right)\bar{i} + \left(\pi - \frac{1}{2}\right)\bar{j}$
(14) $\frac{51}{70}$
(15) $-\frac{7}{6}$
(16) $2a^2$
(17) 8
(18) 8π
(19) 2
(20) $-\frac{19}{2}$
(21) $4\pi + 5$
(22) 0
(23) $\frac{27}{4}$
(24) πa^2
(25) 90

Exercise 2(d)

(10) 0
(11) $4\pi a^3$
(15) $6V$
(16) 5
(17) $2\pi a^2$
(20) 4π
(23) $\frac{8}{3}$
(24) $-\frac{1}{12}$
(27) $\frac{16}{3}a^2$
(28) $\frac{1}{10}$
(34) πa^4
(35) 84π

UNIT-3
ANALYTIC FUNCTIONS

UNIT 3

Analytic Functions

3.1 INTRODUCTION

Before we introduce the concept of complex variable and functions of a complex variable, the definition and properties of complex numbers, which the reader has studied in the lower classes are briefly recalled.

If x and y are real numbers and i denotes $\sqrt{-1}$, then $z = x + iy$ is called a Complex number. x is called *the real part of z* and is denoted by Re(z) or simply $R(z)$; y is called *the imaginary part of z* and is denoted by Im(z) or simply $I(z)$.

The Complex number $x - iy$ is called *the conjugate of the Complex number* $z = x + iy$ and is denoted by $\bar{z}$. Clearly $z\bar{z} = x^2 + y^2 = r^2$, where r is $|z|$, viz., the modulus of the Complex number z. Also $|\bar{z}| = |z|$; $R(z) = \dfrac{z + \bar{z}}{2}$, $I(z) = \dfrac{z - \bar{z}}{2i}$.

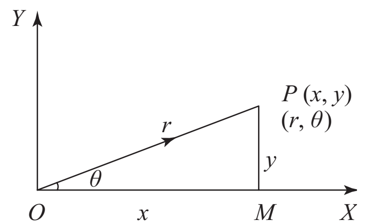


Fig. 3.1

The complex number $z = x + iy$ is geometrically represented by the point $P(x, y)$ with reference to a pair of rectangular coordinate axes OX and OY, which are called the real axis and the imaginary axis respectively (Fig. 3.1). Corresponding to each complex number, there is a unique point in the XOY–plane and conversely, corresponding to each point in the XOY-plane, there is a unique complex number. The XOY-plane, the points in which represent complex numbers, is called the *Complex plane* or *Argand plane* or *Argand diagram.*

If the polar coordinates of the point P are (r, θ), then $r = OP = \sqrt{x^2 + y^2} =$ modulus of z or $|z|$ and $\theta = \angle MOP = \tan^{-1}\left(\frac{y}{x}\right) =$ amplitude of z or amp (z).

Since $x = r\cos\theta$ and $y = r\sin\theta$,

$$z = x + iy = r(\cos\theta + i\sin\theta) \quad \text{or} \quad re^{i\theta}.$$

$r(\cos\theta + i\sin\theta)$ is called the *modulus-amplitude form of* z and $re^{i\theta}$ is called *the polar form of* z.

3.2 THE COMPLEX VARIABLE

The quantity $z = x + iy$ is called a complex variable, when x and y are two independent real variables.

The Argand plane in which the variables z are represented by points is called the z-plane. The point that represents the complex variable z is referred to as the point z.

3.2.1 Function of a Complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables such that there exists one or more values of w, corresponding to each value of z in a certain region R of the z-plane, then w is called *a function of* z and is written as $w = f(z)$ or $w = f(x + iy)$. When $w = u + iv = f(z) = f(x + iy)$, clearly u and v are functions of the variables x and y. For example, if $w = z^2$, then

$$u + iv = (x + iy)^2$$
$$= (x^2 - y^2) + i(2xy)$$

Thus $\quad u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$

$\therefore \quad w = f(z) = u(x, y) + iv(x, y)$

If z is expressed in the polar form, u and v are functions of r and θ. If, for every value of z, there corresponds a unique value of w, then w is called *a single-valued function of* z.

For example, $w = z^2$ and $w = \frac{1}{z}$ are single-valued functions of z.

If, for every value of z, there correspond more than one value of w, then w is called *a multiple valued function of* z. For example, $w = z^{1/4}$ and $w = \text{amp}(z)$ are multiple valued functions of z. $w = z^{1/4}$ is four valued and $w = \text{amp}(z)$ is infinitely many valued for $z \neq 0$.

3.2.2 Limit of a Function of a Complex Variable

The single valued function $f(z)$ is said to have the limit $l\ (= \alpha + i\beta)$ as z tends to z_0, if $f(z)$ is defined in a neighbourhood of z_0 (except perhaps at z_0) such that the values of $f(z)$ are as close to l as desired for all values of z that are sufficiently close to z_0, but different from z_0. We express this by writing $\lim_{z \to z_0} \{f(z)\} = l$.

Note ☑ Neighbourhood of z_0 is the region of the z-plane consisting of the set of points z for which $|z - z_0| < \rho$, where ρ is a positive real number, viz., the set of points z lying inside the circle with the point z_0 as centre and ρ as radius.

Mathematically we say $\lim_{z \to z_0} \{f(z)\} = l$, if, for every positive number $\in$ (however small it may be), we can find a positive number δ, such that $|f(z) - l| < \in$, whenever $0 < |z - z_0| < \delta$.

Note ☑ In real variables, $x \to x_0$ implies that x approaches x_0 along the x-axis or a line parallel to the x-axis (in which x is varying) either from left or from right. In complex variables, $z \to z_0$ implies that z approaches z_0 along any path (straight or curved) joining the points z and z_0 that lie in the z-plane, as shown in the Fig. 3.2.

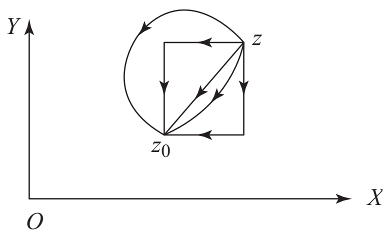


Fig. 3.2

Thus, in order that $\lim_{z \to z_0} \{f(z)\}$ may exist, $f(z)$ should approach the same value l, when z approaches z_0 along all paths joining z and z_0.

3.2.3 Continuity of $f(z)$

The single valued function $f(z)$ is said to be continuous at a point z_0, if

$$\lim_{z \to z_0} \{f(z)\} = f(z_0).$$

This means that if a function $f(z)$ is to be continuous at the point z_0, the value of $f(z)$ at z_0 and the limit of $f(z)$ as $z \to z_0$ must exist (as per the definition given above) and these two values must be equal.

A function $f(z)$ is said to be *continuous in a region R* of the z-plane, if it is continuous at every point of the region.

If $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$, then $u(x, y)$ and $v(x, y)$ will be continuous at (x_0, y_0) and conversely. The function $\phi(x, y)$ is said to be continuous at the point (x_0, y_0), if $\lim_{\substack{x \to x_0 \\ y \to y_0}} [\phi(x, y)] = \phi(x_0, y_0)$, in whatever manner $x \to x_0$ and $y \to y_0$.

3.2.4 Derivative of $f(z)$

The single valued function $f(z)$ is said to be *differentiable* at a point z_0, if

$$\lim_{\Delta z \to 0}\left[\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\right] \text{ exists} \tag{1}$$

This limit is called *the derivative of $f(z)$ at z_0 and is denoted as $f'(z_0)$*. On putting $z_0 + \Delta z = z$ or $\Delta z = z - z_0$, we may write

$$f'(z_0) = \lim_{z \to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right].$$

On putting $z_0 = z$ in (1), we get

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right]$$

3.2.5 Analytic Function

A single valued function $f(z)$ is said to be analytic at the point z_0, if it possesses a derivative at z_0 and at every point in some neighbourhood of z_0.

A function $f(z)$ is said to be analytic in a region R of the z-plane, if it is analytic at every point of R.

An analytic function is also referred to as *a regular function* or *a holomorphic function*.

A point, at which a function $f(z)$ is not analytic, viz., does not possess a derivative, is called *a singular point* or *singularity* of $f(z)$.

3.2.6 Cauchy-Riemann Equations

We shall now derive two conditions (usually referred to as necessary conditions) that are necessarily satisfied when a function $f(z)$ is analytic in a region R of the z-plane.

Theorem

If the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a region R of the z-plane, then (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist and (ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at every point in that region.

Proof

$f(z) = u(x, y) + iv(x, y)$ is analytic in R.

$\therefore$ $f'(z)$ exists at every point z in R(by definition)

i.e.,
$$L = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right] \text{ exists} \tag{1}$$

i.e., L takes the same value, when $\Delta z \to 0$ along all paths.

In particular, L takes the same value when $\Delta z \to 0$ along two specific paths QRP and QSP shown in Fig. 3.3. (2)

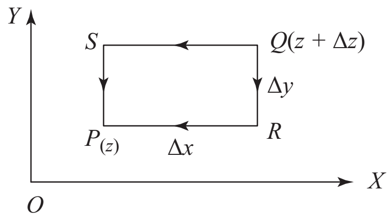


Fig. 3.3

It is evident that when z takes the increment Δz, x and y take the increments Δx and Δy respectively; hence we may write

$$L = \lim_{(\Delta x + i\Delta y)\to 0}\left[\frac{\{u(x+\Delta x,\, y+\Delta y) + iv(x+\Delta x,\, y+\Delta y)\} - \{u(x,\, y) + iv(x,\, y)\}}{\Delta x + i\Delta y}\right]$$

Let us now find the value of L, say L_1, corresponding to the path QRP, viz., by letting $\Delta y \to 0$ first and then by letting $\Delta x \to 0$.

$$\text{Thus} \qquad L_1 = \lim_{\Delta x\to 0}\left[\frac{\{u(x+\Delta x,\, y) + iv(x+\Delta x,\, y)\} - \{u(x\; y) + iv(x\; y)\}}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\left\{\frac{u(x+\Delta x,\, y) - u(x,\, y)}{\Delta x}\right\} + i\left\{\frac{v(x+\Delta x,\, y) - v(x,\, y)}{\Delta x}\right\}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{3}$$

(by the definition of partial derivatives.)

Now we shall find the value of L, say L_2, corresponding to the path QSP, viz., by letting $\Delta x \to 0$ first and then by letting $\Delta y \to 0$.

$$\text{Thus} \qquad L_2 = \lim_{\Delta y\to 0}\left[\frac{\{u(x,\, y+\Delta y) + iv(x,\, y+\Delta y)\} - \{u(x,\, y) + iv(x,\, y)\}}{i\Delta y}\right]$$

$$= \lim_{\Delta y\to 0}\left[\frac{1}{i}\left\{\frac{u(x,\, y+\Delta y) - u(x,\, y)}{\Delta y}\right\} + \left\{\frac{v(x,\, y+\Delta y) - v(x,\, y)}{\Delta y}\right\}\right]$$

$$= -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{4}$$

(by the definition of partial derivatives.)

L exists [by (1)]

$\therefore$ L_1 and L_2 exist at every point in R.

i.e. $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ exist at every point in R (i)

L exists uniquely [by (2)]

$\therefore \qquad L_1 = L_2$

i.e., $\qquad \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}$, by (3) and (4).

$\therefore \qquad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

and $\qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ at every point in R (ii)

Note ☑

1. The two equations given in (ii) above are called *Cauchy-Riemann equations* which will be hereafter referred to as C.R. equations.
2. When $f(z)$ is analytic, $f'(z)$ exists and is given by L, viz., by L_1 or L_2. Thus $f'(z) = u_x + iv_x$ or $v_y - iu_y$, where u_x, u_y, v_x, v_y denote the partial derivatives.
3. If $w = f(z)$, then $f'(z)$ is also denoted as $\dfrac{dw}{dz}$.

   Thus $$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u + iv) = \frac{\partial w}{\partial x}$$

   Also $$\frac{dw}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial}{\partial y}(u + iv) = -i\frac{\partial w}{\partial y}$$

4. In the above theorem, we have assumed that $w = f(z)$ is analytic and then derived the two conditions that necessarily followed. However the two conditions do not ensure the analyticity of the function $w = f(z)$. In other words, the two conditions are *not sufficient* for the analyticity of the function $w = f(z)$.
   The sufficient conditions for the function $w = f(z)$ to be analytic in a region R are given in the following theorem.

Theorem

The single valued continuous function $w = f(z) = u(x, y) + iv(x, y)$ is analytic in a region R of the z-plane, if the four partial derivatives u_x, v_x, u_y and v_y have the following features: (i) They exist, (ii) They are continuous and (iii) They satisfy the C.R. equations $u_x = v_y$ and $u_y = -v_x$ at every point of R.

Proof

Consider $\Delta u = u\,(x + \Delta x, y + \Delta y) - u\,(x, y)$

$$= [u\,(x + \Delta x, y + \Delta y) - u\,(x, y + \Delta y) + u\,(x, y + \Delta y) - u\,(x, y)]$$

$$= \Delta x \cdot u_x(x + \theta_1 \Delta x, y + \Delta y) + \Delta y \cdot u_y(x, y + \theta_2 \Delta y),$$

where $0 < \theta_1, \theta_2 < 1$, (by using the mean-value theorem.)

u_x is a continuous function of x and y, by (ii).

$\therefore$ $u_x(x + \theta_1 \Delta x, y + \Delta y) = u_x(x, y) + \in_1$, where

$\in_1 \to 0$ as Δx and $\Delta y \to 0$ or as $\Delta z \to 0$.

u_y is a continuous function of x and y, by (ii).

$\therefore$ $u_y\,(x, y + \theta_2 \Delta y) = u_y\,(x, y) + \in_2$, where

$\in_2 \to 0$ as $\Delta z \to 0$

$\therefore$ $\Delta u = \Delta x\,[u_x\,(x, y) + \in_1] + \Delta y[u_y\,(x, y) + \in_2]$,

where $\in_1$ and $\in_2 \to 0$ as $\Delta z \to 0$. (1)

Similarly, using the continuity of the partial derivatives v_x and v_y, we have

$$\Delta v = \Delta x\,[v_x(x, y) + \in_3] + \Delta y\,[v_y(x, y) + \in_4],$$

where $\in_3$ and $\in_4 \to 0$ as $\Delta z \to 0$ (2)

Now $\Delta w = \Delta u + i\Delta v$

$$= (u_x + iv_x)\,\Delta x + (u_y + iv_y)\,\Delta y + (\in_1 + i\in_3)\,\Delta x + (\in_2 + i\in_4)\,\Delta y,$$

using (1) and (2).

$$= (u_x + iv_x)\,\Delta x + (u_y + iv_y)\,\Delta y + \eta_1 \Delta x + \eta_2 \Delta y, \text{ where}$$

$\eta_1 = (\in_1 + i\in_3)$ and $\eta_2 = (\in_2 + i\in_4) \to 0$ as $\Delta z \to 0$ (3)

Using (iii) in (3), i.e., putting $v_y = u_x$ and $u_y = -v_x$ in (3), we have

$$\Delta w = (u_x + iv_x)\,\Delta x + (-v_x + iu_x)\,\Delta y + \eta_1 \Delta x + \eta_2 \Delta y$$

$$= u_x(\Delta x + i\Delta y) + iv_x(\Delta x + i\Delta y) + \eta_1 \Delta x + \eta_2 \Delta y$$

$$= (u_x + iv_x)\,\Delta z + \eta_1 \Delta x + \eta_2 \Delta y$$

$\therefore$ $$\frac{\Delta w}{\Delta z} = u_x + iv_x + \eta_1 \frac{\Delta x}{\Delta z} + \eta_2 \frac{\Delta y}{\Delta z} \quad (4)$$

Now $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$

$\therefore$ $$\left|\frac{\Delta x}{\Delta z}\right| \le 1 \quad \text{and} \quad \left|\frac{\Delta y}{\Delta z}\right| \le 1$$

$\therefore$ $$\left|\eta_1 \frac{\Delta x}{\Delta z} + \eta_2 \frac{\Delta y}{\Delta z}\right| \le |\eta_1| + |\eta_2|$$

$$\therefore \qquad \left(\eta_1 \frac{\Delta x}{\Delta z} + \eta_2 \frac{\Delta y}{\Delta z}\right) \to 0 \text{ as } \Delta z \to 0 \; [\because \; \eta_1 \text{ and } \eta_2 \to 0 \text{ as } \Delta z \to 0] \qquad (5)$$

Now taking limits on both sides of (4) and using (5), we have

$$\frac{dw}{dz} = u_x + iv_x .$$

i.e., the derivative of $w = f(z)$ exists at every point in R.
i.e., $w = f(z)$ is analytic in the region R.

3.2.7 C.R. Equations in Polar Coordinates

When z is expressed in the polar form $re^{i\theta}$, we have already observed that u and v, where $w = u + iv$, are functions of r and θ. In this case, we shall derive the C.R. equations satisfied by $u(r, \theta)$ and $v(r, \theta)$, assuming that $w = u(r, \theta) + iv(r, \theta)$ is analytic.

Theorem
If the function $w = f(z) = u(r, \theta) + iv(r, \theta)$ is analytic in a region R of the z-plane, then (i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exit and (ii) they satisfy the C.R. equations, viz.,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ at every point in that region.}$$

Proof

$$f(z) = u(r, \theta) + iv(r, \theta) \text{ is analytic in } R.$$

$\therefore$ $f'(z)$ exists at every point z in R (by definition)

$$\text{i.e., } L = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] \text{ exists} \qquad (1)$$

$$\text{i.e., } L = \lim_{\Delta(re^{i\theta}) \to 0} \left[\frac{\{u(r+\Delta r, \theta+\Delta\theta) + iv(r+\Delta r, \theta+\Delta\theta)\} - \{u(r,\theta) + iv(r,\theta)\}}{\Delta\left(re^{i\theta}\right)}\right]$$

exists.

i.e., L take the same value, in whatever manner $\Delta z = \Delta(re^{i\theta}) \to 0$.

In particular, L takes the same value, corresponding to the two ways given below in which $\Delta z \to 0$. (2)

$$\begin{aligned} \Delta z &= \Delta(re^{i\theta}) \\ &= e^{i\theta} \cdot \Delta r, \text{ if } \theta \text{ is kept fixed} \end{aligned}$$

$\therefore$ When $\Delta z \to 0$, $\Delta r \to 0$, if θ is kept fixed. Let us find the value of L, say L_1, corresponding to this way of Δz tending to zero.

Thus $L_1 = \lim_{\Delta r \to 0}\left[\left\{\dfrac{u(r+\Delta r,\theta)-u(r,\theta)}{e^{i\theta}\Delta r}\right\}+i\left\{\dfrac{v(r+\Delta r,\theta)-v(r,\theta)}{e^{i\theta}\Delta r}\right\}\right]$

$$= e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right) \quad (3)$$

(by the definition of partial derivatives.)

Now $\Delta z = \Delta\,(re^{i\theta})$

$= re^{i\theta}\, i\Delta\theta$, if r is kept fixed.

$\therefore$ When $\Delta z \to 0$, $\Delta\theta \to 0$, if r is kept fixed.

We shall now find the value of $L = L_2$, corresponding to this way of Δz tending to zero.

Thus $L_2 = \lim_{\Delta\theta \to 0}\left[\left\{\dfrac{u(r,\theta+\Delta\theta)-u(r,\theta)}{r\,e^{i\theta}\cdot i\Delta\theta}\right\}+i\left\{\dfrac{v(r,\theta+\Delta\theta)-v(r,\theta)}{r\,e^{i\theta}\cdot i\Delta\theta}\right\}\right]$

$$= \frac{1}{r}e^{-i\theta}\left(-i\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial\theta}\right) \quad (4)$$

(by the definition of partial derivatives.)

Since L exists [by (1)], L_1 and L_2 exist at every point in R.

i.e. $\dfrac{\partial u}{\partial r},\dfrac{\partial u}{\partial\theta},\dfrac{\partial v}{\partial r},\dfrac{\partial v}{\partial\theta}$ exists at every point in R. (i)

Since L exists uniquely [by (2)],

$$L_1 = L_2$$

i.e. $e^{-i\theta}\left(\dfrac{\partial u}{\partial r}+i\dfrac{\partial v}{\partial r}\right)=\dfrac{1}{r}e^{-i\theta}\left(-i\dfrac{\partial u}{\partial\theta}+\dfrac{\partial v}{\partial\theta}\right)$, from (3) and (4).

$\therefore$ $$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta}$$

and $\dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}$ at every point in R (ii)

Note ☑

1. When $f(z)$ is analytic, $f'(z)$ exists and is given by L, viz., by L_1 or L_2. Thus, when $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic,

   $f'(z) = e^{-i\theta}(u_r + iv_r)$ or $\dfrac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)$,

   where u_r, u_θ, v_r, v_θ denote the partial derivatives.
2. If $w = f(z)$, then

$$\frac{dw}{dz} = e^{-i\theta} \cdot \frac{\partial}{\partial r}(u+iv) = e^{-i\theta}\frac{\partial w}{\partial r}.$$

Also
$$\frac{dw}{dz} = -\frac{i}{r}e^{-i\theta}(u_\theta + iv_\theta)$$
$$= -\frac{i}{r}e^{-i\theta}\frac{\partial}{\partial \theta}(u+iv)$$
$$= -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}.$$

3. The two conditions derived in the theorem do not ensure the analyticity of $w = f(z)$. The sufficient conditions for the analyticity of $w = f(z) = u(r, \theta) + iv(r, \theta)$ in R are given below without proof.
   (i) $u_r, u_\theta, v_r, v_\theta$ must exist
   (ii) they must be continuous;
   (ii) they must satisfy the C.R. equations in polar co-ordinates at every point in the region R.

WORKED EXAMPLE 3(a)

Example 3.1 Find $\lim_{z\to 0} f(z)$, when $f(z) = \dfrac{x^2 y}{x^2+y^2}$.

Let us find the limit of $f(z)$ corresponding to any one manner, say, by letting $y \to 0$ first and then by letting $x \to 0$.

Thus
$$\lim_{\substack{y\to 0\\ x\to 0}}[f(z)] = \lim_{x\to 0}\left(\frac{0}{x^2}\right) = \lim_{x\to 0}(0) = 0.$$

This does not mean that $\lim_{z\to 0}[f(z)]$ exists and is equal to 0.

However, we proceed to verify whether $\lim_{z\to 0}[f(z)]$ can be 0, as per the mathematical definition, which states that $\lim_{z\to 0}[f(z)] = 0$, if we can find a δ such that $|f(z) - 0| < \in$, whenever $0 < |z-0| < \delta$.

Now, using polar coordinates,

$$|f(z)-0| = \left|\frac{r^3\cos^2\theta\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)}\right|$$
$$= r\,|\cos^2\theta||\sin\theta|$$
$$< r, \text{ since } |\cos\theta| < 1 \text{ and } |\sin\theta| < 1$$

$\therefore$
$$|f(z)-0| < \in, \text{when } r < \frac{\in}{2}$$

i.e., when $\quad |z-0| < \dfrac{\in}{2}$

Thus, taking $\delta \le \dfrac{\in}{2}$, we have proved that

$$|f(z)-0| < \in, \text{ when } |z-0| < \delta.$$

$\therefore\ \lim\limits_{z\to 0}\left(\dfrac{x^2 y}{x^2+y^2}\right)$ exists and is equal to 0.

Example 3.2 If $f(z)=\dfrac{x^3 y(y-ix)}{x^6+y^2}\ (z\ne 0)$ and $f(0)=0$, prove that $\dfrac{\{f(z)-f(0)\}}{z}$ $\to 0$ as $z\to 0$ along any radius vector, but not as $z\to 0$ in any manner.

Now $\quad \dfrac{f(z)-f(0)}{z} = \dfrac{x^3 y(y-ix)}{(x^6+y^2)(x+iy)} = \dfrac{-ix^3 y}{x^6+y^2} \qquad (1)$

A radius vector is a line through the pole (origin) and hence its equation is $y = mx$. To take the limit of (1) as $z\to 0$ along any radius vector, we put $y = mx$ in (1) and then let $x\to 0$.

Thus $\lim\limits_{\substack{y=mx\\ x\to 0}}\left[\dfrac{-i x^3 y}{x^6+y^2}\right] = \lim\limits_{x\to 0}\left[\dfrac{-imx^4}{x^2\left(x^4+m^2\right)}\right]$

$$= \lim_{x\to 0}\left[\frac{-imx^2}{x^4+m^2}\right] = 0 \text{, for various values of } m.$$

$\therefore\quad \left\{\dfrac{f(z)-f(0)}{z}\right\}\to 0$, as $z\to 0$ along any radius vector.

Now let us find the limit of (1) by moving along the curve $y = x^3$ and approaching the origin.

Thus $\quad \lim\limits_{z\to 0}\left\{\dfrac{f(z)-f(0)}{z}\right\} = \lim\limits_{\substack{y=x^3\\ x\to 0}}\left[\dfrac{-i x^3 y}{x^6+y^2}\right]$

$$= \lim_{x\to 0}\left[\frac{-i x^6}{x^6+x^6}\right] = -\frac{i}{2} \ne 0.$$

Note ☑ Since the limiting values are not unique, the limit does not exist.

Example 3.3 Prove that the function $f(z)$, where $f(z)=\dfrac{x^2(1+i)-y^2(1-i)}{x+y}$, when $z \neq 0$ and $f(z)=0$, when $z=0$ is continuous at $z=0$.

$$f(z)=\left(\frac{x^2-y^2}{x+y}\right)+i\left(\frac{x^2+y^2}{x+y}\right)$$

Consider $$L_1=\lim_{z\to 0}[f(z)]=\lim_{\substack{y\to 0\\ x\to 0}}[f(z)]=\lim_{x\to 0}[(1+i)x]=0$$

$$L_2=\lim_{z\to 0}[f(z)]=\lim_{\substack{x\to 0\\ y\to 0}}[f(z)]=\lim_{y\to 0}[(-1+i)y]=0$$

$$L_3=\lim_{z\to 0}[f(z)]=\lim_{\substack{y\to mx\\ x\to 0}}[f(z)]=\lim_{x\to 0}\left[\left(\frac{1-m^2}{1+m}+i\frac{1+m^2}{1+m}\right)x\right]=0$$

$$L_4=\lim_{z\to 0}[f(z)]=\lim_{\substack{y\to \phi(x)\\ x\to 0}}[f(z)] \tag{1}$$

where $y=\phi(x)$ is any curve through the origin.

$\therefore$ $\phi(0)=0$

By Taylor's series expansion, we get

$$y=\phi(x)=\frac{\phi'(0)}{1!}x+\frac{\phi''(0)}{2!}x^2+\frac{\phi'''(0)}{3!}x^3+\ldots\infty$$

since $\phi(0)=0$.

$$=x\left[\frac{\phi'(0)}{1!}+\frac{\phi''(0)}{2!}x+\frac{\phi'''(0)}{3!}x^2+\ldots\infty\right]$$

$$=x\cdot\psi(x),\text{ where }\psi(x)\to\phi'(0)\text{ as }x\to 0 \tag{2}$$

Using (2) in (1), we have

$$L_4=\lim_{x\to 0}\left[\left\{\frac{1-\psi^2(x)}{1+\psi(x)}+i\frac{1+\psi^2(x)}{1+\psi(x)}\right\}x\right]$$

$$=0$$

Thus $\lim\limits_{z\to 0}[f(z)]$ takes the same value 0, in whatever manner $z\to 0$.

$\therefore$ $\lim\limits_{z\to 0}[f(z)=0=f(0)$.

$\therefore$ $f(z)$ is continuous at $z=0$.

Example 3.4 Show that the function $f(z)$ is discontinuous at the origin $(z=0)$, given that

$$f(z)=\frac{xy(x-2y)}{x^3+y^3},\text{ when } z\neq 0$$

$$= 0, \text{ when } z = 0.$$

Consider

$$\lim_{z\to 0}[f(z)] = \lim_{\substack{y\to mx\\x\to 0}}[f(z)] = \lim_{x\to 0}\left[\frac{m(1-2m)x^3}{(1+m^3)x^3}\right] = \frac{m(1-2m)}{1+m^3}$$

Thus $\lim\limits_{z\to 0}$ $[f(z)]$ depends on the value of m and hence does not take a unique value.

$\therefore$ $\lim\limits_{z\to 0}$ $[f(z)]$ does not exist.

$\therefore$ f(z) is discontinuous at the origin.

Example 3.5 Show that the function $f(z)$ is discontinuous at $z = 0$, given that

$$f(z) = \frac{2xy^2}{x^2+3y^4}, \text{ when } z \neq 0 \quad \text{and} \quad f(0) = 0.$$

Consider
$$\lim_{z\to 0}[f(z)] = \lim_{\substack{y=mx\\x\to 0}}[f(z)]$$
$$= \lim_{x\to 0}\left(\frac{2m^2x}{1+3m^4x^2}\right) = 0$$

Note ☑

Just because the value of $\lim\limits_{z\to 0}$ $[f(z)]$, when $z \to 0$ along the line $y = mx$ is 0, we should not conclude that the limit exists and hence $f(z)$ is continuous at $z = 0$.)

Now let us take the limit by approaching 0 along the curve $x = y^2$.

Then
$$\lim_{z\to 0}[f(z)] = \lim_{\substack{x=y^2\\y\to 0}}[f(z)]$$
$$= \lim_{y\to 0}\left[\frac{2y^4}{y^4+3y^4}\right] = \frac{1}{2} \neq 0.$$

$\therefore$ $\lim\limits_{z\to 0}$ $[f(z)]$ does not exist and hence $f(z)$ is not continuous.

Example 3.6 Show that the function $|z|^2$ is continuous and differentiable at the origin, but it is not analytic at any point.

Let $f(z) = |z|^2$. Then $f(0) = 0$.

Now $|f(z) - 0| = |z|^2 = r^2$, where $z = re^{i\theta}$

$< \in$, whenever $r < \in$, if

$\in$ is sufficiently small and positive

i.e. $|f(z) - 0| < \in$, whenever $0 < |z| < \in$.

$\therefore \lim\limits_{z\to 0} \ [f(z)]$ exists and is equal to $0 = f(0)$

$\therefore f(z)$ is continuous at the origin.

Also
$$f'(0) = \lim_{\Delta z\to 0}\left\{\frac{f(0+\Delta z)-f(0)}{\Delta z}\right\}$$
$$= \lim_{\Delta z\to 0}\left\{\frac{|\Delta z|^2}{\Delta z}\right\}$$
$$= \lim_{\Delta z\to 0}\left\{\Delta \bar{z}\right\} \qquad [\because |\Delta z|^2 = \Delta z \cdot \Delta \bar{z}]$$
$$= 0 \qquad \{\because \text{ When } \Delta x + i\Delta y \to 0,\ \Delta x - i\Delta y \to 0\}$$

$\therefore f(z)$ is differentiable at the origin.

Now
$$\lim_{\Delta z\to 0}\left\{\frac{f(z+\Delta z)-f(z)}{\Delta z}\right\} = \lim_{\Delta z\to 0}\left\{\frac{|z+\Delta z|^2-|z|^2}{\Delta z}\right\}$$
$$= \lim_{\Delta z\to 0}\left[\frac{(z+\Delta z)(\bar{z}+\Delta \bar{z})-z\bar{z}}{\Delta z}\right]$$
$$= \lim_{\Delta z\to 0}\left[z\frac{\Delta \bar{z}}{\Delta z}+\bar{z}+\Delta \bar{z}\right]$$
$$= \lim_{\Delta z\to 0}\left[(x+iy)\frac{(\Delta x-i\Delta y)}{\Delta x+i\Delta y}+(x-iy)+(\Delta x-i\Delta y)\right]$$

Let us find the value of this limit by making $\Delta z \to 0$ in two different manners.

$$L_1 = \lim_{\substack{\Delta y\to 0\\ \Delta x\to 0}}\left[(x+iy)\frac{(\Delta x-i\Delta y)}{\Delta x+i\Delta y}+(x-iy)+(\Delta x-i\Delta y)\right]$$

$$= \lim_{\Delta x\to 0}\left[(x+iy)+(x-iy)+\Delta x\right] = 2x.$$

$$L_2 = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[(x+iy)\frac{(\Delta x-i\Delta y)}{\Delta x+i\Delta y}+(x-iy)+(\Delta x-i\Delta y)\right]$$

$$= \lim_{\Delta y\to 0}\left[-(x+iy)+(x-iy)-i\Delta y\right] = -2iy$$

$L_1 \neq L_2$ for all values of x and y.

$\therefore f(z)$ is not differentiable at any point $z \neq 0$.
$\therefore f(z)$ is not analytic at any point $z \neq 0$.

Though $f(z)$ is differentiable at $z = 0$, it is not differentiable at any point in the neighbourhood of $z = 0$.

$\therefore f(z)$ is not analytic even at the origin.

Hence $f(z) = |z|^2$ is not analytic at any point.

Example 3.7 Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied at the origin.

$$f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|} \,. \quad \therefore \quad u(x, y) = \sqrt{|xy|}; v(x, y) = 0 \,.$$

$$u_x(0,0) = \left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{\Delta x \to 0}\left[\frac{u(\Delta x, 0) - u(0,0)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{0-0}{\Delta x}\right] = 0$$

$$u_y(0,0) = \lim_{\Delta y \to 0}\left[\frac{u(0, \Delta y) - u(0,0)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{0-0}{\Delta y}\right] = 0$$

$$v_x(0,0) = \lim_{\Delta x \to 0}\left[\frac{v(\Delta x, 0) - v(0,0)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{0-0}{\Delta x}\right] = 0$$

$$v_y(0,0) = \lim_{\Delta y \to 0}\left[\frac{v(0, \Delta y) - v(0,0)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0}\left[\frac{0-0}{\Delta y}\right] = 0 \cdot$$

Clearly $u_x = v_y$ and $u_y = -v_x$ at the origin.

i.e., C.R. equations are satisfied at the origin.

Now
$$\lim_{\Delta z \to 0}\left[\frac{f(0+\Delta z) - f(0)}{\Delta z}\right] = \lim_{\Delta z \to 0}\left[\frac{\sqrt{|\Delta x \cdot \Delta y|} - 0}{\Delta x + i\Delta y}\right]$$

$$= \lim_{\substack{\Delta y = m\Delta x \\ \Delta x \to 0}}\left[\frac{\sqrt{|m|\Delta x^2|}}{\Delta x(1+im)}\right]$$

$$= \frac{\sqrt{|m|}}{1+im}$$

The limit is not unique, since it depends on m. $\therefore f'(0)$ does not exist.
Hence $f(z)$ is not regular at the origin.

Note ☑ This problem means that C.R. equations are not sufficient for the analyticity of the function.

Example 3.8 Prove that the following functions are analytic and also find their derivatives using the definition.

(i) z^3; (ii) e^{-z}; (iii) $\sin z$; (iv) $\cosh z$;
(v) z^n (n is a positive integer); (vi) $\log z$.

(i) Let $f(z) = u + iv = z^3 = (x + iy)^3$

$$= (x^3 - 3xy^2) + i\,(3x^2y - y^3)$$

$$\therefore \quad u = x^3 - 3xy^2\ ;\ v = 3x^2y - y^3$$

$$u_x = 3x^2 - 3y^2\ ;\ v_x = 6xy$$

$$u_y = -\,6xy\ ;\ v_y = 3x^2 - 3y^2$$

Obviously u_x, u_y, v_x, v_y exist for finite values of x and y (i.e., everywhere in the finite plane).

They are continuous everywhere, since they are polynomials in x and y.

Also $u_x = v_y = 3x^2 - 3y^2$ and $v_x = -\,u_y = 6xy$ i.e., C.R. equations are satisfied for all finite values of x and y.

All the three sufficient conditions that ensure the analyticity of $f(z)$ are satisfied everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

Now $f'(z) = u_x + iv_x$

$$= 3x^2 - 3y^2 + i6xy$$

$$= 3(x + iy)^2$$

$$= 3\,z^2.$$

(ii) Let $f(z) = u + iv = e^{-z} = e^{-(x\,+\,iy)}$

$$= e^{-x}\,(\cos y - i \sin y)$$

$$\therefore \quad u = e^{-x} \cos y\ ;\ v = -\,e^{-x} \sin y$$

$$u_x = -\,e^{x} \cos y;\ v_x = e^{-x} \sin y$$

$$u_y = -\,e^{-x} \sin y;\ v_y = -\,e^{-x} \cos y$$

u_x, u_y, v_x, v_y exist everywhere, are continuous everywhere ($\because$ e^{-x}, $\sin y$ and $\cos y$ are continuous functions) and satisfy C.R. equations everywhere.

$\therefore f(z)$ is analytic everywhere.

Now $f'(z) = u_x + iv_x$

$$= -e^{-x}\cos y + ie^{-x}\sin y$$
$$= -e^{-x}(\cos y - i\sin y)$$
$$= -e^{-x}\,.\,e^{-iy} = -e^{-(x+iy)} = -e^{-z}.$$

(iii) Let $\quad f(z) = \sin z$

i.e. $\quad u + iv = \sin(x + iy)$

$$= \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i\cos x \sinh y.$$

$\therefore \quad u = \sin x \cosh y \,;\, v = \cos x \sinh y$

$$u_x = \cos x \cosh y \,;\, v_x = -\sin x \sinh y$$
$$u_y = \sin x \sinh y \,;\, v_y = \cos x \cosh y$$

The four partial derivatives are products of circular and hyperbolic functions.

$\therefore$ They exist, are continuous and satisfy C.R. equations everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

Now $\quad f'(z) = u_x + iv_x$

$$= \cos x \cosh y - i\sin x \sinh y$$
$$= \cos x \cos iy - \sin x \sin iy$$
$$= \cos(x + iy)$$
$$= \cos z.$$

(iv) Let $\quad f(z) = \cosh z = \cos iz$

i.e. $\quad u + iv = \cos(ix - y)$

$$= \cosh x \cos y + i\sinh x \sin y$$

$\therefore \quad u = \cosh x \cos y \,;\, v = \sinh x \sin y$

$$u_x = \sinh x \cos y \,;\, v_x = \cosh x \sin y$$
$$u_y = -\cosh x \sin y \,;\, v_y = \sinh x \cos y$$

$\therefore$ u_x, u_y, v_x, v_y exist, are continuous and satisfy C.R. equations everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

Now $\quad f'(z) = u_x + iv_x$

$$= \sinh x \cos y + i\cosh x \sin y$$
$$= \sinh(x + iy) = \sinh z.$$

(v) Let $\quad f(z) = u(r, \theta) + iv(r, \theta) = z^n = (re^{i\theta})^n$

$$= r^n(\cos n\theta + i\sin n\theta)$$

$\therefore \quad u = r^n \cos n\theta \,; \quad v = r^n \sin n\theta$

$$u_r = nr^{n-1}\cos n\theta \,; \quad v_r = nr^{n-1}\sin n\theta$$
$$u_\theta = -nr^n \sin n\theta \,; \quad v_\theta = nr^n \cos n\theta$$

$u_r, u_\theta, v_r, v_\theta$ exist and are continuous for finite values of r and hence everywhere.

Also $$u_r = \frac{1}{r} v_\theta = nr^{n-1} \cos n\theta$$

and $$v_r = -\frac{1}{r} u_\theta = nr^{n-1} \sin n\theta$$

i.e., C.R. equations are also satisfied everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

Now $$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta} \cdot nr^{n-1} (\cos n\theta + i \sin n\theta) \\ &= nr^{n-1} e^{-i\theta} \cdot e^{in\theta} \\ &= nr^{n-1} e^{i(n-1)\theta} \\ &= n[re^{i\theta}]^{n-1} = nz^{n-1}. \end{aligned}$$

(vi) Let $$\begin{aligned} f(z) &= u + iv = \log z = \log (r\, e^{i\theta}) \\ &= \log r + i\theta \end{aligned}$$

$\therefore$ $$u = \log r; \quad v = \theta$$

$$u_r = \frac{1}{r}; \quad v_r = 0$$

$$u_\theta = 0; \quad v_\theta = 1$$

$\therefore$ $u_r, u_\theta, v_r, v_\theta$ exist, are continuous and satisfy C.R. equations everywhere except at $r = 0$ i.e. $z = 0$

$\therefore$ $f(z)$ is analytic everywhere except at $z = 0$.

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta}\left(\frac{1}{r} + i \cdot 0\right) \\ &= \frac{1}{re^{i\theta}} = \frac{1}{z},\ z \neq 0. \end{aligned}$$

3.2.8 Important Note

In the above problems, we observe that the derivatives of some of the elementary functions of a complex variable are similar to those of corresponding functions of a real variable. This is true with respect to other elementary functions also.

This is due to the fact that the definitions of $f'(z)$ in Complex Calculus and $f'(x)$ in Real Calculus are identical, except that there is a slight difference in the interpretation of the concerned limits.

Also, due to the same reason, the rules of differentiation, such as the sum rule, product rule, quotient rule and function of function rule are the same as those in Real Calculus.

Hence, when a function $f(z)$ is known to be analytic, it can be differentiated in the ordinary manner as if z is a real variable.

Further since integration is the inverse operation of differentiation, $\int f(z)\mathrm{d}z$ can be evaluated as in Real Calculus, using the usual formulas and rules of integration. However the arbitrary constant of integration need not be a real constant; it may be a complex constant.

Example 3.9 Find where each of the following functions ceases to be analytic.

(i) $\dfrac{z}{(z^2-1)}$; (ii) $\dfrac{z^2-4}{z^2+1}$; (iii) $\dfrac{z+i}{(z-i)^2}$; (iv) z^3-4z-1;

(v) $\tan^2 z$.

(i) Let $$f(z)=\frac{z}{z^2-1}$$

$\therefore$ $$f'(z)=\frac{(z^2-1)\cdot 1-z\cdot 2z}{(z^2-1)^2}=\frac{-(z^2+1)}{(z^2-1)^2}$$

$f(z)$ is not analytic, where $f'(z)$ does not exist, i.e., where $f'(z)\to\infty$.

$$f'(z)\to\infty,\quad \text{if } (z^2-1)^2=0,\quad \text{i.e., if } z=\pm 1.$$

$\therefore$ $f(z)$ is not analytic at the points $z=\pm 1$.

(ii) Let $$f(z)=\frac{z^2-4}{z^2+1}$$

$\therefore$ $$f'(z)=\frac{(z^2+1)\cdot 2z-(z^2-4)\cdot 2z}{(z^2+1)^2}=\frac{10z}{(z^2+1)^2}$$

$\therefore$ $f(z)$ is not analytic where $z^2+1=0$

i.e. at the points $z=\pm i$.

(iii) Let $$f(z)=\frac{z+i}{(z-i)^2}$$

$\therefore$ $$f'(z)=\frac{(z-i)^2\cdot 1-(z+i)\cdot 2(z-i)}{(z-i)^4}$$

$$=\frac{-(z+3i)}{(z-i)^3}\to\infty,\ \text{at } z=i$$

$\therefore$ $f(z)$ is not analytic at $z=i$.

(iv) Let $$f(z)=z^3-4z-1$$

$\therefore$ $f'(z)=3z^2-4$, that exists everywhere.

$\therefore$ $f(z)$ is analytic everywhere.

(v) Let $$f(z) = \tan^2 z$$

$\therefore$ $$f'(z) = 2\tan z \sec^2 z = \frac{2\sin z}{\cos^3 z}$$

$\therefore f'(z) \to \infty$, when $\cos^3 z = 0$, i.e., when $z = \frac{(2n-1)\pi}{2}$

$\therefore f(z)$ is not analytic at $z = \frac{(2n-1)\pi}{2}$; $n = 1, 2, 3, \ldots$.

Example 3.10 Prove that every analytic function $w = u(x, y) + iv(x, y)$ can be expressed as a function of z alone.

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy$$

$\therefore$ $$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}$$

Hence u and v and also w may be considered as a function of z and $\bar{z}$

Consider
$$\begin{aligned}
\frac{\partial w}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} \\
&= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) \\
&= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right) \\
&= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x) \\
&= 0, \text{ by C.R. equations, as } w \text{ is analytic.}
\end{aligned}$$

This means that w is independent of $\bar{z}$, i.e. w is a function of z alone.

This means that if $w = u(x, y) + iv(x, y)$ is analytic, it can be rewritten as a function of $(x + iy)$. Equivalently a function of $\bar{z}$ cannot be an analytic function of z.

Note ☑ Let the analytic function w be given by

$$w = u(x, y) + iv(x, y) \tag{1}$$

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \tag{2}$$

In order to get w as a function of z alone, we may replace $\bar{z}$ by z in the R.H.S. of (2).

Then $$w = u(z, 0) + iv(z, 0) \tag{3}$$

Equation (3) can be obtained from (1) by replacing x by z and y by 0.

Thus we get the following rule known as *Milne-Thomson rule.* If a function of x and y can be expressed as a function of $z = x + iy$, it can be done by simply replacing x by z and y by 0 in the given function.

Example 3.11 Show that $u + iv = \dfrac{x-iy}{x-iy+a}$ $(a \neq 0)$ is not an analytic function of z, whereas $u - iv$ is such a function at all points where $z \neq -a$.

Now $u + iv = \dfrac{\bar{z}}{\bar{z}+a}$ = a function of $\bar{z}$. Since a function of $\bar{z}$ cannot be analytic, $(u + iv)$ is not an analytic function of z.

$u - iv$ = conjugate of $(u + iv) = \dfrac{z}{z+a} = f(z)$, say.

$f(z)$ is a function of z alone and $f'(z) = \dfrac{a}{(z+a)^2}$, that exists everywhere except at $z = -a$.

$\therefore$ $f(z)$ is analytic, except at $z = -a$.

Example 3.12 Show that an analytic function with

(i) constant real part is a contant; and
(ii) constant modulus is a constant.

Let $f(z) = u + iv$ be the analytic function.

(i) Given u = constant = c, say

$\therefore$ $u_x = 0$ and $u_y = 0$.

By C.R. equations,

$$v_y = u_x = 0 \text{ and}$$
$$v_x = -u_y = 0$$

Since the partial derivatives of v with respect to both x and y are zero,

v is a constant = c', say

$$\therefore \quad f(z) = c + ic'$$
$$= \text{constant.}$$

(ii) Given $|f(z)| = \sqrt{u^2 + v^2} = c$

$$\therefore \quad u^2 + v^2 = c^2 \tag{1}$$

Differentiating (1) partially with respect to x and y, we get

$$2u\, u_x + 2v\, v_x = 0$$

and

$$2u\, u_y + 2v\, v_y = 0$$

i.e.,

$$u\, u_x + v\, v_x = 0 \tag{2}$$

and

$$-u\, v_x + v\, u_x = 0 \qquad \text{(by C.R. equations)}$$

or $$v\,u_x - u\,v_x = 0 \tag{3}$$

(2) and (3) form a system of two homogeneous algebraic equations in the unknowns u_x and v_x.

The system possesses only a trivial solution, since $\begin{vmatrix} u & v \\ v & -u \end{vmatrix} = -(u^2 + v^2) \neq 0$

$\therefore$ Solution of equations (2) and (3) is $u_x = 0$ and $v_x = 0$

Now $$f'(z) = u_x + iv_x = 0$$

$\therefore$ $$f(z) = \text{a constant.}$$

EXERCISE 3(a)

Part A

(Short Answer Questions)

1. Explain briefly the concept of the limit of a function of a complex variable.
2. What is the basic difference between the limit of a function of a real variable and that of a complex variable.
3. Define the continuity of a function of a complex variable.
4. When is a function of a complex variable said to be differentiable at a point?
5. Define analytic function of a complex variable.
6. State the Cauchy-Riemann equations in Cartesian Coordinates satisfied by an analytic function.
7. State the sufficient conditions that will ensure the analyticity of a function
$$w = f(z) = u(x, y) + iv(x, y)$$
8. State the Cauchy-Riemann equations in polar coordinates satisfied by an analytic function.
9. If $w = u\,(x, y) + iv\,(x, y)$ is an analytic function of z, prove that
$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}\,.$$
10. If $w = u\,(r, \theta) + iv\,(r, \theta)$ is an analytic function of z, prove that $e^{i\theta}\dfrac{\mathrm{d}w}{\mathrm{d}z}$
$$= \frac{\partial w}{\partial r} = -\frac{i}{r}\frac{\partial w}{\partial \theta}\,.$$

Prove that the following functions are not analytic:

11 $f(z) = \bar{z}$;

12. $f(z) = \dfrac{x + iy}{x^2 + y^2}$;

13. $f(z) = e^x(\cos y - i \sin y)$

14. $f(z) = \cos x \cosh y + i \sin x \sinh y$

15. $f(z) = \log\left(x^2 + y^2\right) + i\,2\cot^{-1}\left(\dfrac{y}{x}\right).$

Determine where the Cauchy-Riemann equations are satisfied for the following functions:

16. $x^2 + iy^2$;

17. $(x^3 - 3y^2x) + i(3x^2y - y^3)$;

18. $\dfrac{x}{x^2 + y^2} + i\dfrac{y}{x^2 + y^2}$;

19. $xy^2 + iyx^2$;

20. $\dfrac{z-1}{z+1}$.

21. Find the value of a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i\,(cx^2 + dxy + y^2)$ may be analytic.
22. Determine p such that the function $f(z) = \dfrac{1}{2}\log\left(x^2 + y^2\right) + i\tan^{-1}\left(\dfrac{px}{y}\right)$ is analytic.
23. If $u + iv$ is analytic, show that $v - iu$ and $-v + iu$ are also analytic.
24. If $u + iv$ is analytic, show that $v + iu$ is not analytic.

Part B

25. Given that $f(z) = \dfrac{xy(x-y)}{x^2 + y^2}$ and $f(0) = 0$, show that $\lim\limits_{z\to 0}\left[f(z)\right]$ exists.
26. Show that $\lim\limits_{z\to 0}\left[f(z)\right]$ does not exist, if $f(z) = \dfrac{xy}{x^2 + 4y^2}$ and $f(0) = 0$.
27. Show that $\lim\limits_{z\to 0}\left[\dfrac{xy^2}{x^2 + y^4}\right]$ does not exist, even though the function approaches the same limit along every straight line through the origin.
28. Prove that the function $f(z) = \dfrac{xy(y-ix)}{x^2 + y^2}$, when $z \neq 0$ and $f(z) = 0$, when $z = 0$ is continuous at the origin.
29. Given that $f(z) = \dfrac{x^3 - y^3}{x^3 + y^3}$ and $f(0) = 0$, show that $f(z)$ is not continuous at $z = 0$.
30. If $f(z) = \dfrac{x^2y}{x^4 + y^2}$, $z \neq 0$ and $f(0) = 0$, show that $f(z)$ is not continuous at $z = 0$.
31. Show that the function $f(z)$ is not continuous at $z = 0$, if $f(z) = \dfrac{xy}{2x^2 + y^2}$, $z \neq 0$ and $f(0) = 0$.
32. Show that the function $f(x, y)$ is discontinuous at $(0, 0)$, given that

$$f(x, y) = \frac{x^4\,y(y-x)}{\left(x^8 + y^2\right)(y+x)} \text{ and } f(0, 0) = 0.$$

33. Show that the function $f(z)$ defined by

$$f(z)=\frac{xy(y-ix)}{x^2+y^2},\ z\neq 0 \qquad \text{and} \qquad f(0)=0$$

is not analytic at the origin, though it satisfies Cauchy-Riemann equations at the origin.

34. Show that function $f(z)$ defined by $f(z)=\dfrac{x^2y^3(x-iy)}{x^6+y^{10}}$, $z\neq 0$ and $f(0)=0$

is not analytic at the origin, though it satisfies Cauchy-Riemann equations at the origin.

35. Prove that the function $f(z)$ defined by $f(z)=\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$, $z\neq 0$ and

$f(0) = 0$ is continuous at the origin. Prove also that the Cauchy-Riemann equations are satisfied by $f(z)$ at $z = 0$ and yet $f'(z)$ does not exist at $z = 0$.
Prove that the following functions are analytic and also find their derivatives using definition.

36. $f(z) = z^2$; 37. $f(z) = e^z$; 38. $f(z) = \cos z$; 39. $f(z) = \sinh z$.

40. If $f(z)$ and $\overline{f(z)}$ are both analytic, show that $f(z)$ is a constant.

3.3 PROPERTIES OF ANALYTIC FUNCTIONS

3.3.1 Definition

$\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$ is known as *Laplace equation* in two dimensions.

$\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ is called *the Laplacian operator* and is denoted as ∇^2.

Using this operator, the Laplace equation is usually written as $\nabla^2\phi=0$.
It is recalled, from Vector Calculus, that

$$\nabla\equiv\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z} \text{ and hence}$$

$$\nabla^2=\nabla\cdot\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \text{ in three dimensions.}$$

The Laplace equation in polar co-ordinates is defined as $\dfrac{\partial^2\phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}=0$.

Property 1

The real and imaginary parts of an analytic function $w=u+iv$ satisfy the Laplace equation in two dimensions, viz. $\nabla^2u=0$ and $\nabla^2v=0$.

Since $w = u + iv$ is analytic in some region of the z-plane, u and v satisfy Cauchy-Riemann equations.

i.e. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

Differentiating both sides of (1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{3}$$

Differentiating both sides (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{4}$$

The second order mixed partial derivatives $\dfrac{\partial^2 v}{\partial x \partial y}$ and $\dfrac{\partial^2 v}{\partial y \partial x}$ are equal, when they are continuous.

Assuming the continuity of $\dfrac{\partial^2 v}{\partial y \partial x}$ and $\dfrac{\partial^2 v}{\partial y \partial x}$ and adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

i.e. u satisfies Laplace equation or u is a solution of the Laplace equation $\nabla^2 \phi = 0$. Similarly, differentiating (1) partially with respect to y and (2) partially with respect to x and adding $\left(\text{with the assumption of continuity of } \dfrac{\partial^2 u}{\partial x \partial y} \text{ and } \dfrac{\partial^2 u}{\partial y \partial x}\right)$, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

i.e. v also satisfies Laplace equation or v is also a solution of the Laplace equation $\nabla^2 \phi = 0$.

Note ☑ A real function of two real variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a *harmonic* function.

If we assume the continuity of the second order partial derivatives of u and v, the above property means that when $w = u + iv$ is analytic, u and v are harmonic. Conversely, when u and v are any two harmonic functions, chosen at random, $u + iv$ need not be analytic.

If u and v are harmonic functions such that $u + iv$ is analytic, then each is called *the conjugate harmonic function* of the other.

Property 2

The real and imaginary parts of an analytic function $w = u(r, \theta) + iv(r, \theta)$ satisfy the Laplace equation in polar coordinates.

Since $w = u(r, \theta) + iv(r, \theta)$ is analytic in some region of the z-plane, u and v satisfy Cauchy-Riemann equations in polar coordinates.

i.e. $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{1}$$

and $$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \tag{2}$$

or $$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \tag{3}$$

Differentiating (1) partially with respect to r,

we get $$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial v}{\partial \theta} \tag{4}$$

Differentiating (3) partially with respect to θ,

we get $$\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\partial^2 v}{\partial \theta\,\partial r} \tag{5}$$

Using (1), (4) and (5), we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\left(\frac{\partial^2 v}{\partial r\,\partial \theta} - \frac{\partial^2 v}{\partial \theta\,\partial r}\right) = 0,$$

assuming the continuity of the mixed derivatives.

Thus u satisfies Laplace equation in polar coordinates.

Similarly we can prove that v also satisfies Laplace equation in polar co-ordinates.

Property 3

If $w = u(x, y) + iv(x, y)$ is an analytic function, the curves of the family $u(x, y) = a$ and the curves of the family $v(x, y) = b$, cut orthogonally, where a and b are varying constants.

Consider a representative member of the family $u(x, y) = a$, corresponding to $a = a_1$.

i.e. $$u(x, y) = a_1$$

Taking differentials on both sides, we get

$$\mathrm{d}u = 0$$

i.e. $$\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y = 0$$

$\therefore$ $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\left(\dfrac{\partial u}{\partial x}\right)}{\left(\dfrac{\partial u}{\partial y}\right)} = m_1,$$ say, where m_1 is the slope of the curve $u(x, y) = a_1$ at (x, y).

Similarly, considering a typical member of the second family whose equation is $v(x, y) = b_1$, we can get

$$m_2 = \frac{dy}{dx} = -\frac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)}$$, where m_2 is the slope of the curve $v(x, y) = b_1$ at (x, y).

Now
$$m_1\, m_2 = \frac{\left(\dfrac{\partial u}{\partial x}\right)}{\left(\dfrac{\partial u}{\partial y}\right)} \cdot \frac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)}$$

$$= \frac{\left(\dfrac{\partial v}{\partial y}\right)}{-\left(\dfrac{\partial v}{\partial x}\right)} \cdot \frac{\left(\dfrac{\partial v}{\partial x}\right)}{\left(\dfrac{\partial v}{\partial y}\right)}$$, by C.R. equations, since $(u + iv)$ is analytic.

$$= -1$$

This is true at the point of intersection of the two curves $u(x, y) = a_1$ and $v(x, y) = b_1$ also.

Thus a typical member of the family $u(x, y) = a$ cuts orthogonally a typical member of the family $v(x, y) = b$.

$\therefore$ Every member of the family $u(x, y) = a$ cuts orthogonally every member of the family $v(x, y) = b$.

Note ☑ The two families are said to be *orthogonal trajectories* of each other.

Property 4

If $w = u(r, \theta) + iv(r, \theta)$ is an analytic function, the curves of the family $u(r, \theta) = a$ cut orthogonally the curves of the family $v(r, \theta) = b$, where a and b are arbitrary constants.

Proceeding as in property (3), we get

$$\left(\frac{d\theta}{dr}\right)_1 = -\frac{\left(\dfrac{\partial u}{\partial r}\right)}{\left(\dfrac{\partial u}{\partial \theta}\right)} \tag{1}$$

and
$$\left(\frac{d\theta}{dr}\right)_2 = -\frac{\left(\dfrac{\partial v}{\partial r}\right)}{\left(\dfrac{\partial v}{\partial \theta}\right)} \tag{2}$$

In polar coordinates the condition for orthogonality of two curves is

$$\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)_1 \cdot \left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)_2 = -1 .$$

From (1) and (2), we have

$$\left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)_1 \cdot \left(r\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)_2 = \frac{r\dfrac{\partial u}{\partial r}}{\dfrac{\partial u}{\partial \theta}} \cdot \frac{r\dfrac{\partial v}{\partial r}}{\dfrac{\partial v}{\partial \theta}}$$

$$= \frac{r\cdot\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}}{\dfrac{\partial u}{\partial \theta}} \cdot \frac{r\left(-\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\right)}{\dfrac{\partial v}{\partial \theta}},$$

(by C.R. equations in polar coordinates.)

$$= -1.$$

Hence the property follows.

3.3.2 Construction of an Analytic Function, When Its Real or Imaginary Part is Known

Method 1

Let $u\,(x, y)$, the real part of the analytic function $f(z) = u\,(x, y) + iv\,(x, y)$ be known. In this method, we first find $v\,(x, y)$ and then find $u\,(x, y) + iv\,(x, y)$ as a function of z. Since $u\,(x, y)$ is given, $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ can be found out.

$$\mathrm{d}v = \frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y$$

$$= \left(-\frac{\partial u}{\partial y}\right)\mathrm{d}x + \left(\frac{\partial u}{\partial x}\right)\mathrm{d}y \tag{1}$$

The expression $(M\mathrm{d}x + N\mathrm{d}y)$ is an exact differential if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

This condition for exactness is satisfied by the R.H.S. expression of (1), as

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

i.e.,
$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which is true by property (1) discussed above.

$\therefore$ R.H.S. of (1) is an exact differential. Now, integrating both sides of (1), we get

$$v = \int \left[\left(-\frac{\partial u}{\partial y} \right) \mathrm{d}x + \left(\frac{\partial u}{\partial x} \right) \mathrm{d}y \right] + c,$$ where c is an arbitrary (real) constant of integration.

Then $f(z) = u(x, y) + iv(x, y)$ is found out by using Milne-Thomson rule.

Note ☑ If $v(x, y)$ is given, we can find $u(x, y)$ first by a similar procedure and then find $f(z)$.

Method 2 (Milne-Thomson method)
Let $u(x, y)$ be the real part of the analytic function $f(z) = u(x, y) + iv(x, y)$. In this method, we first find $f'(z)$ as a function of z and then find $f(z)$ by ordinary integration. The imaginary part of $f(z)$ gives $v(x, y)$.

Since $u(x, y)$ is given, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can be found out.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad [\because f(z) \text{ is analytic}]$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= u_x(x, y) - iu_y(x, y)$$

$$= u_x(z, 0) - iu_y(z, 0), \text{ by Milne-Thomson rule.}$$

$\therefore\ f(z) = \int \left[u_x(z, 0) - iu_y(z, 0) \right] \mathrm{d}z + c$, where c is an arbitrary (imaginary) contant of integration.

Separating the real and imaginary parts of $f(z)$, we can find $v(x, y)$.

Note ☑

1. The real part of $f(z)$ obtained should be identical to the given $u(x, y)$.
2. If $v(x, y)$ is given, we can first find

$$f(z) = \int \left[v_y(z, 0) + iv_x(z, 0) \right] \mathrm{d}z + c,$$

by a similar procedure and then find $u(x, y)$ by separation of $f(z)$.

3.3.3 Applications

Properties (1) and (3) of analytic functions discussed above provide solutions to a number of flow and field problems.

If we consider two dimensional steady flow such as fluid flow, electric current flow and heat flow, the paths of fluid particles are called *stream lines* and their orthogonal trajectories are called *equipotential lines.* In the study of two dimensional irrotational motion of an incompressible fluid in planes parallel to the xy-plane, if $\bar{v}$ represents the velocity of a fluid particle, we can find a function $\phi(x, y)$ such that

$$\bar{v} = \frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j}$$

The function $\phi(x, y)$ which gives the velocity components is called the *velocity potential.* The function $\psi(x, y)$, which is such that $\phi(x, y) + i\psi(x, y)$ is analytic, is called the *stream function.*

The function $w = f(z) = \phi(x, y) + i\psi(x, y)$, which represents the flow pattern is called the *Complex potential.*

The curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are called *equipotential lines* and *stream lines* respectively.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = a$ and $\Psi /(x, y) = b$ are respectively called *equipotential lines* and *lines of force.*

In heat flow problems, the curves $\phi(x, y) = a$ and $\Psi(x, y) = b$ are respectively called *isothermals* and *heat flow lines.*

WORKED EXAMPLE 3(b)

Example 3.1 Prove that the following function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Also find the conjugate harmonic function v and the corresponding analytic function $(u + iv)$

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$u_x = 3x^2 - 3y^2 + 6x;\ u_{xx} = 6x + 6$$

$$u_y = -6xy - 6y\ ;\ u_{yy} = -6x - 6$$

$\therefore$ $$u_{xx} + u_{yy} = 0 \text{ and so } u \text{ is a harmonic function.}$$

Since v is the conjugate harmonic of u, $u + iv$ is analytic.

$\therefore$ By C.R. equations, $u_x = v_y$ and $u_y = -v_x$

Now
$$\begin{aligned} dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \\ &= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ &= (6xy + 6y)\,dx + (3x^2 - 3y^2 + 6x)\,dy \end{aligned}$$

$\therefore$
$$\begin{aligned} v &= \int\left[(6xy + 6y)\,dx + (3x^2 - 3y^2 + 6x)\,dy\right] + c \\ &= \int(M\,dx + N\,dy) + c, \text{ say.} \end{aligned}$$

To evaluate the integral in the R.H.S., we integrate all the terms in M partially with respect to x, also integrate only those terms in N not containing x and add them.

Thus $$v = 3x^2y + 6xy - y^3 + c$$

Let $\quad w = u + iv$

$$= (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + \text{i}\,(3x^2y + 6xy - y^3 + c)$$

$$= z^3 + 3z^2 + 1 + ic, \text{ by Milne-Thomson rule.}$$

Example 3.2 In a two dimensional fluid flow, find if $xy(x^2 - y^2)$ can represent the stream function. If so, find the corresponding velocity potential and also the complex potential.

If $\psi = xy(x^2 - y^2)$ represents the stream function, it should be the imaginary part of an analytic function and hence harmonic.

$$\psi = x^3y - xy^3$$

$$\psi_x = 3x^2y - y^3; \qquad \psi_{xx} = 6xy$$

$$\psi_y = x^3 - 3xy^2; \qquad \psi_{yy} = -6xy$$

$$\therefore\ \psi_{xx} + \psi_{yy} = 0.$$

i.e., ψ is a harmonic function.

$\therefore$ ψ can represent the stream function. Let ϕ be the corresponding velocity potential. Then $\phi + i\psi$ is analytic.

$$\therefore \qquad \phi_x = \psi_y \quad \text{and} \quad \phi_y = -\psi_x \qquad \text{(by C.R. equations)}$$

Now
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$= \frac{\partial \psi}{\partial y}dx - \frac{\partial \psi}{\partial x}dy$$

$$= (x^3 - 3xy^2)\,dx - (3x^2y - y^3)\,dy$$

$\therefore$
$$\phi = \int \left[(x^3 - 3xy^2)\,dx - (3x^2y - y^3)\,dy\right]$$

$$= \frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} + c$$

If $f(z)$ represents the complex potential,

$$f(z) = \phi + i\psi$$

$$= \left(\frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} + c\right) + i(x^3y - xy^3)$$

$$= \frac{1}{4}z^4 + c \qquad \text{(by Milne-Thomson rule)}$$

Example 3.3 Find if $\phi = (x-y)(x^2+4xy+y^2)$ can represent the equipotential for an electric field. Find the corresponding complex potential $w = \phi + i\psi$ and also ψ, if possible.

If ϕ represents the equipotential for an electric field, it should be the real part of an analytic function and hence harmonic.

$$\phi = (x-y)(x^2+4xy+y^2)$$
$$\phi_x = (x-y)(2x+4y) + (x^2+4xy+y^2)$$
$$\phi_{xx} = 4x + 2y + 2x + 4y = 6x + 6y$$
$$\phi_y = (x-y)(4x+2y) + (x^2+4xy+y^2)(-1)$$
$$\phi_{yy} = -4y - 2x - 4x - 2y = -6x - 6y$$

$\therefore$ $\phi_{xx} + \phi_{yy} = 0$

i.e., ϕ is a harmonic function.

$\therefore$ ϕ can represent the equipotential of an electric field.

The corresponding complex potential

$$w = \phi + i\,\psi \text{ is analytic}$$

$\therefore$
$$\frac{dw}{dz} = \phi_x + i\psi_x$$
$$= \phi_x - i\phi_y$$
$$= [(x-y)(2x+4y) + (x^2+4xy+y^2)] - i[(x-y)(4x+2y) - (x^2+4xy+y^2)]$$
$$= 3z^2 - i\cdot 3z^2$$

$\therefore$
$$w = \int 3(1-i)z^2 dz + ic$$
$$= (1-i)z^3 + ic$$

Now
$$w = \phi + i\psi = (1-i)(x+iy)^3 + ic$$
$$= (1-i)(x^3 + 3ix^2y - 3xy^2 - iy^3) + ic$$
$$= (x^3 - 3xy^2 + 3x^2y - y^3) + i(3x^2y - y^3 - x^3 + 3xy^2 + c)$$

$\therefore$
$$\psi = 3(x^2y + xy^2) - (x^3 + y^3) + c.$$

Note ☑ The value of ϕ obtained from w is the same as the given value of ϕ.

Example 3.4 Prove that $v = \log[(x-1)^2 + (y-2)^2]$ is harmonic in every region which does not include the point (1, 2). Find the corresponding analytic function $w = u + iv$ and also u.

$$v = \log[(x-1)^2 + (y-2)^2]$$

$$v_x = \frac{2(x-1)}{(x-1)^2 + (y-2)^2}; \qquad v_{xx} = \frac{\left[(x-1)^2 + (y-2)^2\right].2 - 4(x-1)^2}{\left[(x-1)^2 + (y-2)^2\right]^2}$$

$$v_y=\frac{2(y-2)}{(x-1)^2+(y-2)^2};\quad v_{yy}=\frac{\left[(x-1)^2+(y-2)\right]^2\cdot 2-4(y-2)}{\left[(x-1)^2+(y-2)^2\right]^2}$$

v_x, v_y, v_{xx} and v_{yy} do not exist at the point (1, 2).
But in every region not containing (1, 2),

$$v_{xx}+v_{yy}=\frac{2\left\{(y-2)^2-(x-1)^2\right\}}{\left[(x-1)^2+(y-2)^2\right]^2}+\frac{2\left\{(x-1)^2-(y-2)^2\right\}}{\left[(x-1)^2+(y-2)^2\right]^2}$$
$$=0$$

$\therefore$ v is harmonic in every region not containing the point (1, 2).
$w = u + iv$ is the corresponding analytic function.

$\therefore$
$$\frac{dw}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$
$$=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}\qquad\text{(by C.R. equations)}$$
$$=\frac{2(y-2)+i2(x-1)}{(x-1)^2+(y-2)^2}$$
$$=\frac{-4+i2(z-1)}{(z-1)^2+4},\qquad\text{(by Milne-Thomson rule.)}$$
$$=\frac{2i\{(z-1)+2i\}}{(z-1+2i)\,(z-1-2i)}$$
$$=\frac{2i}{z-1-2i}$$

$\therefore$
$$w=\int\frac{2i}{z-1-2i}\,dz+c$$
$$=2i\log(z-1-2i)+c.$$

i.e.
$$u+iv=2i\log\{(x-1)+i(y-2)\}+c$$
$$=2i\left[\frac{1}{2}\log\left\{(x-1)^2+(y-2)^2\right\}+i\tan^{-1}\left(\frac{y-2}{x-1}\right)\right]+c$$
$$=\left[-2\tan^{-1}\left(\frac{y-2}{x-1}\right)+c\right]+i\log\left\{(x-1)^2+(y-2)^2\right\}$$

$\therefore \quad u = -2\tan^{-1}\left(\dfrac{y-2}{x-1}\right) + c$, where c is a real constant. Also we note that v is the same as the given value.

Example 3.5 Find v such that $w = u + iv$ is an analytic function of z, given that $u = e^{x^2-y^2}\cdot\cos 2xy$. Hence find w.

$$u = e^{(x^2-y^2)}\cdot\cos 2xy$$

$$u_x = e^{(x^2-y^2)}(-2y\sin 2xy) + 2x\,e^{(x^2-y^2)}\cdot\cos 2xy$$

$$u_y = e^{(x^2-y^2)}(-2x\sin 2xy) - 2y\,e^{(x^2-y^2)}\cdot\cos 2xy$$

$$\begin{aligned} dv &= (v_x\,dx + v_y\,dy) \\ &= -u_y\,dx + u_x\,dy \end{aligned}$$

$$\therefore \quad v = \int\left[e^{(x^2-y^2)}\cdot 2x\sin 2xy + e^{(x^2-y^2)}\cdot 2y\cos 2xy\right]dx$$
$$+\left[-e^{(x^2-y^2)}\cdot 2y\sin 2xy + e^{(x^2-y^2)}\cdot 2x\cos 2xy\right]dy + c$$

To evaluate $\int[M\mathrm{d}x + N\mathrm{d}y]$, where $M\mathrm{d}x + N\mathrm{d}y$ is an exact differential in which N does not contain any term independent of x, it is enough to evaluate just $\int M\mathrm{d}x$ treating y as a constant.

$$\therefore \quad v = \int e^{(x^2-y^2)}\,2x\sin 2xy\,dx + \int e^{(x^2-y^2)}\cdot 2y\cos 2xy\,dx + c$$

$$= \int e^{(x^2-y^2)}\cdot 2x\sin 2xy\,dx + \int e^{(x^2-y^2)}\cdot d\{\sin 2xy\} + c,$$

treating y as a constant.

$$= \int e^{(x^2-y^2)}\cdot 2x\sin 2xy\,dx + e^{(x^2-y^2)}\sin 2xy - \int \sin 2xy\cdot e^{(x^2-y^2)}\cdot 2x\,dx + c,$$

integrating by parts.

$$= e^{(x^2-y^2)}\sin 2xy + c$$

Now $\quad w = u + iv$

$$= e^{(x^2-y^2)}\cos 2xy + i\,\{e^{(x^2-y^2)}\sin 2xy + c\}$$

$$= e^{z^2} + c \qquad \text{(by Milne-Thomson rule.)}$$

Example 3.6 Find the analytic function $w = u + iv$,

if $v = e^{2x}(x \cos 2y - y \sin 2y)$. Hence find u.

$$v = e^{2x}(x \cos 2y - y \sin 2y)$$

$$v_x = e^{2x} . \cos 2y + 2e^{2x}(x \cos 2y - y \sin 2y)$$

$$v_y = e^{2x}(-2x \sin 2y - 2y \cos 2y - \sin 2y)$$

$$\frac{dw}{dz} = u_x + iv_x = v_y + iv_x, \text{ (by C.R. equations.)}$$

$$= 0 + i[2ze^{2z} + e^{2z}], \text{ (by Milne-Thomson rule.)}$$

$\therefore$ $$w = i\int\left(2ze^{2z} + e^{2z}\right) dz + c, \text{ where } c \text{ is real}$$

$$= i\left[\left(2z \cdot \frac{e^{2z}}{2} - 2 \cdot \frac{e^{2z}}{4}\right) + \frac{e^{2z}}{2}\right] + c, \text{ by Bernoulli's formula}$$

$$= iz\, e^{2z} + c$$

i.e., $$u + iv = i(x + iy)\, e^{2(x+iy)} + c$$

$$= (ix - y)\, e^{2x}(\cos 2y + i \sin 2y) + c$$

$$= [-(x \sin 2y + y \cos 2y)\, e^{2x} + c] + i(x \cos 2y - y \sin 2y)\, e^{2x}$$

$\therefore$ $$u = -(x \sin 2y + y \cos 2y)\, e^{2x} + c.$$

Example 3.7 Determine the analytic function $f(z) = u + iv$, given that $3u + 2v = y^2 - x^2 + 16xy$.

$$3u + 2v = y^2 - x^2 + 16xy$$

$\therefore$ $$3u_x + 2v_x = -2x + 16y \qquad (1)$$

and $$3u_y + 2v_y = 2y + 16x$$

i.e., $$2u_x - 3v_x = 2y + 16x \qquad (2)$$

(by C.R. equations.)

Solving (1) and (2), we get

$$u_x = 2x + 4y \quad \text{and} \quad vx = -4x + 2y$$

$$f'(z) = u_x + iv_x$$

$$= (2x + 4y) + i(-4x + 2y)$$

$$= 2z - i4z, \qquad \text{(by Milne-Thomson rule.)}$$

$\therefore$ $$f(z) = \int (2z - i4z)\, dz + c_1 + ic_2$$

$$= z^2 - 2iz^2 + c_1 + ic_2$$

$$= (1 - 2i)z^2 + c_1 + ic_2$$

Since $3u + 2v$ does not contain any constant,

$$3c_1 + 2c_2 = 0 \quad \therefore c_2 = -\frac{3}{2}c_1$$

$\therefore f(z) = (1-2i)z^2 + \left(1 - \frac{3}{2}i\right)c_1$, where c_1 is a real constant.

Example 3.8 Determine the analytic function $f(z) = P + iQ$, given that $P - Q = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ and $f(\pi/2) = 0$.

$$f(z) = P + iQ$$

$\therefore$ $$if(z) = -Q + iP$$

$\therefore$ $$(1+i)f(z) = (P-Q) + i(P+Q)$$

i.e., $$\phi(z) = u + iv, \text{ say.}$$

Thus, if we construct the analytic function $\phi(z)$ with $u = P - Q$ as the real part,

then $$f(z) = \frac{1}{1+i}\phi(z).$$

$$u = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

$\therefore$ $$u_x = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\cdot(\cos x - \sin x) + \sin x \cdot (\cos x + \sin x - e^{-y})}{(\cos x - \cosh y)^2}\right]$$

$$u_y = \frac{1}{2}\left[\frac{(\cos x - \cosh y)e^{-y} + \sinh y\,(\cos x + \sin x - e^{-y})}{(\cos x - \cosh y)^2}\right]$$

Now $$\phi'(z) = u_x + iv_x$$

$$= u_x - iu_y, \qquad \text{(by C.R. equations)}$$

$$= \frac{(\cos z - 1)(\cos z - \sin z) + \sin z\,(\cos z + \sin z - 1) - i(\cos z - 1)}{2(\cos z - 1)^2}$$

$$= \frac{(1+i)(1-\cos z)}{2(1-\cos z)^2} = \frac{1+i}{2(1-\cos z)} = \left(\frac{1+i}{4}\right)\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

$\therefore$ $$\phi(z) = \left(\frac{1+i}{4}\right)\int \operatorname{cosec}^2\left(\frac{z}{2}\right) + dz + c$$

$$= -\left(\frac{1+i}{4}\right)2\cot\left(\frac{z}{2}\right) + c$$

i.e., $$f(z) = \frac{1}{2}\cot\frac{z}{2} + c'$$

Since $$f\left(\frac{\pi}{2}\right)=0, \quad c'-\frac{1}{2}\cot\frac{\pi}{4}=0 \qquad \therefore\ c'=\frac{1}{2}$$

$\therefore$ $$f(z)=\frac{1}{2}\left(1-\cot\frac{z}{2}\right)$$

Example 3.9 Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, when $w = u + iv = z^3$.

$$u + iv = z^3 = (x + iy)^3$$
$$= x^3 + 3ix^2y - 3xy^2 - iy^3$$

$\therefore$ $$u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y - y^3$$

Consider the family of curves $u = c_1$

i.e. $$x^3 - 3xy^2 = c_1 \tag{1}$$

Differentiating (1) with respect to x, we get

$$3x^2 - 3\left(y^2 + 2xy\frac{dy}{dx}\right) = 0$$

$\therefore$ $$\frac{dy}{dx} = m_1 = \frac{x^2 - y^2}{2xy}$$

Consider the family of curves $v = c_2$.

i.e. $$3x^2y - y^3 = c_2 \tag{2}$$

Differentiating (2) with respect to x, we get

$$3\left(2xy + x^2\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0$$

$\therefore$ $$\frac{dy}{dx} = m_2 = -\frac{2xy}{x^2 - y^2}$$

$$m_1m_2 = -1$$

$\therefore$ The families of curves $u = c_1$ and $v = c_2$ cut orthogonally.

Example 3.10 If $u = x^2 - y^2$ and $v = -\dfrac{y}{x^2 + y^2}$ prove that both u and v satisfy Laplace equations, but that $(u + iv)$ is not a regular function of z.

$$u = x^2 - y^2$$

$\therefore$ $$u_x = 2x; \quad u_{xx} = 2; \quad u_y = -2y; \quad u_{yy} = -2$$

$\therefore$ $$u_{xx} + u_{yy} = 0$$

i.e., u satisfies Laplace equation.

$$v = -\frac{y}{x^2 + y^2}$$

$$v_x = \frac{2xy}{\left(x^2 + y^2\right)^2}; \qquad v_{xx} = 2y\left[\frac{\left(x^2 + y^2\right)\cdot 1 - x\cdot 2\left(x^2 + y^2\right)\cdot 2x}{\left(x^2 + y^2\right)^4}\right]$$

$$= \frac{2y\left(y^2 - 3x^2\right)}{\left(x^2 + y^2\right)^3}$$

$$v_y = -\left[\frac{\left(x^2 + y^2\right)\cdot 1 - 2y^2}{\left(x^2 + y^2\right)^2}\right] = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$v_{yy} = \frac{\left(x^2 + y^2\right)^2 \cdot 2y - \left(y^2 - x^2\right)\cdot 2\left(x^2 + y^2\right)\cdot 2y}{\left(x^2 + y^2\right)^4}$$

$$= \frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}$$

$\therefore \quad v_{xx} + v_{yy} = 0$

i.e. v satisfies Laplace equation.

Now $\quad u_x \neq v_y \quad$ and $\quad u_y \neq -v_x$

i.e. C.R. equations are not satisfied by u and v.

Hence $u + iv$ is not a regular (analytic) function of z.

Note ☑ The reason for the above situation is that u and v are not the real and imaginary parts of the same analytic function. In fact, u is the real part of z^2 and v is the imaginary part of $\frac{1}{z}$.

Example 3.11 If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function of $z = x + iy$.

u and v are harmonic functions.

$\therefore u_{xx}, u_{xy}, u_{yy}$ and v_{xx}, v_{xy}, v_{yy} are all continuous (and hence exist) (1)

and $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ (2)

Consider $P + iQ = (u_y - v_x) + i(u_x + v_y)$

$$P_x = u_{yx} - v_{xx} \quad \text{and} \quad P_y = u_{yy} - v_{xy}$$

$$Q_x = u_{xx} + v_{yx} \quad \text{and} \quad Q_y = u_{xy} + v_{yy}.$$

The four partial derivatives P_x, P_y, Q_x and Q_y exist and are continuous in the region R [by (1)].

$$P_x = Q_y, \text{ if } u_{yx} - v_{xx} = u_{xy} + v_{yy}$$

i.e. if $v_{xx} + v_{yy} + (u_{xy} - u_{yx}) = 0$

i.e. if $v_{xx} + v_{yy} = 0$ [by (1)],

which is true, by (2).

$$P_y = - Q_x, \text{ if } u_{yy} - v_{xy} = - (u_{xx} + v_{yx})$$

i.e. if $u_{xx} + u_{yy} - (v_{xy} - v_{yx}) = 0$

i.e. if $u_{xx} + u_{yy} = 0$ [by (1)],

which is true, by (2).

Thus the C.R. equations are satisfied by P and Q (3)

$\therefore$ By (1) and (3), $P + iQ$ is analytic.

Example 3.12 Show that the families of curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ intersect orthogonally, where α and β are arbitrary constants.

The given equations can be rewritten as $r^n \cos n\theta = \alpha$ and $r^n \sin n\theta = \beta$.

i.e. $u(r, \theta) = \alpha$ and $v(r, \theta) = \beta$, say.

Now
$$\begin{aligned} u(r, \theta) + iv(r, \theta) &= r^n (\cos n\theta + i \sin n\theta) \\ &= r^n e^{in}\theta \\ &= (re^{i\theta})^n \text{ or } z^n, \text{ which is an analytic function.} \end{aligned}$$

$\therefore$ By property (4), the families of curves $u(r, \theta) = \alpha$ and $v(r, \theta) = \beta$ cut orthogonally.

Example 3.13 If $f(z)$ is a regular function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Let $f(z) = u+iv$

$\therefore$ $|f(z)|^2 = u^2 + v^2$ and $|f'(z)|^2 = u_x^2 + v_x^2$

Since $f(z)$ is a regular (analytic) function,

$$u_x = v_y \text{ and } u_y = - v_x \text{ (C.R. equations)} \qquad (1)$$

and
$$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0 \text{ (Laplace equations)} \qquad (2)$$

Now
$$\frac{\partial}{\partial x}(u^2) = 2u\, u_x$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2(u\, u_{xx} + u_x^2)$$

Similarly,
$$\frac{\partial^2}{\partial y^2}(u^2) = 2(u\, u_{yy} + u_y^2),$$

$$\frac{\partial^2}{\partial x^2}(v^2) = 2(vv_{xx} + v_x^2) \quad \text{and} \quad \frac{\partial^2}{\partial y^2}(v^2) = 2(vv_{yy} + v_y^2)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v^2$$

$$= 2u\,(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) + 2v(v_{xx} + v_{yy}) + 2(v_x^2 + v_y^2)$$

$$= 2(u_x^2 + u_y^2 + v_x^2 + v_y^2), \text{ by} \qquad (2)$$

$$= 2(u_x^2 + v_x^2 + v_x^2 + u_x^2), \text{ by} \qquad (1)$$

$$= 4(u_x^2 + v_x^2)$$

$$= 4|f'(z)|^2.$$

Example 3.14 If $f(z) = u + iv$ is a regular function of z, prove that $\nabla^2\{\log |f(z)|\} = 0$.

$f(z) = (u + iv)$ is analytic.

$\therefore \quad u_x = v_y \quad$ and $\quad u_y = -v_x$ (C.R. equations) (1)

and $\quad u_{xx} + u_{yy} = 0 \quad$ and $\quad v_{xx} + v_{yy} = 0$ (Laplace equations)

$$\log|f(z)| = \frac{1}{2}\log(u^2 + v^2) \qquad (2)$$

$$\therefore \quad \frac{\partial}{\partial x}\log|f(z)| = \frac{1}{2}\cdot\left(\frac{2uu_x + 2v\cdot v_x}{u^2 + v^2}\right) = \frac{uu_x + v\,v_x}{u^2 + v^2}$$

$$\frac{\partial^2}{\partial x^2}\log|f(z)| = \frac{\left[(u^2 + v^2)\{uu_{xx} + u_x^2 + vv_{xx} + v_x^2\} - (uu_x + vv_x)(2uu_x + 2vv_x)\right]}{(u^2 + v^2)^2}$$

$$= \frac{1}{u^2 + v^2}\{u\,u_{xx} + v\,v_{xx} + u_x^2 + v_x^2\} - \frac{2}{(u^2 + v^2)^2}(u\,u_x + v\,v_x)^2 \qquad (3)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\log|f(z)|| = \frac{1}{u^2 + v^2}\{u\,u_{yy} + vv_{yy} + u_y^2 + v_y^2\} - \frac{2}{(u^2 + v^2)^2}(u\,u_y + v\,v_y)^2 \qquad (4)$$

Adding (3) and (4), we get

$$\nabla^2\{\log|f(z)|\} = \frac{1}{u^2 + v^2}[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2]$$

$$- \frac{2}{(u^2 + v^2)^2}[(u\,u_x + v\,v_x)^2 + (u\,u_y + v\,v_y)^2]$$

$$= \frac{1}{u^2 + v^2}[2(u_x^2 + v_x^2)] - \frac{2}{(u^2 + v^2)^2}[(u\,u_x + v\,v_x)^2 + (-u\,v_x + v\,u_x)^2],$$

by (1) and (2).

$$= \frac{2\left(u_x^2 + v_x^2\right)}{u^2 + v^2} - \frac{2}{\left(u^2 + v^2\right)^2} \{u^2 \left(u_x^2 + v_x^2\right) + v^2 \left(u_x^2 + v_x^2\right)\}$$

$$= \frac{2\left(u_x^2 + v_x^2\right)}{u^2 + v^2} - \frac{2\left(u^2 + v^2\right)\left(u_x^2 + v_x^2\right)}{\left(u^2 + v^2\right)^2}$$

$$= 0$$

Example 3.15 Find the equation of the orthogonal trajectories of the family of curves given by $3x^2y + 2x^2 - y^3 - 2y^2 = a$, where a is an arbitrary constant.

If $w = u + iv$ is analytic, the families of curves $u = a$ and $v = b$ are known as the orthogonal trajectories of each other. The given family can be assumed as $u = a$. We have to find the equation $v = b$, such that $u + iv$ is analytic.

$$u = 3x^2y + 2x^2 - y^3 - 2\,y^2$$

$$u_x = 6_{xy} + 4x\,; \quad u_y = 3x^2 - 3y^2 - 4y$$

$$\mathrm{d}v = v_x\,\mathrm{d}x + v_y\,\mathrm{d}y$$

$$= - u_y\,\mathrm{d}x + u_x\,\mathrm{d}y, \qquad \text{(by C.R. equations)}$$

$$= (3y^2 + 4y - 3x^2)\,\mathrm{d}x + (6xy + 4x)\,\mathrm{d}y$$

$$\therefore \qquad v = \int\left[\left(3y^2 + 4y - 3x^2\right)\mathrm{d}x + \left(6xy + 4x\right)\mathrm{d}y\right]$$

$$= 3xy^2 + 4xy - x^3$$

$\therefore$ The required equation of the orthogonal trajectories is $3xy^2 + 4xy - x^3 = b$, where b is an arbitrary constant.

EXERCISE 3(b)

Part A

(Short Answer Questions)

1. State any two properties of an analytic function.
2. Define a harmonic function and give an example.
3. How are analytic function and harmonic function related?
4. Write down the Laplace equations in two-dimensional cartesian and polar coordinates.
5. What do you mean by conjugate harmonic function? Find the conjugate harmonic of x.

Verify whether the following function's are harmonic.

6. xy 7. $e^x \sin y$ 8. $x^2 + y^2$ 9. $\cos x \sinh y$

10. $e^y \cosh x$.

The following functions are harmonic. Find the corresponding conjugate harmonic functions:

11. $x^2 - y^2$ 12. $e^x \cos y$ 13. $\sin x \cosh y$ 14. $2x(1 - y)$
15. $\log (x^2 + y^2)$

Find the analytic function $f(z) = u + iv$, given that

16. $v = \text{amp}(z)$ 17. $u = y^2 - x^2$ 18. $u = e^y \cos x$ 19. $v = \sinh x \sin y$
20. $u = \dfrac{x}{x^2 + y^2}$.

Part B

21. Prove that the function $u = x(x^2 - 3y^2) + (x^2 - y^2) + 2xy$ is harmonic. Also find the conjugate harmonic function v and the corresponding analytic function $(u + iv)$.
22. Prove that the function $v = 3x^2y + x^2 - y^3 - y^2$ is harmonic. Also find the conjugate harmonic function u and the corresponding analytic function $(u + iv)$.
23. Show that $\phi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ can represent the velocity potential in an incompressible fluid flow. Also find the corresponding stream function and complex potential.
24. Show that $\psi = x^2 - y^2 - 3x - 2y + 2xy$ can represent the stream function of an incompressible fluid flow. Also find the corresponding velocity potential and complex potential.
25. Show that the equation $x^3y - xy^3 + xy + x + y = c$ can represent the path of electric current flow in an electric field. Also find the complex electric potential and the equation of the potential lines.
26. Find the analytic function $w = u + iv$, if $u = e^x (x \sin y + y \cos y)$. Hence find v.
27. Find the analytic function $w = u + iv$, if $v = e^{-x} (x \cos y + y \sin y)$. Hence find u.
28. Find the analytic function $w = u + iv$, if $u = e^{-2xy} \cdot \sin (x^2 - y^2)$. Hence find v.
29. Find the analytic function $w = u + iv$, if $v = e^{-2y} (y \cos 2x + x \sin 2x)$. Hence find u.
30. Find the analytic function $f(z) = u + iv$, given that $u + v = \dfrac{2x}{x^2 + y^2}$ and $f(1) = i$.
31. Find the analytic function $f(z) = u + iv$, given that $2u - 3v = 3y^2 - 2xy - 3x^2 + 3y - x$ and $f(0) = 0$.
32. Find the analytic function $f(z) = P + iQ$, if $P - Q = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.
33. Find the analytic function $f(z) = P + iQ$, if $Q = \dfrac{\sin x \sinh y}{\cos 2x + \cosh 2y}$, if $f(0) = 1$.
34. Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, when $w = u + iv = z^4$.

35. Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally, when $w = u + iv = \dfrac{1}{z}$.

36. Show that the families of curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut orthogonally, where a and b are arbitrary constants.

37. Find the equation of the orthogonal trajectories of the family of curves given by $2x - x^3 + 3xy^2 = a$.

38. Prove that $u = e^{-y} \cos x$ and $v = e^{-x} \sin y$ satisfy Laplace equations, but that $(u + iv)$ is not an analytic function of z.

39. If $f(z)$ is an analytic function of z in any domain, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 |f'(z)|^2 |f(z)|^{p-2}.$$

40. If $f(z)$ is an analytic function of z in a region R, prove that

$$\nabla^2 \{Re\, f(z)\}^2 = \nabla^2 \{\mathrm{Im}\, f(z)\}^2 = 2|f'(z)|^2.$$

3.4 CONFORMAL MAPPING

3.4.1 Mapping

A continuous real function $y = f(x)$ can be represented graphically by a curve in the Cartesian xy-plane. Similarly a continuous real function $z = f(x, y)$ is represented graphically by a surface in three dimensional space.

In the same fashion, if we wish to represent a function of the complex variable $w = f(z)$ or $u + iv = f(x + iy)$, a four-dimensional space is required, since $w = f(z)$ involves four real variables two independent variables x and y and two dependent variables u and v. As it is not possible, we make use of two complex planes for the two variables z and w. These are called the z-plane and the w-plane respectively. In the z-plane, the point $z = x + iy$ is plotted and in the w-plane, the point $w = u + iv$ is plotted.

A function $w = f(z)$ is not, as usual, represented by a locus of points in the four-dimensional space, but by a correspondence between points of the z-plane and points of the w-plane. To each point (x, y) in the z-plane, the function $w = f(z)$ determines a point (u, v) in the w-plane if $f(z)$ is a single-valued function. If the point z moves along some curve C in the z-plane, the corresponding point w will, in general, move along a curve C' in the w-plane. Similarly if the point z moves over a region R in the Z-plane, the corresponding point w moves over a region R' in the w-plane. The correspondence thus defined is called a *mapping* or *transformation* of elements (points, curves or regions) in the z-plane onto elements in the w-plane. The function $w = f(z)$ is called the *mapping* or *transformation function.* The corresponding points, curves or regions in the two planes are called the *image* of each other.

To visualise the nature of a function $f(z)$, we study the properties of the mapping defined by $w = f(z)$. To get a clear idea of the mapping given by $w = f(z)$, we usually consider the images of lines parallel to either co-ordinate axis, of concurrent lines passing through the origin, of concentric circles $|z| =$ constant and of regions enclosed by such curves in the z-plane. Also we can investigate the maps onto the z-plane of lines parallel to the u-axis and v-axis of the w-plane. The images of $u = c_1$ and $v = c_2$ that lie in the z-plane are called *the level curves* of u and v.

3.4.2 Conformal Mapping

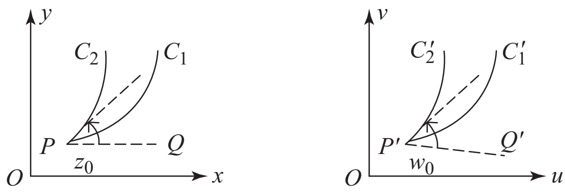


Fig. 3.4

Consider the transformation $w = f(z)$, where $f(z)$ is a single valued function of z. Under this transformation, a point z_0 and any two curves c_1 and c_2 passing through z_0 in the z-plane will be mapped onto a point w_0 and two curves c'_1 and c'_2 in the w-plane. If the angle between c_1 and c_2 at z_0 is the same as the angle between c'_1, and c'_2 at w_0, both in magnitude and sense, then the transformation $w = f(z)$ is said to be *conformal* at the point z_0. The formal definition is given as follows.

3.4.3 Definition

A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is said to be *conformal* at that point. A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be *isogonal* at that point.

The conditions under which the transformation $w = f(z)$ is conformal are given by the following theorem:

Theorem

If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z-plane, then the mapping performed by $w = f(z)$ is conformal at all points of R.

Proof

Let z_0 be a point in the region R of the z-plane where $f(z)$ is analytic and let $f'(z_0) \neq 0$.

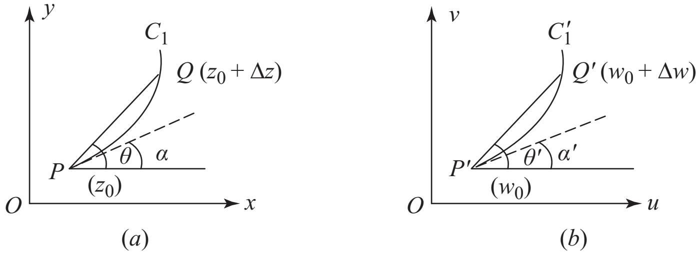


Fig. 3.5

Let c_1 be a continuous curve through z_0 and c'_1 be its image through w_0.

Let Q be a neighbouring point $z_0 + \Delta z$ on c_1 and Q' be its image $w_0 + \Delta w$ on c'_1. Let the chords PQ and $P'Q'$ make angles θ and θ' with the x-axis and u-axis respectively. Then Δz is a complex number whose modulus is the length PQ $(= r)$ and amplitude is θ.

$\therefore$ $$\Delta z = r\, e^{i\theta}$$

Similarly, $$\Delta w = r'\, e^{i\theta'}$$

Since $f(z)$ is analytic at z_0,

$$f'(z_0) = \lim_{\Delta z \to 0} \left\{ \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right\} = \lim_{\Delta z \to 0} \left(\frac{\Delta w}{\Delta z} \right) \text{exists} \tag{1}$$

Since $f'(z_0) \neq 0$, it can be expressed in the polar form as $R\, e^{i\phi}$.

$\therefore$ From (1), we get

$$R\, e^{i\phi} = \lim_{\Delta z \to 0} \left[\frac{r'\, e^{i\theta'}}{r\, e^{i\theta}} \right] = \lim_{r \to 0} \left[\left(\frac{r'}{r} \right) e^{i(\theta' - \theta)} \right]$$

$\therefore$ $$R = \lim_{r \to 0} \left(\frac{r'}{r} \right) \tag{2}$$

and $$\phi = \lim_{r \to 0} (\theta' - \theta) \tag{3}$$

Let the angle made by the tangent to c_1 at z_0 with the x-axis be α and that to c'_1 at w_0 with the u-axis be α'.

When $r \to 0$ or $\Delta z \to 0$, Q approaches P and hence the chord PQ tends to be the tangent at z_0 to the curve c_1 and so $\theta \to \alpha$. Similarly $\theta' \to \alpha'$.

Hence, from (3), we get

$$\phi = \alpha' - \alpha \tag{4}$$

If c_2 is another curve through z_0 in the z-plane and c'_2 is its image through w_0 in the w-plane and if the tangents to c_2 and c'_2 at z_0 and w_0 make angles β and β' with the x-axis and the u-axis respectively, then

$$\phi = \beta' - \beta \tag{5}$$

From (4) and (5), we get

$$\alpha' - \alpha = \beta' - \beta \quad \text{or} \quad \beta - \alpha = \beta' - \alpha'.$$

i.e. the angle between c_1 and c_2 is equal to the angle between c'_1 and c'_2, both in magnitude and sense. This means that the mapping $w = f(z)$ preserves angles between any two curves through the point z_0.

$\therefore$ The mapping $w = f(z)$ is conformal at z_0.

Note ☑

1. If $f(z_0) = 0$, it cannot be expressed in the polar form, since the amp $\{f(z_0)\}$ is undefined. Hence the proof of the theorem is not valid. Thus, though $f(z)$ may be analytic at z_0, the mapping $w = f(z)$ will not be conformal at z_0, if $f(z_0) = 0$.

[See Worked Example 3.3 in Section 3(*c*)]

2. The point, at which the mapping $w = f(z)$ is not conformal, i.e. $f'(z) = 0$, is called *a critical point* of the mapping.

   It is known that if the transformation $w = f(z)$ is conformal at a point, the inverse transformation $z = f^{-1}(w)$ is also conformal at the corresponding point.

   The critical points of $z = f^{-1}(w)$ are given by $\frac{dz}{dw} = 0$. Hence the critical points of the transformation $w = f(z)$ are given by $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

   [See Worked Example (3.3) below].

3. From equation (2) in the proof of the theorem, we get $R = \frac{r'}{r}$ approximately, i.e. $r' = Rr$

   or $\qquad |\Delta w| = |f'(z_0)| \cdot |\Delta z|$.

   This means that, under the transformation, an infinitesimal length Δz in the neighbourhood of z_0 is magnified by the factor $|f'(z_0)|$. Consequently, infinitesimal areas near z_0 in the z-plane are magnified by the factor $|f'(z_0)|^2$.

4. From equation (4) in the proof of the theorem, we get

   $$\alpha' = \alpha + \phi \quad \text{or} \quad \alpha + \text{amp}\,[f'(z_0)]$$

   This means that, under the transformation, the tangent to a curve through z_0 is rotated through an angle $\phi = \text{amp}\,[f'(z_0)]$, i.e. the direction of a curve through z_0 is rotated through an angle $\phi = \text{amp}\,[f'(z_0)]$.

5. From Notes (3) and (4), we observe that the image of a small figure near z_0 under the mapping $w = f(z)$ can be obtained by rotating it through an angle $= \text{amp}[f'(z_0)]$ and by magnifying it by a factor $= |f'(z_0)|$. Hence the shape of the image of a small figure near z_0 is approximately the same as that of the original figure under a conformal transformation.

3.5 SOME SIMPLE TRANSFORMATIONS

1. Translation

The transformation $w = z + c$, where c is a complex constant, represents a translation.

Let $z = x + iy$, $\quad w = u + iv \quad$ and $\quad c = a + ib$

Then $\qquad u + iv = x + iy + a + ib$

$\therefore \qquad u = x + a \quad$ and $\quad v = y + b$

These two equations may be called the transformation equations.

Hence the image of any point (x, y) in the z-plane is the point $(x + a, y + b)$ in the w-plane.

If we assume that the w-plane is super-imposed on the z-plane, we observe that the point (x, y) and hence any figure is shifted by a distance $|c| = \sqrt{a^2 + b^2}$ in the direction of c i.e., translated by the vector representing c. Hence this transformation transforms a circle into an equal circle. Also the corresponding regions in the z- and w-planes will have the same shape, size and orientation.

2. Magnification

The transformation $w = cz$, where c is a real constant, represents magnification. The transformation equations are given by

$$u + iv = c\,(x + iy)$$

i.e., $$u = cx \quad \text{and} \quad v = cy$$

$\therefore$ The image of the point (x, y) is the point (cx, cy).

Hence the size of any figure in the z-plane is magnified c times, but there will be no change in the shape and orientation. This transformation also transforms circles into circles.

3. Magnification and Rotation

The transformation $w = cz$, where c is a complex constant, represents both magnification and rotation.

Let $$z = r\,e^{i\theta}, \quad w = R\,e^{i\phi} \quad \text{and} \quad c = \rho e^{i\alpha},$$

Then $$Re^{i\phi} = (\rho\, e^{i\alpha})(r\, e^{i\theta})$$

$$= (\rho\, r) \cdot e^{i\,(\theta + \alpha)}$$

$\therefore$ The transformation equations are

$$R = \rho r \quad \text{and} \quad \phi = \theta + \alpha.$$

Thus the point (r, θ) in the z-plane is mapped onto the point $(\rho r, \theta + \alpha)$. This means that the magnitude of the vector representing z is magnified by $\rho = |c|$ and its direction is rotated through angle $\alpha = \text{amp}\,(c)$. Hence the transformation consists of a magnification and a rotation. Clearly circles in the z-plane are mapped onto circles by this transformation. Also every region in the z-plane is mapped onto a similar region by this transformation.

4. Magnification, Rotation and Translation

The general linear transformation $w = az + b$, where a and b are complex constants, represents magnification, rotation and translation. The transformation $w = az + b$ can be considered as the combination of the two simple transformations $w_1 = az$ and $w = w_1 + b$.

$w_1 = az$ represents magnification by $|a|$ and rotation through amp (a).

$w = w_1 + b$ represents translation by the vector representing b.

Thus any figure in the z-plane will undergo magnification, rotation and translation by the transformation $w = az + b$. In particular, circles will be mapped into circles by this transformation.

5. Inversion and Reflection

The transformation $w = \frac{1}{z}$ represents inversion with respect to the unit circle $|z| = 1$, followed by reflection in the real axis.

[The *inverse* of a point P with respect to a circle with centre O and radius r is defined as the point P' on OP such that $OP \cdot OP' = r^2$]

Let $\quad z = re^{i\theta} \quad$ and $\quad w = Re^{i\phi}$

Then $\quad w = \frac{1}{z} \quad$ gives $\quad Re^{i\phi} = \frac{1}{r}e^{-i\theta}$

$\therefore$ The transformation equations are

$$R = \frac{1}{r} \quad \text{and} \quad \phi = -\theta.$$

Thus the image of the point (r, θ) in the z-plane is $\left(\frac{1}{r}, -\theta\right)$ under this transformation

If we assume that the w-plane is super-imposed on the z-plane and that P is (r, θ) and P' is $\left(\frac{1}{r}, \theta\right)$, then $OP' = \frac{1}{OP}$.

i.e., $\quad OP \cdot OP' = 1$

$\therefore$ P' is the inverse of P with respect to the unit circle $|z| = 1$, as shown in Fig. 3.6.

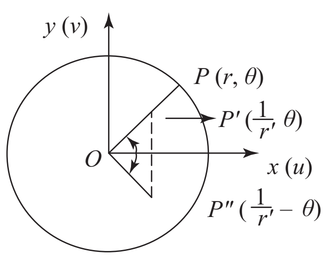


Fig. 3.6

If we consider the point $P''\left(\frac{1}{r}, -\theta\right)$, it is the reflection of the point $P'\left(\frac{1}{r}, \theta\right)$ in the real axis. Thus the transformation $w = \frac{1}{z}$ consists of an inversion of z with respect to the unit circle $|z| = 1$, followed by reflection in the real axis.

Also it is observed that the interior (exterior) of the unit circle $|z| = 1$ is mapped onto the exterior (interior) of the unit circle $|w| = 1$.

3.6 SOME STANDARD TRANSFORMATIONS

1. The Transformation $w = z^2$

$$w = z^2 \quad \therefore \quad u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$$

Hence the transformation equations are

$$u = x^2 - y^2 \tag{1}$$

and
$$v = 2xy \tag{2}$$

Consider a line parallel to the x-axis, given by the equation

$$y = b \tag{3}$$

The equation of the image of (3), which will be an equation in u and v, is got by eliminating x and y from (1), (2) and (3).

Using (3) in (1) and (2), we have

$$u = x^2 - b^2 \tag{4}$$

and
$$v = 2bx \tag{5}$$

Eliminating x from (4) and (5), we have

$$u = \left(\frac{v}{2b}\right)^2 - b^2, \text{ i.e. } v^2 = 4b^2 (u + b^2) \tag{6}$$

Equation (6) represents in the w-plane a parabola whose vertex is $(-b^2, 0)$, focus is (0, 0) and axis lies along the u-axis and which is open to the right. [Fig. 3.7]

If b is regarded as an arbitrary constant or parameter, (3) represents a family of lines parallel to the x-axis. In this case, (6) represents a system of parabolas, all having the origin as the common focus, i.e. equation (6) represents a family of confocal parabolas.

Consider the equation $x = a$ (7)

This represents a line parallel to the y-axis. The image of line (7) is got by eliminating x and y from (1), (2) and (7).

Thus the image of (7) is given by the following equations

$$u = a^2 - y^2$$

and
$$v = 2ay$$

i.e. by the equation

$$u = a^2 - \left(\frac{v}{2a}\right)^2$$

or
$$v^2 = -4a^2(u - a^2) \tag{8}$$

Equation (8) represents in the w-plane a parabola, whose vertex is $(a^2, 0)$, focus is (0, 0) and axis lies along the u-axis and which is open to the left. [Fig. 3.7]

If a is regarded as an arbitrary constant or parameter, (7) represents a family of lines parallel to the y-axis and (8) represents a family of confocal parabolas with the common focus at the origin.

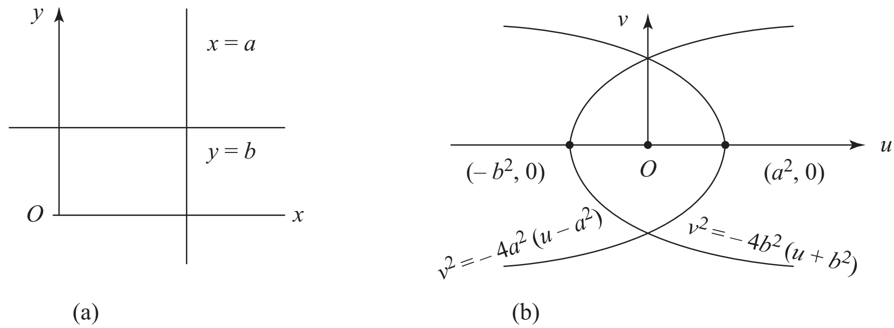


Fig. 3.7

Thus a system of lines parallel to either coordinate axis is mapped onto a system of confocal parabolas under the transformation $w = z^2$, with the exceptions $x = 0$ and $y = 0$.

The map of $y = 0$ is given by $u = x^2$ and $v = 0$, i.e., by $v = 0$, where $u > 0$.

i.e., the map of the entire x-axis is the positive part or the right half of the u-axis.

The map of $x = 0$ is given by $u = -y^2$ and $v = 0$, i.e., by $v = 0$, where $u < 0$.

i.e., the map of the entire y-axis is the negative part or the left half of the u-axis.

Using polar forms of z and w, i.e. putting

$z = re^{i\theta}$ and $w = Re^{i\phi}$ in $w = z^2$, we have

$$R\, e^{i\phi} = (r\, e^{i\theta})^2 = r^2 e^{i2\theta}$$

$\therefore$ The transformation equations are

$$R = r^2 \quad \text{and} \quad \phi = 2\theta.$$

Now $r = a$ represents a family of concentric circles in the z-plane. Its map is given by $R = a^2$, that represents a family of concentric circles in the w-plane. $\theta = \alpha$ represents a family of concurrent lines through the origin in the z-plane. Its map is given by $\phi = 2\alpha$, that represents a family of concurrent lines through the origin in the w-plane.

Consider now $u = c$, that represents a family of lines parallel to the v-axis. The image of $u = c$ is given by $x^2 - y^2 = c^2$, that represents a family of rectangular hyperbolas whose principal axes lie along the coordinate axes in the z-plane. Consider $v = d$, that represents a family of lines parallel to the u-axis. The image of $u = d$ is given by $xy = \frac{d}{2}$, that represents a family of rectangular hyperbolas whose asymptotes are the coordinate axes in the z-plane.

Finally for the mapping function $w = z^2$,

$$\frac{dw}{dz} = 2z$$

$$= 0, \text{ at } z = 0.$$

Hence the transformation $w = z^2$ is conformal at all points in the z-plane except at the origin, i.e. $z = 0$ is the critical point of the transformation $w = z^2$.

2. The Transformation $w = e^z$

$$w = e^z \;\therefore\; \frac{dw}{dz} = e^z \neq 0 \text{ for any } z.$$

$\therefore$ The transformation $w = e^z$ is conformal at all points in the z-plane.

Putting $z = x + iy$ and $w = R\,e^{i\phi}$ in $w = e^z$, we get

$$R\,e^{i\phi} = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$\therefore$ The transformation equations are

$$R = e^x \tag{1}$$

and

$$\phi = y \tag{2}$$

The image of the system of parallel lines $x = a$ in the z-plane is given by $R = e^a$, that represents a family of concentric circles in the w-plane.

The image of the system of parallel lines $y = b$ in the z-plane is given by $\phi = b$, that represents a family of concurrent lines through the origin in the w-plane. In particular, the image of the y-axis i.e. $x = 0$ is the unit circle $R = 1$ or $|w| = 1$. The image of $x > 0$ is given by $R > 1$ or $|w| > 1$ and the image of $x < 0$ is given by $R < 1$ or $|w| < 1$. i.e. the region lying on the right side of the y-axis in the z-plane is mapped onto the exterior of the unit circle $|w| = 1$ in the w-plane.

Similarly, the region lying on the left side of the y-axis in the z-plane is mapped onto the interior of the unit circle $|w| = 1$ in the w-plane.

The image of the entire x-axis, i.e. $y = 0$ is given by $\phi = 0$, i.e., the positive part of the u-axis.

Similarly the image of the line $y = \pi$ is given by $\phi = \pi$, i.e., the negative part of the u-axis.

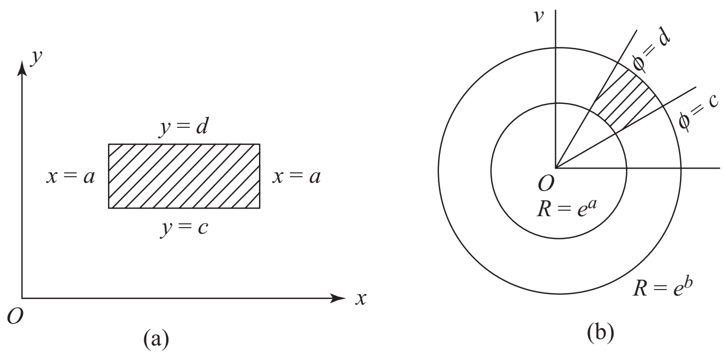


Fig. 3.8

Finally we note that the image of the rectangular region in the z-plane defined by $a \le x \le b$ and $c \le y \le d$ is the annular region in the w-plane defined by $e^a \le R \le e^b$ and $c \le \phi \le d$. The corresponding regions are shaded in the Fig. 3.8.

3. The Transformation $w = \sin z$

$$w = \sin z \quad \therefore \quad \frac{dw}{dz} = \cos z$$

$$= 0,$$

$$\text{when } z = \frac{(2n-1)\pi}{2}, \text{ where } n \text{ is an integer.}$$

$\therefore$ The transformation $w = \sin z$ is conformal at all points except at $z = \dfrac{(2n-1)\pi}{2}$.

i.e. the critical points of the transformation are $z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$

Putting $z = x + iy$ and $w = u + iv$ in $w = \sin z$, we get

$$u + iv = \sin(x + iy)$$

$$= \sin x \cosh y + i \cos x \sinh y.$$

$\therefore$ The transformation equations are

$$u = \sin x \cosh y \tag{1}$$

and
$$v = \cos x \sinh y \tag{2}$$

Consider the family of lines parallel to the x-axis, given by

$$y = b \tag{3}$$

where b is an arbitrary constant.

Image of (3) is given by

$$u = \sin x \cosh b \tag{3a}$$

and
$$v = \cos x \sinh b \tag{4}$$

Eliminating x from (3a) and (4), we get the equation of the image of (3) as

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1 \tag{5}$$

Equation (5) represents a family of ellipses whose principal axes lie along the u- and v-axes, centres are at the origin and semi axes are of lengths $\cosh b$ and $|\sinh b|$.

The foci of these ellipses are at the points $(\pm 1, 0)$ [$\because$ The foci of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are $(\pm ae, 0)$, i.e. $(\pm\sqrt{a^2 - b^2}, 0)$]

The coordinates of the foci of the family of ellipses (5) are independent of the parameter b.

This means that all the members of the family (5) have the same two points as foci.

Hence (5) represents a family of confocal ellipses. [Fig. 3.9]

Thus a family of lines parallel to the x-axis in the z-plane is mapped onto a family of confocal ellipses in the w-plane, with the exception of $y = 0$, i.e, the x-axis itself.

Now the map of $y = 0$ is given by

$$u = \sin x \quad \text{and} \quad v = 0$$

i.e. by $\quad -1 \le u \le 1 \quad$ and $\quad v = 0$

i.e. the map of the entire x-axis is the segment of the u-axis, lying between -1 and $+1$.

Consider again the segment of the line $y = b$ within the range $-\pi/2 \le x \le \pi/2$. For these values of x, $\cos x$ is positive.

Hence, from (4) i.e. $v = \cos x \sinh b$, we see that $v > 0$ when $b > 0$ [as $\sinh b > 0$ when $b > 0$] and $v < 0$ when $b < 0$ [as $\sinh b < 0$ when $b < 0$].

Thus the image of the segment of the line $y = b$ within $-\pi/2 \le x \le \pi/2$ is the upper or lower half of the ellipse $\dfrac{u^2}{\cosh^2 b} + \dfrac{v^2}{\sinh^2 b} = 1$, according as $b > 0$ or < 0.

Now consider the family of lines parallel to the y-axis, given by $x = a$ (6)
where a is an arbitrary constant.

Image of (6) is given by

$$u = \sin a \cosh y \tag{7}$$

and

$$v = \cos a \sinh y \tag{8}$$

Eliminating y from (7) and (8), we get the image of (6) as

$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1 \tag{9}$$

(9) Represents a family of confocal hyperbolas with common foci at $(\pm 1, 0)$ [$\because$ the foci of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are $(\pm ae, 0)$, i.e. $(\pm\sqrt{a^2 + b^2},\ 0)$] [Fig. 3.9]

Thus a family of lines parallel to the y-axis in the z-plane is mapped onto a family of confocal hyperbolas in the w-plane, with the exceptions of the y-axis and the lines $x = \pm\dfrac{\pi}{2}$.

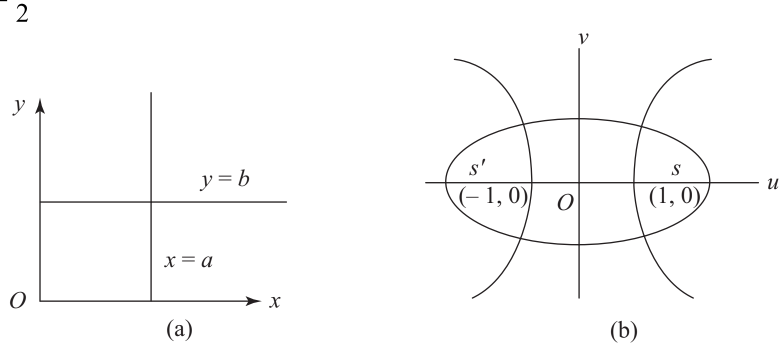


Fig. 3.9

The map of $x = 0$ is given by

$$u = 0 \quad \text{and} \quad v = \sinh y$$

i.e. $\quad u = 0 \quad$ and $\quad v > 0$ (when $y > 0$) and

$u = 0 \quad$ and $\quad v < 0$ (when $y < 0$)

Thus the upper and lower halves of the y-axis are mapped onto the upper and lower halves of the v-axis under the transformation $w = \sin z$.

Consider the map of the line $x = \pi/2$, which is given by $u = \cosh y$ and $v = 0$ i.e. by $u \geq 1$ and $v = 0$

Similarly, the map of the line $x = -\pi/2$ is given by $u = -\cosh y$ and $v = 0$ i.e. by $u \leq -1$ and $v = 0$

Thus the part of the u-axis for which $u \geq 1$ and the part of the u-axis for which $u \leq -1$ are the images of the lines $x = \dfrac{\pi}{2}$ and $x = -\dfrac{\pi}{2}$ respectively.

In conclusion, if we note the family of ellipses (5) and the family of hyperbolas (9) have the same two points ($\pm 1, 0$) as common foci, we can say that in general the transformation $w = \sin z$ maps the system of lines parallel to either coordinate axis in the z-plane into a system of confocal conics in the w-plane.

4. The Transformation $w = \cosh z$

$$w = \cosh z \quad \therefore \quad \frac{dw}{dz} = \sinh z$$

$$= -i \sin iz$$

$$= 0, \text{ when } z = i.\, n\pi, \text{ where } n \text{ is an integer.}$$

$\therefore$ The transformation $w = \cosh z$ is conformal at all points except the critical points $z = \pm i\pi, \pm i2\pi, \ldots$

Putting $z = x + iy$ and $w = u + iv$ in

$$w = \cosh z, \text{ we get}$$

$$u + iv = \cos i(x + iy)$$

$$= \cos(ix - y)$$

$$= \cosh x \cos y + i \sinh x \sin y$$

$\therefore$ The transformation equations are

$$u = \cosh x \cos y \tag{1}$$

and

$$v = \sinh x \sin y \tag{2}$$

The image of the family of parallel lines $y = b$ (3)

is the family of hyperbolas

$$\frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b} = 1 \tag{4}$$

The foci of (4) are ($\pm 1, 0$)

Thus the family of lines parallel to the x-axis in the z-plane is mapped onto a family of confocal hyperbolas in the w-plane with the exception $y = 0$, $y = \pi$ and $y = \pi/2$.

The map of $y = 0$ is given by

$$u = \cosh x \quad \text{and} \quad v = 0.$$

When $x > 0$, $\cosh x$ takes values from 1 to ∞.

$\therefore$ The image of the positive part of the x-axis is the portion of the u-axis for which $u \geq 1$.

The map of $y = \pi$ is given by

$$u = -\cosh x \quad \text{and} \quad v = 0$$

$\therefore$ The image of the positive part of the line $y = \pi$ is the portion of the u-axis for which $u \leq -1$.

The map of $y = \pi/2$ is given by

$$u = 0 \quad \text{and} \quad v = \sinh x$$

$\therefore$ The images of the positive and negative parts of the line $y = \dfrac{\pi}{2}$ are the positive and negative parts of the v-axis respectively.

The image of the family of parallel lines $x = a$ (5), is the family of ellipses

$$\frac{u^2}{\cosh^2 a} + \frac{v^2}{\sinh^2 a} = 1 \tag{6}$$

The foci of (6) are $(\pm 1, 0)$.

Thus the family of lines parallel to the y-axis in the z-plane is mapped onto a family of confocal ellipses in the w-plane with the exception of the y-axis itself.

The image of $x = 0$ is given by

$$u = \cos y \quad \text{and} \quad v = 0$$

i.e. by $$-1 \leq u \leq 1 \quad \text{and} \quad v = 0$$

i.e. the image of the entire y-axis is the segment of the u-axis, for which $-1 \leq u \leq 1$. Thus, in general, the transformation $w = \cosh z$ maps the system of lines parallel to either coordinate axis in the z-plane into a system of confocal conics in the w-plane.

5. The Transformation $w = z + \dfrac{k^2}{z}$, where k is real and positive

$$w = z + \frac{k^2}{z} \qquad \therefore \quad \frac{dw}{dz} = 1 - \frac{k^2}{z^2}$$

$$= 0, \qquad \text{when } z = \pm k$$

$\therefore$ The transformation $w = z + \dfrac{k^2}{z}$ is conformal at all points of the z-plane except at $z = \pm k$.

Putting $z = r e^{i\theta}$ and $w = u + iv$ in $w = z + \dfrac{k^2}{z}$, we get

$$u + iv = re^{i\theta} + \frac{k^2}{r} e^{-i\theta}$$

$$= \left(r + \frac{k^2}{r}\right)\cos\theta + \left(r - \frac{k^2}{r}\right)\sin\theta$$

$\therefore$ The transformation equations are

$$u = \left(r + \frac{k^2}{r}\right)\cos\theta \qquad (1)$$

and
$$v = \left(r - \frac{k^2}{r}\right)\sin\theta \qquad (2)$$

Consider a family of concentric circles with centre at the origin in the z-plane, given by the polar equation $r = a$, (3)
where a is a parameter. The image of (3) is given by

$$u = \left(a + \frac{k^2}{a}\right)\cos\theta \text{ and } v = \left(a - \frac{k^2}{a}\right)\sin\theta.$$

Eliminating θ from these equations, the equation of the image of the family (3) is

$$\frac{u^2}{\left(a + \frac{k^2}{a}\right)^2} + \frac{v^2}{\left(a - \frac{k^2}{a}\right)^2} = 1 \qquad (4)$$

Equation (4) represents a family of ellipses whose centres are at the origin, principal axes lie along the u- and v-axes and foci are at the points

$$\left(\pm\sqrt{\left(a + \frac{k^2}{a}\right)^2 - \left(a - \frac{k^2}{a}\right)^2}, 0\right), \quad \text{i.e. } (\pm 2k, 0).$$

The co-ordinates of the foci do not depend on a.

Hence (4) represents a family of confocal ellipses

Thus a family of concentric circles with centre at the origin in the z-plane is mapped onto a family of confocal ellipses, with the exception of $r = k$.

The image of the circle $r = k$ is given by $u = 2k \cos\theta$ and $v = 0$ (from (1) and (2) i.e. $-2k \le u \le 2k$ and $v = 0$.

Thus the image of the circle $r = k$ is the segment of the u–axis, given by $-2k \le u \le 2k$.

Consider a family of concurrent lines through the origin in the z- plane, given by the polar equation $\theta = \alpha$ (5)
where α is a parameter. The image of (5) is given by

$$u = \left(r + \frac{k^2}{r}\right)\cos\alpha \quad \text{and} \quad v = \left(r - \frac{k^2}{r}\right)\sin\alpha.$$

Eliminating r from these equations, the equation of the image of the family (5) is

$$\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = 4k^2$$

or
$$\frac{u^2}{4k^2\cos^2\alpha} - \frac{v^2}{4k^2\sin^2\alpha} = 1 \qquad (6)$$

Equation (6) represents a family of hyperbolas whose centres are at the origin, principal axes lie along the u- and v-axes and foci are at the points

$$\left(\pm\sqrt{4k^2\cos^2\alpha+4k^2\sin^2\alpha}\,,0\right),\ \text{i.e. } (\pm\, 2k,\, 0)$$

The coordinates of the foci do not depend on α. Hence (6) represents a family of confocal hyperbolas.

Thus a family of concurrent lines through the origin in the z-plane is mapped onto a family of confocal hyperbolas, with the exceptions of $\theta = 0$, $\theta = \pi$ and $\theta = \pi/2$. The image of $\theta = 0$ is given by

$$u = r + \frac{k^2}{r} \quad \text{and} \quad v = 0$$

i.e. $$u = \left(\sqrt{r} - \frac{k}{\sqrt{r}}\right)^2 + 2k \quad \text{and} \quad v = 0$$

i.e. $$u > 2k \quad \text{and} \quad v = 0$$

Thus the image of the positive part of the x-axis is the part of the u-axis for which $u > 2k$.

Similarly the image of $\theta = \pi$, i.e. the negative part of the x-axis is that part of the u-axis for which $u < -\,2k$.

The images of the lines $\theta = \pi/2$ and $\theta = \dfrac{3\pi}{2}$ are given by $u = 0$.

Hence the image of the y-axis is the v-axis.

WORKED EXAMPLE 3(c)

Example 3.1 Find the image of the circle $|z| = 2$ under the transformation (i) $w = z + 3 + 2i$,

(ii) $w = 3\,z$, (iii) $w = \sqrt{2}\,e^{i\pi/4}\,z$ and (iv) $w = (1 + 2i)\,z + (3 + 4i)$

(i) The equation of the given circle $|z| = 2$ in the Cartesian from is $\sqrt{x^2 + y^2} = 2$ or $x^2 + y^2 = 4$ (1)

The mapping function is $w = z + 3 + 2i$

i.e. $$u + iv = x + iy + 3 + 2i$$

$\therefore$ The transformation equations are

$$u = x + 3 \tag{2}$$

and $$v = y + 2 \tag{3}$$

Eliminating x and y from (1), (2) and (3), we get the equation of the image. From (2), $x = u - 3$; from (3), $y = v - 2$.

Using these values of x and y in (1), the required equation of the image is

$$(u-3)^2+(v-2)^2=4.$$

(ii) The transformation is $w = 3z$

i.e. $u + iv = 3\,(x + iy)$

$\therefore \quad u = 3x \quad (4)$

and $\quad v = 3y \quad (5)$

Eliminating x and y from (1), (4) and (5), the equation of the image of (1) is obtained as

$$\left(\frac{u}{3}\right)^2+\left(\frac{v}{3}\right)^2=4 \quad \text{or} \quad u^2+v^2=36.$$

(iii) The transformation is $w=\sqrt{2}\cdot e^{i\pi/4}\cdot z$

i.e., $u+iv=\sqrt{2}(\cos\pi/4+i\sin\pi/4)z$

$\qquad =(1+i)(x+iy)$

$\therefore \quad u=x-y \quad (6)$

and $\quad v=x+y \quad (7)$

From (6) and (7), w get $x=\dfrac{u+v}{2}$ and $y=\dfrac{v-u}{2}$.

Using these values of x and y in (1), the image of (1) is obtained as

$$\left(\frac{v+u}{2}\right)^2+\left(\frac{v-u}{2}\right)^2=4$$

i.e. $\quad u^2+v^2=8.$

(iv) The transformation is $w = (1 + 2i)\,z + (3 + 4i)$

i.e. $u + iv = (1 + 2i)\,(x + iy) + 3 + 4i$

$\qquad = (x - 2y + 3) + \mathrm{i}\,(2x + y + 4)$

$\therefore \quad u = x - 2y + 3 \quad (8)$

and $\quad v = 2x + y + 4 \quad (9)$

Solving (8) and (9), we get

$$x=\frac{u+2v-11}{5} \quad \text{and} \quad y=\frac{v-2u+2}{5}$$

Using these values of x and y in (1), the image of (1) is obtained as

$$(u-3)^2+(v-4)^2=20 \qquad (10)$$

Aliter

The transformation is $w - (3 + 4i) = (1 + 2i)z$

$\therefore \qquad |w - (3 + 4i)| = |1 + 2i|\,|z|$

$\therefore$ The map of $|z| = 2$ is given by

$|w-(3+4i)| = 2\sqrt{5}$, which is a circle whose centre is the point $(3 + 4i)$ and radius equal to $2\sqrt{5}$ and which is the same as the circle given by (10).

Example 3.2

(a) Find the image of the triangular region in the z-plane bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ under the transformation (i) $w = 2z$ and (ii) $w = e^{i\pi/4}.\, z$

(b) Find the image of the rectangular region in the z-plane bounded by the lines $x = 0$, $y = 0$, $x = 2$ and $y = 1$ under the transformation

(i) $\qquad w = z + 2 - i \quad$ and $\quad$ (ii) $\; w = (1 + 2i)z + (1 + i)$.

(a) (i) $\qquad w = 2z$ i.e., $u + iv = 2(x + iy)$

$\therefore \qquad u = 2x \quad$ and $\quad v = 2y$

$\therefore$ The images of $x = 0$, $y = 0$ and $x + y = 1$ are respectively $u = 0$, $v = 0$ and $u + v = 2$. The corresponding regions in the z-and w-planes are shown in the Figs 3.10 (a) and (b) respectively.

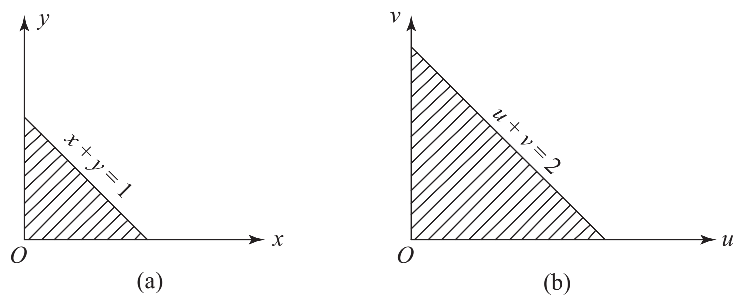


Fig. 3.10

(a) (ii) $\qquad w = e^{i\pi/4} \cdot z \quad$ i.e., $u + iv = \dfrac{1}{\sqrt{2}}(1+i)\,(x+iy)$

$$\therefore \qquad u = \frac{1}{\sqrt{2}}(x-y) \quad \text{and} \quad v = \frac{1}{\sqrt{2}}(x+y)$$

On solving, we get,

$$x = \frac{1}{\sqrt{2}}(u+v) \quad \text{and} \quad y = \frac{1}{\sqrt{2}}(v-u)$$

$\therefore$ The maps of $x = 0$ and $y = 0$ are, respectively, $u + v = 0$ and $u = v$.

The map of $x + y = 1$ is $v = \dfrac{1}{\sqrt{2}}$.

The corresponding regions in the z-plane and w-plane are shown in the Figs 3.11 (a) and (b) respectively.

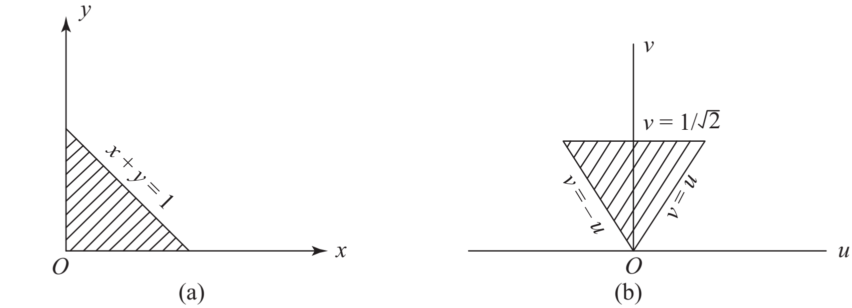


Fig. 3.11

(b) (i) $$w = z + 2 - i, \text{ i.e. } u + iv = x + iy + 2 - i$$

i.e. $$u = x + 2 \quad \text{and} \quad v = y - 1$$

The vertices of the given rectangle in the z-plane are (0, 0), (2, 0), (2, 1) and (0, 1).

The images of these points in the w-plane are (2, –1), (4, –1), (4, 0) and (2, 0) respectively. The corresponding regions in the two planes are shown in the Figs 3.12 (a) and (b)

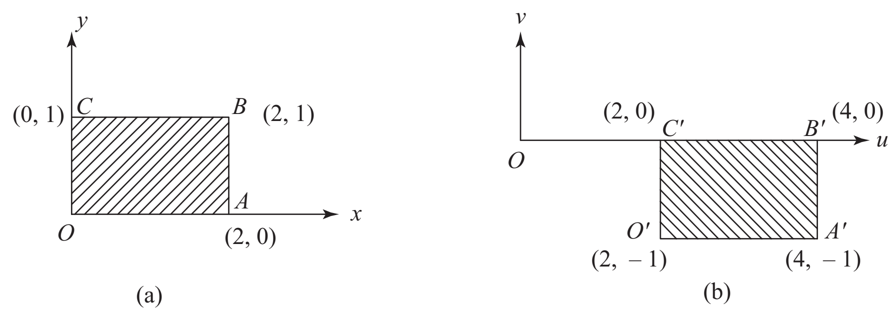


Fig. 3.12

(b) (ii) $$w = (1 + 2i)z + (1 + i)$$

i.e. $$u + iv = (1 + 2i)(x + iy) + (1 + i)$$

The image of (0, 0) is given by

$$u + iv = (1 + 2i)(0 + i0) + 1 + i$$
$$= 1 + i, \text{ i.e. the point } (1, 1)$$

The image of (2, 0) is given by

$$u + iv = (1 + 2i)(2 + i0) + 1 + i$$
$$= 3 + 5i, \text{ i.e. the point } (3, 5)$$

The image of (2, 1) is given by

$$u + iv = (1 + 2i)(2 + i) + 1 + i$$
$$= 1 + 6i, \text{ i.e. the point } (1, 6)$$

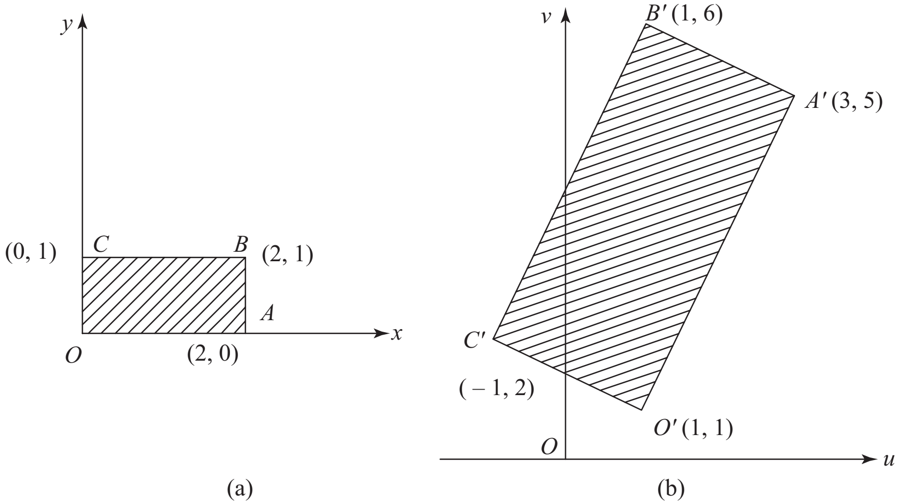


Fig. 3.13

The image of (0, 1) is given by

$$u + iv = (1 + 2i)\,(0 + i) + 1 + i$$

$$= -\,1 + 2i, \text{ i.e. the point } (-1, 2)$$

The corresponding regions in the two planes are shown in the Figs 3.13 (a) and (b).

Example 3.3 Find the critical points of the transformations

(i) $w = z^2$ and (ii) $w = \dfrac{1}{z}$. Give one example each to show that the mapping given by these functions is not conformal at the critical points.

(i) $$w = z^2 \quad \therefore \quad \frac{dw}{dz} = 2z$$

$$\therefore \quad \frac{dw}{dz} = 0 \quad \text{at} \quad z = 0$$

$\therefore$ $z = 0$ is the critical point of the transformation $w = z^2$.

Using polar coordinates, $w = z^2$ becomes

$$Re^{i\phi} = r^2 e^{i2\theta}$$

$$\therefore \quad R = r^2 \tag{1}$$

$$\text{and} \quad \phi = 2\theta \tag{2}$$

Consider two lines passing through the origin in the z-plane, given by the polar equations $\theta = \alpha$ and $\theta = \beta$ $(\beta > \alpha)$

Angle between these two lines = $\beta - \alpha$

The images of these two lines are

$$\phi = 2\alpha \quad \text{and} \quad \phi = 2\beta \qquad \text{[from (2)]}$$

Angle between these two image lines = $2\beta - 2\alpha$. Thus angle between any two curves, i.e. lines through $z = 0$ is not preserved in magnitude.

$\therefore$ The mapping $w = z^2$ is not conformal at $z = 0$.

(ii) $$w = \frac{1}{z} \quad \therefore \frac{dw}{dz} = -\frac{1}{z^2} \to \infty \text{ as } z \to 0.$$

$\therefore$ $w = 1/z$ is not analytic at $z = 0$ and hence the mapping $w = 1/z$ is not conformal at $z = 0$.

i.e. $z = 0$ is the critical point of the transformation.

Using polar co-ordinates, $w = \frac{1}{z}$ becomes

$$Re^{i\phi} = \frac{1}{r} e^{i\theta}$$

$\therefore$ $$R = \frac{1}{r} \tag{3}$$

and $$\phi = -\theta \tag{4}$$

From (4), we see that the images of the two lines $\theta = \alpha$ and $\theta = \beta$ ($\beta > \alpha$) in the z-plane are $\phi = -\alpha$ and $\phi = -\beta$.

Angle between the lines $\theta = \alpha$ and $\theta = \beta$ is $(\beta - \alpha)$ and the angle between the image lines is $(\alpha - \beta)$.

Thus the angle between any two curves, i.e. the lines through $z = 0$ is not preserved in sense.

$\therefore$ The mapping $w = \frac{1}{z}$ is not conformal at $z = 0$.

Example 3.4 Show that the transformation $w = \frac{1}{z}$ transforms, in general, circles and straight lines into circles and straight lines. Point out the circles and straight lines that are transformed into straight lines and circles respectively.

$$w = \frac{1}{z} \quad \therefore z = \frac{1}{w} \text{ i.e. } x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$\therefore$ The transformation equations are

$$x = \frac{u}{u^2 + v^2} \tag{1}$$

and $$y = \frac{-v}{u^2 + v^2} \tag{2}$$

Consider the equation

$$a(x^2 + y^2) + bx + cy + d = 0 \tag{3}$$

Equation (3) represents a circle not passing through the origin if $a \neq 0$ and $d \neq 0$, a circle passing through the origin if $a \neq 0$ and d = 0, a straight line not passing through the origin if $a = 0$ and d $\neq 0$ and a straight line passing through the origin, if $a = 0$ and d = 0.

Using (1) and (2) in (3), the image of (3) in the w-plane is given by

$$a\left\{\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right\}+\frac{bu}{u^2+v^2}-\frac{cv}{u^2+v^2}+\text{d}=0$$

i.e. $$\text{d}\,(u^2+v^2)+bu-cv+a=0 \tag{4}$$

Now Eq. (4) represents a circle not passing through the origin if $a \neq 0$ and d $\neq 0$, a straight line not passing through the origin if $a \neq 0$ and d $= 0$, a circle passing through the origin if $a = 0$ and d $\neq 0$ and a straight line passing through the origin if $a = 0$ and d $= 0$.

Thus circles not passing through the origin and straight lines passing through the origin are mapped onto similar circles and straight lines respectively.

But circles passing through the origin are mapped onto straight lines not passing the origin.

Straight lines not passing through the origin are mapped onto circles passing through the origin.

Example 3.5 Find the image of the following regions under the transformation $w = \dfrac{1}{z}$:

(i) the half-plane $x > c$, when $c > 0$

(ii) the half-plane $y > c$, when $c < 0$

(iii) the infinite strip $\dfrac{1}{4} \leq y \leq \dfrac{1}{2}$

Also show the corresponding regions graphically.

$$w=\frac{1}{z} \quad \therefore \quad z=\frac{1}{w} \quad \text{i.e. } x+iy=\frac{u-iv}{u^2+v^2}$$

$\therefore$ The transformation equations are

$$x=\frac{u}{u^2+v^2} \tag{1}$$

and $$y=-\frac{v}{u^2+v^2} \tag{2}$$

(i) The image of the region $x > c$ is given by $\dfrac{u}{u^2+v^2} > c$ from (1).

i.e. $$c(u^2+v^2) < u \quad \text{or} \quad u^2+v^2 < \frac{u}{c} \qquad [\because c > 0]$$

i.e. $$\left(u-\frac{1}{2c}\right)^2+v^2 < \left(\frac{1}{2c}\right)^2 \tag{3}$$

Equation (3) represents the interior of the circle

$$\left(u-\frac{1}{2c}\right)^2+v^2 = \left(\frac{1}{2c}\right)^2, \text{ whose centre is } \left(\frac{1}{2c}, 0\right) \text{ and radius is } \frac{1}{2c}.$$

The corresponding regions are shown in Figs 3.14 (a) and (b)

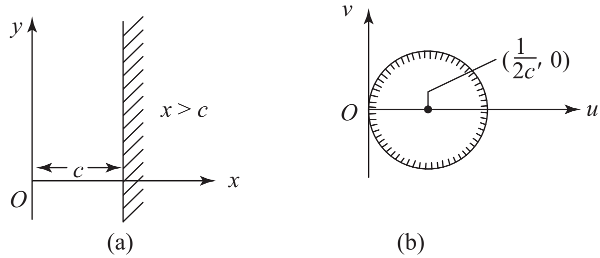


Fig. 3.14

The image of the region $y > c$ is given by

$$-\frac{v}{u^2+v^2} > c \quad \text{from (2)}$$

i.e. $\quad c\,(u^2+v^2) < -v \quad$ or $\quad u^2+v^2 > \dfrac{-v}{c} \quad [\because c < 0]$

i.e. $$u^2 + \left(v + \frac{1}{2c}\right)^2 > \left(\frac{1}{2c}\right)^2 \tag{4}$$

Equation (4) represents the exterior of the circle

$u^2 + \left(v + \dfrac{1}{2c}\right)^2 = \left(\dfrac{1}{2c}\right)^2$, whose centre is $\left(0, -\dfrac{1}{2c}\right)$ and radius is $\dfrac{1}{2|c|}$.

The corresponding regions are shown in the Figs 3.15 (a) and (b)

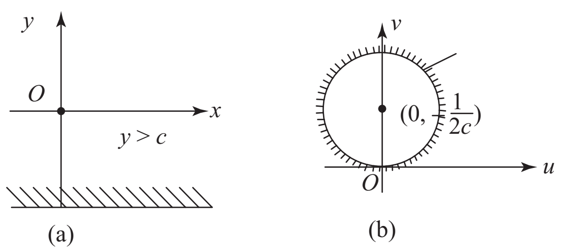


Fig. 3.15

(iii) The image of $y \geq \dfrac{1}{4}$ is given by

$$-\frac{v}{u^2+v^2} \geq \frac{1}{4}$$

i.e. $\quad u^2 + v^2 \leq -4v \quad$ or $\quad u^2 + (v+2)^2 \leq 2^2$

i.e. the interior of the circle $u^2 + (v+2)^2 = 2^2$.

The image of $y \leq 1/2$ is given by

$$-\frac{v}{u^2+v^2} \leq \frac{1}{2}$$

i.e. $\quad u^2 + v^2 \geq -2v \quad$ or $\quad u^2 + (v+1)^2 \geq 1$

i.e. the exterior of the circle $u^2 + (v+1)^2 = 1$.

The corresponding regions are shown in the Figs 3.16 (a) and (b)

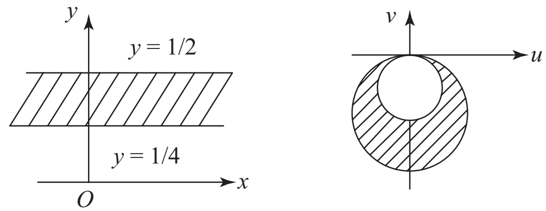


Fig. 3.16

Example 3.6

(i) Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \dfrac{1}{z}$ is the lemniscate $R^2 = \cos 2\phi$.

(ii) Show that the image of the circle $|z - 1| = 1$ under the transformation $w = z^2$ is the cardioid $R = 2(1 + \cos\phi)$.

(i) $$w = \frac{1}{z} \quad \text{or} \quad z = \frac{1}{w}$$

i.e. $$x + iy = \frac{1}{R e^{i\phi}} = \frac{1}{R}(\cos\phi - i\sin\phi)$$

$\therefore$ The transformation equations are

$$x = \frac{1}{R}\cos\phi \tag{1}$$

and $$y = -\frac{1}{R}\sin\phi \tag{2}$$

The given hyperbola is $$x^2 - y^2 = 1 \tag{3}$$

Using (1) and (2) in (3), we get the image of (3) in the w-plane in polar co-ordinates

as $\dfrac{1}{R^2}\cos^2\phi - \dfrac{1}{R^2}\sin^2\phi = 1$

i.e. $R^2 = \cos 2\phi$, which is a lemniscate.

(ii) The given circle is $|z - 1| = 1$

i.e. $|x - 1 + iy| = 1$

i.e. $(x - 1)^2 + y^2 = 1$ or $x^2 + y^2 - 2x = 0$

i.e. $$r^2 - 2r\cos\theta = 0 \quad \text{or} \quad r = 2\cos\theta \tag{4}$$

The transformation is $w = z^2$

i.e. $R e^{i\phi} = r^2 e^{i2\theta}$

$$\therefore \quad R = r^2 \tag{5}$$

and $$\phi = 2\theta \tag{6}$$

Eliminating r and θ from (4), (5) and (6), we get the polar equation of the image of (4).

From (4), $r^2 = 4\cos^2\theta$

$= 2(1 + \cos 2\theta)$

i.e. $R = 2(1 + \cos\phi)$, which is a cardioid.

Example 3.7 Find the image in the w-plane of the region of the z-plane bounded by the straight lines $x = 1$, $y = 1$ and $x + y = 1$ under the transformation $w = z^2$.

$$w = z^2 \quad \text{i.e. } u + iv = (x + iy)^2 = x^2 - y^2 + i2\,xy$$

$\therefore$ The transformation equations are

$$u = x^2 - y^2 \tag{1}$$

and
$$v = 2xy \tag{2}$$

From the discussion of the transformation $w = z^2$, we get the image of the line $x = 1$ as the parabola $v^2 = -4(u - 1)$ and the image of the line $y = 1$ as the parabola $v^2 = 4(u + 1)$.

The image of the line $x + y = 1$ (3)

is got by eliminating x and y from (1), (2) and (3).

Using (3) in (1) and (2), we have

$$u = x^2 - (1 - x)^2$$

and
$$v = 2x(1 - x)$$

i.e.
$$u = 2x - 1 \tag{4}$$

and
$$v = 2x(1 - x) \tag{5}$$

Eliminating x from (4) and (5), we get

$$v = (u + 1)\left\{1 - \frac{u+1}{2}\right\}$$

i.e. $\quad v = \dfrac{1 - u^2}{2} \quad$ or $\quad u^2 = -2(v - 1/2)$

which represents a parabola in the w-plane.

Thus the image of the region bounded by $x = 1$, $y = 1$ and $x + y = 1$ is the region bounded by the three parabolas $v^2 = -4(u - 1)$, $v^2 = 4(u + 1)$ and $u^2 = -2(v - 1/2)$ The corresponding regions in the two planes are shown in the Figs 3.17 (a) and (b).

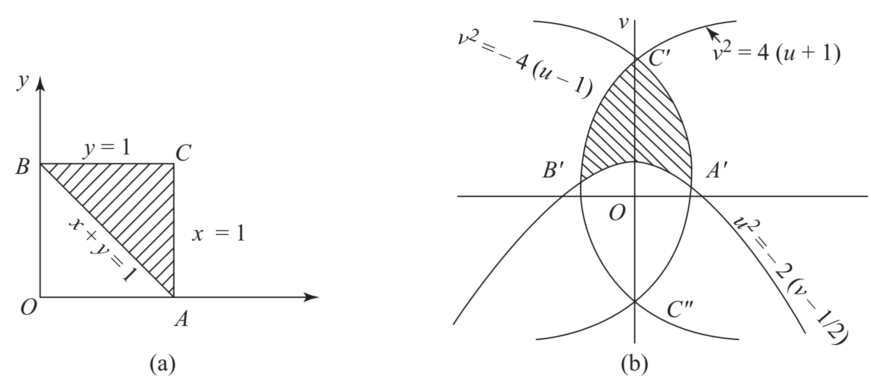


Fig. 3.17

Note ☑ The region $A'\,B'\,C''$ is also bounded by the three parabolas, but the corresponding region is that which contains the images of the points A (1, 0), B (0, 1) and C (1, 1), namely the points $A'(1, 0)$, $B'(-1, 0)$ and $C'(0, 2)$.

Example 3.8 Find the image of the rectangular region bounded by the lines (i) $x = 1$, $x = 3$, $y = 1$ and $y = 2$ and (ii) $u = 1$, $u = 3$, $v = 1$ and $v = 2$ under the transformation $w = z^2$.

(i) Proceeding as in the discussion of the transformation $w = z^2$, we find that the images of the lines $x = 1$, $x = 3$, $y = 1$ and $y = 2$ are respectively the parabolas $v^2 = -4(u-1)$, $v^2 = -36(u-9)$, $v^2 = 4(u+1)$ and $v^2 = 16(u+4)$.

The given rectangular region in the z–plane lies in the first quadrant, bounded by $\theta = 0$ and $\theta = \pi/2$ in polar form.

Hence the image region lies in the upper half of the w-plane, since the images of $\theta = 0$ and $\theta = \pi/2$ are respectively $\phi = 0$ and $\phi = \pi$, as one of the transformation equations of $w = z^2$ in the polar form is $\phi = 2\theta$. The corresponding regions are shown in the Figs 3.18 (a) and (b)

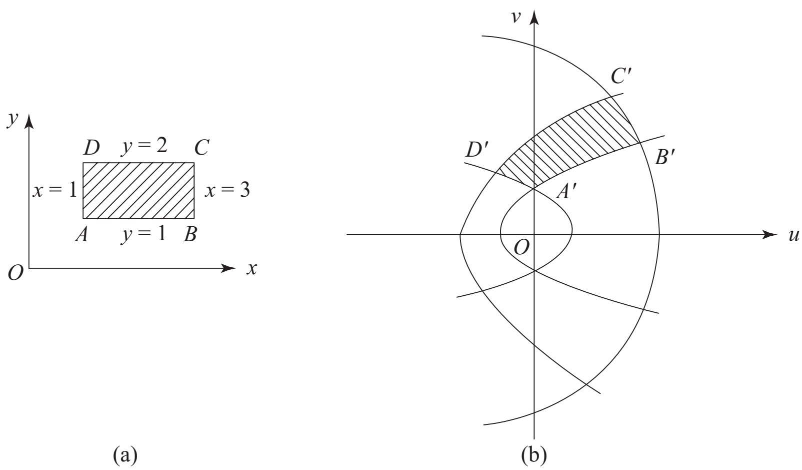


Fig. 3.18

(ii) The transformation equations corresponding to $w = z^2$ are $u = x^2 - y^2$ and $v = 2xy$.

$\therefore$ The images of $u = 1$, $u = 3$, $v = 1$ and $v = 2$ are respectively the rectangular hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 3$, $xy = 1/2$ and $xy = 1$ in the z-plane.

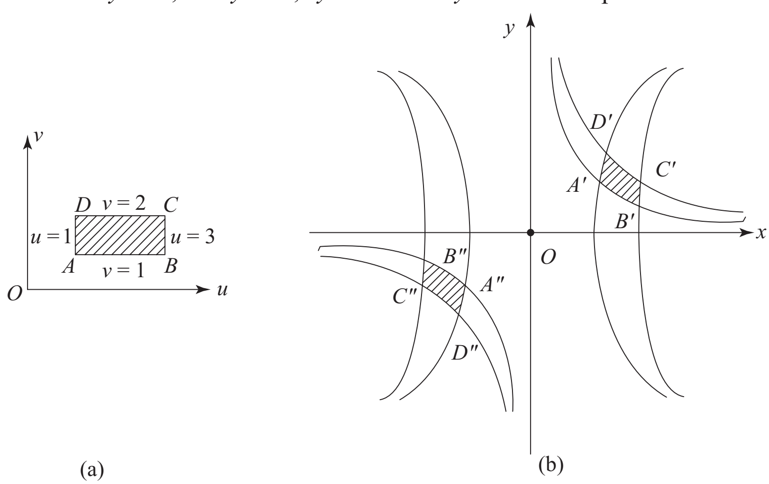


Fig. 3.19

The required image region is that bounded by these four hyperbolas. There are two regions bounded by these hyperbolas as shown in the Figs 3.19 (a) and (b).

Both the region in the z-plane are valid images of the given rectangular region in the w-plane.

This is because we are transforming from the w-plane to the z-plane by means of the transformation $z = w^{1/2}$, which is a two valued function.

Example 3.9 Find the images of the following under the transformation $w = e^z$:

(i) the line $y = x$, (ii) the segment of the y-axis, given by $0 \le y \le \pi$, (iii) the left half of the strip $0 \le y \le \pi$ and (iv) the right half of the strip $0 \le y \le \pi$.

The transformation equations of the mapping $w = e^z$ are given by

$$Re^{i\phi} = e^{x+iy} = e^x \,.\, e^{iy}$$

i.e. $$R = e^x \quad (1)$$

and $$\phi = y \quad (2)$$

(i) The image of the line $y = x$ is

$$\log R = \phi \quad \text{or} \quad R = e^{\phi},$$

which is the polar equation of an equiangular spiral.

(ii) The image of the y-axis i.e., $x = 0$ is $R = 1$ [from (1)] i.e., the unit circle $|w| = 1$. The image of the $y = 0$ is $\phi = 0$ and that of the line $y = \pi$ is $\phi = \pi$.

$\therefore$ The image of the region $0 \le y \le \pi$ is the region defined by $0 \le \phi \le \pi$, i.e. the upper half of the w-plane.

Hence the image of the segment of $x = 0$, between $y = 0$ and $y = \pi$ is the semicircle $|w| = 1$, $v \ge 0$, as shown in the Figs 3.20 (a) and (b).

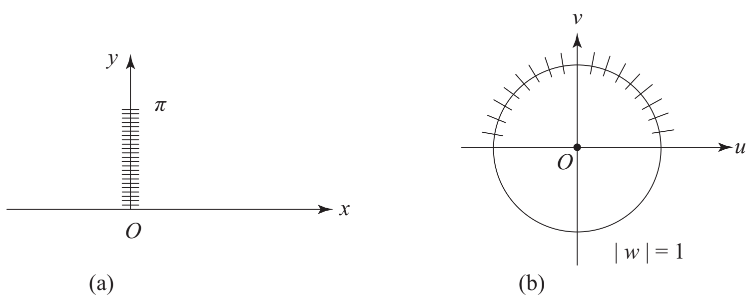


Fig. 3.20

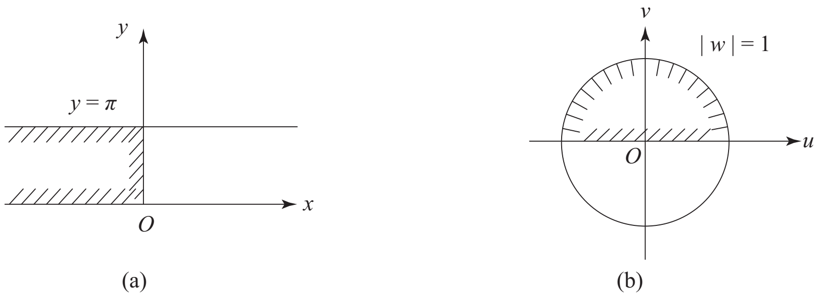


Fig. 3.21

(iii) The left half of the strip $0 \le y \le \pi$ is given by $0 \le y \le \pi$ and $x \le 0$.

$\therefore$ The image is given by $0 \le \phi \le \pi$ and $R \le 1$ or $|w| \le 1$, i.e. the interior of the unit circle $|w| = 1$ lying in the upper half of the w–plane.

The corresponding images are shown in the Figs 3.21 (a) and (b)

(iv) Similarly the right half of the strip $0 \le y \le \pi$ is mapped onto the exterior of the unit circle $|w| = 1$ lying in the upper half of the w-plane Figs 3.22 (a) and (b).

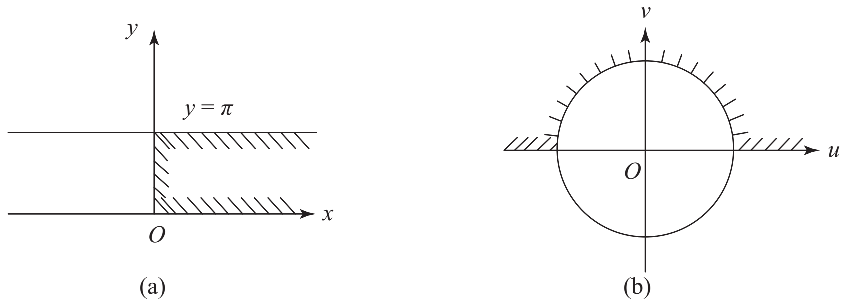


Fig. 3.22

Example 3.10 Find the maps of the boundary and interior of the rectangle formed by $x = \pm\dfrac{\pi}{2}$ and $y = \pm\, c$ in the z-plane under the transformation $w = \sin z$.

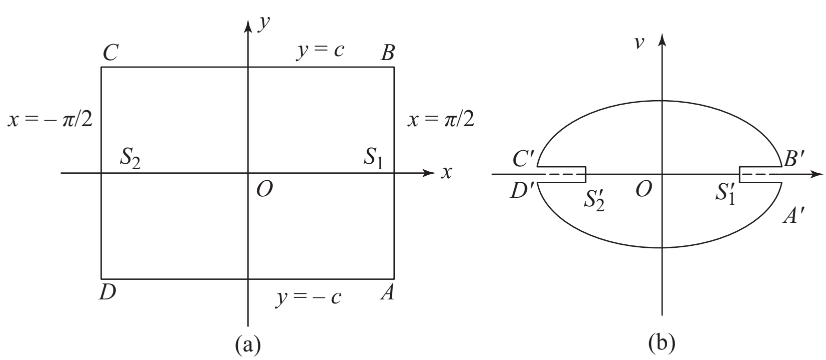


Fig. 3.23

The transformation equations of the mapping function $w = \sin z$ are

$$u = \sin x \cosh y \tag{1}$$

and
$$v = \cos x \sinh y \tag{2}$$

The part AS_1 of the side AB of the given rectangle is given by $x = \pi/2$ and $-\,c \le y \le 0$. Its image is given by

$$u = \cosh y \quad \text{and} \quad v = 0 \text{ [Fig. 3.23]}$$

Now u is a decreasing function of y in $-\,c \le y \le 0$, since $\dfrac{\mathrm{d}u}{\mathrm{d}y} = \sinh y < 0$, when $y < 0$. Hence the image of AS_1 is given by $v = 0$ and $\cosh c \ge u \ge 1$, i.e. by the segment $A'S'_1$ of the u-axis in the w-plane. Similarly the image of $S_1 B$ is given by

$v = 0$ and $1 \le u \le \cosh c$, i.e. by the segment S'_1B' (which is the same as $S'_1\,A'$) of the u-axis.

The image of the line segment BC, i.e. $y = c$, $\frac{\pi}{2} \ge x \ge -\pi/2$ is given by

$$u = \sin x \cosh c \text{ and}$$

$$v = \cos x \sinh c, -\pi/2 \le x \le \pi/2$$

i.e. the image of BC is the elliptic arc

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1, v > 0$$

[When $-\pi/2 \le x \le \pi/2$ and $c > 0$, $\cos x > 0$ and $\sinh c > 0$].

i.e. the image of BC is the upper half $B'\,C'$ of the ellipse

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1 \qquad (3)$$

Similarly the images of the line segments CS_2, S_2D and DA of the z-plane are the line segments $C'S'_2$, S'_2D' (which is the same as S'_2C') and the lower half $D'A'$ of the ellipse (3).

Thus the image of the boundary of the rectangle consists of the ellipse and the two segments of the u-axis.

Hence the image of the rectangular region is the interior of the ellipse in the w-plane.

Example 3.11 Show that the transformation $w = \cos z$ maps the segment of the x-axis given by $0 \le x \le \pi/2$ into the segment $0 \le u \le 1$ of the u-axis. Show also that it maps the strip $y \ge 0$, $0 \le x \le \pi/2$ into the fourth quadrant of the w-plane.

The transformation equation of $w = \cos z$ are given by $u + iv = \cos(x + iy)$

$$= \cos x \cosh y - i \sin x \sinh y$$

i.e.
$$u = \cos x \cosh y \qquad (1)$$

$$v = -\sin x \sinh y \qquad (2)$$

The image of the x-axis, i.e. $y = 0$ is given by

$$u = \cos x \quad \text{and} \quad v = 0$$

Since we consider the segment of the x–axis given by $0 \le x \le \pi/2$, $0 \le u \le 1$

Thus the image of the segment of $y = 0$, $0 \le x \le \pi/2$, is the segment of $v = 0$, $0 \le u \le 1$. The boundaries of the given strip are $x = 0$, $x = \pi/2$ and $y = 0$.

The image of $x = 0$ is given by $u = \cosh y$ and $v = 0$, i.e. $u \ge 1$ and $v = 0$.

The image of $x = \pi/2$ is given by $u = 0$ and $v = \sinh y$.

Since $y \ge 0$ for the given strip, $v \le 0$

Thus the image of $x = \pi/2$, $y \ge 0$ is $u = 0$, $v \le 0$

The image of $y = 0$, $0 \le x \le \pi/2$ is $v = 0$, $0 \le u \le 1$.

Thus the image of the given region is the region bounded by

$$v = 0, 0 \le u \le 1; \quad v = 0, u \ge 1 \text{ and } u = 0, v \le 0$$

i.e.
$$v = 0, u \ge 0 \quad \text{and} \quad u = 0, v \le 0$$

i.e. the fourth quadrant in the w-plane. The corresponding regions are shown in the Figs. 3.24 (a) and (b)

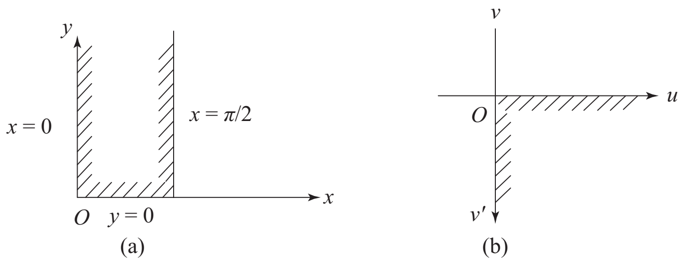


Fig. 3.24

Example 3.12 Find the image of the region defined by $0 \le x \le 3$, $-\pi/4 < y < \pi/4$ under the transformation $w = \sinh z$.

The transformation equations of $w = \sinh z$ are given by $u + iv = \sinh(x + iy)$

$$= -i \sin i(x + iy)$$
$$= -i \sin i(ix - y)$$
$$= \sinh x \cos y + i \cosh x \sin y$$

i.e. $$u = \sinh x \cos y \tag{1}$$

and $$v = \cosh x \sin y \tag{2}$$

The image of $x = 0$ is given by $u = 0$ and

$v = \sin y$ i.e. $u = 0$ and

$-1/\sqrt{2} \le v \le 1/\sqrt{2}$ $\qquad (\because -\pi/4 \le y \le \pi/4)$

The image of $x = 3$ is given by

$$u = \sinh 3 \cos y \text{ and } v = \cosh 3 \cdot \sin y$$

i.e. the ellipse $\dfrac{u^2}{\sinh^2 3} + \dfrac{v^2}{\cosh^2 3} = 1$, $u > 0$, since $\cos y \ge 0$.

The image of $y = -\pi/4$ is given by

$$u = \frac{1}{\sqrt{2}} \sinh x \quad \text{and} \quad v = -\frac{1}{\sqrt{2}} \cosh x$$

i.e. the lower part of rectangular hyperbola $v^2 - u^2 = \dfrac{1}{2}$ $(\because v < 0)$

Similarly the image of $y = \pi/4$ is the upper part of the rectangular hyperbola $v^2 - u^2 = 1/2$. Thus the image of the given region is the region in the right part of the v-axis, bounded by the segment of the v-axis, the ellipse and the rectangular hyperbola as shown in the Figs 3.25 (a) and (b).

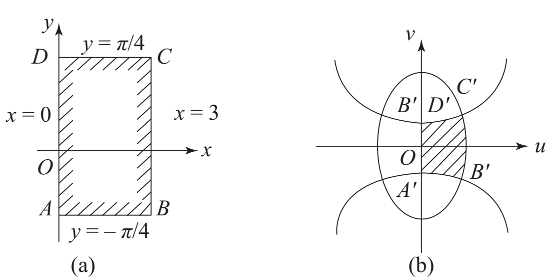


Fig. 3.25

Example 3.13 Show that the transformation $w = z + \dfrac{a^2 - b^2}{4z}$ transforms the circle $|z| = \dfrac{1}{2}(a+b)$ in the z-plane into an ellipse of semi-axes a, b in the w-plane.

In the discussion of the transformation $w = z + \dfrac{k^2}{z}$, we have proved that the image of the circle $|z| = c$, i.e., $r = c$ is the ellipse whose equation is

$$\frac{u^2}{\left(c + \dfrac{k^2}{c}\right)^2} + \frac{v^2}{\left(c - \dfrac{k^2}{c}\right)^2} = 1 \qquad (1)$$

Putting $\qquad k^2 = \dfrac{a^2 - b^2}{4}$ and $c = \dfrac{1}{2}(a+b)$ in (1), we get

$$c + \frac{k^2}{c} = \frac{1}{2}(a+b) + \frac{1}{2}(a-b) = a \text{ and}$$

$$c - \frac{k^2}{c} = \frac{1}{2}(a+b) - \frac{1}{2}(a-b) = b$$

$\therefore$ The image of $|z| = \dfrac{1}{2}(a+b)$, i.e., $r = \dfrac{1}{2}(a+b)$ under the transformation $w = z + \left(\dfrac{a^2 - b^2}{4}\right)/z$ is the ellipse $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$, whose semi-axes are a and b.

Example 3.14 Prove that the region outside the circle $|z| = 1$ maps onto the whole of the w-plane under the transformation $w = z + \dfrac{1}{z}$.

Proceeding as in the discussion of the transformation $w = z + \dfrac{k^2}{z}$, we can prove that the image of the circle $|z| = 1$, i.e. $r = 1$ is the segment of the real axis, given by $-2 \le u \le 2$ and that the image of the circle $|z| = a$, i.e. $r = a$ is the ellipse

$$\frac{u^2}{\left(a + \dfrac{1}{a}\right)^2} + \frac{v^2}{\left(a - \dfrac{1}{a}\right)^2} = 1 \qquad (1)$$

The semi axes of this ellipse are $f(a) = a + 1/a$ and $g(a) = a - \dfrac{1}{a}$.

Now $f'(a) = 1 - 1/a^2 = \dfrac{a^2 - 1}{a^2} > 0$, when $a > 1$ and $g'(a) = 1 + 1/a^2 > 0$, for all a and hence when $a > 1$.

$\therefore$ $f(\text{a})$ and $g(a)$ are increasing functions of a, when $a > 1$.

Hence, when a increases, the semi axes of the ellipse (1) increase.

The region outside the circle $r = 1$ may be regarded as the area swept by the circle $r = a$, where $1 < a < \infty$.

Similarly the area swept by the ellipse $\dfrac{u^2}{\left(a+\dfrac{1}{a}\right)^2}+\dfrac{v^2}{\left(a-\dfrac{1}{a}\right)^2}=1, 1<a<\infty$, is the entire w-plane.

Hence the region outside the circle $r = 1$ maps onto the entire w-plane.

Example 3.15 Show that the transformation $w=\dfrac{1}{2}\left(ze^{-\alpha}+\dfrac{e^{\alpha}}{z}\right)$, where α is real, maps the upper half of the interior of the circle $|z| = e^{\alpha}$ onto the lower half of the w-plane.

The transformation equations of the mapping function $w=\dfrac{1}{2}\left(ze^{-\alpha}+\dfrac{1}{ze^{-\alpha}}\right)$ are given by

$$u+iv=\frac{1}{2}\left(re^{i\theta}\,e^{-\alpha}+\frac{1}{re^{-\alpha}}\,e^{-i\theta}\right)$$

i.e. $$u=\frac{1}{2}\left(r\,e^{-\alpha}+\frac{1}{re^{-\alpha}}\right)\cos\theta \tag{1}$$

and $$v=\frac{1}{2}\left(r\,e^{-\alpha}+\frac{1}{re^{-\alpha}}\right)\sin\theta \tag{2}$$

The boundary of interior of the circle $|z| = e^{\alpha}$ or $r = e^{\alpha}$ lying in the upper half of the z-plane consists of $\theta = 0$, $\theta = \pi$ and $r = e^{\alpha}$.

The image of $\theta = 0$ is given by

$$u=\frac{1}{2}\left(r\,e^{-\alpha}+\frac{1}{re^{-\alpha}}\right)\text{ and } v=0\text{, from (1) and (2).}$$

i.e. $$u=\frac{1}{2}\left[\left(\sqrt{r}\;e^{-\alpha/2}-\frac{1}{\sqrt{r}\;e^{-\alpha/2}}\right)^2+2\right]\text{ and } v=0$$

i.e. $u\geq\dfrac{1}{2}(0+2)$ and $v = 0$ or $u \geq 1$ and $v = 0$.

Similarly the image of $\theta = \pi$ is given by $u \leq -1$ and $v = 0$.

The image of $r = e^{\alpha}$ is given by

$u = \cos\theta$ and $v = 0$, from (1) and (2).

i.e., $-1 \leq u \leq 1$ and $v = 0$.

Thus the boundary of the semi-circle $r = e^{\alpha}$ lying in the upper half of the z-plane is the entire u-axis.

The interior of the semi-circle is given by $r < e^{\alpha}$ and $0 < \theta < \pi$.

When $r < e^{\alpha}$, $re^{-\alpha} < 1$

$$\therefore \qquad -\frac{1}{re^{-\alpha}}<-1$$

$$\therefore \qquad \left(re^{-\alpha} - \frac{1}{re^{-\alpha}}\right) < 0$$

$$\therefore \qquad \frac{1}{2}\left(re^{-\alpha} - \frac{1}{re^{-\alpha}}\right)\sin\theta < 0, \text{ since } \sin\theta \text{ is positive when } 0 < \theta < \pi.$$

i.e. $\quad v < 0.$

Hence the interior of the semi-circle $|z| = e^{\alpha}$ lying in the upper half of the z-plane maps onto the lower half of the w-plane. The corresponding images are shown in the Figs 3.26 (a) and (b).

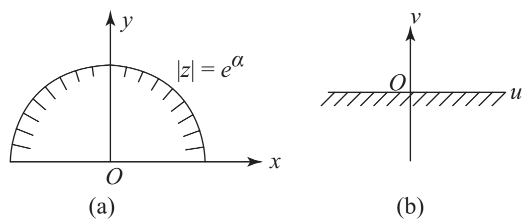


Fig. 3.26

EXERCISE 3(c)

Part A

(Short Answer Questions)

1. What do you mean by conformal mapping?
2. When is a transformation said to be isogonal? Prove that the mapping $w = \bar{z}$ is isogonal.
3. State the conditions for the transformation $w = f(z)$ to be conformal at a point.
4. Define critical point of a transformation.
5. Find the critical points of the transformation
$$w = \frac{1}{2}\left(ze^{-\alpha} + \frac{e^{\alpha}}{z}\right).$$
6. Find the critical points of the transformation $w^2 = (z - \alpha)(z - \beta)$.
7. Find the magnification factor of small lengths near $z = \pi/4$ under the transformation $w = \sin z$.

Find the image of the circle $|z| = a$ under the following transformations

8. $w = z + 2 + 3i$ 9. $w = 2z$ 10. $w = (3 + 4i)z$
11. $w = (1 + i)z + 2 - i$ 12. $w = \dfrac{i}{2z}$
13. Find the image of the infinite strip $0 \le x \le 2$ under the transformation $w = iz$.
14. Find the image of the region that lies on the right of the y-axis under the transformation $w = iz + i$.

15. Find the image of the region $y > 1$ under the transformation $w = (1 - i)z$.
16. Find the image of the hyperbola $xy = 1$ under the transformation $w = 3z$.
17. Find the image of the hyperbola $x^2 - y^2 = a^2$ under the transformation $w = (1 + i)z$.
18. Find the image of the half-plane $y > c$, when $c > 0$ under the transformation $w = \frac{1}{z}$.
19. Find the image of the half-plane $x > c$, when $c < 0$ under the transformation $w = 1/z$.
20. Find the image of the line $y = mx$ under the transformation $w = \frac{i}{z}$.
21. Find the image of the region $0 < \theta < \pi/n$ in the z-plane under the transformation $w = z^n$, where n is a positive integer.
22. Find the images of the lines $x = 1$ and $y = 1$ under the transformation $w = iz^2$.
23. Find the image of the region $2 < |z| < 3$ under the transformation $w = z^2$.
24. Find the image of the region $\frac{\pi}{4} < \arg z < \frac{\pi}{2}$ under the transformation $w = z^2$.
25. Find the level curves of u and v under the transformation $w = -iz^2$.
26. Find the image of the infinite strip $0 \le x \le 1$ under the transformation $w = e^z$.
27. Find the image of the x-axis under the transformation $w = \cos z$.
28. Find the image of the y-axis under the transformation $w = \cos z$.
29. Find the image of the x-axis under the transformation $w = \sinh z$.
30. Find the image of the y-axis under the transformation $w = \sinh z$.
31. Find the equations of transformation for the mapping given by $w = \frac{a}{2}\left(z + \frac{1}{z}\right)$.
32. Find the images of the points whose polar co-ordinates are $(1, 0)$ and $(1, \pi)$ under the transformation $w = z + \frac{1}{z}$.

Part B

33. Find the image of the triangle with vertices at $z = i$, $z = 1 - i$, $z = 1 + i$ under the transformation (i) $w = 3z + 4 - 2i$ and (ii) $w = iz + 2 - i$.
34. Find the map of the square whose vertices are $z = 1 + i, -1 + i, -1 - i$ and $1 - i$ by the transformation $w = az + b$, where $a = \sqrt{2}\,(1 + i)$ and $b = 3 + 3i$.
35. Prove that the interior of the circle $|z| = a$ maps onto the interior of an ellipse in the w-plane under the transformation $w = x + \frac{iby}{a}$, $0 < b < a$. Is the transformation conformal?
36. Show that the transformation $w = \frac{1}{z}$ maps the circle $|z - 3| = 5$ onto the circle $\left|w + \frac{3}{16}\right| = \frac{5}{16}$. What is the image of the interior of the given circle in the z- plane?

37. Find the image of (*i*) the infinite strip $0 < y < \frac{1}{2c}$ and (ii) the quadrant $x > 0$, $y > 1$, under the transformation $w = \frac{1}{z}$.
38. Find the image of the triangle formed by the lines $y = x$, $y = -x$ and $y = 1$ under the transformation $w = z^2$.
39. Find the image of the region of the square whose vertices are $z = 0, 1, 1 + i$ and i under the transformation $w = z^2$.
40. When $a > 0$, show that $w = \exp(\pi z/a)$ transforms the infinite strip $0 \le y \le a$ onto the upper half of the w-plane.
41. Discuss the transformation $w = \log z$. Also prove that the whole of the z-plane maps onto the horizontal strip $-\pi \le v \le \pi$. (The principal value of $\log z$ is to be considered.)

    Note ☑ The properties of the transformation $w = \log z$ are identical with those of $w = e^z$, if we interchange z and w.
42. Discuss the transformation $w = \cos z$.
43. Discuss the transformation $w = \sinh z$.
44. Show that the transformation $w = \sin \frac{\pi z}{a}$ maps the region of the z-plane given by $y \ge 0$ and $-a/2 \le x \le a/2$ onto the upper half of the w-plane.
45. Find the image of the semi-infinite strip $0 \le x \le \pi, y \ge 0$ under the transformation $w = \cos z$.
46. Find the image of the region defined by $-\pi/4 \le x \le \pi/4$, $0 \le y \le 3$ under the transformation $w = \sin z$.
47. Show that the transformation $w = \cosh z$ maps (*i*) the segment of the y-axis from 0 to $\frac{i\pi}{2}$ onto the segment $0 \le u \le 1$ of the u-axis, (ii) the semi-infinite strip $x > 0$, $0 \le y \le \pi/2$ onto the first quadrant of the w-plane.
48. Find the image of the region in the z-plane given by $1 \le x \le 2$ and $-\pi/2 \le y \le \pi/2$ under the transformation $w = \sinh z$.
49. Show that the transformation $w = \frac{a}{2}\left(z + \frac{1}{z}\right)$ where a is a positive constant maps (i) the semi-circle $|z| = 1$ in the upper z-plane onto the segment $-a \le u \le a$ of the u-axis, (ii) the exterior of the unit circle in the upper z-plane onto the upper w-plane.
50. Show that the transformation

    $w = \frac{1}{2}\left(ze^{-\alpha} + \frac{e^{\alpha}}{z}\right)$, where α is real, maps the interior of the circle $|z| = 1$ onto the exterior of an ellipse whose major and minor axes are of lengths 2 cosh α and 2 sinh α respectively.

[*Hint*: The image of the circle $|z| = 1$ is the ellipse $\dfrac{u^2}{\cosh^2\alpha}+\dfrac{v^2}{\sinh^2\alpha}=1$. The image of any point inside $|z| = 1$, for example, the point (a, θ), where $a < 1$ is a point outside the ellipse, since $|u\ (a, \theta)| > |u\ (1, \theta)|$ and $|v(a, \theta)| > |v(1, \theta)|$].

3.7 BILINEAR AND SCHWARZ–CHRISTOFFEL TRANSFORMATIONS

3.7.1 Bilinear Transformation

The transformation $w=\dfrac{az+b}{cz+d}$, where a, b, c, d are complex constants such that $ad - bc \neq 0$ is called *a bilinear transformation.* It is also called *Mobius* or *linear fractional transformation.*

Now $\dfrac{\mathrm{d}w}{\mathrm{d}z}=\dfrac{ad-bc}{(cz+d)^2}$. If $ad - bc = 0$, every point of the z-plane becomes a critical point of the bilinear transformation.

The transformation can be rewritten as

$$w=\frac{a}{c}-\frac{(ad-bc)}{c(cz+d)}. \text{ Hence, if } ad - bc = 0,$$

the transformation takes the form $w=\dfrac{a}{c}$, which has no meaning as a mapping function. Due to these reasons, we assume that $ad - bc \neq 0$. The expression $(ad - bc)$ is called the *determinant* of the bilinear transformation.

The inverse of the transformation $w=\dfrac{az+b}{cz+d}$ is $z=\dfrac{-dw+b}{cw-a}$, which is also a bilinear transformation.

The images of all points in the z-plane are uniquely found, except for the point $z=-\dfrac{d}{c}$. Similarly the image of every point in the w-plane is a unique point, except for the point $w=\dfrac{a}{c}$. If we assume that the images of the points $z=-\dfrac{d}{c}$ and $w=\dfrac{a}{c}$ are the points at infinity in the w- and z-planes respectively, the bilinear transformation becomes one-to-one between all the points in the two planes.

To discuss the transformation $w=\dfrac{az+b}{cz+d}$, (1)

we express it as the combination of simple transformations discussed in the previous section.

When $c \neq 0$, (1) can be expressed as

$$w = \frac{a}{c} + \left(\frac{bc-ad}{c}\right) \cdot \frac{1}{cz+d}$$

If we make the substitutions

$$w_1 = cz + d \tag{2}$$

$$w_2 = \frac{1}{w_1} \tag{3}$$

then $$w = Aw_2 + B \tag{4}$$

where $$A = \frac{bc-ad}{c} \quad \text{and} \quad B = \frac{a}{c}.$$

The substitutions (2), (3) and (4) can be regarded as transformations from the z-plane onto the w_1-plane, from the w_1-plane onto the w_2-plane and from the w_2-plane onto the w-plane respectively.

We know that each of the transformations (2), (3) and (4) maps circles and straight lines into circles and straight lines (since straight lines may be regarded as circles of infinite radii).

Hence the bilinear transformation (1) maps circles and straight lines onto circles and straight lines, in general. When $c = 0$, (1) becomes $w = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)(d \neq 0)$ i.e. (1) reduces to the form $w = Az + B$. This transformation also maps circles into circles.

Thus the bilinear transformation always maps circles into circles with lines as limiting cases.

3.7.2 Definition

If the image of a point z under a transformation $w = f(z)$ is itself, then the point is called *a fixed point* or *an invariant point* of the transformation.

Thus a fixed point of the transformation $w = f(z)$ is given by $z = f(z)$.

The fixed points of the bilinear transformation $w = \frac{az+b}{cz+d}$ are given by $\frac{az+b}{cz+d} = z$.

As this is a quadratic equation in z, we will get two fixed points for the bilinear transformation.

Note ☑

1. A bilinear transformation can be uniquely found, if the images w_1, w_2, w_3 of any three points z_1, z_2, z_3 of the z-plane are given.

   Let the bilinear transformation required be

$$w = \frac{az+b}{cz+d} \tag{1}$$

   (1) can be re-written as $w = \dfrac{\left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)}{\left(\frac{c}{d}\right)z + 1}$ or $\dfrac{Az+B}{Cz+1}$

Since the images of z_1, z_2 and z_3 are w_1, w_2 and w_3 respectively, we have

$$w_1 = \frac{Az_1 + B}{Cz_1 + 1} \tag{2}$$

$$w_2 = \frac{Az_2 + B}{Cz_2 + 1} \tag{3}$$

and
$$w_3 = \frac{Az_3 + B}{Cz_3 + 1} \tag{4}$$

Equations (2), (3) and (4) are three equations in three unknowns A, B, C. Solving them we get the values of A, B, C uniquely and hence the bilinear transformation (1) uniquely.

2. If a set of three points and their images by a bilinear transformation are given, it can be found out by using the cross-ratio property of the bilinear transformation, which is given below.

3.7.3 Definition

If z_1, z_2, z_3, z_4 are four points in the z-plane, then $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ is called the *cross-ratio of these points.*

Cross-ratio property of a bilinear transformation The cross-ratio of four points is invariant under a bilinear transformation.

i.e. if w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively under a bilinear transformation, then

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

Proof

Let the bilinear transformation be $w = \dfrac{az + b}{cz + d}$

Then
$$w_i - w_j = \frac{az_i + b}{cz_i + d} - \frac{az_j + b}{cz_j + d} = \frac{(ad - bc)(z_i - z_j)}{(cz_i + d)(cz_j + d)}$$

$\therefore$
$$(w_1 - w_2)(w_3 - w_4) = \frac{(ad - bc)^2 (z_1 - z_2)(z_3 - z_4)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}$$

and
$$(w_1 - w_4)(w_3 - w_2) = \frac{(ad - bc)^2 (z_1 - z_4)(z_3 - z_2)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}$$

$\therefore$
$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Note ☑ To get the bilinear transformation that maps z_2, z_3, z_4 of the z-plane onto w_2, w_3, w_4, we assume that the image of z under this transformation is w and make use of the invariance of the cross-ratio of the four points z, z_2, z_3, z_4. Thus

$$\frac{(w-w_2)(w_3-w_4)}{(w-w_4)(w_3-w_2)}=\frac{(z-z_2)(z_3-z_4)}{(z-z_4)(z_3-z_2)} \tag{1}$$

(1) ensures that the images of $z = z_2$, z_3, z_4 are respectively $w = w_2$, w_3, w_4. Now, simplifying (1) and solving for w, we get the required bilinear transformation in the form $w=\dfrac{az+b}{cz+d}$.

3.8 SCHWARZ-CHRISTOFFEL TRANSFORMATION

3.8.1 Definition

The transformation that maps the boundary of a given polygon in the w-plane onto the x-axis (and hence maps the vertices of the polygon onto points on the x-axis) and the interior of the polygon onto the upper half of the z-plane is called *Schwarz-Christoffel transformation.*

Specifically, if $x_1, x_2, \ldots, x_n$ that are points on the x-axis such that $x_1 < x_2 < x_3 < \ldots < x_n$, are the images of the vertices $w_1, w_2, \ldots, w_n$ of a polygon in the w-plane and $\alpha_1, \alpha_2, \ldots \alpha_n$ are the corresponding interior angles of the polygon, then the required Schwarz-Christoffel transformation is given by

$$\frac{dw}{dz}=A(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_n)^{\frac{\alpha_n}{\pi}-1},$$

where A is an arbitrary complex constant.

Proof

[The proof is in the nature of verification of the mapping of the transformation.]

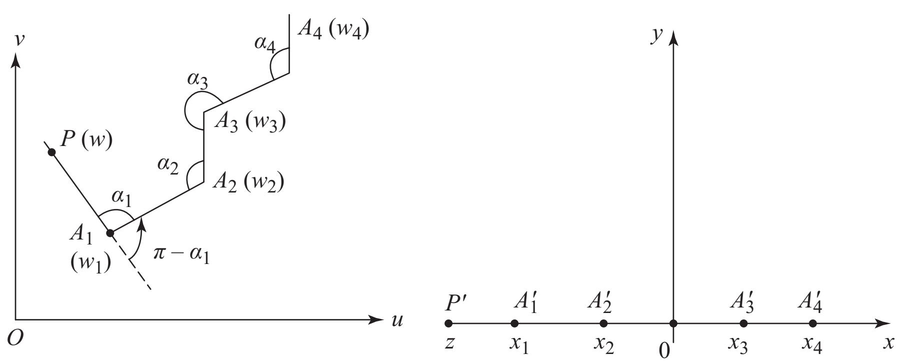


Fig. 3.27

The given transformation is

$$dw=A(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_n)^{\frac{\alpha_n}{\pi}-1}\cdot dz$$

$$\therefore\ \text{amp}\,(\text{d}w) = \text{amp}\,(A) + \left(\frac{\alpha_1}{\pi} - 1\right)\text{amp}\,(z - x_1) + \left(\frac{\alpha_2}{\pi} - 1\right)\text{amp}\,(z - x_2) + \cdots +$$

$$\left(\frac{\alpha_n}{\pi} - 1\right)\text{amp}\,(z - x_n) \tag{1}$$

[$\because$ amp $(z_1 z_2 \ldots)$ = amp (z_1) + amp (z_2) + ... and amp $(z^k) = k$ amp (z)]

Let the image of $P(w)$ be $P'(z)$ as shown in Fig. 3.27.

When P moves towards A_1, P' moves towards A'_1. As long as $P'(z)$ is on the left of A'_1 (x_1), $z - x_1, z - x_2, \ldots, z - x_n$ are all negative real numbers.

$\therefore$ amp $(z - x_1)$ = amp $(z - x_2) = \ldots =$ amp $(z - x_n) = \pi$

But once $P'(z)$ has crossed $A'_1(x_1)$, i.e., when $x_1 < z < x_2$, $z - x_1$ is a positive real number and hence amp $(z - x_1) = 0$, while amp $(z - x_2)$ = amp $(z - x_3) = \ldots =$ amp $(z - x_n) = \pi$

Also amp(A) and amp(dz) do not change.

[$\because$ A is a constant and dz is positive hence amp (dz) = 0]

Thus when z croses A'_1, amp $(z - x_1)$ suddenly changes from π to 0 or undergoes an increment of $-\pi$.

$\therefore$ From (1), we get

$$\text{Increase in amp } (\text{d}w) = \left(\frac{\alpha_1}{\pi} - 1\right)(-\pi) = \pi - \alpha_1 .$$

This increase is in the anticlockwise direction. This means that when $P(w)$, moving along PA_1, reaches A_1, it changes its direction through an angle $\pi - \alpha_1$ in the anticlockwise sense and then starts moving along A_1A_2. Similarly when $P(w)$ reaches A_2, it turns through an angle $\pi - \alpha_2$ and then starts moving along $A_2 A_3$.

Proceeding further, we find that as $P(w)$ moves along the boundary of the polygon, $P'(z)$ moves along the x-axis and conversely. Now when a person walks along the boundary of the polygon in the w-plane in the anticlockwise sense, the interior of the polygon lies to the left of the person. Hence the corresponding area in the z-plane should lie to the left of the person, when he or she walks along the corresponding path in the z-plane, i.e. along the x-axis from left to right. Clearly the corresponding area is the upper half of the z-plane.

Thus the interior of the polygon in the w-plane is mapped onto the upper half of the z-plane.

Note ☑

1. Integrating (1), the Schwarz-Christoffel transformation can also be expressed as $w = A\int (z - x_1)^{\frac{\alpha_1}{\pi} - 1} \cdot (z - x_1)^{\frac{\alpha_2}{\pi} - 1} \cdots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} \cdot \text{d}z + B$

   where B is a complex constant of integration.
2. The transformation which maps a polygon in the z-plane onto the real axis of the w-plane is got by interchanging z and w in the above transformation.
3. It is known that not more than three of the points $x_1, x_2, \ldots, x_n$ can be chosen arbitrarily.

4. It is advantageous to choose one point, say x_n, at infinity, as explained below.

   If we take $A=\dfrac{\lambda}{(-x_n)^{\frac{\alpha_n}{\pi}-1}}$, where λ is a constant, the transformation can be written as

   $$\frac{dw}{dz}=\lambda(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots(z-x_{n-1})^{\frac{\alpha_{n-1}}{\pi}-1}\cdot\left(\frac{x_n-z}{x_n}\right)^{\frac{\alpha_n}{\pi}-1}$$

   As $x_n\to\infty$, the transformation reduces to

   $$\frac{dw}{dz}=\lambda(z-x_1)^{\frac{\alpha_1}{\pi}-1}\cdot(z-x_2)^{\frac{\alpha_2}{\pi}-1}\cdots\cdot(z-x_{n-1})^{\frac{\alpha_{n-1}}{\pi}-1}$$

   This means that, if x_n is at infinity, the factor $(z-x_n)^{\frac{\alpha_n}{\pi}-1}$ is absent in the transformation. Thus the R.H.S. of the transformation contains one factor less than the original form.
5. Infinite open polygons can be considered as limiting cases of closed polygons.

WORKED EXAMPLE 3(d)

Example 3.1 Find the invariant points of the transformation $w=-\dfrac{2z+4i}{iz+1}$. Prove also that these two points together with any point z and its image w, form a set of four points having a constant cross ratio.

The invariant points of the transformation are given by

$$z=-\frac{2z+4i}{iz+1}$$

i.e. $iz^2+3z+4i=0$ or $z^2-3iz+4=0$

i.e. $(z-4i)(z+i)=0$

$\therefore$ The invariant points are $4i$ and $-i$. Taking $z_1=z, z_2=w=-\dfrac{2z+4i}{iz+1}$, $z_3=4i$ and $z_4=-i$, the cross-ratio of the four points z_1, z_2, z_3 and z_4 is given by

$$(z_1,z_2,z_3,z_4)=\frac{\left(z+\dfrac{2z+4i}{iz+1}\right)(4i+i)}{(z+i)\left(4i+\dfrac{2z+4i}{iz+1}\right)}$$

$$=\frac{5i(iz^2+3z+4i)}{(z+i)(-2z+8i)}$$

$$= \frac{5i\left(iz^2 + 3z + 4i\right)}{2i\left(iz^2 + 3z + 4i\right)}$$

$= 5/2$

$=$ a constant, independent of z.

Example 3.2 Find the bilinear transformation that maps the points $1 + i, -i, 2 - i$ of the z-plane into the points $0, 1, i$ of the w-plane.

Taking $z_1 = 1 + i$, $z_2 = -i$, $z_3 = 2 - i$ and $w_1 = 0$, $w_2 = 1$, $w_3 = i$ and using the invariance of the cross-ratio (z, z_1, z_2, z_3), we have

$$\frac{(w-0)(1-i)}{(w-i)(1-0)} = \frac{(z-1-i)(-2)}{(z-2+i)(-1-2i)}$$

i.e.
$$\frac{w-i}{w} = \frac{(z-2+i)(-1-2i)(1-i)}{(z-1-i)(-2)}$$

i.e.
$$1 - \frac{i}{w} = \frac{(3+i)(z-2+i)}{2(z-1-i)}$$

$\therefore$
$$\frac{i}{w} = 1 - \frac{(3+i)z - 7 + i}{2z - 2 - 2i}$$
$$= \frac{-(1+i)z + 5 - 3i}{2z - 2 - 2i}$$

$\therefore$
$$w = \frac{(2z - 2 - 2i)}{-i\{-(1+i)z + 5 - 3i\}}$$

i.e. the required bilinear transformation is

$$w = \frac{2z - 2 - 2i}{(i-1)z - 3 - 5i}$$

Example 3.3 Find the bilinear transformation which maps the points (i) $i, -1, 1$ of the z-plane into the points $0, 1, \infty$ of the w-plane respectively (ii) $z = 0$, $z = 1$ and $z = \infty$ into the points $w = i$, $w = 1$ and $w = -i$.

(i) $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$

i.e.,
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \tag{1}$$

To avoid the substitution of $w_3 = \infty$ in (1) directly, we put $w_3 = \dfrac{1}{w_3'}$ simplify and then put $w'_3 = 0$. Thus (1) becomes

$$\frac{(w-w_1)(w_2\,w_3'-1)}{(w\,w_3'-1)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \tag{2}$$

Using the given values $z_1 = i$, $z_2 = -1$, $z_3 = 1$, $w_1 = 0$, $w_2 = 1$ and $w'_3 = 0$, we get

$$\frac{w(-1)}{(-1)1}=\frac{(z-i)(-2)}{(z-1)(-1-i)}, \text{ i.e., } w=\frac{(1-i)(z-i)}{z-1}$$

(ii)
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$=\frac{(z-z_1)(z_2\,z_3'-1)}{(z z_3'-1)(z_2-z_1)}, \text{ where } z_3'=\frac{1}{z_3}$$

Using the values $z_1 = 0$, $z_2 = 1$, $z'_3 = 0$ and $w_1 = i$, $w_2 = 1$ and $w_3 = -i$, we get

$$\frac{(w-i)(1+i)}{(w+i)(1-i)}=\frac{z(-1)}{(-1)\cdot 1}$$

i.e.
$$\frac{w-i}{w+i}=\frac{(1-i)z}{1+i}$$

i.e.
$$\frac{2w}{2i}=\frac{(1-i)z+(1+i)}{(1+i)-(1-i)z}\left(=\frac{Nr+Dr}{Dr-Nr}\right)$$

$\therefore$
$$w=\frac{(1+i)z+(i-1)}{(1+i)-(1-i)z} \quad \text{or} \quad w=\frac{z+i}{iz+1}.$$

Example 3.4 If a, b are the two fixed points of a bilinear transformation, show that it can be written in the form (i) $\dfrac{w-a}{w-b}=k\left(\dfrac{z-a}{z-b}\right)$, where k is a constant, if $a \neq b$;

(ii) $\dfrac{1}{w-a}=\dfrac{1}{z-a}+c$, where c is a constant, if $a = b$.

(i) Since the images of a and b are a and b respectively, $(w, a, w_3, b) = (z, a, z_3, b)$

i.e.
$$\frac{(w-a)(w_3-b)}{(w-b)(w_3-a)}=\frac{(z-a)(z_3-b)}{(z-b)(z_3-a)}$$

i.e.
$$\frac{w-a}{w-b}=\left[\frac{(z_3-b)(w_3-a)}{(z_3-a)(w_3-b)}\right]\left(\frac{z-a}{z-b}\right)$$

i.e. $$\frac{w-a}{w-b} = k\left(\frac{z-a}{z-b}\right), \text{ where } k \text{ is a constant.}$$

(ii) Since the image of $z = a$ is $w = a$, the bilinear transformation can assumed as

$$w - a = \frac{z-a}{cz+d} \tag{1}$$

The fixed points of (1) are given by

$(z - a)\ [(cz + \mathrm{d}) - 1] = 0$

i.e. $$c\,(z-a)\left[z + \frac{d-1}{c}\right] = 0 \tag{2}$$

Since both the roots of (2) are equal to a,

$$\frac{d-1}{c} = -a \quad \therefore \quad d = -ca + 1 \tag{3}$$

Using (3) in (1), the required bilinear transformation is $w - a = \dfrac{z-a}{cz - ca + 1}$

i.e. $$\frac{1}{w-a} = \frac{c\,(z-a)+1}{z-a}$$

i.e. $$\frac{1}{w-a} = \frac{1}{z-a} + c, \text{ where } c \text{ is a constant.}$$

Example 3.5 Show that the transformation $w = \dfrac{z-1}{z+1}$ maps the unit circle in the w-plane onto the imaginary axis in the z-plane. Find also the images of the interior and exterior of the unit circle.

The image of the unit circle $|w| = 1$ is given by

$$\left|\frac{z-1}{z+1}\right| = 1, \text{ i.e. } |z - 1| = |z + 1|$$

i.e. $|(x - 1) + iy| = |(x + 1) + iy|$

i.e. $(x - 1)^2 + y^2 = (x + 1)^2 + y^2$

i.e. $-2x = 2x$ or $x = 0$, which is the imaginary axis.

The image of the interior of the unit circle i.e. $|w| < 1$ is given by $|z - 1| < |z + 1|$ i.e. $-2x < 2x$ or $x > 0$, which is the right half of the z-plane.

Similarly the image of the exterior of the circle $|w| = 1$ is the left half of the z-plane.

Example 3.6 Show that the transformation $w = \dfrac{z-i}{1-iz}$ maps (i) the interior of the circle $|z| = 1$ onto the lower half of the w-plane and (ii) the upper half of the z-plane onto the interior of the circle $|w| = 1$.

$$w = \frac{z - i}{1 - iz} \tag{1}$$

$\therefore$ $$w - iwz = z - i$$

i.e. $$z(1 + iw) = w + i$$

$\therefore$ $$z = \frac{w + i}{1 + iw} \tag{2}$$

(2) is the inverse transformation of (1). The interior of the circle $|z| = 1$ is given by $|z| < 1$.

From (2), the image of $|z| < 1$ is given by

$$\left|\frac{w + i}{1 + iw}\right| < 1, \text{ i.e. } |w + i| < |1 + iw|$$

i.e. $$|u + i(v + 1)| < |(1 - v) + iu|$$

i.e. $$u^2 + (v + 1)^2 < (1 - v)^2 + u^2$$

i.e. $$2v < -2v \quad \text{or} \quad 4v < 0 \quad \text{or} \quad v < 0,$$

i.e., the lower half of the w-plane.

(2) can be written as

$$x + iy = \frac{u + i(v + 1)}{(1 - v) + iu}$$
$$= \frac{[u + i(v + 1)][(1 - v) - iu]}{(1 - v)^2 + u^2}$$
$$= \frac{u[1 - v + v + 1] + i(1 - v^2 - u^2)}{u^2 + (1 - v)^2}$$

$\therefore$ $$x = \frac{2u}{u^2 + (1 - v)^2} \tag{3}$$

and $$y = \frac{1 - u^2 - v^2}{u^2 + (1 - v)^2} \tag{4}$$

The upper half of the z-plane is given by $y > 0$. Its image is given by

$$\frac{1 - u^2 - v^2}{u^2 + (1 - v)^2} > 0 \quad \text{[from (4)]}$$

i.e. $u^2 + v^2 < 1$ or $|w|^2 < 1$

i.e. $|w| < 1$

i.e. the interior of the circle $|w| = 1$.

Example 3.7 Find the most general bilinear transformation that maps the upper half of the z-plane onto the interior of the unit circle in the w-plane.

Let the required bilinear transformation be

$$w = \frac{az+b}{cz+d} \tag{1}$$

Since the image of $y > 0$ has to be $|w| < 1$, the boundaries of the two regions must correspond i.e. the image of $y = 0$ must be $|w| = 1$. Since three points determine a circle uniquely, we shall make three convenient points lying on $y = 0$ map into three points on $|w| = 1$. Let us assume that the three points $z = 0$, $z = \infty$ and $z = 1$ map onto points on the circle $|w| = 1$. Thus, when $z = 0$, $|w| = 1$.

$\therefore$ From (1), we get $\quad 1 = \left|\frac{b}{d}\right|$, i.e. $|b| = |d| \qquad (2)$

Rewriting (1), we have $\quad w = \dfrac{a + b/z}{c + d/z} \qquad (1)'$

When $\quad z = \infty, \quad |w| = 1.$

$\therefore$ From (1)′, we get $\quad \left|\frac{a}{c}\right| = 1, \quad$ i.e. $|a| = |c| \qquad (3)$

If $\quad a = 0$, then, from (3), we see that $c = 0$

In this case, the transformation (1) becomes $w = \dfrac{b}{d}$, which will map the whole of z-plane onto a single point $w = \dfrac{b}{d}$, which is not true. Hence $a \neq 0$ and so $c \neq 0$, from (3).

$\therefore \quad \dfrac{a}{c} \neq 0$, such that $\left|\dfrac{a}{c}\right| = 1$, from (3)

$\therefore \quad \dfrac{a}{c}$ may be taken as $e^{i\theta}$, where θ is real.

Again, re-writing (1), $w = \dfrac{a}{c}\dfrac{(z + b/a)}{(z + d/c)}$

i.e. $$w = e^{i\theta}\left(\frac{z + b/a}{z + d/c}\right)$$

Putting $\dfrac{b}{a} = -\alpha$ and $\dfrac{d}{c} = -\beta$, where α and β are complex, the required transformation becomes

$$w = e^{i\theta}\left(\frac{z-\alpha}{z-\beta}\right) \tag{4}$$

From (2) and (3), we have $\left|\frac{b}{a}\right| = \left|\frac{d}{c}\right|$

i.e. $|\alpha| = |\beta|$ (5)

We have assumed that the point $z = 1$ also maps onto a point on the circle $|w| = 1$.

$\therefore$ From (4), $1 = |e^{i\theta}| \left|\frac{1-\alpha}{1-\beta}\right|$

i.e. $|1 - \alpha| = |1 - \beta|$ (6)

From (5) and (6), we find that either $\alpha = \beta$ or $\bar{\alpha} = \beta$.

If we assume that $\alpha = \beta$, the transformation reduces to $w = e^{i\theta}$, which will map the whole of the z-plane into a single point, which is not true. Hence $\beta = \bar{\alpha}$.

$\therefore$ The required transformation is $w = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$. Since α is arbitrary, it can be taken as any point in the upper half of the z-plane.

The image of $z = \alpha$ is $w = 0$, which lies inside $|w| = 1$.

Thus the upper half of the z-plane maps onto $|w| < 1$ by the transformation

$w = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$, where α is any point in $y > 0$.

Example 3.8 Find the transformation that will map the strip $x \geq 1$ and $0 \leq y \leq 1$ of the z-plane into the half-plane (i) $v \geq 0$ and (ii) $u \geq 0$.

(i) Consider the isosceles triangle BAC, which is a three sided polygon. [Fig. 3.28]

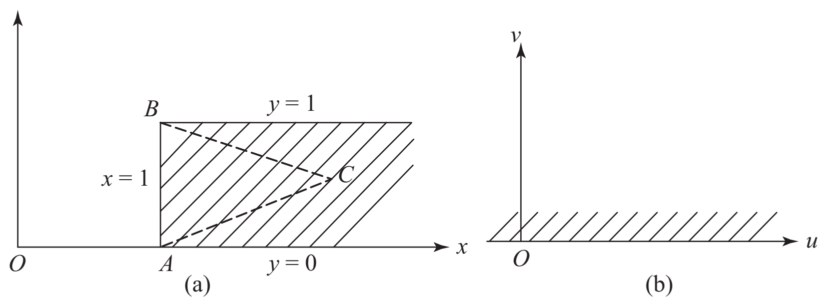


Fig. 3.28

When $C \to \infty$, the interior of this triangle becomes the region of the given strip.

In the limit, the interior angles of the polygon are $\frac{\pi}{2}, \frac{\pi}{2}$ and 0.

Let us assume that the images of B, A, C are $w = -1, 1$ and ∞ respectively. Then the required Schwarz-Christoffel transformation is

$$\frac{dz}{dw} = A\,(w+1)^{\frac{\pi}{2\pi}-1} \cdot (w-1)^{\frac{\pi}{2\pi}-1},$$

omitting the factor corresponding to $w = \infty$, as per Note (4) under the discussion of the transformation.

Note ☑ z and w are interchanged in the Schwarz-Christoffel transformation formula, as we are mapping a polygon in the z-plane onto the upper half of the w-plane.

i.e. $$\frac{dz}{dw} = A(w+1)^{-1/2} \cdot (w-1)^{-1/2}$$

$\therefore$ $$= \frac{A}{\sqrt{w^2-1}}$$

Integrating with respect to w, we get

$$z = A \cosh^{-1} w + B$$

When $z = 1, \quad w = 1, \quad \therefore B = 1$

When $z = 1 + i, \quad w = -1 \quad \therefore 1 + i = \text{A} \cosh^{-1}(-1) + 1$

i.e. $i = A \, . \, i\pi$ $\qquad [\because \cosh i\pi = \cos \pi = -1]$

$$A = \frac{1}{\pi}$$

$\therefore$ Required transformation is $z = \frac{1}{\pi} \cosh^{-1} w + 1$

i.e. $\cosh^{-1} w = \pi(z-1)$ or $w = \cosh \pi (z-1)$

(ii) Put $w' = u' + iv' = i(u + iv) = iw$

This means that $u - v$ system is rotated about the origin through $\frac{\pi}{2}$ in the positive direction giving $u' - v'$ system. Hence $v' \geq 0$ corresponds to $u \geq 0$.

$\therefore$ The required transformation that maps the given region in the z-plane onto $u \geq 0$ is

$$i\, w = \cosh \pi (z-1)$$

i.e. $$w = -i \cosh \pi (z-1)$$

Example 3.9 Find the transformation that maps the semi-infinite strip $y \geq 0, -a \leq x \leq a$ onto the upper half of the w-plane. Make the points $z = -a$ and $z = a$ correspond to the points $w = -1$ and $w = 1$.

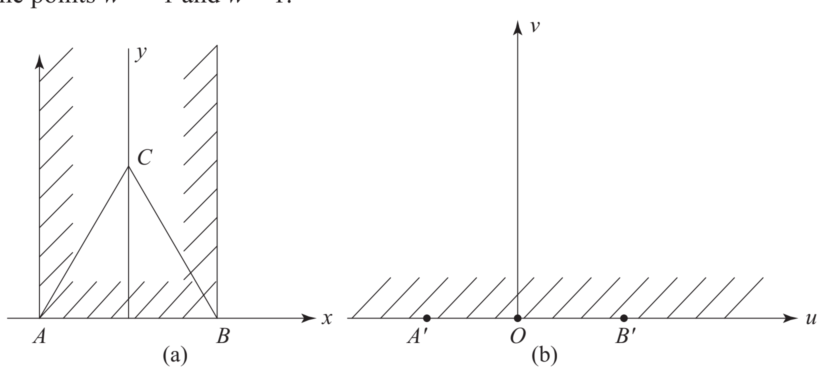


Fig. 3.29

Proceeding as in the previous example, the required transformation is

$$\frac{dz}{dw}=\frac{A}{\sqrt{w^2-1}}=\frac{iA}{i\sqrt{w^2-1}}=\frac{k}{\sqrt{1-w^2}} \quad \text{[Fig. 3.29]}$$

Note ☑ This change is made to simplify the evaluation of constants after integration.

$\therefore$ $$z=k\int\frac{dw}{\sqrt{1-w^2}}+B$$

$$= k\sin^{-1} w + B \qquad (1)$$

When $z = -a$, $w = -1$ and when $z = a$, $w = 1$

$\therefore$ $k(-\pi/2) + B = -a$ and $k(\pi/2) + B = a$

Solving, we get $k=\dfrac{2a}{\pi}$ and $B = 0$.

Using these values in (1), the required transformation is $z=\dfrac{2a}{\pi}\sin^{-1} w$ or

$$w=\sin\left(\frac{\pi z}{2a}\right)$$

Example 3.10 Find the transformation that maps the infinite strip $0 \le v \le \pi$ of the w-plane onto the upper half of the z-plane.

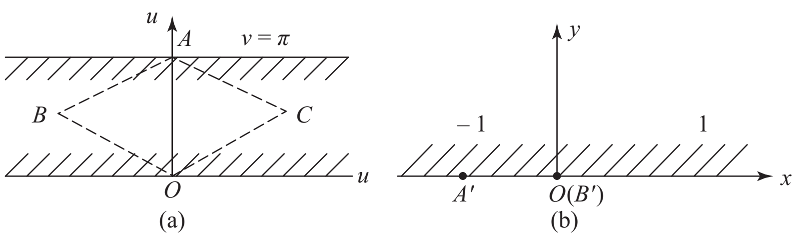


Fig. 3.30

Consider the rhombus $ABOC$, which is a four sided polygon. [Fig. 3.30]

When B and $C \to \infty$, the interior of the this rhombus becomes the region of the given strip. In the limit, the interior angles at A, B, O and C are π, O, π and O respectively.

Let us assume that the images of A, B, O, C are the points -1, 0, 1 and ∞ of the z-plane.

Then the required transformation is

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi}{\pi}-1}\cdot(z-0)^{\frac{0}{\pi}-1}\cdot(z-1)^{\frac{\pi}{\pi}-1}$$

i.e. $$\frac{dw}{dz}=\frac{A}{z} \quad \therefore\ w = A\log z + B \qquad (1)$$

When $w = 0$, $z = 1$ $\therefore B = 0$

When $w = \pi i$, $z = -1$ $\therefore$ $A\log(-1) = \pi i$

i.e. $A\log(e^{i\pi}) = \pi i$

i.e. $A(i\pi) = i\pi$ or $A = 1$

Using these values in (1), the required transformation is $w = \log z$.

EXERCISE 3(d)

Part A

(Short Answer Questions)

1. Define a bilinear transformation and its determinant.
2. Find the value of the determinant of the transformation $w = \dfrac{1-iz}{z-i}$.
3. What do you mean by the fixed point of a transformation? Does the fixed point exist only for a bilinear transformation?
4. Define the cross ratio of four points in a complex plane.
5. State the cross ratio property of a bilinear transformation.
6. What is Schwarz-Christoffel transformation? State the formula for the same.
7. What is the advantage of choosing the image of one vertex of the polygon at infinity in Schwarz-Christoffel transformation?
8. Find the invariant points of the transformations
   (i) $w = iz^2$ and (ii) $w = z^3$

Find the invariant points of the transformations

9. $w = \dfrac{2z+6}{z+7}$ 10. $w = \dfrac{3z-5i}{iz-1}$ 11. $w = \dfrac{z-1-i}{z+2}$

12. Find the condition for the invariant points of the transformation $w = \dfrac{az+b}{cz+d}$ to be equal.
13. Find all linear fractional transformations whose fixed points are -1 and 1.
14. Find all linear fractional transformations whose fixed points are $-i$ and i.
15. Find all linear fractional transformations without fixed points in the finite plane.
16. Find the image of the real axis of the z-plane by the transformation $w = \dfrac{1}{z+i}$.

Part B

17. Find the bilinear transformation which maps the points
    (i) $z = 0, -i, -1$ into $w = i, 1, 0$ respectively,
    (ii) $z = -i, 0, i$ into $w = -1, i, 1$ respectively.
18. Find the bilinear transformation that maps the points (i) $z = 0, -1, \infty$, into the points $w = -1, -2-i, i$ respectively, (ii) $z = 0, -i, 2i$ into the points $w = 5i$, $\infty - i/3$ respectively.
19. Prove that $w = \dfrac{z}{1-z}$ maps the upper half of the z-plane onto the upper half of the w-plane. What is the image of the circle $|z| = 1$ under this transformation?
20. When the point z moves along the real axis of the z-plane from $z = -1$ to $z = +1$, find the corresponding movement of the point w in the w-plane, if $w = \dfrac{1-iz}{z-i}$.

21. Prove that, under the transformation $w = \dfrac{z-i}{iz-1}$, the region Im(z) ≥ 0 is mapped onto the region $|w| \leq 1$. Into what region is Im (z) ≤ 0 mapped by this transformation?

22. Find the images of (i) the segment of the real axis between $z = +1$ and $z = -1$ (ii) the interior of the circle $|z| = 1$ and (iii) the exterior of the circle $|z| = 1$ under the transformation $w = \dfrac{1+iz}{z+i}$.

23. Show that the transformation $w = \dfrac{i-z}{i+z}$ maps the circle $|z| = 1$ onto the imaginary axis of the w-plane. Find also the images of the interior and exterior of this circle.

24. Find the bilinear transformation that maps the upper half of the z-plane onto the interior of the unit circle of the w-plane in such a way that the points $z = i, \infty$ are mapped onto $w = 0, -1$.

    [***Hint:*** Use Worked Example (3.7).]

25. Find the transformation which maps the area in the z-plane within an infinite sector of angle $\dfrac{\pi}{m}$ onto the upper half of the w-plane.

    $\Bigg[$***Hint:*** The sectoral region bounded by OX and OP may be regarded as an open polygon with vertex at O and interior angle $\dfrac{\pi}{m}$. Make the origin of the z-plane correspond to the origin of the w-plane. $\Bigg]$

26. Find the transformation which maps the semi-infinite strip

    (i) $x \geq 0,\ 0 \leq y \leq c$ onto $v \geq 0$

    (ii) $u \geq 0,\ 0 \leq v \leq \pi$ onto $y \geq 0$

27. Find the transformation which maps the semi-infinite strip $y \geq 0,\ 0 \leq x \leq a$ onto the upper half of the w-plane. Make the points $z = 0$ and $z = a$ correspond to $w = +1$ and $w = -1$.

28. Find the transformation that maps the infinite strip $0 \leq y \leq k$ onto the upper half of the w-plane. Make the points $z = 0$, $z = ik$ correspond to $w = 1, -1$.

29. Find the transformation that maps the region in the z-plane above the line $y = b$, when $x < 0$ and that above the x-axis, when $x > 0$ into the upper half of the w-plane. Make the points $z = ib$ and $z = 0$ correspond to $w = -1$ and $w = 1$.

30. Find the transformation that maps the region in the w-plane above the u-axis when $u < 0$ and that above $v = b$ when $u > 0$ into the upper half of the z-plane. Make the points $w = 0$ and $w = ib$ correspond to $z = 0$ and $z = 1$.

ANSWERS

Exercise 3(a)

16. at all points on the line $y = x$ 17. at all points
18. nowhere 19. only at the origin
20. at all points except $z = -1$ 21. $a = 2, b = -1, c = -1, d = 2$
22. $p = -1$ 36. $2z$ 37. e^z 38. $-\sin z$
39. $\cosh z$

Exercise 3(b)

5. $y + c$ 6. Yes 7. Yes 8. No
9. Yes 10. No 11. $2xy$ 12. $e^x \sin y$
13. $\cos x \sinh y$ 14. $x^2 - y^2 + 2y$; 15. $2\tan^{-1}\left(\dfrac{y}{x}\right)$ 16. $\log z$
17. $-z^2$ 18. e^{iz} 19. $\cosh z$ 20. $\dfrac{1}{z}$
21. $v = 3x^2y - y^3 + 2xy + y^2 - x^2 + c;$ $f(z) = z^3 + z^2 - iz^2 + ic$
22. $u = x^3 - 3xy^2 - 2xy + c;$ $f(z) = z^3 + iz^2 + c$
23. $\psi = 2xy - \dfrac{y}{x^2 + y^2} + c;$ $f(z) = z^2 + \dfrac{1}{z} + ic$
24. $\phi = x^2 - y^2 - 2x + 3y - 2xy + c$ $f(z) = z^2 - 2z + i\,(z^2 - 3z)$
25. $\dfrac{z^4}{4} + \dfrac{z^2}{2} + (1 + i)\,z$ $x^4 - 6x^2y^2 + y^4 + 2\,(x^2 - y^2) + 4\,(x - y) = c'$
26. $w = -iz \cdot e^z + ic\,;\ v = e^x\,(y \sin y - x \cos y) + c$
27. $w = ize^{-z} + c\,;\ u = e^{-x}\,(x \sin y - y \cos y) + c$
28. $w = -ie^{iz^2} + ic\,;\ v = e^{-2xy} \cos\,(x^2 - y^2) + c$
29. $w = ze^{i2z} + c\,;\ u = e^{-2y}\,(x \cos 2x - y \sin 2x) + c$
30. $f(z) = \dfrac{1+i}{z} - 1$ 31. $f(z) = iz^2 - z$
32. $f(z) = \dfrac{\cot z}{1+i} + c$ 33. $f(z) = \dfrac{1}{2}(1 + \sec z)$
37. $2y + y^3 - 3x^2y = b$

Exercise 3(c)

5. $z = \pm\, e^{\alpha}$ 6. $z = \alpha, \beta$ and $\dfrac{1}{2}(\alpha + \beta)$ 7. $\dfrac{1}{\sqrt{2}}$
8. $(u - 2)^2 + (v - 3)^2 = a^2$ 9. $|\,w\,| = 2a$ 10. $|\,w\,| = 5a$

11. $(u-2)^2+(v+1)^2=2a^2$ 12. $|w|=\dfrac{1}{2a}$ 13. $0<v<2$

14. $v>1$ 15. $u+v>2$ 16. $u\,v=9$ 17. $u\,v=a^2$

18. Interior of the circle $u^2+\left(v+\dfrac{1}{2c}\right)^2=\left(\dfrac{1}{2c}\right)^2$

19. Exterior of the circle $\left(u-\dfrac{1}{2c}\right)^2+v^2=\left(\dfrac{1}{2c}\right)^2$

20. $u=mv$ 21. Upper half of the w-plane

22. $u^2=-4(v-1)$; $u^2=4(v+1)$ 23. $4<|w|<9$

24. $\dfrac{\pi}{2}<\arg(w)<\pi$ 25. $x\,y=c_1$ and $y^2-x^2=c_2$

26. The annular region $1\le R\le e$
27. Segment of $v=0$, given by $-1\le u\le 1$
28. Part of $v=0$, given by $u\ge 1$
29. u-axis 30. Segment of $u=0$, given by $-1\le v\le 1$.

31. $u=\dfrac{a}{2}\left(r+\dfrac{1}{r}\right)\cos\theta,\ v=\dfrac{a}{2}\left(r-\dfrac{1}{r}\right)\sin\theta$

32. $(\pm 2, 0)$

33. (i) Triangle with vertices $w=4+i,\ 7+i,\ 7-5i$
    (ii) Triangle with vertices $w=1-i,\ 1,\ 3-2i$

34. Square with vertices $w=3+\left(3\pm 2\sqrt{2}\right)i,\ \left(3\mp 2\sqrt{2}\right)+3i$

35. No
36. The exterior of the image circle
37. (i) $u^2+(v+c)^2>c^2,\ v<0$
    (ii) $u^2+v^2+v<0,\ u>0$
38. The region enclosed by $v^2=4(u+1)$ and $u=0$
39. The region above the u-axis, bounded by the u-axis, $v^2=4(1-u)$ and $v^2=4(1+u)$.
45. The lower half of the w-plane
46. The region in the upper half of the w-plane bounded by the part of the u-axis given by $-\dfrac{1}{\sqrt{2}}\le u\le\dfrac{1}{\sqrt{2}}$, the ellipse $\dfrac{u^2}{\cosh^2 3}+\dfrac{v^2}{\sinh^2 3}=1$ and the hyperbola $u^2-v^2=\dfrac{1}{2}$.
48. The elliptic annular region bounded by $\dfrac{u^2}{\sinh^2 1}+\dfrac{v^2}{\cosh^2 1}=1$ and $\dfrac{u^2}{\sinh^2 2}+\dfrac{v^2}{\cosh^2 2}=1$, lying in the right half of the w-plane.

Exercise 3(d)

2. -2 3. No

8. (i) $z = 0$ and $z = -i$
   (ii) $z = 0, \pm 1$.

9. $z = 1, -6$ 10. $z = i, -5i$ 11. $z = -i$, $z = -1 + i$

12. $(a - d)^2 + 4bc = 0$ 13. $w = \dfrac{az + b}{bz + a}$ 14. $w = \dfrac{az + b}{a - bz}$

15. $w = z + a$ 16. $u^2 + v^2 + v = 0$

17. (i) $w = -i\left(\dfrac{z+1}{z-1}\right)$

    (ii) $w = \dfrac{i - iz}{z + 1}$

18. (i) $w = \dfrac{iz - 2}{z + 2}$

    (ii) $w = \dfrac{3z - 5i}{iz - 1}$

19. The line $u = -1/2$

20. w moves along the upper half of the circle $|w| = 1$ from $w = -1$ to $w = +1$ in the clockwise sense.

21. $|w| \geq 1$

22. The lower half of the circle $|w| = 1$; $v < 0$; $v > 0$

23. $u > 0$; $u < 0$ 24. $w = \dfrac{i - z}{i + z}$ 25. $w = kz^m$

26. (i) $w = \cosh \dfrac{z}{c}$

    (ii) $z = \cosh w$

27. $w = \cos \dfrac{\pi z}{a}$ 28. $w = e^{\pi z/k}$

29. $z = \dfrac{b}{\pi}\left(\sqrt{w^2 - 1} + \cosh^{-1} w\right)$ 30. $w = \dfrac{2ib}{\pi}\left(\sin^{-1}\sqrt{z} + \sqrt{z(1-z)}\right)$

UNIT-4
COMPLEX INTEGRATION

UNIT 4

Complex Integration

4.1 INTRODUCTION

The concept of a real line integral, with which the reader is familiar is extended to that of a complex line integral as given below in Fig. 4.1.

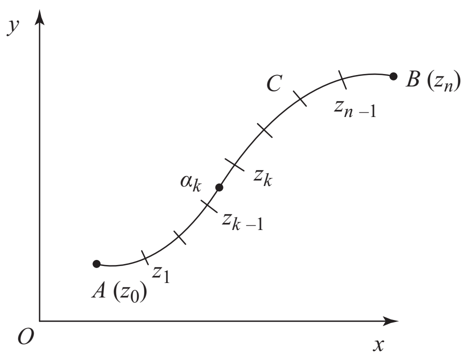


Fig. 4.1

Let $f(z)$ be a continuous function of the complex variable z (= $x + iy$) and bet C be any continuous curve connecting two points $A(z_0)$ and $B(z_n)$ on the z-plane. Let C be divided into n parts by means of the points $z_1, z_2, ...z_{n-1}$. Let $z_k - z_{k-1} = \Delta z_k$. Let α_k be an arbitrary point in the arc $z_{k-1} z_k$ ($k = 1, 2, ..., n$).

Let $$S_n = \sum_{k=1}^{n} f(\alpha_k)(z_k - z_{k-1}) \quad \text{or} \quad \sum_{k=1}^{n} f(\alpha_k)\Delta z_k.$$

If the limit of S_n exists as $n \to \infty$ in such a way that each $\Delta z_k \to 0$ and if the limit is independent of the mode of subdivision of C and the choice of the points α_k, then it is called *the complex line integral* of $f(z)$ along C from A to B and denoted as $\int_C f(z)\,dz$. Practically a complex line integral is expressed in terms of two real line integrals and evaluated.

i.e. If $f(z) = u(x, y) + iv(x, y)$, then

$$\int_C f(z)\mathrm{d}z = \int_C (u + iv)(\mathrm{d}x + i\mathrm{d}y)$$

$$= \int_C (u\mathrm{d}x - v\mathrm{d}y) + i \int_C (v\mathrm{d}x + u\mathrm{d}y)$$

Note ☑ When C is an open arc (of finite length) of a continuous curve, the sense of description of C is obvious, as it is traversed from $A(z_0)$ to $B(z_n)$. When C is a simple closed curve, i.e. a continuous closed curve which does not intersect itself and which encloses a finite region in the Argand plane, then C is traversed in the direction indicated by the arrows drawn on C. In this case, the complex line integral is called contour integral and denoted by the special symbol $\oint_C f(z)\mathrm{d}z$.

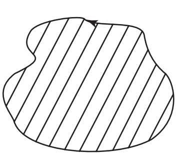
If the sense of description of C is not indicated by arrows, C is traversed in the positive sense or direction, i.e. the direction in which a person, walking along C, has the interior region of C to his or her left. Practically we shall take the anticlockwise direction of traversal of C as the positive direction.

4.1.1 Simply and Multiply Connected Regions

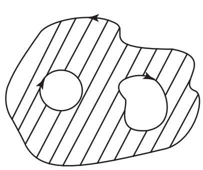
A region R is called *simply connected*, if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply connected is called *multiply connected*.

Obviously, a simply connected region is one which does not have any "holes" in it, whereas a multiply connected region is one which has.

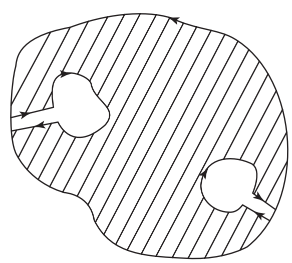
A multiply connected region can be converted into a simply connected region by introducing cross-cuts as shown in Figs 4.2 (a), (b) and (c)



(a) Simply connected region



(b) Multiply connected region



(c) Multiply connected region converted into simply connected region by cross-cuts

Fig. 4.2

4.2 CAUCHY'S INTEGRAL THEOREM OR CAUCHY'S FUNDAMENTAL THEOREM

If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points on and inside a simple closed curve C, then $\int_C f(z)\mathrm{d}z = 0$.

Proof

Let $$f(z) = u(x, y) + iv(x, y) = u + iv$$

Then $$\int_C f(z)\,\mathrm{d}z = \int_C (u + iv)(\mathrm{d}x + i\mathrm{d}y)$$

$$= \int_C (u\mathrm{d}x - v\mathrm{d}y) + i\int_C (v\mathrm{d}x + u\mathrm{d}y) \qquad (1)$$

Since $f'(z)$ is continuous, the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous on C and in the region R enclosed by C.

Hence Green's theorem in a plane, namely,

$$\int_C (P\mathrm{d}x + Q\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

can be applied to each of the lines integral in the R.H.S. of (1).

$$\therefore \qquad \int_C f(z)\mathrm{d}z = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

Since the function $f(z) = u + iv$ is analytic, u and v satisfy the Cauchy-Riemann equations in R.

i.e $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e $$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \text{ and } -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (3)$$

Using (3) in (2), we get

$$\int_C f(z)\mathrm{d}z = 0 + i0 = 0$$

Note ☑

1. The above theorem can be proved without assuming that $f'(z)$ is continuous, as was done by a French mathematician E. Goursat. We state below the modified form of the above theorem, called *Cauchy-Goursat theorem* without proof.

   "If $f(z)$ is analytic at all points on and inside a simple closed curve C, then $\int_C f(z)\mathrm{d}z = 0$"

2. We have proved Cauchy's integral theorem for a simply connected region. It can be extended to a multiply connected region as follows.

4.2.1 Extension of Cauchy's Integral Theorem

If $f(z)$ is analytic on and inside a multiply connected region whose outer boundary is C and inner boundaries are $C_1, C_2, \ldots, C_n$, then

$\int\limits_C f(z)\mathrm{d}z = \int\limits_{C_1} f(z)\mathrm{d}z + \int\limits_{C_2} f(z)\mathrm{d}z + \cdots + \int\limits_{C_n} f(z)\mathrm{d}z$ where all the integrals are taken in the same sense. We shall prove the extension for a doubly connected region for simplicity.

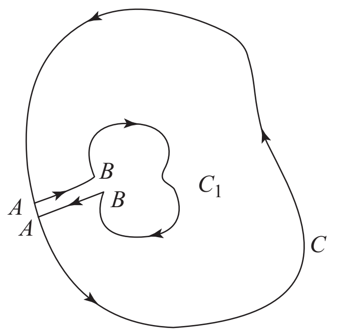


Fig. 4.3

We convert the doubly connected region into a simply connected region by introducing the cross-cut AB (Fig. 4.3).

By Cauchy's integral theorem, $\oint\limits_{C'} f(z)\mathrm{d}z = 0$ where C' includes C described in anticlockwise sense, C_1 described in clockwise sense, AB and BA

$$\therefore \qquad \oint\limits_{C'\circlearrowleft} f(z)\mathrm{d}z = \oint\limits_{C\circlearrowleft} f(z)\mathrm{d}z + \oint\limits_{C_1\circlearrowright} f(z)\mathrm{d}z + \oint\limits_{AB} f(z)\mathrm{d}z + \oint\limits_{BA} f(z)\mathrm{d}z = 0$$

The last two integrals in the R.H.S. are equal in value but opposite in sign and hence cancel each other.

$$\therefore \qquad \oint\limits_{C\circlearrowleft} f(z)\mathrm{d}z + \oint\limits_{C_1\circlearrowright} f(z)\mathrm{d}z = 0$$

$$\text{i.e.} \qquad \oint\limits_{C\circlearrowleft} f(z)\mathrm{d}z = -\oint\limits_{C_1\circlearrowright} f(z)\mathrm{d}z$$

$$= \oint\limits_{C_1\circlearrowleft} f(z)\mathrm{d}z$$

Note ☑ By introducing as many cross-cuts as the number of inner boundaries, we can give the proof in a similar manner for the extension of Cauchy's integral theorem stated above.

4.2.2 Cauchy's Integral Formula

If $f(z)$ is analytic inside and on a simple closed curve C that encloses a simply connected region R and if 'a' is any point in R, then $f(a) = \dfrac{1}{2\pi i}\oint\limits_C \dfrac{f(z)}{z-a}\mathrm{d}z$, where C is described in the anticlockwise sense.

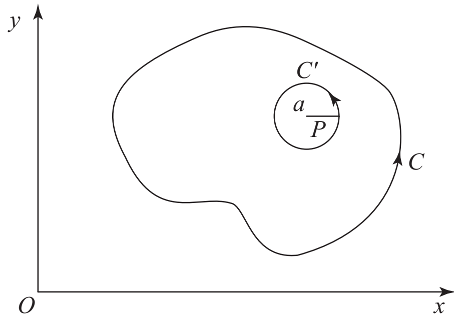


Fig. 4.4

Proof

Since $f(z)$ is analytic on and inside C, $\dfrac{f(z)}{z-a}$ is also analytic on and inside C, except at the point $z = a$. Now we draw a small circle C' with centre at $z = a$ and radius ρ, lying completely inside C. The function $\dfrac{f(z)}{z-a}$ is analytic in the doubly connected region bounded by C and C'.

$\therefore$ By Cauchy's Extended theorem, we have

$$\therefore \quad \oint_C \frac{f(z)}{z-a}\,dz = \oint_{C'} \frac{f(z)}{z-a}\,dz \tag{1}$$

If z is any point on C', then $|z-a| = \rho$ and hence $z - a = \rho e^{i\theta}$ or $z = a + \rho e^{i\theta}$

$$\therefore \quad dz = i\rho e^{i\theta}\,d\theta$$

$$\therefore \quad \oint_C \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\,d\theta$$

($\because$ When z moves around C' once completely, θ varies from 0 to 2π)

$$= i\int_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta \tag{2}$$

(2) is true, however small the circle C' is and hence true when $\rho \to 0$. Taking limits of (2) as $\rho \to 0$, we get

$$\oint_{C'} \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta = if(a)\cdot 2\pi \tag{3}$$

Using (3) in (1), we have

$$\oint_C \frac{f(z)}{z-a}\,\mathrm{d}z = 2\pi i\, f(a)$$

$$\therefore \qquad f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,\mathrm{d}z.$$

4.2.3 Extension of Cauchy's Integral Formula to a Doubly Connected Region

If $f(z)$ is analytic on C_1 and C_2 (C_2 lies completely within C_1) and in the annular region R between C_1 and C_2 and if 'a' is any point in R, then

$$f(a) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,\mathrm{d}z - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-a}\,\mathrm{d}z$$

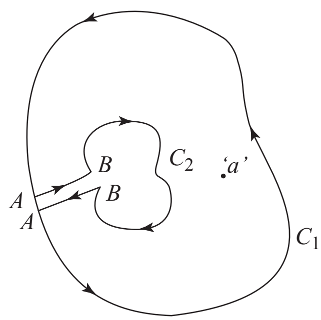


Fig. 4.5

Proof

We convert the doubly connected region to a simply connected region by introducing the cross-cut AB. 'a' lies in this region (Fig. 4.5).
By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,\mathrm{d}z,$$

where C includes C_1 described in the anticlockwise sense, C_2 in the clockwise sense, AB and BA.

$$\therefore \quad f(a) = \frac{1}{2\pi i}\left[\oint_{C_1} \frac{f(z)}{z-a}\,\mathrm{d}z + \oint_{C_1} \frac{f(z)}{z-a}\,\mathrm{d}z + \int_{AB} \frac{f(z)}{z-a}\,\mathrm{d}z + \int_{BA} \frac{f(z)}{z-a}\,\mathrm{d}z\right]$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,\mathrm{d}z + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-a}\,\mathrm{d}z$$ ($\because$ the last two integrals cancel each other)

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-a}\,dz$$

4.2.4 Cauchy's Integral Formulas for the Derivatives of an Analytic Function

By Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \qquad (1)$$

Differentiating partially both sides of (1) with respect to 'a' and performing the differentiation within the integration symbol in the R.H.S., we get

$$f'(a) = \frac{1!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz \qquad (2)$$

Proceeding further, we get

$$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz \text{ etc.}$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

WORKED EXAMPLE 4(a)

Example 4.1 If $f(z)$ is analytic in a simply connected region R, show that $\int_{z_0}^{z_1} f(z)dz$ is independent of the path joining the points z_0 and z_1 in R and lying within R. Verify this by evaluating $\int_C (z^2+3z)dz$ along (i) the circle $|z| = 2$ from (2, 0) to (0, 2) in the anticlockwise direction (ii) the straight line from (2, 0) to (0, 2) and (iii) the straight lines (2, 0) to (2, 2) and then from (2, 2) to (0, 2).

Let C_1 (ADB) and C_2 (AEB) be any two curves joining $A(z_0)$ and $B(z_1)$ in the region R. (Fig. 4.6)

Now $ADBEA$ may be regarded as a simple closed curve in R.

∴ By Cauchy's integral theorem,

$$\oint_{ADBEA} f(z)dz = 0, \text{ i.e. } \int_{ADB} f(z)dz + \int_{BEA} f(z)dz = 0$$

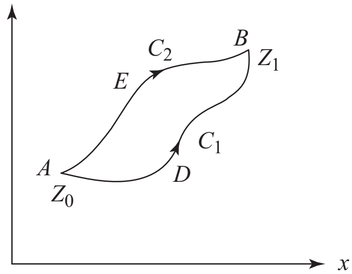


Fig. 4.6

i.e. $$\int_{ADB} f(z)\mathrm{d}z = -\int_{BEA} f(z)\mathrm{d}z = \int_{AEB} f(z)\mathrm{d}z$$

i.e. $$\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z$$

$$\int_{C_1} (z^2+3z)\mathrm{d}z = \int_{|z|=2} (z^2+3z)\mathrm{d}z = \int_0^{\pi/2} \left[(2e^{i\theta})^2 + 3.2e^{i\theta}\right]2e^{i\theta} i\mathrm{d}\theta$$

($\because$ on the circle $|z| = 2$, $z = 2e^{i\theta}$ and the end points are given by $\theta = 0$ and $\theta = \pi/2$)

$$= \left[8i\frac{e^{3i\theta}}{3i} + 12i\cdot\frac{e^{2i\theta}}{2i}\right]_0^{\pi/2}$$

$$= \frac{8}{3}(e^{i3\pi/2}-1) + 6(e^{i\pi}-1)$$

$$= \frac{8}{3}(-i-1) + 6(-2+i.0) = -\frac{44}{3} - \frac{8}{3}i$$

The equation of C_2, the line joining (2, 0) and (0, 2) is $x + y = 2$. [Fig. 4.7]

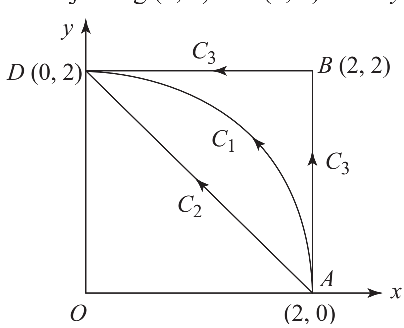


Fig. 4.7

$$\int_{C_2} (z^2+3z)\mathrm{d}z = \int_{C_2(x+y=2)} [x^2 - y^2 + i2xy + 3(x+iy)](\mathrm{d}x + i\mathrm{d}y)$$

$$= \int_2^0 [x^2 - (2-x)^2 + i2x(2-x) + 3\{x + i(2-x)\}](\mathrm{d}x - i\mathrm{d}x)$$

$$= \int_2^0 [(-2x^2 + 8x + 2) + i(-2x^2 - 6x + 10)]\mathrm{d}x$$

$$= \left[-2\frac{x^3}{3} + 4x^2 + 2x + i\left(-\frac{2}{3}x^3 - 3x^2 + 10x \right) \right]_2^0$$

$$= -\frac{44}{3} \frac{-8}{3} i$$

$$\int_{C_3} (z^2 + 3z)\mathrm{d}z = \int_{\substack{AB \\ (x=2)}} + \int_{\substack{BD \\ (y=2)}} (z^2 + 3z)\mathrm{d}z$$

$$= \int_0^2 [(2+iy)^2 + 3(2+iy)]i\mathrm{d}y + \int_2^0 [(x+i2)^2 + 3(x+i2)]\mathrm{d}x$$

[$\because$ on AB, $\mathrm{d}x = 0$ and on BD, $\mathrm{d}y = 0$]

$$= \left[\frac{(2+iy)^3}{3} + \frac{3(2+iy)^2}{2} \right]_0^2 + \left[\frac{(x+i2)^3}{3} + \frac{3(x+i2)^2}{2} \right]_2^0$$

$$= \left(-\frac{8}{3} - 6 \right) - \frac{8}{3}i - 6 = -\frac{44}{3} - \frac{8}{3}i.$$

The values of the given integral are the same, irrespective of the curve joining the two points, since $f(z) = z^2 + 3z$ is analytic everywhere.

If the curve is not specified, the integral can be evaluated easily as follows, provided the integrand is an analytic function.

$$\int_{(2,0)}^{(0,2)} f(z)\mathrm{d}z = \int_{2+i0}^{0+2i} (z^2 + 3z)\mathrm{d}z = \left(\frac{z^3}{3} + \frac{3z^2}{2} \right)_2^{2i} = -\frac{44}{3} - \frac{8}{3}i.$$

Example 4.2 Evaluate $\int_0^{1+i} (x - y + ix^2)\mathrm{d}z$ along (i) the line joining z = 0 and $z = 1 + i$, (ii) the parabola $y = x^2$ and (iii) the curve $x = t$, $y = 2t - t^2$.

(i) The line joining the points $z = 0$ and $z = 1 + i$, i.e. the points (0, 0) and (1, 1) is $y = x$

$$\int_0^{1+i} (x - y + ix^2)(\mathrm{d}z + i\mathrm{d}y) = \int_0^1 (x - x + ix^2)(1+i)\mathrm{d}x$$

$$= (-1+i)\left(\frac{x^3}{3} \right)_0^1 = \frac{1}{3}(-1+i).$$

(ii) When $y = x^2$, $\mathrm{d}y = 2x\ \mathrm{d}x$

$\therefore$ The given integral $= \int_0^1 (x - x^2 + ix^2)(\mathrm{d}x + i2x\mathrm{d}x)$

$$= \int_0^1 [(x - x^2 - 2x^3) + i(3x^2 - 2x^3)]\mathrm{d}x$$

$$= \left(\frac{1}{2} - \frac{1}{3} - \frac{2}{4}\right) + i\left(1 - \frac{2}{4}\right) = -\frac{1}{3} + \frac{i}{2}$$

(iii) (0, 0) corresponds to $t = 0$ and (1, 1) corresponds to $t = 1$.

$\therefore$ The given integral

$$= \int_0^1 (t - 2t + t^2 + i \cdot t^2)[\mathrm{d}t + i(2 - 2t)\mathrm{d}t]$$

$$= \int_0^1 [(2t^3 - t^2 - t) + i\ (-2t^3 + 5t^2 - 2t)]\mathrm{d}t$$

$$= \left(\frac{2}{4} - \frac{1}{3} - \frac{1}{2}\right) + i\left(-\frac{2}{4} + \frac{5}{3} - 1\right) = -\frac{1}{3} + \frac{1}{6}i$$

Note ☑ The values of the integral along three different curves are different, as the integrand is not an analytic function of z.

Example 4.3 Evaluate

(i) $\int_C \frac{\mathrm{d}z}{z-2}$ and (ii) $\int_C (z-2)^n \,\mathrm{d}z$

($n \neq -1$), where C is the circle whose centre is 2 and radius 4.

(i) The equation of the circle whose centre is 2 and radius 4 is $|z - 2| = 4$

$\therefore$ $z - 2 = 4e^{i\theta}$ and $\mathrm{d}z = 4e^{i\theta}\, i\mathrm{d}\theta$

To describe C once completely, θ has to vary from 0 to 2π.

$$\therefore \quad \int_C \frac{\mathrm{d}z}{z-2} = \int_0^{2\pi} \frac{4e^{i\theta} i\mathrm{d}\theta}{4e^{i\theta}}. = i[\theta]_0^{2\pi} = 2\pi i$$

(ii) $\int_C (z-2)^n \,\mathrm{d}z = \int_0^{2\pi} 4^n e^{in\theta}.4e^{i\theta} i\mathrm{d}\theta$

$$= 4^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \mathrm{d}\theta$$

$$= 4^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} = 0, \text{since } n \neq -1$$

Aliter for (ii)

$(z-2)^n$ is analytic on and inside C.

$\therefore$ By Cauchy's integral theorem, $\int\limits_C (z-2)^n \,\mathrm{d}z = 0$

Example 4.4 Evaluate $\oint\limits_C \frac{z+2}{z}\mathrm{d}z$, where (i) C is the semicircle $|z| = 2$ in the upper half of the z-plane, (ii) C is the semicircle $|z| = 2$ in the lower half of the z-plane and (iii) C is the entire circle $|z| = 2$.

(i) On the semicircle $|z| = 2$, $z = 2e^{i\theta}$ and θ varies from 0 to π in the upper half.

$$\therefore \quad \int\limits_C \frac{z+2}{z}\mathrm{d}z = \int\limits_0^{\pi} \frac{(2e^{i\theta}+2)}{2e^{i\theta}} \cdot 2e^{i\theta} i\mathrm{d}\theta$$

$$= 2i\left(\frac{e^{i\theta}}{i}+\theta\right)_0^{\pi} = 2(e^{i\pi}-1)+2\pi i$$

$$= -4 + 2\pi i$$

(ii) On the semicircle in the lower half, θ varies from π to 2π

$$\therefore \quad \int\limits_C \frac{z+2}{z}\mathrm{d}z = 2i\left(\frac{e^{i\theta}}{i}+\theta\right)_{\pi}^{2\pi} = 4+2\pi i$$

(iii) On the entire circle, θ varies from 0 to 2π

$$\therefore \quad \int\limits_C \frac{z+2}{z}\mathrm{d}z = 2i\left(\frac{e^{i\theta}}{i}+\theta\right)_0^{2\pi} = 4\pi i$$

= The sum of values of the integral along the two semi-circles.

Aliter for (iii)

$\int\limits_{|z|=2} \frac{z+2}{z}$ is of the form $\int\limits_C \frac{f(z)}{z-a}\mathrm{d}z$, where $f(z)$ is analytic on and inside C that contains the point a.

Here $f(z) = z + 2$ is analytic on and inside $|z| = 2$, contains the point $z = 0$.

$\therefore$ By Cauchy's integral formula, we have

$$f(0) = \frac{1}{2\pi i}\int\limits_C \frac{f(z)}{z}\mathrm{d}z$$

$$\therefore \quad \int\limits_C \frac{z+2}{z}\mathrm{d}z = 2\pi i(0+2) = 4\pi i$$

Example 4.5 Evaluate $\int_C \frac{ze^z}{(z-a)^3}\,dz$, where $z = a$ lies inside the closed curve C, using Cauchy's integral formula.

By Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a) \tag{1}$$

where $f(z)$ is analytic on and inside C and the point $z = a$ lies inside C.

Comparing the given integral with the L.H.S. of (1), we find that $f(z) = ze^z$. ze^z is analytic everywhere and hence analytic on and inside C. Also 'a' lies inside C.

$$\therefore \qquad \int_C \frac{ze^z}{(z-a)^3}\,dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\left(ze^z\right)\right]_{z=a}$$

$$= \pi i\,\{(z+2)e^z\}_{z=a}$$

$$= (a+2)e^a \pi i.$$

Example 4.6 Evaluate $\int_C \frac{\tan\frac{z}{2}}{(z-a)^2}\,dz = (-2 < a < 2)$, where C is the boundary of the square whose sides lie along $x = \pm 2$ and $y = \pm 2$ described in the positive sense (Fig. 4.8).

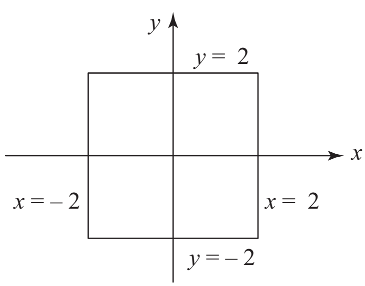


Fig. 4.8

Let $\int_C \frac{\tan\frac{z}{2}}{(z-a)^2}\,dz \equiv \int_C \frac{f(z)}{(z-a)^2}\,dz$

$f(z) = \tan\frac{z}{2}$ is analytic on and inside C, since $f'(z) = \frac{1}{2}\sec^2\frac{z}{2}$ does not exist at $z = \pm \pi, \pm 3\pi$ etc. which lie outside C.

Also, since $-2 < a < 2$, the point $z = a$ lies inside C.

Hence Cauchy’s integral formula holds good.

$$\therefore \qquad \int \frac{\tan\frac{z}{2}}{(z-a)^2}\, dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}\left(\tan\frac{z}{2}\right)\right]_{(z=a)}$$

$$= 2\pi i \cdot \left(\frac{1}{2}\sec^2\frac{z}{2}\right)_{z=a} = \pi i \sec^2\frac{a}{2}.$$

Example 4.7 Use Cauchy's integral formula to evaluate

$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$, where C is the circle $|z| = 4$.

$$\frac{1}{(z-2)(z-3)} = \frac{1}{(z-3)} - \frac{1}{(z-2)}$$

$$\therefore \qquad \text{Given integral } = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3} dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$$= \int_C \frac{f(z)}{z-3} dz - \int_C \frac{f(z)}{z-2} dz, \text{say} \tag{1}$$

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic on and inside C.

The points $z = 2$ and $z = 3$ lie inside C (Fig.4.9).

$\therefore$ By Cauchy's integral formula, form (1), we get

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz = 2\pi i\left(\sin \pi z^2 + \cos z^2\right)_{z=3}$$

$$- 2\pi i(\sin \pi z^2 + \cos \pi z^2)_{z=2}$$

$$= 2\pi i\ (\sin 9\pi + \cos 9\pi) - 2\pi i\ (\sin 4\pi + \cos 4\pi)$$

$$= -\ 2\pi i - 2\pi i = -4\pi i.$$

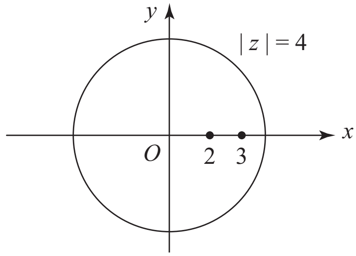


Fig. 4.9

Example 4.8 Evaluate $\int_C \frac{7z-1}{z^2-3z-4}\mathrm{d}z$, where C is the ellipse $x^2 + 4y^2 = 4$, using Cauchy's integral formula. The ellipse $x^2+4y^2=4$ or $\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$ is the standard ellipse as shown in the Fig. 4.10.

$$\int_C \frac{7z-1}{z^2-3z-4}\mathrm{d}z = \int_C \frac{7z-1}{(z-4)(z+1)}\mathrm{d}z$$

Of the two points $z = 4$ and $z = -1$ the point $z = -1$ only lies inside C. Re-writing the given integral, we have

$$\int_C \frac{7z-1}{z^2-3z-4}\mathrm{d}z = \int_C \frac{\left(\frac{7z-1}{z-4}\right)}{(z+1)}\mathrm{d}z \equiv \int_C \frac{f(z)}{z+1}\mathrm{d}z \tag{1}$$

$f(z)=\frac{7z-1}{z-4}$ is analytic inside C and the point $z = -1$ lies inside C.

$\therefore$ By Cauchy's integral formula, form (1),

$$\int_C \frac{7z-1}{z^2-3z-4}\mathrm{d}z = 2\pi i\, f(-1) = 2\pi i\left(\frac{-8}{-5}\right) = \frac{16}{5}\pi i.$$

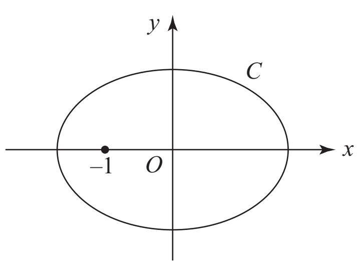


Fig. 4.10

Example 4.9 Evaluate $\int_C \frac{z\mathrm{d}z}{(z-1)(z-2)^2}$, where C is the circle $|z-2|=\frac{1}{2}$, using Cauchy's integral formula.

$|z-2|=\frac{1}{2}$ is the circle with centre at $z = 2$ and radius equal to 1/2. The point $z = 2$ lies inside this circle (Fig. 4. 11).

The given integral can be re-written as

$$\int_C \frac{\left(\dfrac{z}{z-1}\right)}{(z-2)^2} \equiv \int_C \frac{f(z)}{(z-2)^2}\,dz, \text{ say,}$$

$f(z)=\dfrac{z}{z-1}$ is analytic on and inside C and the point $z = 2$ lies inside C.

$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{z}{(z-1)(z-2)^2}\,dz = \frac{2\pi i}{1!} f'(2)$$

$$= 2\pi i\left\{\frac{d}{dz}\left(\frac{z}{z-1}\right)\right\}_{z=2}$$

$$= 2\pi i\left\{\frac{-1}{(z-1)^2}\right\}_{z=2} = -2\pi i$$

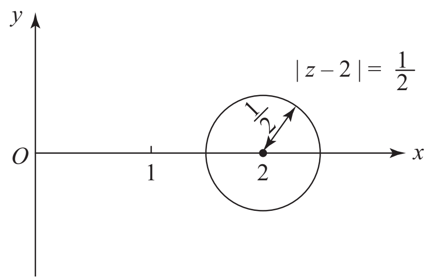


Fig. 4.11

Example 4.10 Use Cauchy's integral formula evaluate $\int_C \dfrac{z+1}{z^3-2z^2}\,dz$, where C is the circle $|z - 2 - i| = 2$.

The circle $|z - (2 + i)| = 2$ is the circle whose centre is the point $z = 2 + i$ and radius is 2, as shown in Fig. 4.12. The point $z = 2$ lies inside this circle.

The given integral can be re-written as

$$\int_C \frac{\left(\dfrac{z+1}{z^2}\right)}{z-2}\,dz \equiv \int_C \frac{f(z)}{z-2}\,dz, \text{say.}$$

$f(z)=\dfrac{z+1}{z^2}$ is analytic on and inside C and the point $z = 2$ lies inside C.

$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{z+1}{z^3-2z^2}\,dz = 2\pi i\, f(2) = 2\pi i\left(\frac{z+1}{z^2}\right)_{z=2} = \frac{3}{2}\pi i.$$

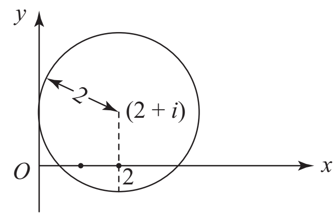


Fig. 4.12

Example 4.11 Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1+i| = 2$, using Cauchy's integral formula.

$|z+1+i| = 2$ is the circle whose centre is $z = -1 - i$ and radius is 2.

$$\frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1+2i)(z+1-2i)}$$

$\therefore$ The integrand is not analytic at $z = -1 - 2i$ and $z = -1 + 2i$.

Of these, the point $z = -1 - 2i$ lies inside C. (Fig. 4.13)

Noting this, we rewrite the given integral as

$$\int_C \frac{\left(\dfrac{z+4}{z+1-2i}\right)}{z+1+2i}\,dz \equiv \int_C \frac{f(z)}{z-(-1-2i)}\,dz, \text{say.}$$

$f(z)$ is analytic on and inside C and the point $(-1-2i)$ lies inside C.

$\therefore$ By Cauchy's integral formula.

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\, f(-1-2i)$$

$$= 2\pi i\left\{\frac{-1-2i+4}{-1-2i+1-2i}\right\}$$

$$= -\frac{\pi}{2}(3-2i)$$

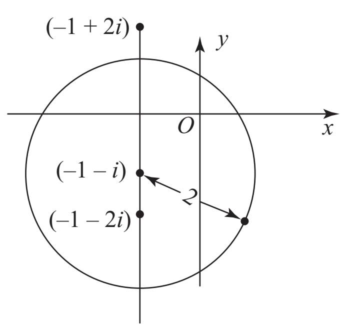


Fig. 4.13

Example 4.12 If $f(a)=\int_C \frac{3z^2+7z+1}{z-a}\,dz$, where C is the circle $|z|=2$, find the values of $f(3)$, $f'(1-i)$ and $f''(1-i)$

Note ☑ In Cauchy's integral formula, $f(z)$ was used to denote the numerator of the integrand and $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$, but in this problem $f(a)$ is used to denote the value of the integeral.

$$f(a)=\int_C \frac{\phi(z)}{z-a}\,dz, \text{ where } \phi(z) = 3z^2 + 7z + 1$$

$\therefore$ $$f(3)=\int_C \frac{\phi(z)}{z-3}\,dz \text{ and } \frac{\phi(z)}{z-3} \text{ is analytic on and inside } C.$$

$\therefore$ By Cauchy's integral theorem, $f(3) = 0$

Now $$f'(a)=\int_C \frac{\phi(z)}{(z-a)^2}\,dz$$

$$= \frac{2\pi i}{1!}\phi'(a), \text{ if the point '}a\text{' lies inside } C$$

(by Cauchy's integral formula)

The point $(1-i)$ lies within the circle C (Fig. 4.14).

$\therefore$ $$f'(1-i) = 2\pi i\left[\frac{d}{dz}(3z^2+7z+1)\right]_{z=1-i}$$

$$= 2\pi i\,[(6(1-i)+7] = 2\pi\,(6+13i)$$

Also $$f''(a) = 2\int_C \frac{\phi(z)}{(z-a)^3}\,dz = 2.\frac{2\pi i}{2!}\phi''(a)$$

$$= 2\pi i\left[\frac{d^2}{dz^2}(3z^2+7z+1)\right]_{z=1-i} = 12\pi i$$

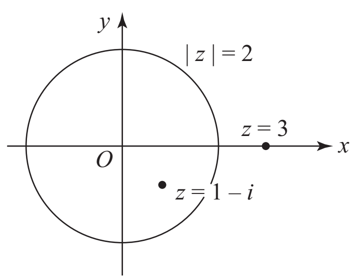


Fig. 4.14

EXERCISE 4(a)

Part A

(Short Answer Questions)

1. Define simply and multiply connected regions.
2. State Cauchy's integral theorem.

3. State Cauchy-Goursat theorem.
4. State Cauchy's extended integral theorem as applied to a multiply connected region.
5. State Cauchy's integral formula.
6. State Cauchy's extended integral formula, as applied to a doubly connected region.
7. State Cauchy's integral formula for the nth derivative of an analytic function.
8. Evaluate $\int_0^{3+i} |z|^2 \, dz$, along the lines $3y = x$.
9. Evaluate $\int (x^2 - iy^2)\, dz$ along the straight line from (0, 0) to (0, 1) and then from (0, 1) to (2, 1).
10. Evaluate $\int_C \bar{z}^2 dz,$ where C is circle $|z - 1| = 1$.
11. Evaluate $\int_i^{2-i} \left(3xy + iy^2\right) dz$ along the line joining the points $z = i$ and $z = 2 - i$.
12. Evaluate $\int_C \frac{1}{z} dz,$ where C is the semi-circular are $|z| = 1$ above the real axis.
13. Evaluate $\int_0^{1+i} \left(x^2 + iy\right) dz$ along the parabola (i) $y = x^2$ and (ii) $x = y^2$.
14. Evaluate $\int_C \log z \, dz,$ where C is the circle $|z| = 2$.
15. Evaluate $\int_C \bar{z} \, dz$ along the curve $z = t^2 + it$ from 0 to $4 + 2i$.

Part B

16. Evaluate $\int_C \left(5z^4 - z^3 + 2\right) dz$ around (a) the circle $|z| = 1$, (b) the square with vertices at (0, 0), (1, 0) (1, 1) and (0,1) (c) the parabola $y = x^2$ from (0, 0) to (1, 1) and then the parabola $y^2 = x$ from (1, 1) to (0, 0).
17. Evaluate $\int_C \left(x^2 - iy^2\right) dz$ (i) the parabola $y = 2x^2$ from (1, 1) to (2, 8), (ii) the straight lines from (1, 1) to (1, 8) and then from (1, 8) to (2, 8), (iii) the straight line from (1, 1) to (2, 8).
18. Evaluate $\int_C \left(z^2 + 1\right)^2 dz,$ along the arc of the cycloid $x = a\,(\theta - \sin\,\theta)$, $y = a\,(1 - \cos\,\theta)$ from the point $\theta = 0$ to the point $\theta = 2\pi$.

(***Hint:*** The end points are $z = 0$ and $z = 2\pi a$. The integrand is an analytic function and hence the value of the integral does not depend on the curve joining the end points.)

19. Evaluate $\int_C \frac{dz}{z-2-i}$, where C is the boundary of (i) the square bounded by the real and imaginary axes and the lines $x = 1$ and $y = 1$, (ii) the rectangle bounded by the real and imaginary axes and the lines $x = 3$ and $y = 2$, described in the counter clockwise sense.

Evaluate the following integrals using Cauchy's integral formula.

20. $\int_C \frac{\sinh 2z}{z^4}dz$ where C is the boundary of the square whose sides lie along $x = \pm 2$ and $y = \pm 2$, described in the positive sense.

21. $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)}dz$, where C is $|z| = 3$.

22. $\int_C \frac{z^2+1}{z^2-1}dz$, where C is the circle of unit radius with centre at (i) $z = 1$ and (ii) $z = i$.

23. $\int_C \frac{z^3+1}{z^2-3iz}dz$, where C is $|z| = 1$.

24. $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz$, where C is $|z| = 3$.

25. $\int_C \frac{e^{2z}}{(z+1)^4}dz$, where C is $|z| = 2$.

26. $\int_C \frac{dz}{\left(z^2+4\right)^2}$, where C is $|z - i| = 2$.

27. $\int_C \frac{z+4}{z^2+2z+5}dz$, where C is $|z + 1 - i| = 2$.

28. $\int_C \frac{z^3+z+1}{z^2-7z+6}dz$, where C is the ellipse $4x^2 + 9y^2 = 1$.

29. If $f(a) = \int_C \frac{4z^2+z+5}{z-a}dz$, where C is $|z| = 2$, find the values of $f(1)$, $f(i)$, $f'(-1)$ and $f''(-i)$.

30. If $f(z)$ is analytic on and inside a simple closed curve C and z_0 is a point not lying on C, show that $\oint_C \frac{f'(z)}{z-z_0}\mathrm{d}z = \oint_C \frac{f(z)\mathrm{d}z}{(z-z_0)^2}$.

4.3 SERIES EXPANSIONS OF FUNCTIONS OF COMPLEX VARIABLE-POWER SERIES

4.3.1 Power Series

A series of the form $\sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$ is called *a power series* in powers of $(z - a)$, where z is a complex variable, the constants $c_0, c_1, c_2, \ldots$ are called the *coefficients* and the constant a is called the *centre* of the series.

Most of the definitions and theorems relating to convergence of infinite series of real terms, with which the reader is familiar, hold good for series of complex terms also.

It can be proved that there exists a positive number R such that the power series given above converges for $|z - a| < R$ and diverges for $|z - a| > R$, while it may or may not converge for $|z - a| = R$. This means that the power series converges at all points inside the circle $|z - a| = R$, diverges at all points outside the circle and may or may not converge on the circle. Due to this interpretation. R is called the *radius of convergence* of the above series and the circle $|z - a| = R$ is called the *circle of convergence*.

Power series play an important role in complex analysis, since they represent analytic functions and conversely every analytic function has a power series representation, called *Taylor series* that are similar to Taylor series in real calculus.

Analytic functions can also be represented by another type of series, called *Laurent's series*, which consist of positive and negative integral powers of the independent variable. They are useful for evaluating complex and real integrals, as will be seen later.

4.3.2 Taylor's Series (Taylor's Theorem)

If $f(z)$ is analytic inside a circle C_0 with centre at 'a' and radius r_0, then at each point z inside C_0,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots \infty$$

Proof

Let C_1 be any circle with centre at a and radius $r_1 < r_0$, containing the point z (Fig. 4.15). Let w be any point on C_1. Then, by Cauchy's integral formula,

$$f(z)=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,\mathrm{d}w \qquad (1)$$

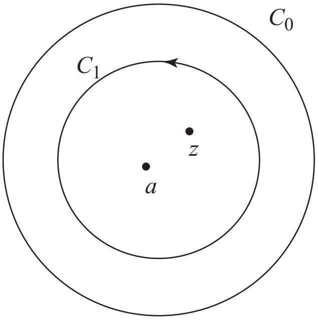


Fig. 4.15

and $$f^{(n)}(a)=\frac{n!}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w \qquad (2)$$

Now $$\frac{1}{w-z}=\frac{1}{(w-a)-(z-a)}$$

$$=\frac{1}{(w-a)}\left\{1-\left(\frac{z-a}{w-a}\right)\right\}^{-1}$$

$$=\frac{1}{w-a}\left[\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}\right\}\right.$$

$$\left.+\left(\frac{z-a}{w-a}\right)^n\cdot\frac{1}{1-\left(\frac{z-a}{w-a}\right)}\right]$$

$$\left(\because 1+\alpha+\alpha^2+\cdots\alpha^{n-1}=\frac{1-\alpha^n}{1-\alpha}\text{ and so}\right.$$

$$\left.1+\alpha+\alpha^2+\cdots\alpha^{n-1}+\frac{\alpha^n}{1-\alpha}=\frac{1}{1-\alpha}\right)$$

i.e. $$\frac{1}{w-z}=\frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\cdots+\frac{(z-a)^{n-1}}{(w-a)^n}+\left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z}$$

$$\therefore \quad \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,\mathrm{d}w=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)}\,\mathrm{d}w+(z-a)\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^2}\,\mathrm{d}w$$

$$+\cdots+(z-a)^{n-1}\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^n}\,\mathrm{d}w+R_n,$$

where $$R_n=\frac{1}{2\pi i}\int_{C_1}\left(\frac{z-a}{w-a}\right)^n.\frac{f(w)}{(w-z)}\,\mathrm{d}z$$

i.e. $$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\ldots$$

$$+\frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1}+R_n,\text{by}(1)\text{and}(2) \qquad (3)$$

Let $|z-a|=r$ and $|w-a|=r_1$.

$\therefore$ $$|w - z| = |\,(w - a) - (z - a)|$$
$$\geq r_1 - r$$

Let M be the maximum value of $|f(w)|$ on C_1

Then $$\left|R_n\right| \leq \frac{r^n}{2\pi}\int_0^{2\pi}\frac{M}{r_1 - r}\frac{r_1 d\theta}{r_1^{\,n}}\left(\because w = r_1 e^{i\theta}\right)$$

i.e. $$= \frac{r_1 M}{r_1 - r}\left(\frac{r}{r_1}\right)^n$$

Since $$\frac{r}{r_1} < 1, \lim_{n\to\infty}\left(\frac{r}{r_1}\right)^n = 0 \therefore \left|Rn\right| \to 0 \text{ and so } Rn \to 0 \text{ as } n \to \infty.$$

Taking limits on both sides of (3) as n $\to \infty$, we get

$$f(z) = f(a) + \frac{f'(a)}{1!}(z - a) + \frac{f''(a)}{2!}(z - a)^2 + \cdots\infty \tag{4}$$

Note ☑ For any point z inside C_0, we can always find C_1. So the Taylor series representation of $f(z)$ is valid for any z inside C_0.

The largest circle with centre at 'a' such that $f(z)$ is analytic at every point inside it is the circle of convergence of the Taylor's series and its radius is the radius of convergence of the Taylor's series. Clearly the radius of convergence is the distance between 'a' and the nearest singularity of $f(z)$.

2. Putting $a = 0$ in the Taylor's series, we get

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots\infty \tag{5}$$

This series is called *the Maclaurin's series* of $f(z)$.

3. The Maclaurin's series of some elementary functions, which can be derived by using (5), are given below:

(i) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots, \text{when } |z| < \infty.$

(ii) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!}\ldots, \text{when } |z| < \infty.$

(iii) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}\ldots, \text{when } |z| < \infty.$

(iv) $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!}\ldots+, \text{when } |z| < \infty.$

(v) $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!}\ldots, \text{when } |z| < \infty.$

(vi) $(1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots$, when $|z| < 1$

(vii) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \ldots$, when $|z| < 1$.

4.3.3 Laurent's Series (Laurent's Theorem)

If C_1 and C_2 are two concentric circles with centre at *'a'* and radii r_1 and r_2 ($r_1 > r_2$) and if $f(z)$ is analytic on C_1 and C_2 and throughout the annular region R between them, then at each point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

where
$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w, n = 0, 1, 2, \ldots$$

and
$$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, n = 1, 2, \ldots$$

Proof

By Extension of Cauchy's integral formula to a doubly connected region,

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-z)} \mathrm{d}w - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z} \mathrm{d}w \,[\text{Fig. } 4.16] \tag{1}$$

$$= I_1 + I_2, \text{ say}$$

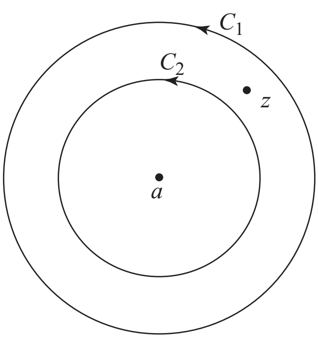


Fig. 4.16

In I_1, $\dfrac{1}{w-z} = \dfrac{1}{(w-a)-(z-a)}$

$$= \frac{1}{w-a}\left\{1 - \left(\frac{z-a}{w-a}\right)\right\}^{-1}$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \frac{(z-a)^n}{(w-a)^n}\frac{1}{w-z} \tag{2}$$

since $|z-a| < |w-a|$ when w is on C_1

In I_2,
$$-\frac{1}{w-z} = \frac{1}{(z-a)-(w-a)}$$

$$= \frac{1}{(z-a)}\left\{1 - \left(\frac{w-a}{z-a}\right)\right\}^{-1}$$

$$= \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \cdots + \frac{(w-a)^{n-1}}{(z-a)^n} + \frac{(w-a)^n}{(z-a)^n (z-w)}, \tag{3}$$

since $|w-a| < |z-a|$, when w is on C_2.

Using the expansion (2) in I_1, we have

$$I_1 = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2}\,dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^n}\,dw + R_n$$

$$= a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + R_n,$$

where
$$R_n = \frac{(z-a)^n}{2\pi i}\oint_{C_1} \frac{f(w)\,dw}{(w-a)^n (w-z)} \tag{4}$$

Using the expansion (3) in I_2, we have

$$I_2 = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{z-a}\,dw + \frac{1}{2\pi i}\oint_{C_2} \frac{(w-a)}{(z-a)^2} f(w)\,dw + \cdots$$

$$\cdots + \frac{1}{2\pi i}\oint_{C_2} \frac{(w-a)^{n-1} f(w)}{(z-a)^n}\,dw + S_n.$$

$$= b_1(z-a)^{-1} + b_2(z-a)^{-2} + \cdots + b_n(z-a)^{-n} + S_n$$

where
$$S_n = \frac{1}{2\pi i (z-a)^n}\oint_{C_2} \frac{(w-a)^n f(w)}{z-w}\,dw \tag{5}$$

If $|z-a| = r$, then $r_2 < r < r_1$.

$R_n \to 0$ as $n \to \infty$, as in Taylor's theorem.

Let M be the maximum value of $|f(w)|$ on C_2.

Then $$|S_n| \le \frac{1}{2\pi r^n}\int_0^{2\pi} \frac{r_2^n n}{r - r_2} r_2\, d\theta \left(\because \text{on } C_2, w = r_2 e^{i\theta}\right)$$

i.e $$= \frac{M r_2}{r - r_2}\left(\frac{r_2}{r}\right)^n$$

$\therefore$ $|S_n| \to 0$ and so $S_n \to 0$ as $n \to \infty$, since $r_2 < r$.

Using (4) and (5) in (1) and taking limits as $n \to \infty$, we have $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$.

Note ☑

1. If $f(z)$ is analytic inside C_2, then the Laurent's series reduces to the Taylor series of $f(z)$ with centre a, since in this case all the coefficients of negative powers in Laurent's are zero.
2. As the Taylor's and Laurent's expansions in the given regions are unique, they are not usually found by the theorems given above, but by other simpler methods such as use of binomial series.
3. The part $\sum_{n=0}^{\infty} a_n (z-a)^n$, consisting of positive integral powers of $(z-a)$, is called *the analytic part* of the Laurent's series, while $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$, consisting of negative integral powers of $(z-a)$ is called the *principal part* of the Laurent's series.

4.4 CLASSIFICATION OF SINGULARITIES

We have stated, in Chapter 3, that a point at which $f(z)$ is not analytic is called a *singular point or singularity* of $f(z)$. We now consider various types of singularities.

1. Isolated Singularity

The point $z = a$ is called an *isolated singularity* of $f(z)$, if there is no other singularity in its neighbourhood. In other words, $z = a$ is called an isolated singularity of $f(z)$, if we can find a $\delta > 0$ such that the circle $|z-a| = \delta$ encloses no singularity other than a. If we cannot find any such δ, then 'a' is called a non-isolated singularity.

[If *'a'* is not a singularity and we can find $\delta > 0$ such that $|z-a| = \delta$ encloses no singularity, then *'a'* is called a *regular point* or *ordinary point* of $f(z)$.]

Note ☑

1. If a function has only a finite number of singularities in a region, those singlarities are necessarily isolated.

For example, $z = 1$ is an isolated singularity of $f(z) = \dfrac{1}{(z-1)^2}$ and $z = 0$ and $\pm\, i$ are isolated singularities of $f(z) = \dfrac{z+1}{z^3(z^2+1)}$.

2. If *'a'* is an isolated singularity of $f(z)$, then $f(z)$ can be expanded in Laurent's series valid throughout some neighbouhood of $z = a$ (except at $z = a$ itself), i.e., valid in $0 < |z-a| < r_1$. Here r_2 is chosen arbitrarily small.

2. Pole

If $z = a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's expansion of $f(z)$ at $z = a$ valid in $0 < |z-a| < r_1$ has only a finite number of terms, then $z = a$ is called a *pole*.

i.e. if in $$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$
$b_m \neq 0$, $b_{m+1}=0=b_{m+2}\ldots$, then $z = a$ is called a *pole of order m*.
A pole of order 1 is called a *simple pole*.

For example, $z = 0$ is a simple pole of $f(z)=\dfrac{1}{z(z-1)^2}$, as the Laurent's expansion.

of $f(z)$ valid in $0 < |z| < 1$ is given by $f(z)=\dfrac{1}{z}(1-z)^{-2}$

i.e. $$f(z)=\frac{1}{z}\left(1+2z+3z^2+\cdots\infty\right)$$
$$=\left(2+3z+4z^2+\cdots\infty\right)+\frac{1}{z}$$

It has only one term i.e. $1/z$ in the principal part.
Similarly, $z = 1$ is a pole of order 2 of $f(z)$, as the Laurent's expansion of $f(z)$, valid in $0 < |z - 1| < 1$, is given by

$$f(z)=\frac{1}{(z-1)^2}\{1+(z-1)\}^{-1}$$
$$=\frac{1}{(z-1)^2}\left\{1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right\}$$
$$=\left\{1-(z-1)+(z-1)^2-\cdots\infty\right\}+\left\{-\frac{1}{z-1}+\frac{1}{(z-1)^2}\right\}$$

Here $b_2 \neq 0$ and $b_3 = b_4 = \cdots = 0$.

3. Essential Singularity

If $z = a$ is an isolated singularity of $f(z)$ such that principal part of the Laurent's expansion of $f(z)$ at $z = a$, valid in $0 < |z - a| < r_1$, has an infinite number of terms, then $z = a$ is called an *essential singularity*.

For example, $z = 1$ is an essential singularity of $f(z) = e^{1/(z-1)}$, as the Laurent's

expansion is given by $f(z)=1+\dfrac{1}{1!}\dfrac{1}{z-1}+\dfrac{1}{2!}\dfrac{1}{(z-1)^2}+\dfrac{1}{3!(z-1)^3}+\cdots\infty.$

4. Removable Singularity

If a single-valued function $f(z)$ is not defined at $z = a$, but $\left[\lim\limits_{z\to a} f(z)\right]$ exists, then $z = a$ is called a *removable singularity*. For example, $z=0$ is a removable singularity of

$f(z) = \dfrac{\sin z}{z}$, as $f(0)$ is not defined, but $\lim_{z \to 0}\left(\dfrac{\sin z}{z}\right) = 1$. The Laurent's expansion of

f (z) is given by

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \infty\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \infty$$

Note ☑

1. If $f(z)$ has a pole of order m at $z = a$, then its Laurent's expansion is

$$f(z). = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m},$$

where $b_m \neq 0$.

$$= \frac{1}{(z-a)^m}\left[\sum_{n=0}^{\infty} a_n (z-a)^{n+m} + b_m + b_{m-1}(z-a) + \cdots + b_1 (z-a)^{m-1}\right]$$

$$= \frac{1}{(z-a)^m}\phi(z), \text{say.}$$

Clearly, $\phi(z)$ is analytic everywhere that includes $z = a$ and $\phi(a) = b_m \neq 0$.

Thus for a function of the form $\dfrac{\phi(z)}{(z-a)^m}$, $z = a$ is a pole of order m, provided that $\phi(z)$ is analytic everywhere and $\phi(a) \neq 0$.

2. A function $f(z)$ which is analytic everywhere in the finite z-plane is called an *entire function* or *integral function*. An entire function can be represented by a Taylor's series whose radius of convergence is ∞ and conversely a power series whose radius of convergence is ∞ represents an entire function. The functions e^z, sin z, cosh z are examples of entire functions.

3. A function $f(z)$ which is analytic everywhere in the finite plane except at a finite number of poles is called a *meromorphic function*.

For example, $f(z) = \dfrac{1}{z(z-1)^2}$ is a mermorphic function, as it has only two poles —a simple pole at $z = 0$ and a double pole at $z = 1$.

4.4.1 Residues and Evaluation of Residues

If *'a'* is an isolated singularity of any type for the function $f(z)$, then the coefficient of $\dfrac{1}{z-a}$ (viz. b_1) in the Laurent's expansion of $f(z)$ at $z = a$ valid in $0 < |z - a| < r_1$ is called *the residue* of $f(z)$ at $z = a$.

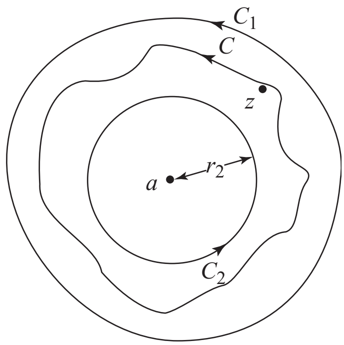


Fig. 4.17

We know, from Laurent's theorem, that $b_1 = \dfrac{1}{2\pi i}\oint_{C_2} f(w)\mathrm{d}w$, where C_2 is any circle $|z-a| = r_2 < r_1$, described in the anticlockwise sense. [Fig. 4.17]

Now if C is any closed curve around *'a'* such that $f(z)$ is analytic on and inside C except at *'a'* itself, then by Extension of Cauchy's integral theorem,

$$\oint_{C_2} f(w)\mathrm{d}w = \oint_{C} f(w)\mathrm{d}w.$$

Hence the residue of $f(z)$ *at* $z = a$ is also given by $[\text{Res.} f(z)]_{z=a} = \dfrac{1}{2\pi i}\oint_{C} f(z)\mathrm{d}z$, where C is any closed curve around *'a'* such that $f(z)$ is analytic on and inside it except at $z = a$ itself.

4.4.2 Formulas for the Evaluation of Residues

1. If $z = a$ is a simple pole of $f(z)$, then

$$\left[\text{Res.} f(z)\right]_{z=a} = \lim_{z\to a}\left\{(z-a)f(z)\right\}.$$

Since $z = a$ is a simple pole of $f(z)$, then the Laurent's expansion of $f(z)$ is of the following form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$$

$$\therefore \qquad (z-a)f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1$$

$$\therefore \qquad \lim_{z\to a}\left\{(z-a)f(z)\right\} = b_1 = \left[\text{Res.} f(z)\right]_{z=a}.$$

2. If $z = a$ is a simple pole of $f(z) = \dfrac{P(z)}{Q(z)}$ then [Ref.$f(z)]_{z=a}$

$$= \lim_{z \to a}\left(\frac{P(z)}{Q'(z)}\right).$$

By the previous formula,

$$[\text{Res}.f(z)]_{z=a} = \lim_{z \to a}\left[\frac{(z-a)P(z)}{Q(z)}\right] \to \frac{0}{0} \text{ form}$$

$$= \lim_{z \to a}\left[\frac{(z-a)P'(z)+P(z)}{Q'(z)}\right], \text{ by L'Hospital's rule}$$

$$= \frac{P(a)}{Q'(a)} \quad \text{or} \quad \lim_{z \to a}\left(\frac{P(z)}{Q'(z)}\right)$$

3. If $z = a$ is a pole of order m of $f(z)$, then

$$\left[\text{Res}.f(z)\right]_{z=a} = \frac{1}{(m-1)!}\lim_{z \to a}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right]$$

Since $z = a$ is a pole of order m of $f(z)$, then the Laurent's expansion of $f(z)$ is of the following form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_{m-1}}{(z-a)^{m-1}} + \frac{b_m}{(z-a)^m}$$

$$\therefore (z-a)^m f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+m} + b_1 (z-a)^{m-1} + b_2 (z-a)^{m-2} + \cdots +$$

$$b_{m-1}(z-a) + b_m.$$

$$\therefore \lim_{z \to a}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right] = (m-1)!b_1$$

$$\therefore b_1 = \left[\text{Res}.f(z)\right]_{z=a} = \frac{1}{(m-1)!}, \lim_{z \to a}\left[D^{m-1}\left\{(z-a)^m f(z)\right\}\right]$$

where $\quad D \equiv \dfrac{d}{dz}.$

Note ☑ The residue at an essential singularity of $f(z)$ is found out using the Laurent's expansion of $f(z)$ directly.

4.4.3 Cauchy's Residue Theorem

If $f(z)$ is analytic on and inside a simple closed curve C, except for a finite number of singularities $a_1, a_2, \ldots, a_n$ lying inside C and if $R_1, R_2, \ldots, R_n$ are the residues of $f(z)$ at these singularities respectively, then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i\left(R_1 + R_2 + \cdots + R_n\right).$$

Proof

We enclose the singularities a_i by simple closed curves C_i, which are not overlapping.
$f(z)$ is analytic in the multiply connected region bounded by the outer curve C and the inner curves $C_1, C_2, \ldots C_n$. (Fig. 4.18)
$\therefore$ By Extension of Cauchy's Integral theorem, we have

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z + \oint_{C_2} f(z)\,\mathrm{d}z + \cdots \oint_{C_n} f(z)\,\mathrm{d}z$$

$$= 2\pi i\left(R_1 + R_2 + \cdots + R_n\right), \text{ by the property of residue at a point.}$$

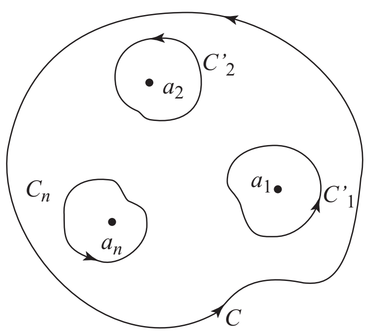


Fig. 4.18

Note ☑ Cauchy's integral formulas for $f(a)$ and $f^{(n)}(a)$ can be deduced from Cauchy's residue theorem.

Consider, $\int_C \frac{f(z)}{z-a}\,\mathrm{d}z$, where $f(z)$ is analytic on and inside C enclosing the point 'a'.

Now $\frac{f(z)}{z-a}$ has a simple pole at $z = a$ with residue $= \lim_{z\to a}\left\{(z-a)\frac{f(z)}{(z-a)}\right\} = f(a)$.

$\therefore$ By Cauchy's residue theorem, $\int_C \frac{f(z)}{(z-a)}\,\mathrm{d}z = 2\pi i f(a)$.

$\frac{f(z)}{(z-a)^{n+1}}$ has a ploe of order $(n + 1)$ at $z = a$

$$\text{with residue } = \frac{1}{n!}\lim_{z\to a} D^n\left[(z-a)^{n+1}\frac{f(z)}{(z-a)^{n+1}}\right]$$

$$= \frac{1}{n!} f^{(n)}(a)$$

$$\therefore \qquad \int_C \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z = 2\pi i \frac{f^{(n)}(a)}{n!}$$

or
$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z.$$

WORKED EXAMPLE 4(b)

Example 4.1 Expand each of the following functions in Taylor's series about the indicated point and also determine the region of convergence in each case:

(i) e^{2z} about $z = 2i$; (ii) $\cos z$ about $z = -\pi/2$;

(iii) $\dfrac{z+3}{(z-1)(z-4)}$ at $z = 2$; (iv) $\log\left(\dfrac{1+z}{1-z}\right)$ at $z = 0$.

(i) $e^{2z} = e^{2(z-2i)+4i} = e^{4i}.\ e^{2(z-2i)}$

Let $f(z) = e^{2(z-2i)}$. Then $f^{(n)}(z) = 2^n e^{2(z-2i)}$

$\therefore$ $f^{(n)}(2i) = 2^n;\ n = 1, 2, \ldots\infty.$

Also $f(2i) = 1.$

Taylor's series of $f(z)$ at $z = 2i$ is given by

$$f(z) = f(2i) + \frac{f'(2i)}{1!}(z-2i) + \frac{f''(2i)}{2!}(z-2i)^2 + \cdots$$

$$\therefore \qquad e^{2z} = e^{4i}\left[1 + \frac{2(z-2i)}{1!} + \frac{2^2(z-2i)^2}{2!} + \frac{2^3(z-2i)^3}{3!} + \cdots\right]$$

The series converges in a circle whose centre is $2i$ and radius is the distance between $2i$ and the nearest singularity of $f(z)$ which is ∞.

$\therefore$ The region of convergence of the Taylor's series in this case is $|z - 2i| < \infty$.

(ii) Let $f(z) = \cos z;\ f'(z) = -\sin z;\ f''(z) = -\cos z;$

$$f'''(z) = \sin z;\ f^{\text{iv}}(z) = \cos z, \text{ etc.}$$

$$\therefore \qquad f(-\pi/2) = 0;\ f'(-\pi/2) = 1;\ f''(-\pi/2) = 0;$$

$$f'''(-\pi/2) = -1;\ f^{\text{iv}}(-\pi/2) = 0, \text{etc.}$$

Taylor's series of $f(z)$ at $z = -\pi/2$ is given by

$$f(z) = f(-\pi/2) + \frac{f'(-\pi/2)}{1!}(z+\pi/2) + \frac{f''(-\pi/2)}{2!}(z+\pi/2)^2 + \cdots$$

$$\therefore \qquad \cos z = \frac{(z+\pi/2)}{1!} - \frac{(z+\pi/2)^3}{2!} + \frac{(z+\pi/2)^5}{5!} - \frac{(z+\pi/2)^7}{7!} + \cdots$$

The region of convergence is $|z + \pi/2| < \infty$.

(iii) $$f(z) = \frac{z+3}{(z-1)(z-4)}$$

The Taylor's series of $f(z)$ at $z = 2$ is a series of powers of $(z - 2)$.
Putting $z - 2 = u$ or $z = u + 2$, we have

$$f(z) = \frac{u+5}{(u+1)(u-2)} = \frac{-4/3}{u+1} + \frac{7/3}{u-2},$$

on resolving into partial fractions

$$= -\frac{4}{3}(1+u)^{-1} - \frac{7}{6}(1-u/2)^{-1}$$

$$= -\frac{4}{3}\left\{1 - u + u^2 - u^3 + \cdots\right\} - \frac{7}{6}\left\{1 + \frac{u}{2} + \frac{u^2}{2^2} + \cdots\right\}$$

$$= -\frac{4}{3}\sum_{n=0}^{\infty}(-1)^n u^n - \frac{7}{6}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\left\{\frac{4}{3}(-1)^{n+1} - \frac{7}{6}.\frac{1}{2^n}\right\}(z-2)^n$$

The Taylor's series expansion was obtained using binomial series, which are convergent in $|u| < 1$ and $|u/2| < 1$, i.e., in $|u| < 1$.
$\therefore$ The region of convergence of the Taylor's series of $f(z)$ is $|z - 2| < 1$.

Note ☑ The singularities of $f(z)$ are at $z = 1$ and $z = 4$ which lie outside the region of convergence.

(iv) $$\log\left(\frac{1+z}{1-z}\right) = \log(1+z) - \log(1-z)$$

Log $(1 \pm z)$ are many valued functions. We consider only those branches (values) which take the value zero when $z = 0$.
Now consider $f(z) = \log(1 + z)$

$$f'(z) = (1+z)^{-1};\ f''(z) = -(1+z)^{-2};\ f'''(z) = (-1)(-2)(1+z)^{-3} \text{ etc.}$$

$$\therefore \qquad f(0) = 0; f'(0) = 1; f''(0) = -1; f'''(0) = 2!, \text{ etc.}$$

Taylor's series of $f(z)$ at $z = 0$ [or Maclaurin's series of $f(z)$] is

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots$$

$$\therefore \qquad \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \qquad (1)$$

The only singularity of log $(1 + z)$ is -1 and its distance from the origin, i.e. the radius of convergence of series (1) is 1.

$\therefore$ The region of convergence of series (1) is given by $|z| < 1$.

Changing z into $-z$ in (1), the Taylor series of log $(1 - z)$ at $z = 0$ is given by

$$\log(1 - z) = -z - z^2/2 - z^3/3 - z^4/4 - \cdots$$

The region of convergence being $|z| < 1$

Using (1) and (2), we get

$$\log\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right), \text{ which converges for } |z| < 1.$$

Example 4.2 Expand each of the following functions in Laurent's series about $z = 0$. Identify the type of singularity also.

(i) ze^{-z^2} (ii) $(1 - \cos z)/z$ (iii) $z^{-1} e^{-2z}$ (iv) $\frac{1}{z^3} e^{z^2}$

(v) $(z - 1) \sin \frac{1}{z}$

(i)
$$ze^{-z^2} = z\left\{1 - \frac{z^2}{1!} + \frac{z^4}{2!} - \frac{z^6}{3!} + \cdots\right\}$$
$$= z - \frac{z^3}{1!} + \frac{z^5}{2!} - \frac{z^7}{3!} + \cdots \tag{1}$$

The Laurent's series (1) does not contain negative powers of z and the circle of convergence $|z| = \infty$ does not include any singularity $\therefore z = 0$ is an ordinary point of ze^{-z^2}

(ii)
$$\frac{1 - \cos z}{z} = \frac{1}{z}\left[1 - \left\{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right\}\right]$$
$$= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots \tag{2}$$

Though $z = 0$ appears to be a singularity of $\frac{1 - \cos z}{z}$, the Laurent's series of $\frac{1 - \cos z}{z}$ at $z = 0$ does not contain negative powers of z.

$\therefore z = 0$ is a removable singularity of $\frac{1 - \cos z}{z}$.

Note ☑ The value of $\left(\frac{1 - \cos z}{z}\right)$ at $z = 0$ is not defined; but $\lim_{z \to 0}\left(\frac{1 - \cos z}{z}\right) = 0$.

(iii)
$$z^{-1}e^{-2z} = \frac{1}{z}\left\{1 - \frac{2z}{1!} + \frac{2^2 z^2}{2!} - \frac{2^3 z^3}{3!} + \cdots\right\}$$
$$= \frac{1}{z} - 2 + \frac{2^2 z}{2!} - \frac{2^3 z^2}{3!} + \cdots \tag{3}$$

The principal part in the Laurent's series (3) contains the only term $\frac{1}{z}$.

$\therefore$ $z = 0$ is a simple pole of $z^{-1} e^{-2z}$.

(iv)
$$\frac{1}{z^3} e^{z^2} = \frac{1}{z^3}\left\{1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \frac{z^6}{3!} + \cdots\right\}$$
$$= \frac{1}{z^3} + \frac{1}{z} + \frac{z}{2!} + \frac{z^3}{3!} + \cdots \tag{4}$$

The last non-vanishing term with negative powers of z in the principal part of (4) is $\frac{1}{z^3}$.

$\therefore$ $z = 0$ is pole of order 3 for the given function.

(v)
$$(z-1)\sin\left(\frac{1}{z}\right) = (z-1)\left[\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots\right]$$
$$= 1 - \frac{1}{z} - \frac{1}{3!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^4} - \cdots \tag{5}$$

The principal part consists of an infinite number of terms in the Laurent's series (5).

$\therefore$ $z = 0$ is an essential singularity.

Example 4.3 Find the Laurent's series of $f(z) = \frac{1}{z(1-z)}$ valid in the region (i) $|z + 1| < 1$, (ii) $1 < |z + 1| < 2$ and (iii) $|z + 1| > 2$.

$f(z) = \frac{1}{z(1-z)}$; Since Laurent's series in powers of $(z + 1)$ are required, put $z + 1 = u$ or $z = u - 1$.

$$\therefore \qquad f(z) = \frac{1}{(u-1)(2-u)} = \frac{1}{u-1} + \frac{1}{2-u} \tag{1}$$

(i) Since the region of convergence of required Laurent's series is $|u| < 1$, the two terms in the R.H.S. of (1) should be rewritten as standard binomials whose first term is 1 and the numerator of the second term is u. Accordingly.

$$f(z) = -\frac{1}{1-u} + \frac{1}{2\left(1-\frac{u}{2}\right)} = -(1-u)^{-1} + \frac{1}{2}\left(1-\frac{u}{2}\right)^{-}$$
$$= -\sum_{n=0}^{\infty} u^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$
$$= \sum_{n=0}^{\infty}\left(-1 + \frac{1}{2^{n+1}}\right)(z+1)^n \tag{2}$$

The Laurent's expansion (2) is valid, if $|u| < 1$ and $|u| < 2$, i.e. $|z + 1| < 1$.

(ii) Since the region of convergence of the required Laurent's series is given by $|u| > 1$ and $|u| < 2$, the first term in the R.H.S. (1) should be re-written as a standard binomial in such a way that u occurs in the denominator of the second member and the second term as a standard binomial in such a way that u occurs in the numerator of the second member.

Accordingly,

$$f(z) = \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{1}{2\left(1-\frac{u}{2}\right)}$$

$$= \frac{1}{u}(1-1/u)^{-1} + \frac{1}{2}\left(1-\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z+1)^n \tag{3}$$

The Laurent's expansion (3) is valid, if

$$\left|\frac{1}{u}\right| < 1 \text{ and } \left|\frac{u}{2}\right| < 1 \text{ i.e., } |u| > 1 \text{ and } |u| < 2,$$

i.e. $1 < |z+1| < 2$.

(iii) Since the region of convergence of the required Laurent's series is $|u| > 2$, both the terms in the R.H.S. of (1) should be rewritten as standard binomials in which u occurs in the denominator of the second member. Accordingly,

$$f(z) = \frac{1}{u\left(1-\frac{1}{u}\right)} - \frac{1}{u\left(1-\frac{2}{u}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1-\frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} - \frac{1}{u}\sum_{n=0}^{\infty}\frac{2^n}{u^n}$$

$$= \sum_{n=0}^{\infty}(1-2^n)\frac{1}{(z+1)^{n+1}} \tag{4}$$

The Laurent's expansion (4) is valid, if

$$\left|\frac{1}{u}\right| < 1 \text{ and } \left|\frac{2}{u}\right| < 1, \text{ i.e., } |u| > 1 \text{ and } |u| > 2 \text{ i.e., } |z+1| > 2$$

Example 4.4 Find all possible Laurent's expansions of the function $f(z)=\dfrac{4-3z}{z(1-z)(2-z)}$ about $z=0$. Indicate the region of convergence in each case. Find also the residue of $f(z)$ at $z=0$, using the appropriate Laurent's series.

$$f(z)=\frac{4-3z}{z(1-z)(2-z)}=\frac{2}{z}+\frac{1}{1-z}+\frac{1}{2-z},$$

on resolving into partial fractions. Various Laurent's series about $z=0$ will be series of ascending and descending powers of z. They can be obtained by keeping $\dfrac{2}{z}$ unaltered and rewriting the other two terms as standard binomials in three different ways and expanding in powers of z.

Case (i)

$$\begin{aligned} f(z)&=\frac{2}{z}+(1-z)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}\\ &=\frac{2}{z}+\sum_{n=0}^{\infty}z^n+\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \qquad (1)\\ &=\frac{2}{z}+\sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right)z^n \text{ if } |z|<1 \text{ and } |z|<2, \text{ i.e. } |z|<1. \end{aligned}$$

But $|z|<1$ includes $z=0$, which is a singularity of $f(z)$. Laurent's series expansion is valid in an annular region which does not contain any singularity of the function expanded.

Hence the region of convergence of the Laurent's series, in this case, is $0<|z|<1$.

Case (ii)

$$\begin{aligned} f(z)&=\frac{2}{z}-\frac{1}{z}(1-1/z)^{-1}+\frac{1}{2}(1-z/2)^{-1}\\ &=\frac{2}{z}-\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}\\ &=\frac{2}{z}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \qquad (2) \end{aligned}$$

if $\left|\dfrac{1}{z}\right|<1$ and $\left|\dfrac{z}{2}\right|<1$, i.e. $|z|>1$ and $|z|<2$

i.e. $1<|z|<2$, which represents the region of convergence of series (2).

Case (iii)

$$f(z)=\frac{2}{z}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=\frac{2}{z}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

$$=\frac{2}{z}-\sum_{n=0}^{\infty}\left(1-2^n\right)\frac{1}{z^{n+1}} \qquad (3)$$

if $\left|\frac{1}{z}\right|<1$ and $\left|\frac{2}{z}\right|<1$, i.e. $|z|>1$ and $|z|>2$

i.e. $|z| > 2$ or $2 < |z| < \infty$, which represents the region of convergence of series (3).

Since $z = 0$ is an isolated singularity of $f(z)$, the residue at $z = 0$ is the coefficient of $\frac{1}{z}$ in the Laurent's expansion of $f(z)$,valid in the region $0 < |z| < 1$, i.e. in the series (1) above.

Thus $\qquad [\text{Res.} f(z)]_{z=0} = 2.$

Example 4.5 Find the residues of $f(z)=\frac{z^2}{(z-1)(z+2)^2}$ at its isolated singularities using Laurent's series expansions.

$$f(z)=\frac{z^2}{(z-1)(z+2)^2}=\frac{1/9}{z-1}+\frac{8/9}{z+2}-\frac{4/3}{(z+2)^2}$$, on resolving into partial fractions.

Both $z = 1$ and $z = -2$ are isolated singularities of $f(z)$.

To find the residue of $f(z)$ at $z = 1$, we have to expand $f(z)$ in series of powers of $(z - 1)$, valid in $0 < |z - 1| < r$ and find the coefficient of $\frac{1}{z-1}$ in it.

Thus $$f(z)=\frac{1/9}{z-1}+\frac{8/9}{3+(z-1)}-\frac{4/3}{\{3+(z-1)\}^2}$$

$$=\frac{1/9}{z-1}+\frac{8}{27}\left\{1+\frac{z-1}{3}\right\}^{-1}-\frac{4}{27}\left\{1+\frac{z-1}{3}\right\}^{-2}$$

$$=\frac{1/9}{z-1}+\frac{8}{27}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n}-\frac{4}{27}\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{(z-1)^n}{3^n} \qquad (1)$$

The expansion (1) is valid in the region $\left|\frac{z-1}{3}\right|<1$, i.e. $0<|z-1|<3$.

[Residue of $f(z)$ at z =1] = coefficient of $\frac{1}{(z-1)}$ in (1)

$$=\frac{1}{9}.$$

To find the residue of $f(z)$ at $z = -2$, we have to expand $f(z)$ in series of powers of $(z + 2)$, valid in $0 < |z + 2| < r$ and find the coefficient of $\dfrac{1}{z+2}$ in it.

Thus
$$f(z)=\frac{1/9}{(z+2)-3}+\frac{8/9}{z+2}-\frac{4/3}{(z+2)^2}$$
$$=-\frac{1}{27}\left\{1-\frac{z+2}{3}\right\}^{-1}+\frac{8/9}{z+2}-\frac{4/3}{(z+2)^2}$$
$$=-\frac{1}{27}\sum_{n=0}^{\infty}\frac{(z+2)^n}{3^n}+\frac{8/9}{z+2}-\frac{4/3}{(z+2)^2} \tag{2}$$

The expansion (2) is valid in the region
$$\left|\frac{z+2}{3}\right|<1, \text{i.e.} 0<|z+2|<3.$$

[Residue of $f(z)$ at $z = -2$] = Coefficient of $\dfrac{1}{z+2}$ in (2)
$$=\frac{8}{9}.$$

Example 4.6 Find the singularities of $f(z)=\dfrac{z^2+4}{z^3+2z^2+2z}$ and the corresponding residues.

The singularities of $f(z)$ are given by
$$z^3+2z^2+2z=0,$$
i.e
$$z\left(z^2+2z+2\right)=0 \text{ i.e, } z=0,-1\pm i$$

For these values of z, the numerator $(z^2 + 4)$ does not vanish. Also it is analytic everywhere.

$\therefore z=0,-1\pm i$ are simple poles of $f(z)$.

$$\left[\text{Res.} f(z)\right]_{z=0}=\left[z\cdot f(z)\right]_{z=0}=\left(\frac{z^2+4}{z^2+2z+2}\right)=2.$$
$$\left[\text{Res.} f(z)\right]_{z=-1+i}=\left[(z+1-i)f(z)\right]_{z=-1+i}=\left\{\frac{z^2+4}{z(z+1+i)}\right\}_{z=-1+i}$$
$$=\frac{2-i}{-1-i}=\frac{1}{2}(-1+3i)$$

Since $z = -1 - i$ is the conjugate of $z = -1 + i$,
$$\left[\text{Res.} f(z)\right]_{z=-1-i}=\frac{-1}{2}(1+3i)$$

Example 4.7 Identify the singularities of $f(z)=\dfrac{z^2}{(z-2)^2\left(z^2+9\right)}$ and also find the residue at each singularity. The singularities of $f(z)$ are given by

$$(z-2)^2\left(z^2+9\right)=0, \text{ i.e. } z=2, \pm i3.$$

Of these $z = 2$ is a double pole and $z = \pm\, i3$ are simple poles.

$$\left[\text{Res.} f(z)\right]_{z=2} = \frac{1}{1!}\left[\frac{d}{dz}(z-2)^2 f(z)\right]$$

$$= \left[\frac{d}{dz}\left(\frac{z^2}{z^2+9}\right)\right]_{z=2} = \left[\frac{18z}{\left(z^2+9\right)^2}\right]_{z=2} = \frac{36}{169}$$

$$\left[\text{Res.} f(z)\right]_{z=i3} = \left[(z-i3) f(z)\right]_{z=i3} = \left[\frac{z^2}{(z-2)^2 (z+i3)}\right]_{z=i3}$$

$$= \frac{-9}{i6(i3-2)^2} = -\frac{3i}{2(5+12i)}$$

$$= -\frac{3}{338}(12+5i)$$

$$\left[\text{Res.} f(z)\right]_{z=-i3} = \text{conjugate of } -\frac{3}{338}(12+5i)$$

$$= \frac{-3}{338}(12-5i)$$

Example 4.8 Find the singularities of $f(z) = \dfrac{z^2}{z^4+a^4}$ and also find the residue at each singularity. The singularities of $f(z)$ are given by $z^4 + a^4 = 0$, i.e., $z^4 = (-1)\, a^4 = e^{i(2r+1)\pi} \cdot a^4$

i.e. $\quad z = e^{i(2r+1)\pi/4}\, a$, where $r = 0, 1, 2, 3.$

$\therefore$ The singularities are $z = a \cdot e^{i\pi/4}, a \cdot e^{3\pi i/4}, a \cdot e^{5\pi i/4}$ and $a \cdot e^{7\pi i/4}$, all of which are simple poles.

$$\left[\text{Res.} f(z)\right]_{z=a \cdot e^{i\pi/4}} = \lim_{z \to e^{i\pi/4}} \left\{\frac{P(z)}{Q'(z)}\right\}, \text{as } f(z) \text{ is of the form } \frac{P(z)}{Q(z)}.$$

$$= \left(\frac{1}{4z}\right)_{z=ae^{i\pi/4}} = \frac{1}{4a} e^{-i\pi/4} = \frac{1}{4\sqrt{2}a}(1-i)$$

$$\left[\text{Res.} f(z)\right]_{z=ae^{i3\pi/4}} = \frac{1}{4a} e^{-i3\pi/4} = -\frac{1}{4\sqrt{2}a}(1+i)$$

$$\left[\text{Res.} f(z)\right]_{z=ae^{i5\pi/4}} = \frac{1}{4a} e^{-i5\pi/4} = \frac{1}{4\sqrt{2}a}(-1+i)$$

$$\left[\text{Res.} f(z)\right]_{z=ae^{i7\pi/4}} = \frac{1}{4a} e^{-i7\pi/4} = \frac{1}{4\sqrt{2}a}(1+i)$$

Example 4.9 Evaluate the following integrals, using Cauchy's residue theorem.

(i) $\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz$, where C is $|z| = 3$.

(ii) $\int \frac{e^z dz}{(z^2+\pi^2)^2}$, where C is $|z| = 4$.

(iii) $\int_C \frac{dz}{z \sin z}$, where C is $|z| = 1$

(i) $$\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz = \int_C f(z) dz, \text{ say.}$$

The singularities of $f(z)$ are $z = -1$ and $z = -2$, which are simple poles lying within the circle $|z| = 3$.

$$[\text{Res.} f(z)]_{z=-1} = \left\{\frac{\cos \pi z^2 + \sin \pi z^2}{z+2}\right\}_{z=-1} = -1$$

$$[\text{Res.} f(z)]_{z=-2} = \left\{\frac{\cos \pi z^2 + \sin \pi z^2}{z+1}\right\}_{z=-2} = -1$$

By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \times \text{sum of the residues}$$

$$= -4\pi i.$$

(ii) Let $$\int_C f(z) dz = \int_C \frac{e^z dz}{(z^2+\pi^2)^2} = \int_C \frac{e^z dz}{(z-i\pi)^2(z+i\pi)^2}$$

The singularities of $f(z)$ are $z = i\pi$ and $-i\pi$ each of which is a double pole of $f(z)$ and lies within the circle $|z| = 4$.

$$[\text{Res.} f(z)]_{z=i\pi} = R_1 = \frac{1}{1!}\left[\frac{d}{dz}(z-i\pi)^2 f(z)\right]_{z=i\pi}$$

$$= \left[\frac{d}{dz}\left\{\frac{e^z}{(z+i\pi)^2}\right\}\right]_{z=i\pi}$$

$$= \left[\frac{(z+i\pi)^2 e^z - 2e^z(z+i\pi)}{(z+i\pi)^4}\right]_{z=i\pi}$$

$$= \frac{e^{i\pi}(2i\pi-2)}{8i^3\pi^3} = \frac{(-1)(i\pi-1)}{-4i\pi^3} = \frac{1}{4\pi^3}(\pi+i)$$

$$\left[\text{Res. } f(z)\right]_{z=-i\pi} = R_2 = \frac{1}{4\pi^3}(\pi - i)$$

By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\left(R_1 + R_2\right)$$

$$= \frac{2\pi i}{4\pi^3} \times 2\pi = \frac{i}{\pi}$$

(iii) Let $\displaystyle\int_C \frac{\mathrm{d}z}{z\sin z} = \int_C f(z)\,\mathrm{d}z$

The singularity of $f(z)$ is given by

$$z\sin z = 0, \text{i.e.} z\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right\} = 0$$

i.e. $$z^2\left\{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right\} = 0$$

i.e. $z = 0$, which is a double pole of $f(z)$ lying within the circle $|z| = 1$.

$$\left[\text{Res. } f(z)\right]_{z=0} = \frac{1}{1!}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^2}{z\sin z}\right)\right\}_{z=0}$$

$$= \left(\frac{\sin z - z\cos z}{\sin^2 z}\right)_{z=0} \rightarrow \left(\frac{0}{0}\text{ form}\right)$$

$$= \left(\frac{z}{2\cos z}\right)_{z=0}, \text{ by L'Hospital's rule}$$

$$= 0$$

By Cauchy's residue theorem.

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \times 0 = 0.$$

Example 4.10 Evaluate the following integrals, using Cauchy's residue theorem.

(i) $\displaystyle\int_C \frac{(z+1)\,\mathrm{d}z}{z^2 + 2z + 4}$, where C is $|z + 1 + i| = 2$

(ii) $\displaystyle\int_C \frac{(12z - 7)\,\mathrm{d}z}{(z-1)^2(2z+3)}$, where C is $|z + i| = \sqrt{3}$

(iii) $\displaystyle\int_C \frac{\mathrm{d}z}{\left(z^2 + 9\right)^3}$, where C is $|z - i| = 3$

(i) Let $\int_C f(z)\,\mathrm{d}z = \int_C \frac{(z+1)\,\mathrm{d}z}{z^2+2z+4}$

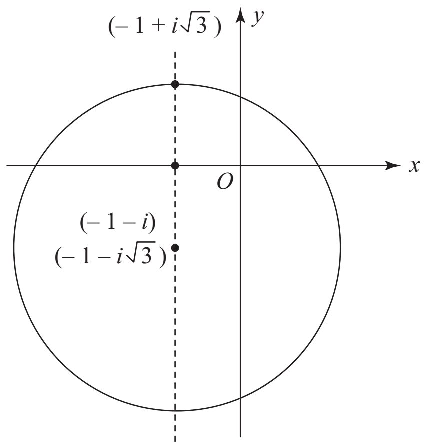


Fig. 4.19

The singularities of $f(z)$ are given by

$z^2 + 2z + 4 = 0$, i.e. $(z+1)^2 = -3$, i.e. $z = -1 \pm i\sqrt{3}$, each of which is a simple pole. Of the two poles $z = -1 - i\sqrt{3}$ alone lies inside the circle $|z-(-1-i)| = 2$, as shown in the Fig.4.19.

$$\{\text{Res.} f(z)\}_{z=-1-i\sqrt{3}} = \left\{\frac{z+1}{z+1-i\sqrt{3}}\right\}_{z=-1-i\sqrt{3}} = \frac{1}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \times \frac{1}{2} = \pi i$$

(ii) Let $\int_C \frac{(12z-7)}{(z-1)^2(2z+3)}\,\mathrm{d}z = \int_C f(z)\,\mathrm{d}z.$

The singularities of $f(z)$ are $z = 1$ and $z = -3/2$. $z = 1$ is a double pole and $z = -3/2$ is a simple pole. Only the pole $z = 1$ lies inside the circle $|z-(-i)| = \sqrt{3}$, as shown in the Fig. 4.20.

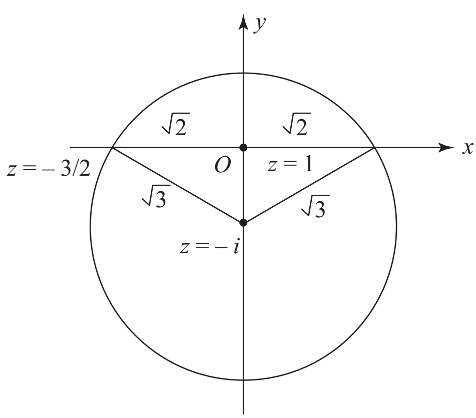


Fig. 4.20

$$[\text{Res.} f(z)]_{z=1} = \frac{1}{1!}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{12z-7}{2z+3}\right)\right\}_{z=1}$$

$$= \left[\frac{50}{(2z+3)^2}\right]_{z=1} = 2.$$

By Cauchy's residue theorem,

$$\int_C f(z)\mathrm{d}z = 2\pi i \times 2 = 4\pi i.$$

(iii) Let $\int_C f(z)\mathrm{d}z = \int_C \frac{\mathrm{d}z}{(z^2+9)^3}$

The singularities of $f(z)$ are $z = \pm 3i$, of which $z = 3i$ lies inside the circle $|z - i| = 3$, as shown in Fig. 4.21.

$z = 3i$ is a triple pole of $f(z)$.

$$\therefore \qquad [\text{Res.} f(z)]_{z=3i} = \frac{1}{2!}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{1}{(z+3i)^3}\right]_{z=3i}$$

$$= \frac{1}{2!}\left[\frac{12}{(z+3i)^5}\right]_{z=3i}$$

$$= \frac{6}{6^5 i^5} = \frac{1}{1296i}$$

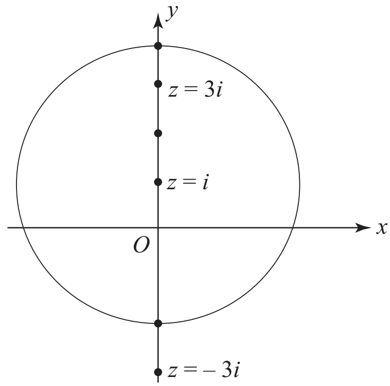


Fig. 4.21

By Cauchy's residue theorem,

$$\int_C f(z)\mathrm{d}z = 2\pi i \times \frac{1}{1296i} = \frac{\pi}{648}$$

EXERCISE 4(b)

Part A

(Short Answer Questions)

1. Define radius and circle of convergence of a power series.
2. State Taylor's theorem.
3. State Laurent's theorem.
4. What do you mean by analytic part and principal part of Laurent's series of a function of z?
5. Define regular point and isolated singularity of $f(z)$. Give one example for each.
6. Define simple pole and multiple pole of a function $f(z)$. Give one example for each.
7. Define essential singularity with an example.
8. Define removable singularity with an example.
9. Define entire function with an example.
10. Define meromorphic function with an example.
11. Define the residue of a function at an isolated singularity.
12. Express the residue of a function at an isolated singularity as a contour integral.

13. State two different formulas for finding the residue of a function at a simple pole.
14. State the formula for finding the residue of a function at a multiple pole.
15. State Cauchy's residue theorem.
16. Derive Cauchy's integral formula as a particular case of Cauchy's residue theorem.

Find the Taylor series for each of the following functions about the indicated point:

17. $\sin z$ about $z = \pi/4$
18. $\cos z$ about $z = \pi/3$
19. e^z about $z = -i$
20. e^{-z} about $z = 1$
21. $\dfrac{z-1}{z^2}$ about $z = 1$

If each of the following functions is expanded as a Taylor's series about the indicated point, find the region of convergence in each case, without actually expanding.

22. $\dfrac{z-1}{z+1}$ about $z = 0$
23. $\dfrac{\sin z}{z^2+4}$ about $z = 0$
24. $\sec \pi z$ about $z = 1$
25. $\dfrac{z+3}{(z-1)(z-4)}$ about $z = 2$
26. $\dfrac{e^z}{z(z-1)}$ about $z = 4i$

(***Hint:*** The radius of convergence is the distance between the centre of the Taylor's series and the nearest singularity of the concerned function.)

Find the Laurent's series of each of the following functions valid in the indicated regions:

27. $\dfrac{1}{z^2(z-2)}$, valid in $0 < |z| < 2$
28. $\dfrac{1}{z(z-1)}$, valid in $0 < |z-1| < 1$
29. $\dfrac{1}{z^3(1-z)}$, valid in $|z| > 1$
30. $\dfrac{z-1}{z^2}$, valid in $|z-1| > 1$

Find the residue at the essential singularity of each of the following functions, using Laurent's expansions:

31. $e^{1/z}$
32 $\dfrac{1-e^z}{z^2}$
33. $\dfrac{\cos z}{z}$
34. $\dfrac{\sinh z}{z^4}$
35. $\dfrac{1-\cosh z}{z^3}$

Find the residues at the isolated singularities of each of the following functions:

36. $\dfrac{z}{(z+1)(z-2)}$

37. $\dfrac{z^{2}}{z^2+a^2}$

38. $\cot z$ (at $z = 0$)

39. $\dfrac{ze^z}{(z-1)^2}$

40. $\dfrac{z\sin z}{(z-\pi)^3}$.

Evaluate the following integrals using Cauchy's residue theorem.

41. $\int_C \dfrac{z+1}{z(z-1)}\,dz$, where C is $|z| = 2$

42. $\int_C \dfrac{z^2}{z^2+1}\,dz$, where C is $|z| = 2$

43. $\int_C \dfrac{dz}{\sin z}\,dz$, where C is $|z| = 1$

44. $\int_C \tan z\,dz$, where C is $|z| = 2$

45. $\int_C \dfrac{e^{-z}}{z^2}dz$, where C is $|z| = 1$

Part B

46. Find the Taylor's series expansion of $f(z) = \dfrac{1}{z(1-z)}$ about $z = -1$. State also the region of convergence of the series.

47. Find the Taylor's series expansion of $f(z) = \dfrac{z}{z(z+1)(z+2)}$ about $z = i$. State also the region of convergence of the series.

48. Find the Laurent's series of $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ valid in the region (i) $|z| < 2$, (ii) $2 < |z| < 3$, and (iii) $|z| > 3$.

49. Find the Laurent's series of $f(z) = \dfrac{z}{(z-1)(z-2)}$, valid in the region (i) $|z + 2| < 3$, (ii) $3 < |z + 2| < 4$, and (iii) $|z + 2| > 4$.

50. Find all possible Laurent's expansions of the function $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$ about $z = -1$. Indicate the region of convergence in each case. Find also the residue of $f(z)$ at $z = -1$.

51. Find all possible Laurent's expansion of the function $f(z) = \dfrac{z^3 - 6z - 1}{(z-1)(z-3)(z+2)}$ about $z = 3$. Indicate the region of convergence in each case. Find also the residue of $f(z)$ at $z = 3$.

52. Find the residues of $f(z) = \dfrac{z^2}{(z-1)^2(z-2)}$ at its isolated singularities, using Laurent's series expansions.

53. By finding appropriate Laurent's expansions for $f(z) = \dfrac{1}{z^2(z^2+1)}$, find the residue at the poles of $f(z)$.

54. Indentify the singularities of $f(z)$ and find the corresponding residues, when

    (i) $f(z) = \dfrac{z}{z^2 + 2z + 5}$ (ii) $f(z) = \dfrac{1}{z^2(z^2 + 2z + 2)}$

55. Find the singularities of the following: (i) $f(z) = \dfrac{z}{(z+1)(z^2-1)}$

    (ii) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$; also find the corresponding residues.

56. Indentify the singularities of $f(z)$ and the corresponding residues, when

    (i) $f(z) = \dfrac{1}{z^4 + 16}$ (ii) $f(z) = \dfrac{z}{z^4 + a^4}$

57. Evaluate (i) $\displaystyle\int_C z e^{1/z} \mathrm{d}z$; and (ii) $\displaystyle\int_C \frac{\mathrm{d}z}{z^2 \sin z}$,

    using Cauchy's residue theorem, where C is the circle $|z| = 1$ in both cases.

58. Use Cauchy's residue theorem to evaluate the following

    (i) $\displaystyle\int_C \frac{\mathrm{d}z}{\sinh z}$, where C is the circle $|z| = 4$

    (ii) $\displaystyle\int_C \frac{z}{\cos z} \mathrm{d}z$, where C is the circle $\left|z - \dfrac{\pi}{2}\right| = \dfrac{\pi}{2}$

59. Use Cauchy's residue theorem to evaluate the following:

    (i) $\displaystyle\int_C \frac{z\mathrm{d}z}{(z-1)(z-2)^2}$, where C is circle $|z - 2| = 1/2$

    (ii) $\displaystyle\int_C \frac{(3z^2 + z)\mathrm{d}z}{(z-1)(z^2+9)}$, where C is the circle $|z - 2| = 2$

    (iii) $\displaystyle\int_C \frac{(3z^2 + z + 1)\mathrm{d}z}{(z^2-1)(z+3)}$, where C is the circle $|z - i| = 2$

60. Use Cauchy's residue theorem to evaluate the following:

(i) $\int_C \frac{(z-3)\mathrm{d}z}{z^2+2z+5}$, where C is the circle $|z+1-i|=2$

(ii) $\int_C \frac{\mathrm{d}z}{(z^2+4)^3}$, where C is the circle $|z-i|=2$

(iii) $\int_C \frac{(z-1)\mathrm{d}z}{(z+1)^2(z-2)}$, where C is the circle $|z+i|=2$

4.5 CONTOUR INTEGRATION—EVALUATION OF REAL INTEGRALS

The evaluation of certain types of real definite integrals can be done by expressing them in terms of integrals of complex functions over suitable closed paths or contours and applying Cauchy's residue theorem. This process of evaluation of definite integrals is known as contour integration.

We shall consider only three types of definite integrals which are commonly used in applications.

4.5.1 Type 1

Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\mathrm{d}x$, where $P(x)$ and $Q(x)$ are polynomials in x. This integral converges (exists), if

(i) The degree of $Q(x)$ is at least 2 greater than the degree of $P(x)$.

(ii) $Q(x)$ does not vanish for any real value of x.

To evaluate this integral, we evaluate $\int_C \frac{P(z)}{Q(z)}\mathrm{d}z$,

where C is the closed contour consisting of the real axis from $-R$ to $+R$ and the semicircle S above the real axis having this line as diameter, by using Cauchy's residue theorem and then letting $R \to \infty$. The contour C is shown in Fig. 4.22.

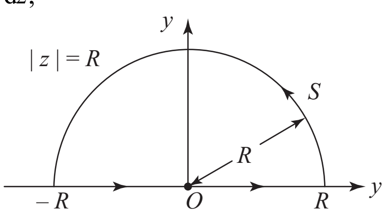


Fig. 4.22

Now we consider a result, known as Cauchy's Lemma, which will be used in evaluation of $\int_S f(z)\mathrm{d}z$, where S is the semicircle $|z| = R$ above the real axis.

Cauchy's Lemma

If $f(z)$ is a continuous function such that

$|zf(z)| \to 0$ uniformly as $|z| \to \infty$ on S, then $\int_S f(z)\mathrm{d}z \to 0$, as $R \to \infty$, where S is the semicircle $|z| = R$ above the real axis.

Proof

$$\left|\int_S f(z)\mathrm{d}z\right| = \left|\int_S zf(z)\cdot\frac{1}{z}\mathrm{d}z\right|$$

$$\leq \int_S |zf(z)|\cdot\frac{1}{|z|}\cdot|\mathrm{d}z|$$

i.e. $$\int_0^{\pi} |zf(z)|\cdot\frac{1}{R}\cdot R\mathrm{d}\theta$$

[$\because$ When $|z| = R$, $z = Re^{i\theta}$ and $\mathrm{d}z = Rie^{i\theta}\mathrm{d}\theta$]

$\therefore$ $$\lim_{R\to\infty}\left|\int_S f(z)\mathrm{d}z\right| \leq \pi\times\lim_{R\to\infty}|zf(z)|$$

i.e. ≤ 0, by the given condition.

$\therefore$ $$\left|\int_S f(z)\mathrm{d}z\right| = 0 \text{ and hence } \int_S f(z)\mathrm{d}z = 0, \text{as } R\to\infty.$$

Note ☑ Similarly, we can shown that $\int_{S_1} f(z)\mathrm{d}z \to 0$ as $r \to 0$, where S_1 is the semicircle $|z - a| = r$ above the real axis, provided that $f(z)$ is a continuous function such that $|(z - a) f(z)| \to 0$ as $r \to 0$ uniformly.

4.5.2 Type 2

Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)\sin mx}{Q(x)}\mathrm{d}x$ or $\int_{-\infty}^{\infty} \frac{P(x)\cos mx}{Q(x)}\mathrm{d}x$, where $P(x)$ and $Q(x)$ are polynomials in x.

This integral converges (exists), if

(i) $m > 0$
(ii) the degree of $Q(x)$ is greater than the degree of $P(x)$
(iii) $Q(x)$ does not vanish for any real value of x. To evaluate this integral, we evaluate $\int_C \frac{P(z)}{Q(z)}e^{imz}\mathrm{d}z$, where C is the same contour as in Type 1, by using Cauchy's residue theorem and letting $R\to\infty$. After getting $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}e^{imx}\mathrm{d}x$, we find the value of the real part or the imaginary part of this integral as per the requirement.

A result, known as Jordan's Lemma, which will be used in the evaluation of $\int_S f(z)e^{imz}\mathrm{d}z$, where S is the semicircle $|z| = R$ above the real axis, is given as follows.

Jordan's Lemma

If $f(z)$ is a continuous function such that $|f(z)| \to 0$ uniformly as $|z| \to \infty$ on S, then $\int_S e^{imz} f(z)\mathrm{d}z \to 0$ as $R \to \infty$, where S is he semicircle $|z| = R$ above the real axis and $m > 0$.

Proof

On the circle $|z|\ R,\ z = Re^{i\theta}$.

$$\therefore \quad \left|\int_S e^{imz} f(z)\mathrm{d}z\right| = \left|\int_S e^{imR(\cos\theta + i\sin\theta)}.f(z).iRe^{i\theta}\mathrm{d}\theta\right|$$

$$\le \int_0^{\pi} \left|e^{imR(\cos\theta+i\sin\theta)}\right| |f(z)|\, R\mathrm{d}\theta$$

$$\text{i.e.} \quad \le \int_0^{\pi} e^{-mR\sin\theta}\, |f(z)|\, R\mathrm{d}\theta$$

$$\text{i.e.} \quad \le 2\int_0^{\pi/2} e^{-mR\sin\theta}\, |f(z)|\, R\mathrm{d}\theta \tag{1}$$

$$\text{Now} \quad \sin\theta \ge \frac{2\theta}{\pi}, \text{for } 0 \le \theta \le \pi/2$$

$$\therefore \quad -mR\sin\theta \le -\frac{2mR\theta}{\pi}, \text{for } 0 \le \theta \le \pi/2$$

$$\therefore \quad e^{-mR\sin\theta} \le e^{-2mR\theta/\pi}, \text{ for } 0 \le \theta \le \pi/2 \tag{2}$$

Using (2) in (1), we have

$$\left|\int_S e^{imz} f(z)\mathrm{d}z\right| \le 2\int_0^{\pi/2} e^{-2mR\theta/\pi}\, |f(z)|\, R\mathrm{d}\theta$$

$$\therefore \quad \lim_{R\to\infty}\left|\int_S e^{imz} f(z)\mathrm{d}z\right| \le \int_0^{\pi/2} \lim_{R\to\infty}(2Re^{-2mR\theta/\pi}) \lim_{R\to\infty}[f(z)]\mathrm{d}\theta$$

$$\text{i.e.} \quad \le \lim_{R\to\infty}[f(z)] \times \lim_{R\to\infty}\int_0^{\pi/2} 2Re^{-2mR\theta/\pi}\mathrm{d}\theta$$

$$\text{i.e.} \quad \le \lim_{R\to\infty}[f(z)] \times \lim_{R\to\infty}\left[\frac{\pi}{m}\left(1 - e^{-mR}\right)\right]$$

i.e. $$\leq \frac{\pi}{m} \times 0, \text{ by the given condition.}$$

$\therefore$ $$\left| \int_S e^{imz} f(z) \mathrm{d}z \right| = 0 \text{ and hence } \int_S e^{imz} f(z) \mathrm{d}z = 0$$

as $R \to \infty$.

4.5.3 Type 3

Integrals of the form $\int_0^{2\pi} \frac{P(\sin\theta, \cos\theta)}{Q(\sin\theta, \cos\theta)} \mathrm{d}\theta$, where P and Q are polynomials in sin θ and cos θ.

To evaluate this integral, we take the unit circle $|z| = 1$ as the contour.

When $|z| = 1$, $z = e^{i\theta}$ and so $\sin\theta = \frac{z - z^{-1}}{2i}$ and $\cos\theta = \frac{z + z^{-1}}{2}$. Also $\mathrm{d}z = e^{i\theta} i \mathrm{d}\theta$ or $\mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$. When θ varies from 0 to 2π, the point z moves once around the unit circle $|z| = 1$.

Thus $$\int_0^{2\pi} \frac{P(\sin\theta, \cos\theta)}{Q(\sin\theta, \cos\theta)} \mathrm{d}\theta = \int_C f(z)\, \mathrm{d}z,$$

where $f(z)$ is a rational function of z and C is $|z| = 1$.

Now applying Cauchy's residue theorem, we can evaluate the integral on the R.H.S.

WORKED EXAMPLE 4(c)

Example 4.1 Evaluate $\int_{-\infty}^{\infty} \frac{x^2 \mathrm{d}x}{(x^2 + a^2)(x^2 + b^2)}$, using contour integration, where $a > b > 0$.

Consider $\int_C \frac{z^2 \mathrm{d}z}{(z^2 + a^2)(z^2 + b^2)}$, where C is the contour consisting of the segment of the real axis from $-R$ to $+R$ and the semicircle S above the real axis having this segment as diameter. The singularities of the integrand are given by

$$(z^2 + a^2)(z^2 + b^2) = 0$$

i.e. $z = \pm ia$ and $z = \pm ib$, which are simple poles.

Of these poles, only $z = ia$ and $z = ib$ lie inside C. (Fig. 4.23)

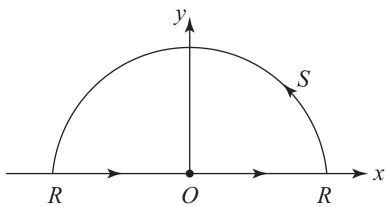


Fig. 4.23

Now $\quad R_1 = (\text{Res. of the integrand})_{z=ia} = \left[\dfrac{z^2}{(z+ia)(z^2+b^2)}\right]_{z=ia}$

$$= \frac{-a^2}{2ia(b^2-a^2)} = \frac{a}{2i(a^2-b^2)}$$

Similarly $\quad R_2 = \text{Res. at } (z = ib) = -\dfrac{b}{2i(a^2-b^2)}$

By Cauchy's residue theorem,

$$\int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz = 2\pi i(R_1+R_2)$$

$$= \frac{\pi(a-b)}{a^2-b^2} = \frac{\pi}{a+b}$$

i.e.,
$$\int_{-R}^{R} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} + \int_S \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)} = \frac{\pi}{a+b} \qquad (1)$$

($\because$ on the real axis, $z = x$ and so $dz = dx$)

Now $\quad |z^2+a^2| \geq |z|^2 - a^2 = R^2 - a^2$ on $|z| = R$.

Similarly $\quad |z^2+b^2| \geq R^2 - b^2$ on $|z| = R$.

$$\therefore \quad \left|z.\frac{z^2}{(z^2+a^2)(z^2+b^2)}\right| \leq \frac{R^3}{(R^2-a^2)(R^2-b^2)}$$

Since the limit of the R.H.S. is zero as $R \to \infty$,

$$\lim_{R\to\infty}\left|z.\frac{z^2}{(z^2+a^2)(z^2+b^2)}\right| = 0, \text{on } |z| = R.$$

$\therefore$ By Cauchy's lemma, $\displaystyle\int_S \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)} = 0 \qquad (2)$

Using (2) in (1) and letting $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$

Example 4.2 Evaluate $\int_{-\infty}^{\infty} \frac{x^2 \mathrm{d}x}{(x^2+1^2)(x^2+2x+2)}$, using contour integration.

Consider $\int_C \frac{z^2 \mathrm{d}z}{(z^2+1^2)(z^2+2z+2)}$, where C is the same contour as in the previous example. The singularities of $f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$ are given by $(z^2 + 1)^2 (z^2 + 2z + 2) = 0$

i.e. $z = \pm\, i$ and $z = -1 \pm i$, of which $z = i$ and $z = -1 + i$ lie inside C.

$z = i$ is double pole and $z = -1 + i$ is a simple pole.

$$R_1 = [\text{Res.} f(z)_{z=i} = \frac{1}{1!}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{(z-i)^2 \cdot z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\}\right]_{z=i}$$

$$= \left[\frac{(z+i)^2(z^2+2z+2).2z - z^2\{2(z+i)(z^2+2z+2)+(z+i)^2(2z+2)\}}{(z+i)^4(z^2+2z+2)^2}\right]_{z=i}$$

$$= \frac{9i-12}{100}$$

$$R_2 = [\text{Res.} f(z)]_{z=-1+i} = \left[\frac{z^2}{(z^2+1)^2(z+1+i)}\right]_{z=-1+i} = \frac{3-4i}{25}$$

By Cauchy's residue theorem,

$$\int_C \frac{z^2 \mathrm{d}z}{(z^2+1)^2(z^2+2z+2)} = 2\pi i(R_1 + R_2)$$

$$= 2\pi i\left\{\frac{9i-12}{100} + \frac{3-4i}{25}\right\}$$

$$= \frac{7\pi}{50}$$

i.e.
$$\int_{-R}^{R} \frac{x^2 \mathrm{d}x}{(x^2+1)^2(x^2+2x+2)} + \int_S \frac{z^2 \mathrm{d}z}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50} \qquad (1)$$

$$\lim_{R\to\infty}\left|\frac{z.z^2}{(z^2+1)^2(z^2+2z+2)}\right| = 0 \text{ on } |z| = R.$$

$\therefore$ By Cauchy's Lemma, when $R \to \infty$

$$\int_S \frac{z^2 \mathrm{d}z}{(z^2+1)^2 (z+2z+2)} = 0 \tag{2}$$

Letting $R \to \infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{x^2 \mathrm{d}x}{(x^2+1)^2 (x^2+2x+2)} = \frac{7\pi}{50}$$

Example 4.3 Evaluate $\int_0^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^3}$, using contour integration, where $a > 0$.

Consider $\int_C \frac{\mathrm{d}z}{(z^2+a^2)^3}$, where C is the same contour as in example 4.1.

The singularities of $f(z) = \frac{1}{(z^2+a^2)^3}$ are $z = \pm\, ia$,which are poles of order 3. Of the two poles, $z = ia$ alone lies inside C.

$$R = [\text{Res.} f(z)]_{z=ia} = \frac{1}{2!}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left\{\frac{1}{(z+ia)^3}\right\}\right]_{z=ia}$$

$$= \left[\frac{6}{(z+ia)^5}\right]_{z=ia} = \frac{3}{16a^5 i}$$

$\therefore$ By Cauchy's residue theorem,

$$\int_C \frac{\mathrm{d}z}{(z^2+a^2)^3} = 2\pi i R = 2\pi i \,.\, \frac{3}{16a^5 i} = \frac{3\pi}{8a^5}$$

i.e.
$$\int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+a^2)^3} + \int_S \frac{\mathrm{d}z}{(z^2+a^2)^3} = \frac{3\pi}{8a^5} \tag{1}$$

Now
$$\left| z.\frac{1}{(z^2+a^2)^3} \right| \le \frac{R}{(R^2-a^2)^3} \text{ and so}$$

$$\lim_{R\to\infty} \left| z.\frac{1}{(z^2+a^2)^3} \right| = 0 \text{ on } |z| = R$$

$\therefore$ By Cauchy's Lemma, when $R \to \infty$,

$$\int_S \frac{\mathrm{d}z}{(z^2+a^2)^3} = 0 \tag{2}$$

Letting $R \to \infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{8a^5}.$$

Since the integrand $\frac{1}{(x^2+a^2)^3}$ is an even function of x, we have $\int_0^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{16a^5}$

Example 4.4 Use contour integration to prove that

$$\int_0^{\infty} \frac{x^2 dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a}, \text{ where } a > 0.$$

Consider $\int_C \frac{z^2 dz}{z^4+a^4}$, where C is the same contour as in Example 4.1.

The singularities of $f(z) = \frac{z^2}{z^4+a^4}$ are given by $z^4 = -a^4 = e^{i(2r+1)\pi}.\ a^4$

i.e. $z = e^{i(2r+1)\pi/4}.\ a$, where $r = 0, 1, 2, 3$.

i.e. $z = ae^{i\pi/4}, ae^{i3\pi|4}, ae^{i5\pi/4}, ae^{i7\pi/4}$, all of which are simple poles.

Of these poles, only $z = ae^{i\pi/4}$ and $z = ae^{i3\pi/4}$ lie inside C.

$$R_1 = [\text{Res}.f(z)]_{z=aei\pi/4} = \lim_{z \to ae^{i\pi/4}} \left[\frac{z^2}{4z^3}\right]\left[= \lim_{z\to a} \frac{P(z)}{Q^1(z)}\right]$$

$$= \frac{1}{4a} e^{-i\pi/4} = \frac{1}{4\sqrt{2}a}(1-i)$$

$$R_2 = [\text{Res}.f(z)_{z=aei3\pi/4} = \frac{1}{4a} e^{-i3\pi/4} = \frac{1}{4\sqrt{2}a}(-1-i)$$

$\therefore$ By Cauchy's residue theorem,

$$\int_C \frac{z^2 dz}{z^4+a^4} = 2\pi i \cdot \frac{1}{4\sqrt{2}a}(1-i-1-i) = \frac{\pi}{\sqrt{2}a}$$

i.e. $$\int_{-R}^{R} \frac{x^2 dx}{x^4+a^4} + \int_S \frac{z^2 dz}{z^4+a^4} = \frac{\pi}{\sqrt{2}a} \tag{1}$$

Now $$\lim_{r\to\infty} \left| z.\frac{z^2}{z^4+a^4} \right| = 0 \text{ on } |z| = R.$$

$\therefore$ By Cauchy's Lemma, when $R \to \infty$,

$$\int_s \frac{z^2 dz}{z^4+a^4} = 0 \tag{2}$$

Letting $R \to \infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 + a^4} = \frac{\pi}{\sqrt{2}a}$$

Since the integrand is even,

$$\int_{0}^{\infty} \frac{x^2 dx}{x^4 + a^4} = \frac{\pi}{2\sqrt{2}a}$$

Example 4.5 Evaluate $\int_{-\infty}^{\infty} \frac{x^4}{x^6 - a^6} dx$, using contour integration where $a > 0$.

Consider $\int_C \frac{z^4}{z^6 - a^6} dz$, where C is the same contour as in example 4.1, with some modifications explained below.

The singularities of $f(z) = \frac{z^4}{z^6 - a^6}$ are given by

$$z^6 = a^6 = e^{i2r\pi} a^6, \text{ i.e. } z = ae^{\frac{r\pi}{3}i}, \; r = 0, 1, 2, 3, 4, 5.$$

i.e. $z = a, ae^{\pi i/3}, ae^{2\pi i/3}, -a, ae^{4\pi i/3}, ae^{5\pi i/3}$.

Of these singularities (simple poles),

$z = ae^{\pi i/3}$ and $z = ae^{2\pi i/3}$ lie inside C, but $z = a$ and $z = -a$ lie on the real axis. But for the evaluation of the integrals of type 1, no singularity of $f(z)$ should lie on the real axis. To avoid them, we modify C by introducing small indents, i.e., semi-circle of small radius at $z = \pm a$, as shown in Fig. 4.24.

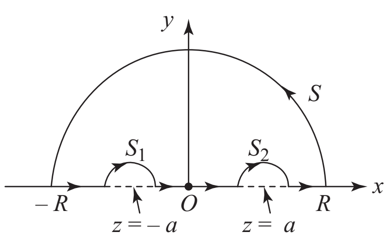


Fig. 4.24

Now the modified contour C contains only the simple poles

$$z = ae^{\pi i/3} \quad \text{and} \quad ae^{2\pi i/3}$$

$$R_1 = [\text{Res. } f(z)]_{z = ae^{\pi/3i}}$$

$$= \left(\frac{z^4}{6z^5}\right) = \frac{1}{6a} e^{-\pi i/3}$$

$$= \frac{1}{6a}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

Similarly, $$R_2 = \frac{1}{6a} e^{-2\pi i/3} = \frac{1}{6a}\left(-1/2 - i\frac{\sqrt{3}}{2}\right)$$

By Cauchy's residue theorem,

$$\int_C \frac{z^4}{z^6 - a^6} \mathrm{d}z = \frac{2\pi i}{6a}\left(1/2 - i\frac{\sqrt{3}}{2} - 1/2 - i\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{\sqrt{3}a}$$

i.e. $$\int_{-R}^{-a-r} \frac{x^4}{x^6 - a^6} \mathrm{d}x + \oint_{S_1} \frac{z^4}{z^6 - a^6} \mathrm{d}z + \int_{-a+r}^{a-r'} \frac{x^4}{x^6 - a^6} \mathrm{d}x + \oint_{S_2} \frac{z^4}{z^6 - a^6} \mathrm{d}z$$

$$+ \int_{a+r'}^{R} \frac{x^4}{x^6 - a^6} \mathrm{d}x + \oint_S \frac{z^4}{z^6 - a^6} \mathrm{d}z = \frac{\pi}{\sqrt{3}a} \qquad (1)$$

where r and r' are the radii of the semicircles S_1 and S_2 whose equations are $|z + a| = r$ and $|z - a| = r'$. These two integrals taken along S_1 and S_2 vanish as $r \to 0$ and $r' \to 0$, by the note under Cauchy's Lemma.

$$\int_S \frac{z^4}{z^6 - a^6} = 0 \text{, as } R \to \infty \text{, by Cauchy's lemma.}$$

Now, letting $r \to 0$, $r' \to 0$ and $R \to \infty$ in (1),

We get $$\int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\infty} \frac{x^4 \mathrm{d}x}{x^6 - a^6} = \frac{\pi}{\sqrt{3}a}$$

i.e. $$\int_{-\infty}^{\infty} \frac{x^4}{x^6 - a^6} \mathrm{d}x = \frac{\pi}{\sqrt{3}a}$$

Example 4.6 Evaluate $\int_0^\infty \frac{x \sin x}{x^2 + a^2} \mathrm{d}x$, by contour integration.

Consider $\int_C \frac{ze^{iz}}{z^2 + a^2} \mathrm{d}z$, where C is the same contour as in Example 4.1.

The singularities of $f(z) = \frac{ze^{iz}}{z^2 + a^2}$. are $z = \pm ia$, which are simple poles. Of these poles, only $z = ia$ lies inside C.

$$[\text{Res.} f(z)]_{z = ia} = \left(\frac{ze^{iz}}{z + ia}\right)_{z = ia} = \frac{1}{2} e^{-a}.$$

By Cauchy's residue theorem,

$$\int_C \frac{ze^{iz}}{z^2+a^2}\,dz = 2\pi i \,.\, \frac{1}{2}e^{-a} = \pi i e^{-a}$$

i.e. $$\int_{-R}^{R} \frac{xe^{ix}}{x^2+a^2}\,dx + \int_S \frac{ze^{iz}}{z+a^2}\,dz = \pi i e^{-a} \qquad (1)$$

Now $$\left|\frac{z}{z^2+a^2}\right| \le \frac{R}{R^2-a^2}$$

Since the limit of the R.H.S. is zero as $R \to \infty$,

$$\lim_{R\to\infty}\left|\frac{z}{z^2+a^2}\right| = 0 \text{ on } |z| = R.$$

$\therefore$ By Jordan's Lemma, $\displaystyle\int_S \frac{ze^{iz}}{z^2+a^2}\,dz \to 0$ as $R \to \infty$, (2)

Letting $R\to\infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}\,dx = i\pi e^{-a}$$

i.e. $$\int_{-\infty}^{\infty} \frac{x}{x^2+a^2}(\cos x + i\sin x)\,dx = i\pi e^{-a}.$$

Equating the imaginary parts on both sides,

we get $$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \pi e^{-a}.$$

Since the integrand is an even function of x,

$$\int_0^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-a}.$$

Example 4.7 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$., using contour integration, where a > b > 0.

Consider $\displaystyle\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz$, where C is the same contour as in Example 4.1.

The singularities of the integrand $f(z)$ are given by $z = \pm\, ia$ and $z = \pm\, ib$, which are simple poles. Of these poles, $z = ia$ and $z = ib$ only lie inside C.

$$R_1 = [\text{Res.} f(z)]_{z=ia} = \left[\frac{e^{iz}}{(z+ia)(z^2+b^2)}\right]_{z=ia}$$

$$= \frac{e^{-a}}{-2ia(a^2-b^2)}$$

$$R_2 = [\text{Res.} f(z)]_{z=ib} = \frac{e^{-b}}{2ib(a^2-b^2)}$$

By Cauchy's residue theorem,

$$\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,\mathrm{d}z = 2\pi i(R_1+R_2)$$

$$= \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

i.e.

$$\int_{-R}^{R} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x + \int_S \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,\mathrm{d}z$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \tag{1}$$

Now

$$\left|\frac{1}{(z^2+a^2)(z^2+b^2)}\right| \le \frac{1}{(R^2-a^2)(R^2-b^2)}$$

R.H.S. $\to 0$ as $R \to \infty$. Hence

$$\lim_{R\to\infty}\left|\frac{1}{(z^2+a^2)(z^2+b^2)}\right| = 0 \text{ on } |z| = R.$$

$\therefore$ By Jordan's Lemma, $\displaystyle\int_S \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,\mathrm{d}z \to 0$, as $R \to \infty$ (2)

Letting $R\to\infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x = \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Equating the real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x = \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Example 4.8 Evaluate $\int_0^\infty \frac{\cos ax}{(x^2+b^2)}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}$, where $a > 0$ and $b > 0$.

Consider $\int_C \frac{e^{iaz}}{(z^2+b^2)^2}\,dz$, where C is the same contour as in Example 4.1.

The singularities of $f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$ are $z = \pm\, ib$, which are double poles. Of these poles, only $z = ib$ lies inside C.

$$[\text{Res. } f(z)]_{z=ib} = \frac{1}{1!}\frac{d}{dz}\left[\left(\frac{e^{iaz}}{(z+ib)^2}\right)\right]_{z=ib}$$

$$= \left[\frac{(z+ib)ia\,e^{iaz} - 2e^{iaz}}{(z+ib)^3}\right]_{z=ib}$$

$$= \frac{1}{4ib^3}(ab+1)e^{-ab}.$$

By Cauchy's residue theorem,

$$\int_{-R}^{R} \frac{e^{iax}}{(x^2+b^2)^2}\,dx + \int_S \frac{e^{iaz}}{(z^2+b^2)^2}\,dz = 2\pi i \times \frac{1}{4ib^3}(ab+1)e^{-ab}$$

$$= \frac{\pi}{2b^3}(ab+1)e^{-ab} \qquad (1)$$

Now $$\left|\frac{1}{(z^2+b^2)^2}\right| \le \frac{1}{(R^2-b^2)^2}$$

Since the R.H.S.$\to 0$ as $R \to \infty$, L.H.S. also $\to 0$ as $R \to \infty$ on $|z| = R$.

$\therefore$ By Jordan's Lemma,

$$\int_S \frac{e^{iaz}}{(z^2+b^2)^2}\,dz \to 0 \text{ as } R \to \infty, \text{ since } a > 0 \qquad (2)$$

Letting $R \to \infty$ in (1) and using (2), we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b^3}(ab+1)e^{-ab}.$$

Equating the real parts on both sides and noting that $\dfrac{\cos ax}{\left(x^2+b^2\right)^2}$ is an even function of x, we get

$$\int_0^\infty \frac{\cos ax}{\left(x^2+b^2\right)^2}\,\mathrm{d}x = \frac{\pi}{4b^3}(ab+1)e^{-ab}.$$

Example 4.9 Use contour integration to prove that

$$\int_0^\infty \frac{\sin mx}{x}\,\mathrm{d}x = \frac{\pi}{2}, \text{ when } m > 0.$$

Consider $\int_C \dfrac{e^{imz}}{z}\,\mathrm{d}z$, where C is the usual semicircular contour, but with an indent i.e. a small semicircle at the origin, which is introduced to avoid the singularity $z = 0$, which lies on the real axis. The modified contour is shown in Fig. 4.25.

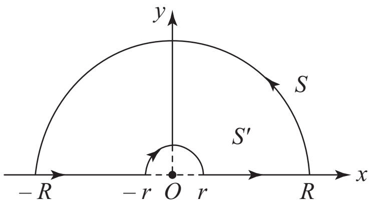


Fig. 4.25

The modified contour C does not include any singularity of $f(z) = \dfrac{e^{imz}}{z}$.

$\therefore$ By Cauchy's residue theorem,

$$\int_{-R}^{-r} \frac{e^{imx}}{x}\,\mathrm{d}x + \oint_{S'} \frac{e^{imz}}{z}\,\mathrm{d}z + \int_r^R \frac{e^{imx}}{x}\,\mathrm{d}x + \oint_S \frac{e^{imz}}{z}\,\mathrm{d}z = 0 \qquad (1)$$

The Eqn. of S' is $|z| = r$. $\therefore$ $z = re^{i\theta}$ and $\mathrm{d}z = re^{i\theta}\,i\mathrm{d}\theta$.

As S' is described in the clockwise sense, θ varies from π to 0.

Thus $$\int_{S'} \frac{e^{imz}}{z}\,\mathrm{d}z = \int_\pi^0 \frac{e^{imre^{i\theta}}}{re^{i\theta}} re^{i\theta}\,i\mathrm{d}\theta$$

$\therefore$ $$\lim_{r\to 0}\int_{S'} \frac{e^{imz}}{z}\,\mathrm{d}z = \int_\pi^0 \left[\lim_{r\to 0}\left(e^{imre^{i\theta}}\right)\right] i\mathrm{d}\theta = -i\pi \qquad (2)$$

$$\left|\frac{1}{z}\right| = \frac{1}{R} \to 0 \text{ as } R \to \infty \text{ on } |z| = R.$$

$\therefore$ By Jordan's Lemma, $\int_S \dfrac{e^{imz}}{z}\,\mathrm{d}z \to 0$ as $R \to \infty$, since $m > 0$ (3)

Letting $r \to 0$ and $R \to \infty$ in (1) and using (2) and (3),

we get $$\int_{-\infty}^{0} \frac{e^{imx}}{x} \mathrm{d}x - i\pi + \int_{0}^{\infty} \frac{e^{imx}}{x} \mathrm{d}x = 0$$

i.e. $$\int_{-\infty}^{\infty} \frac{e^{imx}}{x} \mathrm{d}x = i\pi$$

Equating the imaginary parts on both sides and noting that $\frac{\sin mx}{x}$ is an even function of x, we get $\int_{0}^{\infty} \frac{\sin mx}{x} \mathrm{d}x = \frac{\pi}{2}$.

Example 4.10 Evaluate $\int_{0}^{\infty} \frac{\sin x \, \mathrm{d}x}{x\left(x^2 + a^2\right)}$, using contour integrations where $a > 0$.

Consider $\int_{C} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z$, where C is the same modified contour as in Example 4.9, which includes the only pole $z = ia$ of $f(z) = \frac{e^{iz}}{z\left(z^2 + a^2\right)}$

$$[\text{Res.} f(z)]_{z=ia} = \left[\frac{e^{iz}}{z(z+ia)}\right]_{z=ia} = \frac{e^{-a}}{-2a^2}$$

By Cauchy's residue theorem,

$$\int_{-R}^{-r} \frac{e^{ix}}{x\left(x^2 + a^2\right)} \mathrm{d}x + \oint_{S'} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z + \int_{r}^{R} \frac{e^{ix}}{x\left(x^2 + a^2\right)} \mathrm{d}x + \oint_{S} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z$$

$$= -\frac{\pi i}{a^2} e^{-a} \tag{1}$$

$$\oint_{S'} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z = \int_{\pi}^{0} \frac{e^{ir e^{i\theta}} . re^{i\theta} i\mathrm{d}\theta}{re^{i\theta}\left(r^2 e^{i2\theta} + a^2\right)}$$

$\therefore$ $$\lim_{r \to 0} \int_{S'} \frac{e^{iz}}{z\left(z^2 + a^2\right)} \mathrm{d}z = \int_{\pi}^{0} \lim_{r \to 0} \left[\frac{e^{ir e^{i\theta}}}{r^2 e^{i2\theta} + a^2}\right] i\mathrm{d}\theta$$

$$= -\frac{\pi i}{a^2} \tag{2}$$

$$\left|\frac{1}{z\left(z^2 + a^2\right)}\right| \le \frac{1}{R\left(R^2 - a^2\right)}$$

Since R.H.S. $\to 0$ as $R \to \infty$ on $|z| = R$, L.H.S. also tends to 0 as $R \to \infty$.

$\therefore$ By Jordan's Lemma,

$$\oint_S \frac{e^{iz}}{z(z^2+a^2)}\,dz \to 0, \text{ as } R \to \infty \tag{3}$$

Letting $r \to 0$ and $R \to \infty$ in (1) and using (2) and (3), we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+a^2)}\,dx = \frac{\pi i}{a^2} - \frac{\pi i}{a^2} e^{-a}.$$

Equating imaginary parts on both sides and noting that $\dfrac{\sin x}{x(x^2+a^2)}$ is an even function of x, we get,

$$\int_0^{\infty} \frac{\sin x}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1 - e^{-a}).$$

Example 4.11 Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2x\sin\theta + x^2}$ $(0 < x < 1)$, using contour integration.

On the circle $|z| = 1$, $z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$ or $d\theta = \dfrac{dz}{iz}$ and $\sin\theta = \dfrac{z - \frac{1}{z}}{2i} = \dfrac{z^2-1}{2iz}$.

$\therefore$ The given integral $I = \displaystyle\int_C \frac{dz/iz}{1 - 2x\left(\frac{z^2-1}{2iz}\right) + x^2}$, where C is $|z| = 1$

i.e. $$I = \int_C \frac{dz}{iz - xz^2 + x + ix^2 z}$$

$$= -\frac{1}{x}\int_C \frac{dz}{z^2 - i\left(x + \frac{1}{x}\right)z - 1} = -\frac{1}{x}\int_C \frac{dz}{(z - ix)\left(z - \frac{i}{x}\right)} \tag{1}$$

The singularities of $\dfrac{1}{(z-ix)\left(z - \frac{i}{x}\right)}$ are $z = ix$ and $z = \dfrac{i}{x}$, which are simple poles

Now $|ix| = |x| < 1$, as $0 < x < 1$

$\therefore$ The pole $z = ix$ lies inside C, but $z = \dfrac{i}{x}$ lies outside C.

$$\left[\text{Res.}\left\{\frac{1}{(z-ix)(z-i/x)}\right\}\right]_{z=ix} = \frac{1}{i\left(x - \frac{1}{x}\right)} = \frac{ix}{1-x^2}$$

Using Cauchy's residue theorem, from (1) we get

$$I = -\frac{1}{x} \times 2\pi i \times \frac{ix}{1-x^2} = \frac{2\pi}{1-x^2}.$$

Note ☑ We have integrated w.r.t. θ and x is a parameter.

Example 4.12 Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} (a > b > 0)$, using contour integration.

Deduce the value of $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$. On the circle $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ and

$\cos\ \theta = \frac{z^2+1}{2z}$.

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{a+b\ \cos\theta} = \int_C \frac{dz/iz}{a+b\frac{z^2+1}{2z}} = -\frac{2i}{b}\int_C \frac{dz}{\left(z^2+\frac{2a}{b}z+1\right)} \tag{1}$$

where C is $|z| = 1$.

The singularities of the integrand are given by $z^2 + \frac{2a}{b}z + 1 = 0$,

i.e. $z = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$, which are simple poles.

Since $a > b, \frac{a}{b} > 1$ and hence $\left| -\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1} \right| > 1$ and so $z = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}$ lies outside $|z| = 1$.

$$z = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1} \text{ lies inside } |z| = 1.$$

$$\left[\text{Res. of}\left(\frac{1}{z^2+\frac{2a}{b}z+1}\right)\right]_{z=-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}}$$

$$= \left(\frac{1}{2\left(z+\frac{a}{b}\right)}\right)_{z=-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}} \tag{2}$$

$$= \frac{b}{2\sqrt{a^2-b^2}}$$

By Cauchy's residue theorem and using (2) in (1), we get

$$I = -\frac{2i}{b} \times 2\pi i \frac{b}{2\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

i.e. $$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} \qquad (3)$$

Differentiating both sides of (3) partially with respect to 'a', we get

$$\int_0^{2\pi} -\frac{d\theta}{(a + b\cos\theta)^2} = -\frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

$\therefore$ $$\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

Example 4.13 Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 3\cos\theta} d\theta,$ using contour integration.

On the circle $|z| = 1$, $z = e^{i\theta}$, $d\theta = \dfrac{dz}{iz}$, $\sin\theta = \dfrac{z^2 - 1}{2iz}$ and $\cos\theta = \dfrac{z^2 + 1}{2z}$.

$\therefore$ $$I = \int_0^{2\pi} \frac{\sin^2\theta}{5 - 3\cos\theta} d\theta = \int_C \frac{\dfrac{(z^2 - 1)^2}{-4z^2} \cdot \dfrac{dz}{iz}}{5 - \dfrac{3(z^2 + 1)}{2z}}, \text{ where } C \text{ is } |z| = 1$$

$$= -\frac{i}{6} \int_C \frac{(z^2 - 1)^2 \, dz}{z^2 (z - 3)(z - 1/3)} \qquad (1)$$

The singularities of the integrand which lie within C are $z = 0$ and $z = 1/3$.

$z = 0$ is a double pole and $z = \dfrac{1}{3}$ is a simple pole.

$$R_1 = [\text{Res. of the integrand } f(z)]_{z=0}$$

$$= \frac{1}{1!} \left[\frac{d}{dz} \left\{ \frac{(z^2 - 1)^2}{z^2 - \frac{10}{3} z + 1} \right\} \right]_{z=0} = \frac{10}{3}$$

$$R_2 = [\text{Res.} f(z)]_{z=1/3} = \left[\frac{(z^2 - 1)^2}{z^2 (z - 3)} \right]_{z=1/3} = -\frac{8}{3}$$

By Cauchy's residue theorem, from (1), we get

$$I = -\frac{i}{6} \times 2\pi i (R_1 + R_2) = \frac{\pi}{3}\left(\frac{10}{3} - \frac{8}{3}\right) = \frac{2\pi}{9}.$$

Example 4.14 Evaluate $\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2}$, using contour integration, where $a^2 < 1$.

On the circle $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$ and $\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{z^4+1}{2z^2}$.

$\therefore$ The given integral $I = \int_C \frac{\frac{z^4+1}{2z^2} \frac{dz}{iz}}{1 - a\left(\frac{z^2+1}{z}\right) + a^2}$, where C is $|z| = 1$

$$= \frac{i}{2a} \int_C \frac{(z^4+1)\,dz}{z^2\left\{z^2 - \left(a + \frac{1}{a}\right)z + 1\right\}}$$

$$= \frac{i}{2a} \int_C \frac{(z^4+1)\,dz}{z^2 (z-a)(z-1/a)} \qquad (1)$$

The singularities of the integrand which lie inside C are $z = 0$ and $z = a$ ($\because a^2 < 1$)
$z = 0$ is a double pole and $z = a$ is a simple pole.
$R_1 = [\text{Res. of the integrand}]_{z=0}$

$$= \frac{1}{1!}\left[\frac{d}{dz}\left\{\frac{z^4+1}{z^2 - (a+1/a)z + 1}\right\}\right]_{z=0}$$

$$= a + \frac{1}{a}$$

$$R_2 = (\text{Res. of the integrand})_{z=a} = \frac{a^4+1}{a^2(a-1/a)}$$

$$\therefore \qquad R_1 + R_2 = a + \frac{1}{a} + \frac{a^4+1}{a^2(a-1/a)}$$

$$= \frac{2a^3}{a^2-1}$$

By Cauchy's residue theorem and from (1), we get

$$I = \frac{i}{2a} \times 2\pi i \times \frac{2a^3}{a^2-1} = \frac{2\pi a^2}{1-a^2}$$

Example 4.15 Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta$, using contour integration

On the circle $|z| = 1$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$ and $\cos 3\theta = \frac{e^{i3\theta}+e^{-i3\theta}}{2} = \frac{z^6+1}{2z^3}$.

$\therefore$ The given integral $I = \int_C \frac{\frac{z^6+1}{2z^3} \cdot \frac{dz}{iz}}{5+4\left(\frac{z^2+1}{2z}\right)}$, where C is $|z| = 1$.

i.e.
$$I = -\frac{i}{4}\int_C \frac{(z^6+1)\,dz}{z^3(z+2)(z+1/2)}$$

The singularities of the integrand $f(z)$ which lie inside C are the simple pole $z = -1/2$ and the triple pole $z = 0$.

R_1 = (Res. of the integrand)$_{z=-1/2}$ $= \left\{\frac{z^6+1}{z^3(z+2)}\right\}_{z=-1/2} = -\frac{65}{12}$

R_2, the residue at $z = 0$ is found out as the coefficient of $\frac{1}{z}$ in the Laurent's expansion of the integrand.

$$f(z) = \left(z^3 + \frac{1}{z^3}\right)\left(1+\frac{z}{2}\right)(1+2z)$$

$$= \left(z^3 + \frac{1}{z^3}\right)\left\{1 - \frac{z}{2} + \frac{z^2}{4} - \cdots\right\}\{1 - 2z + 4z^2 - \cdots\}$$

$$= \left(z^3 + \frac{1}{z^3}\right)\left\{1 - \frac{5}{2}z + \frac{21}{4}z^2 - \cdots\right\}$$

$\therefore R_2$ = Coefficient of $\frac{1}{z}$ in this expansion $= \frac{21}{4}$.

By Cauchy's residue theorem.

$$I = -i/4 \times 2\pi i\left(-\frac{65}{12} + \frac{21}{4}\right) = -\frac{\pi}{12}$$

EXERCISE 4(c)

Part A

(Short Answer Questions)

1. Give the forms of the definite integrals which can be evaluated using the infinite semi-circular contour above the real axis.

2. Explain how to convert $\int_0^{2\pi} f(\sin\theta, \cos\theta)\mathrm{d}\theta$ into a contour integral, where f is a rational function.

3. Sketch the contour to be used for the evaluation of $\int_0^\infty \frac{\sin mx}{x}\mathrm{d}x$.

Part B

Evaluate the following integrals by contour integration technique.

4. $\int_0^\infty \frac{x^2\mathrm{d}x}{(x^2+1)(x^2+4)}$

5. $\int_0^\infty \frac{x^2\mathrm{d}x}{(x^2+4)^2(x^2+9)}$

6. $\int_{-\infty}^\infty \frac{x^2-x+2}{x^4+10x^2+9}\mathrm{d}x$

7. $\int_0^\infty \frac{\mathrm{d}x}{x^4+a^4}\mathrm{d}x$

8. $\int_0^\infty \frac{x^2\mathrm{d}x}{(x^2+1)^3}$

9. $\int_0^\infty \frac{x^2}{x^6+1}\mathrm{d}x$

10. $\int_0^\infty \frac{x^4\mathrm{d}x}{x^6+1}$

11. $\int_{-\infty}^\infty \frac{\cos ax}{x^2+b^2}\mathrm{d}x\,(a>0; b>0)$

12. $\int_{-\infty}^\infty \frac{\cos x\,\mathrm{d}x}{(x^2+4)(x^2+9)}$

13. $\int_0^\infty \frac{x\sin x}{x^4+1}\mathrm{d}x$

14. $\int_{-\infty}^\infty \frac{x\sin x}{(x^2+1)(x^2+4)}\mathrm{d}x$

15. $\int_{-\infty}^\infty \frac{x\sin x}{(x^2+1)^2}\mathrm{d}x$

16. $\int_{-\infty}^\infty \frac{\sin x\,\mathrm{d}x}{x^2+4x+5}$

17. $\int_{-\infty}^\infty \frac{x\cos \pi x}{x^2+2x+5}\mathrm{d}x$

18. $\int_0^\infty \frac{\sin x}{x(x^2+1)^2}\mathrm{d}x$

19. $\int_0^{2\pi} \frac{\mathrm{d}\theta}{1-2p\cos\theta+p^2}, 0<p<1$

20. $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}(a>b>0)$

21. $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta(a>b>0)$

22. $\int_0^{2\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}(a>0)$

23. $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta}$

24. $\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$

25. $\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta} d\theta$

ANSWERS

Exercise 4(a)

8. $\frac{10}{3}(3+i)$

9. $3-i2$

10. $4\pi i$

11. $-\frac{4}{3}+\frac{8}{3}i$

12. πi

13. (i) $-1/6 + i5/6$; (ii) $-1/6+i\frac{13}{15}$

14. $4\pi i$

15. $10-i\frac{8}{3}$

16. 0 in all cases

17. (i) $\frac{511}{3}-\frac{49}{5}i$; (ii) $\frac{518}{3}-57i$; (iii) $\frac{518}{3}-8i$

18. $\frac{1}{15}\left(96\pi^5a^5+80\pi^3a^3+30\pi a\right)$

19. (i) 0; (ii) $2\pi i$

20. $\frac{8\pi i}{3}$

21. $-4\pi i$

22. (i) $2\pi i$; (ii) 0

23. $-\frac{2\pi}{3}$

24. $\sin t$

25. $\frac{8\pi i}{3e^2}$

26. $\frac{\pi}{16}$

27. $\frac{\pi}{2}(3+2i)$

28. 0

29. $20\pi i$; $2\pi(i-1)$; $-14\pi i$; $16\pi i$

Exercise 4(b)

17. $\sin z = \frac{1}{\sqrt{2}}\left\{1+(z-\pi/4)-\frac{(z-\pi/4)^2}{2!}-\frac{(z-\pi/4)^3}{3!}+\cdots\right\}$

18. $\cos z = \frac{1}{2}\left[1 - \sqrt{3}\left(z - \pi/3\right) - \frac{1}{2!}\left(z - \pi/3\right)^2 + \frac{\sqrt{3}}{3!}\left(z - \pi/3\right)^3 + \frac{1}{4!}\left(z - \pi/3\right)^4 - \cdots\right]$

19. $e^{-i}\left\{1 + \frac{(z+i)}{1!} + \frac{(z+i)^2}{2!} + \cdots\right\}$

20. $e^{-1}\left\{1 - \frac{(z-1)}{1!} + \frac{(z-1)^2}{2!} - \frac{(z-1)^3}{3!} + \cdots\right\}$

21. $(z-1) - 2(z-1)^2 + 3(z-1)^3 - 4(z-1)^4 + \cdots$

22. $|z| < 1$

23. $|z| < 2$

24. $|z-1| < \frac{1}{2}$

25. $|z-2| < 1$

26. $|z-4i| < 4$

27. $-\frac{1}{2z^2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right)$

28. $\frac{1}{z-1}\left\{1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\right\}$

29. $-\frac{1}{z^4}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$

30. $\frac{1}{z-1}\left[1 - \frac{2}{z-1} + \frac{2}{(z-1)^2} - \cdots\right]$

31. 1

32. -1

33. 1

34. $\frac{1}{6}$

35. $-\frac{1}{2}$

36. $R_{(z=-1)} = \frac{1}{3}; R_{(z=2)} = \frac{2}{3}$

37. $\pm\frac{ia}{2}$

38. 1

39. $2e$

40. -1

41. $2\pi i$

42. 0

43. $2\pi i$

44. $-4\pi i$

45. $-2\pi i$

46. $\sum\left(\frac{1}{2^{n+1}} - 1\right)(z+1)^n ; |z+1| < 1$

47. $\sum(-1)^n\left\{\frac{2}{(2+i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n$.

48. (i) $1+\sum(-1)^n\left\{\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right\}z^n$

(ii) $1+3\sum(-1)^n\,\frac{2^n}{z^n+1}-8\sum(-1)^n\,\frac{z^n}{3^{n+1}}$

(iii) $1+\sum(-1)^n\left\{3.2^n-8.3^n\right\}\frac{1}{z^{n+1}}$

49. (i) $\sum\left(-\frac{1}{2.4^n}+\frac{1}{3^{n+1}}\right)(z+2)^n$

(ii) $-\frac{1}{2}\sum\frac{(z+2)^n}{4^n}-\sum\frac{3^n}{(z+2)^{n+1}}$ (iii) $\sum\left(2.4^n-3^n\right)\frac{1}{(z+2)\,z^{n+1}}$

50. (i) $-\frac{3}{z+1}-\sum\left(1+\frac{2}{3^{n+1}}\right)(z+1)^n\,;0<|z+1|<1$

(ii) $-\frac{3}{z+1}+\sum\frac{1}{(z+1)^{n+1}}-2\sum\frac{(z+1)^n}{3^{n+1}};1<|z+1|<3$

(iii) $-\frac{3}{z+1}+\sum\left(1+2.3^n\right)\big/z^{n+1},|\,z+1\,|>3.$ Res. at $(z=-1)=-3$

51. (i) $1+\frac{4/5}{z-3}+\sum(-1)^n\left\{\frac{1}{2^{n+1}}+\frac{1}{5^{n+2}}\right\}(z-3)^n$, in $0<|z-3|<2$

(ii) $1+\frac{4/5}{z-3}+\sum(-1)^n\left\{\frac{2^n}{(z-3)^{n+1}}+\frac{1}{5^{n+2}}.(z-3)^n\right\}$, in $2<|z-3|<5$

(iii) $1+\frac{4/5}{z-3}+\sum(-1)^n\left\{2^n+5^{n-1}\right\}\frac{1}{(z-3)^n}$, in $|z-3|>5.$ Res. at $(z=3)=\frac{4}{5}$

52. (i) $-\frac{3}{z-1}-\frac{1}{(z-2)^2}-4\sum(z-1)^n$; Res. at $(z=1)=-3$; $0<|z-1|<1$

(ii) $\frac{4}{z-2}-\sum(-1)^n(n+4)(z-2)^n$; Res. at $(z=2)=4$; $0<|z-2|<1$

53. (i) $\frac{1}{z^2}-\frac{1}{2}\sum\left\{(-1)^n+1\right\}i^n z^n$,; Res. at $(z=0)=0$

(ii) $\frac{i/2}{z-i}-\sum\left\{(n+1)+\frac{1}{2^{n+2}}\right\}i^n\;(z-i)^n$; Res. at $(z=i)=\frac{i}{2}$

(iii) $\frac{-i/2}{z+i}-\sum\left\{(n+1)+\frac{1}{2^{n+2}}\right\}(-1)^n(z+i)^n$; Res. at $(z=-i)=-\frac{i}{2}$

54. (i) Res. at the simple pole $(z=-1+2i)=\frac{1}{4}(2+i)$

and that at $(z=-1-2i)=\frac{1}{4}(2-i)$

(ii) $z = 0$ is a double pole with Res. $= -1/2$; $z = -1 + i$ is a simple pole with Res. $= 1/4$; $z = -1 - i$ is a simple pole with Res. $= 1/4$.

55. (i) $z = 1$ is a simple pole; Res $= 1/4$; $z = -1$ is a double pole; Res. $= -1/4$

(ii) $z = -1$ is a double pole; Res $= -\frac{14}{25}$; $z = 2i$ is a simple pole with Res. $= \frac{1}{25}(7+i)$; $z = -2i$ is a simple pole with Res. $= \frac{1}{25}(7-i)$

56. (i) $z = 2e^{i\pi/4}, 2e^{i3\pi/4}, 2e^{i5\pi/4}, 2e^{i7\pi/4}$ are simple poles. Cor. residues are,

$$\frac{1}{32\sqrt{2}}(-1-i),\ \frac{1}{32\sqrt{2}}(1-i),\ \frac{1}{32\sqrt{2}}(1+i),\ \frac{1}{32\sqrt{2}}(-1+i)$$

(ii) $z = ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}$ are simple poles. Cor. residues are

$$\frac{-i}{4a^2}, \frac{i}{4a^2}, \frac{-i}{4a^2}, \frac{i}{4a^2}.$$

57. (i) πi; (ii) $\frac{\pi i}{3}$;

58. $-2\pi i$; (ii) $-\frac{\pi^2}{2}i$

59. (i) $-2\pi i$; (ii) πi; (iii) $-\frac{\pi i}{4}$

60. (i) $\pi(i-2)$; (ii) $\frac{3\pi}{256}$

(iii) $\frac{-2\pi i}{9}$

Exercise 4(c)

4. $\frac{\pi}{6}$

5. $\frac{\pi}{200}$

6. $\frac{5\pi}{12}$

7. $\frac{\pi}{2\sqrt{2}a^3}$

8. $\frac{\pi}{16}$

9. $\frac{\pi}{6}$

10. $\frac{\pi}{3}$

11. $\frac{\pi}{b}e^{-ab}$

12. $\frac{\pi}{30}\left(\frac{3}{e^2} - \frac{2}{e^3}\right)$

13. $\frac{\pi}{2}e^{-1/\sqrt{2}}\sin\left(\frac{1}{\sqrt{2}}\right)$

14. $\frac{\pi}{3e^2}(e-1)$

15. $\frac{\pi}{2e}$

16. $-\frac{\pi}{e}\sin 2$

17. $\frac{\pi}{2}e^{-2\pi}$

18. $\frac{\pi}{2} - \frac{3\pi}{4e}$

19. $\frac{2\pi}{1-p^2}$

20. $\frac{2\pi}{\sqrt{a^2-b^2}}$

21. $\frac{2\pi}{b^2}\left\{a - \sqrt{a^2-b^2}\right\}$

22. $\frac{2\pi}{\sqrt{1+a^2}}$

23. $\frac{\pi}{6}$

24. $\frac{\pi}{12}$

25. $\frac{3\pi}{8}$

UNIT-5
LAPLACE TRANSFORMS

UNIT 5

Laplace Transforms

5.1 INTRODUCTION

The Laplace transform is a powerful mathematical technique useful to the engineers and scientists, as it enables them to solve linear differential equations with given initial conditions by using algebraic methods. The Laplace transform technique can also be used to solve systems of differential equations, partial differential equations and integral equations. Starting with the definition of Laplace transform, we shall discuss below the properties of Laplace transforms and derive the transforms of some functions which usually occur in the solution of linear differential equations.

5.1.1 Definition

If $f(t)$ is a function of t defined for all $t \geq 0$, $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is defined as the *Laplace transform of f(t)*, provided the integral exists.

Clearly the integral is a function of the parameters s. This function of s is denoted as $\bar{f}(s)$ or F(s) or $\phi(s)$. Unless we have to deal with the Laplace transforms of more than one function, we shall denote the Laplace transform of $f(t)$ as $\phi(s)$. Sometimes the letter 'p' is used in the place of s.

The Laplace transform of $f(t)$ is also denoted as $L\{f(t)\}$, where L is called the Laplace transform operator.

Thus $$L\{f(t)\} = \phi(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

The operation of multiplying $f(t)$ by e^{-st} and integrating the product with respect to t between 0 and ∞ is called *Laplace transformation*.

The function $f(t)$ is called the *Laplace inverse transform* of $\phi(s)$ and is denoted by $L^{-1}\{\phi(s)\}$.

Thus $$f(t) = L^{-1}\{\phi(s)\}, \text{ when } L\{f(t)\} = \phi(s)$$

Note ☑

1. The parameter s used in the definition of Laplace transform is a real or complex number, but we shall assume it to be a real positive number sufficiently large to ensure the existence of the integral that defines the Laplace transform.
2. Laplace transforms of all functions do not exist. For example, L (tan t) and $L(e^{t^2})$ do not exist. We give below the sufficient conditions (without proof) for the existence of Laplace transform of a function $f(t)$:

Conditions for the existence of Laplace transform If the function $f(t)$ defined for $t \geq 0$ is

(i) piecewise continuous in every finite interval in the range $t \geq 0$, and
(ii) of the exponential order, then $L\{f(t)\}$ exists.

Note ☑

1. A function $f(t)$ is said to be piecewise continuous in the finite interval $a \leq t \leq b$, if the interval can be divided into a finite number of sub-intervals such that (i) $f(t)$ is continuous at every point inside each of the sub-intervals and (ii) $f(t)$ has finite limits as t approaches the end points of each sub-interval from the interior of the sub-interval.
2. A function $f(t)$ is said to be of the exponential order, if $|f(t)| \leq M e^{\alpha t}$, for all $t \geq 0$ and some constants M and α or equivalently, if $\lim_{t\to\infty}\{e^{-\alpha t} f(t)\}$ = a finite quantity.

Most of the functions that represent physical quantities and that we encounter in differential equations satisfy the conditions stated above and hence may be assumed to have Laplace transforms.

5.2 LINEARITY PROPERTY OF LAPLACE AND INVERSE LAPLACE TRANSFORMS

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1 L\{f_1(t)\} \pm k_2 L\{f_2(t)\},$$

where k_1 and k_2 are constants.

Proof:
$$\begin{aligned} L\{k_1 f_1(t) \pm k_2 f_2(t)\} &= \int_0^\infty \{k_1 f_1(t) \pm k_2 f_2(t)\} e^{-st}\, \mathrm{d}t \\ &= k_1 \int_0^\infty f_1(t)\cdot e^{-st}\, \mathrm{d}t \pm k_2 \int_0^\infty f_2(t)\cdot e^{-st}\, \mathrm{d}t \\ &= k_1 \cdot L\{f_1(t)\} \pm k_2 \cdot L\{f_2(t)\}. \end{aligned}$$

Thus L is a linear operator.

As a particular case of the property, we get

$L\{kf(t)\} = kL\{f(t)\}$, where k is a constant.

If we take $L\{f_1(t)\} = \phi_1(s)$ and $L\{f_2(t)\} = \phi_2(s)$,
the above property can be written in the following form.

$$L\{k_1 f_1(t) \pm k_2 f_2(t)\} = k_1\phi_1(s) \pm k_2\phi_2(s).$$

$$\therefore \quad L^{-1}\{k_1\phi_1(s) \pm k_2\ \phi_2(s)\} = k_1 \cdot f_1(t) \pm k_2 \cdot f_2(t)$$
$$= k_1 \cdot L^{-1}\{\phi_1(s) \pm k_2 \cdot L^{-1}\{\phi_2(s)\}$$

Thus L^{-1} is also a linear operator

As a particular case of this property, we get

$$L^{-1}\{k\phi(s)\} = k\,L^{-1}\{\phi(s)\}, \text{ where } k \text{ is a constant.}$$

Note ☑

1. $L\{f_1(t) \cdot f_2(t)\} \neq L\{f_1(t)\} \cdot L\{f_2(t)\}$ and
   $L^{-1}\{\phi_1(s) \cdot \phi_2(s)\} \neq L^{-1}(\phi_1(s)\} \times L^{-1}\{\phi_2(s)\}$
2. Generalising the linearity properties,

we get (i) $L\left\{\sum_{r=1}^{n} k_r\ f_r\ (t)\right\} = \sum_{r=1}^{n} k_r \cdot L\{f_r(t)\}$

(ii) $L^{-1}\left\{\sum_{r=1}^{n} k_r\,\phi_r\ (s)\right\} = \sum_{r=1}^{n} k_r \cdot L^{-1}\{\phi_r(s)\}$

Using (i), we can find Laplace transform of a function which can be expressed as a linear combination of elementary functions whose transforms are known.

Using (ii), we can find inverse Laplace transform of a function which can be expressed as a linear combination of elementary functions whose inverse transforms are known.

5.3 LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

1. $L\{k\} = \dfrac{k}{s}$, s > 0, where k is a constant,

$$L\{k\} = \int_0^\infty k\,e^{-st}\,\mathrm{d}t \text{, by definition.}$$

$$= k\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{k}{-s}(0-1)[\because e^{-st} \to 0 \text{ as } t \to \infty, \text{if } s > 0]$$

$$= \frac{k}{s}\cdot$$

In particular, $L(0) = 0$ and $L(1) = \dfrac{1}{s}$

$$\therefore \qquad L^{-1}\left\{\frac{1}{s}\right\} = 1\,.$$

2. $L\{e^{-at}\} = \dfrac{1}{s+a}$, where a is a constant,

$$L\{e^{-at}\} = \int_0^\infty e^{-st} \cdot e^{-at}\, \mathrm{d}t$$

$$= \int_0^\infty e^{-(s+a)t}\, \mathrm{d}t = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{-(s+a)}(0-1), \text{ if } (s+a) > 0$$

$$= \frac{1}{s+a}, \text{ if } s > -a.$$

Inverting, we get $L^{-1}\left(\dfrac{1}{s+a}\right) = e^{-at}$.

3. $L\{e^{at}\} = \dfrac{1}{s-a}$, where a is a constant, if $s - a > 0$ or $s > a$.

Changing a to $-\,a$ in (2), this result follows. The corresponding inverse result is

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

4. $L(t^n) = \dfrac{\overline{|(n+1)}}{s^{n+1}}$, if $s > 0$ and $n > -\,1$.

$$L(t^n) = \int_0^\infty e^{-st} \cdot t^n\, \mathrm{d}t$$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \cdot \frac{\mathrm{d}x}{s}, \text{ on putting } st = x$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x}\, x^n\, \mathrm{d}x$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } s > 0 \text{ and } n+1 > 0$$

[by definition of Gamma function]

In particular, if n is a positive integer,

$$\Gamma(n+1) = n!$$

$\therefore$ $$L(t^n) = \frac{n!}{s^{n+1}}, \text{ if } s > 0 \text{ and } n \text{ is a positive integer.}$$

Inverting, we get $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$ or

$$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{1}{n!}t^n$$

Changing n to $n-1$, we get $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!}t^{n-1}$, if n is a positive integer.

If $n > 0$, then $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{\Gamma(n)}t^{n-1}$

In particular, $L(t) = \frac{1}{s^2}$ or $L^{-1}\left(\frac{1}{s^2}\right) = t.$

5. $L(\sin at) = \frac{a}{s^2+a^2}$

$$L(\sin at) = \int_0^\infty e^{-st}\sin at\, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$$= -\frac{s}{s^2+a^2}\left(e^{-st}\sin at\right)_0^\infty - \frac{a}{s^2+a^2}\left(e^{-st}\cos at\right)_0^\infty$$

$$= \frac{a}{s^2+a^2}$$

[$\because e^{-st}\sin at$ and $e^{-st}\cos at$ tend to zero at $t \to \infty$, if $s > 0$]

Inverting this result we get $L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at.$

6. $L(\cos at) = \dfrac{s}{s^2 + a^2}$

$$L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^\infty$$

$$= -\frac{s}{s^2 + a^2} \left(e^{-st} \cos at \right)_0^\infty + \frac{a}{s^2 + a^2} \left(e^{-st} \sin at \right)_0^\infty$$

$$= \frac{s}{s^2 + a^2} \text{, as per the results stated above.}$$

Inverting the above result we get $L^{-1}\left(\dfrac{s}{s^2 + a^2} \right) = \cos at$.

Aliter

$$L(\cos at + i \sin at) = L(e^{iat})$$

$$= \frac{1}{s - ia} \text{, by result (3).}$$

$$= \frac{s + ia}{s^2 + a^2}$$

i.e., $L(\cos at) + iL(\sin at) = \dfrac{s}{s^2 + a^2} + i \dfrac{a}{s^2 + a^2}$, by linearity property.

Equating the real parts, we get $L(\cos at) = \dfrac{s}{s^2 + a^2}$.

Equating the imaginary parts, we get $L(\sin at) = \dfrac{a}{s^2 + a^2}$

7. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

$$L(\sinh at) = L\left[\frac{1}{2} (e^{at} - e^{-at}) \right]$$

$$= \frac{1}{2} \left[L(e^{at}) - L(e^{-at}) \right] \text{, by linearity property.}$$

$$= \frac{1}{2} \left(\frac{1}{s - a} - \frac{1}{s + a} \right) \text{, if } s > a \text{ and } s > -a.$$

$$= \frac{a}{s^2 - a^2} \text{, if } s > |a|$$

Aliter

$$L(\sinh at) = -\,iL(\sin i\,at) \;[\because\; \sin i\theta = i \sinh \theta]$$

$$= -i \cdot \frac{ia}{s^2 + i^2 a^2}, \text{ by result (5)}$$

$$= \frac{a}{s^2 - a^2}$$

Inverting the above result we get $L^{-1}\left\{\dfrac{a}{s^2 - a^2}\right\} = \sinh at.$

8. $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

$$L(\cosh at) = L\left[\frac{1}{2}\left(e^{at} + e^{-at}\right)\right]$$

$$= \frac{1}{2}\left[L\left(e^{at}\right) + L\left(e^{-at}\right)\right], \text{ by linearity property,}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right), \text{ if } s > |a|.$$

$$= \frac{s}{s^2 - a^2}, \text{ if } s > |a|.$$

Aliter

$$L(\cosh at) = L(\cos iat) \;[\because \cos i\theta = \cosh \theta]$$

$$= \frac{s}{s^2 + i^2 a^2}, \text{ by result (6)}$$

$$= \frac{s}{s^2 - a^2}$$

Inverting the above result we get $L^{-1}\left\{\dfrac{s}{s^2 - a^2}\right\} = \cosh at.$

5.4 LAPLACE TRANSFORMS OF SOME SPECIAL FUNCTIONS

5.4.1 Definition

The function $f(t) = \begin{cases} 0, \text{ when } t < a \\ 1, \text{ when } t > a, \text{ where } a \geq 0 \end{cases}$, is called *Heavyside's unit step function* and is denoted by $u_a(t)$ or $u(t - a)$

In particular, $u_0(t) = \begin{cases} 0, & \text{when } t < 0 \\ 1, & \text{when } t > 0 \end{cases}$

Now $$L\{u_a(t)\} = \int_0^\infty e^{-st}\, u_a(t)\, \mathrm{d}t$$

$$= \int_0^a e^{-st} u_a(t)\, \mathrm{d}t + \int_a^\infty e^{-st} u_a(t)\, \mathrm{d}t$$

$$= \int_a^\infty e^{-st}\, \mathrm{d}t, \text{ by the definition of } u_a(t)$$

$$= \left(\frac{e^{-st}}{-s}\right)_a^\infty = \frac{e^{-as}}{s}, \text{ assuming that } s > 0.$$

In particular, $L\{u_0(t)\} = \dfrac{1}{s}$, which is the same as $L(1)$.

Inverting the above result, we get $L^{-1}\left\{\dfrac{e^{-as}}{s}\right\} = u_a(t)$.

5.4.2 Definition

$\lim\limits_{h \to 0}\{f(t)\}$, where $f(t)$ is defined by

$$f(t) = \begin{cases} \dfrac{1}{h}, & \text{when } a - \dfrac{h}{2} \le t \le a + \dfrac{h}{2} \\ 0, & \text{otherwise} \end{cases}$$ is called *Unit Impulse Function or Dirace Delta Function* and is denoted by $\delta_a(t)$ or $\delta\,(t - a)$.

Now $L\{\delta_a(t)\} = L\left[\lim\limits_{h\to 0}\{f\,(t)\}\right]$,where $f(t)$ is taken as given in the definition

$$= \lim_{h\to 0} L\{f\,(t)\}$$

$$= \lim_{h\to 0} \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

$$= \lim_{h\to 0} \int_{a-\frac{h}{2}}^{a+\frac{h}{2}} e^{-st} \cdot \frac{1}{h}\, \mathrm{d}t$$

$$= \lim_{h\to 0} \left[\frac{1}{h} \cdot \left(\frac{e^{-st}}{-s}\right)_{a-\frac{h}{2}}^{a+\frac{h}{2}}\right]$$

$$= \lim_{h\to 0}\left[\frac{e^{-s\left(a-\frac{h}{2}\right)} - e^{-s\left(a+\frac{h}{2}\right)}}{sh}\right]$$

$$= e^{-as}\lim_{h\to 0}\left[\frac{e^{sh/2} - e^{-sh/2}}{sh}\right]$$

$$= e^{-as}\lim_{h\to 0}\left[\frac{2\sinh\left(\frac{sh}{2}\right)}{sh}\right]$$

$$= e^{-as}\lim_{h\to 0}\left[\frac{s\cosh\left(\frac{sh}{2}\right)}{s}\right], \text{ by L' Hospital's rule.}$$

$$= e^{-as}\lim_{h\to 0}\left[\cosh\frac{sh}{2}\right] = e^{-as}$$

Aliter

$$f(t) = \frac{1}{h}\left[u_{a-\frac{h}{2}}(t) - u_{a+\frac{h}{2}}(t)\right], \text{ since}$$

$u_{a-\frac{h}{2}}(t) = 1$, when $t > a - \dfrac{h}{2}$ and $u_{a+\frac{h}{2}}(t) = 0$,

when $t < a + \dfrac{h}{2}$ and hence $u_{a-\frac{h}{2}}(t) - u_{a+\frac{h}{2}}(t) = 1$

when $a - \dfrac{h}{2} < t < a + \dfrac{h}{2}$

$$\therefore \quad L\{f(t)\} = \frac{1}{h}\left[L\left\{u_{a-\frac{h}{2}}(t)\right\} - L\left\{u_{a+\frac{h}{2}}(t)\right.\right.$$

$$= \frac{1}{h}\left[\frac{e^{-\left(a-\frac{h}{2}\right)s}}{s} - \frac{e^{-\left(a+\frac{h}{2}\right)s}}{s}\right]$$

$$= e^{-as}\,\frac{2\sinh\left(\frac{sh}{2}\right)}{sh}$$

$$\therefore \qquad L\{\delta_a(t)\} = \lim_{h\to 0} L\{f(t)\} = e^{-as}\cdot \lim_{h\to 0}\left[\frac{2\sinh\left(\frac{sh}{2}\right)}{sh}\right]$$

$$= e^{-as}$$

Inverting the above result, we get

$L^{-1}\{e^{-as}\} = \delta_a(t)$. When $a \to 0$, $L^{-1}\{1\} = \delta(t)$.

5.5 PROPERTIES OF LAPLACE TRANSFORMS

1. Change of Scale Property

If $L\{f(t)\} = \phi(s)$, then $L\{f(at)\} = \frac{1}{a}\,\phi\left(\frac{s}{a}\right)$ and $L\left\{f\left(\frac{t}{a}\right)\right\} = a\,\phi(as)$.

Proof

By definition, $\phi(s) = L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\,\mathrm{d}t$ (1)

and $L\{f(at)\} = \int_0^\infty e^{-st}\, f(at)\,\mathrm{d}t$ [$\because f(at)$ is a function of t]

$= \int_0^\infty e^{-s\frac{x}{a}}\, f(x)\frac{\mathrm{d}x}{a}$, putting $x = at$ and making necessary changes.

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)x}\cdot f(x)\,\mathrm{d}x \qquad (2)$$

$= \frac{1}{a}\int_0^\infty e^{-(s/a)t}\cdot f(t)\,\mathrm{d}t$, changing the dummy variable x as t.

Now, comparing (1) and (2), we note that the integral in (2) is the same as the integral in (1) except that 's' in integral in (1) is replaced by $\left(\frac{s}{a}\right)$ in the integral in (2).

$\therefore$ When the integral in (1) is equal to $\phi(s)$, that in (2) is equal to $\phi(s/a)$.

Thus $\qquad L\{f(at)\} = \frac{1}{a}\phi(s/a) \qquad (3)$

Changing a to $\frac{1}{a}$ in (3) or proceeding as in the proof given above, we have

$$L\{f(t/a)\} = a\,\phi(as)\cdot$$

2. First Shifting Property

If $L\{f(t)\} = \phi(s)$, then $L\{e^{-at}f(t)\} = \phi(s+a)$

and $L\{e^{at}f(t)\} = \phi(s-a)$.

Proof

By definition, $$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\,\mathrm{d}t = \phi(s) \tag{1}$$

and $$L\{e^{-at}f(t)\} = \int_0^\infty e^{-st}\,[e^{-at}f(t)]\,\mathrm{d}t$$

[$\because e^{-at}f(t)$ is a function of t]

$$= \int_0^\infty e^{-(s+a)t} f(t)\,\mathrm{d}t \tag{2}$$

$= \phi(s+a)$, comparing the integrals in (1) and (2)

Changing a to $-a$ in the above result,
we get $L\{e^{at}\, f(t)\} = \phi(s-a)$.

Note ☑

1. The above property can be rewritten as a working rule (formula) in the following way:

$$L\{e^{-at}f(t)\} = \phi(s+a)$$
$$= [\phi(s)]_{s\to s+a}$$
$$= L\{f(t)\}_{s\to s+a}$$

'$s \to s + a$' means that s is replaced by $(s + a)$.
Thus, to find the Laplace transform of the product of two factors, one of which is e^{-at}, we ignore e^{-at}, find the Laplace transform of the other factor as a function of s and change s into $(s + a)$ in it.
Similarly,
$L\{e^{at}f(t)\} = L\{f(t)_{s\to s-a}$

2. The above property can be stated in terms of the inverse Laplace operator as follows:
If $L^{-1}\{\phi(s)\} = f(t)$, then $L^{-1}\{\phi(s+a)\} = e^{-at}\cdot f(t)$
From this form of the property, we get the following working rule:

$$L^{-1}\{\phi(s+a)\} = e^{-at}\cdot L^{-1}\{\phi(s)\}$$

This means that if we wish to find the Laplace inverse transform of a function that can be identified as a function of $(s + a)$, we have to find the Laplace inverse transform of the corresponding function of s and multiply it with e^{-at}.
Similarly,

$$L^{-1}\{\phi(s-a)\} = e^{at}L^{-1}\{\phi(s)\}.$$

3. The above property is called so, as it concerns shifting on the s-axis by a (or $-a$), i.e., replacing s by $s + a$ (or $s - a$).
   The second shifting property, that follows, concerns shifting on the t-axis by $-a$ i.e., replacing t by $t - a$.

3. Second Shifting Property

If $\quad L\{f(t)\} = \phi(s)$, then $L\{f(t-a)u_a(t)\} = e^{-as}\phi(s)$,

where a is a positive constant and $u_a(t)$ is the unit step function.

Proof

By definition, $L\{f(t-a)u_a(t)\} = \int_0^\infty e^{-st} f(t-a)u_a(t)\,dt$

$$= \int_0^a e^{-st} f(t-a)u_a(t)\,dt + \int_a^\infty e^{-st} f(t-a)u_a(t)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt \text{, by the definition of } u_a(t).$$

$$= \int_0^\infty e^{-s(x+a)} f(x)\,dx \text{, putting } t - a = x \text{ and effecting consequent changes}$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt \text{, changing the dummy variable } x \text{ as } t.$$

$$= e^{-as}\phi(s)$$

Note ☑

1. Rewriting the above property, we get the following working rule:
$$L\{f(t-a)\,u_a(t)\} = e^{-as}\,L\{f(t)\}$$
2. The above property can be stated in terms of the inverse Laplace operator as follows:

If $\quad L^{-1}\{\phi(s)\} = f(t)$, then $L^{-1}\{e^{-as}\,\phi(s)\} = f(t-a)\,u_a(t)$.

From this form of the property, we get the following working rule.

$$L^{-1}\{e^{-as}\,\phi(s)\} = L^{-1}\{\phi(s)\}_{t \to t-a} \cdot u_a(t).$$

Thus if we wish to find the Laplace inverse transform of the product of two factors, one of which is e^{-as}, we ignore e^{-as}, find the Laplace inverse transform of the other factor as a function of t, replace t by $(t-a)$ in it and multiply by $u_a(t)$.

WORKED EXAMPLE 5(a)

Example 5.1 Find the Laplace transforms of the following functions:

(i) $$f(t)=\begin{cases}(t-1)^2, & \text{for } t>1\\ 0, & \text{for } 0<t<1\end{cases}$$

(ii) $$f(t)=\begin{cases}e^{kt}, & \text{for } 0<t<a\\ 0, & \text{for } t>a\end{cases}$$

(iii) $$f(t)=\begin{cases}\sin\omega t, & \text{for } 0<t<\dfrac{\pi}{\omega}\\ 0, & \text{for } t>\dfrac{\pi}{\omega}\end{cases}$$

(iv) $$f(t)=\begin{cases}t, & \text{for } 0<t<4\\ 5, & \text{for } t>4\end{cases}$$

$$L\{f(t)\}=\int_0^\infty e^{-st} f(t)\,dt$$

(i) $$L\{f(t)\}=\int_0^1 e^{-st}\cdot o\,dt+\int_1^\infty (t-1)^2 e^{-st}\,dt$$

$$=\left[(t-1)^2\cdot\left(\frac{e^{-st}}{-s}\right)-2(t-1)\left(\frac{e^{-st}}{s^2}\right)+2\left(\frac{e^{-st}}{-s^3}\right)\right]_1^\infty$$

$$=0+\frac{2}{s^3}e^{-s}=\frac{2}{s^3}e^{-s}$$

Aliter

$$(t-1)^2 u_1(t)=\begin{cases}0, & \text{for } 0<t<1\\ (t-1)^2 & \text{for } t>1\end{cases}$$

Thus $$f(t) = (t-1)^2 u_1(t)$$

$\therefore$ $L\{f(t)\} = e^{-s}\cdot L(t^2)$, by the second shifting property.

$$=\frac{2}{s^3}e^{-s}$$

(ii) $$L\{f(t)\}=\int_0^a e^{-st}\cdot e^{kt}\,dt+\int_a^\infty e^{-st}\cdot 0\,dt$$

$$=\left[\frac{e^{-(s-k)t}}{-(s-k)}\right]_0^a=\frac{1}{s-k}\{1-e^{-a(s-k)}\}$$

Aliter

Consider $e^{kt}\{1-u_a(t)\}=\begin{cases} e^{kt}, & \text{for } 0<t<a \\ 0, & \text{for } t>a \end{cases}$

Thus $f(t) = e^{kt} - e^{kt}u_a(t)$

$$= e^{kt} - e^{k(t-a)+ka} \cdot u_a(t)$$

$\therefore$ $L\{f(t)\} = L(e^{kt}) - e^{ak} \cdot L\{e^{k(t-a)} \cdot u_a(t)\}$

$$= \frac{1}{s-k} - e^{ak} \cdot e^{-as} \cdot L\{e^{kt}\}$$, by the second shifting property

$$= \frac{1}{s-k} - e^{-a(s-k)} \cdot \frac{1}{s-k}$$

$$= \frac{1}{s-k}\{1-e^{-a(s-k)}\}.$$

(iii) $L\{f(t)\} = \int_0^{\pi/\omega} e^{-st}\sin\omega t\, dt$

$$= \left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{\omega}{s^2+\omega^2}\left(1+e^{-\pi s/\omega}\right)$$

(iv) $L\{f(t)\} = \int_0^4 te^{-st}\, dt + \int_4^\infty 5e^{-st}\, dt.$

$$= \left[t\cdot\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_0^4 + 5\left(\frac{e^{-st}}{-s}\right)_4^\infty$$

$$= -\frac{4}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2} + \frac{5}{s}e^{-4s}$$

$$= \frac{1}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2}$$

Aliter

$$t\{1-u_4(t)\}+5u_4(t)=\begin{cases} t, & \text{for } 0<t<4 \\ 5, & \text{for } t>4 \end{cases}$$

Thus $f(t) = t - (t-5)\, u_4(t)$

$$= t - (t-4)\, u_4(t) + u_4(t)$$

$$\therefore \quad L\{f(t)\} = L(t) - e^{-4s}\cdot L(t) + L\{u_4(t)\}$$
$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-4s} + \frac{1}{s}e^{-4s}.$$

Example 5.2 Find the Laplace transforms of the following functions:

(i) $\dfrac{1+2t}{\sqrt{t}}$, (ii) $\sin\sqrt{t}$, (iii) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$.

(i)
$$L\left\{\frac{1+2t}{\sqrt{t}}\right\} = L(t^{-1/2}) + 2L(t^{1/2})$$
$$= \frac{\Gamma(1/2)}{s^{1/2}} + 2\frac{\Gamma(3/2)}{s^{3/2}}$$
$$= \frac{\sqrt{\pi}}{\sqrt{s}} + \frac{2\cdot\frac{1}{2}\sqrt{\pi}}{s\sqrt{s}} \quad \left[\because \Gamma(1/2) = \sqrt{\pi} \quad \text{and} \quad \Gamma(n+1) = n\Gamma(n)\right]$$
$$= \sqrt{\frac{\pi}{s}}\left(1 + \frac{1}{s}\right).$$

(ii)
$$\sin\sqrt{t} = \sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3!} + \frac{\left(\sqrt{t}\right)^5}{5!} - \ldots.\infty$$
$$= t^{1/2} - \frac{1}{3!}t^{3/2} + \frac{1}{5!}t^{5/2} - \ldots.\infty$$

$$\therefore \quad L\left(\sin\sqrt{t}\right) = L(t^{1/2}) - \frac{1}{3!}L(t^{3/2}) + \frac{1}{5!}L(t^{5/2}) - \ldots.\infty$$
$$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \ldots.\infty$$

$$= \frac{1}{s^{3/2}}\left[\frac{1}{2}\Gamma(1/2) - \frac{1}{3!}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)\cdot\frac{1}{s} + \frac{1}{5!}\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)\cdot\frac{1}{s^2} - \ldots.\infty\right]$$
$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{1!}\cdot\left(\frac{1}{4s}\right) + \frac{1}{2!}\cdot\left(\frac{1}{4s}\right)^2 - \ldots..\infty\right]$$
$$= \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$$

(iii)
$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left[1 - \frac{\left(\sqrt{t}\right)^2}{2!} + \frac{\left(\sqrt{t}\right)^4}{4!} - \ldots..\infty\right]$$
$$= t^{-1/2} - \frac{1}{2!}t^{1/2} + \frac{1}{4!}t^{3/2} - \ldots..\infty$$

$$\therefore \qquad L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = L(t^{-1/2}) - \frac{1}{2!}L\left(t^{1/2}\right) + \frac{1}{4!}L\left(t^{3/2}\right) - \ldots..\infty$$

$$= \frac{\left|\overline{(1/2)}\right.}{s^{1/2}} - \frac{1}{2!}\frac{\left|\overline{(3/2)}\right.}{s^{3/2}} + \frac{1}{4!}\frac{\left|\overline{(5/2)}\right.}{s^{5/2}} - \ldots..\infty$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{2!\,2s} + \frac{1}{4!}\frac{3\cdot 1}{2\cdot 2s^2} - \ldots..\infty\right]$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{1!}\cdot\left(\frac{1}{4s}\right) + \frac{1}{2!}\cdot\left(\frac{1}{4s}\right)^2 - \ldots..\infty\right]$$

$$= \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

Example 5.3 Find the Laplace transforms of the following functions:

(i) $(t^3 + 3e^{2t} - 5\sin 3t)e^{-t}$ (ii) $(1 + te^{-t})^3$

(iii) $e^{-2t}\cosh^3 2t$ (iv) $\cosh at \cos at$

(v) $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$

(i) $L\{(t^3 + 3e^{2t} - 5\sin 3t)e^{-t}\}$

$= L(t^3 + 3e^{2t} - 5\sin 3t)_{s\to s+1}$, by first shifting property.

$$= \left(\frac{3!}{s4} + \frac{3}{s-2} - 5\cdot\frac{3}{s^2+9}\right)_{s\to s+1}$$

$$= \frac{6}{(s+1)^4} + \frac{3}{(s-1)} - \frac{15}{s^2+2s+10}$$

(ii) $L(1 + te^{-t})^3 = L(1 + 3t\,e^{-t} + 3t^2 e^{-2t} + t^3 e^{-3t})$

$= L(1) + 3L(t)_{s\to s+1} + 3L(t^2)_{s\to s+2} + L(t^3)_{s\to s+3}$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

(iii) $$L(e^{-2t}\cosh^3 2t) = L\left\{e^{-2t}\cdot\left(\frac{e^{2t}+e^{-2t}}{2}\right)^3\right\}$$

$$= \frac{1}{8}L\{e^{-2t}(e^{6t} + 3e^{2t} + 3e^{-2t} + e^{-6t})\}$$

$$= \frac{1}{8}L\{e^{4t} + 3 + 3e^{-4t} + e^{-8t}\}$$

$$= \frac{1}{8}\left(\frac{1}{s-4} + \frac{3}{s} + \frac{3}{s+4} + \frac{1}{s+8}\right)$$

(iv) $$L(\cosh at \cos at) = L\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at\right\}$$

$$= \frac{1}{2}\left[L(\cos at)_{s\to s-a} + L(\cos at)_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{s^2+2a^2-2as} + \frac{s+a}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)\left(s^2+2a^2+2as\right)+(s+a)\left(s^2+2a^2-2as\right)}{\left(s^2+2a^2\right)^2-4a^2s^2}\right]$$

$$= \frac{s^3}{s^4+4a^4}.$$

(v) $$L\left(\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right)$$

$$= L\left\{\left(\frac{e^{t/2}-e^{-t/2}}{2}\right)\sin\frac{\sqrt{3}}{2}t\right\}$$

$$= \frac{1}{2}\left[L\left(\sin\frac{\sqrt{3}}{2}t\right)_{s\longrightarrow s-\frac{1}{2}} - L\left(\sin\frac{\sqrt{3}}{2}t\right)_{s\longrightarrow s+\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s-1/2)^2+3/4} - \frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right]$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{s}{s^4+s^2+1}$$

Example 5.4 Find the Laplace transforms of the following functions:

(i) $e^{at}\cos(bt+c)$ (ii) $e^{-2t}\cos^2 3t$ (iii) $e^{t}\sin^3 2t$

(iv) $e^{-t}\sin 2t\cos 3t$ (v) $e^{3t}\sin 2t\sin t$

(i) $$L\{e^{at}\cos(bt+c)\} = L\{\cos(bt+c)\}_{s\longrightarrow s-a}$$

$$= L\{\cos c\cos bt - \sin c\sin bt\}_{s\longrightarrow s-a}$$

$$= \frac{(s-a)\cos c}{(s-a)^2+b^2} - \frac{b\sin c}{(s-a)^2+b^2}$$

(ii) $L\{e^{-2t}\cos^2 3t\} = L(\cos^2 3t)_{s\to s+2}$

$$= L\left\{\frac{1}{2}(1+\cos 6t)\right\}_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{1}{s+2} + \frac{s+2}{(s+2)^2+36}\right]$$

(iii) $L\{e^t \sin^3 2t\} = L(\sin^3 2t)_{s\to s-1}$

$$= L\left\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\right\}_{s\to s-1}$$

$$= \left(\frac{3}{4}\cdot\frac{2}{s^2+4} - \frac{1}{4}\cdot\frac{6}{s^2+36}\right)_{s\to s-1}$$

$$= \frac{3}{2}\left(\frac{1}{s^2-2s+5} - \frac{1}{s^2-2s+37}\right)$$

(iv) $L\{e^{-t}\sin 2t \cos 3t\} = L(\sin 2t \cos 3t)_{s\to s+1}$

$$= L\left\{\frac{1}{2}(\sin 5t - \sin t)\right\}_{s\to s+1}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]_{s\to s+1}$$

$$= \frac{1}{2}\left\{\frac{5}{s^2+2s+26} - \frac{1}{s^2+2s+2}\right\}$$

(v) $L\{e^{3t}\sin 2t \sin t\} = L(\sin 2t \sin t)_{s\to s-3}$

$$= L\left\{\frac{1}{2}(\cos t - \cos 3t)\right\}_{s\to s-3}$$

$$= \frac{1}{2}\left\{\frac{s}{s^2+1} - \frac{s}{s^2+9}\right\}_{s\to s-3}$$

$$= \frac{1}{2}\left\{\frac{s-3}{s^2-6s+10} - \frac{s-3}{s^2-6s+18}\right\}$$

Example 5.5 Find the Laplace transforms of the following functions:
(i) $(t-1)^2 \cdot u_1(t)$ (ii) $\sin t \cdot u_\pi(t)$ (iii) $e^{-3t}\cdot u_2(t)$

(i) $L\{(t-1)^2 u_1(t)\} = e^{-s}\cdot L\{t^2\}$, by the second shifting property

$$= \frac{2}{s^3}e^{-s}$$

(ii) $L\{\sin t \cdot u_\pi(t)\} = L\{\sin(t-\pi+\pi)\cdot u_\pi(t)\}$

$$= -L\{\sin(t-\pi)\cdot u_\pi(t)\}$$

$$= -e^{-s\pi} L(\sin t)$$

$$= -\frac{e^{-s\pi}}{s^2+1}$$

(iii) $L\{e^{-3t}u_2(t)\} = L\{e^{-3(t-2)-6} \cdot u_2(t)\}$

$$= e^{-6} \cdot e^{-2s} \cdot L\{e^{-3t}\} = \frac{e^{-(2s+6)}}{s+3}$$

Example 5.6 Find the Laplace transforms of the following functions:

(i) $t \sin at$ (ii) $t \cos at$ (iii) $te^{-4t} \sin 3t$

(i) $L(t \sin at) = L\{t \times \text{I.P. of } e^{iat}\}$

$$= \text{I.P. of } L\{t\, e^{iat}\}$$

$$= \text{I.P. of } L(t)_{s \to s-ia}$$

$$= \text{I.P. of } \frac{1}{(s-ia)^2} = \text{I.P. of } \frac{(s+ia)^2}{(s^2+a^2)^2}$$

$$= \text{I.P. of} \left[\frac{s^2-a^2}{(s^2+a^2)^2} + i\frac{2as}{(s^2+a^2)^2}\right]$$

$$= \frac{2as}{(s^2+a^2)^2}$$

(ii) $L(t \cos at) = \text{R.P. of } L\{te^{iat}\}$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

(iii) $L(t\, e^{-4t} \sin 3t) = \text{I.P. of } L\ \{t\, e^{-4t}\, e^{i3t}\}$

$$= \text{I.P. of } L\ \{t \cdot e^{-(4-i3)t}\}$$

$$= \text{I.P. of } L(t)_{s \to (s+4-i3)}$$

$$= \text{I.P. of } \frac{1}{(s+4-i3)^2} = \text{I.P. of } \frac{(s+4+i3)^2}{\{(s+4)^2+9\}^2}$$

$$= \text{I.P. of } \frac{\{(s+4)^2-9\} + i\,6(s+4)}{\{(s+4)^2+9\}^2}$$

$$= \frac{6(s+4)}{\{(s+4)^2+9\}^2}$$

Example 5.7

(i) Assuming $L(\sin t)$ and $L(\cos t)$, find the Laplace transforms of $L\left(\sin\frac{t}{2}\right)$ and $L(\cos 3t)$.

(ii) Given that $L(t\sin t)=\frac{2s}{\left(s^2+1\right)^2}$ and $L(t\cos t)=\frac{s^2-1}{\left(s^2+1\right)^2}$, find $L(t \sin at)$ and

$L\left(t\cos\frac{t}{a}\right)$.

(i) $L(\sin t)=\frac{1}{s^2+1}$

$$\therefore \quad L\left(\sin\frac{t}{2}\right)=2\cdot\frac{1}{(2s)^2+1}$$

$$\left[\because L\left\{f\left(\frac{t}{a}\right)\right\}=a\cdot L\{f(t)\}_{s\to as}, \text{by the charge of scale property}\right]$$

$$=\frac{2}{4s^2+1}$$

$$L(\cos t)=\frac{s}{s^2+1}$$

$$\therefore \quad L(\cos 3t)=\frac{1}{3}\cdot\frac{s/3}{(s/3)^2+1}$$

$$\left[\because L\{f(at)\}=\frac{1}{a}L\{f(t)\}_{s\to\frac{s}{a}}, \text{by the charge of scale property}\right]$$

$$=\frac{s}{s^2+9}.$$

(ii) Given that $L(t\sin t)=\frac{2s}{\left(s^2+1\right)^2}$

$$\therefore \quad L(t\sin at)=\frac{1}{a}L(at\sin at)$$

$$=\frac{1}{a}\cdot\frac{1}{a}\cdot\frac{2s/a}{\left(s^2/a^2+1\right)^2}, \text{by change of scale property}\cdot$$

$$=\frac{2as}{\left(s^2+a^2\right)^2}$$

Given that $L(t\cos t)=\dfrac{s^2-1}{\left(s^2+1\right)^2}$

$$\therefore \qquad L\left(t\cos\frac{t}{a}\right)=a\cdot L\left(\frac{t}{a}\cos\frac{t}{a}\right)$$

$$=a\cdot a\cdot\frac{(as)^2-1}{\left[(as)^2+1\right]^2}\text{, by change of scale property}$$

$$=\frac{a^4s^2-a^2}{\left(a^2s^2+1\right)^2}\quad\text{or}\quad\frac{s^2-\dfrac{1}{a^2}}{\left(s^2+\dfrac{1}{a^2}\right)^2}.$$

Example 5.8 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{1}{(s+2)^{5/2}}e^{-s}$ (ii) $\dfrac{e^{-2s}}{(2s-3)^3}$ (iii) $\dfrac{s\,e^{-s}}{(s-3)^5}$ (iv) $\dfrac{(3a-4s)}{s^2+a^2}e^{-bs}$

(v) $\dfrac{(s+4)}{s^2-4}e^{-4s}$

(i) From the second shifting property, we have

$$L^{-1}\{e^{-as}\,\phi(s)\}=L^{-1}\{\phi(s)_{t\to t-a}\cdot u_a(t)$$

$$\therefore L^{-1}\left\{e^{-s}\cdot\frac{1}{(s+2)^{5/2}}\right\}=L^{-1}\left\{\frac{1}{(s+2)^{5/2}}\right\}_{t\longrightarrow t-1}\cdot u_1(t) \tag{1}$$

Now $L^{-1}\left\{\dfrac{1}{(s+2)^{5/2}}\right\}=e^{-2t}\cdot L^{-1}\left\{\dfrac{1}{s^{5/2}}\right\}$

[$\because L^{-1}\{\phi(s+2)=e^{-2t}L^{-1}\{\phi(s)\}$, by the first shifting property.]

$$=e^{-2t}\cdot\frac{1}{\left|\!\overline{\,(5/2)}\right.}\cdot t^{3/2} \qquad \left[\because L^{-1}\left\{\frac{1}{s^n}=\frac{1}{\left|\!\overline{\,(n)}\right.}t^{n-1}\right\}\right]$$

$$=e^{-2t}\cdot\frac{1}{\frac{3}{2}\cdot\frac{1}{2}\cdot\left|\!\overline{\,(1/2)}\right.}\cdot t^{3/2} \qquad \left[\because \left|\!\overline{\,(n)}\right.=(n-1)\left|\!\overline{\,(n-1)}\right.\right]$$

$$=\frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}}e^{-2t} \qquad \left[\because \left|\!\overline{\,(1/2)}\right.=\sqrt{\pi}\right] \tag{2}$$

Inserting (2) in (1), we have

$$L^{-1}\left\{e^{-s}\frac{1}{(s+2)^{5/2}}\right\}=\frac{4}{3\sqrt{\pi}}(t-1)^{3/2}e^{-2(t-1)}\cdot u_1(t)$$

or $$= \begin{cases} 0, & \text{when } t < 1 \\ \dfrac{4}{3\sqrt{\pi}}(t-1)^{3/2}\cdot e^{-2(t-1)}, & \text{when } t > 1 \end{cases}$$

(ii) $$L^{-1}\left\{\frac{e^{-2s}}{(2s-3)^3}\right\} = L^{-1}\left\{\frac{1}{(2s-3)^3}\right\}_{t\longrightarrow t-2} \cdot u_2(t) \qquad (1)$$

Now $$L^{-1}\left\{\frac{1}{(2s-3)^3}\right\} = \frac{1}{8}L^{-1}\left\{\frac{1}{(s-3/2)^3}\right\}$$

$$= \frac{1}{8}e^{\frac{3}{2}t}\, L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{1}{8}e^{\frac{3}{2}t}\cdot\frac{1}{2!}t^2 = \frac{1}{16}e^{\frac{3}{2}t}\, t^2 \qquad (2)$$

Using (2) in (1), we get

$$L^{-1}\left\{\frac{e^{-2s}}{(2s-3)^3}\right\} = \frac{1}{16}\, e^{\frac{3}{2}(t-2)}\,(t-2)^2\cdot u_2(t)$$

(iii) $$L^{-1}\left\{\frac{s\,e^{-s}}{(s-3)^5}\right\} = L^{-1}\left\{\frac{s}{(s-3)^5}\right\}_{t\longrightarrow\, t-1} \cdot u_1(t) \qquad (1)$$

Now $$L^{-1}\left\{\frac{s}{(s-3)^5}\right\} = L^{-1}\left\{\frac{(s-3)+3}{(s-3)^5}\right\}$$

$$= e^{3t}L^{-1}\left\{\frac{s+3}{s^5}\right\}$$

$$= e^{3t}L^{-1}\left\{\frac{1}{s^4}+\frac{3}{s^5}\right\}$$

$$= e^{3t}\left\{\frac{1}{3!}t^3 + 3\cdot\frac{1}{4!}t^4\right\}$$

$$= \frac{1}{24}e^{3t}(4t^3+3t^4) \qquad (2)$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{s}{(s-3)^5}e^{-s}\right\} = \frac{1}{24}e^{3(t-1)}\,\{4(t-1)^3+3(t-1)^4\}\cdot u_1(t)$$

(iv) $$L^{-1}\left\{\frac{(3a-4s)}{s^2+a^2}e^{-bs}\right\} = L^{-1}\left\{\frac{3a-4s}{s^2+a^2}\right\}_{t\longrightarrow\, t-b} \cdot u_b(t) \qquad (1)$$

Now $L^{-1}\left\{\frac{3a-4s}{s^2+a^2}\right\} = 3L^{-1}\left\{\frac{a}{s^2+a^2}\right\} - 4L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$

$$= 3 \sin at - 4 \cos at \tag{2}$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{(3a-4s)}{s^2+a^2}e^{-bs}\right\} = [3\sin a(t-b) - 4\cos a(t-b)]u_b(t)$$

(v) $L^{-1}\left\{\frac{(s+4)}{s^2-4}e^{-4s}\right\} = L^{-1}\left\{\frac{s+4}{s^2-4}\right\}_{t\to t-4} \cdot u_4(t)$ (1)

Now $L^{-1}\left\{\frac{s+4}{s^2-4}\right\} = L^{-1}\left\{\frac{s}{s^2-4}\right\} + 2L^{-1}\left\{\frac{2}{s^2-4}\right\}$

$$= \cosh 2t + 2 \sinh 2t \tag{2}$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{(s+4)}{s^2-4}e^{-4s}\right\} = \{\cosh 2(t-4) + 2\sinh 2(t-4)\} \cdot u_4(t)$$

Example 5.9 Find the inverse Laplace transforms of the following functions:

(i) $\frac{e^{-s}}{(s-2)(s+3)}$ (ii) $\frac{e^{-2s}}{s^2(s^2+1)}$ (iii) $\frac{e^{-s}}{s(s^2+4)}$ (iv) $\frac{e^{-3s}}{s^2+4s+13}$

(v) $\frac{(s+1)e^{-\pi s}}{s^2+2s+5}$

(i) $L^{-1}\left\{\frac{e^{-s}}{(s-2)(s+3)}\right\} = L^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\}_{t\to t-1} \cdot u_1(t)$ (1)

Now to find $L^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\}$, we resolve $\frac{1}{(s-2)(s+3)}$ into partial fractions and then use the linearity property of L^{-1} operator.

Let $\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$

Then $A(s+3) + B(s-2) = 1$

By the usual procedure, we get $A = \frac{1}{5}$, $B = -\frac{1}{5}$

$$\therefore\ L^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\} = L^{-1}\left\{\frac{1/5}{s-2} - \frac{1/5}{s+3}\right\}$$

$$=\frac{1}{5}L^{-1}\left(\frac{1}{s-2}\right)-\frac{1}{5}L^{-1}\left(\frac{1}{s+3}\right)$$

$$=\frac{1}{5}(e^{2t}-e^{-3t}) \tag{2}$$

Putting (2) in (1), we have

$$L^{-1}\left\{\frac{e^{-s}}{(s-2)(s+3)}\right\}=\frac{1}{5}\left\{e^{2(t-1)}-e^{-3(t-1)}\right\}\cdot u_1(t)$$

(ii) $L^{-1}\left\{\frac{e^{-2s}}{s^2(s^2+1)}\right\}=L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}_{t\longrightarrow t-2}\cdot u_2(t)$ (1)

Now $\frac{1}{s^2(s^2+1)}=\frac{1}{u(u+1)}=\frac{1}{u}-\frac{1}{u+1}=\frac{1}{s^2}-\frac{1}{s^2+1}$

$\therefore$ $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}=L^{-1}\left(\frac{1}{s^2}\right)-L^{-1}\left(\frac{1}{s^2+1}\right)$

$$= t - \sin t \tag{2}$$

Inserting (2) in (1), we get

$$L^{-1}\left\{\frac{e^{-2s}}{s^2(s^2+1)}\right\}=\{(t-2)-\sin(t-2\}u_2(t)$$

(iii) $L^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\}=L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}_{t\longrightarrow t-1}\cdot u_1(t)$ (1)

Let $\frac{1}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}$

Then $A(s^2+4)+s\,(Bs+c)=1$

$\therefore$ $A=\frac{1}{4}, B=-\frac{1}{4}$ and $C=0$

$\therefore$ $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}=L^{-1}\left[\frac{\frac{1}{4}}{s}-\frac{\frac{1}{4}s}{s^2+4}\right]$

$$=\frac{1}{4}(1-\cos 2t) \tag{2}$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\}=\frac{1}{4}\{1-\cos 2(t-1)\}\cdot u_1(t)$$

(iv) $$L^{-1}\left\{\frac{e^{-3s}}{s^2+4s+13}\right\}=L^{-1}\left\{\frac{1}{s^2+4s+13}\right\}_{t\longrightarrow t-3}\cdot u_3(t) \quad (1)$$

Now $$L^{-1}\left\{\frac{1}{s^2+4s+13}\right\}=L^{-1}\left\{\frac{1}{(s+2)^2+3^2}\right\}$$

$$=e^{-2t}\cdot L^{-1}\left\{\frac{1}{s^2+3^2}\right\}, \text{ by the shifting property}$$

$$=\frac{1}{3}e^{-2t}\,L^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$=\frac{1}{3}e^{-2t}\sin 3t \quad (2)$$

Using (2) in (1), we get

$$L^{-1}\left\{\frac{e^{-3s}}{s^2+4s+13}\right\}=\frac{1}{3}e^{-2(t-3)}\cdot\sin 3(t-3)\cdot u_3(t)$$

(v) $$L^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+2s+5}\right\}=L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\}_{t\longrightarrow t-\pi}\cdot u_\pi(t) \quad (1)$$

Now $$L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\}=L^{-1}\left\{\frac{(s+1)}{(s+1)^2+2^2}\right\}$$

$$=e^{-t}\cdot L^{-1}\left\{\frac{s}{s^2+2^2}\right\}, \text{ by the shifting property}$$

$$= e^{-t}\cos 2t \quad (2)$$

Using (2) in (1), we have

$$L^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+2s+5}\right\}=e^{-(t-\pi)}\cdot\cos 2(t-\pi)\cdot u_\pi(t)$$

Example 5.10 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s^2+s-2}{s(s+3)(s-2)}$ (ii) $\dfrac{2s^2+5s+2}{(s-2)^4}$ (iii) $\dfrac{s}{(s+1)^2(s^2+1)}$

(iv) $\dfrac{1}{s^2(s^2+1)(s^2+9)}$ (v) $\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$.

(i) To find $L^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\}$, we resolve the given function of s into partial fractions and then use the linearity property of L^{-1} operator.

Let $\dfrac{s^2+s-2}{s(s+3)(s-2)}=\dfrac{A}{s}+\dfrac{B}{s+3}+\dfrac{C}{s-2}$

We find, by the usual procedure, that

$$A=\frac{1}{3},\ B=\frac{4}{15}\quad\text{and}\quad C=\frac{2}{5}.$$

$$\therefore\qquad L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\}=L^{-1}\left[\frac{1/3}{s}+\frac{4/15}{s+3}+\frac{2/5}{s-2}\right]$$

$$=\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$$

(ii) To resolve $\dfrac{2s^2+5s+2}{(s-2)^4}$ into partial fractions, we put $s-2=x$, so that

$s=x+2.$

Then
$$\frac{2s^2+5s+2}{(s-2)^4}=\frac{2(x+2)^2+5(x+2)+2}{x^4}$$

$$=\frac{2x^2+13x+20}{x^4}$$

$$=\frac{2}{x^2}+\frac{13}{x^3}+\frac{20}{x^4}$$

$$=\frac{2}{(s-2)^2}+\frac{13}{(s-2)^3}+\frac{20}{(s-2)^4}$$

$$\therefore\ L^{-1}\left\{\frac{2s^2+5s+2}{(s-2)^4}\right\}=2L^{-1}\left\{\frac{1}{(s-2)^2}\right\}+13L^{-1}\left\{\frac{1}{(s-2)^3}\right\}+20L^{-1}\left\{\frac{1}{(s-2)^4}\right\}$$

$$=2e^{2t}\cdot L^{-1}\left(\frac{1}{s^2}\right)+13e^{2t}\cdot L^{-1}\left(\frac{1}{s^3}\right)+20e^{2t}\cdot L^{-1}\left(\frac{1}{s^4}\right)$$

$$=e^{2t}\left[2\cdot t+\frac{13}{2!}t^2+\frac{20}{3!}t^3\right]$$

$$=e^{2t}\left(2t+\frac{13}{2}t^2+\frac{10}{3}t^3\right)$$

$$= \frac{1}{6} e^{2t} (12t + 39t^2 + 20t^3)$$

(iii) Let $$\frac{s}{(s+1)^2 \left(s^2+1\right)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C\,s+D}{s^2+1}$$

$\therefore$ $$A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 = s$$

By the usual procedure, we find that

$$A = 0; \quad B = -\frac{1}{2}; \quad C = 0 \quad \text{and} \quad D = \frac{1}{2}$$

$\therefore$ $$L^{-1}\left\{\frac{s}{(s+1)^2 \left(s^2+1\right)}\right\} = L^{-1}\left\{\frac{-1/2}{(s+1)^2} + \frac{1/2}{s^2+1}\right\}$$

$$= -\frac{1}{2} e^{-t} \cdot L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{t}{2} e^{-t} + \frac{1}{2} \sin t$$

(iv) Since $\dfrac{1}{s^2\left(s^2+1\right)\left(s^2+9\right)}$ is a function of s^2, we put $s^2 = u$, we resolve $\dfrac{1}{u(u+1)(u+9)}$ into partial fractions and then replace u by s^2.

Now let $$\frac{1}{u(u+1)(u+9)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u+9}$$

By the usual procedure, we find that $A = \dfrac{1}{9}$, $B = -\dfrac{1}{8}$ and $C = \dfrac{1}{72}$

$\therefore$ $$L^{-1}\left\{\frac{1}{s^2\left(s^2+1\right)\left(s^2+9\right)}\right\} = L^{-1}\left\{\frac{1/9}{s^2} - \frac{1/8}{s^2+1} + \frac{1/72}{s^2+9}\right\}$$

$$= \frac{1}{9} t - \frac{1}{8} \sin t + \frac{1}{216} \sin 3t$$

(v) To resolve $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$ into partial fractions, we first resolve $\dfrac{1}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$ into partial fractions as shown in (iv).

Thus $$\frac{1}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)} = \frac{1}{(u+1)(u+4)(u+9)}$$

$$= \frac{A}{u+1} + \frac{B}{u+4} + \frac{C}{u+9}, \text{ say.}$$

We find that $A = \frac{1}{24}$, $B = -\frac{1}{15}$ and $C = \frac{1}{40}$.

$$\therefore \quad \frac{1}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1/24}{s^2+1} - \frac{1/15}{s^2+4} + \frac{1/40}{s^2+9}$$

$$\therefore \quad \frac{s}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1}{24}\cdot\frac{s}{s^2+1} - \frac{1}{15}\cdot\frac{s}{s^2+4} + \frac{1}{40}\cdot\frac{s}{s^2+9}$$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\} = \frac{1}{24}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{15}L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{40}L^{-1}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{1}{24}\cos t - \frac{1}{15}\cos 2t + \frac{1}{40}\cos 3t$$

Example 5.11 Find the inverse Laplace transforms of the following functions:

(i) $\frac{14s+10}{49s^2+28s+13}$ (ii) $\frac{2s^3+4s^2-s+1}{s^2(s^2-s+2)}$

(iii) $\frac{1}{s^3-a^3}$ (iv) $\frac{1}{s^4+4}$

(v) $\frac{s}{s^4+s^2+1}$

(i)
$$L^{-1}\left\{\frac{14s+10}{49s^2+28s+13}\right\} = \frac{14}{49}L^{-1}\left\{\frac{s+\frac{5}{7}}{s^2+\frac{4}{7}s+\frac{13}{49}}\right\}$$

$$= \frac{2}{7}L^{-1}\left\{\frac{s+\frac{5}{7}}{\left(s+\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2}\right\}$$

$$= \frac{2}{7}L^{-1}\left\{\frac{\left(s+\frac{2}{7}\right)+\frac{3}{7}}{\left(s+\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2}\right\}$$

$$= \frac{2}{7}e^{-\frac{2}{7}t}\cdot L^{-1}\left\{\frac{s+\frac{3}{7}}{s^2+\left(\frac{3}{7}\right)^2}\right\}$$

$$= \frac{2}{7} e^{-\frac{2}{7}t} \left(\cos \frac{3}{7} t + \sin \frac{3}{7} t \right)$$

(ii) Let $$\frac{2s^3 + 4s^2 - s + 1}{s^2 (s^2 - s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 - s + 2}$$

$\therefore$ $$As(s^2 - s + 2) + B(s^2 - s + 2) + (Cs + D)s^2 = 2s^3 + 4s^2 - s + 1.$$

By the usual procedure, we find that

$$A = \frac{1}{4}, \quad B = \frac{1}{2}, C = \frac{9}{4} \text{ and } D = \frac{13}{4}$$

$\therefore$ $$L^{-1} \left\{ \frac{2s^3 + 4s^2 - s + 1}{s^2 \left(s^2 - s + 2 \right)} \right\} = -\frac{1}{4} L^{-1} \left(\frac{1}{s} \right) + \frac{1}{2} L^{-1} \left(\frac{1}{s^2} \right) + L^{-1} \left\{ \frac{\frac{9}{4} s + \frac{13}{4}}{s^2 - s + 2} \right\}$$

$$= -\frac{1}{4} + \frac{t}{2} + L^{-1} \left\{ \frac{\frac{9}{4} \left(s - \frac{1}{2} \right) + \frac{35}{8}}{\left(s - \frac{1}{2} \right)^2 + \left(\frac{\sqrt{7}}{2} \right)^2} \right\}$$

$$= -\frac{1}{4} + \frac{t}{2} + e^{t/2} L^{-1} \left\{ \frac{9}{4} \cdot \frac{s}{s^2 + \left(\frac{\sqrt{7}}{2} \right)^2} + \frac{\frac{5\sqrt{7}}{4} \cdot \frac{\sqrt{7}}{2}}{s^2 + \left(\frac{\sqrt{7}}{2} \right)^2} \right\}$$

$$= -\frac{1}{4} + \frac{t}{2} + e^{t/2} \left(\frac{9}{4} \cos \frac{\sqrt{7}}{2} t + \frac{5\sqrt{7}}{4} \sin \frac{\sqrt{7}}{2} t \right)$$

(iii) $$\frac{1}{s^3 - a^3} = \frac{1}{(s - a) \left(s^2 + as + a^2 \right)}$$

$\therefore$ Let $$\frac{1}{s^3 - a^3} = \frac{A}{s - a} + \frac{Bs + C}{s^2 + as + a^2}$$

$\therefore$ $$A(s^2 + as + a^2) + (s - a)(Bs + C) = 1$$

By the usual procedure, we find that

$$A = \frac{1}{3a^2}, B = -\frac{1}{3a^2} \text{ and } C = -\frac{2}{3a}.$$

$$\therefore \qquad L^{-1}\left\{\frac{1}{s^3-a^3}\right\}=\frac{1}{3a^2}L^{-1}\left(\frac{1}{s-a}\right)+L^{-1}\left\{\frac{-\frac{1}{3a^2}s-\frac{2}{3a}}{s^2+as+a^2}\right\}$$

$$=\frac{1}{3a^2}e^{at}-\frac{1}{3a^2}L^{-1}\left\{\frac{s+2a}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\}$$

$$=\frac{1}{3a^2}e^{at}-\frac{1}{3a^2}e^{-\frac{at}{2}}\cdot L^{-1}\left\{\frac{s+\frac{3}{2}a}{s^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right\}$$

$$=\frac{1}{3a^2}\left[e^{at}-e^{-\frac{at}{2}}\left\{\cos\frac{\sqrt{3}}{2}at+\sqrt{3}\,\sin\frac{\sqrt{3}}{2}at\right\}\right]$$

(iv) $$\frac{1}{s^4+4}=\frac{1}{\left(s^4+4s^2+4\right)-4s^2}=\frac{1}{\left(s^2+2\right)^2-(2s)^2}$$

$$=\frac{1}{\left(s^2+2s+2\right)\left(s^2-2s+2\right)}$$

$\therefore$ Let $$\frac{1}{s^4+4}=\frac{As+B}{s^2+2s+2}+\frac{Cs+D}{s^2-2s+2}$$

$$\therefore \qquad (As+B)\,(s^2-2s+2)+(Cs+D)(s^2+2s+2)=1$$

By the usual procedure, we get

$$A=1/8,\quad B=1/4,\quad C=-1/8\quad \text{and}\quad D=1/4$$

$$\therefore \qquad L^{-1}\left\{\frac{1}{s^4+4}\right\}=L^{-1}\left\{\frac{\frac{1}{8}s+\frac{1}{4}}{s^2+2s+2}\right\}-L^{-1}\left\{\frac{\frac{1}{8}s-\frac{1}{4}}{s^2-2s+2}\right\}$$

$$=\frac{1}{8}L^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+1}\right\}-\frac{1}{8}L^{-1}\left\{\frac{(s-1)-1}{(s-1)^2+1}\right\}$$

$$=\frac{1}{8}\left[e^{-t}\cdot L^{-1}\left\{\frac{s+1}{s^2+1}\right\}-e^{t}\cdot L^{-1}\left\{\frac{s-1}{s^2+1}\right\}\right]$$

$$= \frac{1}{8}\left[e^{-t}\ (\cos t + \sin t) - e^{t}\ (\cos t - \sin t)\right]$$

$$= \frac{1}{4}\left[\left(\frac{e^{t}+e^{-t}}{2}\right)\sin t - \left(\frac{e^{t}-e^{-t}}{2}\right)\cos t\right]$$

$$= \frac{1}{4}\ (\sin t\ \cosh t - \cos t\ \sinh t)$$

(v) $$\frac{s}{s^4+s^2+1} = \frac{s}{\left(s^4+2s^2+1\right)-s^2} = \frac{s}{\left(s^2+1\right)^2-s^2}$$

$$= \frac{s}{\left(s^2+s+1\right)\left(s^2-s+1\right)}$$

$\therefore$ Let $$\frac{s}{s^4+s^2+1} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1}$$

$\therefore$ $$(As + B)\ (s^2 - s + 1) + (Cs + D)\ (s^2 + s + 1) = s$$

By the usual procedure, we get

$$A = 0,\quad B = \frac{-1}{2},\quad C = 0 \quad \text{and} \quad D = \frac{1}{2}.$$

$\therefore$ $$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s^2-s+1}\right\}\frac{-1}{2}L^{-1}\left\{\frac{1}{s^2+s+1}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}\left[e^{t/2}L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - e^{-t/2}\cdot L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\left\{\left(e^{t/2}-e^{-t/2}\right)\sin\frac{\sqrt{3}}{2}t\right\}$$

$$= \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\ \sinh\ \frac{t}{2}.$$

Example 5.12

(i) If $L^{-1}\left\{\dfrac{s}{\left(s^2+1\right)^2}\right\} = \dfrac{t}{2}\sin t$, find $L^{-1}\left\{\dfrac{s}{\left(s^2+a^2\right)^2}\right\}$

(ii) Given that $L^{-1}\left\{\dfrac{s^2+4}{\left(s^2-4\right)^2}\right\}=t\cosh 2t$, find $L^{-1}\left\{\dfrac{s^2+1}{\left(s^2-1\right)^2}\right\}$

(iii) Given that $L^{-1}\left\{\dfrac{1}{\left(s^2+4\right)^2}\right\}=\dfrac{1}{16}(\sin 2t-2t\cos 2t)$, find $L^{-1}\left\{\dfrac{1}{\left(s^2+9\right)^2}\right\}$.

(i) By change of scale property,

$$L\{f(at)\}=\frac{1}{a}\phi\left(\frac{s}{a}\right)$$

$$\therefore \qquad L^{-1}\{\phi(s/a)\}=a\,L^{-1}\{\phi(s)\}_{t\to at} \qquad (1)$$

Given
$$L^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\}=\frac{t}{2}\sin t$$

$$\therefore \qquad L^{-1}\left\{\frac{s/a}{\left(\dfrac{s^2}{a^2}+1\right)^2}\right\}=a\cdot\frac{at}{2}\sin at,\text{ by } (1)$$

i.e.,
$$a^3L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{a^2}{2}t\sin at$$

$$\therefore \quad L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}=\frac{t}{2a}\sin at.$$

(ii) By change of scale property,

$$L\{f(t/a)\}=a\,\phi(as)$$

$$\therefore \qquad L^{-1}\{\phi(as)\}=\frac{1}{a}L^{-1}(\phi(s)\}_{t\to t/a} \qquad (2)$$

Given
$$L^{-1}\left\{\frac{s^2+4}{\left(s^2-4\right)^2}\right\}=t\cosh 2t$$

$$\therefore \qquad L^{-1}\left\{\frac{(2s)^2+4}{[(2s)^2-4]^2}\right\}=\frac{1}{2}\cdot\frac{t}{2}\cosh t,\text{ by } (2)$$

i.e.,
$$\frac{1}{4}L^{-1}\left\{\frac{s^2+1}{\left(s^2-1\right)^2}\right\}=\frac{t}{4}\cosh t$$

$$\therefore \qquad L^{-1}\left\{\frac{s^2+1}{\left(s^2-1\right)^2}\right\}=t\cosh t.$$

(iii) Given $L^{-1}\left\{\frac{1}{\left(s^2+4\right)^2}\right\}=\frac{1}{16}(\sin 2t-2t\cos 2t)$

$$\therefore \quad L^{-1}\left\{\frac{1}{\left[\left(\frac{2s}{3}\right)^2+4\right]^2}\right\}=\frac{3}{2}L^{-1}\left\{\frac{1}{\left(s^2+4\right)^2}\right\}_{t\to\frac{3}{2}t}, \text{ by (2)}$$

i.e. $$\frac{81}{16}L^{-1}\left\{\frac{1}{\left(s^2+9\right)^2}\right\}=\frac{3}{32}(\sin 3t-3t\cos 3t)$$

$$\therefore \quad L^{-1}\left\{\frac{1}{\left(s^2+9\right)^2}\right\}=\frac{1}{54}(\sin 3t-3t\cos 3t)$$

EXERCISE 5(a)

Part A

(Short Answer Questions)

1. Define Laplace transform.
2. State the conditions for the existence of Laplace transform of a function.
3. Give two examples for a function for which Laplace transform does not exist.
4. State the change of scale property in Laplace transformation.
5. State the first shifting property in Laplace transformation.
6. State the second shifting property in Laplace transformation.
7. Find the Laplace transform of unit step function.
8. Find the Laplace transforms of unit impulse function.
9. Find $L\{f(t)\}$, if $f(t)=\begin{cases} e^{2t}, & \text{for } 0<t<1 \\ 0, & \text{for } t>1 \end{cases}$
10. Find $L\{f(t)\}$, if $f(t)=\begin{cases} 0, & \text{for } 0<t<1 \\ t, & \text{for } 1<t<2 \\ 0, & \text{for } t>2 \end{cases}$
11. Find $L\{f(t)\}$, if $f(t)=\begin{cases} \sin 2t, & \text{for } 0<t<\pi \\ 0, & \text{for } t>\pi \end{cases}$
12. Find $L\{f(t)\}$, if $f(t)=\begin{cases} \cos t, & \text{for } 0<t<2\pi \\ 0, & \text{for } t>2\pi \end{cases}$

13. Find $L\{f(t)\}$, if $f(t)=\begin{cases} 0, & \text{for } 0<t<\dfrac{2\pi}{3} \\ \cos\left(t-\dfrac{2\pi}{3}\right), & \text{for } t>\dfrac{2\pi}{3} \end{cases}$

14. Find $L\{f(t)\}$, if $f(t)=\begin{cases} \sin t, & \text{for } 0<t<\pi \\ t, & \text{for } t>\pi \end{cases}$

15. Find $L\left(\sqrt{t}\right)$ and $L\left(\dfrac{1}{\sqrt{\pi t}}\right)$

16. Find $L\{(t-3)\,u_3(t)\}$ and $L\{u_1(t)\sin \pi(t-1)\}$.

17. Find $L\{t^2 u_2(t)\}$

Find the Laplace transforms of the following functions:

18. $(at+b)^3$
19. $\sin(\omega t+\theta)$
20. $\sin^2 3t$
21. $\cos^3 2t$
22. $\sin 2t \cos t$
23. $\cos 3t \cos 2t$
24. $\sinh^3 t$
25. $\cosh^2 2t$
26. $(t+1)^2 e^{-t}$
27. $e^{-2t}\left(\cos 3t-\dfrac{2}{3}\sin 3t\right)$
28. $e^{t}\left(\cosh 2t+\dfrac{1}{2}\sinh 2t\right)$
29. $\sin t \sinh t$
30. $t^2 \cosh t$
31. $\sqrt{e^{3(t+2)}}$

Find the Laplace inverse transforms of the following functions:

32. $e^{-as}/s^2\ (a>0)$.
33. $(e^{-2s}-e^{-3s})/s$
34. $\dfrac{e^{-2s}}{s-3}$
35. $\dfrac{se^{-s}}{s^2+9}$
36. $\dfrac{1+e^{-\pi s}}{s^2+1}$

Find $f(t)$ if $L\{f(t)\}$ is given by the following functions:

37. $\dfrac{1}{(s+1)^{3/2}}$
38. $\dfrac{s^2+2s+3}{s^3}$
39. $\dfrac{1}{(2s-3)^4}$
40. $\dfrac{s}{(s-2)^5}$
41. $\dfrac{2s+3}{s^2+4}$
42. $\dfrac{s+6}{s^2-9}$
43. $\dfrac{1}{s(s+a)}$
44. $\dfrac{1}{s^2+2s+5}$
45. $\dfrac{s-3}{s^2-6s+10}$

Part B

Find the Laplace transforms of the following functions:

46. $e^{3t}(2t+3)^3$
47. $e^{at}(1+2at)/\sqrt{t}$
48. $e^{-t}\sinh^3 t$
49. $\sin at \cosh at - \cos at \sinh at$
50. $(e^t \sin t)^2$
51. $\left(\dfrac{\cos 2t}{e^t}\right)^3$
52. $e^{-kt}\sin(\omega t+\theta)$
53. $e^{-t}\sin 3t\cos t$
54. $e^t \text{cost} \cos 2t \cos 3t$
55. $e^{-2t}\sin 2t \sin 3t \sin 4t$
56. $t\cos 2t$
57. $te^{-t}\cos t$
58. $t\,e^{2t}\sin 3t$
59. Given that $L(t\sin 2t)=\dfrac{4s}{\left(s^2+4\right)^2}$, find $L(t \sin t)$.
60. Given that $L(t\cos 3t)=\dfrac{s^2-9}{\left(s^2+9\right)^2}$, find $L(t \cos 2t)$.

Find the Laplace inverse transforms of the following function:

61. $\dfrac{s^2+1}{s^3+3s^2+2s}$
62. $\dfrac{4s^2-3s+5}{(s+1)\left(s^2-3s+2\right)}$
63. $\dfrac{s^2-3s+5}{(s+1)^3}$
64. $\dfrac{7s-11}{(s+1)(s-2)^2}$
65. $\dfrac{1}{\left(s^2+s\right)^2}$
66. $\dfrac{s^4-8s^2+31}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$
67. $\dfrac{2s}{\left(s^2+1\right)\left(s^2+2\right)\left(s^2+3\right)}$
68. $\dfrac{2s-9}{s^2+6s+34}$
69. $\dfrac{s+1}{s^2+s+1}$
70. $\dfrac{ls+m}{as^2+bs+c}$
71. $\dfrac{3s^2-16s+26}{s\left(s^2+4s+13\right)}$
72. $\dfrac{1}{s^3+1}$
73. $\dfrac{1}{s^4+4a^4}$
74. $\dfrac{s}{s^4+4}$
75. $\dfrac{s^2}{s^4+64}$
76. $\dfrac{4s^3}{4s^4+1}$
77. $\dfrac{s^2+1}{s^4+s^2+1}$

78. If $L^{-1}\left\{\frac{s}{\left(s^2-1\right)^2}\right\}=\frac{1}{2}t\sinh t$, find $L^{-1}\left\{\frac{s}{\left(s^2-a^2\right)^2}\right\}$

79. If $L^{-1}\left\{\frac{1}{\left(s^2+9\right)^2}\right\}=\frac{1}{54}(\sin 3t-3t\cos 3t)$, find $L^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\}$.

80. Given that $L^{-1}\left\{\frac{s}{\left(s^2+9\right)^2}\right\}=\frac{t}{6}\sin 3t$, find $L^{-1}\left\{\frac{s}{\left(s^2+4\right)^2}\right\}$

5.6 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

5.6.1 Definition

A function $f(t)$ is said to be a *periodic function*, if there exists a constant P (> 0) such that $f(t+P)=f(t)$, for all values of t. Now $f(t+2P)=f(t+P+P)=f(t+P)=f(t)$, for all t. In general, $f(t+nP)=f(t)$, for all t, when n is an integer (positive or negative).

P is called the *period of the function*.

Unlike other functions whose Laplace transforms are expressed in terms of an integral over the semi-infinite interval $0 \le t < \infty$, the Laplace transform of a periodic function $f(t)$ with period P can be expressed in terms of the integral of $e^{-st}f(t)$ over the finite interval $(0, P)$, as established in the following theorem.

Theorem

If $f(t)$ is a piecewise continuous periodic function with period P, then

$$L\{f(t)\}=\frac{1}{1-e^{-Ps}}\cdot\int_0^P e^{-st}f(t)\,\mathrm{d}t.$$

Proof:

By definition, $L\{f(t)\}=\int_0^\infty e^{-st}f(t)\,\mathrm{d}t$

$$=\int_0^P e^{-st}f(t)\,\mathrm{d}t+\int_P^\infty e^{-st}f(t)\,\mathrm{d}t \tag{1}$$

In the second integral in (1), put $t=x+P$, $\therefore$ $\mathrm{d}t=\mathrm{d}x$ and the limits for x become 0 and ∞.

$$\therefore \quad \int_P^\infty e^{-st}f(t)\,\mathrm{d}t=\int_0^\infty e^{-s(x+P)}f(x+P)\,\mathrm{d}x$$

$$=e^{-sP}\cdot\int_0^\infty e^{-sx}f(x)\,\mathrm{d}x \qquad [\because f(x+P)=f(x)]$$

$$= e^{-sP} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \text{, on changing the dummy variable } x \text{ to } t.$$

$$= e^{-sP} \cdot L\{f(t)\} \qquad (2)$$

By putting (2) in (1), we have

$$L\{f(t)\} = \int_0^P e^{-st} f(t)\,\mathrm{d}t + e^{-sP} \cdot L\{f(t)\}$$

$$\therefore \quad (1 - e^{-Ps})\, L\{f(t)\} = \int_0^P e^{-st} f(t)\,\mathrm{d}t$$

$$\therefore \quad L\{f(t)\} = \frac{1}{1 - e^{-Ps}} \int_0^P e^{-st} f(t)\,\mathrm{d}t$$

5.7 DERIVATIVES AND INTEGRALS OF TRANSFORMS

The following two theorems, in which we differentiate and integrate the transform function $\phi(s) = L\{f(t)\}$ with respect to s, will help us to find $L\{t\,f(t)\}$ and $L\left\{\frac{1}{t} f(t)\right\}$ respectively. Repeated differentiation and integration of $\phi(s)$ will enable us to find $L\{t^n f(t)\}$ and $L\left\{\frac{1}{t^n} f(t)\right\}$, where n is a positive integer.

Theorem

If $\quad L\{f(t)\} = \phi(s), \quad$ then $L\{t\,f(t)\} = -\,\phi'(s)$.

***Proof*:**

Given: $\quad L\{f(t)\} = \phi(s)$

i.e. $$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \phi(s) \qquad (1)$$

Differentiating both sides of (1) with respect to s,

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}s} \phi(s) \qquad (2)$$

Assuming that the conditions for interchanging the two operations of integration with respect to t and differentiation with respect to s in (2) are satisfied, we have

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} \left\{e^{-st}\right\} f(t)\,\mathrm{d}t = \phi'(s)$$

i.e. $$\int_0^\infty -t\, e^{-st} f(t)\,\mathrm{d}t = \phi'(s)$$

i.e. $$\int_0^\infty e^{-st}[t\, f(t)]\,\mathrm{d}t = -\phi'(s)$$

i.e. $$L\{t\, f(t)\} = -\phi'(s)$$

Corollary

Differentiating both sides of (1) n times with respect to s, we get

$$L\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\phi(s) \quad \text{or} \quad (-1)^n \phi^{(n)}(s)\,.$$

Note ☑

1. The above theorem can be rewritten as a working rule in the following manner

$$\begin{aligned} L\{t\, f(t)\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\phi(s) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s} L\{f(t)\} \end{aligned}$$

Thus, to find the Laplace transform of the product of two factors, one of which is 't', we ignore 't' and find the Laplace transform of the other factor as a function of s; then we differentiate this function of s with respect to s and multiply by (− 1).

Extending the above rule,

$$L\{t^2\, f(t)\} = (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} L\{f(t)\} \text{ and in general}$$

$$L\{t^n\, f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} L\{f(t)\}\,.$$

2. The above theorem can be stated in terms of the inverse Laplace operator as follows:

If $\quad L^{-1}\{\phi(s)\} = f(t)$,

then $\quad L^{-1}\{\phi'(s)\} = -t\, f(t)$.

From this form of the theorem, we get the following working rule:

$$L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$$

This rule is applied when the inverse transform of the derivative of the given function can be found out easily. In particular, the inverse transforms of functions of s that contain logarithmic functions and inverse tangent and cotangent functions can be found by the application of this rule.

Theorem

If $L\{f(t)\} = \phi(s)$, then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^{\infty}\phi(s)\,\mathrm{d}s$, provided $\lim_{t\to 0}\left\{\frac{1}{t}f(t)\right\}$ exists.

Proof:

Given: $L\{f(t)\} = \phi(s)$

i.e. $$\int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t = \phi(s) \qquad (1)$$

Integrating both sides of (1) with respect to s between the limits s and ∞, we have

$$\int_s^{\infty}\left[\int_0^{\infty} e^{-st}\, f(t)\,\mathrm{d}t\right]\mathrm{d}s = \int_s^{\infty}\phi(s)\,\mathrm{d}s \qquad (2)$$

Assuming that the conditions for the change of order of integration in the double integral on the left side of (2) are satisfied, we have

$$\int_0^{\infty}\left[\int_s^{\infty} e^{-st}\,\mathrm{d}s\right] f(t)\,\mathrm{d}t = \int_s^{\infty}\phi(s)\,\mathrm{d}s$$

i.e. $$\int_0^{\infty}\left[\frac{e^{-st}}{-t}\right]_{s=s}^{s=\infty} f(t)\,\mathrm{d}t = \int_s^{\infty}\phi(s)\,\mathrm{d}s$$

i.e. $$\int_0^{\infty} -\frac{1}{t}(0 - e^{-st})\, f(t)\,\mathrm{d}t = \int_s^{\infty}\phi(s)\,\mathrm{d}s,$$

assuming that $s > 0$

i.e. $$\int_0^{\infty} e^{-st}\left[\frac{f(t)}{t}\right]\mathrm{d}t = \int_s^{\infty}\phi(s)\,\mathrm{d}s$$

i.e. $$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty}\phi(s)\,\mathrm{d}s$$

Corollary

$$L\left\{\frac{1}{t^2}f(t)\right\} = L\left[\frac{1}{t}\left\{\frac{1}{t}f(t)\right\}\right]$$

$$= \int_s^{\infty}\left[\int_s^{\infty}\phi(s)d\,s\right]\mathrm{d}s$$

$$= \int_s^{\infty}\int_s^{\infty}\phi(s)\mathrm{d}s\,\mathrm{d}s$$

Generalising this result, we get

$$L\left\{\frac{1}{t^n}f(t)\right\} = \int_s^{\infty}\int_s^{\infty}\cdots\int_s^{\infty}\phi\,(s)\,(\mathrm{d}s)^n$$

Note ☑

1. The above theorem can be rewritten as a working rule as given below:

$$L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty L\{f(t)\}\mathrm{d}s$$

Thus, to find the Laplace transform of the product of two factors, one of which is $\frac{1}{t}$, we ignore $\frac{1}{t}$, find the Laplace transform of the other factor as a function of s and integrate this function of s with respect to s between the limits s and ∞.
Extending the above rule. We get;

$$L\left\{\frac{1}{t^2}f(t)\right\} = \int_s^\infty\int_s^\infty L\{f(t)\}\mathrm{d}s\,\mathrm{d}s \text{ and in general}$$

$$L\left\{\frac{1}{t^n}f(t)\right\} = \int_s^\infty\int_s^\infty\ldots\int_s^\infty L\{f(t)\}(\mathrm{d}s)^n.$$

2. The above theorem can be stated in terms of the inverse Laplace operator as follows:

If $\quad L^{-1}\{\phi(s)\} = f(t)$,

then $\quad L^{-1}\left[\int_s^\infty \phi(s)\,\mathrm{d}s\right] = \frac{1}{t}f(t)$.

From this form of the theorem, we get the following working rule:

$$L^{-1}\{\phi(s)\} = t\cdot L^{-1}\left[\int_s^\infty \phi(s)\,\mathrm{d}s\right]$$

This rule is applied when the inverse transform of the integral of the given function with respect to s between the limits s and ∞ can be found out easily.

In particular, the inverse transforms of proper rational functions whose numerators are first degree expressions in s and denominators are squares of second degree expressions in s can be found by applying this rule

WORKED EXAMPLE 5(b)

Example 5.1 Find the Laplace transform of the "saw-tooth wave" function $f(t)$ which is periodic with period 1 and defined as $f(t) = kt$, in $0 < t < 1$.

The graph of $f(t)$ is shown in Fig. 5.1 below. If the period of the function $f(t)$ is P, the function will be defined as $f(t) = \frac{k}{P}t$ in $0 < t < P$.

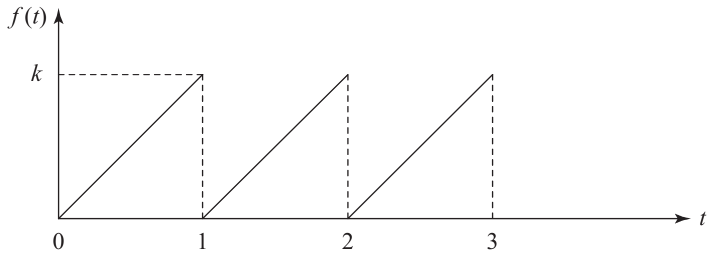


Fig. 5.1

By the formula for the Laplace transform of a periodic function $f(t)$ with period P,

$$L\{f(t)\} = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\,\mathrm{d}t$$

$\therefore$ For the given function,

$$L\{f(t)\} = \frac{1}{1-e^{-s}} \int_0^1 kt\,e^{-st}\,\mathrm{d}t$$

$$= \frac{k}{1-e^{-s}} \left[t\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right) \right]_0^1$$

$$= \frac{k}{1-e^{-s}} \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{k}{1-e^{-s}} \left[\frac{\left(1-e^{-s}\right)}{s^2} - \frac{e^{-s}}{s} \right]$$

$$= \frac{k}{s^2} - \frac{ke^{-s}}{s(1-e^{-s})}$$

Example 5.2 Find the Laplace transform of the "square wave" function $f(t)$ defined by

$$f(t) = k \text{ in } 0 \le t \le a$$

$$= -k \text{ in } a \le t \le 2a$$

and $\quad f(t+2a) = f(t)$ for all t.

$f(t+2a) = f(t)$ means that $f(t)$ is periodic with period $2a$. The graph of the function is shown in Fig. 5.2.

For a periodic function $f(t)$ with period P,

$$L\{f(t)\} = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\,\mathrm{d}t$$

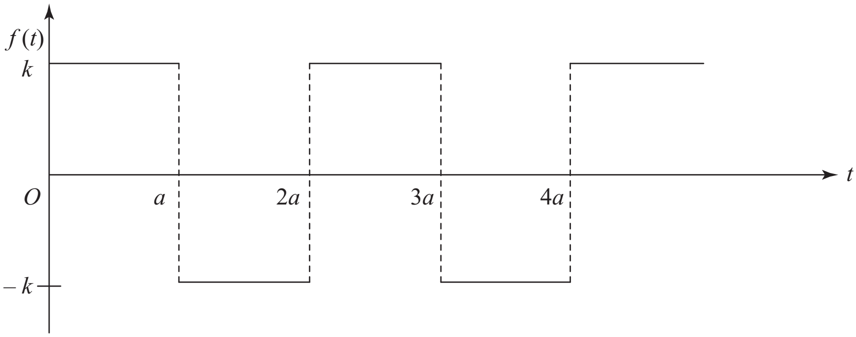


Fig. 5.2

$\therefore$ For the given function;

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\left[\int_0^a k\, e^{-st}\, dt + \int_a^{2a} (-k)\, e^{-st} dt\right]$$

$$= \frac{k}{1-e^{-2as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{k}{s(1-e^{-2as})}[1 - e^{-as} - e^{-as} + e^{-2as}]$$

$$= \frac{k(1 - e^{-as})^2}{s(1 - e^{-as})(1+e^{-as})}$$

$$= \frac{k(1 - e^{-as})}{s(1+e^{-as})} = \frac{k}{s}\frac{(e^{as/2} - e^{-as/2})}{(e^{as/2} + e^{-as/2})}$$

$$= \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

Example 5.3 Find the Laplace transform of “triangular wave function $f(t)$ whose graph is given below in Fig. 5.3.

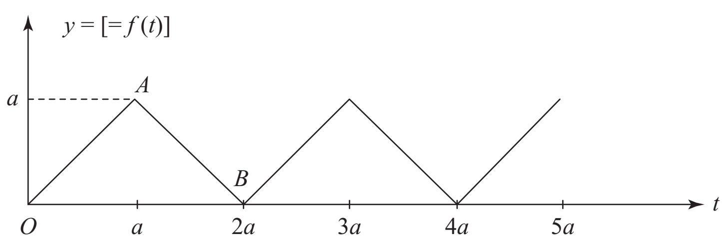


Fig. 5.3

From the graph it is obvious that $f(t)$ is periodic with period $2a$.

Let us find the value of $f(t)$ in $0 \le t \le 2a$, by finding the equations of the lines OA and AB.

OA passes through the origin and has a slope 1.

$\therefore$ Equation of OA is $y = t$, in $0 \le t \le a$

AB passes through the point $B(2a, 0)$ and has a slope -1.

$\therefore$ Equation of AB is $y - 0 = (-1)(t - 2a)$

or $y = 2a - t$ in $a \le t \le 2a$.

Thus the definition of $f(t)$ $[= y]$ can be taken as

$$f(t) = t, \text{ in } 0 \le t \le a$$

$$= 2a - t, \text{ in } a \le t \le 2a$$

and $f(t + 2a) = f(a)$.

$$\text{Now} \quad L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a te^{-st}\,dt + \int_a^{2a} (2a-t)e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[t\left\{ \left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right) \right\}_0^a + \left\{ (2a-t)\left(\frac{e^{-st}}{-s}\right) + 1\cdot\left(\frac{e^{-st}}{s^2}\right) \right\}_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{a}{s}e^{-as} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{a}{s}e^{-as} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1 - 2e^{-as} + e^{-2as}}{s^2(1-e^{-2as})} = \frac{(1-e^{-as})^2}{s^2\,(1-e^{-as})\,(1+e^{-as})}$$

$$= \frac{1\,(1-e^{-as})}{s^2\,(1+e^{-as})} = \frac{1}{s^2}\left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right)$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

Example 5.4 Find the Laplace transform of the "half-sine wave rectifier" function $f(t)$ whose graph is given in Fig. 5.4.

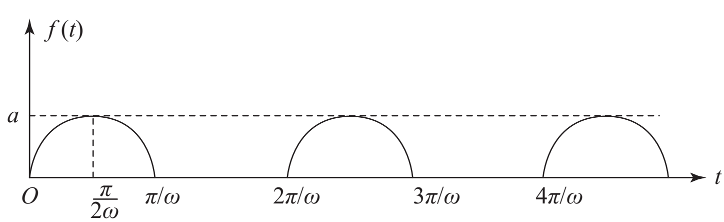


Fig. 5.4

From the graph, it is obvious that $f(t)$ is a periodic function with period $2\pi/\omega$. The graph of $f(t)$ in $0 \le t \le \pi/\omega$ is a sine curve that passes through (0, 0), $\left(\dfrac{\pi}{2\omega}, a\right)$ and $\left(\dfrac{\pi}{\omega}, 0\right)$

$\therefore$ The definition of $f(t)$ is given by

$$f(t) = a \sin \omega t, \text{ in } 0 \le t \le \pi/\omega$$
$$= 0, \text{ in } \pi/\omega \le t \le 2\pi/\omega$$

and $$f\left(t + \frac{2\pi}{\omega}\right) = f(t).$$

Now $$L\{f(t)\} = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{a}{1 - e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t\,dt$$

$$= \frac{a}{1 - e^{-2\pi s/\omega}} \cdot \left[\frac{e^{-st}}{s^2 + \omega^2}(-s \sin \omega t - \omega \cos \omega t\right]_0^{\pi/\omega}$$

$$= \frac{a}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})}\left[\omega e^{-\pi s/\omega} + \omega\right]$$

$$= \frac{\omega a\left(1 + e^{-\pi s/\omega}\right)}{(s^2 + \omega^2)\left(1 - e^{-2\pi s/\omega}\right)} = \frac{\omega a}{(s^2 + \omega^2)\left(1 - e^{-\pi s/\omega}\right)}$$

Example 5.5 Find the Laplace transform of the "full-sine wave rectifier" function $f(t)$, defined as

$$f(t) = |\sin \omega t|, \; t \ge 0$$

We note that $$f(t + \pi/\omega) = |\sin \omega (t + \pi/\omega)|$$
$$= |\sin \omega t|$$
$$= f(t)$$

$\therefore f(t)$ is periodic with period π/ω.

Also $f(t)$ is always positive. The graph of $f(t)$ is the sine curve as shown in Fig. 5.5.

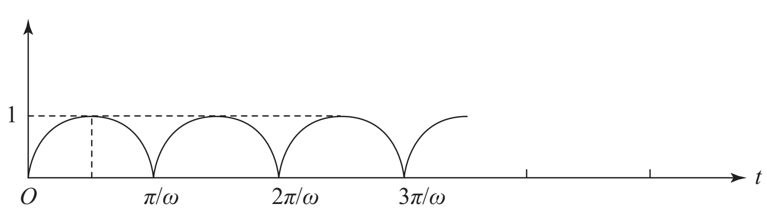


Fig. 5.5

Now $$L\{f(t)\}=\frac{1}{1-e^{-\pi s/\omega}}\int_0^{\pi/\omega} e^{-st}\,|\sin\omega t|\,\mathrm{d}t$$

$$=\frac{1}{1-e^{-\pi s/\omega}}\int_0^{\pi/\omega} e^{-st}\sin\omega t\,\mathrm{d}t\,[\because \sin\omega t>0 \text{ in } 0\le t\le \pi/\omega]$$

$$=\frac{1}{1-e^{-\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$=\frac{1}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}\left(\omega e^{-\pi s/\omega}+\omega\right)=\frac{\omega}{s^2+\omega^2}\left(\frac{1+e^{-\pi s/\omega}}{1-e^{-\pi s/\omega}}\right)$$

$$=\frac{\omega}{s^2+\omega^2}\left(\frac{e^{\pi s/2\omega}+e^{-\pi s/2\omega}}{e^{\pi s/2\omega}-e^{-\pi s/2\omega}}\right), \text{ on integration and simplification}$$

$$=\frac{\omega}{s^2+\omega^2}\coth\left(\frac{\pi s}{2\omega}\right)$$

Example 5.6 Find the Laplace transforms of the following functions:

(i) $t\cosh^3 t$ (ii) $t\cos 2t\cos t$ (iii) $t\sin^3 t$ (iv) $(t\sin at)^2$

(i) $$L\{t\cosh^3 t\}=L\left\{t\left(\frac{e^t+e^{-t}}{2}\right)^3\right\}$$

$$=\frac{1}{8}L\{t(e^{3t}+3e^t+3e^{-t}+e^{-3t})\} \tag{1}$$

$$=-\frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}s}L(e^{3t}+3e^t+3e^{-t}+e^{-3t})$$

$$=-\frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{s-3}+\frac{3}{s-1}+\frac{3}{s+1}+\frac{1}{s+3}\right\}$$

$$=\frac{1}{8}\left\{\frac{1}{(s-3)^2}+\frac{3}{(s-1)^2}+\frac{3}{(s+1)^2}+\frac{1}{(s+3)^2}\right\}$$

Note ☑ After getting step (1), we could have applied the first shifting property and got the same result.

(ii) $$L(t\cos 2t\cos t)=L\left\{\frac{t}{2}(\cos 3t+\cos t)\right\}$$

$$=\frac{1}{2}\left[-\frac{\mathrm{d}}{\mathrm{d}s}L(\cos 3t+\cos t)\right]$$

$$=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+9}+\frac{s}{s^2+1}\right)$$

$$= -\frac{1}{2}\left[\frac{s^2+9-2s^2}{(s^2+9)^2}+\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-9}{(s^2+9)^2}+\frac{s^2-1}{(s^2+1)^2}\right]$$

(iii) $$L(t\sin^3 t) = L\left\{\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right\}$$

$$= -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}s}\{L(3\sin t - \sin 3t)\}$$

$$= -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+1}-\frac{3}{s^2+9}\right)$$

$$= -\frac{3}{4}\left\{-\frac{2s}{(s^2+1)^2}+\frac{2s}{(s^2+9)^2}\right\}$$

$$= \frac{3}{2}s\left\{-\frac{1}{(s^2+1)^2}-\frac{1}{(s^2+9)^2}\right\}$$

(iv) $$L\{(t\sin at)^2\} = L\left[t^2\left(\frac{1-\cos 2at}{2}\right)\right]$$

$$= \frac{1}{2}(-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\{L(1-\cos 2at)\}$$

$$= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left\{\frac{1}{s}-\frac{s}{s^2+4a^2}\right\}$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left\{-\frac{1}{s^2}+\frac{s^2-4a^2}{(s^2+4a^2)^2}\right\}$$

$$= \frac{1}{2}\left[\frac{2}{s^3}+\frac{(s^2+4a^2)^2\cdot 2s-(s^2-4a^2)\cdot 2(s^2+4a^2)\cdot 2s}{(s^2+4a^2)^4}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3}+\frac{2s(12a^2-s^2)}{(s^2+4a^2)^3}\right]$$

$$= \frac{1}{s^3}+\frac{s(12a^2-s^2)}{(s^2+4a^2)^3}$$

Example 5.7 Use Laplace transforms to evaluate the following;

(i) $\int_0^\infty te^{-2t}\sin 3t\,\mathrm{d}t$ (ii) $\int_0^\infty te^{-3t}\cos 2t\,\mathrm{d}t$

(i) $$\int_0^\infty e^{-st}(t\sin 3t)\,\mathrm{d}t = L(t\sin 3t) \quad \text{(by definition)} \tag{1}$$

Now $$L(t\sin 3t) = -\frac{d}{ds}L(\sin 3t)$$

$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2} \tag{2}$$

Putting (2) in (1), we have

$$\int_0^\infty te^{-st}\sin 3t\, dt = \frac{6s}{(s^2+9)^2}, s > 0 \tag{3}$$

Putting $s = 2$ in (3), we get

$$\int_0^\infty te^{-2t}\sin 3t\, dt = \frac{12}{169}.$$

(ii) $$\int_0^\infty e^{-st}(t\cos 2t)\, dt = L(t\cos 2t), \text{ by definition} \tag{1}$$

Now $$L(t\cos 2t) = -\frac{d}{ds}L(\cos 2t)$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{s^2-4}{(s^2+4)^2} \tag{2}$$

Inserting (2) in (1), we have

$$\int_0^\infty te^{-st}\cos 2t\, dt = \frac{s^2-4}{(s^2+4)^2}, s > 0 \tag{3}$$

Putting $s = 3$ in (3), we get

$$\int_0^\infty te^{-3t}\cos 2t\, dt = \frac{5}{169}.$$

Example 5.8 Find the Laplace transforms of the following functions:

(i) $te^{-4t}\sin 3t$; (ii) $t\cosh t\cos t$; (iii) $te^{-2t}\sinh 3t$ (iv) $t^2e^{-t}\cos t$.

(i) $L\{te^{-4t}\sin 3t\} = [L(t\sin 3t)]_{s\to s+4}$ (1)

(by the first shifting property)

Now $$L(t\sin 3t) = -\frac{d}{ds}L(\sin 3t)$$

$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2} \qquad (2)$$

Using (2) in (1), we have

$$L\{te^{-4t}\sin 3t\} = \left[\frac{6s}{(s^2+9)^2}\right]_{s\to s+4}$$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

Note ☑ The same problem has been solved by using an alternative method in Worked Example (6) in Section 5(a).

(ii) $$L\{t\cosh t\cos t\} = L\left\{\frac{t}{2}(e^t + e^{-t})\cos t\right\}$$

$$= \frac{1}{2}[L(t\cos t)_{s\to s-1} + L(t\cos t)_{s\to s+1}] \qquad (1)$$

Now $$L(t\cos t) = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{s^2-1}{(s^2+1)^2} \qquad (2)$$

Using (2) in (1), we have

$$L(t\cosh t\cos t) = \frac{1}{2}\left[\frac{(s-1)^2-1}{(s^2-2s+2)^2} + \frac{(s+1)^2-1}{(s^2+2s+2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-2s}{(s^2-2s+2)^2} + \frac{s^2+2s}{(s^2+2s+2)^2}\right]$$

(iii) $$L\{t\,e^{-2t}\sinh 3t\} = L\{t\sinh 3\,t\}_{s\to s+2} \qquad (1)$$

Now $$L(t\sinh 3t) = -\frac{d}{ds}L(\sinh 3t)$$

$$= -\frac{d}{ds}\left(\frac{3}{s^2-9}\right)$$

$$= \frac{6s}{(s^2-9)^2} \qquad (2)$$

Using (2) in (1), we have

$$L\{te^{-2t}\sinh 3t\}=\frac{6(s+2)}{\{(s+2)^2-9\}^2}=\frac{6(s+2)}{(s-1)^2\,(s+5)^2}$$

Aliter

$$L\{te^{-2t}\sinh 3t\}=L\left\{te^{-2t}\cdot\frac{1}{2}(e^{3t}-e^{-3t})\right\}$$

$$=\frac{1}{2}L\{te^{t}-te^{-5t}\}$$

$$=\frac{1}{2}\left\{\frac{1}{(s-2)^2}-\frac{1}{(s+5)^2}\right\}$$

$$=\frac{1}{2}\left\{\frac{12s+24}{(s-1)^2\,(s+5)^2}\right\}=\frac{6(s+2)}{(s-1)^2\,(s+5)^2}$$

(iv) $L\{t^2 e^{-t}\cos t\} = [L\,(t^2\cos t)]_{s\to s+1}$ (1)

Now $$L(t^2\cos t)=(-1)^2\frac{d^2}{ds^2}L(\cos t)$$

$$=\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$=\frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}$$

$$=\frac{2(s^3-3s)}{(s^2+1)^3} \qquad (2)$$

Using (2) in (1), we have

$$L\{t^2 e^{-t}\cos t\}=\frac{2\{(s+1)^3-3(s+1)\}}{(s^2+2s+2)^3}$$

Example 5.9 Find the inverse Laplace transforms of the following functions:

(i) $\log\left(1-\frac{a}{s}\right)$ (ii) $\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

(iii) $\log\frac{s^2+1}{s(s+1)}$ (iv) $s\log\left(\frac{s-1}{s+1}\right)+k$ (k is a constant)

(i) $L^{-1}\{\phi(s)\}=-\frac{1}{t}L^{-1}\{\phi'(s)\}$ (1)

$$\therefore\ L^{-1}\log\left(1-\frac{a}{s}\right)=L^{-1}\log\left(\frac{s-a}{s}\right)=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s-a}{s}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\frac{d}{ds}\{\log(s-a) - \log s\}$$

$$= -\frac{1}{t}L^{-1}\left(\frac{1}{s-a} - \frac{1}{s}\right)$$

$$= -\frac{1}{t}(e^{at} - 1) = \frac{1}{t}(1 - e^{at})$$

(ii) By rule (1),

$$L^{-1}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) = -\frac{1}{t}L^{-1}\frac{d}{ds}[\log(s^2+a^2) - \log(s^2+b^2)]$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\}$$

$$= \frac{2}{t}(\cos bt - \cos at)$$

(iii) By rule (1),

$$L^{-1}\log\left[\frac{s^2+1}{s(s+1)}\right] = -\frac{1}{t}L^{-1}\frac{d}{ds}[\log(s^2+1) - \log s - \log(s+1)]$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= -\frac{1}{t}(2\cos t - 1 - e^{-t})$$

$$= \frac{1}{t}(1 + e^{-t} - 2\cos t)$$

(iv) By rule (1),

$$L^{-1}\left[s\log\left(\frac{s-1}{s+1}\right) + k\right] = -\frac{1}{t}L^{-1}\frac{d}{ds}[s\log(s-1) - s\log(s+1)] + L^{-1}(k)$$

$$= -\frac{1}{t}L^{-1}\left[\frac{s}{s-1} + \log(s-1) - \frac{s}{s+1} - \log(s+1)\right] + k\delta(t)$$

$$= -\frac{1}{t}L^{-1}\left[\frac{s}{s-1} - \frac{s}{s+1}\right] - \frac{1}{t}L^{-1}[\log(s-1) - \log(s+1)] + k\delta(t)$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2-1}\right] - \frac{1}{t}\left(-\frac{1}{t}\right)L^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] + k\delta(t)$$

Note ☑

$\left[[L^{-1}\left(\frac{s}{s-1}\right)\right.$ and $L^{-1}\left(\frac{s}{s+1}\right)$ do not exist, as $\frac{s}{s-1}$ and $\frac{s}{s+1}$ are improper rational functions. Hence we have simplified $\left(\frac{s}{s-1}-\frac{s}{s+1}\right)$ as $\frac{2s}{s^2-1}$, which is a proper rational function.$]$

$$= -\frac{2}{t}\cosh t + \frac{1}{t^2}(e^t - e^{-t}) + k\,\delta(t)$$
$$= \frac{2}{t^2}\sinh t - \frac{2}{t}\cosh t + k\,\delta(t)$$

Example 5.10 Find the inverse Laplace transforms of the following functions:

(i) $\cot^{-1}(as)$ (ii) $\tan^{-1}\left(\frac{s+a}{b}\right)$

(iii) $\cot^{-1}\left(\frac{2}{s+1}\right)$ (iv) $\tan^{-1}\left(\frac{2}{s^2}\right)$

(i) $$L^{-1}\{\phi(s)\} = -\frac{1}{t}L^{-1}\{\phi'(s)\} \tag{1}$$

$\therefore$
$$L^{-1}\{\cot^{-1}(as)\} = -\frac{1}{t}L^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}s}\cot^{-1}(as)\right]$$
$$= -\frac{1}{t}L^{-1}\left(\frac{-a}{1+a^2s^2}\right)$$
$$= \frac{a}{t}L^{-1}\left(\frac{1}{1+a^2s^2}\right)$$
$$= \frac{1}{t}L^{-1}\left\{\frac{1/a}{s^2+(1/a)^2}\right\} = \frac{1}{t}\sin\frac{t}{a}.$$

(ii) $$L^{-1}\left\{\tan^{-1}\left(\frac{s+a}{b}\right)\right\} = -\frac{1}{t}L^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}s}\tan^{-1}\left(\frac{s+a}{b}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1/b}{1+\left(\frac{s+a}{b}\right)^2}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$$

$$= -\frac{1}{t} e^{-at} \sin bt$$

(iii) $$L^{-1}\left\{\cot^{-1}\left(\frac{2}{s+1}\right)\right\} = -\frac{1}{t} L^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}s}\cot^{-1}\left(\frac{2}{s+1}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\left\{-\frac{1}{1+\frac{4}{(s+1)^2}}\right\}\left\{-\frac{2}{(s+1)^2}\right\}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{2}{(s+1)^2+4}\right]$$

$$= -\frac{1}{t} e^{-t} \sin 2t.$$

(iv) $$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = -\frac{1}{t} L^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}s}\tan^{-1}\left(\frac{2}{s^2}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+\frac{4}{s^4}} \cdot \left(\frac{-4}{s^3}\right)\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{s^4+4}\right] \tag{2}$$

Consider $$\frac{s}{s^4+4} = \frac{s}{(s^2+2)^2-(2s)^2}$$

$$= \frac{s}{(s^2-2s+2)(s^2+2s+2)}$$

$$= \frac{1}{4}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right],$$

by resolving into partial fractions

$$= \frac{1}{4}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore\ L^{-1}\left(\frac{s}{s^4+4}\right) = \frac{1}{4}\left[e^{t}\sin t - e^{-t}\sin t\right]$$

$$= \frac{1}{2}\sin t \sinh t \tag{3}$$

Using (3) in (2), we have

$$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = \frac{2}{t}\sin t \sinh t$$

Example 5.11 Find the Laplace transforms of the following functions:

(i) $\dfrac{\sinh t}{t}$; (ii) $\dfrac{e^{-at}-e^{-bt}}{t}$; (iii) $\dfrac{e^{at}-\cos bt}{t}$; (iv) $\dfrac{2\sin 2t \sin t}{t}$;

(v) $\left(\dfrac{\sin t}{t}\right)^2$.

(i) $$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} L\{f(t)\}\,\mathrm{d}s \tag{1}$$

$$\therefore \quad L\left\{\frac{\sinh t}{t}\right\} = \int_s^{\infty} L(\sinh t)\,\mathrm{d}s$$

$$= \int_s^{\infty} \frac{1}{s^2-1}\,\mathrm{d}s$$

$$= \left[\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right]_s^{\infty}$$

$$= \left[\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right]_{s\to\infty} - \frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$$

$$= \left[\frac{1}{2}\log\left(\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)\right]_{s\to\infty} + \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

$$= \frac{1}{2}\log 1 + \frac{1}{2}\log\left(\frac{s+1}{s-1}\right) = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

(ii) $$L\left\{\frac{e^{-at}-e^{bt}}{t}\right\} = \int_s^{\infty} L(e^{-at}-e^{-bt})\,\mathrm{d}s\text{, by rule} \tag{1}$$

$$= \int_s^{\infty}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)\mathrm{d}s$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_{s\to\infty} - \log\left(\frac{s+a}{s+b}\right)$$

$$= \left[\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_{s\to\infty} + \log\left(\frac{s+b}{s+a}\right)$$

$$= \log 1 + \log\left(\frac{s+b}{s+a}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

(iii) $$L\left\{\frac{e^{at} - \cos bt}{t}\right\} = \int_s^\infty L\,(e^{at} - \cos bt)\,\mathrm{d}s, \text{ by rule (1)}$$

$$= \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right)\mathrm{ds}$$

$$= \left[\log(\mathrm{s-a}) - \frac{1}{2}\log(\mathrm{s}^2+b^2)\right]_s^\infty$$

$$= \left[\log\left(\frac{\mathrm{s-a}}{\sqrt{\mathrm{s}^2+b^2}}\right)\right]_{s\to\infty} - \log\left(\frac{\mathrm{s-a}}{\sqrt{\mathrm{s}^2+b^2}}\right)$$

$$= \left[\log\left(\frac{1-\mathrm{a}/s}{\sqrt{1+b^2/s^2}}\right)\right]_{s\to\infty} + \log\left(\frac{\sqrt{\mathrm{s}^2+b^2}}{\mathrm{s-a}}\right)$$

$$= \log 1 + \log\sqrt{\frac{\mathrm{s}^2+b^2}{(\mathrm{s-a})^2}}$$

$$= \frac{1}{2}\log\left\{\frac{\mathrm{s}^2+b^2}{(\mathrm{s-a})^2}\right\}$$

(IV) $$L\left\{\frac{2\sin 2t\sin t}{t}\right\} = \int_s^\infty L(2\sin 2t\sin t)\,\mathrm{d}s, \text{ by rule (1)}$$

$$= \int_s^\infty L(\cos t - \cos 3t)\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{s}{s^2+1} - \frac{s}{s^2+9}\right)\mathrm{d}s$$

$$= \left[\frac{1}{2}\log\left(\frac{s^2+1}{s^2+9}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2+9}\right)\right]_{s\to\infty} - \frac{1}{2}\log\left(\frac{s^2+1}{s^2+9}\right)$$

$$= \frac{1}{2}\left[\log\left(\frac{1+\frac{1}{s^2}}{1+9/s^2}\right)\right]_{s\to\infty} + \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$$

$$= \frac{1}{2}\log 1 + \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$$

(v) $$L\left\{\frac{f(t)}{t^2}\right\} = \int_s^\infty \int_s^\infty L\{f(t)\}\,\mathrm{d}s\,\mathrm{d}s \tag{2}$$

$\therefore$ $$L\left\{\frac{\sin^2 t}{t^2}\right\} = \int_s^\infty \int_s^\infty L(\sin^2 t)\,\mathrm{d}s\,\mathrm{d}s$$

$$= \int_s^\infty \int_s^\infty L\left\{\frac{1-\cos 2t}{2}\right\}\mathrm{d}s\,\mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s\,\mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)\mathrm{d}s,$$

by putting $a = 0$ and $b = 2$ in (iii) above.

$$= \frac{1}{4}\left[\left\{s\log\left(\frac{s^2+4}{s^2}\right)\right\}_s^\infty - \int_s^\infty s\left\{\frac{2s}{s^2+4} - \frac{2}{s}\right\}\mathrm{d}s\right]$$

by integrating by parts.

$$= \left[\frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right)\right]_{s\to\infty} - \frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right) + \frac{1}{4}\int_s^\infty \frac{8}{s^2+4}\mathrm{d}s$$

$$= L + \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) - \left(\cot^{-1}\frac{s}{2}\right)_s^\infty, \text{ say}$$

$$= L + \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + \cot^{-1}\left(\frac{s}{2}\right) \tag{3}$$

Now $$L = \log\left(1+\frac{4}{s^2}\right)^{s/4}$$

$$= \log\left[\left(1+\frac{4}{s^2}\right)^{s^2/4}\right]^{\frac{1}{s}}$$

$$\lim_{s\to\infty}(L) = \log\left[\lim_{s\to\infty}\left\{\left(1+\frac{4}{s^2}\right)^{s^2/4}\right\}^{\frac{1}{s}}\right]$$

$$= \log(e^0) = \log 1 = 0 \tag{4}$$

Using (4) in (3), we have

$$L\left(\frac{\sin^2 t}{t^2}\right) = \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + \cot^{-1}\left(\frac{s}{2}\right).$$

Example 5.12 Use Laplace transforms to evaluate the following:

(i) $\displaystyle\int_0^\infty \frac{e^{-t}\sin\sqrt{3}\,t}{t}\,dt$ (ii) $\displaystyle\int_0^\infty \frac{\sin^2 t}{t e^t}\,dt$

(iii) $\displaystyle\int_0^\infty \left(\frac{\cos at - \cos bt}{t}\right)dt$ (iv) $\displaystyle\int_0^\infty \left(\frac{e^{-2t}-e^{-4t}}{t}\right)dt.$

(i) $$\int_0^\infty \frac{e^{-t}\sin\sqrt{3}\,t}{t}\,dt = \left[\int_0^\infty e^{-st}\left(\frac{\sin\sqrt{3}\,t}{t}\right)dt\right]_{s=1}$$

$$= L\left(\frac{\sin\sqrt{3}\,t}{t}\right)_{s=1} \tag{1}$$

Now $$L\left(\frac{\sin\sqrt{3}\,t}{t}\right) = \int_s^\infty L(\sin\sqrt{3}\,t)\,ds$$

$$= \int_s^\infty \frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\,ds$$

$$= \left[\sqrt{3}\left(-\frac{1}{\sqrt{3}}\right)\cot^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_s^\infty$$

$$= 0 + \cot^{-1}\left(\frac{s}{\sqrt{3}}\right) \tag{2}$$

Using (2) in (1), we have

$$\int_s^\infty \frac{e^{-1}\sin\sqrt{3}t}{t}\,dt = \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}.$$

(ii) $$\int_0^\infty \frac{\sin^2 t}{t\,e^t}\,\mathrm{d}t = \int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)\mathrm{d}t$$

$$= \left[\int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t}\right)\mathrm{d}t\right]_{s=1}$$

$$= L\left(\frac{\sin^2 t}{t}\right)_{s=1} \qquad (1)$$

Now $$L\left(\frac{\sin^2 t}{t}\right) = L\left(\frac{1-\cos 2t}{2t}\right)$$

$$= \frac{1}{2}\int_s^\infty L(1-\cos 2t)\,\mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left(\log\sqrt{\frac{s^2}{s^2+4}}\right)_{s\to\infty} - \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\left(\log\sqrt{\frac{1}{1+4/s^2}}\right)_{s\to\infty} + \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

$$= \frac{1}{2}\log 1 + \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right) \qquad (2)$$

Using (2) in (1), we have

$$\int_0^\infty \frac{\sin^2 t}{t\,e^t}\,\mathrm{d}t = \frac{1}{2}\log\sqrt{5} = \frac{1}{4}\log 5$$

(iii) $$\int_0^\infty \left(\frac{\cos at - \cos bt}{t}\right)\mathrm{d}t = \left[\int_0^\infty e^{-st}\left(\frac{\cos at - \cos bt}{t}\;\mathrm{d}t\right)\right]_{s=0}$$

$$= L\left\{\frac{\cos at - \cos bt}{t}\right\}_{s=0} \qquad (1)$$

Now $$L\left\{\frac{\cos at-\cos bt}{t}\right\}=\int_s^\infty L(\cos at-\cos bt)\,\mathrm{d}s$$

$$=\int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)\mathrm{d}s$$

$$=\left[\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$=\left[\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_{s\to\infty}-\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$=\log\sqrt{\frac{s^2+b^2}{s^2+a^2}} \tag{2}$$

Using (2) in (1), we have

$$\int_0^\infty\left(\frac{\cos at-\cos bt}{t}\right)\mathrm{d}t=\log\left(\frac{b}{a}\right)$$

(iv) $$\int_0^\infty\left(\frac{e^{-2t}-e^{-4t}}{t}\right)\mathrm{d}t=\left[\int_0^\infty e^{-st}\left(\frac{e^{-2t}-e^{-4t}}{t}\right)\mathrm{d}t\right]_{s=0}$$

$$=L\left(\frac{e^{-2t}-e^{-4t}}{t}\right)_{s=0} \tag{1}$$

Now $$L\left(\frac{e^{-2t}-e^{-4t}}{t}\right)=\int_s^\infty L(e^{-2t}-e^{-4t})\,\mathrm{d}s$$

$$=\int_s^\infty\left(\frac{1}{s+2}-\frac{1}{s+4}\right)\mathrm{d}s$$

$$=\left[\log\left(\frac{s+2}{s+4}\right)\right]_s^\infty$$

$$=\left[\log\left(\frac{s+2}{s+4}\right)\right]_{s\to\infty}-\log\left(\frac{s+2}{s+4}\right)$$

$$=\log\left(\frac{s+4}{s+2}\right) \tag{2}$$

Using (2) in (1), we have

$$\int_0^\infty\left(\frac{e^{-2t}-e^{-4t}}{t}\right)\mathrm{d}t=\log 2$$

Example 5.13 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s}{(s^2+a^2)^2}$ (ii) $\dfrac{s}{(s^2-4)^2}$

(iii) $\dfrac{4(s-1)}{(s^2-2s+5)^2}$ (iv) $\dfrac{s^2-3}{(s^2+4s+5)^2}$

(v) $\dfrac{s+1}{(s^2+2s-8)^2}$

(i) $$L^{-1}\{\phi(s)\} = t\cdot L^{-1}\int_s^\infty \phi(s)\,\mathrm{d}s \tag{1}$$

$\therefore$ $$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = t\cdot L^{-1}\int_s^\infty \frac{s}{(s^2+a^2)^2}\,\mathrm{d}s$$

$$= tL^{-1}\int_{s^2+a^2}^\infty \frac{1}{2}\frac{\mathrm{d}x}{x^2}, \text{ on putting } s^2+a^2=x$$

$$= \frac{t}{2}L^{-1}\left(-\frac{1}{x}\right)_{s^2+a^2}^\infty$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$= \frac{t}{2a}\sin at.$$

(ii) $$L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\} = t\cdot L^{-1}\int_s^\infty \frac{s}{(s^2-4)^2}\,\mathrm{d}s$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{s^2-4}\right), \text{ as in (i) above.}$$

$$= \frac{t}{4}\sinh 2t$$

Note ☑ The inverse transform in this case can also found out by resolving the given function into partial fractions.

(iii) $$L^{-1}\left\{\frac{4(s-1)}{(s^2-2s+5)^2}\right\} = 4L^{-1}\left\{\frac{s-1}{\left[(s-1)^2+2^2\right]^2}\right\}$$

$$= 4e^t L^{-1}\left\{\frac{s}{(s^2+2^2)^2}\right\}, \text{ by the first shifting property}$$

$$= 4e^t \frac{t}{4} \sin 2t \text{ , as in problem (i)}$$

$$= te^t \sin 2t.$$

(iv) $$L^{-1}\left\{\frac{s^2-3}{(s^2+4s+5)^2}\right\} = L^{-1}\left[\frac{(s^2+4s+5)-(4s+8)}{(s^2+4s+5)^2}\right]$$

$$= L^{-1}\left\{\frac{1}{s^2+4s+5}\right\} - 4L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} - 4L^{-1}\left[\frac{s+2}{\{(s+2)^2+1\}^2}\right]$$

$$= e^{-2t}\sin t - 4e^{-2t}\frac{t}{2}\sin t, \text{ as in problem (i)}$$

$$= e^{-2t}(1-2t)\sin t.$$

(v) $$L^{-1}\left\{\frac{s+1}{(s^2+2s-8)^2}\right\} = L^{-1}\left\{\frac{s+1}{[(s+1)^2-3^2]^2}\right\}$$

$$= e^{-t}\cdot L^{-1}\left\{\frac{s}{(s^2-3^2)^2}\right\} ,$$

$$= e^{-t}\cdot\frac{t}{6}\sinh 3t, \text{proceeding as in problem (ii)}$$

$$= \frac{t}{6}e^{-t}\sinh 3t$$

EXERCISE 5(b)

***Part** A*

(Short Answer Questions)

1. State the formula for the Laplace transform of a periodic function.
2. Find the Laplace transform of $f(t) = t$, in $0 < t < 1$ if $f(t+1) = f(t)$
3. State the relation between the Laplace transforms of $f(t)$ and $t \cdot f(t)$.
4. State the relation between the inverse Laplace transforms of $\phi(s)$ and $\phi'(s)$.
5. State the relation between the Laplace transforms of $f(t)$ and $\frac{1}{t}f(t)$.
6. State the relation between the inverse Laplace transform of $\phi(s)$ and its integral.

   Find the Laplace transforms of the following functions:

7. $\frac{1}{2a}t\sin at$ 8. $t\cos at$ 9. $\sin kt - kt\cos kt$

10. $\sin kt + kt\cos kt$ 11. $\dfrac{t}{a^2}(1-\cos at)$

12. $\cos kt - \dfrac{1}{2}kt\sin kt.$

Find the inverse Laplace transforms of the following functions:

13. $\log\left(\dfrac{s+1}{s-1}\right)$ 14. $\log\left(\dfrac{s+1}{s}\right)$ 15. $\log\left(1+\dfrac{a}{s}\right)$

16. $\log\left(\dfrac{s+a}{s+b}\right)$ 17. $\log\left(\dfrac{s}{s-1}\right)$ 18. $\log\left(\dfrac{s^2+1}{s^2+4}\right)$

19. $\cot^{-1}s$ 20. $\tan^{-1}\dfrac{a}{s}$

Find the Laplace transforms of the following functions:

21. $\dfrac{\sin at}{t}$ 22. $\dfrac{1-e^{-t}}{t}$ 23. $\dfrac{1-e^{t}}{t}$

24. $\dfrac{1-\cos at}{t}$ 25. $\dfrac{\sin^2 t}{t}$

Part B

Find the Laplace transforms of the following periodic functions:

26. $$f(t) = E, \text{ in } 0 \le t < \frac{1}{E}$$
$$= 0, \text{ in } \frac{1}{E} \le t < \frac{2\pi}{n}$$

given that $f\left(t+\dfrac{2\pi}{n}\right) = f(t)$

27. $$f(t) = E, \text{ in } 0 \le t < \frac{T}{2}$$
$$= -E, \text{ in } T/2 \le t < T$$

given that $f(t+T) = f(t)$

28. $f(t) = e^t$, in $0 < t < 2\pi$ and $f(t+2\pi) = f(t)$

29. $f(t) = \sin\left(\dfrac{t}{2}\right)$, in $0 < t < 2\pi$ and $f(t+2\pi) = f(t)$

30. $f(t) = |\cos\omega t|,\ t \ge 0$

$$\left[\text{Hint: } f(t) \text{ is periodic with period } (\pi/\omega) \text{ and } \int_0^{\pi/\omega} e^{-st}\,|\cos\omega t|\,dt = \int_0^{\pi/2\omega} e^{-st}\cos\omega t\,dt + \int_{\pi/2\omega}^{\pi/\omega} e^{-st}(-\cos\omega t)\,dt\right]$$

31. $$f(t) = t, \text{ in } 0 < t < \pi$$
$$= 0, \text{ in } \pi < t < 2\pi,$$
given that $f(t + 2\pi) = f(t)$

32. $$f(t) = \sin t, \text{ in } 0 < t < \pi$$
$$= 0, \text{ in } \pi < t < 2\pi,$$
given that $f(t + 2\pi) = f(t)$.

33. $$f(t) = 0, \text{ in } 0 < t < \frac{\pi}{\omega}$$
$$= -\sin \omega t, \text{ in } \frac{\pi}{\omega} < t < \frac{2\pi}{\omega},$$
given that $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$.

34. $$f(t) = t, \text{ in } 0 < t < \pi$$
$$= 2\pi - t, \text{ in } \pi < t < 2\pi,$$
given that $f(t + 2\pi) = f(t)$.

35. $$f(t) = t, \text{ in } 0 < t < \pi$$
$$= \pi - t, \text{ in } \pi < t < 2\pi,$$
given that $f(t + 2\pi) = f(t)$.

Find the Laplace transforms of the following functions:

36. $t \sinh^3 t$
37. $t \cos^3 2t$
38. $t \sin 3t \sin 5t$
39. $t \sin 5t \cos t$
40. $(t \cos 2t)^2$
41. $t^2 \sin t \cos 2t$
42. $t^2 e^{-2t} \sin 3t$
43. $te^{3t} \cos 4t$
44. $t^2 e^{-3t} \cosh 2t$
45. $t \sinh 2t \sin 3t$

Find the inverse Laplace transforms of the following functions:

46. $\log\left(1 + \frac{a^2}{s^2}\right)$
47. $\log \frac{s^2 + a^2}{(s+b)^2}$
48. $\log \frac{(s-2)^2}{s^2+1}$
49. $s \log\left(\frac{s-a}{s+a}\right) + a$
50. $\tan^{-1}\left(\frac{1}{2s}\right)$
51. $\tan^{-1}\left(\frac{s+2}{3}\right)$
52. $\cot^{-1}\left(\frac{a}{s+b}\right)$
53. $\tan^{-1}(s^2)$

Find the values of the following integrals, using Laplace transforms:

54. $\int_0^\infty t\, e^{-2t} \cos 2t \, dt$
55. $\int_0^\infty t^2\, e^{-t} \sin t \, dt$
56. $\int_0^\infty \left(\frac{e^{-t} - e^{-3t}}{t}\right) dt$

57. $\int_0^\infty \frac{(1-\cos t)\, e^{-t}}{t}\,dt$ 58. $\int_0^\infty \left(\frac{e^{-at}-\cos bt}{t}\right) dt$ 59. $\int_0^\infty \frac{e^{-\sqrt{2}t}\sin t\,\sinh t}{t}\,dt$

Find the Laplace transforms of the following functions:

60. $\frac{1-e^{-t}}{t}$ 61. $\frac{1-\cos at}{t}$ 62. $\left(\frac{\sin 2t}{\sqrt{t}}\right)^2$

63. $\frac{\sin 3t\,\sin t}{t}$ 64. $\frac{1-\cos t}{t^2}$

Find the Laplace inverse transforms of the following functions:

65. $\frac{s}{(s^2+1)^2}$ 66. $\frac{s}{(s^2-a^2)^2}$ 67. $\frac{s-2}{(s^2-4s+5)^2}$

68. $\frac{(s-a)^2}{(s^2-a^2)^2}$ 69. $\frac{s^2+8s+16}{(s^2+6s+10)^2}$ 70. $\frac{s+4}{(s^2+8s+15)^2}$

5.8 LAPLACE TRANSFORMS OF DERIVATIVES AND INTEGRALS

In the following two theorems we find the Laplace transforms of the derivatives and integrals of a function $f(t)$ in terms of the Laplace transform of $f(t)$. These results will be used in solving differential and integral equations using Laplace transforms.

Theorem

If $f(t)$ is continuous in $t \geq 0$, $f'(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and $f(t)$ and $f'(t)$ are of the exponential order, then
$L\{f'(t)\} = sL\{f(t)\} - f(0)$

Proof:

The given conditions ensure the existence of the Laplace transforms of $f(t)$ and $f'(t)$.

By definition, $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

$$= \int_0^\infty e^{-st}\,d\,[f(t)]$$

$$= \left[e^{-st}\cdot f(t)\right]_0^\infty - \int_0^\infty (-s)\,e^{-st}\, f(t)\,dt, \text{on integration by parts.}$$

$$= \lim_{t\to\infty}[e^{-st} f(t)] - f(0) + s\cdot L\{f(t)$$

$$= 0 - f(0) + sL\,\{f(t)\} \quad [\because f(t) \text{ is of the exponential order}]$$

$$= sL\,\{f(t)\} - f(0) \tag{1}$$

Corollary 1

In result (1) if we replace $f(t)$ by $f'(t)$ we get

$$L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$

$$= s[sL\{f(t)\} - f(0)] - f'(0), \qquad \text{again by (1)}$$

$$= s^2 L\{f(t)\} - sf(0) - f'(0) \qquad (2)$$

Note ☑

1. Result (2) holds good, if $f(t)$ and $f'(t)$ are continuous in $t \geq 0$, $f''(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and $f(t)$, $f'(t)$ and $f''(t)$ are of the exponential order.

Corollary 2

Repeated application of (1) gives the following result:

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \qquad (3)$$

Note ☑

2. Result (3) holds good, if $f(t)$ and its first $(n-1)$ derivatives are continuous in $t \geq 0$, $f^{(n)}(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and $f(t), f'(t), \ldots, f^{(n)}(t)$ are of the exponential order.

3. If we take $L\{f(t)\} = \phi(s)$, result (1) becomes

$$L\{f'(t)\} = s\phi(s) - f(0) \qquad (4)$$

If we further assume that $f(0) = 0$, the result becomes

$$\mathrm{L}\{f'(t)\} = s\phi(s) \qquad (5)$$

In terms of the inverse Laplace operator, (5) becomes

$$L^{-1}\{s\phi(s)\} = f'(t) \qquad (6)$$

From result (6), we get the following working rule:

$$L^{-1}\{s\phi(s) = \frac{\mathrm{d}}{\mathrm{d}t} L^{-1}\{\phi(s), \text{ provided that}$$

$$f(0) = L^{-1}\{\phi(s)\}_{t=0} = 0.$$

Thus, to find the inverse transform of the product of two factors, one of which is 's', we ignore 's', and find the inverse transform of the other factor; we call it $f(t)$, verify that $f(0) = 0$ and get $f'(t)$, which is the required inverse transform.

4. In a similar manner, from result (2) we get

$$L^{-1}\{s^2\phi(s)\} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} L^{-1}\{\phi(s)\}, \text{ provided that}$$

$$f(0) = 0 \text{ and } f'(0) = 0, \text{ where}$$

$$f(t) = L^{-1}\{\phi(s)\}.$$

Theorem

If $f(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and is of the exponential order, then

$$L\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\}.$$

Proof:

Let $$g(t) = \int_0^t f(t)\,\mathrm{d}t$$

$\therefore$ $g'(t) = f(t)$

Under the given conditions, it can be shown that the Laplace transforms of both $f(t)$ and $g(t)$ exist.

Now by the previous theorem,

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

i.e., $$s\cdot L\left[\int_0^t f(t)\,\mathrm{d}t\right] - \int_0^0 f(t)\,\mathrm{d}t = L\{f(t)\}$$

$\therefore$ $$L\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\} \tag{1}$$

Corollary

$L\left[\int_0^t\int_0^t f(t)\,\mathrm{d}t\,\mathrm{d}t\right] = \frac{1}{s^2}L\{f(t)\}$, as explained below.

Let $$\int_0^t f(t)\,\mathrm{d}t = g(t).$$

Then, by result (1) above,

$$L\int_0^t g(t)\,\mathrm{d}t = \frac{1}{s}L\{g(t)\}$$

i.e., $$L\left[\int_0^t\int_0^t f(t)\,\mathrm{d}t\,\mathrm{d}t\right] = \frac{1}{s}L\int_0^t f(t)\mathrm{d}t$$

$$= \frac{1}{s}\cdot\frac{1}{s}L\{f(t)\}\text{, again by} \quad (1)$$

$$= \frac{1}{s^2}L\{f(t)\} \tag{2}$$

Generalising (2), we get

$$L\left[\int_0^t\int_0^t\cdots\int_0^t f(t)\,(\mathrm{d}t)^n\right] = \frac{1}{s^n}L\{f(t)\} \tag{3}$$

Note ☑

1. If we put $L\{f(t)\} = \phi(s)$, result (1) becomes

$$L\left[\int_0^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}\phi(s) \tag{4}$$

Result (4) can be expressed, in terms of L^{-1} operator, as

$$L^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t f(t)\,\mathrm{d}t \tag{5}$$

From (5), we get the following rule:

$$L^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t L^{-1}\{\phi(s)\}\,\mathrm{d}t \quad .$$

Thus, to find the inverse Laplace transform of the product of two factors, one of which is $\frac{1}{s}$, we ignore $\frac{1}{s}$, find the inverse transform of the other factor and integrate it with respect to t between the limits 0 and t.

2. In a similar manner, from (2) above, we get

$$L^{-1}\left\{\frac{1}{s^2}\phi(s)\right\} = \int_0^t\int_0^t L^{-1}\{\phi(s)\}\,\mathrm{d}t\,\mathrm{d}t\,.$$

3. $$L\left[\int_a^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\} + \frac{1}{s}\int_a^0 f(t)\,\mathrm{d}t$$

If we let $\quad g(t) = \int_a^t f(t)\,\mathrm{d}t$ and $g'(t) = f(t)$,

we get $\quad L\{g'(t)\} = sL\{g(t)\} - g(0)$

i.e., $\quad L\{f(t)\} = sL\left[\int_a^t f(t)\,\mathrm{d}t\right] - \int_a^0 f(t)\,\mathrm{d}t$

or $\quad L\left[\int_a^t f(t)\,\mathrm{d}t\right] = \frac{1}{s}L\{f(t)\} + \frac{1}{s}\int_a^0 f(t)\,\mathrm{d}t.$

5.9 INITIAL AND FINAL VALUE THEOREMS

We shall now consider two results, which are derived by applying the theorem on Laplace transform of the derivative of a function.

The first result, known as the initial value theorem, gives a relation between $\lim_{t\to 0}[f(t)]$ and $\lim_{s\to\infty}[s\phi(s)]$, where $\phi(s) = L\{f(t)\}$

The second result, known as the final value theorem, gives a relation between
$\lim_{t\to\infty}[f(t)]$ and $\lim_{s\to 0}[s\phi(s)]$.

5.9.1 Initial Value Theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L\{f(t)\} = \phi(s)$, then

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[s\phi(s)]$$

Proof:

We know that $L\{f'(t)\} = s\phi(s) - f(0)$

$\therefore$ $s\phi(s) = L\{f'(t)\} + f(0)$

$$= \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t + f(0)$$

$\therefore$
$$\lim_{s\to\infty}[s\phi(s) = \lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,\mathrm{d}t + f(0),$$

$$= \int_0^\infty \lim_{s\to\infty}\{e^{-st} f'(t)\}\,\mathrm{d}t + f(0),$$

assuming that the conditions for the interchange of the operations of integration and taking limit hold.

i.e. $\lim_{s\to\infty}[s\phi(s)] = 0 + f(0)$

$$= \lim_{t\to 0}[f(t)].$$

5.9.2 Final Value Theorem

If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L\{f(t)\} = \phi(s)$, then
$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\phi(s)]$, provided all the singularities of $\{s\phi(s)]$ are in the left half plane $Rl(s) < 0$.

Proof:

We know that $L\{f'(t)\} = s\phi(s) - f(0)$

$\therefore$ $s\phi(s) = L\{f'(t)\} + f(0)$

$$= \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t + f(0)$$

$$\therefore \qquad \lim_{s\to 0}[s\phi(s)] = \lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0$$

$$= \int_0^\infty \lim_{s\to 0}\{e^{-st} f'(t)\}\mathrm{d}t + f(0)$$, assuming that the conditions for the interchange of the operations of integration and taking limit hold.

i.e.
$$\lim_{s\to 0}[s\phi(s)] = \int_0^\infty f'(t)\mathrm{d}t + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= \lim_{t\to\infty}[f(t)] - f(0) + f(0)$$

Thus
$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\phi(s)]$$

5.10 THE CONVOLUTION

Another result, which is of considerable practical importance, is the convolution theorem that enables us to find the inverse Laplace transform of the product of $\bar{f}(s)$ and $\bar{g}(s)$ in terms of the inverse transforms of $\bar{f}(s)$ and $\bar{g}(s)$.

Definition The *convolution* or *convolution integral* of two function $f(t)$ and $g(t)$, defined in $t \geq 0$, is defined as the integral

$$\int_0^t f(u)\, g(t-u)\, \mathrm{d}u$$

It is denoted as $f(t) * g(t)$ or $(f * g)(t)$

i.e.
$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, \mathrm{d}u$$

$$= \int_0^t f(t-u)\, g[t-(t-u)]\, \mathrm{d}u$$, on using the result

$$\int_0^t \phi(u)\,\mathrm{d}u = \int_0^t \phi(t-u)\,\mathrm{d}u$$

$$= \int_0^t g(u) f(t-u)\,\mathrm{d}u$$

$$= g(t) * f(t).$$

Thus the convolution product is commutative.

5.10.1 Convolution Theorem

If $f(t)$ and $g(t)$ are Laplace transformable,

then $$L\{f(t) * g(t)\} = L\{f(t)\} \cdot L\{g(t)\}$$

Proof:

By definition,
$$L\{f(t) * g(t)\} = \int_0^\infty e^{-st}\{f(t) * g(t)\}\,\mathrm{d}t \qquad ,$$
$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)\mathrm{d}u\right]\mathrm{d}t,$$

by the definition of convolution.

$$= \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\mathrm{d}u\,\mathrm{d}t \tag{1}$$

The region of integration for the double integral (1) is bounded by the lines $u = 0$, $u = t$, $t = 0$ and $t = \infty$ and is shown in the Fig. 5.6.

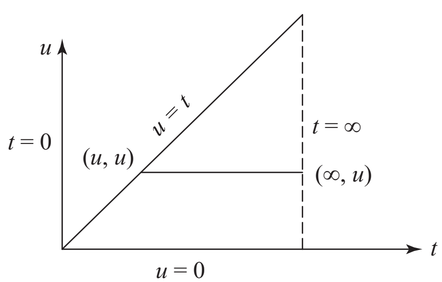


Fig. 5.6

Changing the order of integration in (1), we get,

$$L\{f(t) * g(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\,g(t-u)\,\mathrm{d}t\,\mathrm{d}u \tag{2}$$

In the inner integral in (2), on putting $t - u = v$ and making the consequent changes, we get,

$$L\{f(t) * g(t)\} = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u)g(v)\,\mathrm{d}v\,\mathrm{d}u$$
$$= \int_0^\infty e^{-su} f(u)\left[\int_0^\infty e^{-sv} g(v)\mathrm{d}v\right]\mathrm{d}u$$

$$= \int_0^\infty e^{-su} f(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv}\; g(v)\mathrm{d}v$$

$$= \int_0^\infty e^{-st} f(t)\mathrm{d}t \cdot \int_0^\infty e^{-st} g(t)\mathrm{d}t,$$

on changing the dummy variables u and v.

$$= L\{f(t)\} \cdot L\{g(t)\}$$

Note☑ If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$, the convolution theorem can be put as

$$L\{f(t) * g(t)\} = \bar{f}(s) \cdot \bar{g}(s) \tag{3}$$

In terms of the inverse Laplace operator, result (3) can be written in the following way.

$$L^{-1}\{\bar{f}(s) \cdot \bar{g}(s)\} = f(t) * g(t)$$

$$= \int_0^t f(u)\, g(t-u)\, \mathrm{d}u \tag{4}$$

Result (4) means that the inverse Laplace transform of the ordinary product of two functions of s is equal to the convolution product of the inverses of the individual functions.

WORKED EXAMPLE 5(c)

Example 5.1 Using the Laplace transforms of derivatives, find the Laplace transforms of

(i) e^{-at} (ii) $\sin at$

(iii) $\cos^2 t$ (iv) t^n (n is a positive integer)

(i) $$L\{f'(t)\} = sL\{f(t)\} - f(0) \tag{1}$$

Putting $f(t) = e^{-at}$ in (1), we get

$$L(-ae^{-at}) = sL(e^{-at}) - 1$$

i.e. $(s + a)\, L(e^{-at}) = 1$

$$\therefore \quad L(e^{-at}) = \frac{1}{s+a}$$

(ii) $$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0) \tag{2}$$

Putting $f(t) = \sin at$ in (2), we get

$$L(-a^2 \sin at) = s^2 L(\sin at) - s \times 0 - a$$

i.e. $(s^2 + a^2)\, L(\sin at) = a$

$\therefore$ $$L(\sin at) = \frac{a}{s^2 + a^2}$$

(iii) Putting $f(t) = \cos^2 t$ in (1), we get

$$L(-2\cos t \sin t) = sL(\cos^2 t) - 1$$

i.e. $$s \cdot L(\cos^2 t) = 1 - L(\sin 2t)$$

$$= 1 - \frac{2}{s^2 + 4}$$

$$= \frac{s^2 + 2}{s^2 + 4}$$

$\therefore$ $$L(\cos^2 t) = \frac{s^2 + 2}{s(s^2 + 4)}$$

(iv) $$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{(n-1)}(0) \qquad (3)$$

Putting $f(t) = t^n$ in (3) and nothing that $f^{(n)}(\text{t}) = n!$

and $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, we get

$$L\{n!\} = s^n L(t^n)$$

i.e. $$n! L(1) = \text{s}^n L(t^n)$$

i.e. $$n! \frac{1}{s} = s^n L(t^n)$$

$\therefore$ $$L(t^n) = \frac{n!}{s^{n+1}}$$

Example 5.2 Find the Laplace transform of $\sqrt{\dfrac{t}{\pi}}$ and hence find $L\left\{\dfrac{1}{\sqrt{\pi t}}\right\}$.

$$L\left\{\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{\sqrt{\pi}} L(t^{1/2}) = \frac{1}{\sqrt{\pi}} = \frac{\boxed{(3/2)}}{s^{3/2}}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\frac{1}{2}\boxed{(1/2)}}{s^{3/2}} = \frac{1}{2s^{3/2}} \qquad \left(\because \boxed{(1/2)} = \sqrt{\pi}\right)$$

In the result $L\{f'(t)\} = sL\{f(t)\} - f(0)$, we put

$$f(t) = \sqrt{\frac{t}{\pi}}, \text{ we get}$$

$$L\left\{\frac{1}{2\sqrt{\pi t}}\right\} = s \cdot \frac{1}{2s^{3/2}} - 0$$

$$= \frac{1}{2\sqrt{s}}$$

$$\therefore \qquad L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}.$$

Aliter

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = L\left\{\frac{1}{t}\cdot\sqrt{t/\pi}\right\}$$

$$= \int_s^\infty L(\sqrt{t/\pi})\,\mathrm{d}s$$

$$= \int_s^\infty \frac{1}{2s^{3/2}}\,\mathrm{d}s$$

$$= \left(-\frac{1}{\sqrt{s}}\right)_s^\infty = \frac{1}{\sqrt{s}}.$$

Example 5.3 Using the Laplace transforms of the derivatives, find

(i) $L(t\cos at)$ and hence $L(\sin at - at\cos at)$ and $L(\cos at - at\sin at)$

(ii) $L(t\sinh at)$ and hence $L(\sinh at + at\cosh at)$ and $L\left(\cosh at + \frac{1}{2}at\sinh at\right)$

(i) $$L\{f''(t)\} = s^2L\{f(t)\} - sf(0)\text{-}f'(0) \tag{1}$$

Put $f(t) = t\cos at$ in (1)

Then $f'(t) = \cos at - at\sin at$ and $f'(0) = 1$

$$f''(t) = -a^2 t\cos at - 2a\sin at.$$

$$\therefore \qquad L\{-a^2 t\cos at - 2a\sin at\} = s^2 L\{t\cos at\} - 1$$

$[\because f(0) = 0]$ and $f'(0) = 1$

i.e. $$(s^2 + a^2)L(t\cos at) = 1 - 2aL(\sin at)$$

$$= 1 - \frac{2a^2}{s^2 + a^2}$$

$$= \frac{s^2 - a^2}{s^2 + a^2}$$

$$\therefore \qquad L(t\cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

Now $$L(\sin at - at\cos at) = \frac{a}{s^2 + a^2} - \frac{a(s^2 - a^2)}{(s^2 + a^2)^2}$$

$$= \frac{a\left\{(s^2+a^2)-(s^2-a^2)\right\}}{(s^2+a^2)^2}$$

$$= \frac{2a^3}{(s^2+a^2)^2}$$

Taking $f(t) = t \cos at$ in the result

$L\{f'(t)\} = sL\{f(t)\} - f(0)$, we get

$$L\{\cos at - at \sin at\} = sL\,(t \cos at) - 0$$

$$= \frac{s(s^2-a^2)}{(s^2+a^2)^2}$$

(ii) Put $f(t) = t \sinh at$ in (1).

Then $f'(t) = \sinh at + at \cosh at$ and $f'(0) = 0$

$f''(t) = a^2 t \sinh at + 2a \cosh at$

$\therefore L\{a^2 t \sinh at + 2a \cosh at\}$

$= s^2 L(t \sinh at)$ $[\because f(0) = 0 = f'(0)]$

i.e. $(s^2 - a^2)\, L(t \sinh at) = 2a\, L(\cosh at)$

$$= \frac{2as}{s^2-a^2}$$

$$\therefore \quad L(t \sinh at) = \frac{2as}{(s^2-a^2)^2}$$

In the result $L\{f'(t)\} = sL\,\{f(t)\} - f(0)$, if we put $f(t) = t \sinh at$, we have

$$L(\sinh at + at \cosh at) = \frac{2as^2}{(s^2-a^2)^2} \quad [\because f(0) = 0]$$

Now
$$L\left(\cosh at + \frac{1}{2} at \sinh at\right)$$

$$= L(\cosh at) + \frac{a}{2} L(t \sinh at)$$

$$= \frac{s}{s^2-a^2} + \frac{a}{2} \cdot \frac{2as}{(s^2-a^2)^2}$$

$$= \frac{s(s^2-a^2)+a^2 s}{(s^2-a^2)^2} = \frac{s^3}{(s^2-a^2)^2}.$$

Example 5.4 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s}{(s+2)^4}$ (ii) $\dfrac{s^2}{(s-2)^3}$

(iii) $\dfrac{s}{s^2+4s+5}$ (iv) $\dfrac{s}{(s+2)(s+3)}$

(v) $\dfrac{s}{(s^2+1)(s^2+4)}$

(i) $L^{-1}\{s\phi(s)\} = \dfrac{d}{dt}L^{-1}\{\phi(s)\}$, provided $L^{-1}\{\phi(s)\}$ vanishes at $t=0$ (1)

To find $L^{-1}\left\{\dfrac{s}{(s+2)^4}\right\}$, let us first find $f(t) = L^{-1}\left\{\dfrac{1}{(s+2)^4}\right\}$ and then apply rule (1)

Now
$$f(t) = e^{-2t}L^{-1}\left(\frac{1}{s^4}\right)$$
$$= e^{-2t}\frac{1}{3!}t^3 = \frac{1}{6}t^3e^{-2t}$$

We note that $f(0) = 0$

$\therefore$ By (1),
$$L^{-1}\left\{\frac{s}{(s+2)^4}\right\} = \frac{d}{dt}\left(\frac{1}{6}t^3e^{-2t}\right)$$
$$= \frac{1}{6}(-2t^3e^{-2t} + 3t^2e^{-2t})$$
$$= \frac{1}{6}t^2e^{-2t}(3-2t).$$

(ii) $L^{-1}\{s^2\phi(s)\} = \dfrac{d^2}{dt^2}L^{-1}\{\phi(s)\}$, provided

$f(0) = 0\, f'(0) = 0$, where $f(t) = L^{-1}\{\phi(s)\}$. (2)

To find $L^{-1}\left\{\dfrac{s^2}{(s-2)^3}\right\}$, we shall find $f(t) = L^{-1}\left\{\dfrac{1}{(s-2)^3}\right\}$ and then apply rule (2).

Now
$$f(t) = e^{2t}L^{-1}\left(\frac{1}{s^3}\right)$$
$$= e^{2t}\cdot\frac{1}{2!}t^2 = \frac{1}{2}t^2e^{2t}$$
$$f'(t) = \frac{1}{2}(2t^2e^{2t} + 2te^{2t})$$

We note that $f(0) = 0$ and $f'(0) = 0$

$\therefore$ By (2),
$$L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = \frac{d^2}{dt^2}\left\{\frac{1}{2}t^2e^{2t}\right\}$$

$$= \frac{d}{dt}[(t^2 + t)e^{2t}]$$

$$= 2(t^2 + t)e^{2t} + (2t + 1)e^{2t}$$

$$= (2t^2 + 4t + 1)e^{2t}$$

(iii) To find $L^{-1}\left\{\frac{s}{s^2 + 4s + 5}\right\}$,

we shall find $f(t) = L^{-1}\left\{\frac{1}{s^2 + 4s + 5}\right\}$ and then apply the rule (1).

Now $f(t) = L^{-1}\left\{\frac{1}{(s+2)^2 + 1}\right\}$

$$= e^{-2t} \sin t$$

and $f(0) = 0.$

$\therefore$ By (l), $L^{-1}\left\{\frac{s}{s^2 + 4s + 5}\right\} = \frac{d}{dt}(e^{-2t} \sin t)$

$$= e^{-2t} \cos t - 2e^{-2t} \sin t$$

$$= e^{-2t} (\cos t - 2 \sin t)$$

(iv) To find $L^{-1}\left\{\frac{s}{(s+2)(s+3)}\right\}$, we shall find $f(t) = L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$ and then apply the rule (1).

Now $f(t) = L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$

$$= L^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+3}\right\},$$

by resolving the function into partial fractions.

$$= e^{-2t} - e^{-3t}$$

and $f(0) = 0.$

$\therefore$ By (1), $L^{-1}\left\{\frac{s}{(s+2)(s+3)}\right\} = \frac{d}{dt}(e^{-2t} - e^{-3t})$

$$= 3e^{-3t} - 2e^{-2t}$$

(v) To find $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$, we shall find $f(t) = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$

and then apply the rule (1).

Now $$f(t) = L^{-1}\left\{\frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}\right\},$$

by resolving the function into partial fractions.

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$

and $f(0) = 0$

$\therefore$ By (1), $$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\} = \frac{d}{dt}\left\{\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right\}$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

Note ☑ We have solved the problems in the above example by using the working rule derived from the theorem on Laplace transforms of derivatives. They can be solved by elementary methods, such as partial fraction methods, discussed in Section 5(a) also.

Example 5.5 Find the inverse Laplace transforms of the following functions.

(i) $\dfrac{s^2}{(s^2+a^2)^2}$ (ii) $\dfrac{s^3}{(s^2+a^2)^2}$

(iii) $\dfrac{(s+1)^2}{(s^2+2s+5)^2}$ (iv) $\dfrac{s^2}{(s^2-4)^2}$

(v) $\dfrac{(s-3)^2}{(s^2-6s+5)^2}$

(i) Let $$f(t) = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$= \frac{t}{2a}\sin at$$ [Refer to Worked Example (13) (i) in Section 5(b)]

We note that $f(0) = 0$

Now $$L^{-1}\{s\phi(s)\} = \frac{d}{dt}L^{-1}\{\phi(s)\}, \text{ provided } f(0) = 0,$$

where $$f(t) = L^{-1}\{\phi(s)\} \quad (1)$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{s \cdot \frac{s}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{dt}\left(\frac{t}{2a}\sin at\right), \text{ by rule (1)}$$

$$= \frac{1}{2a}(\sin at + at \cos at) \tag{2}$$

(ii) Let $$f(t) = L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{1}{2a}(\sin at + at \cos at)\text{, by (2)}$$

we note that $f(0) = 0$

Now $$L^{-1}\left\{\frac{s^3}{(s^2+a^2)^2}\right\} = L^{-1}\left\{s \cdot \frac{s^2}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{dt}\left[\frac{1}{2a}(\sin at + at \cos at)\right]\text{, by rule (1)}$$

$$= \frac{1}{2}(2\cos at - at \sin at).$$

(iii) $$L^{-1}\left\{\frac{(s+1)^2}{(s^2+2s+5)^2}\right\} = L^{-1}\left[\frac{(s+1)^2}{\{(s+1)^2+2^2\}^2}\right]$$

$$= e^{-t}L^{-1}\left\{\frac{s^2}{(s^2+2^2)^2}\right\}\text{, by the first shifting property}$$

$$= \frac{1}{4}e^{-t}(\sin 2t + 2t \cos 2t)\text{, by (2)}$$

(iv) Let $$f(t) = L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\}$$

$$= \frac{t}{4}\sinh 2t$$

[Refer to Worked Example (13) (ii) in Section 5(b)]

We note that $f(0) = 0$.

Now $$L^{-1}\left\{\frac{s^2}{(s^2-4)^2}\right\} = L^{-1}\left\{s \cdot \frac{s}{(s^2-4)^2}\right\}$$

$$= \frac{d}{dt}L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\}\text{, by rule (1)}$$

$$= \frac{d}{dt}\left(\frac{t}{4}\sinh 2t\right)$$

$$= \frac{1}{4}(\sinh 2t + 2t \cosh 2t) \tag{3}$$

(v) $$L^{-1}\left[\frac{(s-3)^2}{(s^2-6s+5)^2}\right]=L^{-1}\left\{\frac{(s-3)^2}{\{(s-3)^2-4\}^2}\right\}$$

$$=e^{3t}L^{-1}\left[\frac{s^2}{(s^2-4)^2}\right], \text{ by the first shifting property.}$$

$$=\frac{1}{4}e^{3t}(\sinh 2t+2t\cosh 2t)\text{ , by (3).}$$

Example 5.6 Find the Laplace transforms of the following functions:

(i) $\int_0^t te^{-4t}\sin 3t\,\mathrm{d}t$ (ii) $e^{-4t}\int_0^t t\sin 3t\,\mathrm{d}t$

(iii) $t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t$ (iv) $\int_0^t \frac{e^{-t}\sin t}{t}\mathrm{d}t$

(v) $e^{-t}\int_0^t \frac{\sin t}{t}\mathrm{d}t$ (vi) $\frac{1}{t}\int_0^t e^{-t}\sin t\,\mathrm{d}t$

(i) $$L\left[\int_0^t f(t)\,\mathrm{d}t\right]=\frac{1}{s}L\{f(t)\}\quad (1),$$

by the theorem on Laplace transform of integral

$$\therefore\quad L\left[\int_0^t te^{-4t}\sin 3t\,\mathrm{d}t\right]$$

$$=\frac{1}{s}L\{te^{-4t}\sin 3t\}$$

$$=\frac{6(s+4)}{s(s^2+8s+25)^2}$$

[Refer to Worked Example 8(i) in Section 5(b)].

(ii) $$L\left[e^{-4t}\int_0^t t\sin 3t\,\mathrm{d}t\right]=\left[L\int_0^t t\sin 3t\,\mathrm{d}t\right]_{s\to s+4}\quad (2), \text{ by the first shifting property}$$

Now $$L\int_0^t t\sin 3t\,\mathrm{d}t=\frac{1}{s}L(t\sin 3t)\text{ , by rule (1)}$$

$$=\frac{1}{s}\left[-\frac{\mathrm{d}}{\mathrm{d}s}L(\sin 3t)\right]$$

$$=-\frac{1}{s}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right)$$

$$= -\frac{1}{s} \times \frac{-3 \times 2s}{(s^2+9)^2} = \frac{6}{(s^2+9)^2} \tag{3}$$

Using (3) in (2), we get

$$L\left[e^{-4t}\int_0^t t\sin 3t \ \mathrm{d}t\right] = \frac{6}{\{(s+4)^2+9\}^2}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

(iii) $L\left[t \cdot \int_0^t e^{-4t} \ \sin 3t \ \mathrm{d}t\right]$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left[L\int_0^t e^{-4t} \sin 3t \,\mathrm{d}t\right] \tag{4}$$

Now $\qquad L\int_0^t e^{-4t}\sin 3t \ \mathrm{d}t = \frac{1}{s} L(e^{-4t} \ \sin 3t)$, by (1)

$$= \frac{1}{s}[L(\sin 3t)]_{s \to s+4}$$

$$= \frac{1}{s} \cdot \frac{3}{(s+4)^2+9}$$

$$= \frac{3}{s^3+8s^2+25s} \tag{5}$$

Using (5) in (4), we get

$$L\left[t\int_0^t e^{-4t}\sin 3t \ \mathrm{d}t\right] = -\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{3}{s^3+8s^2+25s}\right\}$$

$$= \frac{3(3s^2+16s+25)}{(s^3+8s+25s)^2}$$

$$= \frac{3(3s^2+16s+25)}{s^2(s^2+8s+25)^2}$$

(iv) $$L\int_0^t \frac{e^{-t}\sin t}{t} \ \mathrm{d}t = \frac{1}{s} L\left(\frac{e^{-t}\sin t}{t}\right), \text{ by (1)} \tag{6}$$

Now $\qquad L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L(e^{-t}\sin t) \ \mathrm{d}s$

$$= \int_s^\infty \frac{\mathrm{d}s}{(s+1)^2+1}$$

$$= \{-\cot^{-1}(s+1)\}_s^{\infty}$$
$$= \cot^{-1}(s+1) \tag{7}$$

Using (7) in (6), we get

$$L\int_0^t \frac{e^{-t}\sin t}{t}\,dt = \frac{1}{s}\cot^{-1}(s+1)$$

(v)
$$L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right] = \left[L\int_0^t \frac{\sin t}{t}\,dt\right]_{s\to s+1} \tag{8}$$

Now
$$L\int_0^t \frac{\sin t}{t}\,dt = \frac{1}{s}L\left(\frac{\sin t}{t}\right), \text{ by (1)}$$

$$= \frac{1}{s}\int_s^{\infty} L(\sin t)\,ds$$

$$= \frac{1}{s}\int_s^{\infty}\frac{ds}{s^2+1} = \frac{1}{s}\cot^{-1}s \tag{9}$$

Using (9) in (8), we get

$$L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s+1}\cot^{-1}(s+1)$$

(vi)
$$L\left[\frac{1}{t}\int_0^t e^{-t}\sin t\,dt\right] = \int_s^{\infty} L\left[\int_0^t e^{-t}\sin t\,dt\right]ds \tag{10}$$

Now
$$L\int_0^t e^{-t}\sin t\,dt = \frac{1}{s}L\{e^{-t}\sin t\}, \text{ by (1)}$$

$$= \frac{1}{s}\cdot\frac{1}{(s+1)^2+1} \tag{11}$$

Using (11) in (10), we get

$$L\left[\frac{1}{t}\int_0^t e^{-t}\sin t\,dt\right] = \int_s^{\infty}\frac{ds}{s(s^2+2s+2)}$$

$$= \int_s^{\infty}\frac{1}{2}\left[\frac{1}{s} - \frac{s+2}{s^2+2s+2}\right]ds,$$

on resolving the integrand into partial fractions.

$$= \frac{1}{2}\int_s^{\infty}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log\{(s+1)^2+1\} + \cot^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+2s+2}}\right) + \cot^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{4}\log\left(\frac{s^2+2s+2}{s^2}\right) - \frac{1}{2}\cot^{-1}(s+1).$$

Example 5.7 Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{1}{s(s+2)^3}$; (ii) $\dfrac{54}{s^3(s-3)}$; (iii) $\dfrac{1}{s(s^2+4s+5)}$.

(iv) $\dfrac{1}{s^2(s^2+a^2)}$; (v) $\dfrac{1}{s^2}\left(\dfrac{s+1}{s^2+1}\right)$; (vi) $\dfrac{5s-2}{s^2(s-1)(s+2)}$;

(vii) $\dfrac{1}{(s+2)(s^2+4s+13)}$

Note ☑ All the problems in this example may be solved by resolving the given functions into partial fractions and applying elementary methods. However we shall solve them by applying the following working rule and its extensions.

$$L^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t L^{-1}(\phi(s)\,\mathrm{d}t \qquad (1)$$

(i) $$L^{-1}\left[\frac{1}{s(s+2)^3}\right] = \int_0^t L^{-1}\left\{\frac{1}{(s+2)^3}\right\}\mathrm{d}t, \text{ by (1)}$$

$$= \int_0^t e^{-2t} L^{-1}\left(\frac{1}{s^3}\right)\mathrm{d}t$$

$$= \int_0^t e^{-2t}\,\frac{1}{2}t^2\,\mathrm{d}t$$

$$= \frac{1}{2}\left[t^2\left(\frac{e^{-2t}}{-2}\right) - 2t\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$$

by Bernoulli's formula.

$$= \frac{1}{2}\left[-e^{-2t}\left(\frac{t^2}{2}+\frac{t}{2}+\frac{1}{4}\right)+\frac{1}{4}\right]$$

$$= \frac{1}{8}[1-(2t^2+2t+1)e^{-2t}]$$

(ii) $L^{-1}\left[\dfrac{54}{s^3(s-3)}\right] = 54\int_0^t\int_0^t\int_0^t L^{-1}\left(\dfrac{1}{s-3}\right)\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$, by the extension of rule (1).

$$= 54\int_0^t\int_0^t\int_0^t e^{3t}\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$$

$$= 54\int_0^t\int_0^t\left(\frac{e^{3t}}{3}\right)_0^t \mathrm{d}t\,\mathrm{d}t$$

$$= 18\int_0^t\int_0^t (e^{3t}-1)\,\mathrm{d}t\,\mathrm{d}t$$

$$= 18\int_0^t\left(\frac{e^{3t}}{3}-t\right)_0^t \mathrm{d}t$$

$$= 6\int_0^t (e^{3t}-3t-1)\,\mathrm{d}t$$

$$= 6\left(\frac{e^{3t}}{3}-\frac{3t^2}{2}-t\right)_0^t$$

$$= 2e^{3t}-9t^2-6t-2.$$

Aliter

We can avoid the multiple integration by using the following alternative method.

$$L^{-1}\left[\frac{54}{s^3(s-3)}\right] = L^{-1}\left[\frac{54}{\left(\overline{s-3}+3\right)^3(s-3)}\right]$$

$$= 54e^{3t}L^{-1}\left\{\frac{1}{s(s+3)^3}\right\},$$

by the first shifting property.

$$= 54e^{3t}\cdot\int_0^t L^{-1}\left\{\frac{1}{(s+3)^3}\right\}\mathrm{d}t, \text{ by rule (1)}$$

$$= 54e^{3t}\int_0^t e^{-3t}\cdot L^{-1}\left\{\frac{1}{s^3}\right\}\mathrm{d}t$$

$$= 54e^{3t}\int_0^t \frac{1}{2}t^2\,e^{-3t}\,\mathrm{d}t$$

$$= 27e^{3t}\left[t^2\left(\frac{e^{-3t}}{-3}\right) - 2t\left(\frac{e^{-3t}}{9}\right) + 2\left(\frac{e^{-3t}}{-27}\right)\right]_0^t$$

$$= e^{3t}[2 - e^{-3t}(9t^2 + 6t + 2)]$$

$$= 2e^{3t} - 9t^2 - 6t - 2$$

(iii) $$L^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\} = \int_0^t L^{-1}\left\{\frac{1}{s^2+4s+5}\right\} dt,$$

by rule (1).

$$= \int_0^t L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} dt$$

$$= \int_0^t e^{-2t}\sin t\, dt$$

$$= \left[\frac{-e^{-2t}}{5}(2\sin t + \cos t)\right]_0^t$$

$$= \frac{1}{5}[1 - e^{-2t}(2\sin t + \cos t)]$$

(iv) $$L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \int_0^t\int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right) dt\, dt,$$

by the extension of rule (1).

$$= \int_0^t\int_0^t \frac{1}{a}\sin at\, dt\, dt$$

$$= \frac{1}{a}\int_0^t \left(\frac{-\cos at}{a}\right)_0^t dt$$

$$= \frac{1}{a^2}\int_0^t (1 - \cos at)\, dt$$

$$= \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)_0^t$$

$$= \frac{1}{a^3}(at - \sin at).$$

(v) $$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)\right\} = \int_0^t\int_0^t L^{-1}\left(\frac{s+1}{s^2+1}\right) dt\, dt,$$

by the extension of rule (1).

$$= \int_0^t \int_0^t (\cos t + \sin t)\, \mathrm{d}t\, \mathrm{d}t$$

$$= \int_0^t (\sin t - \cos t)_0^t\, \mathrm{d}t$$

$$= \int_0^t (\sin t - \cos t + 1)\, \mathrm{d}t$$

$$= (-\cos t - \sin t + t)_0^t$$

$$= 1 + t - \cos t - \sin t.$$

(vi) $$L^{-1}\left\{\frac{5s-2}{s^2(s-1)(s+2)}\right\} = \int_0^t \int_0^t L^{-1}\left\{\frac{5s-2}{(s-1)(s+2)}\right\} \mathrm{d}t\, \mathrm{d}t,$$

by the extension of rule (1).

$$= \int_0^t \int_0^t L^{-1}\left(\frac{1}{s-1} + \frac{4}{s+2}\right) \mathrm{d}t\, \mathrm{d}t,$$

by resolving the function into partial fractions.

$$= \int_0^t \int_0^t (e^t + 4e^{-2t})\, \mathrm{d}t\, \mathrm{d}t$$

$$= \int_0^t (e^t - 2e^{-2t})_0^t\, \mathrm{d}t$$

$$= \int_0^t (e^t - 2e^{-2t} + 1)\, \mathrm{d}t$$

$$= (e^t + e^{-2t} + t)_0^t = e^t + e^{-2t} + t - 2$$

(vii) $$L^{-1}\left[\frac{1}{(s+2)(s^2+4s+13)}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)\{(s+2)^2+9\}}\right]$$

$$= e^{-2t} \cdot L^{-1}\left[\frac{1}{s(s^2+9)}\right],$$ by the first shifting property.

$$= e^{-2t} \int_0^t L^{-1}\left(\frac{1}{s^2+9}\right) \mathrm{d}t,$$ by rule (1),

$$= \frac{1}{3} e^{-2t} \int_0^t \sin 3t\, \mathrm{d}t$$

$$= \frac{1}{3} e^{-2t} \left(\frac{-\cos 3t}{3} \right)_0^t$$

$$= \frac{1}{9} e^{-2t} (1 - \cos 3t)$$

Example 5.8 Find the inverse Laplace transforms of the following functions:

(i) $\frac{1}{(s^2+a^2)^2}$; (ii) $\frac{1}{s(s^2+a^2)^2}$; (iii) $\frac{1}{(s^2+2s+5)^2}$;

(iv) $\frac{1}{(s^2-4)^2}$; (v) $\frac{1}{s(s^2-4)^2}$; (vi) $\frac{1}{(s^2-2s-3)^2}$.

(i) $$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int_0^t L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} \mathrm{d}t, \text{as}$$

$$L^{-1}\left\{\frac{1}{s}\phi(s)\right\} = \int_0^t L^{-1}\{\phi(s)\}\,\mathrm{d}t \tag{1}$$

$$= \int_0^t \frac{t}{2a} \sin at\, \mathrm{d}t$$

[Refer to Worked Example 13(i) in Section 5(b)]

$$= \frac{1}{2a}\left[t\left(\frac{-\cos at}{a}\right) - \left(\frac{-\sin at}{a^2}\right)\right]_0^t$$

$$= \frac{1}{2a^3}(\sin at - at\cos at) \tag{2}$$

(ii) $$L^{-1}\left[\frac{1}{s(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s^2}\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int_0^t \int_0^t L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} \mathrm{d}t\, \mathrm{d}t,$$

by the extension of rule (1)

$$= \int_0^t \int_0^t \frac{t}{2a} \sin at\, \mathrm{d}t\, \mathrm{d}t$$

$$= \int_0^t \frac{1}{2a^3}(\sin at - at\cos at)\, \mathrm{d}t, \text{ by (2)}$$

$$= \frac{1}{2a^3}\left[\frac{-\cos at}{a} - a\left\{t\left(\frac{\sin at}{a}\right) - \left(\frac{-\cos at}{a^2}\right)\right\}\right]_0^t$$

$$= \frac{1}{2a^4}\,(-2\cos at - at\sin at)_0^t$$

$$= \frac{1}{2a^4}\,(2 - 2\cos at - at\sin at).$$

(iii) $$L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\} = L^{-1}\left\{\frac{1}{\left[(s+1)^2+4\right]^2}\right\}$$

$$= e^{-t}\cdot L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{1}{16}\,e^{-t}\,(\sin 2t - 2t\cos 2t), \text{ by problem (i)}$$

(iv) $$L^{-1}\left\{\frac{1}{(s^2-4)^2}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{s}{(s^2-4)^2}\right\}$$

$$= \int_0^t L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\} \mathrm{d}t, \text{ by rule (1)}$$

$$= \int_0^t \frac{t}{4}\sinh 2t\,\mathrm{d}t$$

[Refer to Worked Example 13 (ii) in Section 5(b)]

$$= \frac{1}{4}\left[t\left(\frac{\cosh 2t}{2}\right) - \left(\frac{\sinh 2t}{4}\right)\right]_0^t$$

$$= \frac{1}{16}\,(2t\cosh 2t - \sinh 2t) \qquad (3)$$

(v) $$L^{-1}\left\{\frac{1}{s(s^2-4)^2}\right\} = L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2-4)^2}\right\}$$

$$= \int_0^t\int_0^t L^{-1}\left\{\frac{s}{(s^2-4)^2}\right\} \mathrm{d}t\,\mathrm{d}t,$$

by the extension of rule (1).

$$= \int_0^t\int_0^t \frac{t}{4}\sinh 2t\,\mathrm{d}t\,\mathrm{d}t$$

$$= \int_0^t \frac{1}{16}\,(2t\cosh 2t - \sinh 2t)\,\mathrm{d}t, \text{ by (3)}$$

$$= \frac{1}{16}\left[2\left(t\frac{\sinh 2t}{2} - \frac{\cosh 2t}{4}\right) - \frac{\cosh 2t}{2}\right]_0^t$$

$$= \frac{1}{16}[t\sinh 2t - \cosh 2t]_0^t$$

$$= \frac{1}{16}(1 + t\sinh 2t - \cosh 2t)$$

(vi) $$L^{-1}\left\{\frac{1}{(s^2 - 2s - 3)^2}\right\} = L^{-1}\left[\frac{1}{\{(s-1)^2 - 4\}^2}\right]$$

$$= e^t \cdot L^{-1}\left[\frac{1}{(s^2 - 4)^2}\right],$$

by the first shifting property.

$$= \frac{1}{16}e^t(2t\cosh 2t - \sinh 2t), \text{ by problem (iv).}$$

Example 5.9

(i) Verify the initial and final value theorems when (a) $f(t) = (t + 2)^2 e^{-t}$; (b) $f(t) = L^{-1}\left\{\frac{1}{s(s+2)^2}\right\}$

(ii) If $L(e^{-t}\cos^2 t) = \phi(s)$, find $\lim_{s\to 0}[s\phi(s)]$ and $\lim_{s\to\infty}[s\phi(s)]$.

(iii) If $L\{f(t)\} = \frac{1}{s(s+1)(s+2)}$, find $\lim_{t\to 0}[f(t)]$ and $\lim_{t\to\infty}[f(t)]$.

(i) (a) $f(t) = (t^2 + 4t + 4)e^{-t}$

$$\therefore \quad \phi(s) = L\{f(t)\} = \frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{s+1}$$

$$\therefore \quad s\phi(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}$$

Now $\lim_{t\to 0}[f(t)] = 4$ and $\lim_{s\to\infty}[s\phi(s)] = 0 + 0 + 4 = 4$

Hence the initial value theorem is verified.

Also $\lim_{t\to\infty}[f(t)] = 0$ and $\lim_{s\to 0}[s\phi(s)] = 0$

Hence the final value theorem is verified.

(i) (b) $$f(t) = L^{-1}\left\{\frac{1}{s(s+2)^2}\right\} = \int_0^t L^{-1}\left\{\frac{1}{(s+2)^2}\right\} dt$$

$$= \int_0^t te^{-21}\, dt$$

$$= \left[t\left(\frac{e^{-2t}}{-2}\right) - \left(\frac{e^{-2t}}{4}\right) \right]_0^t$$

$$= \frac{1}{4}(1 - 2t\,e^{-2t} - e^{-2t})$$

$$s\phi(s) = \frac{1}{(s+2)^2}$$

Now $\quad \lim_{t\to 0}[f(t)] = 0 = \lim_{s\to\infty}[s\phi(s)]$

and $\quad \lim_{t\to\infty}[f(t)] = \frac{1}{4} = \lim_{s\to 0}[s\phi(s)]$

Hence the initial and final value theorems are verified.

(ii) $L(e^{-t}\cos^2 t) = \phi(s)$

i.e., $\quad f(t) = e^{-t}\cos^2 t$

By the final value theorem,

$$\lim_{s\to 0}[s\phi(s)] = \lim_{t\to\infty}[e^{-t}\cos^2 t] = 0$$

By the initial value theorem,

$$\lim_{s\to\infty}[s\phi(s)] = \lim_{t\to 0}[e^{-t}\cos^2 t] = 1$$

(iii) $\quad L\{f(t)\} = \dfrac{1}{s(s+1)(s+2)}$

$\therefore \quad s\phi(s) = \dfrac{1}{(s+1)(s+2)}$

By the initial value theorem,

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[s\phi(s)] = 0$$

By the final value theorem,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\phi(s)] = \frac{1}{2}$$

Example 5.10 Use convolution theorem to evaluate the following

(i) $\displaystyle\int_0^t u^2\, e^{-a(t-u)}\,du$ (ii) $\displaystyle\int_0^t \sin u \cos(t-u)\,du$

(i) $\displaystyle\int_0^t u^2\, e^{-a(t-u)}\,du$ is of the form $\displaystyle\int_0^t f(u)\,g(t-u)\,du$

where $f(t) = t^2$ and $g(t) = e^{-at}$

i.e. $$\int_0^t u^2\, e^{-a(t-u)}\, \mathrm{d}u = (t^2) * (e^{-at})$$

$\therefore$ By convolution theorem,

$$L\left[\int_0^t u^2\, e^{-a(t-u)}\, \mathrm{d}u\right] = L(t)^2 \cdot L(e^{-at})$$

$$= \frac{2}{s^3} \cdot \frac{1}{s+a}$$

$$\int_0^t u^2\, e^{-a(t-u)}\, \mathrm{d}u = L^{-1}\left\{\frac{2}{s^3\,(s+a)}\right\}$$

$$= e^{-at} \cdot L^{-1}\left\{\frac{2}{s\,(s-a)^3}\right\}$$

$$= e^{-at} \int_0^t L^{-1}\left\{\frac{2}{(s-a)^3}\right\} \mathrm{d}\,t$$

$$= e^{-at} \int_0^t t^2\, e^{at}\, \mathrm{d}t$$

$$= e^{-at}\left[t^2\,\frac{e^{at}}{a} - 2t\,\frac{e^{at}}{a^2} + 2\,\frac{e^{at}}{a^3}\right]_0^t$$

$$= e^{-at}\left(\frac{t^2\, e^{at}}{a} - \frac{2t\, e^{at}}{a^2} + \frac{2e^{at}}{a^3} - \frac{2}{a^3}\right)$$

$$= \frac{1}{a^3}\,\{a^2 t^2 - 2at + 2 - 2e^{-at}\}$$

(ii) $\int_0^t \sin u \cos(t-u)\, \mathrm{d}u$ is of the form $\int_0^t f(u)\, g(t-u)\, \mathrm{d}u$,

where $f(t) = \sin t$ and $g(t) = \cos t$

i.e. $$\int_0^t \sin u \cos(t-u)\, \mathrm{d}u = (\sin t) * (\cos t)$$

$\therefore$ By convolution theorem,

$$L\left[\int_0^t \sin u \cos(t-u)\, \mathrm{d}u\right] = L(\sin t) \cdot L(\cos t)$$

$$= \frac{s}{(s^2+1)^2}$$

$$\therefore \quad \int_0^t \sin u \cos(t-u)\, du = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$= \frac{t}{2}\sin t, \text{ by Worked Example 13(i) of Section 5(b).}$$

Example 5.11 Use convolution theorem to find the inverse Laplace transforms of the following functions:

(i) $\dfrac{1}{(s+1)(s+2)}$ (ii) $\dfrac{s}{(s^2+a^2)^2}$ (iii) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

(iv) $\dfrac{4}{(s^2+2s+5)^2}$ (v) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$

(i) $$L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+2}\right\} = L^{-1}\left(\frac{1}{s+1}\right) * L^{-1}\left(\frac{1}{s+2}\right), \text{ by convolution theorem}$$

$$= e^{-t} * e^{-2t}$$

$$= \int_0^t e^{-u}\cdot e^{-2(t-u)}\, du$$

$$= e^{-2t}\int_0^t e^{u}\, du$$

$$= e^{-2t}(e^t - 1) = e^{-t} - e^{-2t}$$

(ii) $$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{s}{s^2+a^2}\right\}$$

$$= L^{-1}\left(\frac{1}{s^2+a^2}\right) * L^{-1}\left(\frac{s}{s^2+a^2}\right), \text{ by convolution theorem}$$

$$= \left(\frac{1}{a}\sin at\right) * (\cos at)$$

$$= \int_0^t \frac{1}{a}\sin au \cos a(t-u)\, du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin(2au - at)]\, du$$

$$= \frac{1}{2a}\left[(\sin at)\, u - \frac{\cos(2au - at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}(\cos at - \cos at)\right]$$

$$= \frac{1}{2a}t\sin at.$$

(iii) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}$

$$= L^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} * L^{-1}\left\{\frac{s}{s^2+b^2}\right\}, \text{ by convolution theorem}$$

$$= (\cos at) * (\cos bt)$$

$$= \int_0^t \cos au \cdot \cos b(t-u)\,\mathrm{d}u$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\}\,\mathrm{d}u$$

$$= \frac{1}{2}\left[\frac{1}{a-b}\sin\{(a-b)u+bt + \frac{1}{a+b}\sin\{(a+b)u-bt\}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{a-b}(\sin at - \sin bt) + \frac{1}{a+b}(\sin at + \sin bt)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{a-b}+\frac{1}{a+b}\right)\sin at + \left(\frac{1}{a+b}+\frac{1}{a-b}\right)\sin bt\right]$$

$$= \frac{1}{2}\left[\frac{2a}{a^2-b^2}\sin at - \frac{2b}{a^2-b^2}\sin bt\right]$$

$$= \frac{1}{a^2-b^2}(a\sin at - b\sin bt).$$

(iv) $L^{-1}\left[\frac{4}{\left(s^2+2s+5\right)^2}\right] = L^{-1}\left[\frac{2}{\left(s^2+2s+5\right)}\cdot\frac{2}{\left(s^2+2s+5\right)}\right]$

$$= L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} * L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\},$$

by convolution theorem.

$$= (e^{-t}\sin 2t) * (e^{-t}\sin 2t)$$

$$= \int_0^t e^{-u}\sin 2u \cdot e^{-(t-u)}\cdot \sin 2(t-u)\,\mathrm{d}u$$

$$= \frac{1}{2} e^{-t} \int_0^t [\cos(4u - 2t) - \cos 2t] \, dv$$

$$= \frac{1}{2} e^{-t} \left[\frac{\sin(4u - 2t)}{4} - (\cos 2t) \cdot u \right]_0^t$$

$$= \frac{1}{2} e^{-t} \left[\frac{1}{4} (\sin 2t + \sin 2t) - t \cos 2t \right]$$

$$= \frac{1}{4} e^{-t} (\sin 2t - 2t \cos 2t).$$

(v) $$L^{-1} \left[\frac{s^2 + s}{(s^2 + 1)(s^2 + 2s + 2)} \right] = L^{-1} \left[\frac{s+1}{s^2 + 2s + 2} \cdot \frac{s}{s^2 + 1} \right]$$

$$= L^{-1} \left\{ \frac{s+1}{(s+1)^2 + 1} \right\} * L^{-1} \left(\frac{s}{s^2 + 1} \right)$$

$$= (e^{-t} \cos t) * (\cos t)$$

$$= \int_0^t e^{-u} \cos u \cos(t - u) \, du$$

$$= \frac{1}{2} \int_0^t e^{-u} [\cos t + \cos(2u - t)] \, du$$

$$= \frac{1}{2} \cos t \left(-e^{-u} \right)_0^t + \frac{1}{2} \cdot \frac{1}{5} \left[e^{-u} \{ -\cos(2u - t) + 2 \sin(2u - t) \} \right]_0^t$$

$$= \frac{1}{2} \cos t (1 - e^{-t}) + \frac{1}{10} \left[e^{-t} (2 \sin t - \cos t) + (2 \sin t + \cos t) \right]$$

$$= \frac{1}{5} e^{-t} (\sin t - 3 \cos t) + \frac{1}{5} (\sin t + 3 \cos t).$$

EXERCISE 5(c)

Part A

(Short Answer Questions)

1. State the relation between the Laplace transforms of $f(t)$ and $f'(t)$. Under what conditions does this relation hold good?
2. Express $L^{-1}\{s\phi(s)\}$ interms of $L^{-1}\{\phi(s)\}$. State the condition for the validity of your answer.
3. Express $L\left[\int_0^t \int_0^t f(t) \, dt \, dt \right]$ in terms of $L\{f(t)\}$.

4. State the relation between $L^{-1}\{\phi(s)\}$ and $L^{-1}\left\{\frac{1}{s^2}\phi(s)\right\}$.
5. State the initial value theorem in Laplace transforms.
6. State the final value theorem in Laplace transforms.
7. Define the convolution product of two functions and prove that it is commutative.
8. Verify whether $1 * g(t) = g(t)$, when $g(t) = t$.
9. State convolution theorem in Laplace transforms.

Using the Laplace transforms of the derivatives find the Laplace transforms of the following functions:

10. e^{at} 11. $\cos at$ 12. $\sin^2 t$

Find the inverse Laplace transforms of the following functions:

13. $\dfrac{s}{(s+2)^3}$ 14. $\dfrac{s^2}{(s-1)^3}$

15. $\dfrac{s}{(s-a)^2+b^2}$ 16. $\dfrac{s}{(s+1)(s+2)}$

Find the Laplace transforms of the following functions:

17. $\displaystyle\int_0^t \frac{\sin t}{t}\,dt$ 18. $\displaystyle\int_0^t \frac{1-e^t}{t}\,dt$ 19. $\displaystyle\int_0^t \frac{1-2\cos t}{t}\,dt$

20. $\displaystyle\int_0^t t e^{-t}\,dt$ 21. $\displaystyle\int_0^t e^{-t}\sin t\,dt$ 22. $\displaystyle\int_0^t t\sin t\,dt$

Find the inverse Laplace transforms of the following functions:

23. $\dfrac{1}{s(s+a)}$ 24. $\dfrac{1}{s^2(s+1)}$

25. $\dfrac{1}{s(s^2-a^2)}$ 26. $\dfrac{1}{s(s^2+1)}$

27. If $L\{f(t)\}=\dfrac{s+3}{(s+1)(s+2)}$, find $\lim\limits_{t\to 0}(f(t)\}$ and $\lim\limits_{t\to\infty}\{f(t)\}$.

28. If $L^{-1}\{\phi(s)\}=\frac{1}{2}(1-2e^{-t}+e^{-2t})$, find $\lim\limits_{s\to 0}(s\phi(s)\}$ and $\lim\limits_{s\to\infty}\{s\phi(s)\}$.

29. Show that $1*1*1*......*1\,(n \text{ times}) = \dfrac{t^{n-1}}{(n-1)!}$, where * denotes Convolution.

30. If $L\{f(t)\}=\dfrac{1}{\sqrt{s^2+1}}$, evaluate $\displaystyle\int_0^t f(u)\,f(t-u)\,du$.

Use convolution theorem to find the inverse Laplace transforms of the following functions:

31. $\dfrac{1}{(s-1)(s+3)}$

32. $\dfrac{1}{(s-1)^2}$

33. $\dfrac{1}{s(s^2+a^2)}$

34. $\dfrac{1}{s^2(s+1)}$

35. $2/(s+1)(s^2+1)$.

Part B

36. Using the Laplace transforms of the derivatives find $L(t \sin at)$ and hence find $L(2\cos at - at\sin at)$ and $L(\sin at + at\cos at)$.
37. Using the Laplace transforms of the derivatives, find $L(t\cosh at)$ and hence find $L(\sinh at + at\cosh at)$ and $L(at\cosh at - \sinh at)$.
38. Find $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$ and hence find $L^{-1}\left\{\dfrac{s}{(s+a)(s+b)}\right\}$.
39. Find $L^{-1}\left\{\dfrac{1}{(s-1)(s-2)(s-3)}\right\}$ and hence find $L^{-1}\left\{\dfrac{s^2}{(s-1)(s-2)(s-3)}\right\}$.
40. Find $L^{-1}\left\{\dfrac{1}{(s^2+a^2)(s^2+b^2)}\right\}$ and hence find $L^{-1}\left\{\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right\}$ and $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$.
41. Given that $L^{-1}\left\{\dfrac{s}{(s^2+4)^2}\right\} = \dfrac{t}{4}\sin 2t$, find $L^{-1}\left\{\dfrac{s^2}{(s^2+4)^2}\right\}$, $L^{-1}\left\{\dfrac{s^3}{(s^2+4)^2}\right\}$ and $L^{-1}\left\{\dfrac{(s+3)^2}{(s^2+6s+13)^2}\right\}$
42. Given that $L^{-1}\left\{\dfrac{s}{(s^2-a^2)^2}\right\} = \dfrac{t}{2a}\sinh at$, find $L^{-1}\left\{\dfrac{s^2}{(s^2-a^2)^2}\right\}$, $L^{-1}\left\{\dfrac{s^3}{(s^2-a^2)^2}\right\}$ and $L^{-1}\left\{\dfrac{s+a}{s(s+2a)}\right\}^2$.
43. Given that $L^{-1}\left\{\dfrac{1}{s^4+4}\right\} = \dfrac{1}{4}(\sin t\cosh t - \cos t\sinh t)$, find $L^{-1}\left\{\dfrac{s}{s^4+4}\right\}$,

$L^{-1}\left\{\dfrac{s^2}{s^4+4}\right\}$ and $L^{-1}\left\{\dfrac{s^3}{s^4+4}\right\}$.

Find the Laplace transforms of the following functions:

44. $\int_0^t t\, e^t \sin t \, dt$

45 . $e^t \int_0^t t \sin t \, dt$

46. $t \int_0^t e^t \sin t \, dt$

47. $\int_0^t \dfrac{e^{-2t} \sin 3t}{t} \, dt$

48. $e^{-2t} \int_0^t \dfrac{\sin 3t}{t} \, dt$

49. $\dfrac{1}{t} \int_0^t e^{-2t} \sin 3t \, dt$

Find the inverse Laplace transforms of the following functions:

50. $\dfrac{1}{s^2}\left(\dfrac{s-1}{s+1}\right)$ $\left[\textbf{Hint}: \text{Consider the function as } \dfrac{s-1}{s(s^2+s)}\right]$

51. $\dfrac{4s+7}{s^2\,(2s+3)\,(3s+5)}$

52. $\dfrac{1}{s\,(s^2+6s+25)}$

53. $\dfrac{1}{(s+1)\,(s^2+2s+2)}$

54. $\dfrac{1}{s^2}\left(\dfrac{s-2}{s^2+4}\right)$

55. $\dfrac{1}{(s^2+9)^2}$

56. $\dfrac{1}{s\,(s^2+9)^2}$

57. $\dfrac{1}{(s^2+6s+10)^2}$

58. $\dfrac{1}{(s^2-a^2)^2}$

59. $\dfrac{1}{s\,(s^2-a^2)^2}$

60. $\dfrac{1}{(s^2+4s)^2}$

Verify the initial and final value theorems when

61. $f(t) = (2t+3)^2 e^{-4t}$

62. $f(t) = L^{-1}\left\{\dfrac{1}{s(s+4)^3}\right\}$

63. Use convolution theorem to evaluate

$$\int_0^t e^{-u} \sin(t-u)\, du$$

64. Evaluate $\int_0^t \cos a\,u \cosh a\,(t-u)\, d\,u$, using convolution theorem.

Use convolution theorem to find the inverse of the following functions:

65. $\dfrac{s}{(s^2+4)\,(s^2+9)}$

66. $\dfrac{s^2}{(s^2+a^2)^2}$

67. $\dfrac{1}{(s^2+4)^2}$

68. $\dfrac{10}{(s+1)\,(s^2+4)}$

69. $\dfrac{1}{s^2(s+1)^3}$ 70. $\dfrac{1}{s^4+4}$

5.11 SOLUTIONS OF DIFFERENTIAL AND INTEGRAL EQUATIONS

As mentioned in the beginning, Laplace transform technique can be used to solve differential (both ordinary and partial) and integral equations. We shall apply this method to solve only ordinary linear differential equations with constant coefficients and a few integral and intergo-differential equations. The advantage of this method is that it gives the particular solution directly. This means that there is no need to first find the general solution and then evaluate the arbitrary constants as in the classical approach.

5.11.1 Procedure

1. We take the Laplace transforms of both sides of the given differential equation in $y\,(t)$, simultaneously using the given initial conditions. This gives an algebraic equation in $\bar{y}\,(s) = L\{y\,(t)\}$.

Note☑ $L\{y^{(n)}(t)\} = s^n\ \bar{y}\,(s) - s^{n-1}y(0) - s^{n-2}y'(0) \ldots\ldots\ y^{(n-1)}\,(0).$

2. We solve the algebraic equation to get $\bar{y}\,(s)$ as a function of s.
3. Finally we take $L^{-1}\,\{\bar{y}\,(s)\}$ to get $y(t)$. The various methods we have discussed in the previous sections will enable us to find $L^{-1}\,\{\bar{y}\,(s)\}$.
   The procedure is illustrated in the worked examples given below:

WORKED EXAMPLE 5(d)

Example 5.1 Using Laplace transform, solve the following equation

$L\dfrac{\mathrm{d}i}{\mathrm{d}t} + Ri = E\,e^{-at}$; $i\,(0) = 0$, where L, R, E and a are constants.

Taking Laplace transforms of both sides of the given equation, we get,

$$L \cdot L\{i'(t)\} + RL\{i\,(t)\} = EL\,\{e^{-at}\}$$

i.e., $$L\{s\,\bar{i}\,(s) - i\,(0)\} + R\bar{i}\,(s) = \frac{E}{s+a}, \text{ where } \bar{i}\,(s) = L\{i\,(t)\}$$

i.e., $$(Ls + R)\,\bar{i}\,(s) = \frac{E}{s+a}$$

$\therefore$ $$\bar{i}\,(s) = \frac{E}{(s+a)\,(Ls+R)}$$

$$= E\left\{\frac{\left(\dfrac{1}{R-aL}\right)}{s+a} + \frac{\left(\dfrac{1}{aL-R}\right)}{s+R/L}\right\}$$

Taking inverse Laplace transforms

$$i\,(t) = \frac{E}{R-aL}\left[L^{-1}\left\{\frac{1}{s+a}\right\} - L^{-1}\left\{\frac{1}{s+R/L}\right\}\right]$$

$$= \frac{E}{R-aL}\,(e^{-at} - e^{-Rt/L})$$

Example 5.2 Solve $y'' - 4y' + 8y = e^{2t}$, $y\,(0) = 2$ and $y'\,(0) = -2$.
Taking Laplace transforms of both sides of the given equation, we get

$$[s^2\,\bar{y}\,(s) - s\,y\,(0) - y'\,(0)] - 4[s\,\bar{y}\,(s) - y\,(0)] + 8\bar{y}\,(s) = \frac{1}{s-2}$$

i.e.,
$$(s^2 - 4s + 8)\,\bar{y}\,(s) = \frac{1}{s-2} + (2s-10)$$

$\therefore$
$$\bar{y}\,(s) = \frac{1}{(s-2)\left(s^2-4s+8\right)} + \frac{2s-10}{s^2-4s+8}$$

$$= \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8} + \frac{2s-10}{s^2-4s+8}$$

$$= \frac{\frac{1}{4}}{s-2} + \frac{-\frac{1}{4}s+\frac{1}{2}}{s^2-4s+8} + \frac{2s-10}{s^2-4s+8}$$

$$= \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}s-\frac{19}{2}}{s^2-4s+8}$$

$$= \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$$

$$y = \frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right) + e^{2t}L^{-1}\left\{\frac{\frac{7}{4}s-6}{s^2+4}\right\}$$

$$= \frac{1}{4}e^{2t} + e^{2t}\left(\frac{7}{4}\cos 2t - 3\sin 2t\right)$$

$$= \frac{1}{4}e^{2t}\,(1 + 7\cos 2t - 12\sin 2t)$$

Example 5.3 Solve $y'' - 2y' + y = (t + 1)^2$, $y(0) = 4$ and $y'(0) = -2$.
Taking Laplace transforms of both sides of the given equation, we get,

$$[s^2 \bar{y}(s) - sy(0) - y'(0)] - 2[s\bar{y}(s) - y(0)] + \bar{y}(s) = L(t^2 + 2t + 1)$$

i.e., $$(s^2 - 2s + 1)\,\bar{y}(s) - 4s + 10 = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

i.e., $$\bar{y}(s) = \frac{4s - 10}{(s-1)^2} + \frac{1}{s(s-1)^2} + \frac{2}{s^2(s-1)^2} + \frac{2}{s^3(s-1)^2}$$

$$= \frac{4}{s-1} - \frac{6}{(s-1)^2} + \frac{1}{s(s-1)^2} + \frac{2}{s^2(s-1)^2} + \frac{2}{s^3(s-1)^2}$$

$$\therefore\ y = 4e^t - 6te^t + \int_0^t t e^t \mathrm{d}t + 2\int_0^t\int_0^t t e^t \,\mathrm{d}t\,\mathrm{d}t + 2\int_0^t\int_0^t\int_0^t t\, e^t \,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$$

$$= 4e^t - 6te^t + (te^t - e^t + 1) + 2\int_0^t (te^t - e^t + 1)\,\mathrm{d}t + 2\int_0^t\int_0^t (te^t - e^t + 1)\,\mathrm{d}t\,\mathrm{d}t$$

$$= 3e^t - 5te^t + 1 + 2(te^t - 2e^t + t + 2) + 2\int_0^t (te^t - 2e^t + t + 2)\,\mathrm{d}t$$

$$= -e^t - 3te^t + 2t + 5 + 2\left(te^t - 3e^t + \frac{t^2}{2} + 2t + 3\right)$$

$$= -7e^t - te^t + t^2 + 6t + 11$$

Example 5.4 Solve $y'' + 4y = \sin \omega t$, $y(0) = 0$ and $y'(0) = 0$.

Taking Laplace transforms of both sides of the equation, we get

$$[s^2 \bar{y}(s) - sy(0) - y'(0)] + 4\bar{y}(s) = \frac{\omega}{s^2 + \omega^2}$$

$$\therefore \qquad \bar{y}(s) = \frac{\omega}{(s^2 + 4)(s^2 + \omega^2)} \tag{1}$$

$$= \frac{\omega}{\omega^2 - 4}\left(\frac{1}{s^2 + 4} - \frac{1}{s^2 + \omega^2}\right)$$

Inverting, we have,

$$y = \frac{1}{\omega^2 - 4}\left(\frac{\omega}{2}\sin 2t - \sin \omega t\right), \text{ if } \omega \neq 2.$$

If $\omega = 2$, from (1), we have

$$\bar{y}(s) = \frac{2}{(s^2 + 4)^2}$$

$$\therefore \qquad y = 2L^{-1}\left\{\frac{1}{(s^2 + 4)^2}\right\}$$

$$= \frac{1}{8}(\sin 2t - 2t\cos 2t)$$

[Refer to Worked Example 8(i) in Section 5(c).

Example 5.5 Solve the equation $y'' + y' - 2y = 3\cos 3t - 11\sin 3t$, $y(0) = 0$ and $y'(0) = 6$.

Taking Laplace transforms and using the giving initial conditions, we get

$$(s^2 + s - 2)\,\bar{y}(s) = \frac{3s}{s^2+9} - \frac{33}{s^2+9} + 6$$

$$= \frac{6s^2 + 3s + 21}{s^2+9}$$

$$\therefore \quad \bar{y}(s) = \frac{6s^2+3s+21}{(s^2+9)(s+2)(s-1)},$$

$$= \frac{As+B}{s^2+9} + \frac{C}{s+2} + \frac{D}{s-1}$$

$$= \frac{3}{s^2+9} - \frac{1}{s+2} + \frac{1}{s-1} \quad \text{by the usual procedure}$$

$$\therefore \quad y = \sin 3t - \mathrm{e}^{-2t} + e^{t}.$$

Example 5.6 Solve the equation $(D^2 + 4D + 13)\,y = e^{-t}\sin t$, $y = 0$ and $D\,y = 0$ at $t = 0$, where $D \equiv \frac{\mathrm{d}}{\mathrm{d}t}$.

Taking Laplace transforms and using the given initial conditions, we get

$$(s^2 + 4s + 13)\,\bar{y}(s) = \frac{1}{s^2 + 2s + 2}$$

$$\therefore \quad \bar{y}(s) = \frac{1}{(s^2+2s+2)(s^2+4s+13)}$$

$$= \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+4s+13}$$

$$= \frac{1}{85}\left[\frac{-2s+7}{s^2+2s+2} + \frac{2s-3}{s^2+4s+13}\right],$$

on finding the constants A, B, C, D by the usual procedure.

$$= \frac{1}{85}\left[\frac{-2(s+1)+9}{(s+1)^2+1} + \frac{2(s+2)-7}{(s+2)^2+9}\right]$$

$$\therefore \quad y = \frac{1}{85}\left[e^{-t}\{-2\cos t + 9\sin t\} + e^{-2t}\left\{2\cos 3t - \frac{7}{3}\sin 3t\right\}\right]$$

Example 5.7 Solve the equation $(D^2 + 6D + 9)\, x = 6\, t^2 e^{-3t}$, $x = 0$ and $Dx = 0$ at $t = 0$.

Taking Laplace transforms and using the initial conditions, use get,

$$(s^2 + 6s + 9)\,\bar{x}(s) = \frac{12}{(s+3)^3}$$

$$\therefore \qquad \bar{x}(s) = \frac{12}{(s+3)^5}$$

$$\therefore \qquad x = e^{-3t} \cdot L^{-1}\frac{12}{s^5}.$$

$$= \frac{1}{2} t^4 e^{-3t}$$

Example 5.8 Solve the equation $y'' + 9y = \cos 2t$, $y(0) = 1$ and $y(\pi/2) = -1$.

Note In all the problems discussed above the values of y and y' at $t = 0$ were given. Hence they are called initial conditions. In fact, the differential equation with such initial conditions is called an *initial value problem.*

But in this problem, the value of y at $t = 0$ and $t = \pi/2$ are given. Such conditions are called boundary conditions and the differential equation itself is called a *boundary value problem.*

As $y'(0)$ is not given, it will be assumed as a constant, which will be evaluated towards the end using the condition $y(\pi/2) = -1$.

Taking Laplace transforms, we get

$$s^2\bar{y}(s) - s\,y(0) - y'(0) + 9\,\bar{y}(s) = \frac{s}{s^2+4}$$

i.e. $$(s^2 + 9)\,\bar{y}(s) = \frac{s}{s^2+4} + s + A, \text{ where } A = y'(0).$$

$$\therefore \qquad \bar{y}(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\left\{\frac{s}{s^2+4} - \frac{s}{s^2+9}\right\} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$\therefore \qquad y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

Given $$y\left(\frac{\pi}{2}\right) = -1$$

i.e. $$-1 = -\frac{1}{5} - \frac{A}{3} \quad \therefore A = \frac{12}{5}$$

$$\therefore \qquad y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t.$$

Example 5.9 Find the general solution of the following equation

$$y'' - 2ky' + k^2y = f(t).$$

Taking Laplace transforms, we get

$$[s^2\bar{y}(s) - sy(0) - y'(0)] - 2k[s\bar{y}(s) - y(0)] + k^2\bar{y}(s) = \bar{f}(s)$$

i.e. $$(s^2 - 2ks + k^2)\,\bar{y}(s) = sy(0) + [y'(0) - 2ky(0)] + \bar{f}(s)$$

i.e. $$(s-k)^2\,\bar{y}(s) = As + B + \bar{f}(s)$$

[*As* $y(0)$ and $y'(0)$ are not given, they are assumed as arbitrary contants]

$\therefore$ $$\bar{y}(s) = \frac{As}{(s-k)^2} + \frac{B}{(s-k)^2} + \frac{\bar{f}(s)}{(s-k)^2}$$

$$= \frac{A(\mathrm{s}-k) + (Ak+B)}{(s-k)^2} + \frac{\bar{f}(s)}{(s-k)^2}$$

$$= \frac{C_1}{s-k} + \frac{C_2}{(s-k)^2} + \frac{\bar{f}(s)}{(s-k)^2}$$, where $C_1 = A$ and $C_2 = Ak + B$

$\therefore$ $$y = C_1 e^{kt} + C_2\, t\, e^{kt} + L^{-1}\left\{\bar{f}(s)\cdot\frac{1}{(s-k)^2}\right\}$$

i.e. $$y = (C_1 + C_2 t)\, e^{kt} + f(t) * t\, e^{kt}$$

i.e. $$y = (C_1 + C_2 t)e^{kt} + \int_0^t f(t-u)\, u e^{ku}\, \mathrm{d}u.$$

Example 5.10 Solve the equation $(D^3 + D)\,x = 2$, $x = 3$, $D\,x = 1$ and $D^2x = -2$ at $t = 0$.

Taking Laplace transforms and using the initial conditions, we get

$$(s^3 + s)\,\bar{y}(s) = 3s^2 + s + 1 + \frac{2}{s}$$

$\therefore$ $$\bar{y}(s) = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s\left(s^2+1\right)} + \frac{2}{s^2\left(s^2+1\right)}$$

$\therefore$ $$y = 3\cos t + \sin t + \int_0^t \sin t\, dt + 2\int_0^t\int_0^t \sin t\, \mathrm{d}t\, \mathrm{d}t$$

$$= 3\cos t + \sin t + (1 - \cos t) + 2\int_0^t (1 - \cos t)\, \mathrm{d}t$$

$$= 2\cos t + \sin t + 1 + 2(t - \sin t)$$

$$= 2\cos t - \sin t + 1 + 2t.$$

Example 5.11 Solve the equation $\frac{d^4 y}{dx^4}-\frac{d^3 y}{dx^3}=0$, $y=\frac{dy}{dx}=2$ and $\frac{d^2 y}{dx^2}=\frac{d^3 y}{dx^3}=1$ at $x = 0$.

Note ☑ Change in the independent variable from t to x makes no difference in the procedure.)

Taking Laplace transforms and using the initial conditions, we get

$$(s^4 - s^3)\bar{y}(s)=2s^3 - s$$

$$\therefore \qquad \bar{y}(s)=\frac{2s^2-1}{s^2(s-1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}.$$

i.e.
$$\bar{y}(s)=\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s-1}$$

$$\therefore \qquad y = 1 + t + e^t$$

Example 5.12 Solve the simultaneous differential equation $\frac{dy}{dt}+2x=\sin 2t$ and $\frac{dy}{dt}+2x=\cos 2t$, $x(0) = 1$, $y(0) = 0$.

Taking Laplace transforms of both sides of the given equation and using the given initial conditions, we get

$$s\bar{y}(s)+2\bar{x}(s)=\frac{2}{s^2+4} \tag{1}$$

and
$$s\bar{x}(s)-2\bar{y}(s)=\frac{s}{s^2+4}+1 \tag{2}$$

Solving (1) and (2), we have

$$\bar{x}(s)=\frac{1}{s^2+4}+\frac{s}{s^2+4} \text{ and } \bar{y}(s)=-\frac{2}{s^2+4}$$

$$\therefore \qquad x=\frac{1}{2}\sin 2t+\cos 2t \text{ and } y = -\sin 2t.$$

Example 5.13 Solve the simultaneous equations

$2x' - y' + 3x = 2t$ and $x' + 2y' - 2x - y = t^2 - t$, $x(0) = 1$ and $y(0) = 1$.

Taking Laplace transforms of both the equations, we get

$$2[s\bar{x}(s)-1]-[s\bar{y}(s)-1]+3\bar{x}(s)=\frac{2}{s^2} \text{ and}$$

$$[s\bar{x}(s)-1]+2[s\,\bar{y}(s)-1]-2\bar{x}(s)-\bar{y}(s)=\frac{2}{s^3}-\frac{1}{s^2}$$

i.e.
$$(2s+3)\bar{x}(s)-s\bar{y}(s)=\frac{2}{s^2}+1 \qquad (1)$$

and
$$(s-2)\,\bar{x}\,(s)+(2s-1)\,\bar{y}(s)=\frac{2}{s^3}-\frac{1}{s^2}+3 \qquad (2)$$

Solving (1) and (2), we have

$$\bar{x}(s)=\frac{3}{s(s+1)(5s-3)}+\frac{5s-1}{(s+1)(5s-3)}$$
$$=\left(\frac{-1}{s}+\frac{3/8}{s+1}+\frac{25/8}{5s-3}\right)+\left(\frac{3/4}{s+1}+\frac{5/4}{5s-3}\right)$$
$$=-\frac{1}{s}+\frac{9/8}{s+1}+\frac{7/8}{s-3/5}$$

$\therefore$
$$x=-1+\frac{9}{8}e^{-t}+\frac{7}{8}e^{3/5t} \qquad (3)$$

Eliminating y' from the given equations,
we get $5x' + 4x - y = t^2 + 3t$

$\therefore$
$$y = 5x' + 4x - t^2 - 3t$$
$$=5\left(-\frac{9}{8}e^{-t}+\frac{21}{40}e^{\frac{3}{5}t}\right)+4\left(-1+\frac{9}{8}e^{-t}+\frac{7}{8}e^{\frac{3}{5}t}\right)$$
$$- t^2 - 3t, \qquad \text{on using (3)}$$

i.e.
$$y=-\frac{9}{8}e^{-t}+\frac{49}{8}e^{\frac{3}{5}t}-t^2-3t-4\,.$$

Example 5.14 Solve the simultaneous equations

$$Dx + Dy = t \quad \text{and } D^2x - y = e^{-t};\ \text{n}x = 3,$$
$$Dx = -2 \text{ and } y = 0 \text{ at } t = 0.$$

Taking Laplace transformed of both the equations, we get

$$s\bar{x}(s)-3+s\bar{y}(s)=\frac{1}{s^2} \quad \text{and}$$

$$s^2\bar{x}(s)-3s+2-\bar{y}(s)=\frac{1}{s+1}$$

i.e.
$$\bar{x}(s)+\bar{y}(s)=\frac{1}{s^3}+\frac{3}{s} \qquad (1)$$

and
$$s^2\bar{x}(s)-\bar{y}(s)=\frac{1}{s+1}+3s-2 \qquad (2)$$

Solving (1) and (2), we have

$$\bar{x}(s)=\frac{1}{(s+1)\left(s^2+1\right)}+\frac{3}{s\left(s^2+1\right)}+\frac{1}{s^3\left(s^2+1\right)}+\frac{3s-2}{s^2+1}$$

$$=\frac{\frac{1}{2}}{s+1}+\frac{\left(\frac{1}{2}-\frac{1}{2}s\right)}{s^2+1}+\frac{3}{s\left(s^2+1\right)}+\frac{1}{s^3\left(s^2+1\right)}+\frac{3s}{s^2+1}-\frac{2}{s^2+1}$$

$$=\frac{\frac{1}{2}}{s+1}-\frac{\frac{3}{2}}{s^2+1}+\frac{\frac{5}{2}s}{s^2+1}+\frac{3}{s\left(s^2+1\right)}+\frac{1}{s^3\left(s^2+1\right)}$$

$\therefore$
$$x=\frac{1}{2}e^{-t}-\frac{3}{2}\ \sin t+\frac{5}{2}\ \cos t+3\int_0^t \sin t\,\mathrm{d}t+\int_0^t\int_0^t\int_0^t \sin t\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$$

$$=\frac{1}{2}e^{-t}-\frac{3}{2}\ \sin t+\frac{5}{2}\ \cos t+3\left(1-\cos t\right)+\frac{t^2}{2}+\cos t-1$$

i.e.
$$x=\frac{1}{2.}e^{-t}-\frac{3}{2}\sin t+\frac{1}{2}\cos t+\frac{t^2}{2}+2 \tag{3}$$

$\therefore$
$$x''=\frac{1}{2}e^{-t}+\frac{3}{2}\sin t-\frac{1}{2}\cos t+1 \tag{4}$$

From the given second equation we have

$$y=x''-e^{-t}$$

$$=1-\frac{1}{2}e^{-t}+\frac{3}{2}\sin t-\frac{1}{2}\cos t.$$

Example 5.15 Solve the simultaneous equations

$D^2x - Dy = \cos t$ and $Dx + D^2y = -\sin t$; $x = 1$,
$Dx = 0$, $y = 0$, $Dy = 1$ at $t = 0$.

Taking the Laplace transforms of both equations, we get

$$s^2\bar{x}(s)-s\bar{y}(s)=\frac{s}{s^2+1}+s \tag{1}$$

and
$$s\bar{x}(s)+s^2\bar{y}(s)=2-\frac{1}{s^2+1} \tag{2}$$

Solving (1) and (2), we have

$$\bar{x}(s)=\frac{s}{s^2+1}+\frac{2}{s\left(s^2+1\right)}+\frac{s}{\left(s^2+1\right)^2}-\frac{1}{s\left(s^2+1\right)^2}$$

$$=\frac{s}{s^2+1}+\frac{2}{s\left(s^2+1\right)}+\frac{s^2-1}{s\left(s^2+1\right)^2}$$

and $$\bar{y}(s) = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$$

$\therefore$ $$x = \cos t + 2\int_0^t \sin t\, dt + \int_0^t \left\{ L^{-1}\left(\frac{1}{s^2+1}\right) - 2L^{-1}\frac{1}{(s^2+1)^2} \right\} dt$$

$$= \cos t + 2(1 - \cos t) + \int_0^t (\sin t - \sin t + t\cos t)\, dt$$

$$= 1 + t \sin t$$

and $$y = \sin t - 2 \times \frac{1}{2}(\sin t - t\cos t)$$

$$= t \cos t.$$

Example 5.16 Show that the solution of the equation

$L\dfrac{di}{dt} + Ri + \dfrac{1}{C}\displaystyle\int_0^t i\, dt = E, i(0) = 0$ [where L, R, E are constants] is given by

$$i = \begin{cases} \dfrac{E}{\omega L} e^{-at} \sin \omega t, \text{if } \omega^2 > 0 \\ \dfrac{E}{L} te^{-at}, \text{if } \omega = 0 \\ \dfrac{E}{kL} e^{-at} \sinh kt, \text{if } \omega^2 < 0 \end{cases}$$

where $$a = \frac{R}{2L},\ \omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \quad \text{and} \quad k^2 = -\omega^2.$$

Note ☑ The given equation is an integro-differential equation, as the unknown (dependent variable) i occurs within the integral and differential operations.]

Taking Laplace transforms of the given equation, we get

$$Ls\bar{i}(s) + R\bar{i}(s) + \frac{1}{Cs}\bar{i}(s) = \frac{E}{s}$$

i.e. $$(LCs^2 + RCs + 1)\bar{i}(s) = EC$$

$\therefore$ $$\bar{i}(s) = \frac{EC}{LCs^2 + RCs + 1}$$

$$= \frac{E}{L} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$= \frac{E}{L} \cdot \frac{1}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$= \frac{E}{L} \cdot \frac{1}{(s+a)^2 + \omega^2}, \text{if } \omega^2 > 0$$

$\therefore$ $$i(t) = \frac{E}{L\omega} \cdot e^{-at} \sin \omega t$$

If $\omega = 0$,

$$\bar{i}(s) = \frac{E}{L} \cdot \frac{1}{(s+a)^2}$$

$\therefore$ $$i(t) = \frac{E}{L} te^{-at}$$

If $\omega^2 < 0$ and $\omega^2 = -k^2$,

$$\bar{i}(s) = \frac{E}{L} \cdot \frac{1}{(s+a)^2 - k^2}$$

$\therefore$ $$i(t) = \frac{E}{Lk} e^{-at} \sinh kt.$$

Example 5.17 Solve the simultaneous equations

$3x' + 2y' + 6x = 0$ and

$y' + y + 3\int_0^t x\,dt = \cos t + 3\sin t$, $x(0) = 2$ and $y(0) = -3$.

Taking Laplace transforms of both the equations, we get

$$3[s\,\bar{x}(s) - 2] + 2[s\,\bar{y}(s) + 3] + 6\,\bar{x}(s) = 0 \text{ and}$$

$$s\,\bar{y}(s) + 3 + \bar{y}(s) + \frac{3}{s}\,\bar{x}(s) = \frac{s}{s^2+1} + \frac{3}{s^2+1}$$

i.e. $$(3s+6)\,\bar{x}(s) + 2s\,\bar{y}(s) = 0 \tag{1}$$

and $$\frac{3}{s}\bar{x}(s) + (s+1)\,\bar{y}(s) = \frac{s+3}{s^2+1} - 3 \tag{2}$$

Solving (1) and (2) for $\bar{x}(s)$, we have

$$[3(s+2)(s+1) - 6]\,\bar{x}(s) = -\frac{2s(s+3)}{s^2+1} + 6s$$

i.e. $$\bar{x}(s) = -\frac{2}{3} \cdot \frac{1}{s^2+1} + \frac{2}{s+3}$$

$\therefore \qquad x = -\dfrac{2}{3}\sin t + 2e^{-3t}.$

Solving (1) and (2) for $\bar{y}(s)$, we have

$$\bar{y}(s) = \frac{3s^2+5s-2}{(s+3)\left(s^2+1\right)} = -\left[\frac{1}{s+3} + \frac{2s-1}{s^2+1}\right]$$

$\therefore \qquad y = -e^{-3t} - 2\cos t + \sin t.$

Example 5.18 Solve the integral equation

$$y(t) = \frac{t^2}{2} - \int_0^t u\, y(t-u)\,\mathrm{d}u$$

Noting that the integral in the given equation is a convolution type integral and taking Laplace transforms, we get

$$\bar{y}(s) = \frac{1}{s^3} - L(t)\cdot L\{y(t)\}$$

$$= \frac{1}{s^3} - \frac{1}{s^2}\,\bar{y}(s)$$

$\therefore \qquad \left(\dfrac{1+s^2}{s^2}\right)\bar{y}(s) = \dfrac{1}{s^3}$ or $\bar{y}(s) = \dfrac{1}{s\left(1+s^2\right)}$

$\therefore \qquad y(t) = \displaystyle\int_0^t \sin t\,\mathrm{d}t$

$$= 1 - \cos t.$$

Example 5.19 Solve the integral equation

$$y(t) = a\sin t - 2\int_0^t y(u)\cos(t-u)\,\mathrm{d}u.$$

Taking Laplace transforms,

$$\bar{y}(s) = \frac{a}{s^2+1} - 2L\{y(t)\}\cdot L(\cos t)$$

i.e. $\qquad \bar{y}(s) = \dfrac{a}{s^2+1} - \dfrac{2s}{s^2+1}\cdot\bar{y}(s)$

i.e. $\qquad \dfrac{(s+1)^2}{s^2+1}\,\bar{y}(s) = \dfrac{a}{s^2+1}$

i.e. $$\bar{y}(s) = \frac{a}{(s+1)^2}$$

$\therefore$ $$y(t) = a\,t\,e^{-t}.$$

Example 5.20 Solve the integro-differential equation

$$y'(t) = t + \int_0^t y(t-u)\cos u \,\mathrm{d}u,\ y(0) = 4$$

Taking Laplace transforms,

$$s\,\bar{y}(s) - 4 = \frac{1}{s^2} + \frac{s}{s^2+1}\bar{y}(s)$$

i.e. $$s\left(1 - \frac{1}{s^2+1}\right)\bar{y}(s) = \frac{1}{s^2} + 4$$

i.e. $$\bar{y}(s) = \frac{(s^2+1)(1+4s^2)}{s^5}$$

$$= \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}$$

$\therefore$ $$y(t) = 4 + \frac{5}{2}t^2 + \frac{1}{24}t^4.$$

EXERCISE 5(d)

Part A

(Short Answer Questions)

Using Laplace transforms, solve the following equations:

1. $x' + x = 2\sin t,\ x(0) = 0$
2. $x' - x = e^t,\ x(0) = 0$
3. $y' - y = t,\ y(0) = 0$
4. $y' + y = 1,\ y(0) = 0$
5. $y + \int_0^t y(t)\,\mathrm{d}t = e^{-t}$
6. $x + \int_0^t x(u)\,\mathrm{d}u = t^2 + 2t$

7. $x + \int_0^t x(t)\,\mathrm{d}t = \cos t + \sin t$

8. $x - 2\int_0^t x(t)\,\mathrm{d}t = 1$

9. $y = 1 + 2\int_0^t e^{-2u}\, y(t-u)\,\mathrm{d}u$

10. $y = 1 + \int_0^t y(u)\sin(t-u)\,\mathrm{d}u$

11. $f(t) = \cos t + \int_0^t e^{-u} f(t-u)\,\mathrm{d}u$

12. $y(t) = t + \int_0^t \sin u\, y(t-u)\,\mathrm{d}u$

Part B

Solve the following differential equations, using Laplace transforms:

13. $x'' + 3x' + 2x = 2(t^2 + t + 1)$, $x(0) = 2, x'(0) = 0$
14. $y'' - 3y' - 4y = 2e^{-t}$, $y(0) = y'(0) = 1$
15. $x'' + 4x' + 3x = 10 \sin t$, $x(0) = x'(0) = 0$
16. $(D^2 + D - 2)\, y = 20 \cos 2t$, $y = -1$, $Dy = 2$ *at* $t = 0$
17. $x'' + 4x' + 5x = e^{-2t}(\cos t - \sin t)$, $x(0) = 1$, $x'(0) = -3$
18. $y'' + 2y' + 2y = 8\, e^t \sin t$, $y(0) = y'(0) = 0$
19. $x'' - 2x' + x = t^2 e^t$, $x(0) = 2$, $x'(0) = 3$
20. $y'' + y = t \cos 2t$, $y(0) = y'(0) = 0$
21. $x'' + 9x = 18t$, $x(0) = 0$, $x(\pi/2) = 0$
22. $y'' + 4y' = \cos 2t$, $y(\pi) = 0$, $y'(\pi) = 0$
23. $(D^2 + a^2)\, x = f(t)$
24. $(D^3 - D)\, y = 2 \cos t$, $x = 3$, $Dx = 2$, $D^2x = 1$ *at* $t = 0$
25. $x''' - 3x'' + 3x' - x = 16\, e^{3t}$, $x(0) = 0$, $x'(0) = 4$, $x''(0) = 6$
26. $(D^4 - a^4)\, y = 0$, $y(0) = 1$, $y'(0) = y''(0) = y'''(0) = 0$

Solve the following simultaneous equations, using Laplace transforms:

27. $x' - y = e^t$; $y' + x = \sin t$, given that $x(0) = 1$ and $y(0) = 0$
28. $x' - y = \sin t$; $y' - x = -\cos t$, given that $x = 2$ and $y = 0$ for $t = 0$
29. $x' + 2x - y = -6t$; $y' - 2x + y = -30t$, given that $x = 2$ and $y = 3$ at $t = 0$
30. $Dx + Dy + x - y = 2$; $D^2x + Dx - Dy = \cos t$, given that $x = 0$, $Dx = 2$ and $y = 1$ at $t = 0$
31. $D^2x + y = -5 \cos 2t$; $D^2y + x = 5 \cos 2t$, given that $x = Dx = Dy = 1$ and $y = -1$ at $t = 0$

32. $x' + y' - x = 2e^t + e^{-t}$; $2x' + y + 3\int_0^t y\,dt = 2e^t(t+3)$, given that $x(0) = -1$ and $y(0) = 2$

Solve the following integral equations, using Laplace transforms:

33. $x' + 3x + 2\int_0^t x\,dt = t, x(0) = 0$

34. $y' + 4y + 5\int_0^t y\,dt = e^{-t}, y(0) = 0$

35. $x' + 2x + \int_0^t x\,dt = \cos t,\ x(0) = 1$

36. $y' + 4y + 13\int_0^t y\,dt = 3e^{-2t}\sin 3t,\ y(0) = 3$

37. $x(t) = 4t - 3\int_0^t x(u)\sin(t-u)\,du$

38. $y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)\,du$

39. $\int_0^t \frac{x(u)}{\sqrt{t-u}}\,du = 1 + t + t^2$

40. $\int_0^t y(u)\,y(t-u)\,du = 2y(t) + t - 2$

ANSWERS

Exercise 5(a)

3. $\tan t; e^{t^2}$

9. $\frac{1}{s-2}\{1 - e^{-(s-2)}\}$

10. $\left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s} - \left(\frac{2}{s} + \frac{1}{s^2}\right)e^{-2s}$

11. $(1 - e^{-\pi s})\cdot\frac{2}{s^2+4}$

12. $(1 - e^{-2\pi s})\cdot\frac{2}{s^2+1}$

13. $\dfrac{s}{s^2+1}\cdot e^{-2\pi s/3}$

14. $\dfrac{1}{s^2+1}(1+e^{-\pi s})+\left(\dfrac{\pi}{s}+\dfrac{1}{s^2}\right)e^{-\pi s}$

15. $\dfrac{1}{2s}\sqrt{\dfrac{\pi}{s}};\ \dfrac{1}{\sqrt{s}}$

16. $\dfrac{1}{s^2}e^{-3s};\ \dfrac{\pi}{s^2+\pi^2}e^{-s}$

17. $\left(\dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}\right)e^{-2s}$

18. $\dfrac{6a^3}{s^4}+\dfrac{6a^2b}{s^3}+\dfrac{3ab^2}{s^2}+\dfrac{b^3}{s}$

19. $\dfrac{1}{s^2+\omega^2}(\omega\cos\theta+s\sin\theta)$

20. $\dfrac{1}{2}\left(\dfrac{1}{s}-\dfrac{s}{s^2+36}\right)$

21. $\dfrac{1}{4}\left(\dfrac{3s}{s^2+4}-\dfrac{s}{s^2+36}\right)$

22. $\dfrac{1}{2}\left(\dfrac{3}{s^2+9}-\dfrac{1}{s^2+1}\right)$

23. $\dfrac{1}{2}\left(\dfrac{s}{s^2+25}+\dfrac{s}{s^2+1}\right)$

24. $\dfrac{1}{8}\left(\dfrac{1}{s-3}-\dfrac{1}{s-1}+\dfrac{3}{s+1}-\dfrac{1}{s+3}\right)$

25. $\dfrac{1}{4}\left(\dfrac{1}{s-2}+\dfrac{3}{s+2}+\dfrac{2}{s}\right)$

26. $\dfrac{2}{(s+1)^3}+\dfrac{2}{(s+1)^2}+\dfrac{1}{s+1}$

27. $\dfrac{s}{(s+2)^2+9}$

28. $\dfrac{1}{4}\left\{\dfrac{3}{s-3}+\dfrac{1}{s+1}\right\}$

29. $\dfrac{2s}{s^4+4}$

30. $\dfrac{1}{(s-1)^3}+\dfrac{1}{(s+1)^3}$

31. $\dfrac{2e^3}{2s-3}$

32. $(t-a)\,u_a(t)$

33. $u_2(t)-u_3(t)$

34. $e^{3(t-2)}.\,u_2(t)$

35. $\cos 3\,(t-1)\,u_1\,(t)$

36. $\sin t+\sin(t-\pi)\,u_\pi(t)$

37. $2\sqrt{\dfrac{t}{\pi}}\,e^{-t}$

38. $\dfrac{1}{2}(2+4t+3t^2)$

39. $\dfrac{1}{96}t^3\,e^{3t/2}$

40. $\dfrac{1}{12}t^3\,(2+t)\,e^{2t}$

41. $2\cos 2t+\dfrac{3}{2}\sin 2t$

42. $\cosh 3t+2\sinh 3t$

43. $\dfrac{1}{a}(1-e^{-at})$

44. $\dfrac{1}{2}e^{-t}\sin 2t$

45. $e^{3t}\cos t$

46. $\dfrac{8}{(s-3)^3}+\dfrac{12}{(s-3)^2}+\dfrac{9}{s-3}$

47. $\sqrt{\dfrac{\pi}{s-a}}\cdot\dfrac{s}{s-a}$

48. $\dfrac{1}{8}\left\{\dfrac{1}{s-2}-\dfrac{3}{s}+\dfrac{3}{s+2}-\dfrac{1}{s+4}\right\}$

49. $\dfrac{4a^3}{s^4+4a^4}$

50. $\dfrac{2}{(s-2)\left(s^2-4s+8\right)}$

51. $\dfrac{(s+3)}{4}\left\{\dfrac{3}{(s-3)^2+4}+\dfrac{1}{(s+3)^2+36}\right\}$

52. $\dfrac{\omega\cos\theta+(s+k)\sin\theta}{(s+k)^2\ \omega^2}$

53. $\dfrac{3\left(s^2+2s+9\right)}{\left(s^2+2s+5\right)\left(s^2+2s+17\right)}$

54. $\dfrac{(s-1)}{4}\left\{\dfrac{1}{(s-1)^2}+\dfrac{1}{(s-1)^2+4}+\dfrac{1}{(s-1)^2+16}+\dfrac{1}{(s+1)^2+36}\right\}$

55. $\dfrac{1}{4}\left\{\dfrac{1}{(s+2)^2}+\dfrac{3}{(s+2)^2+9}+\dfrac{5}{(s+2)^2+25}-\dfrac{9}{(s+2)^2+81}\right\}$

56. $\dfrac{s^2-4}{\left(s^2+4\right)^2}$

57. $\dfrac{s\,(s+2)}{\left(s^2+2s+2\right)^2}$

58. $\dfrac{6\,(s-2)}{\left(s^2-4s+13\right)^2}$

59. $\dfrac{2s}{\left(s^2+1\right)^2}$

60. $\dfrac{s^2-4}{\left(s^2+4\right)^2}$

61. $\dfrac{1}{2}-2e^t+\dfrac{5}{2}e^{-2t}$

62. $2e^{-t}-3e^t+5e^{2t}$

63. $\left(1-5t+\dfrac{9}{2}t^2\right)e^{-t}$

64. $-2e^{-t}+2e^{2t}+te^{2t}$

65. $-2+t+2e^{-t}+t\,e^{-t}$

66. $\sin t-\dfrac{1}{2}\sin 2t+\dfrac{1}{3}\sin 3t$

67. $\cos t-2\cos\sqrt{2}\,t+\cos\sqrt{3}\,t$

68. $e^{-3t}(2\cos 5t-3\sin 5t)$

69. $\dfrac{1}{\sqrt{3}}e^{-t/2}\left(\sqrt{3}\cos\dfrac{\sqrt{3}}{2}t+\sin\dfrac{\sqrt{3}}{2}t\right)$

70. $\dfrac{1}{a}e^{-bt/2a}\cdot\left\{\cos\left(\dfrac{\lambda t}{2a}\right)+\left(\dfrac{m}{l}-\dfrac{b}{2a}\right)\cdot\dfrac{2a}{\lambda}\sin\left(\dfrac{\lambda t}{2a}\right)\right.$

if $b^2-4ac<0$ and $=-\lambda^2$

$\dfrac{1}{a}e^{-bt/2a}\cdot\left\{\cosh\left(\dfrac{kt}{2a}\right)+\left(\dfrac{m}{l}-\dfrac{b}{2a}\right)\cdot\dfrac{2a}{k}\sinh\left(\dfrac{kt}{2a}\right)\right\},$

if $b^2-4ac>0$ and $=k^2$

$\dfrac{1}{a}e^{-bt/2a}\left\{1+\left(\dfrac{m}{l}-\dfrac{b}{2a}\right)t\right\}$, if $b^2-4ac=0$

71. $2 + e^{-2t}(\cos 3t + 2\sin 3t)$

72. $\frac{1}{3}\left[e^{-t} - e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)\right]$

73. $\frac{1}{4a^3}(\sin at \cosh at - \cos at \sinh at)$

74. $\frac{1}{2}\sin t \sinh t$

75. $\frac{1}{4}(\sinh 2t \cos 2t + \cosh 2t \sin 2t)$

76. $\cos\frac{t}{2}\cosh\frac{t}{2}$

77. $\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\cosh\frac{t}{2}$

78. $\frac{1}{2a}t\sinh at$

79. $\frac{1}{2}(\sin t - t\cos t)$

80. $\frac{1}{4}t\sin 2t$

Exercise 5(b)

2. $\frac{1}{s^2} - \frac{1}{s\left(e^s - 1\right)}$

7. $\frac{s}{\left(s^2 - a^2\right)^2}$

8. $\frac{s^2 - a^2}{\left(s^2 + a^2\right)^2}$

9. $\frac{2k^3}{\left(s^2 + k^2\right)^2}$

10. $\frac{2ks^2}{\left(s^2 + k^2\right)^2}$

11. $\frac{3s^2 + a^2}{s^2\left(s^2 + a^2\right)^2}$

12. $\frac{s^3}{\left(s^2 + k^2\right)^2}$

13. $\frac{2\sinh t}{t}$

14. $\frac{1 - e^{-t}}{t}$

15. $\frac{1 - e^{-at}}{t}$

16. $\frac{e^{-bt} - e^{-at}}{t}$

17. $\frac{e^t - 1}{t}$

18. $\frac{2}{t}\left(\cos 2t - \cos t\right)$

19. $\frac{\sin t}{t}$

20. $\frac{\sin at}{t}$

21. $\cot^{-1}(s/a)$

22. $\log\left(\frac{s+1}{s}\right)$

23. $\log\left(\frac{s-1}{s}\right)$

24. $\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)$

25. $\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

26. $\frac{E\left(1-e^{-s/E}\right)}{s\left(1-e^{-2\pi s/n}\right)}$

27. $\frac{E}{s}\tanh\left(\frac{sT}{4}\right)$

28. $[1-e^{-2\pi(s-1)}]/(s-1)(1-e^{-2\pi s})$

29. $\frac{2}{4s^2+1}\coth(\pi s)$

30. $\frac{1}{s^2+\omega^2}\left\{s+\omega\ \text{cosech}\left(\frac{\pi s}{2\omega}\right)\right\}$

31. $\frac{1}{1-e^{-2\pi s}}\left\{\frac{1}{s^2}\left(1-e^{-\pi s}\right)-\frac{\pi}{s}e^{-\pi s}\right\}$

32. $1/(s^2+1)(1-e^{-\pi s})$

33. $\omega/(s^2+\omega^2)(e^{\pi s/\omega}-1)$

34. $\frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right)$

35. $\left[\frac{1}{s^2}\left(e^{-\pi s}-1\right)^2+\frac{\pi}{s}e^{-\pi s}\left(e^{-\pi s}-1\right)\right]/\left(1-e^{-2\pi s}\right)$

36. $\frac{1}{8}\left[\frac{1}{(s-3)^2}-\frac{3}{(s-1)^2}+\frac{3}{(s+1)^2}-\frac{1}{(s+3)^2}\right]$

37. $\frac{1}{4}\left[\frac{3\left(s^2-4\right)}{\left(s^2+4\right)^2}+\frac{\left(s^2-36\right)}{\left(s^2+36\right)^2}\right]$

38. $\frac{1}{2}\left[\frac{\left(s^2-4\right)}{\left(s^2+4\right)^2}-\frac{\left(s^2-64\right)}{\left(s^2+64\right)^2}\right]$

39. $s\left[\frac{s}{\left(s^2+36\right)^2}+\frac{s}{\left(s^2+4\right)^2}\right]$

40. $\frac{1}{s^3}-\frac{s\left(48-s^2\right)}{\left(s^2+16\right)^3}$

41. $\frac{9\left(s^2-3\right)}{\left(s^2+9\right)^3}+\frac{1-3s^2}{\left(s^2+1\right)^3}$

42. $\frac{18(s^2+4s+1)}{(s^2+4s+13)^3}$

43. $\frac{s^2-6s-7}{(s^2-6s+25)^2}$

44. $\frac{1}{(s+1)^3}+\frac{1}{(s+5)^3}$

45. $3\left[\frac{s-2}{(s^2-4s+13)^2}-\frac{s+2}{(s^2+4s+13)^2}\right]$

46. $\dfrac{2}{t}(1-\cos at)$

47. $\dfrac{2}{t}(e^{-bt}-\cos at)$

48. $\dfrac{2}{t}(\cos t-e^{2t})$

49. $\dfrac{2}{t^2}\sinh at-\dfrac{2a}{t}\cosh at+a\,\delta(t)$

50. $\dfrac{1}{t}\sin\left(\dfrac{t}{2}\right)$

51. $-\dfrac{1}{t}\cdot e^{-2t}\sin 3t$

52. $-\dfrac{1}{t}e^{-bt}\sin at$

53. $-\dfrac{2}{t}\sin\dfrac{t}{\sqrt{2}}\sinh\dfrac{t}{\sqrt{2}}$

54. 0

55. $\dfrac{1}{2}$

56. log 3

57. $\dfrac{1}{2}\log 2$

58. $\log\left(\dfrac{b}{a}\right)$

59. $\dfrac{\pi}{8}$

60. $\log\left(\dfrac{s+1}{s}\right)$

61. $\dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2}\right)$

62. $\dfrac{1}{4}\log\left(\dfrac{s^2+16}{s^2}\right)$

63. $\dfrac{1}{4}\log\left(\dfrac{s^2+16}{s^2+4}\right)$

64. $s\log\dfrac{s}{\sqrt{s^2+1}}+\cot^{-1}s$

65. $\dfrac{t}{2}\sin t$

66. $\dfrac{t}{2a}\sinh at$

67. $\dfrac{t}{2}e^{2t}\sin t$

68. $\dfrac{1}{a}\sin at-t\sin at$

69. $e^{-3t}(1+t)\sin t$

70. $\dfrac{t}{2}e^{-4t}\sinh t.$

Exercise 5(c)

8. No; $1*t=\dfrac{t^2}{2}\neq t$

10. $\dfrac{1}{s-a}$

11. $\dfrac{s}{s^2+a^2}$

12. $\dfrac{2}{s(s^2+4)}$

13. $t(1-t)\,e^{-2t}$

14. $e^t\left(\dfrac{t^2}{2}+2t+1\right)$

15. $\dfrac{1}{b}e^{at}(a\sin bt+b\cos bt)$

16. $2e^{-2t}-e^{-t}$

17. $\frac{1}{s}\cot^{-1}s$ 18. $\frac{1}{s}\log\left(\frac{s-1}{s}\right)$ 19. $\frac{1}{s}\log\left(\frac{s^2+1}{s}\right)$

20. $\frac{1}{s(s+1)^2}$ 21. $\frac{1}{s(s^2+2s+2)}$. 22. $\frac{2}{\left(s^2+1\right)^2}$

23. $\frac{1}{a}(1-e^{-at})$. 24. $e^{-at}+t-1$ 25. $\frac{1}{a^2}(\cosh at-1)$

26. $1-\cos t$ 27. 1;0 28. $\frac{1}{2}$; 0 30. $\sin t$

31. $\frac{1}{4}(e^{t}-e^{-3t})$. 32. $t\,e^{t}$ 33. $\frac{1}{a^2}(1-\cos at)$

34. $e^{-t}+t-1$ 35. $\sin t-\cos t+e^{-t}$

36. $\frac{2as}{\left(s^2+a^2\right)^2};\frac{2s^3}{\left(s^2+a^2\right)^2};\frac{2as^2}{\left(s^2+a^2\right)^2}$

37. $\frac{s^2+a^2}{\left(s^2-a^2\right)^2};\frac{2as^2}{\left(s^2-a^2\right)^2};\frac{2a^3}{\left(s^2-a^2\right)^2}$

38. $\frac{1}{a-b}(e^{-bt}-e^{-at});\ \frac{1}{a-b}(a\,e^{-at}-b\,e^{-bt})$

39. $\frac{1}{2}e^{t}-e^{2t}+\frac{1}{2}e^{3t};\frac{1}{2}e^{t}-4e^{2t}+\frac{9}{2}e^{3t}$

40. $\frac{1}{a^2-b^2}\left(\frac{1}{b}\sin bt-\frac{1}{a}\sin at\right);\ \frac{1}{a^2-b^2}(\cos bt-\cos at);$

$\frac{1}{a^2-b^2}(a\sin at-b\sin bt)$

41. $\frac{1}{4}(\sin 2t+2t\sin 2t)$; $(\cos 2t-t\sin 2t)$; $\frac{t}{4}e^{-3t}\sin 2t$

42. $\frac{1}{2a}(\sinh at+at\cosh at);\frac{1}{2}(at\sinh at+2\cosh at);\ \frac{t}{2a}e^{-at}\sinh at$

43. $\frac{1}{2}\sin t\sinh t;\frac{1}{2}(\sin t\cosh t+\cos t\sinh t)$; $\cos t\cosh t$

44. $\frac{2(s-1)}{s\left(s^2-2s+2\right)^2}$ 45. $\frac{2}{\left(s^2-2s+2\right)^2}$ 46. $\frac{3s^2-4s+2}{s^2\left(s^2-2s+2\right)^2}$

47. $\frac{1}{s}\cot^{-1}\left(\frac{s+2}{3}\right)$ 48. $\frac{1}{s+2}\cot^{-1}\left(\frac{s+2}{3}\right)$

49. $\frac{3}{26}\log\left(\frac{s^2+4s+13}{s^2}\right)-\frac{6}{13}\cot^{-1}\left(\frac{s+2}{3}\right)$

50. $2 - t - 2e^{-t}$

51. $\frac{4}{9}e^{-3t/2} - \frac{3}{25}e^{-5t/3} + \frac{7}{15}t - \frac{73}{225}$

52. $\frac{1}{100}[4 - e^{-3t}(3\sin 4t + 4\cos 4t)]$

53. $e^{-t}(1 - \cos t)$

54. $\frac{1}{4}(1 - 2t - \cos 2t + \sin 2t)$

55. $\frac{1}{54}(\sin 3t - 3t\cos 3t)$

56. $\frac{1}{162}(2 - 2\cos 3t - 3t\sin 3t)$

57. $\frac{1}{2}e^{-3t}(\sin t - t\cos t)$

58. $\frac{1}{2a^3}(at\cosh at - \sinh at)$

59. $\frac{1}{2a^4}(2 - 2\cosh at + at\sinh at)$

60. $\frac{1}{16}e^{-2t}(2t\cosh 2t - \sinh 2t)$

63. $\frac{1}{2}(\cos t - \sin t - e^{-t})$

64. $\frac{1}{2a}(\sin at + \sinh at)$

65. $\frac{1}{5}(\cos 2t - \cos 5t)$

66. $\frac{1}{2a}(\sin at + at\cos at)$

67. $\frac{1}{16}(\sin 2t - 2t\cos 2t)$

68. $2e^{-t} + \sin 2t - 2\cos 2t$

69. $\frac{e^{-t}}{2}(t^2 + 4t + 6) + t - 3$

70. $\frac{1}{4}(\sin t\cosh t - \cos t\sinh t)$

Exercise 5(d)

1. $x = e^{-t} - \cos t + \sin t$
2. $x = te^{t}$
3. $y = e^{t} - t - 1$
4. $y = 1 - e^{-t}$
5. $y = (1 - t)e^{-t}$
6. $x = 2t$
7. $x = \cos t$
8. $x = e^{2t}$
9. $y = 1 + 2t$
10. $y = 1 + \frac{t^2}{2}$
11. $f(t) = \cos t + \sin t$
12. $y = t + \frac{t^3}{6}$
13. $x = t^2 - 2t + 3 - e^{-2t}$
14. $y = \frac{1}{25}(13e^{-t} - 10t\,e^{-t} + 12e^{4t})$
15. $x = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} - 2\cos t + \sin t$
16. $y = \frac{2}{3}e^{-2t} + \frac{4}{3}e^{t} - 3\cos 2t + \sin 2t$
17. $x = e^{-2t}\left[\cos t - \frac{3}{2}\sin t + \frac{t}{2}(\sin t + \cos t)\right]$

18. $y = 2\,(\sin t \cosh t - \cos t \sinh t)$

19. $x = \left(\dfrac{t^4}{12} + t + 2\right)e^t$

20. $y = -\dfrac{5}{9}\sin t + \dfrac{4}{9}\sin 2t - \dfrac{t}{3}\cos 2t$

21. $x = 2t + \pi \sin 3t$

22. $y = \dfrac{1}{4}(t - \pi)\sin 2t$

23. $x = A\cos at + B\sin at + \dfrac{1}{a}\displaystyle\int_0^t \sin au \cdot f(t-u)\,\mathrm{d}u$

24. $y = 3\sinh t + \cosh t - \sin t + 2$

25. $x = 2e^{3t} - 5t^2 e^t - 2e^t$

26. $y = \dfrac{1}{2}(\cos at + \cosh at)$.

27. $x = \dfrac{1}{2}(e^t + 2\sin t + \cos t - t\cos t)$;

$y = \dfrac{1}{2}(-e^{-t} - \sin t + \cos t + t\sin t)$

28. $x = 2\cosh t$; $y = 2\sinh t - \sin t$

29. $x = 1 + 2t - 6t^2 + e^{-3t}$; $y = 4 - 2t - 12t^2 - e^{-3t}$

30. $x = t + \sin t$; $y = t + \cos t$

31. $x = \sin t + \cos 2t$; $y = \sin t - \cos 2t$

32. $x = t\,e^t - e^{-t}$; $y = e^t + e^{-t}$

33. $x = \dfrac{1}{2}(1 + e^{-2t}) - e^{-t}$

34. $y = -\dfrac{1}{2}e^{-t} + \dfrac{1}{2}e^{-2t}(\cos t + 3\sin t)$

35. $x = \dfrac{1}{2}\{(1-t)e^{-t} + \cos t\}$

36. $y = e^{-2t}\left\{3\cos 3t - \dfrac{7}{3}\sin 3t + \dfrac{3}{2}t\sin 3t + t\cos 3t\right\}$

37. $x = t + \dfrac{3}{2}\sin 2t$

38. $y(t) = e^{-t}(1-t)^2$

39. $x(t) = \dfrac{1}{\pi}\left(t^{-1/2} + 2t^{1/2} + \dfrac{8}{3}t^{3/2}\right)$

40. $y(t) = 1$