Laplace Transforms

Transform:- Transform is a equalent form of mathematical operations.

Laplace transform:- A function $f(t)$ can be defined and continuous for some (interval) positive values of 't'. Then Laplace transform of $f(t)$ is given by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt, \quad t>0$$

s is an function associated with $f(t)$

Note:-

$$\Gamma n = \int_0^\infty e^{-x} x^n\,dx$$

$$\Gamma n = n\,\Gamma n-1$$

$$= \vdots$$

$$= (n-1)!$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Piece wise continuous func:- A function $f(t)$ is said to be piece wise continuous or sectionally continuous and closed interval $[a,b]$ If the interval can be sub divided into smaller sub intervals in each of which $f(t)$ should be continuous.

Exponential order:- A function $f(t)$ is said to be of exponential order if it satisfies the condition

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

Scanned by CamScanner

Eg:- $f(t) = t^2$

$$\lim_{t \to \infty} e^{-st} t^2$$

$$= \lim_{t \to \infty} \frac{t^2}{e^{st}} \left(\frac{\infty}{\infty}\right) \quad \text{L-Hospital's rule}$$

$$= \lim_{t \to \infty} \frac{2t}{e^{st}(s)}$$

$$= \lim_{t \to \infty} \frac{2}{s^2 e^{st}}$$

$$= 0$$

Eg:- Show that $f(x) = x^n$

$$\lim_{x \to \infty} e^{-sx} x^n$$

$$\lim_{x \to \infty} \frac{x^n}{e^{sx}} \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x \to \infty} \frac{n x^{n-1}}{(s) e^{sx}} \quad \text{(value of } x \text{ decreases by continuation)}$$

$$\lim_{x \to \infty} \frac{n(n-1) \cdots}{s^n e^{sx}}$$

$$\lim_{x \to \infty} \frac{n!}{s^n e^{sx}}$$

Note:-

The p-form of Laplace transforms is given by

$$\int_0^\infty p \cdot e^{-pt} f(t)\, dt$$

Scanned by CamScanner

Conditions for Laplace transform (or)

necessary conditions

(i) $f(t)$ should be continuous or piecewise continuous on $[a,b]$

(ii) $f(t)$ should be of exponential order

Laplace transforms of some standard functions

1) $f(t) = 1$, $\boxed{L[1] = \frac{1}{s}}$

$$Lf(t) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}(1)\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty}$$

$$= \frac{e^{-\infty}}{-s} - \left[\frac{1}{-s}\right]$$

$$L(1) = \frac{-1}{-s}$$

$$\boxed{L[1] = \frac{1}{s}}$$

Similarly $\boxed{L(k) = \frac{k}{s}}$

where k is a constant.

2) $L(t^n) = \frac{n!}{s^{n+1}}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st} t^n\,dt$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \left(\frac{dx}{s}\right)$$

$x = st$

$dx = s\,dt$

Limits

t	0	∞
x	0	∞

Scanned by CamScanner

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$\boxed{L(t^n) = \frac{n!}{s^{n+1}}}$$

* we can derive this by using integration by parts

Let: $u = t^n \qquad dv = \int e^{-st} dt$

$du = nt^{n-1} dt \qquad v = \frac{e^{-st}}{-s}$

$$\int u \, dv = uv - \int v \, du$$

$$= \cancel{t^n \left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty}} - \int_0^\infty \frac{e^{-st}}{-s} (nt^{n-1}) dt$$

cancel

$$= \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt$$

$$L(t^n) = \frac{n}{s} L(t^{n-1})$$

Replace $n \to n-1$

$$L(t^{n-1}) = \frac{n-1}{s} L(t^{n-2})$$

$$L[t^0] = \frac{1}{s} \quad (t^0 = 1)$$

$$\therefore L[t^n] = \frac{n}{s} \cdot \frac{(n-1)}{s} \cdots \frac{1}{s}$$

$$\boxed{L[t^n] = \frac{n!}{s^{n+1}}}$$

Scanned by CamScanner

3) $\boxed{L(e^{at}) = \frac{1}{(s-a)}}$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-st+at}\,dt$$

$$= \int_0^\infty e^{-t(s-a)}\,dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_{t=0}^{\infty}$$

$$= 0 - \frac{1}{-(s-a)}$$

$$\boxed{L(e^{at}) = \frac{1}{s-a}}$$

4) Similarly $L(e^{-at}) = \frac{1}{(s+a)}$

$$\int_0^\infty e^{-st}(e^{-at})\,dt$$

$$\int_0^\infty e^{-st-at}\,dt$$

$$\int_0^\infty e^{-t(s+a)}$$

$$\boxed{L(e^{-at}) = \frac{1}{s+a}}$$

Scanned by CamScanner

5) $e^{i\theta} = \cos\theta + i\sin\theta$

$L[\cos at + i\sin at] =$

$L(e^{iat}) = \frac{1}{(s-ia)}$

$\Rightarrow L[\cos at + i\sin at] = \frac{1}{s-ia}$

$L[\cos at] + iL[\sin at] = \frac{s+ia}{s^2+a^2}$

$L[\cos at] + iL[\sin at] = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$

$L[\cos at] = \frac{s}{(s^2+a^2)}$ $L[\sin at] = \frac{a}{(s^2+a^2)}$

Note:- Additive property

$L[f_1(t)+f_2(t)] = L[f_1(t)] + L[f_2(t)]$

The Laplace transforms of two functions is equal to their individual Laplace trans.

Linearity property,

$L[k \cdot f(t)] = k \cdot L[f(t)]$.

6) $L[\cosh at]$ & $L[\sinh at]$

$\cosh at = \frac{e^{at}+e^{-at}}{2}$

$L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$= \frac{1}{2}[L(e^{at})] + \frac{1}{2}[L(e^{-at})]$

$= \frac{1}{2}\left(\frac{1}{s-a}\right) + \frac{1}{2}\left(\frac{1}{s+a}\right)$

Scanned by CamScanner

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

$$= \frac{s}{s^2-a^2}$$

Similarly:

$$L[\sinh at] = \frac{a}{s^2-a^2}$$

Formula

L.H.S.	R.H.S.
$L[1]$	$\frac{1}{s}$
$L[k]$	$\frac{k}{s}$
$L[t^n]$ $(n>0)$	$\frac{n!}{s^{n+1}}$
$L[t^n]$	$\frac{\Gamma(n+1)}{s^{n+1}}$
$L[e^{at}]$	$\frac{1}{s-a}$
$L[e^{-at}]$	$\frac{1}{s+a}$
$L[\cos at]$	$\frac{s}{s^2+a^2}$
$L[\sin at]$	$\frac{a}{s^2+a^2}$
$L[\cosh at]$	$\frac{s}{s^2-a^2}$
$L[\sinh at]$	$\frac{a}{s^2-a^2}$

Scanned by CamScanner

Problems:-

1. Find the L.T for the following.

(i) $L[t^2+7+\cos^2 t]$

Sol: $L[t^2]+L[7]+L[\cos^2 t]$ $\qquad L[t^n]=\frac{n!}{s^{n+1}}$

$\left[\frac{2!}{s^3}\right]+\frac{7}{s}+L[\cos^2 t]$

$\frac{2!}{s^3}+\frac{7}{s}+L\left[\frac{1+\cos 2t}{2}\right]$ $\qquad L[\cos at]=\frac{s}{s^2+a^2}$

$\frac{2!}{s^3}+\frac{7}{s}+\frac{1}{2}[L(1)+L[\cos 2t]]$

$\frac{2!}{s^3}+\frac{7}{s}+\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+4}\right]$

$\frac{2!}{s^3}+\frac{7}{s}+\frac{1}{2s}+\frac{1}{2s^2+8}$

(ii) $L[t^{3/2}+\cosh 3t+\cos 3t\cdot\cos 5t]$

Sol:- $L[t^{3/2}]+L[\cosh 3t]+L[\cos 3t\cdot\cos 5t]$

$L[t^{3/2}]=\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{3/2+1}}$ $\qquad L[t^n]=\frac{\Gamma(n+1)}{s^{n+1}}$

$=\frac{3\sqrt{\pi}}{4(s^{5/2})}$ $\qquad \Gamma\left(\frac{3}{2}+1\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$=\frac{3}{2}\Gamma\left(1+\frac{1}{2}\right)$

$=\frac{3}{4}\sqrt{\pi}$

$L[\cosh 3t]=\frac{s}{s^2-9}$

Scanned by CamScanner

$\cos 3t \cdot \cos 5t = \frac{1}{2}[\cos(3t+5t) + \cos(3t-5t)]$

$= \frac{1}{2}[\cos 8t + \cos(2t)]$

$\frac{1}{2} L[\cos 8t] + \frac{1}{2} L[\cos 2t]$

$\frac{1}{2}\left[\frac{s}{s^2+64}\right] + \frac{1}{2}\left[\frac{s}{s^2+4}\right]$

$\therefore$ RHS $= \frac{3\sqrt{\pi}}{4(s^{5/2})} + \frac{s}{s^2-9} + \frac{1}{2}\left[\frac{s}{s^2+64} + \frac{s}{s^2+4}\right]$

(iii) $L[\sin^3 2t + \cos^3 t]$

Sol: $L[\sin^3 2t] + L[\cos^3 t]$

$\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$

$\cos^3\theta = \frac{1}{4}[3\cos\theta + \cos 3\theta]$.

$L\left[\frac{1}{4}(3\sin 2t - \sin 6t)\right]$

$+ L\left[\frac{1}{4}(3\cos t + \cos 3t)\right]$

$\frac{1}{4} L[3\sin 2t] - \frac{1}{4} L[\sin 6t] + \frac{3}{4} L[\cos t] + \frac{1}{4} L[\cos 3t]$

$\frac{3}{4}\left[\frac{2}{s^2+4}\right] - \frac{1}{4}\left[\frac{6}{s^2+36}\right] + \frac{3}{4}\left[\frac{s}{s^2+1}\right] + \frac{1}{4}\left[\frac{s}{s^2+9}\right]$

$\frac{6}{4}\left[\frac{1}{s^2+4}\right] - \frac{6}{4}\left[\frac{1}{s^2+36}\right] + \frac{3}{4}\left[\frac{s}{s^2+1}\right] + \frac{1}{4}\left[\frac{s}{s^2+9}\right]$

$\frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right] + \frac{1}{4}\left[\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right]$

(iv) $L[\sin t \cdot \sin 2t \cdot \sin 3t]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$.

$L\left[\frac{1}{2}[\cos(-t) - \cos(3t)] \sin 3t\right]$

$L\left[\frac{1}{2}(\sin 3t \cos t - \cos 3t \sin 3t)\right]$

Scanned by CamScanner

$$\frac{1}{2} L[\sin 3t \cos 2t - \cos 3t \sin 3t]$$

$$\frac{1}{2}\left[L\left[\frac{1}{2}(\sin(4t) - \sin(-2t)\right] - \left[\frac{1}{2}\sin(6t) - \sin(0)\right]\right]$$

$$\frac{1}{4}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right] - \frac{1}{4}\left[\frac{6}{s^2+36}\right]$$

H.W

(v) $L[\cos^4 t]$

(vi) $L[\sin^3 5t + \cosh 2t + e^{-5t}]$

(vii) $L\left[\frac{1}{\sqrt{\pi t}}\right]$

(viii) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 0, & \text{otherwise} \end{cases}$

Sol:- $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f(t)] = \int_0^\pi e^{-st} \cos t\, dt$$

$$\boxed{\int e^{at} \cos bt\, dt = \frac{e^{at}}{a^2+b^2}[a\cos bt + b\sin bt]}$$

$a = -s$

$b = 1$

Scanned by CamScanner

$$= \left[\frac{e^{-st}}{s^2+1}\left[-s\cos t + \sin t\right]\right]_0^{\pi}$$

$$= \left\{\frac{e^{-\pi s}}{s^2+1}\left[-s\cos\pi + \sin\pi\right]\right\}$$

$$- \left\{\frac{e^{0}}{s^2+1}\left[-s\cos 0 + \sin 0\right]\right\}$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$= \frac{e^{-\pi s}}{s^2+1}\left\{-s[(-1)^{\pi}]\right\} + \frac{s}{s^2+1}$$

(ix) $f(t) = \begin{cases} \sin 2t, & (0, \pi) \\ 0, & \end{cases}$

$$L(f(t)) = \int_0^{\pi} e^{-st}\sin 2t\, dt$$

$$\int e^{at}\sin bt\, dt = \frac{e^{at}}{a^2+b^2}\left[a\sin bt - b\cos bt\right]$$

$$= \left[\frac{e^{-st}}{s^2+4}\left[-s\sin 2t - \cos 2t\right]\right]_0^{\pi}$$

$$= \frac{e^{-\pi s}}{s^2+4}\left[-s\sin 2\pi - \cos\pi\right] - \frac{e^{0}}{s^2+4}\left[-s\sin 0 - \cos\right.$$

$$= \frac{e^{-\pi t}}{s^2+4}\left[(+1)\right] + \frac{1}{s^2+4} = \frac{e^{-\pi t}+1}{s^2+4}$$

HW

(x) $f(t) = \begin{cases} e^{-2t}, & 0 < t < 4 \\ 0, & t > 4 \end{cases}$

Scanned by CamScanner

First shifting theorem

If $L[f(t)] = F[s]$ then

(i) $L[e^{at} f(t)] = F(s-a)$

(ii) $L[e^{-at} f(t)] = F(s+a)$

$L[f(t)]$ omit s then by $s \to (s-a)$

(i) $L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\, dt$

$= \int_0^\infty e^{-st+at} f(t)\, dt$

$= \int_0^\infty e^{-t(s-a)} f(t)\, dt$

$\Rightarrow F[s-a]$

H.W.

(1) $L[\cos^4 t]$

Sol:- $L[\cos^4 t]$

$L[(\cos^2 t)^2]$

$L\left[\left(\frac{1+\cos 2t}{2}\right)^2\right]$

$\frac{1}{2} L\left[(1+\cos 2t)^2\right]$

$\frac{1}{4} L\left[(1+\cos^2 2t + 2\cos 2t)\right]$

$\frac{1}{4}\left[L[1] + L[\cos^2 2t] + 2L[\cos 2t]\right]$

$\frac{1}{4}\left\{\frac{1}{s} + L\left[\frac{1+\cos 4t}{2}\right] + 2\left[\frac{s}{s^2+4}\right]\right]$

Scanned by CamScanner

$$\frac{1}{4}\left\{\frac{1}{s}+\frac{1}{2}\left(\frac{1}{s}\left[+\frac{s}{s^2+16}\right)+\frac{2s}{s^2+4}\right\}\right.$$

(ii) $L\left[\sin^3 5t+\cosh 2t+e^{-5t}\right]$

Sol:- $L[\sin^3 5t]+L[\cosh 2t]+L[e^{-5t}]$

$$L\left[\frac{1}{4}(3\sin 5t-\sin 15t)\right]+\left[\frac{s}{s^2-4}\right]+\frac{1}{s+5}$$

$$\frac{1}{4}\left\{L[3\sin 5t]-L[\sin 15t]\right\}+\frac{s}{s^2-4}+\frac{1}{s+5}$$

$$\frac{1}{4}\left\{\frac{15}{s^2+25}-\frac{15}{s^2+225}\right\}+\frac{s}{s^2-4}+\frac{1}{s+5}$$

$$\frac{15}{4}\left\{\frac{1}{s^2+25}-\frac{1}{s^2+225}\right\}+\frac{s}{s^2-4}+\frac{1}{s+5}$$

(iii) $L\left[\cancel{\frac{1}{\sqrt{\pi t}}}\right]$

Sol:- (iv)

$$f(t)=\begin{cases} e^{-3t}, & 0<t<4 \\ 0, & t>4 \end{cases}$$

Sol:- $L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt$

$$\int_0^4 e^{-st}e^{-3t}\,dt$$

$$\int_0^4 e^{-st-3t}\,dt$$

$$\int_0^4 e^{-t(s+3)}\,dt$$

$$F[s+3]$$

Scanned by CamScanner

Q. $L\left[\frac{1}{\sqrt{\pi t}}\right]$

$\frac{L}{\sqrt{\pi}}\left[\frac{1}{t^{1/2}}\right]$

$\frac{1}{\sqrt{\pi}} L\left[t^{-1/2}\right]$

$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} \cdot \frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{s^{1/2}}\right]$

$= \frac{1}{s^{1/2}}$

Second shifting theorem:-

Statement:- If $L[f(t)] = F(s)$ & $G(t) = \begin{cases} f(t-a), & t>a \\ 0, & \text{otherwise} \end{cases}$

$L[G(t)] = e^{-as} F(s)$.

proof:- $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$L[G(t)] = \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$ $\quad 0 \quad t<a \quad t>a$

$= 0 + \int_a^\infty f(t-a)\, e^{-st}\,dt$

$= \int_0^\infty f(u) e^{-s(u+a)}\,du$ $\quad$ Put $u = t-a$, $du = dt$

t	0	∞
u	0	∞

$= \int_0^\infty f(u)\, e^{-su} e^{-sa}\,du$

$= e^{-sa} \int_0^\infty f(u) e^{-su} du$ $\quad u \to t$ dummy variable

$= e^{-sa} \cdot L[f(t)]$ → from statement

$= e^{-sa} F(s)$

Derivative of transform

$L[t f(t)] = -\frac{d}{ds} L[f(t)]$

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [L(f(t))]$.

Q:- $e^{-3t} \cosh 3t$.

Sol:- $L[e^{-3t} \cosh 3t]$ $\quad L[\cosh 3t]\Big|_{s \to s+3}$

$L[e^{at} f(t)] = L[f(t)]\Big|_{s \to s-a}$.

$L[\cosh 3t] = \left[\frac{s}{s^2-9}\right]_{s \to s+3}$

$= \frac{s+3}{(s+3)^2-9} = \frac{s+3}{s^2+9+6s-9}$

$= \frac{s+3}{s^2+6s} = \frac{s+3}{s(s+6)}$.

Q:- $e^{-2t} \sin 3t \cos 5t$.

Sol:- $L[e^{-2t} \sin 3t \cos 5t]$:

$L[\sin 3t \cos 5t]_{s \to s+2}$

$\sin A \cos A = \frac{1}{2}[\sin(A+B) + \sin(A-B)]_{s \to s+2}$

$L\left\{\frac{1}{2}[\sin(8t) + \sin(-2t)]\right\}_{s \to s+2}$

$\frac{1}{2} L\{L[\sin 8t] + L[\sin -2t]\}_{s \to s+2}$

$\frac{1}{2}\{L[\sin 8t - L[\sin 2t]\}_{s \to s+2}$

Scanned by CamScanner

$$\frac{1}{2}\left[\frac{8}{s^2+64} - \frac{2}{s^2+4}\right]_{s \to s+2}$$

$$\frac{1}{2}\left[\frac{8}{(s+2)^2+64} - \frac{2}{(s+2)^2+4}\right]$$

$$\frac{1}{2}\left[\frac{8}{s^2+4+2s+64} - \frac{2}{s^2+4+2s+4}\right]$$

$$\frac{1}{2}\left[\frac{8}{s^2+2s+68} - \frac{2}{s^2+2s+8}\right]$$

Q:- $L\left[e^{-2t}\sin(2t+3)\right]$

Sol:- $L\left[\sin(2t+3)\right]_{s \to s+2}$

$$L\left[\sin 2t \cos 3 + \cos 2t \sin 3\right]_{s \to s+2}$$

$$L\left[\sin 2t \cos 3\right] + L\left[\cos 2t \sin 3\right]_{s \to s+2}$$

$$\cos 3\left[L(\sin 2t)\right] + \sin 3\, L\left[\cos 2t\right]_{s \to s+2}$$

$$\left[\cos 3 \frac{2}{s^2+4} + \sin 3 \frac{s}{s^2+4}\right]_{s \to s+2}$$

$$\cos 3 \frac{(2)}{(s+2)^2+4} + \frac{(s+2)\sin 3}{(s+2)^2+4}$$

$$\frac{2\cos 3}{s^2+4+2s+4} + \frac{(s+2)\sin 3}{s^2+4+2s+4}$$

$$\frac{2\cos 3}{s^2+2s+8} + \frac{(s+2)\sin 3}{s^2+2s+8}$$

$$\frac{2\cos 3 + (s+2)\sin 3}{s^2+2s+8}$$

Scanned by CamScanner

Q: $t^2 \cos 2t$

(1A) $L[t \cdot \sinh 3t] = L[t \sinh 3t]$

$$= \frac{-d}{ds} L[\sinh 3t]$$

$$= \frac{-d}{ds} \frac{3}{s^2-9} \qquad \because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$= -3\left(\frac{-1}{(s^2-9)^2}\right) \times 2s$$

$$= \frac{3}{s^2+81+18s} \times 2s$$

$$= \frac{6s}{s^2+18s+81}$$

(2A) $[t^2 \cos 2t] = (-1)^2 \frac{d^2}{ds^2}[L(\cos 2t)]$

$$= 1 \frac{d^2}{ds^2} L[\cos 2t]$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{d}{ds} \frac{s^2+4 - s(2s)}{(s^2+4)^2}$$

$$= \frac{d}{ds}\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{(-2s)(s^2+4)^2 - (4-s^2)\,2(s^2+4)(2s)}{(s^2+4)^4}$$

$$= \frac{(s^2+4)\left[(-2s)(s^2+4) - (4-s^2)4s\right]}{(s^2+4)^4}$$

$$= \frac{-2s(s^2+4) - 4s(4-s^2)}{(s^2+4)^3}$$

Scanned by CamScanner

$$= \frac{-2s^3 - 8s - 16s + 4s^3}{(s^2+4)^3}$$

$$= \frac{2s^3 - 24s}{(s^2+4)^3}.$$

Q: Find the L.T of $[e^t . t^2 \sin 2t]$

Sol: $L[e^t t^2 \sin 2t]$

$$L[t^2 \sin 2t]_{s \to s-1}$$

$$= \left[(-1)^2 \frac{d^2}{ds^2} [\sin 2t]\right]_{s \to s-1}$$

$$= \left\{\frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right]\right\}_{s \to s-1}$$

$$= \left\{\frac{d}{ds} \frac{-2(2s)}{(s^2+4)^2}\right\}_{s \to s-1}$$

$$= \left\{-4 \frac{d}{ds}\left[\frac{s}{(s^2+4)^2}\right]\right\}_{s \to s-1}$$

$$= \left\{-4\left[\frac{(s^2+4)^2 - s\,2(s^2+4)2s}{(s^2+4)^4}\right]\right\}_{s \to s-1}$$

$$= \left\{-4\left[\frac{(s^2+4)[s^2+4-4s^2]}{(s^2+4)^4}\right]\right\}_{s \to s-1}$$

$$= \left\{-4\left[\frac{s^2+4-4s^2}{(s^2+4)^3}\right]\right\}_{s \to s-1}$$

Scanned by CamScanner

$$= \left[\frac{-4(-3s^2+4)}{(s^2+4)^3}\right]_{s \to s-1}$$

$$= \frac{-4(-3(s-1)^2+4)}{((s-1)^2+4)^3}$$

$$= \frac{-4(-3(s^2+1-2s)+4)}{(s^2+1-2s+4)^3}$$

$$= \frac{-4(-3s^2-3+6s+4)}{(s^2-2s+5)^3}$$

$$= \frac{-4(-3s^2+6s+1)}{(s^2-2s+5)^3}$$

H.W

1. $L[e^{at} \sinh bt]$
2. $L[e^t t^{-1/2}]$
3. $L[te^{-2t} \cos 3t]$
4. $L[t \sin 3t \sin 5t]$
5. $L[t^2 e^{3t} \sin 4t]$

Scanned by CamScanner

Unit step function:-

This is also known as ~~Heavis~~ Heaviside function is denoted by $u(t-a)$

or $H(t-a) = \begin{cases} 0, & t<a \\ 1, & t>0 \end{cases}$

$$L[u(t-a)] = \int_0^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt$$

$$= \int_a^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= 0 + \frac{e^{-sa}}{s}$$

$$\boxed{L[u(t-a)] = \frac{e^{-sa}}{s}}$$

(ii) $L[e^{at} \sinh bt]$

Sol:- $L[e^{at} \sinh bt]$

$L[\sinh bt]_{s \to s-a}$

$$= \frac{b}{s^2 - b^2}$$

$$= \frac{b}{(s-a)^2 - b^2}$$

Scanned by CamScanner

$$= \frac{b}{s^2+a^2-2as-b^2}$$

$$= \frac{b}{s^2-2as+a^2-b^2}$$

(iv) $L[e^t t^{-1/2}]$

Sol: $L[t^{-1/2}]_{s \to s-1}$

$$\left[\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}\right]_{s \to s-1}$$

$$\left[\frac{\Gamma\frac{1}{2}}{s^{1/2}}\right]_{s \to s-1}$$

$$\left[\frac{\sqrt{\pi}}{s^{1/2}}\right]_{s \to s-1}$$

$$\frac{\sqrt{\pi}}{(s-1)^{1/2}} = \sqrt{\pi}(s-1)^{-1/2}$$

(v) $L[t e^{-2t} \cos 3t]$

Sol: $L[t \cos 3t]_{s \to (s+2)}$

$$= \left[-\frac{d}{ds} L[\cos 3t]\right]_{s \to s+2}$$

$$= \left\{-\frac{d}{ds}\left[\frac{s}{s^2+9}\right]\right\}_{s \to s+2}$$

$$= \left\{-\left[\frac{s^2+9-s(2s)}{(s^2+9)^2}\right]\right\}_{s \to s+2}$$

$$= \left\{-\left[\frac{s^2+9-2s^2}{(s^2+9)^2}\right]\right\}_{s \to s+2}$$

$$= \left\{-\left[\frac{-s^2+9}{(s^2+9)^2}\right]\right\}_{s \to s+2}$$

Scanned by CamScanner

$$= \left\{ \frac{s^2-9}{(s^2+9)^2} \right\}_{s \to s+2}$$

$$= \frac{(s+2)^2 - 9}{((s+2)^2+9)^2}$$

$$= \frac{s^2+4+4s-9}{(s^2+4+4s+9)^2} = \frac{s^2+4s-5}{(s^2+4s+13)^2}$$

$$= \cancel{\frac{s^2+4s+13}{}}$$

(iv) $L[t \sin 3t \sin 5t]$

Sol:- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$\sin 3t \sin 5t = \frac{1}{2}[\cos(-2t) - \cos(8t)]$$

$$= \frac{1}{2}[\cos 2t - \cos 8t]$$

$$= L\left[t \cdot \frac{1}{2}[\cos 2t - \cos 8t]\right]$$

$$= \frac{1}{2}\left\{\left[-\frac{d}{ds}(\cos 2t)\right] - \left[-\frac{d}{ds}\cos 8t\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{d}{ds}\frac{s}{s^2+4}\right] - \left[\frac{-d}{ds}\cos\frac{s}{s^2+64}\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{(s^2+4-2s^2)}{(s^2+4)^2} + \frac{s^2+64-2s^2}{(s^2+64)^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{s^2+4}{(s^2+4)^2} - \frac{s^2-64}{(s^2+64)^2}\right\}$$

Scanned by CamScanner

(v) $L[t^2 e^{3t} \sin 4t]$.

Sol: $L[t^2 \sin 4t]_{s \to (s-3)}$

$$= \left[(-1)^2 \frac{d^2}{ds^2}\left[\sin 4t\right]\right]_{s \to s-3}$$

$$= \frac{d^2}{ds^2}\left[\frac{4}{s^2+16}\right]$$

$$= \left\{4\frac{d}{ds}\left[\frac{-1 \times 2s}{(s^2+16)^2}\right]\right\}_{s \to s-3}$$

$$= -4\left[\frac{(0)(s^2+16)^2 - (-1)(s^2+16)^2}{(s^2+16)^4}\right]_{s \to s-3}$$

$$= -4\left[\frac{(s^2+16)^2}{(s^2+16)^4}\right]_{s \to s-3}$$

$$= -4\left[\frac{1}{(s^2+16)^2}\right]_{s \to s-3}$$

$$= -4\left[\frac{1}{((s-3)^2+16)^2}\right]$$

$$= -4\left[\frac{1}{(s^2+9-6s+16)^2}\right]$$

$$= -4\left[\frac{1}{(s^2-6s+25)^2}\right]$$

$$= \frac{-4}{s^2-6s+25}$$

$$= -8\frac{d}{ds}\left[\frac{s}{(s^2+16)^2}\right]_{s \to s-3}$$

$$= -8\left[\frac{(s^2+16)^2 - s\,2(s^2+16)(2s)}{(s^2+16)^4}\right]_{s \to s-3}$$

Scanned by CamScanner

$$= -8\left[\frac{(s^2+16)^2-4s^2(s^2+16)}{(s^2+16)^4}\right]_{s\to s-3}$$

$$= -8\left[\frac{(s-3)^2+16)^2-4s^2((s-3)^2+16)}{((s-3)^2+16)^4}\right]$$

$$= -8\left[\frac{(s^2+9-6s+16)^2-4s^2(s^2+9-6s+16)}{(s^2+9-6s+16)^4}\right]$$

$$= -8\left[\frac{(s^2-6s+25)^2-4s^2(s^2-6s+25)}{(s^2-6s+25)^4}\right]$$

$$= -8\left[\frac{s^2-6s+25-4s^2}{(s^2-6s+25)^3}\right]$$

$$= -8\left[\frac{s^2-6s+25-6s-3s^2}{(s^2-6s+25)^3}\right]$$

$$= 8\left[\frac{3s^2+6s-25}{(s^2-6s+25)^3}\right]$$

$$= -8\left[\frac{(s^2-6s+25)^2-4(s-3)^2(s^2-6s+25)}{(s^2-6s+25)^4}\right]$$

$$= -8\left[\frac{(s^2-6s+25)^2-4(s^2+9-6s)(s^2-6s+25)}{(s^2-6s+25)^4}\right]$$

$$= -8\left[\frac{s^2-6s+25-4s^2-36+24s}{(s^2-6s+25)^3}\right]$$

Scanned by CamScanner

$$= -8\left[\frac{-3s^2-48s-11}{(s^2-6s+25)^3}\right]$$

$$= 8\left[\frac{3s^2+48s-11}{(s^2-6s+25)^2}\right]$$

(iii) $L[t e^{-2t}\cos 3t]$

Q:- $\int_0^a e^{-2t}\, t\cos t \, dt$.

Sol $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

by comparison $f(t) = t\cos t$ $s=2$

$$Lf(t) = L[t\cos t]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s=2}$$

$$= -\left[\frac{s^2+1-s(2s)}{(s^2+1)^2}\right]_{s=2}$$

$$= -\left[\frac{-s^2+1}{(s^2+1)^2}\right]_{s=2} = \left[\frac{-4+1}{(4+1)^2}\right]$$

$$= -\left[\frac{-3}{25}\right] = \frac{3}{25}.$$

Integral of transforms:-

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\, ds$$

$$L\left[\frac{f(t)}{t^n}\right] = \underbrace{\int_s^\infty\int_s^\infty \cdots \int_s^\infty}_{n\text{ times}} L[f(t)]\, ds.$$

Scanned by CamScanner

$$L\left[\frac{1-e^{-2t}}{t}\right]$$

Apply l'opital rule

$$\lim_{t\to 0} \frac{1-e^{-2t}}{t} = \frac{1}{0}$$

$$\lim_{t\to 0} \frac{2e^{-2t}}{1}$$

$= 2 \quad \therefore f(t)$ exits.

$$\int_s^\infty L[1-e^{-2t}]\,ds = \int_s^\infty [L(1) - L(e^{-2t})]\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{1}{s+2}\right] ds$$

$$= \cancel{\int_s^\infty \frac{-1}{s^2} + \frac{1}{(s+2)^2}} \quad [\log s - \log(s+2)]_s^\infty$$

$$= \left[\log\left(\frac{s}{s+2}\right)\right]_s^\infty = \left[\log\left[\frac{\cancel{s}}{\cancel{s}(1+\frac{2}{s})}\right]\right]_s^\infty$$

$$= \log 1 - \log \frac{1}{(1+\frac{2}{s})}$$

$$= -\log\left[\frac{1}{\frac{s+2}{s}}\right]$$

$$= -\log\left(\frac{s+2}{s}\right)^{-1}$$

$$= \log\left(\frac{s+2}{s}\right)$$

Scanned by CamScanner

(c) $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

Sol: Apply Capital rule

$$\lim_{t\to 0}\frac{e^{-at}}{t}-\lim_{t\to 0}\frac{e^{-bt}}{t}$$

$$\lim_{t\to 0}\frac{-ae^{-at}}{1}-\lim_{t\to 0}\frac{-be^{-bt}}{1}$$

$$-a-(-b)=b-a \quad \therefore f(t) \text{ exists.}$$

$$\int_s^\infty L\left[\frac{e^{-at}-e^{-bt}}{t}\right]ds=\int_s^\infty\left(L[e^{-at}]-L[e^{-bt}]\right)ds$$

$$=\int_s^\infty\left(\frac{1}{s+a}\right)ds-\int_s^\infty\left(\frac{1}{s+b}\right)ds$$

$$=\left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$=\left[\log\left[\frac{s+a}{s+b}\right]\right]_s^\infty$$

$$=\left[\log\frac{s(1+\frac{a}{s})}{s(1+\frac{b}{s})}\right]_s^\infty=\left[\frac{\log(1+\frac{a}{s})}{\log(1+\frac{b}{s})}\right]_s^\infty$$

$$=\log\left[\frac{\frac{s+a}{s}}{\frac{s+b}{s}}\right]=-\log\left[\frac{s+a}{s+b}\right]$$

$$=\log\left[\frac{s+b}{s+a}\right].$$

Scanned by CamScanner

Q1. Find $L\left[\frac{\sin at}{t}\right]$ and hence evaluate $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Sol:

$$\lim_{t\to 0} \frac{\sin at}{t} \quad \text{L'optital rule}$$

$$\lim_{t\to 0} a\cos at$$

$$= a = f(t) \text{ exists.}$$

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\,ds.$$

$$= \int_s^\infty \left[\frac{a}{s^2+a^2}\right] ds$$

$$= a\int_s^\infty \frac{1}{s^2+a^2}\,ds \quad \int \frac{ds}{s^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cancel{a}\left[\frac{1}{\cancel{a}}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$= \tan^{-1}\left(\frac{\infty}{a}\right) - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = L\left[\frac{\sin at}{t}\right]$$

$$= \int_0^\infty e^{-st}\left(\frac{\sin at}{t}\right)dt$$

By comparing problems

put $s=0$ and $a=1$

$$\therefore \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}\left(\frac{0}{1}\right)$$

$$= \frac{\pi}{2}$$

Scanned by CamScanner

$= \frac{\pi}{2}$

Q. Find the L.T of $\frac{e^{-3t}\sin 2t}{t}$

Sol

$$\lim_{t\to 0} \frac{e^{-3t}\sin 2t}{t}$$

$$\lim_{t\to 0} \frac{2\cos 2t\,(e^{-3t}) + \sin 2t\,(-3)e^{-3t}}{1}$$

$$= \lim_{t\to 0} e^{-3t}(2\cos 2t - 3\sin 2t)$$

$$= 1(2) = f(t) \text{ exists}$$

$$L\left[\frac{e^{-3t}\sin 2t}{t}\right] = \int_s^\infty L[e^{-3t}\sin 2t]\,ds$$

$$= \int_s^\infty \left[\frac{2}{s^2+4}\right]_{s\to s+3} ds \Big|_{s\to s+3}$$

$$= \int_s^\infty \frac{2}{(s+3)^2+4}\,ds$$

$$= \int_s^\infty \frac{2}{s^2+9+6s+4}\,ds = \int_s^\infty \left(\frac{2}{s^2+6s+13}\right)ds$$

$$= 2\int_s^\infty \frac{1}{s^2+6s+13}\,ds.$$

$\tan^{-1}(s) + \cot^{-1}(s) = \pi/2$

$$= 2\int_s^\infty \frac{1}{(s^2+4)^2}(2s)\,ds \Big|_{s\to s+3}$$

$$= \left[2\,\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty \Big|_{s\to s+3}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{2}\Big|_{s\to s+3} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s+3}{2}\right)$$

$$= \cot^{-1}\left(\frac{s+3}{2}\right)$$

Scanned by CamScanner

$$L\left[\frac{\sin^2 2t}{t}\right]$$

$$\lim_{t\to 0} \frac{\sin^2 2t}{t}$$

$$\lim_{t\to 0} \frac{4\sin 2t \cos 2t}{1}$$

$$= 4(1)(0)$$

$$= 0 = f(t) \text{ exists.}$$

$$L\left[\frac{\sin^2 2t}{t}\right] = \int_s^\infty L\left[\frac{\sin^2 2t}{t}\right] ds$$

$$= \int_s^\infty L\left[\frac{1-\cos 4t}{2}\right] ds$$

$$= \frac{1}{2}\int_s^\infty (1-\cos 4t)\, ds$$

$$= \frac{1}{2}\left[\int_s^\infty ds - \int_s^\infty \cos 4t\, ds\right]$$

$$= \frac{1}{2}\left[\int_s^\infty \left[\frac{1}{s}\right] ds - \int_s^\infty \frac{s}{s^2+16} ds\right]$$

$$= \frac{1}{2}\left[\int_s^\infty \frac{1}{s} ds - \int_s^\infty \frac{s}{s^2+16} ds\right]$$

$$= \frac{1}{2}\left[\left[\log s\right]_s^\infty - \int_s^\infty \frac{2s}{2(s^2+16)}\right]$$

$$= \frac{1}{2}\left[\log s\right]_s^\infty - \frac{1}{4}\log\left[s^2+16\right]_0^\infty$$

$$= \frac{1}{2}\left[\log s\right]$$

Scanned by CamScanner

Q1 prove that $\int_0^\infty \left(\frac{1-\cos 2t}{t^2}\right) dt = \pi$

Soln

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$f(t) = \frac{1-\cos 2t}{t^2}$$

By comparison

$$L\left[\frac{1-\cos 2t}{t^2}\right]_{s=0}$$

$$= L\left[\frac{1-\cos 2t}{t}\right]_s = \int_s^\infty L[1-\cos 2t]\, ds$$

$$= \int_s^\infty (L[1] - L[\cos 2t])\, ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds = \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \int_s^\infty \log\left[\frac{s}{\sqrt{s^2+4}}\right]_s^\infty ds$$

$$= \int_s^\infty \left[\log \frac{s}{(\sqrt{s})\sqrt{1+\frac{4}{s^2}}}\right]_s^\infty ds$$

$$= \int_s^\infty \left(\log 1 - \log\left(\frac{s}{\sqrt{s^2+4}}\right)\right) ds$$

$$= \int_s^\infty \log\left(\frac{\sqrt{s^2+4}}{(\sqrt{s})^2}\right) ds.$$

$$= \int_s^\infty \log\sqrt{\frac{s^2+4}{s}}\, ds = \frac{1}{2}\int_s^\infty \log\left(\frac{s^2+4}{s}\right) ds$$

$$u = \log\left(1+\frac{4}{s^2}\right) \qquad du = \frac{1}{u+\frac{4}{s}}\left(0+4\times\frac{-2}{s^3}\right)$$

$$= \frac{s^2}{s^2+4} \times \frac{-8}{s^3}\, ds = \frac{-8}{s(s^2+4)}$$

$$dv = ds$$

Scanned by CamScanner

$\int u\,dv = uv - \int v\,du$

$\Rightarrow \left[s\log\left[1+\frac{4}{s^2}\right]\right]_s^\infty = \int_s^\infty s \times \frac{-8}{(s^2+4)s}\,ds$

$\Rightarrow 0 - s\log\left[1+\frac{4}{s^2}\right] + \left[8 \times \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= -s\log\left[1+\frac{4}{s^2}\right] + 4\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{2}\right)\right]$

$= -s\log\left[1+\frac{4}{s^2}\right] + \frac{4\pi}{2} - 4\tan^{-1}\left(\frac{s}{2}\right)$

put $s=0$

$\Rightarrow 0 + \frac{4\pi}{2} + 0$

$= 2\pi.$

$\int_0^\infty e^{-t}\, t\cos t\, dt = 0$

$\int_0^\infty e^{-t}\, t\sin t\, dt = \frac{1}{2}$

$L\left[\frac{1-\cos at}{t}\right]$

$L\left[\frac{\sin 3t\cos 2t}{t}\right]$

P.T $L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Scanned by CamScanner

(1) $\int_0^\infty e^{-t}\, t \cos t \, dt = 0$

By comparison $s = 1$

$f(t) = t \cos t$

$L[f(t)] = \frac{d}{ds} L[t \cos t]$

$= \frac{-d}{ds}\left[\frac{s}{s^2+1}\right]\Big|_{s=1}$

$= \frac{-[s^2+1-s(2s)]}{(s^2+1)^2}\Big|_{s=1}$

$= \frac{-s^2-1+2s^2}{(s^2+1)^2}\Big|_{s=1}$

$= \frac{s^2-1}{(s^2+1)^2}\Big|_{s=1} = \frac{1-1}{(1+1)^2}$

$= 0$

(2) $\int_0^\infty e^{-t}\, t \sin t \, dt = \frac{1}{2}$

By comparison $s = 1$

$f(t) = t \sin t$

$L[f(t)] = L[t \sin t]$

$= \frac{-d}{ds}\left[\frac{1}{(s^2+1)}\right]\Big|_{s=1}$

$= -\left[\frac{-1\,(2s)}{(s^2+1)^2}\right]_{s=1}$

$= \frac{2(1)}{(1^2+1)^2} = \frac{1}{4}$

$= \frac{2}{4} = \frac{1}{2}.$

(3) $L\left[\frac{1-\cos at}{t}\right]$

$\lim_{t\to 0}\left(\frac{1-\cos at}{t}\right) = \lim_{t\to 0} \frac{a \sin at}{1}$

$= 0.$

$= f(x)$ exists.

Scanned by CamScanner

$$\int_s^{\infty} L(f(t))\,dt$$

$$\int_s^{\infty} L[1-\cos at]\,dt$$

$$\int_s^{\infty} \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] ds$$

$$\left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^{\infty}$$

$$\left[\log\left[\frac{s}{\sqrt{s^2+a^2}}\right]\right]_s^{\infty}$$

$$\left[\log\left[\frac{s}{(\sqrt{s})^2\sqrt{1+\frac{a^2}{s^2}}}\right]\right]_s^{\infty}$$

$$\left[\log\left[\frac{1}{\sqrt{1+\frac{a^2}{s^2}}}\right]\right]_s^{\infty}$$

$$= \log\left(\frac{1}{1}\right) - \log\frac{1}{\sqrt{1+\frac{a^2}{s^2}}}$$

$$= 0 - \log\frac{s}{\sqrt{s^2+a^2}}$$

$$= \log\sqrt{\frac{s^2+a^2}{s}}$$

$\sqrt{s^2+a^2}$

$\sqrt{s^2\left(1+\frac{a^2}{s^2}\right)}$

$(\sqrt{s})^2\sqrt{1+\frac{a^2}{s}}$

Scanned by CamScanner

(5) $L\left[\dfrac{\cos at - \cos bt}{t}\right]$

$$\lim_{t\to 0} \frac{\cos at - \cos bt}{t} = \lim_{t\to 0} \frac{-a\sin bt + b\sin bt}{(1)}$$

$$= 0 = f(x) \text{ exists.}$$

$$\int_s^\infty L(\cos at - \cos bt)\, ds = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$$

$$= \left[\frac{s^2+a^2-(2s)s}{(s^2+a^2)^2} - \frac{s^2+b^2-2s(s)}{(s^2+b^2)^2}\right]_s^\infty$$

$$= \left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2} - \frac{s^2+b^2-2s^2}{(s^2+b^2)^2}\right]_s^\infty$$

$$= \left[\frac{-s^2+a^2}{(s^2+a^2)^2} - \frac{(-s^2+b^2)}{(s^2+b^2)^2}\right]_s^\infty$$

$$= \left[\frac{-(s^2-a^2)}{s^2}\right]$$

$$= \frac{1}{2}\left[\int_s^\infty \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[(-\log s^2+a^2) - (-\log(s^2+b^2))\right]$$

$$= \frac{1}{2}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$$

$$= \frac{1}{2}\log\left[\frac{s^2+b^2}{s^2+a^2}\right]$$

Scanned by CamScanner

b) $L\left[\frac{\sin 3t \cos 2t}{t}\right]$

$\lim_{t\to 0} \frac{\sin 3t \cos 2t}{t} = \lim_{t\to 0} \frac{3\cos 3t(-2\sin 2t)}{(1)}$

$= \lim_{t\to 0} 0 = f(x)$ constant

$L\left[\frac{\sin 3t \cos 2t}{t}\right] = \int_s^\infty (\sin 3t \cos 2t)\, ds$

$\sin A \cos B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$

$\sin 3t \cos 2t = \frac{1}{2}(\sin(5t) - \sin(t))$

$L[t] = \int_s^\infty L\left[\frac{1}{2}[\sin 5t - \sin t]\right] ds$

$= \frac{1}{2}\int_s^\infty \left(\frac{5}{s^2+25} - \frac{1}{s^2+1}\right) ds$

$= \frac{1}{2}\int_s^\infty \left[5\left(\frac{1}{s^2+25} - \frac{1}{s^2+1}\right)\right] ds$

$= \frac{1}{2}\left[5(\log(s^2+25) - \log(s^2+1))\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{(s^2+25)^5}{(s^2+1)}\right)\right]_s^\infty$

$= \frac{1}{2}\left[5\,\frac{1}{5}\tan^{-1}\left(\frac{s}{5}\right)\right]_s^\infty + \left(\frac{1}{1}\tan^{-1}\frac{s}{1}\right)_s^\infty$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{5}\right) + \tan^{-1}\left(\frac{1}{1}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{\infty}{5}\right) + \tan^{-1}\left(\frac{\infty}{1}\right)\right] - \tan^{-1}\left(\frac{s}{5}\right) + \tan^{-1}(s)$

Scanned by CamScanner

put $s=0$

$$\frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\pi}{2}\right)-\frac{\pi}{4}+0\right]$$

$$= \frac{\pi}{2}$$

Solve. $\int_0^\infty \frac{e^{-\sqrt{2t}}\sin\sqrt{2t}}{t}\,dt$

compare with $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

Put $u=\sqrt{2t}$

$du = \sqrt{2}\frac{1}{2\sqrt{t}}dt$

$du = \frac{1}{\sqrt{2}\sqrt{t}}dt$

$du = \frac{1}{u}dt$

t	0	∞
u	0	∞

$u^2 = 2t$

$t = \frac{u^2}{2}$

$$\int_0^\infty \frac{e^{-\sqrt{2t}}\sin\sqrt{2t}}{t}dt = \int_0^\infty \frac{e^{-u}\sin u}{(u^2/2)}\,u\,du$$

$$= \int_0^\infty \frac{2e^{-u}\sin u}{u}\,du$$

$$\Rightarrow 2\int_0^\infty \frac{e^{-t}\sin t}{t}dt = 2L\left[\frac{e^{-t}\sin t}{t}\right]$$

$$\Rightarrow 2\int_s^\infty L[\sin t]\,ds\Big|_{s=1}$$

$$= 2\int_s^\infty \frac{1}{s^2+1}\,ds\Big|_{s=1}$$

$$= 2\left[\frac{1}{1}\tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty\Big|_{s=1}$$

$$= 2\left[\tan^{-1}(\infty)-\tan^{-1}(s)\right]\Big|_{s=1}$$

$$= 2\left[\frac{\pi}{2}-\tan^{-1}(1)\right]$$

$$= 2\left[\frac{\pi}{2}-\frac{\pi}{4}\right] = 2\left[\frac{2\pi}{4}\right] = \frac{\pi}{2}$$

Scanned by CamScanner

1) If $L[f(t)] = F(s)$

$$L[tf(t)] = \frac{-d}{ds} L[f(t)] = \frac{-d}{ds} F(s) \text{ (or)}$$

$$= -F'(s)$$

$$L[f(t)] = \int_0^\infty f(t) e^{-st} dt = F(s)$$

$$F(s) = L[f(t)]$$

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left(L[f(t)]\right)$$

$$\Rightarrow F'(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t) dt\right]$$

Leibnitz's rule

$$= \int_0^\infty \frac{\partial}{\partial s}[e^{-st} f(t)] dt.$$

$$F'(s) = \int_0^\infty f(t)\, e^{-st}(-t)\, dt.$$

$$= -\int_0^\infty f(t) . t\, e^{-st} dt.$$

$$= -L[t f(t)]$$

$$L[t f(t)] = -F'(s)$$

2) $$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[L[f(t)]]$$

$$= (-1)^n \frac{d^n}{ds^n} F(s)$$

Let $n = 2$

$$L[t^2 f(t)] = L[t\, t f(t)] = \frac{-d}{ds} L[t f(t)]$$

$$= \frac{-d}{ds} \frac{-d}{ds} \times L[f(t)]$$

$$L[t^2 f(t)] = \frac{d^2}{ds^2} L[f(t)]$$

Scanned by CamScanner

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} L[f(t)]$$

3) If $L[f(t)] = F(s)$ and $\lim_{t \to 0} \frac{f(t)}{t}$ exists. then

$$L[f(t)] = \int_s^\infty \underbrace{L[f(t)]}_{F(s)} ds.$$

$$F(s) = L[f(t)]$$

$$\int_s^\infty L[f(t)]\, ds$$

$$\int_s^\infty \int_0^\infty e^{-st} f(t)\, \underbrace{dt\, ds}_{(\text{'t' \& 's' independent})}.$$

$$= \int_0^\infty \left(\int_s^\infty e^{-st} f(t)\, ds \right) dt$$

$$= \int_0^\infty f(t) \left(\frac{e^{-st}}{t} \right)_s^\infty dt$$

$$= \int_0^\infty \frac{f(t)}{t} \left(e^{-\infty} - e^{-st} \right) dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

Change of scale property

1) Statement:- If $a > 0$ and $L[f(t)] = F(s)$ then

$$L[f(at)] = \frac{1}{a} F\left[\frac{s}{a}\right]$$

Proof:- $L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$

t	0	∞
u	0	∞

put $at = u$

$du = a\,dt$

$$= \int e^{-su/a} f(u)\, du$$

Scanned by CamScanner

$$= \frac{1}{a}\int_0^\infty e^{-\frac{su}{a}} f(u)\,du.$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{st}{a}} f(t)\,dt$$

$$\because \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

2) $L[f'(t)] = s.L[f(t)] - f(0).$

Derivation

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\, d f(t)$$

$$u \qquad dv = d[f(t)] \Rightarrow v = f(t)$$

$u = e^{-st}$

$du = -s e^{-st} dt$ $\quad \int u\,dv = uv - \int v\,du$

$$= \left[e^{-st} f(t)\right]_{t=0}^{\infty} - \int_0^\infty f(t) - s e^{-st}\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$L[f'(t)] = -f(0) + s L[f(t)]$$
$$= sL[f(t)] - f(0).$$

3) $$L[f''(t)] = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \int_0^\infty e^{-st}\, d[f'(t)]\,dt$$

$$u \qquad dv$$

$du = -s e^{-st} dt$

$$= \left[e^{-st} f'(t)\right]_{t=0}^{\infty} - \int_0^\infty f'(t)(-s e^{-st})\,dt$$

Scanned by CamScanner

$$= \left[0 - f'(0)\right] + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) - s\,L[f'(t)]$$

$$= -f'(0) + s\left[s \cdot L[f(t)] - f(0)\right]$$

$$= -f'(0) + s^2 L[f(t)] - sf(0).$$

$$= s^2 L[f(t)] - f'0 - sf(0).$$

4) $L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} \cdot L[f(t)].$

$F(t) = \int_0^t f(t)\,dt$; $F(0) = 0$.

$F'(t) = f(t)$

$\Rightarrow L[F'(t)] = s\,L[F(t)] - F(0)$

$= s\,L[F(t)] - 0$

$\Rightarrow \frac{1}{s} L[F'(t)] = L[F(t)]$

$\frac{1}{s}[f(t)] = L\left[\int_0^t f(t)\,dt\right]$

<u>Initial value theorem and final value theorem</u>

<u>IVT:-</u> $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,L[f(t)]$

<u>FVT:-</u> $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,L[f(t)]$

Scanned by CamScanner

Verify IVT for $f(t) = e^{-t}[\cos t + \sin t] + 1$.

$$f(t) = 1 + e^{-t}[\cos t + \sin t]$$

L.H.S :- $\lim_{t \to 0} f(t) = \lim_{t \to 0} 1 + e^{-t}(\cos t + \sin t)$

$$= 1 + e^{-0}(\cos 0 + \sin 0$$

$$= 1 + 1 = 2$$

R.H.S :- $L[f(t)] = L[1 + e^{-t}(\cos t + \sin t)]$

$$= L(1) + L[e^{-t}(\cos t + \sin t)]$$

$$= \frac{1}{s} + L[\cos t + \sin t]\Big|_{s \to s+1}$$

$$= \frac{1}{s} + \left[\frac{s}{s^2+1} + \frac{1}{s^2+1}\right]_{s \to s+1}$$

$$= \frac{1}{s} + \left[\frac{s+1}{(s^2+1)}\right]_{s \to s+1}$$

$$= \frac{1}{s} + \left[\frac{s+1+1}{(s+1)^2+1}\right]$$

$$= \frac{1}{s} + \frac{s+2}{s^2+1+2s+1}$$

$$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$$

$$= \frac{s^2+2s+2+s+2}{s(s^2+2s+2)}$$

$$= \frac{s^2+3s+4}{s(s^2+2s+2)}$$

Scanned by CamScanner

$$\lim_{s\to\infty} s\,L[f(t)] = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s(s+2)}{s^2+2s+2}\right]$$

$$= 1 + \lim_{s\to\infty} \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}$$

$$= 1 + \lim_{s\to\infty}(1)$$

$$= 1+1 = 2$$

LHS = RHS

H.W

(1) $f(t) = e^{-t}(t+2)^2$

(2) $f(t) = 5e^{-3t}$

(3) $f(t) = 1 - e^{-at}$

Sol: 1) IVT $f(t) = e^{-t}(t+2)^2$

L.H.S $\lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-t}(t+2)^2$

$= 4$

R.H.S

$L[f(t)] = L[e^{-t}(t+2)^2]$

$= L[(t+2)^2]_{s\to s+1}$

$= L[t^2+4+4t]\Big|_{s\to s+1}$

$= \left[\frac{2!}{s^3} + \frac{4}{s} + 4\frac{1}{s^2}\right]\Big|_{s\to s+1}$

$= \frac{2!}{(s+1)^3} + \frac{4}{s+1} + \frac{4}{(s+1)^2}$

$= \frac{2}{(s+1)^3} + \frac{4}{s+1} + \frac{4}{(s+1)^2}$

Scanned by CamScanner

$$= \frac{2+4(s+1)^2+4(s+1)}{(s+1)^3}$$

$$= \frac{2+4(s^2+1+2s)+4s+4}{(s+1)^3}$$

$$= \frac{2+4s^2+4+8s+4s+4}{(s+1)^3}$$

$$= \frac{4s^2+12s+10}{(s+1)^3}$$

$$\underset{s\to\infty}{Lt}\, L[f(t)] = \underset{s\to\infty}{Lt}\, \frac{4s^2+12s+10}{(s+1)^3}$$

$$= \underset{s\to\infty}{Lt}\, \frac{2}{(s+3)^3} + \frac{4(s+1)^2}{(s+1)^3} + \frac{4(s+1)}{(s+1)^3}$$

$$= \frac{4s^2+12s+10}{s^3+3s^2+3s+1}$$

$$s\cdot L[f(t)] = \frac{s^3\left[4+\frac{12}{s}+\frac{10}{s^2}\right]}{s^3\left(1+\frac{3}{s}+\frac{3}{s^2}+\frac{1}{s^3}\right)}$$

$$\underset{s\to\infty}{Lt}\, s\cdot L[f(t)] = \underset{s\to\infty}{Lt}\, \frac{4+\frac{12}{s}+\frac{10}{s^2}}{1+\frac{3}{s}+\frac{3}{s^2}+\frac{1}{s^3}}$$

$$= \frac{4}{1}$$

$$= 4$$

L.H.S = R.H.S

FVT:-

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, s\cdot L[f(t)]$$

$$f(t) = e^t(t+2)^2$$

Scanned by CamScanner

$$LHS = \lim_{t\to\infty} e^{-t}(t+2)^2$$

$$= 0$$

$$R.H.S = \lim_{s\to 0} sL[f(t)]$$

$$s\cdot L[f(t)] = \frac{4s^3+12s^2+10s}{s^3+3s^2+3s+1}$$

$$\lim_{s\to 0} sL[f(t)] = \frac{4(0)+12(0)+10(0)}{0^3+3(0)^2+3(0)+1}$$

$$= \frac{0}{1} = 0$$

Q. $f(t) = 5e^{-3t}$

IVT:- $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\cdot L[f(t)]$

LHS:- $\lim_{t\to 0} 5\cdot e^{-3t} = 5\cdot e^{-3(0)}$

$$= 5e^0$$

$$= 5$$

R.H.S $sL[f(t)] = 5sL[e^{-3t}]$

$$= 5s\frac{1}{s+3}$$

$$\lim_{s\to\infty} sL[f(t)] = \lim_{s\to\infty} \frac{5s}{s+3} = \lim_{s\to\infty} \frac{s[5]}{s\left[1+\frac{7}{5}\right]}$$

$$= 5$$

FVT:-

$$\lim_{t\to\infty} = \lim_{s\to 0} s\cdot L[f(t)]$$

LHS:- $\lim_{t\to\infty} f(t) = 5\lim_{t\to\infty} e^{-3t} = 5(0) = 0$

RHS:- $\lim_{s\to 0} s\cdot L[f(t)] = \lim_{s\to 0} \frac{5s}{s+3} = \frac{5(0)}{(0+3)} = 0$

Scanned by CamScanner

$f(t) = 1 - e^{-at}$

IVT:-

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s.L[f(t)]$$

LHS $\lim_{t \to 0} f(t) = \lim_{t \to 0} 1 - e^{-at}$

$= 1 - 1 = 0$

RHS $\lim_{s \to \infty} s.L[f(t)] = \lim_{s \to \infty} s\left[\frac{1}{s} - \frac{1}{s+a}\right]$

$= \lim_{s \to \infty} \left[1 - \frac{s}{s+a}\right]$

$\lim_{s \to \infty} 1 - \frac{s}{s+a} = \lim_{s \to \infty} 1 - \frac{s(1)}{s(1+\frac{a}{s})} = 1 - 1 = 0$

FVT:-

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s.L[f(t)]$$

LHS :- $\lim_{t \to \infty} = \lim_{t \to \infty} (1 - e^{-at}) = 1 - \lim_{t \to \infty} e^{-at} = 1 - 0 = 1$

RHS:- $\lim_{s \to 0} \left(1 - \frac{s}{s+a}\right) = 1 - \lim_{s \to 0} \frac{s}{s+a} = 1 - 0 = 1$

LHS = RHS

Periodic functions:-

A function $f(t)$ is said to have a period 'p'. If $f(t+p) = f(t)$ in some interval where p is the smallest positive integer

Scanned by CamScanner

Eg:- $f(t) = \sin t$

$f(t+\pi) = \sin(t+\pi)$

$= \sin t \cos\pi + \cos t \sin\pi$

$= \sin t$

$= f(t)$

$p = 2\pi, 4\pi, 6\pi \ldots$

$f(t) = \tan t$, $f(t) = \cos t$

$p = \pi, 2\pi, 3\pi \ldots$, $p = 2\pi, 4\pi, 6\pi \ldots$

Laplace transform ~~form~~ of periodic function.

Prove $\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$

$$\left\{ L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \right.$$

split

$$= \int_0^p e^{-st} f(t)\,dt + \int_p^\infty e^{-st} f(t)\,dt.$$

put $t = u+p$

$dt = du$

$u = t-p$

t	p	∞
$u = t-p$	0	∞

$$= \int_0^p e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+p)} f(u+p)\,du$$

$e^{-su} \quad e^{-ps} \quad f(u)$

$$L[f(t)] = \int_0^p e^{-st} f(t)\,dt + e^{-ps} \int_0^\infty e^{-su} f(u)\,du$$

u is dummy

$$= \int_0^p e^{-st} f(t)\,dt + e^{-ps} \left\{ L[f(t)] \right\}$$

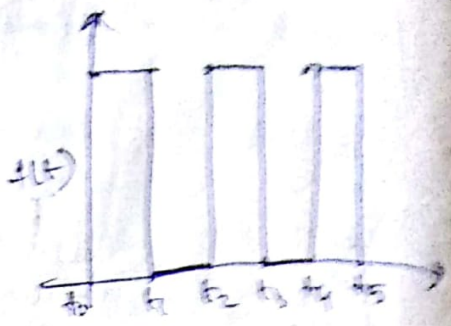
$$L[f(t)] - e^{-ps} \left\{ L[f(t)] \right\} = \int_0^p e^{-st} f(t)\,dt$$

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt.$$

Scanned by CamScanner

Q:- Find the L.T of the rectangular wave function

$$f(t) = \begin{cases} +k, & 0<t<b \\ -k, & b<t<2b \end{cases}$$



Sol:-

$p = 2b$

$$\Rightarrow \frac{1}{(1-e^{-2bs})} \int_0^{2b} e^{-st} f(t)\, dt$$

$$\Rightarrow \frac{1}{1-e^{-2bs}} \left[\int_0^{b} e^{-st} f(t)\, dt + \int_b^{2b} e^{-st} f(t)\, dt \right]$$

$$\Rightarrow \frac{1}{1-e^{-2bs}} \left[\int_0^{b} e^{-st} k\, dt + \int_b^{2b} e^{-st} (-k)\, dt \right]$$

$$\Rightarrow \frac{k}{1-e^{-2bs}} \left[\int_0^{b} e^{-st}\, dt - \int_b^{2b} e^{-st}\, dt \right]$$

$$\Rightarrow \frac{k}{1-e^{-2bs}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^{b} - \left[\frac{e^{-st}}{-s} \right]_b^{2b} \right\}$$

$$\Rightarrow \frac{k}{s(1-e^{-2bs})} \left\{ -\left(e^{-bt} - (-1)\right) - \left[\left(-e^{-2bs}\right) - \left(-e^{-bs}\right) \right] \right\}$$

$$\Rightarrow \frac{k}{s(1-e^{-2bs})} \left[-e^{-bs} + 1 + e^{-2bs} - e^{-bs} \right]$$

$$\Rightarrow \frac{k}{s(1-e^{-2bs})} \left[e^{-2bs} - 2e^{-bs} + 1 \right]$$

$$\Rightarrow \frac{k}{s(1-e^{-2bs})} \left[1 - e^{-bs} \right]^2$$

Scanned by CamScanner

$$\Rightarrow \frac{k}{s(1-e^{-bs})(1+e^{-bs})}\cdot(1-e^{-bs})$$

$$\Rightarrow \frac{k}{s}\left[\frac{1-e^{-bs}}{1+e^{-bs}}\right]$$

$$\tanh\theta = \frac{1-e^{-2\theta}}{1+e^{-2\theta}}$$

$$2\theta = bs$$

$$\theta = \frac{bs}{2}$$

$$\tanh\left(\frac{bs}{2}\right)$$

$$\Rightarrow \frac{k}{s}\left[\cancel{1} - \tanh\left(\frac{bs}{2}\right)\right]$$

$$\Rightarrow \frac{k}{s}\tanh\left(\frac{bs}{2}\right).$$

H.W

$$f(t) = \begin{cases} +E, & 0<t<a/2 \\ -E, & \frac{a}{2}<t<a. \end{cases}$$

Q: L.T of Half wave sine rectifier.

$$f(t) = \begin{cases} \sin\omega t, & 0<t<\frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega}<t<\frac{2\pi}{\omega} \end{cases}$$

Sol: $p = \frac{2\pi}{\omega}$

$$\Rightarrow \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\,\overset{0}{\cancel{\sin\omega t}}\,dt\right].$$

$$\Rightarrow \cancel{\frac{1}{1-e^{-2\pi s/\omega}}} \quad \int e^{at}\sin bt = \frac{e^{at}}{a^2+b^2}\left[a\sin bt - b\cos bt\right]$$

Here $a = -s$, $b = \omega$

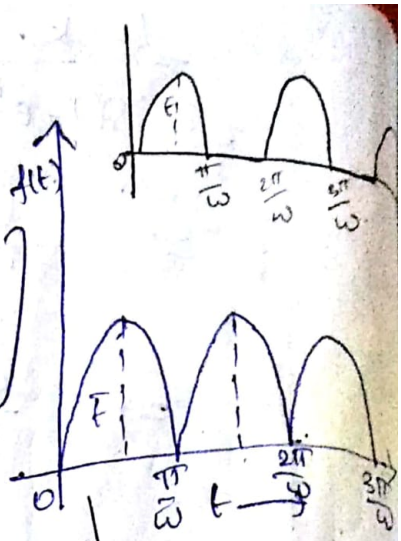
$$\Rightarrow \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{(s^2+\omega^2)}\left[-s\sin\omega t - \omega\cos\omega t\right]_0^{\pi/\omega}\right]_0^{\pi/\omega}$$

$$\Rightarrow \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\left[-s\sin\left(\omega\times\frac{\pi}{\omega}\right) - \omega\cos\left(\omega\times\frac{\pi}{\omega}\right)\right]\right.$$

$$\left. - \left\{\frac{1}{s^2+\omega^2}\left[-s\sin 0 - \omega\cos 0\right]\right\}\right\}$$

Scanned by CamScanner

$$\Rightarrow \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{\omega e^{-s\pi/\omega}+\omega}{(s^2+\omega^2)}\right]$$



$$\Rightarrow \frac{\omega}{1-e^{-2\pi s/\omega}}\left[\frac{1+e^{-s\pi/\omega}}{s^2+\omega^2}\right]$$

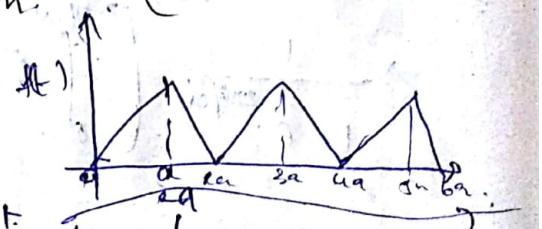
$$\Rightarrow \frac{\omega}{s^2+\omega^2}\left(\frac{1}{1-e^{-\pi s/\omega}}\right)$$

Hw.

① $f(t) = \begin{cases} \sin 10t, & 0<t<\frac{\pi}{10} \\ 0, & \frac{\pi}{10}<t<\frac{\pi}{5} \end{cases}$

② Full wave sine rectifier, $f(t) = |\sin t|$, $P=\pi$.

Q: Find the L.T of $f(t) = \begin{cases} t & 0<t<a \\ 2a-t, & a<t<2a \end{cases}$ triangular wave form.



Sol:

$P = 2a$

$$\Rightarrow \frac{1}{1-e^{-2as}}\left\{\underbrace{\int_0^a e^{-st}\,t\,dt}_{I_1} + \underbrace{\int_a^{2a} e^{-st}(2a-t)\,dt}_{I_2}\right\} \quad -①$$

$\int u\,v\,du = u_1v_1 - u'v_2 + \cdots$

$I_1 \Rightarrow u = t \quad v = e^{-st}$

$u' = 1 \quad v_1 = \frac{e^{-st}}{-s}$

$v_2 = \frac{e^{-st}}{s^2}$

Scanned by CamScanner

$$\therefore I_1 = \left[\frac{t\,e^{-st}}{-s}\right]_0^a - \left[(1)\frac{e^{-st}}{s^2}\right]_0^a$$

$$= \left[\frac{a e^{-sa}}{-s} - 0\right] - \left[\frac{e^{-as}}{s^2} - \frac{1}{s^2}\right]$$

$$= \frac{a e^{-sa}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}$$

$I_2 = u = 2a - t \qquad v = e^{-st}$

$u' = -1 \qquad v_1 = \frac{e^{-st}}{-s}$

$v_2 = \frac{e^{-st}}{s^2}$

$$I_2 = \left[\frac{(2a-t)e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right]_a^{2a}$$

$$= \left[\left(0 + \frac{e^{-2as}}{s^2}\right) - \left(\frac{a e^{-as}}{-s} + \frac{e^{-as}}{s^2}\right)\right]$$

$$I_2 = \frac{e^{-2as}}{s^2} + \frac{a e^{-as}}{s} - \frac{e^{-as}}{s^2}$$

Sub I_1 and I_2 in ①

$$\Rightarrow \frac{1}{1-e^{-2as}}\left[\frac{-2a e^{-sa}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2}\right]$$

$$\Rightarrow \frac{1}{s^2(1-e^{-as})^2}\left[-2a e^{-sa} + 1 + (e^{-as})^2\right]$$

$2\theta = as$

$\theta = \frac{as}{2}$

$$\Rightarrow \frac{1}{s^2(1+e^{-as})}(1-e^{-as}) = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

Scanned by CamScanner

Hw

$$f(t) = \begin{cases} t, & 0<t<1 \\ (2-t), & 1<t<2 \end{cases}$$

Sol:

$p = 2$

$$\Rightarrow \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st}\, t\, dt + \int_1^2 e^{-st}(2-t)\, dt \right\}$$

$$\int u\, dv = uv_1 + u'v_2 + \cdots$$

$I_1 =$ $u = t$ $\quad v = e^{-st}$

$u' = 1$ $\quad v_1 = \frac{e^{-st}}{-s}$

$v_2 = \frac{e^{-st}}{s^2}$

$$\therefore I_1 = \left[\frac{t e^{-st}}{-s}\right]_0^1 - \left[\frac{e^{-st}}{s^2}\right]_0^1$$

$$= \left[\frac{e^{-s}}{-s} - 0\right] - \left[\frac{e^{-s}}{s^2} - \frac{1}{s^2}\right]$$

$$I_1 = \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

I_2 $u = 2-t$ $\quad v = e^{-st}$

$u' = -1$ $\quad v_1 = \frac{e^{-st}}{-s}$

$v_2 = \frac{e^{-st}}{s^2}$

Scanned by CamScanner

$$I_1 = \left[\frac{(2-1-)\,e^{-s(1-)}}{-s}\right]_1^2 + \left[\frac{e^{-s(1-)}}{s^2}\right]_1^2$$

$$= \left[0 - \frac{e^{-s}}{s}\right] + \left[\frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}\right]$$

$$= \cancel{-\frac{2e^{-s}}{s}} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

$$\Rightarrow \frac{1}{1-e^{-2s}}\left\{\cancel{\frac{e^{-s}}{s}} + \frac{e^{-s}}{s^2} - \cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}+1}{s^2}\right\}$$

$$\Rightarrow \frac{1}{1-e^{-2s}}\left\{-\frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\right\} + \frac{1}{s^2}\Big\}$$

$$\Rightarrow \frac{1}{1-e^{-2s}}\left\{\cancel{\frac{-2e^{-s}+e^{-2s}}{s^2}}\right\} + \frac{1}{s^2}\Big\}$$

$$\Rightarrow \frac{1}{1-e^{-2s}}\left\{\cancel{\frac{e^{-s}(-2+e^{-s})}{s^2}}\right\}$$

$$\Rightarrow \frac{1}{1-e^{-2s}}\left\{\frac{-2e^{-s}+e^{-2s}+1}{s^2}\right\}$$

$$\Rightarrow \frac{1}{1-e^{-2s}}\left\{\frac{(1+e^{-s})^2}{s^2}\right\}$$

$2\theta = s$

$\theta = \frac{s}{2}$

$$\Rightarrow \frac{1}{(1-e^{-s})}\frac{(1+e^{-s})}{s^2} = \frac{1}{s}\,\cot h\left(\frac{s}{2}\right)$$

Scanned by CamScanner

① $f(t) = \begin{cases} \sin 10t, & 0 < t < \frac{\pi}{10} \\ 0, & \frac{\pi}{10} < t < \frac{\pi}{5} \end{cases}$

Sol:- $\frac{1}{\cancel{c}}$ $p = \frac{\pi}{5}$

$$\frac{1}{1-e^{-\pi s/5}}\left[\int_0^{\pi/10} e^{-st}\sin 10t\,dt + \int_{\pi/10}^{\pi/5} e^{-st}\,0\,dt\right]$$

$$\frac{1}{1-e^{-\pi s/5}}\left[\int_0^{\pi/10} e^{-st}\sin 10t\,dt\right]$$

$$\frac{1}{1-e^{-\pi s/5}}\left[\int e^{at}\sin bt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)\right]$$

$$\int e^{-st}\sin 10t = \frac{e^{-st}}{s^2+100}(-s\sin 10t - 10\cos 10t)$$

$$\frac{1}{1-e^{-\pi s/5}}\left[\frac{e^{-st}}{s^2+100}(-s\sin 10t - 10\cos 10t)\right]_0^{\pi/10}$$

$$\frac{1}{1-e^{-\pi s/5}}\left\{\left[\frac{e^{-\pi s/10}}{s^2+100}\left[-s\sin 10\frac{\pi}{10} - 10\cos 10\frac{\pi}{10}\right]\right] - \left[\frac{1}{s^2+100}(-\sin 0 - 10\cos 0)\right]\right\}$$

$$\frac{1}{1-e^{-\pi s/5}}\left\{\frac{e^{-\pi s/10}}{s^2+100}(+10) - \left[\frac{1}{s^2+100}(-10)\right]\right\}$$

Scanned by CamScanner

$$\frac{1}{1-e^{-\pi s/5}}\left[\frac{10e^{-\pi s/10}}{s^2+100}+\frac{10}{s^2+100}\right]$$

$$\frac{10}{1-e^{-\pi s/5}}\left[\frac{e^{-\pi s/10}+1}{s^2+100}\right]$$

$$\frac{10}{(1-e^{-\pi s/10})(s^2+100)}$$

(2) $f(t) = \begin{cases} E, & 0<t<a/2 \\ -E, & \frac{a}{2}<t<a. \end{cases}$

Solⁿ $p=a$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}E\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\int_0^{a/2} e^{-st}dt\right] - \left[\int_{a/2}^{a} e^{-st}dt\right]\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-sa/2}}{-s}+\frac{1}{s}\right]-\left[\frac{e^{-as}}{-s}+\frac{e^{-sa/2}}{-s}\right]\right\}$$

$$= \frac{E}{1-e^{-as}}\left[\frac{e^{-sa/2}}{-s}+\frac{1}{s}+\frac{e^{-as}}{s}-\frac{e^{-sa/2}}{s}\right]$$

$$= \frac{E}{s(1-e^{-as})}\left[-\frac{2e^{-sa/2}}{s}+\frac{1}{s}+e^{-as}\right]$$

Scanned by CamScanner

$$= \frac{E}{s(1-e^{-as/2})}\left[1-e^{-as/2}\right]$$

$$= \frac{E}{s}\left[\frac{1-e^{-as/2}}{1+e^{-as/2}}\right]$$

$$= \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

27) $f(t) = |\sin t|$ $\quad p = \pi$

$$f(t) = \begin{cases} +\sin t & 0 < t < \frac{\pi}{2} \\ \sin t & \pi/2 < 0 < \pi. \end{cases}$$

$p = \pi$

$$\frac{1}{1-e^{-\pi s}}\left[\int_0^{\pi/2} e^{-st}\sin t\, dt + \int_{\pi/2}^{\pi} e^{-st}\sin t\, dt\right]$$

$$\int e^{at}\sin bt = \frac{e^{at}}{a^2+b^2}\left[a\sin bt - b\cos bt\right]$$

$$\int_0^{\pi/2} e^{-st}\sin t = \frac{e^{-st}}{s^2+1}\left[-s\sin t - \cos t\right]$$

$$\int_{\pi/2}^{\pi} e^{-st}\sin t =$$

$$\frac{1}{1-e^{-\pi s}}\left\{\left[\frac{e^{-st}}{s^2+1}\left[-s\sin t - \cos t\right]\right]_0^{\pi/2}\right.$$

$$+\left[\frac{e^{-st}}{s^2+1}\left[-s\sin t - \cos t\right]\right]_{\pi/2}^{\pi}$$

Scanned by CamScanner

$$\frac{1}{1-e^{-\pi s}}\left\{\left(\frac{e^{-s\pi/2}}{s^2+1}\left[-s\sin\pi/2-\cos\pi/2\right]-\frac{1}{s^2+1}\left[-s\sin 0-\cos 0\right]\right)+\left(\frac{e^{-s\pi/2}}{s^2+1}\left[-s\sin\pi-\cos\pi\right]-\frac{e^{-s\pi/2}}{s^2+1}\left[-s\sin\pi/2-\cos\pi/2\right]\right)\right\}$$

$$\frac{1}{1-e^{-\pi s}}\left\{\frac{e^{-\pi s/2}}{s^2+1}(-s)+\frac{1}{s^2+1}+\frac{s\,e^{-\pi s/2}}{s^2+1}\right\}$$

$$\frac{1}{1-e^{-\pi s}}\left\{\frac{-s\,e^{-\pi s/2}}{s^2+1}+\frac{s\,e^{-\pi s/2}}{s^2+1}+\frac{1}{s^2+1}\right\}$$

$$\frac{1}{(1-e^{-\pi s})(s^2+1)}$$

Inverse laplace transform

If $L[f(t)] = F(s)$ then

$$f(t) = L^{-1}[F(s)]$$

where L^{-1} is the inverse laplace operator.

Linearity property

$$L^{-1}[c_1F_1(s)+c_2F_2(s)]$$
$$= c_1[L^{-1}[F_1(s)]+c_2\cdot L^{-1}[F_2(s)]$$

Proof:-

$$L[f_1(t)] = F_1(s) \Rightarrow f_1(t) = L^{-1}[F_1(s)]$$
$$L[f_2(t)] = F_2(s) \Rightarrow f_2(t) = L^{-1}[F_2(s)]$$

w.k.T of LT Linear Prop

$$L[c_1f_1(t)+c_2f_2(t)]$$
$$= c_1L[f_1(t)]+c_2L[f_2(t)]$$
$$= c_1F_1(s)+c_2F_2(s)$$

Scanned by CamScanner

$\Rightarrow c_1 f_1(t) + c_2 f_2(t) = L^{-1}[c_1 F_1(s) + c_2 F_2(s)]$

$\Rightarrow c_1 L^{-1}[F_1(s)] + c_2 L^{-1}[F_2(s)] = L^{-1}[c_1 F_1(s) + c_2 F_2(s)]$

* $L^{-1}\left[\frac{1}{s}\right] = 1$

* $L^{-1}\left[\frac{k}{s}\right] = k$

$L[t] = \frac{1!}{s^{1+1}}$

* $L[t^n] = \frac{n!}{s^{n+1}}$

$\frac{t^1}{1!} = L^{-1}\left[\frac{1}{s^2}\right]$

$L[t^2] = \frac{2!}{s^3} = \frac{2!}{s^3}$

$\frac{t^2}{2!} = L^{-1}\left[\frac{1}{s^3}\right]$

$$\boxed{\therefore L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}}$$

* $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

* $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

* $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

* $L^{-1}\left[\frac{a \times a}{as^2+a^2}\right] = \frac{1}{a}\sin at$

* $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

* $L^{-1}\left[\frac{a \times a}{as^2-a^2}\right] = \frac{1}{a}\sinh at$

Scanned by CamScanner

Shifting property

* $L^{-1}[F(s-a)] = e^{at} F(s)_{s-a \to s}$

* $L^{-1}[F(s+a)] = e^{-at} F(s)_{s+a \to s}$

* Find L^{-1} of $\dfrac{s+2}{s^2+4s+8}$.

Sol:- $L^{-1}\left[\dfrac{s+2}{s^2+4s+8}\right]$

$A^2 = s^2 \Rightarrow A = s$

$2AB = 4s$

$\Rightarrow 2sB = 4s$

$B = 2$

Denominator $= s^2+4s+8+4-4$

$= s^2+4s+4+8-4$

$= (s+2)^2+4$

$L^{-1}\left[\dfrac{s+2}{(s+2)^2+4}\right]_{s+2 \to s}$

$\Rightarrow e^{-2t} L^{-1}\left[\dfrac{s}{s^2+4}\right]$

$\Rightarrow e^{-2t} \cos 2t$.

* $L^{-1}\left[\dfrac{s}{(s+2)^2}\right]$

Sol:- $L^{-1}\left[\dfrac{s+2-2}{(s+2)^2}\right]$

$L^{-1}\left[\dfrac{s+2}{(s+2)^2} - \dfrac{2}{(s+2)^2}\right]$

$L^{-1}\left[\dfrac{1}{s+2}\right] - 2L^{-1}\left[\dfrac{1}{(s+2)^2}\right]$

$e^{-2t} - 2\cancel{\tfrac{1}{2}}\cancel{\sin at}\, e^{-2t} L^{-1}\left[\dfrac{1}{s^2}\right]$

Scanned by CamScanner

$= e^{-2t} - 2e^{-2t}\, t$

$= e^{-2t} - 2t e^{-2t}$

$= e^{-2t}(1-2t)$

Method of partial fractions

$\deg(Nr) < \deg(Dr)$

* $\dfrac{F(s)}{(s-a)(s-b)(s-c)} = \dfrac{A}{s-a} + \dfrac{B}{s-b} + \dfrac{C}{s-c}$

* $\dfrac{F(s)}{(s-a)^2(s-b)} = \dfrac{A}{s-a} + \dfrac{B}{(s-a)^2} + \dfrac{C}{s-b}$

* $\dfrac{F(s)}{(s^2+a^2)(s+b)} = \dfrac{(As+B)}{(s^2+a^2)} + \dfrac{C}{(s+b)}$

① $L^{-1}\left[\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right]$

Sol:- $\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \dfrac{A}{(s-1)} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$

Put $s=1$

$2-6+5 = A(1-2)(1-3)$

$A = \frac{1}{2}$

Put $s=2$

$8-12+5 = B(2-1)(2-3)$

$1 = B(1)(-1)$

$B = -1$

Scanned by CamScanner

put $s = 3$

$$18 - 18 + 5 = c(3-1)(3-2)$$

$$5 = c(2)(1)$$

$$c = 5/2$$

$$L^{-1}\left[\frac{1}{2(s-1)} - \frac{1}{s-2} + \frac{5}{2(s-3)}\right]$$

$$\Rightarrow \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$$

H.W

(2) $L^{-1}\left[\frac{1}{(s+2)(s+3)}\right]$

(2) $L^{-1}\left[\frac{4s^2 - 3s + 5}{(s+1)(s^2 - 3s + 2)}\right]$

(1) Soln:

$$\frac{1}{(s+2)(s+3)} \quad \frac{A}{s+2} + \frac{B}{s+3}$$

$$1 = A(s+3) + B(s+2)$$

put $s = -2$

$$1 = A(-2+3)$$

$$A = 1$$

put $s = -3$

$$1 = B(-3+2)$$

$$B = -1$$

$$\frac{1}{(s+2)(s+3)} \quad \frac{1}{s+2} - \frac{1}{s+3}$$

$$L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{s+3}\right]$$

$$e^{-2t} - e^{-3t}$$

Scanned by CamScanner

$$L^{-1}\left[\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right]$$

s^2-3s+2

$s^2-2s-s+2$

$s(s-2)-1(s-2)$

$(s-1)(s-2)$

$$L^{-1}\left[\frac{4s^2-3s+5}{(s+1)(s-1)(s-2)}\right]$$

$$\frac{4s^2-3s+5}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-2)}$$

$$4s^2-3s+5 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)$$

put $s=-1$

$$4+3+5 = A(-1-1)(-1-2)$$

$$12 = A(-2)(-3)$$

$$12 = 6A$$

$$A = 2$$

put $s=1$

$$4-3+5 = B(1+1)(1-2)$$

$$6 = (-1)(2)B$$

$$-2B = 6$$

$$B = -3$$

Put $s=2$

$$16-6+5 = C(2+1)(2-1)$$

$$15 = C(3)(1)$$

$$3C = 15$$

$$C = 5$$

$$\frac{4s^2-3s+5}{(s+1)(s-1)(s-2)} = \frac{2}{s+1} - \frac{3}{s-1} + \frac{5}{s-2}$$

$$L^{-1}\left[\frac{2}{s+1} - \frac{3}{s-1} + \frac{5}{s-2}\right]$$

$$2e^{-t} - 3e^{+t} + 5e^{2t}$$

Scanned by CamScanner

Q:- Find $L^{-1}\left[\frac{2s-1}{s^2(s-1)^2}\right]$

Sol:-

$$\frac{2s-1}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$2s-1 = A(s)(s-1)^2 + B(s-1)^2 + C(s^2)(s-1) + Ds^2$$

Put $s=0$

$\Rightarrow -1 = B(0-1)^2$

$\underline{B = -1}$

Put $s=1$

$2-1 = D$

$\underline{D=1}$

$A=0,\ B=-1,\ C=0,\ D=1$

Coeff of s^3

$A+C=0$

Coeff of s^2

$-2A+B-C+D=0$

$-2A-1-C+1=0$

$-2A-C=0$

$-A=0$

$A=0$

$C=0$

$$\frac{2s-1}{s^2(s-1)^2} = \frac{-1}{s^2} + \frac{1}{(s-1)^2}$$

$$L^{-1}\left[\frac{-1}{s^2}\right] + L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$-\frac{t^{2-1}}{(2-1)!} + L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$-\frac{t}{1} + e^{-t}\left(\frac{1}{s^2}\right)$$

$$-t + e^{-t}(1)\,t$$

$$-t + e^{-t}t$$

$$t(-1+e^{-t})$$

Scanned by CamScanner

Q: Find $L^{-1}\left[\frac{1}{s(s+1)^3}\right]$

Sol: $\frac{1}{s(s+1)^3} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$

$1 = A(s+1)^3 + Bs(s+1)^2 + Cs(s+1) + Ds$

Put $s = -1$

$D = -1$

Put $s = 0$

$1 = A$

Coeff of s^3

$A + B = 0$

$1 + B = 0$

$B = -1$

Coeff of s^2

$3A + 2B + C = 0$

$3(1) + 2(-1) + C = 0$

$3 - 2 + C = 0$

$C = -1$

$C = -3$

$\frac{1}{s(s+1)^3} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}$

$= L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2}\right] - L^{-1}\left[\frac{1}{(s+1)^3}\right]$

$= 1 - e^{-t} - e^{-2t}t - e^{-3t}\frac{t^2}{2}$

$= 1 - e^{-t} - e^{-2t}t - e^{-3t}\frac{t^2}{2}$

HW

(i) $\frac{5s^2 - 15s - 11}{(s+4)(s+2)^3}$

(ii) $\frac{1}{(s^2+4s+4)(s-3)}$

Scanned by CamScanner

Q:- Find the $L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]$.

Sol:-

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

put $s=1$

$4 = A(2)$

$A = 2$.

put $s=0$

$1 = A - C$

$1 = 2 - C$

$-C = 1-2$

$C = 1$

Coeff of s^2

$0 = A + B$

$B = -2$.

put $s=-1$

$-3+1 = (B(-1)+C)(-2)$

$-2 = (-B+C)-2$

$-2 = 2B - 2C$

$B - C = -1$

$B = 0$

$$L^{-1}\left[\frac{2}{s-1}\right] + L^{-1}\left[\frac{-2s+1}{s^2+1}\right]$$

$$2L^{-1}\left[\frac{1}{s-1}\right] - 2\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$2e^t(1) - 2\cos t + \sin t.$$

$$2e^t - 2\cos t + \sin t.$$

Hw

(3) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (5) $\frac{2s^2+5s-4}{s^3+s^2-2s}$

(4) $\frac{1}{s(s^2+2s+2)}$ (6) $\frac{1}{s^3(s^2+1)}$ (7) $\frac{s}{(s+1)^2(s^2+1)}$

Scanned by CamScanner

Q: $L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$

Sol: $\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+13}$

$1-s = A(s^2+4s+13) + (Bs+C)(s+1)$

put $s=-1$

$2 = A(-1)^2+4(-1)+13)$

$2 = A(1-4+13)$

$2 = A(14-4)$

$A = \frac{2}{10}$

$A = \frac{1}{5}$.

$s=0$

$1 = 13A + C$

$1 = \frac{13}{5} + C$

$C = 1 - \frac{13}{5}$

$= \frac{-8}{5}$

$B = \frac{-1}{5}$

Coeff of s^2

$A + B = 0$

$A = -B$

$A = \frac{-1}{5}$.

Scanned by CamScanner

$$L^{-1}\left[\frac{1}{s(s+1)}\right] + L^{-1}\left[\frac{\left(\frac{1}{5}\right)\left(\frac{8}{5}\right)}{(s^2+4+1+3)}\right]$$

$\frac{1}{5} - \frac{8}{5}$

$-\frac{9}{5}$

$$\frac{1}{5} L^{-1}\left[\frac{1}{s+1}\right] + \frac{9}{5}\left[L^{-1}\left(\frac{1}{s^2+4+1+2}\right)\right]$$

$s^2+4+1+2+4$

-4

$(s+2)^2+9$

$$\frac{1}{5} e^{-t}(1) - \frac{9}{5} L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right]$$

$$\frac{1}{5} e^{-t} - \frac{9}{5} e^{-2t}\cos 3t.$$

$$\frac{1}{5} e^{-t} - \frac{9}{5} e^{-2t}\sin 3t.$$

$$\frac{1}{5} e^{-t} - \frac{1}{5} L^{-1}\left[\frac{s+8}{(s+2)^2+9}\right]$$

$\frac{s+2-2}{(s+2)^2+9}$

$\frac{s-2}{s^2-9}$

$\cos 3t - \frac{2}{3}\sin 3t$

$$\frac{1}{5} e^{-t} - \frac{1}{5}\left[e^{-2t}\cos 3t - \frac{8}{3} e^{-2t}\sin 3t\right]$$

$$\frac{1}{5}\left[e^{-t} - e^{-2t}\sin 3t - \frac{8}{3} e^{-2t}\sin 3t\right]$$

$$\frac{1}{5} e^{-t} - \frac{1}{5}\left[e^{-2t}\left(\cos 3t - \frac{2}{3}\sin 3t\right) - \frac{8}{3} e^{-2t}\sin 3t\right]$$

Q:- $L^{-1}\left[\frac{3s^2+2s-1}{(s^2+4)(s^2-2s+1)}\right]$

Sol:- $$\frac{3s^2+2s-1}{(s^2+4)(s^2-2s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$3s^2+2s-1 = As+B(s-1)^2 + C(s^2+4)(s-1) + D(s^2+4)$$

put $s=1$

$3+2-1 = 5D$

$4 = 5D \Rightarrow D = \frac{4}{5}$

Scanned by CamScanner

(1) $L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

Sol:-

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$5s^2-15s-11 = A(s-2)^3 + B(s-2)^2(s+1) + C(s-2)(s+1) + D(s+1)$$

put $s=-1$

$$5+15-11 = A(-3)^3$$

$$9 = -A27$$

$$A = -\frac{1}{3}$$

put $s=2$

$$5s^2-15s-11$$

$$5(2)^2-15(2)-11 = 3D$$

$$20-30-11 = 3D$$

$$-10-11 = 3D$$

$$-21 = 3D$$

$$D = -\frac{7}{7}$$

$$\begin{array}{r} 20 \\ 11 \\ \hline \end{array}$$

$s^2+4-4s(s+1)$

$s^3+4s-4s^2$

$+ s^2+4-4s$

s^3-3s^2

coeff of s^3

$$A+B = 0$$

$$-\frac{1}{3}+B = 0$$

$$B = +\frac{1}{3}$$

$$B = \frac{1}{2}$$

coeff of s^2

$$6A-3B+C = 5$$

$$6\left(-\frac{1}{3}\right)-3\left(\frac{1}{3}\right)+C = 5$$

$$-2-1+C = 5$$

$$-3+C = 5$$

$$C = 8$$

$$L^{-1}\left[\frac{-1}{3(s+1)}\right] + L^{-1}\left[\frac{1}{3}\frac{1}{(s-2)}\right] + L^{-1}\left[\frac{8}{(s-2)^2}\right] + L^{-1}\left[\frac{-1}{7(s-2)^3}\right]$$

$$-\frac{1}{3}L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right] + 8L^{-1}\left[\frac{1}{(s-2)^2}\right] - \frac{1}{7}L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$-\frac{1}{3}e^{-t}(1) + \frac{1}{3}e^{2t}(1) + 8e^{2t}t - \frac{1}{7}e^{2t}\frac{t^2}{2}$$

$$-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 8te^{2t} - \frac{1}{14}t^2e^{2t}.$$

Scanned by CamScanner

(ii) $L^{-1}\left[\frac{1}{(s^2+4s+4)(s-3)}\right]$

Sol:- $\frac{1}{(s+2)^2(s-3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-3}$

$1 = A(s+2)(s-3) + B(s-3) + C(s+2)^2$

$s^2-3s+2s-6$ $\quad$ s^2+4s+4

Put $s=3$ $\quad s^2-s-6$

$1 = C(3+2)^2$

$C = \frac{1}{25}$

Put $s=-2$

$1 = B(-2-3)$

$B = \frac{-1}{5}$

Coeff of s^2

$A + C = 0$

$A + \frac{1}{25} = 0$

$A = \frac{-1}{25}$

Coeff of s

$-A + B + 4C = 0$

$-\frac{1}{25} + B + \frac{4}{25} = 0$

$B + \frac{3}{25} = 0$

$B = \frac{-3}{25}$.

$L^{-1}\left[\frac{-1}{25(s+2)}\right] + L^{-1}\left[\frac{-1}{5(s+2)^2}\right] + L^{-1}\left[\frac{1}{25(s-3)}\right]$

$\frac{-1}{25}e^{-2t} - \frac{1}{5}e^{-2t}\frac{(t)^{2-1}}{2} + \frac{1}{25}e^{3t}(1)$

$\frac{-1}{25}e^{-2t} - \frac{1}{5}te^{-2t} + \frac{1}{25}e^{3t}$.

(iii) $L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]$

Sol:- $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$

Scanned by CamScanner

Denominator = $s^3 - 6s^2 + 11s - 6$

$$\begin{array}{c|cccc} 2 & 1 & -6 & 11 & -6 \\ & 0 & 2 & -8 & 6 \\ \hline 1 & 1 & -4 & 3 & 0 \\ & 0 & 1 & -3 & \\ \hline & x^2 & -4x & +3 & 0 \\ & x^2 & -3 & & \\ & x & -3 & & \end{array}$$

$x - 1 = 0$
$x - 2 = 0$
$x + 3 = 0$

$$\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s+3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$2s^2 - 6s + 5 = A(s-2)(s+3) + B(s-1)(s+3) + C(s-1)(s-2)$$

put $s = 1$

$2 - 6 + 5 = A(1-2)(1+3)$

$-1 + 2 = A(-1)(4)$

$1 = -4A$

$A = -\frac{1}{4}$

$2A = 1$

$A = \frac{1}{2}$

put $s = 2$

$8 - 12 + 5 = B(2-1)(2+3)$

$13 - 12 = B(1)(5)$

$1 = 5B$

$B = \frac{1}{5}$

$-B = 1$

$B = -1$

36
5
41

Put $s = +3$

$18 + 18 + 5 = C(-3-1)(-3-2)$

$31 = (-4)(-5)C$

$C = \frac{31}{20}$

$C = \frac{5}{2}$

$\frac{+1}{2} e^{t} - \frac{1}{5} e^{2t} + \frac{41}{20} e^{+3t}$

Scanned by CamScanner

(4) $L^{-1}\left[\frac{1}{s(s^2+2s+2)}\right]$

Sol:-

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$1 = A(s^2+2s+2) + (Bs+C)(s)$$

put $s=0$

$A = \frac{1}{2}$

Coeff of s^2

$A+B=0$

$B = -\frac{1}{2}$

Coeff of s

$2A+C=0$

$2(\frac{1}{2})+C=0$

$2(\frac{1}{2})+C=0$

$C=-1$

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2s} + \frac{-\frac{1}{2}s-1}{s^2+2s+2}$$

$$= \frac{1}{2s} + \frac{-s-2}{2(s^2+2s+2)}$$

$$= \frac{1}{2s} - \frac{(s+2)}{2(s^2+2s+2)}$$

$$L^{-1}\left[\frac{1}{2s}\right] - L^{-1}\left[\frac{s+2}{2(s^2+2s+2)}\right]$$

$$\frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{s+2}{s^2+2s+2+1-1}\right]$$

$$\frac{1}{2} - \frac{1}{2}\left\{L^{-1}\left(\frac{s+2}{(s+1)^2+1}\right)\right\}$$

$$\frac{1}{2} - \frac{1}{2}L^{-1}\left[\frac{s}{(s+1)^2+1} + \frac{2}{(s+1)^2+1}\right]$$

Scanned by CamScanner

$$\frac{1}{2} - \frac{1}{2} L^{-1}\left[\frac{s+1-1}{(s+1)^2+1} + \frac{2}{(s+1)^2+1}\right]$$

$$\frac{1}{2} - \frac{1}{2}\left\{e^{-t} L^{-1}\left[\frac{s-1}{s+1}\right] + e^{-t} L^{-1}\left[\frac{2}{s^2+1}\right]\right\}$$

$$\frac{1}{2} - \frac{1}{2}\left\{e^{-t} L^{-1}\left[\frac{s-1+1-1}{}\right.\right.$$

$$\frac{1}{2} - \frac{1}{2}\left\{e^{-t} L^{-1}\left[\frac{s-1}{s^2+1}\right] + e^{-t} L^{-1}\left[\frac{2}{s^2+1}\right]\right\}$$

$$\frac{1}{2} - \frac{1}{2} e^{-t}\left\{L^{-1}\left[\frac{s}{s^2+1} - \frac{1}{s^2+1}\right] + 2L^{-1}\left[\frac{1}{s^2+1}\right]\right.$$

$$\frac{1}{2} - \frac{1}{2} e^{-t}\left\{\cos at - \sin at + 2\sin at\right\}$$

$$\frac{1}{2} - \frac{1}{2} e^{-t}\left\{\cos at - \sin at\right\}$$

5) $L^{-1}\left[\frac{2s^2+5s-4}{s^3+s^2-2s}\right]$

Sol:- $\frac{2s^2+5s-4}{s^3+s^2-2s} = \frac{2s^2+5s-4}{s(s^2+s-2)} = \frac{A}{s} + \frac{Bs+C}{s^2+s-2}$

~~Put s=0~~

$2s^2+5s-4 = A(s^2+s-2) + (Bs+C)(s)$

Put $s=0$

$-4 = -2A$

$A = 2$

Coeff of s^2

$2 = A + B$

$2 = 2 + B$

$B = 0$

Coeff of s

$A + C = 5$

$2 + C = 5$

$C = 3$

Scanned by CamScanner

$2, \frac{4}{2}$
$1, -2$

5) $L^{-1}\left[\frac{2s^2+5s-4}{s^3+s^2-2s}\right]$

Sol:- $\frac{2s^2+5s-4}{s^3+s^2-2s} = \frac{2s^2+5s-4}{s(s^2+s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$

$2s^2+5s-4 = A(s-1)(s+2)+B(s)(s+2)+C(s-1)s$

put $s=0$

$-4 = -A(2)$

$A = 2$

put $s=1$

$2+5-4 = B(3)$

$3 = B(3)$

$B = 1$

put $s = -2$

$8-10-4 = C(-2)(-2-1)$

$-2-4 = (-2)(-3)(C)$

$-6 = 6C$

$C = -1$

Scanned by CamScanner

$$L^{-1}\left(\frac{2}{s}\right) + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$2(1) + e^{t} - e^{-2t}$$

(6) $L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$

$$\frac{1}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{(s^2+1)}$$

~~Put s=0~~

$$1 = \underbrace{A(s^2+1)(s^2)}_{s^4+s^2} + \underbrace{B(s)(s^2+1)}_{s^3+s} + \underbrace{C(s^2+1)}_{s^2+1} + \underbrace{(Ds+E)s^3}_{Ds^4+Es^3}$$

Put $s=0$

$C=1$

Coeff of s^4

$A=0$

Put $s=1$

coeff of s^3

$B+D+E=0$

coeff of s^2

$A+C=0$

$B+D+E=0$

$D+E=0$

coeff of $s=1$

$B=0$

$E=0$

$$L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^3}\right] + L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$-(1) + \frac{t^2}{2} + \cos t$$

$$-1 + \frac{t^2}{2} + \cos t$$

Scanned by CamScanner

7) $L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right]$

Sol:- $\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$

$s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2$

$A(s^3+s+s^2+1) + Bs^2+B + (Cs+D)(s^2+2s+1)$

put $s=-1$

$-1 = B(1+1)$

$2B = -1$

$B = \frac{-1}{2}$

Coeff of s^3

$A+C=0$

Coeff of s^2

$A+B+2C+D=0$

$A+2C+D=\frac{1}{2}$

Coeff of s $\quad -C+2C+D=\frac{1}{2}$ $\quad C+D=\frac{1}{2}$

$1 = A+C+2D$

$A+C+2D=1$

$C-D=\frac{-1}{2}$

$2C=0$

$C=0$

$D=\frac{1}{2}$

$A=0$

$-\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right] + L^{-1}\left[\frac{1}{2(s^2+1)}\right]$

$-\frac{1}{2}e^{-t}(t) + \frac{1}{2}\sin t$

Scanned by CamScanner

Convolution method

* The convolution of two functions $f(t), g(t)$ where $t \geq 0$ is given by $f(t) * g(t)$ (or) $f * g$

$$f(t) * g(t) = \int_0^t f(u) g(t-u)\, du$$

where $*$ denotes convolution operator.

* $f * g = g * f$ i.e, convolution operator is commutative.

proof:-

$$f * g = f(t) * g(t) = \int_0^t f(u) g(t-u)\, du$$

$$\int_0^a f(t)\, dt = \int_0^a f(a-t)\, dt \qquad u \to t.$$

$$= \int_0^t f(t-u)\, g[\cancel{t} - (\cancel{t} - u)]\, du$$

$$= \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t g(u) f(t-u)\, du = g(t) * f(t)$$

$$= g * f.$$

Statement:- If $f(t), g(t)$ be two functions defined for $t \geq 0$ then $L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$

This is known as Faltung theorem.

Proof:-

$$f(t) * g(t) = \int_0^t f(u) g(t-u)\, du \quad \text{—— (1)}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \quad \text{—— (2)}$$

Scanned by CamScanner

LHS :-

$$L[f(t)*g(t)] = \int_0^\infty e^{-st}[f(t)*g(t)]\,dt$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(u)\,g(t-u)\,du\right]dt$$

$$= \int_0^\infty\int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt \quad \text{--- (3)}$$

By change the order of integration

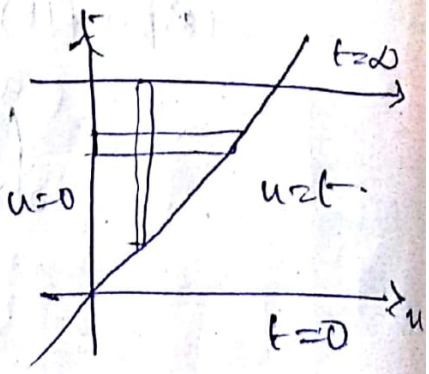
$u=0, \quad u=t$ (var. limit)

$t=0, \quad t=\infty$ (const limit)

$du\,dt \longrightarrow dt\,du$

$u=0 \longrightarrow \infty$

$t=u \longrightarrow \infty$



From (3)

$$\int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)\,g(t-u)\,dt\,du$$

$t=u+v \Rightarrow v=t-u$

$u=t-v \qquad dv=dt$

t	u	∞
v = t-u	0	∞

$$\int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-s(u+v)}\,g(u)\,f(t-v)\,dv\,du$$

$$\int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-su}\cdot e^{-sv}\,f(t-v)\,g(v)\,dv\,du$$

Scanned by CamScanner

$$\int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-s(u+v)} f(u)\, g(v)\, dv\, du$$

$$\int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-su} \cdot e^{-sv} f(u)\, g(v)\, dv\, du.$$

$$\left[\int_{u=0}^{\infty} e^{-su} f(u)\, du\right]\left[\int_{v=0}^{\infty} e^{-sv} g(v)\, dv\right]$$

$u, v \to$ dummy.

$$L[f(t)] \cdot L[g(t)]$$

$\Rightarrow$ RHS.

* $L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

* $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$

$$= e^{-at} * e^{-bt}$$

$t \to u, \quad t \to t-u.$

$$= \int_0^t e^{-au} e^{-b(t-u)}\, du$$

$$= e^{-bt} \int_0^t e^{-au} e^{bu}\, du$$

$$= e^{-bt} \int_0^t e^{-u(a-b)}\, du$$

$$= e^{-bt}\left[\frac{e^{-u(a-b)}}{-(a-b)}\right]_{u=0}^{u=t}$$

$$= \frac{e^{-bt}}{(b-a)}\left[e^{-t(a-b)} - 1\right]$$

$$= \frac{e^{-bt}}{b-a}\left[e^{-t(a-b)} - 1\right]$$

Scanned by CamScanner

$$= \frac{e^{-bt}\left[e^{-at}e^{bt}-1\right]}{b-a}$$

$$= \frac{e^{-at}-e^{-bt}}{b-a}.$$

* Find $L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right]$

$\frac{-2\pm\sqrt{4-20}}{2}$

Denominator $= s^2+2s+5$

$= (s+1)^2+4$

$$L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right] = 4L^{-1}\left[\frac{1}{(s^2+2s+5)^2}\right]$$

$$= 4L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] * L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= 4e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right] * e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= 4e^{-t}\frac{1}{2}\sin 2t * e^{-t}\frac{1}{2}\sin 2t$$

$$= e^{-2t}\sin^2 2t$$

$$= 2e^{-t}\sin 2t \; e^{-t}\sin 2t.$$

$t \to u \qquad t \to t-u.$

$$= 2\int_0^t e^{-u}\sin 2u \; e^{-(t-u)}\sin(2t-u)\,du$$

$$= 2\int_0^t e^{-t}\sin 2u \cdot \sin(2t-2u)\,du$$

$$= 2e^{-t}\int_0^t \sin 2u \, \sin(2t-2u)\,du.$$

Scanned by CamScanner

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$= \cancel{2} e^{-t} \int_0^t \frac{1}{\cancel{2}} [\cos(4u-2t) - \cos 2t]\, du$$

$$= e^{-t} \left[\frac{\sin(4u-2t)}{4} - u\cos 2t \right]_{u=0}^{u=t}$$

$$= e^{-t} \left[\frac{\sin 2t}{4} - \frac{\sin(-2t)}{4} \right\} - \{(t-0)\cos 2t\}$$

$$= e^{-t} \left\{ \frac{2\sin 2t}{4_2} - t\cos 2t \right\}$$

$$= e^{-t} \left[\frac{\sin 2t}{2} - t\cos 2t \right].$$

* $L^{-1}\left[\frac{1}{s^2(s+5)}\right]$

Soln $L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s+5}\right]$

$$t * e^{-5t}$$

$$\downarrow \qquad \downarrow$$

$$t \to u \qquad t \to t-u$$

$$= \int_0^t u\, e^{-5(t-u)}\, du$$

$$= \int_0^t u\, e^{-5t} e^{5u}\, du$$

$$= e^{-5t} \int_0^t u\, e^{5u}\, du$$

$$= e^{-5t} \left[u\frac{e^{5u}}{5} - \frac{e^{5u}}{25} \right]_0^t$$

$$= e^{-5t} \left[\frac{te^{5t}}{5} - \frac{e^{5t}}{25} - \left(0 - \frac{1}{25}\right) \right]$$

$$e^{-5t}\left[\frac{te^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25} \right]$$

$$\frac{e^{-5t}}{5}\left[te^{5t} - \frac{e^{5t}}{5} + \frac{1}{5} \right]$$

$\int u e^{5u} = uv_1 - u'v_2$

$u = u \qquad v = e^{5u}$

$u' = 1 \qquad v_1 = \frac{e^{5u}}{5}$

$v_2 = \frac{e^{5u}}{25}$

Scanned by CamScanner

Q

① $L^{-1}\left[\frac{1}{(s+5)(s-6)}\right]$

② $L^{-1}\left[\frac{10}{(s+1)(s^2+4)}\right]$

③ $L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$

(A) $L^{-1}\left[\frac{1}{(s+5)(s-6)}\right]$

$L^{-1}\left[\frac{1}{s+5}\right] L^{-1}\left[\frac{1}{(s-6)}\right]$

$e^{-5t} * e^{+6t}$

$\downarrow \qquad \downarrow$

$t \to u \qquad t = t - u$

$= \int_0^t e^{-5u} e^{6(t-u)} du$

$= \int_0^t e^{-5u} e^{6t} e^{-6u} du$

$= e^{6t} \int_0^t e^{-5u} e^{-6u} du$

$= e^{6t} \int_0^t e^{-u(6+5)} du$

$= e^{6t} \int_0^t e^{-11u} du$

$= e^{6t} \left(\frac{e^{-11u}}{-11}\right)_0^t$

$= e^{+6t}\left[-\frac{e^{-11t}}{11} + \frac{1}{11}\right]$

$= \frac{e^{+6t}}{11}\left[-e^{-11t} + 1\right]$

Scanned by CamScanner

(2) $L^{-1}\left[\frac{10}{(s+1)(s^2+4)}\right]$

$10\left\{L^{-1}\left[\frac{1}{s+1}\right] L^{-1}\left[\frac{1}{s^2+4}\right]\right\}$

$10\left\{ e^{-t} * \frac{1}{2}\sin 2t \right\}$

$\downarrow \qquad \downarrow$

$t=u \qquad t=t-u$

$\frac{10}{2}\left\{\int_0^t e^{-u} \sin(2(t-u))\, du\right\}$

$5\left\{\int_0^t e^{-u} \sin(2t-2u)\, du\right\}$ $\quad -2dv = du$ $\quad v \quad 0t$ $\quad v \quad 2t \; 0$

$5\left\{\int_0^t e^{-u} \frac{1}{2} \cancel{\sin} \cos(\ldots\right.$

$5\left\{\int_0^t e^{-u} [\sin 2t \cos 2u - \cos 2t \sin 2u]\, du\right\}$

$5\left\{\int_0^t e^{-u} \sin 2t \cos 2u\, du - \int_0^t e^{-u} \cos 2t \sin 2u\, du\right\}$

$5\left\{\sin 2t \int_0^t \cancel{\sin}\, e^{-u} \cos 2u\, du - \cos 2t \int_0^t e^{-u} \sin 2u\, du\right\}$

$\frac{5}{2}\int_{2t}^{0} e^{-v/2} \sin v\, dv$

$= \frac{5}{2} e^{-t} \int_{2t}^{0} e^{v/2} \sin v\, dv$

$= \frac{5}{2} e^{-t} \left(\frac{e^{v/2}}{(1/2)^2+1}\left(\frac{1}{2}\sin v - \cos v\right)\right)_{2t}^{0}$

$\frac{5}{2} e^{-t} \left(\frac{-4}{5} - \frac{4}{5} e^{t} \frac{\sin 2t}{2} + e^{t} \frac{4\cos 2t}{5}\right)$

$e^{-t} + e^{-t}\sin 2t - 2e^{-t}\cos 2t$

Scanned by CamScanner

3) $L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$

$s^2+4s+13 = s^2+4s+4+9$

$= (s+2)^2+9$

$L^{-1}\left[\frac{s+2}{((s+2)^2+9)^2}\right]$

$L^{-1}\left[e^{-2t}L^{-1}\left[\frac{s}{s^2+9}\right]\right]$ $\quad L^{-1}\left[\frac{s+2}{((s+2)^2+9^2)^2}\right]$

$e^{-2t} * \cos 3t$

$L^{-1}\left[\frac{1}{(s+2)^2+9^2}\right] * L^{-1}\left[\frac{s+2}{(s+2)^2+9^2}\right]$

$e^{-2t}\frac{1}{3}\sin 3t * e^{-2t}\cos 3t.$

$\frac{1}{3}\left[e^{-2t}\sin 3t * e^{-2t}\cos 3t\right]$

$\frac{1}{3}\int_0^t e^{-2u}\sin 3u \, e^{-2(t-u)}\cos 3(t-u)\,du$

$\frac{1}{3}e^{-2t}\int_0^t e^{-2u}\sin 3u \; e^{2u}\cos(3t-3u)\,du$

$\frac{1}{3}e^{-2t}\int_0^t \sin 3u \cos(3t-3u)\,du.$

$\frac{1}{3}e^{-2t}\int_0^t \sin 3u\left[\cos 3t\cos 3u+\sin 3t\sin 3u\right]du$

$\frac{1}{3}e^{-2t}\left[\int_0^t \cos 3t\int_0^t\right.$

$\frac{1}{3}e^{-2t}\int_0^t\left[\sin 3t+\sin(6u-3t)\right]du$

Scanned by CamScanner

$$\frac{1}{6} e^{-2t}\left[\sin 3t \cdot [u]_0^t - \frac{1}{6}\left(\frac{\cos(6u-3t)}{6}\right)\right]$$

$$\frac{1}{6} e^{-2t}\left[t\sin 3t - \frac{1}{6}\left[\cos(6t-3t) - \cos 3t\right]\right]$$

$$\frac{1}{6} e^{-2t}\left[t\sin 3t - \frac{1}{6}\left[\cos 3t - \cos 3t\right]\right]$$

$$\frac{1}{6} e^{-2t}\left[t\sin 3t\right]$$

* Find the $L^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$ by convolution method.

Sol:- $L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{(s+1)^2}\right]$

$$t * e^{-t}\frac{t^2}{2}$$

$$t = u \qquad t = t-u$$

$$= \int_0^t u\, e^{-t}\frac{(t-u)^2}{2}\, du$$

$$= \frac{1}{2}\int_0^t u\, e^{-t}(t-u)^2\, du$$

$$= \frac{e^{-t}}{2}\int_0^t u\,(t^2+u^2-2tu)\, du$$

$$= \frac{e^{-t}}{2}\int_0^t (ut^2+u^3-2tu^2)\, du$$

$$= \frac{e^{-t}}{2}\left\{\left[t^2\frac{u^2}{2}+\frac{u^4}{4}-2t\frac{u^3}{3}\right]_0^t\right\}$$

$$= \frac{e^{-t}}{2}\left\{t^2\frac{t^2}{2}+\frac{t^4}{4}-\frac{2t^4}{3}-(0+0+0)\right\}$$

$$= \frac{e^{-t}}{2}\left\{\frac{t^4}{2}+\frac{t^4}{4}-\frac{2t^4}{3}\right\} \Rightarrow \frac{e^{-t}t^4}{2}\left[\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right]$$

$$= e^{-t}\left[\frac{1}{2}-\frac{1}{12}\right]$$

Scanned by CamScanner

Find the $L^{-1}\left[\frac{1}{s^2(s+1)^3}\right]$

Sol:

$$L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{(s+1)^3}\right]$$

$$t * e^{-t} L^{-1}\left[\frac{1}{s^3}\right]$$

$$t * e^{-t}\frac{t^2}{2}$$

$$f * g = g * f$$

$$e^{-t}\frac{t^2}{2} * t$$

$$\downarrow \qquad \downarrow$$

$$t = u \qquad t = t-u$$

$$= \int_0^t e^{-u}\frac{u^2}{2}(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}u^2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}(u^2t - u^3)\,du$$

$$\int uv\,du = uv_1 - u'v_2 + u''v_3 - \cdots$$

$$\text{Let } u = u^2t - u^3 \longrightarrow v_1 = -e^{-u}$$

$$u' = 2ut - 3u^2 \longrightarrow v_2 = e^{-u}$$

$$u'' = 2t - 6u \longrightarrow v_3 = -e^{-u}$$

$$u''' = -6 \longrightarrow v_4 = e^{-u}$$

$$= \frac{1}{2}\left\{(0) - \left[(-t^2e^{-t})\right] + (-4t)(-e^{-t}) - (-6)e^{-t}\right\}$$

$$- \left\{0 - 0 + 2t - 6\right\}$$

Scanned by CamScanner

$= \frac{1}{2}\{t^2e^{-t} + 4te^{-t} + 6e^{-t}) + 2t - 6\}$

$= \frac{1}{2}e^{-t}\{t^2 + 4t + 6\} + \frac{1}{2}(2)(t-3)$

$= \frac{1}{2}e^{-t}\{(t^2+4t+6)\} + (t-3)$

Derivation of some standard functions using some convolutions.

1. $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ $a \neq b$.

$L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$

$\cos at * \cos bt$

$\downarrow$ $t = u$ $\quad$ $\downarrow$ $t = t-u$.

$= \int_0^t \cos au \cos b(t-u)\, du$.

$= \int_0^t \cos au \cos(bt - bu)\, du$

$= \cos bt \int_0^t \cos au (\cos bu - \sin bt \sin bu)\, du$

$= \int_0^t \cos au \cos bt \cos bu - \sin bt \sin bu \cos au\, du$.

$= \int_0^t \cos bt$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$= \frac{1}{2}\int_0^t [\cos(au + bt - bu] + \cos(au - bt + bu)]\, du$

$= \frac{1}{2}\int_0^t \{\cos[u(a-b) - bu] + \cos[u(a+b) - bt]\}\, du$

Scanned by CamScanner

$$= \frac{1}{2}\left[\frac{\sin(a-b)u+bt)}{a-b} + \frac{\sin(a+b)u-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left\{\left[\frac{\sin(a-b)t+bt}{a-b} + \frac{\sin(a+b)t-bt}{a+b}\right] - \left[\frac{\sin bt}{a-b} + \frac{\sin(-bt)}{a+b}\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right\}$$

$$= \frac{1}{2}\left\{\sin at\left[\frac{1}{a-b} + \frac{1}{a+b}\right] + \sin bt\left[\frac{-1}{a-b} + \frac{1}{a+b}\right]\right\}$$

$$= \frac{1}{2}\left\{\sin at\left[\frac{a+b+a-b}{a^2-b^2}\right] + \sin bt\left[\frac{a+b-a+b}{a^2-b^2}\right]\right\}$$

$$= \frac{1}{2}\left\{\sin at\left[\frac{2a}{a^2-b^2}\right] + \sin bt\left[\frac{2b}{a^2-b^2}\right]\right\}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

$$* \quad L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$$

Sol:- $L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+a^2}\right]$

$\cos at * \cos at.$

$$\int_0^t \left[\cos au \cos a(t-u)\right] du$$

$$= \int_0^t \left(\frac{1}{2}\left[\cos[au+at-au] + \cos[au-at+au]\right] du\right.$$

Scanned by CamScanner

$$\frac{1}{2}\int_0^t [\cos at + \cos 2au]\,du$$

$$\frac{1}{2}\left\{\cos at\,[u]_0^t + \int_0^t \cos(2au)\,du\right\}$$

$\cos(2au - at)$

$$\frac{1}{2}\left\{t\cos at + \left[\frac{\sin(2au - at)}{2a}\right]_0^t\right\}$$

$$\frac{1}{2}\left\{t\cos at + \frac{\sin 2at}{2} + \frac{\sin at}{2}\right\}$$

$$\frac{1}{2}\left\{t\cos at - \frac{\sin at}{a}\right\}$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

sol:- $$L^{-1}\left[\frac{1}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$\frac{1}{a}\sin at * \frac{1}{a}\sin at$$

$$\frac{1}{a}\left[\sin at * \sin at\right]$$

$$\frac{1}{a^2}\int_0^t \sin au\,\sin a(t-u)\,du$$

$$\frac{1}{a^2}\int_0^t \frac{1}{2}\left[\cos(au - at + au) - \cos(au + at - au)\right]du$$

$$\frac{1}{2a^2}\int_0^t (\cos(2au - at) - \cos at)\,du.$$

$$\frac{1}{2a^2}\int_0^t \left[\cos at - \cos(2au - at)\right]du$$

$$\frac{1}{2a^2}\left[t\cos at - \int_0^t \cos(2au - at)\,du\right]$$

$$\frac{1}{2a^2}\left\{t\cos at - \left[\frac{\sin(2au - at)}{2a}\right]_0^t\right\}$$

Scanned by CamScanner

$$-\frac{1}{2a^2}\left\{t\cos at - \left(\frac{\sin at}{2a} - \frac{\sin(-at)}{2a}\right)\right\}$$

$$-\frac{1}{2a^2}\left\{t\cos at - \frac{\sin at}{2a} + \frac{\sin at}{2a}\right\}$$

$$-\frac{1}{2a^2}\left\{t\cos at + \frac{\cancel{2}\sin at}{\cancel{2}a}\right\}$$

$$\frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right].$$

H.W

$$L^{-1}\left[\frac{1}{(s^2+25)(s^2+36)}\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+9)(s^2+81)}\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+49)}\right]$$

$$L^{-1}\left[\frac{1}{(s^2+64)^2}\right]$$

1A)

$$L^{-1}\left[\frac{1}{(s^2+25)(s^2+36)}\right]$$

$$L^{-1}\left[\frac{1}{s^2+25}\right] * L^{-1}\left[\frac{1}{s^2+36}\right]$$

$$\frac{1}{5}\sin 5t * \frac{1}{6}\cdot\sin 6t$$

Scanned by CamScanner

$$\frac{1}{30}\left[\sin 5t * \sin 6t\right]$$

$$\frac{1}{30}\int_0^t \sin 5u \, \sin(6t-6u)\,du$$

$$\frac{1}{60}\int_0^t \left[\cos(5u-6t+6u) - \cos(5u+6t-6u)\right]du$$

$$\frac{1}{60}\left[\left[\cos(11u-6t) - \cos(6t-u)\right]du\right.$$

$$\frac{1}{60}\left[\frac{\sin 11u-6t}{11} - \frac{\sin 6t-u}{(-1)}\right]_0^t$$

$$\frac{1}{60}\left\{\left(\frac{\sin 11t-6t}{11} + \sin(6t-t)\right) - \left(\frac{\sin(-6t)}{11} + \sin 6t\right)\right\}$$

$$\frac{1}{60}\left[\frac{\sin 5t}{11} + \sin 5t + \frac{\sin 6t}{11} - \sin 6t\right]$$

$$= \frac{1}{60}\left[\frac{\sin 5t + \sin 6t}{11} + \sin 5t - \sin 6t\right]$$

2. $$L^{-1}\left[\frac{s^2}{(s^2+9)(s^2+81)}\right]$$

Sol:- $$L^{-1}\left[\frac{s}{s^2+9}\right] * L^{-1}\left[\frac{s}{s^2+81}\right]$$

$$\cos 3t * \cos 9t.$$

$$\int_0^t \left[\cos 3u \, \cos(9t-9u)\right]du.$$

$$= \frac{1}{2}\int_0^t \cos(3u+9t-9u) + \cos(3u-9t+9u)\,du.$$

$$= \frac{1}{2}\int_0^t \left[\cos(9t-6u) + \cos(12u-9t)\right]du$$

Scanned by CamScanner

$$\frac{1}{2}\left\{\left[\frac{\sin(9t-6u)}{-6}+\frac{\sin(12t-9t)}{12}\right]_0^t\right\}$$

$$\frac{1}{2}\left\{\left[\frac{\sin(9t-6t)}{-6}+\sin\frac{(12t-9t)}{12}\right]-\left[\frac{\sin 9t}{-6}-\frac{\sin 9t}{12}\right]\right\}$$

$$\frac{1}{2}\left[\frac{\sin 3t}{-6}+\frac{\sin 3t}{12}\right]+\left[\frac{\sin 9t}{6}+\frac{\sin 9t}{12}\right]$$

$$\frac{1}{2}\left[\frac{3\sin 9t}{12}-\frac{\sin 3t}{12}\right]=\frac{1}{2}\left\{\frac{3\sin 9t-\sin 3t}{12}\right\}$$

3. $L^{-1}\left[\frac{s^2}{(s^2+49)^2}\right]$

$$L^{-1}\left[\frac{s}{s^2+7^2}\right]*L^{-1}\left[\frac{s}{s^2+7^2}\right]$$

$$\cos 7t * \cos 7t$$

$$\int_0^t \cos 7u \cdot \cos(7t-7u)\,du$$

$$\frac{1}{2}\int_0^t \left[\cos(7u-7t-7u)+\cos(7u-7t+7u)\right]du$$

$$\frac{1}{2}\int_0^t \cos 7u+\cos(14u-7t)\,du$$

$$\frac{1}{2}\left[\cos 7t(u)+\frac{\sin(14u-7t)}{14}\right]_0^t$$

$$\frac{1}{2}\left[\left[\cos 7t(t)+\frac{\sin 7t}{14}\right]-\left[0+\frac{\sin(-7t)}{14}\right]\right]$$

$$\frac{1}{2}\left[t\cos 7t+\frac{\sin 7t}{14}+\frac{\sin 7t}{14}\right]$$

Scanned by CamScanner

$$\frac{1}{2}\left[t\cos 7t + \frac{\sin 7t}{7}\right]$$

4. $L^{-1}\left[\frac{1}{(s^2+64)^2}\right]$

$$L^{-1}\left[\frac{1}{s^2+8^2}\right] * L^{-1}\left[\frac{1}{s^2+8^2}\right]$$

$$\frac{1}{8}\sin 8t * \frac{1}{8}\sin 8t$$

$$\frac{1}{8}\left[\sin 8t * \sin 8t\right]$$

$$\frac{1}{8}\int_0^t \sin 8u \, \sin(8t-8u)\, du$$

$$\frac{1}{16}\int_0^t \left[\cos(8u-8t+8u) + \cos(8u+8t-8u)\right] du$$

$$\frac{1}{16}\left[\frac{\sin(16u-8t)}{16} + \cos 8t(u)\right]_0^t$$

$$\frac{1}{16}\left\{\left[\frac{\sin 8t}{16} - t\cos 8t\right] - \left[-\frac{\sin 8t}{16} + 0\right]\right\}$$

$$\frac{1}{16}\left\{2\frac{\sin 8t}{16} - t\cos 8t\right\}$$

$$\frac{1}{16}\left\{\frac{\sin 8t}{8} - t\cos 8t\right\}$$

<u>Solution</u> of ODE <u>using</u> <u>Laplace Transforms</u>

To find $L[y(t)] = ?$

$$L[y'(t)] = s\,L[y(t)] - y(0)$$

$$L[y''(t)] = s^2 L[y(t)] - s\,y(0) - y'(0)$$

$$L[y'''(t)] = s^3 L[y(t)] - s^2 y(0) - s\,y'(0) - y''(0)$$

Scanned by CamScanner

Q. Solve $(D^2-3D+2)y=4$, $y(0)=2$, $y'(0)=-3$

Sol:-

$$y''(t)-3y'(t)+2y(t)=4$$

Taking $L.T$ on both sides

$$L[y''(t)]-3L[y'(t)]+2L[y(t)]=L[4]$$

$$\left[s^2L[y(t)]-sy(0)-y'(0)\right]-3\left[sL[y(t)]-y(0)\right]$$
$$+2L[y(t)]=\frac{4}{s}$$

$$L[y(t)]\left[s^2-3s+2\right]-2s+3+6=\frac{4}{s}$$

$$L[y(t)]\cdot[s^2-3s+2]=\frac{4}{s}+(2s-9)$$

$$L[y(t)]\,[s^2-3s+2]=\frac{4+s(2s-9)}{s}$$

$$L[y(t)]\,[s^2-3s+2]=\frac{4+2s^2-9s}{s(s^2-3s+2)}$$

$$L[y(t)]=\frac{2s^2-9s+4}{s(s^2-3s+2)}$$

$$y(t)=L^{-1}\left[\frac{2s^2-9s+4}{s(s^2-3s+2)}\right]$$

$$y(t)=L^{-1}\left[\frac{2s^2-9s+4}{s(s-1)(s-2)}\right]$$

$$\frac{2s^2-9s+4}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}$$

Put $s=0$

Scanned by CamScanner

$2s^2-9s+4 = A(s-1)(s-2)+B(s)(s-2)+C(s-1)s$

put $s=0$

$4 = A(-1)(-2)$

$2A = 4$

$A = 2$

put $s=1$

$2-9+4 = B(1)(1-2)$

$-5+2 = (-1)B$

$-B = -3$

$B = 3$

put $s=2$.

$8-18+4 = C(2-1)(2)$

$-10+4 = C(1)(2)$

$-6 = 2C$

$C = -3$.

$$L^{-1}\left[\frac{2}{s}\right]+L^{-1}\left[\frac{3}{s-1}\right]+L^{-1}\left[\frac{-3}{s-2}\right]$$

$$2(1)+3e^{+t}-3e^{+2t}$$

$$2+3e^{+t}-3e^{+2t}$$

Q: $(D+2)^2y = 4e^{-2t}$. $y(0)=-1$, $y'(0)=4$

Sol. $(y'(t)+2)^2 \cdot D^2+4D+4 = 4e^{-2t}$

$y''(t)+4y'(t)+4y(t) = 4e^{-2t}$

$L[y''(t)]+4L(y'(t))+4L[y(t)] = 4L[e^{-2t}]$

Scanned by CamScanner

$$\left[s^2 L[y(t)] - s y(0) - y'(0)\right] + 4\left[s L[y(t)] - y(0)\right] + 4 L[y(t)] = \frac{4}{s+2}$$

$$L[y(t)](s^2+4s+4) + 2s - 4 + 16 = \frac{4}{s+2}$$

$$(s^2+4s+4) L[y(t)] = \frac{1}{s+2} - s$$

$$L[y(t)] = \frac{4 - s^2 - 2s}{s+2\,(s^2+4s+4)}$$

$$L[y(t)] = \frac{(4 - s^2 - 2s)}{(s+2)^3}$$

$$y(t) = L^{-1}\left(\frac{4 - s^2 - 2s}{(s+2)^3}\right)$$

$$\frac{4 - s^2 - 2s}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$

$$4 - s^2 - 2s = A(s+2)^2 + B(s+2) + C$$

$$A(s^2+4s+4)$$

Put $s = -2$

$$4 - 4 + 4 = C$$

$$C = 4$$

Coeff of s^2

$$-1 = A$$

Coeff of s

$$4A + B = -2$$

$$4(-1) + B = -2$$

$$B = -2 + 4$$

$$B = 2$$

Scanned by CamScanner

$$L^{-1}\left[\frac{4-s^2-2s}{(s+2)^2}\right] = -e^{-2t} + 2te^{-2t} + 4e^{-2t}\frac{t^2}{2}$$

$$= -e^{-2t} + 2te^{-2t} + 2e^{-2t}t^2$$

Q: Solve $x'' + 4x' + 3x = 10\sin t$. $x(0) = x'(0) = 0$

Sol: $y''(t) + 4y'(t) + 3y(t) = 10\sin t$ $y(0) = y'(0) = 0$

$$L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = 10L[\sin t]$$

$$\left[s^2 L[y(t)] - sy(0) - y'(0)\right] + 4\left[sL[y(t)] - y(0)\right] + 3L[y(t)] = 10\frac{1}{s^2+1}$$

$$L[y(t)]\left[s^2+4s+3\right] = \frac{10}{s^2+1}$$

$$L[y(t)] = \frac{10}{(s^2+1)(s^2+4s+3)}$$

$$y(t) = L^{-1}\left[\frac{10}{(s^2+1)(s^2+4s+3)}\right]$$

$$\frac{10}{(s^2+1)(s^2+4s+3)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$10 = A(s+3)(s^2+1) + B(s+1)(s^2+1) + (Cs+D)(s+1)(s+3)$$

s^3+s+3s^2+3 $\quad s^3+s+s^2+1$ $\quad (s^2+4s+3)$

put $s=-1$

$10 = A(-1+3)(1+1)$

$2A(2) = 10$

$4A = 10$

$A = \frac{5}{2}$

Put $s=-3$

$10 = B(-3+1)(9+1)$

$(-2)(10)B = 10$

$B = -\frac{1}{2}$

Coeff of s^2

$3A + B + 4C = 0$

$C = -2$

$D = +1$

$\frac{15}{2} - \frac{1}{2} + 4C = 0$

$\frac{14}{2} + 4C = 0$

$4C = -\frac{14}{2}$

$C = -\frac{14}{8}$

$3\left(\frac{5}{2}\right) + B\left(\frac{-1}{2}\right) + 4C = 0$

Scanned by CamScanner

$$\left[\frac{5}{2} L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{s+3}\right] + L^{-1}\left[\frac{-2s+1}{s^2+1}\right]\right]$$

$$\left\{\frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t} + L^{-1}\left[\frac{-2s+1}{s^2+1}\right]\right\}$$

$$\left\{\frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t} + L^{-1}\left[\frac{-2s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]\right\}$$

$$\left[\frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t} + \sin t - 2\cos t\right].$$

$$D^2y + 2Dy + 2y = 5\sin t. \quad y(0) = y'(0) = 0$$

$$y''(t) + 2y'(t) + 2y(t) = 5\sin t$$

$$L[y''(t)] + 2L[y'(t)] + 2L[y(t)] = 5L[\sin t]$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)]$$
$$+ 2L[y(t)] = 5\frac{1}{s^2+1}$$

$$L[y(t)][s^2+2s+2] = \frac{5}{s^2+1}$$

$$L[y(t)] = \frac{5}{(s^2+1)(s^2+2s+2)}$$

$$y(t) = \frac{5}{(s^2+1)(s^2+2s+2)}$$

$$\frac{5}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

$$5 = (As+B)(s^2+2s+B) + (s^2+1)(Cs+D)$$

$$As^3 + 2As^2$$

Scanned by CamScanner

$$\frac{5}{(s^2+1)(s^2+2s+2)} \qquad \frac{5}{(s^2+1)} * \frac{1}{s^2+2s+2}$$

$$5\,L^{-1}\left(\frac{1}{s^2+1}\right) * L^{-1}\left[\frac{1}{s^2+2s+2}\right] \qquad s^2+2s+1+1 \quad (s+1)^2+1$$

$$5\sin t * L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$5\sin t * e^{-t}\sin t$$

$$5\int_0^t \sin e^{-u}\sin u \;\sin(t-u)\,du$$

$$\frac{5}{2}\int_0^t e^{-u}[\cos(u-t+u) - \cos(u+t-u)]\,du$$

$$\frac{5}{2}\int_0^t e^{-u}[\cos(2u-t) - \cos t]\,du$$

$$\frac{5}{2}\int_0^t e^{-u}\cos(2u-t) - e^{-u}\cos t\,du$$

$$\int e^{-at}\cos(bt)\,dt \qquad \int e^{-u}\cos(2u-t) \qquad a=-1 \quad b=2u-t$$

$$\frac{5}{2}\left[\frac{e^{-ut}}{u^2+(2u-t)^2}(-u\cos(2u-t))\right]$$

with partial eqns.

$A+C=0$ $\qquad A=-2$

$2A+B+D=0$ $\qquad B=1$

$2A+2B+C=0$ $\qquad C=2$

$2B+D=5$ $\qquad D=3$

$$L^{-1}\left[\frac{-2s+1}{s^2+1}\right] + L^{-1}\left[\frac{2s+3}{s^2+2s+2}\right]$$

$$= -2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] + 2L^{-1}\left[\frac{s+1-1}{(s+1)^2+1}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -2e^{-t}\cos t - 2e^{-t}\sin t + 3e^{-t}\sin t.$$

Scanned by CamScanner

① $(D^2+2D+5)y=0$, $y=2$, $\frac{dy}{dt}=-4$ at $t=0$

② $(D^2-2D+1)y=e^t$, $y(0)=2$, $y'(0)=1$

③ $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=2$, $x=0$, $\frac{dx}{dt}=5$ at $t=0$

④ $\ddot{x}+4\dot{x}+13x=26$, $x(0)=3$ $\dot{x}(0)=4$.

1(a) $(D^2+2D+5)y=0$

$y''(t)+y'(t)+5y(t)=0$

$L[y''(t)]+L[y'(t)]+5L[y(t)]=0$

$[s^2L[y(t)]-sy(0)-y'(0)]+2[sL[y(t)]-y(0)]$
$+5L[y(t)]=L[0]$

$(s^2+2s+5)L[y(t)]-2s+4-2=0$

$L[y(t)](s^2+s+5)=2s-4+2$

$L[y(t)]=\frac{2s-\cancel{4}+\cancel{2}}{s^2+s+5}$

$L[y(t)]=\frac{2s-2}{s^2+s+5}$

$y(t)=L^{-1}\left[\frac{2s-2}{s^2+s+5}\right]$

$\frac{2s-2}{s^2+s+5}=\frac{As+B}{s^2+s+5}$

$2s-2=(As+B)(s^2+s+5)$

$=As^3+As^2+5As+Bs^2+Bs+5B$

Coeff of s^3
$A=0$

Coeff of s^2
$A+B=0$
$0+B=0$
$B=0$

$5A+B=2$

$5B=-2$
$B=\frac{-2}{5}$

Scanned by CamScanner

$$y(t) = 2L^{-1}\left[\frac{s}{s^2+2s+1+4}\right]$$

$$= 2L^{-1}\left[\frac{s}{(s+1)^2+4}\right]$$

$$= 2e^{-t}L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$y(t) = 2e^{-t}\cos 2t.$$

2. $(D^2-2D+1)y = e^t$. $y(0)=2$ $y'(0)=1$

$$y''(t) - 2y'(t) + y(t) = e^t$$

$$L[y''(t) - 2y'(t) + y(t)] = L[e^t]$$

$$[s^2 L[y(t)] - sy(0) - y'(0)] - 2[sL[y(t)] - y(0)] + L[y(t)] = \frac{1}{s-1}$$

$$L[y(t)]\{s^2-2s+1\} - 2s-1+4 = \frac{1}{s-1}$$

$$L[y(t)][s-1]^2 = \frac{1}{s-1} + 2s-3$$

$$= \frac{(2s-3)(s-1)}{s-1}$$

$$L[y(t)] = \frac{2s^2-5s+4}{(s-1)^3}$$

$$\text{Let } y(t) = L^{-1}\left[\frac{2s^2-5s+4}{(s-1)^3}\right]$$

By partial fraction method,

$$\frac{2s^2-5s+4}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$$2s^2-5s+4 = A(s-1)^2 + B(s-1) + C$$

Scanned by CamScanner

put $s=1$

$2-5+4=C$

$C=1$

Comparing the coeff

coeff of s^2

$2=A$

$A=2$

coeff of s

$-5=2A+B$

$2(2)+B=-5$

$B=-5-4$

$B=-9$

$$L^{-1}\left[\frac{2}{s-1}\right]+L^{-1}\left[\frac{-9}{(s-1)^2}\right]+L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$2e^{t}-9L^{-1}\left[\frac{1}{(s-1)^2}\right]+L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$2e^{t}-9e^{t}t+\frac{t^2}{2}e^{t}$$

3a) $(D^2x+3Dx+2x)=2$

$y''(t)+3y'(t)+2y(t)=2$

$L[y''(t)+3y'(t)+2y(t)]+3[sL[y(t)]-y(0)]$
$+2L[y(t)]=2L[1]$

$[s^2L[y(t)]-s(0)-5]+3[sL(y(t)]-0]+2L(y(t)$

$=\frac{2}{s}$

$L[y(t)]\{s^2+3s+2\}-5=\frac{2}{s}$

s^2+2s+s
s^2+3s

$L[y(t)]=\frac{2-5s}{s(s+1)(s+2)}$

$y(t)=L^{-1}\left[\frac{2-5s}{s(s+1)(s+2)}\right]$

Scanned by CamScanner

$$\frac{2-6s}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$2-6s = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Put $s=0$

$$2 = A(1)(2) + 0 + 0$$

$$A = 1$$

Put $s=-1$

$$2+6 = B(-1)(1)$$

$$B = -7$$

Put $s=-2$

$$2-6(-2) = C(-2)(-2+1)$$

$$14 = 2C$$

$$C = 6$$

$$L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{-7}{s+1}\right] + L^{-1}\left[\frac{6}{s+2}\right]$$

$$1 - 7e^{-t} + 6e^{-2t}$$

(4) $y''(t) + 4y'(t) + 3y(t) = 26$

$$L[y''(t) + 4y'(t) + 3y(t)] = L[26]$$

$$[s^2 L[y(t)] - sy(0) - y'(0)] + 4[sL[y(t)] - y(0)] + 3L[y(t)] = L[26]$$

$$L[y(t)] = \frac{26-16s}{s(s^2+4s+3)}$$

$$\frac{26-16s}{s(s^2+4s+3)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+3}$$

$$26-16s = A(s^2+4s+3) + (Bs+C)s$$

Scanned by CamScanner

Put $s=0$

$16=13A$

$A=2$

Coeff of s^2

$0=A+B$

$B=-2$

Coeff of s

$-16=4A+C$

$-16=8+C$

$C=-8$

$$L^{-1}\left[\frac{2}{s}\right]+L^{-1}\left[\frac{2s-8}{s^2+4s+13}\right]$$

$$2(1)-2L^{-1}\left[\frac{s}{s^2+4s+13}\right]-8L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$2-2L^{-1}\left[\frac{s}{(s+2)^2+3^2}\right]-8L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$2-2e^{-2t}\cos 3t-8e^{-2t}\sin 3t.$$

* $(D^2+2D+5)y=e^{-t}\sin t$. $y(0)=0$ $y'(0)=1$

$$y''(t)+2y'(t)+5y(t)=e^{-t}\sin t$$

$$L[y''(t)]+2L[y'(t)]+5L[y(t)]=L[e^{-t}\sin t].$$

$$\left[s^2L[y(t)]-s[y(0)-y'(0)]\right]+2\left[L[y(t)]-y(0)\right]$$

$$+5[y(t)]=L[e^{-t}\sin t]$$

$$L[y(t)](s^2+2s+5)-1 \qquad 2\left[\frac{1}{s^2+1}\right]\Big|_{s\to s+1}$$

$$L[y(t)][s^2+2s+5]=\frac{1}{(s+1)^2+1}$$

$$=\frac{1}{s^2+2s+1+1}$$

$$=\frac{1}{s^2+2s+2}+1$$

Scanned by CamScanner

$$L[y(t)](s^2+2s+5) = \frac{1+s^2+2s+2}{s^2+2s+2}$$

$$L[y(t)] = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

$$= As^3+2As^2+5As+Bs^2+2Bs+5B$$
$$+ Cs^3+2Cs^2+2Cs$$
$$+ Ds^2+2Ds+2D$$

Coeff of s^3 coeff of s^2

$A+C=0$ $\quad 2A+B+2C+D=1$

Coeff of s $\quad$ ~~Coeff of constant~~

$5A+2B+2C+2D=2$ $\quad 3=5B+2D$

$5B+2D=3$

$B+D=1$

$2A+B-2A+D=1$

$B=0$

$B+D=1$ $\quad D=0$

$5A+2B-2A+2D=2$

$3A+2B+2D=2$ $\quad A+C=0$

$A=0$ $\quad C=0$

$B=1/3$

$D=2/3$

$$L^{-1}\left[\frac{1}{3(s^2+2s+1)}\right] + L^{-1}\left[\frac{2}{3(s^2+2s+5)}\right]$$

$$\frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$\frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\sin 2t$$

Scanned by CamScanner

* $y'' + y' = t^2 + 2t$ where $y = 4$ $y' = 2$ at $t = 0$

$$y''(t) + y'(t) = t^2 + 2t$$

$$L[y''(t)] + L[y'(t)] = L[t^2 + 2t]$$

$$[s^2 L[y(t)] - s y(0) - y'(0)] + s L[y(t) - y(0)]$$

$$= \frac{2}{s^3} + \frac{2}{s^2}$$

$$L[y(t)](s^2 + s) - 4s + 2 - 4 = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L[y(t)](s^2 + s) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2$$

$$L[y(t)]\, s(s+1) = \frac{2 + 2s + 4s^4 + 2s^3}{s^3}$$

$$L[y(t)] = \frac{4s^4 + 2s^3 + 2s + 2}{s^3(s)(s+1)}$$

$$L[y(t)] = \frac{4}{s+1} + \frac{2}{s(s+1)} + \frac{2}{s^3(s+1)} + \frac{2}{s^4(s+1)}$$

$$y(t) = L^{-1}\left[\frac{4}{s+1}\right] + L^{-1}\left[\frac{2}{s(s+1)}\right] + L^{-1}\left[\frac{2}{s^3(s+1)}\right] + \left[\frac{2}{s^4(s+1)}\right]$$

$$= \cancel{L}$$

$$s(s+1)\, L[y(t)] = \frac{2 + 2s + 2s^3 + 4s^4}{s^3}$$

$$s(s+1)\, L[y(t)] = \frac{2(s+1)}{s^3} + \frac{2s^3(1+2s)}{s^3}$$

Scanned by CamScanner

$$L[y(t)] = \frac{2(s+1)}{s^4(s+1)} + \frac{2(3(1+2s))}{s(s)(s+1)}$$

$$L[y(t)] = \frac{2}{s^4} + \frac{2(1+2s)}{s(s+1)}$$

$$L[y(t)] = \frac{2}{s^4} + \frac{2}{s(s+1)} + \frac{4s}{(s^2+1)}$$

$$L^{-1}\left[\frac{2}{s^4}\right] + L^{-1}\left[\frac{2}{s(s+1)}\right] + L^{-1}\left[\frac{4s}{s^2+1}\right]$$

$$2\frac{t^3}{3!} + L^{-1}\left[\frac{2}{s(s+1)}\right] + 4L^{-1}\left[\frac{1}{s+1}\right]$$

$$2\frac{t^3}{3!} + 4e^{-t} + \frac{2}{2} = 2e^{-t}.$$

$$\frac{2t^3}{3!} + 2e^{-t} + 2.$$

$2 = \frac{A}{s} + \frac{B}{s+1}$

$A(s+1) + B(s)$

$A(1) = 2$

$A = 2$

$\frac{1}{2s} + \frac{-2}{s+1}$

$B = 2$

$B = -2$

* $(D^2+4)y = \sin 2t$. $y(0) = 3$, $y'(0) = 4$.

Sol: $y''(t) + 4y(t) = \sin 2t$

$$[s^2 L[y(t)] - s y(0) - y'(0)] + 4[L[y(t)]] = L[\sin 2t]$$

$$L[y(t)](s^2+4) - 3s - 4 = \frac{2}{s^2+4}$$

$$L[y(t)] = \frac{2}{(s^2+4)^2} + \frac{3s}{s^2+4} + \frac{4}{s^2+4}$$

(1)

$$\frac{2}{(s^2+4)^2} = \cancel{\frac{As^3+Bs^2+Cs+D}{s^4+8s^2+16}}$$

$s^4+16+8s^2$

Q:-

By convolution method

$$\frac{1}{4}\int_0^t \sin 4u \; \sin(4t-4u)\, du$$

$$\frac{1}{4}\int_0^t \frac{1}{2}\left(\cos(4t+4u-4t-4u) - \cos(4u+4t-4u)\right) du$$

$$\frac{1}{4}\int_0^t \frac{1}{2}\left(\cos(4u-4t) - \cos 4t\right) du$$

$$\frac{1}{4}\times\frac{1}{2}\left[\frac{\sin(8u-4t)}{8} - u\cos 4t\right]_0^t$$

$$\frac{1}{8}\left[\frac{\sin 4t}{8} - t\cos 4t - \frac{\sin 4t}{8} - 0\right]$$

$$= \frac{1}{8}\left[t\cos 4t - \frac{2\sin 4t}{8}\right]$$

$$= 2\times\frac{1}{8}\left[\frac{\sin 4t}{8} - t\cos 4t\right]$$

$$3U^{-1}\left[\frac{s}{s^2+4}\right] = 3\cos 4t.$$

$$\cancel{e^{-t}(\cos 4t + \tfrac{1}{2}\sin 4t)}$$

from ①

$$\cancel{\frac{s}{s^2+4} + \frac{1}{s^2}}$$

$$3\cos 4t + \sin 4t.$$

Scanned by CamScanner

① $(D^2+D-2)y = 3\cos 3t - 11\sin 3t \quad y(0)=0,\ y'(0)=$

② $y''' - 3y'' + 3y' - y = t^2 e^t \quad y(0)=1,\ y'(0)=0$

$y''(0) = -2$

③ $(D^2+1)x = t\cos 2t \quad x(0) = x'(0) = 0.$

1A) $y''(t) + y'(t) - 2y(t) = 3\cos 3t - 11\sin 3t$

$L[y''(t)] + [y'(t)] - 2L[y(t)] = L[3\cos 3t] - 11L[\sin 3t]$

$[s^2L[y(t)] - sy(0) - y'(0)] + [sL[y(t)] - y(0)]$

$- 2L[y(t)]$

$= \frac{3s}{s^2+9} - \frac{33}{s^2+9}$

$L[y(t)][s^2+s-2] - 0 - 6 = \frac{3s-33}{s^2+9}$

$(s^2+s-2)L[y(t)] = \frac{3s-33}{s^2+9} + 6$

$(s^2+s-2)L[y(t)] = \frac{3s-33+6s^2+9}{s^2+9}$

$L[y(t)] = \frac{6s^2-3s-24}{(s^2+9)(s^2+s-2)}$

$\frac{6s^2-3s-24}{(s^2+9)(s^2+s-2)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+s-2} + \frac{C}{s-1} + \frac{D}{s+2}$

$6s^2-3s-24 = (As+B)(s^2+9) + (Cs+D)(s^2+s-2)$

$6s^2-3s-24 = As^3+9As+Bs^2+9B + Cs^3+Cs^2-2Cs$

$+Ds^2+Ds-2D$

Coeff of s^3	Coeff of s^2	Coeff of s	Const
$A+C=0$	$B+C+D=6$	$9A-2C+D=-3$	$9B-2D=-24$
		$-9C-2C+D=-3$	
$B=\frac{-57}{43}$	$C=\frac{111}{86}$	$-7C+D=-3$	

Scanned by CamScanner

$$\frac{6s^2+3s+21}{(s^2+9)(s-1)(s+2)} = \frac{As+B}{s^2+9} + \frac{C}{s-1} + \frac{D}{s+2}$$

Put $s=1$

$6+3+21 = C(10)(-1)$

$C = -3$

Coeff put $s=-2$

$6(4)+3(-2)+21 = D(-3)(13)$

$D = -1$

Coeff of s^3

$A + C + D = 0$

$A - 3 - 1 = 0 \Rightarrow A = 4$

Coeff of s^2

$6 = B + 2C + 2D$

$B = 11$

$$L^{-1}\left[\frac{4s+11}{s^2+s^2}\right] - 3L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$4\cos 3t + \frac{1}{3}\sin 3t - 3e^{t} - e^{-2t}$$

$(D^2+1)x = t\cos 2t$

$y''(t) + y(t) = t\cos 2t$

$s^2 L[y(t)] - sy(0) - y'(0) + Ly(t) = L(t\cos 2t)$

$L[y(t)][s^2+1] = \frac{d}{ds}\frac{s}{s^2+4}$

$$L[y(t)] = \frac{s^2+4-s(2s)}{(s^2+4)^2(s^2+1)}$$

$$= \frac{s^2-2s^2+4}{(s^2+4)^2(s^2+1)}$$

$$L[y(t)] = \frac{(s^2+4)^2}{(s^2+4)(s^2+1)}$$

$$\frac{(s-2)^2}{(s^2+4)(s^2+1)^2} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

Scanned by CamScanner

$(s-2)^2 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$

$s^2-4s+4 = As^3+As+Bs^2+B + Cs^3+4Cs+Ds^2+4D$

Eq coeff of s^3 $\quad s^2 \quad s$

$A+C=0$ $\quad B+D=1$ $\quad A+4C=-4$

$A+4A=-4$

$-3A=-4$

$A=\frac{4}{3}$ $\quad B+4D=4$

$C=\frac{-4}{3}$ $\quad B+D=1$ $\quad B+4B=4$

$B+4D=4$ $\quad -3B=4$

$B=-\frac{4}{3}$

$B=0$ $\quad D=1$ $\quad D=\frac{4}{3}$

$B+4(1-B)=4$

$B+1-B=$

$$L^{-1}\left[\frac{-\frac{4}{3}s}{s^2+4}\right] + L^{-1}\left[\frac{-\frac{4}{3}s+1}{s^2+1}\right]$$

$$-\frac{4}{3}\cos 2t + \frac{1}{3}L^{-1}\left[\frac{-4s+3}{s^2+1}\right]$$

$$-\frac{4}{3}\cos 2t + \frac{1}{3}\{-4\cos t + 3\sin t\}$$

* $y'''(t) - 3y''(t) + 3y'(t) - y(t) = t^2e^t$

$L[y'''(t)] - 3L[y''(t)] + 3L[y'(t)] - L[y(t)] = t^2e^t$

$\Rightarrow s^3L[y(t)] - s^2y(0) - sy'(0) - y''(0)$

$[s^2L[y(t)] - sy(0) - y'(0)] +$

$3[sL[y(t)] - y(0)] - L[y(t)] = \frac{2}{(s-1)^3}$

$(s^3-3s^2+3s-1)\,L[y(t)] - s^2+3s+1 = \frac{2}{(s-1)^3}$

Scanned by CamScanner

$$(s-1)^3 L[y(t)] = \frac{2}{(s-1)^3} + s^2 - 2s + 1$$

$$L[y(t)] = \frac{2}{(s-1)^6} + \frac{s^2-2s+1}{(s-1)^3}$$

$$\frac{2}{(s-1)^6} + \frac{(s-1)^2}{(s-1)^3}$$

$$\frac{2}{(s-1)^6} + \frac{1}{s-1} - \frac{s}{(s-1)^3}$$

$$\frac{2}{(s-1)^6} + \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

$$y(t) = L^{-1}\left[\frac{2}{(s-1)^6}\right] + L^{-1}\left[\frac{1}{s-1}\right]$$

$$\frac{t^5 e^t}{60} + e^t - te^t - \frac{t^2}{2} e^t$$

Solution of integral equations

An integral eqn is of the form

$$y = f(t) + \int_0^t F(u) G(t-u)\, du$$

$$= f(t) + F(t) * G(t)$$

Scanned by CamScanner

* Solve the integral eqn

$$y'(t) = \sin t + \int_0^t y(t-u)\cos u \, du \quad \text{where } y(0)=0$$

Sol:

$$y'(t) = \sin t + y(t) * \cos t$$

$$L[y'(t)] = L[\sin(t)] + L[y(t) * \cos t]$$

$$L[y'(t)] = L[\sin t] + L[y(t)] \cdot L[\cos t]$$

$$s\,L[y(t)] - \cancel{y(0)}^{\,0} = \frac{1}{s^2+1} + L[y(t)] \frac{s}{s^2+1}$$

$$s\,L[y(t)] - \frac{s}{s^2+1} L[y(t)] = \frac{1}{s^2+1}$$

$$L[y(t)] \left\{ s - \frac{s}{s^2+1} \right] = \frac{1}{s^2+1}$$

$$L[y(t)] \cdot \left[\frac{s^3+s-s}{s^2+1} \right] = \frac{1}{s^2+1}$$

$$L[y(t)] \left\{ \frac{s^3}{s^2+1} \right] = \frac{1}{s^2+1}$$

$$L[y(t)] = \frac{1}{s^3}$$

$$y(t) = L^{-1}\left[\frac{1}{s^3}\right]$$

$$y(t) = \frac{t^2}{2}$$

* $$f'(t) = t + \int_0^t f(t-u)\cos u \, du \quad f(0)=4$$

Sol: $$y'(t) = t + \int_0^t y(t-u)\cos u \, du \quad y(0)=4$$

$$L[y'(t)] = L[t] + y(t) * \cos t$$

$$L[y'(t)] = L[t] + L[y(t) * \cos t]$$

Scanned by CamScanner

$$s^2 L[y(t)] - y(0) = L[t] + L[y(t)] \cdot L[\cos t]$$

$$s L[y(t)] - 4 = \frac{1}{s^2} + L[y(t)] \frac{s}{s^2+1}$$

$$s L[y(t)] - \frac{s}{s^2+1} L[y(t)] = \frac{1}{s^2} + 4$$

$$L[y(t)] \left[s - \frac{s}{s^2+1} \right] = \frac{1+4s^2}{s^2}$$

$$L[y(t)] \left[\frac{s^3+s-s}{s^2+1} \right] = \frac{1+4s^2}{s^2}$$

$$L[y(t)] \left[\frac{s^3}{s^2+1} \right] = \frac{1+4s^2}{s^2}$$

$$L[y(t)] = \frac{1+4s^2}{s^2} \times \frac{s^2+1}{s^3}$$

$$L[y(t)] = \frac{(1+4s^2)(s^2+1)}{s^5}$$

$$= \frac{s^2+1+4s^4+4s^2}{s^5}$$

$$L[y(t)] = \frac{4s^4+5s^2+1}{s^5}$$

$$= \frac{4s^4}{s^5} + \frac{5s^2}{s^5} + \frac{1}{s^5}$$

$$L[y(t)] = \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}$$

$$y(t) = 4(1) + \frac{5t^2}{2} + \frac{t^4}{24}$$

Scanned by CamScanner

$$y(t) = at + \int_0^t y(u)\sin(t-u)\,du$$

$$y(t) = at + y(t) * \sin(t)$$

$$L[y(t)] = L[at] + L[y(t)] \cdot L[\sin t]$$

$$L[y(t)] = a\frac{1}{s^2} + L[y(t)]\frac{1}{s^2+1}$$

$$L[y(t)] - L[y(t)]\frac{1}{s^2+1} = \frac{a}{s^2}$$

$$L[y(t)]\left[1 - \frac{1}{s^2+1}\right] = \frac{a}{s^2}$$

$$L[y(t)]\left[\frac{s^2+1-1}{s^2+1}\right] = \frac{a}{s^2}$$

$$L[y(t)] = \frac{a}{s^2} \times \frac{s^2+1}{s^2}$$

$$L[y(t)] = \frac{a(s^2+1)}{s^4}$$

$$y(t) = a\left\{L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s^4}\right]\right.$$

$$= at + \frac{at^3}{6} = a\left(t + \frac{t^3}{6}\right)$$

4.10. $y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)\,du$

$$y(t) = 1 - e^{t}\int_0^t \sin(t-u)\, y(u)\,du.$$

$$y'(t) = \sin t + \int_0^t y(t-u)\cos 2u\,du \quad y(0) = 0.$$

Scanned by CamScanner

* Solve the integral eqn :

$$y(t) = t + \frac{1}{6}\int_0^t (t-u)^3\, y(u)\, du$$

Sol

$$y(t) = t + \frac{1}{6}\, t^3 * y(t)\, du$$

$$L[y(t)] = L[t] + \frac{1}{6}\left\{L(t^3) \cdot L[y(t)]\right\}$$

$$L[y(t)] = \frac{1}{s^2} + \frac{1}{6}\left\{\frac{6}{s^4} \times L[y(t)]\right.$$

$$L[y(t)] - \frac{L[y(t)]}{s^4} = \frac{1}{s^2}$$

$$L[y(t)]\left[1 - \frac{1}{s^4}\right] = \frac{1}{s^2}$$

$$L[y(t)]\,\frac{s^4 - 1}{s^4 \cdot s^2} = \frac{1}{s^2}$$

$$L[y(t)] = \frac{s^2}{s^4 - 1}$$

$$L[y(t)] = L^{-1}\left[\frac{s^2}{s^4 - 1}\right]$$

$$L[y(t)] = \left(\frac{s^2}{(s^2-1)(s^2+1)}\right)$$

$$s^2 + 1 \quad s^2$$

$$L^{-1}\left(\frac{s^2+1}{(s^2-1)(s^2+1)}\right) - L^{-1}\left(\frac{1}{(s^2+1)(s^2-1)}\right)$$

$$L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{(s^2+1)(1^2-1)}$$

Scanned by CamScanner

$$\frac{s^2}{(s^2+1)(s^2-1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1}$$

$$s^2 = (As+B)(s^2-1) + (Cs+D)(s^2+1)$$

$$s^2 = As^3 - As + Bs^2 - B + Cs^3 + Cs + Ds^2 + D$$

Coeff of s^3	s^2	s	const
$A+C=0$	$B+D=1$	$-A+C=0$	$-B+D=0$
$B-C=0$	$B+D=1$, $D=1-B$, $D=1/2$	$-A-A=0$, $-2A=0$, $A=0$	$-B+B=0$, $-2B=0$, $B=0$

$$\frac{2s+2}{s^2+1} + \frac{-2s-1}{s^2-1} \qquad \frac{1/2}{s^2+1} + \frac{1/2}{s^2-1}$$

$$\frac{2(s+1)}{s^2+1} - \frac{2s+1}{s^2-1} \qquad y(t) = \frac{1}{2}[\sin t + \sinh t]$$

$$y(t) = 2\{\cos t + \sin t\} - \{2\cosh t + \sinh t\}$$

Solution of simultaneous ODE.

* $2x' - y' + 3x = 2t$, $x(0) = 1$ — (1)

$x' + 2y' - 2x - y = t^2 - t$, $y(0) = 1$ — (2)

Sol: for (1) $2[L[x(t)] - x(0)] - [sL[y(t)] - y(0)] + 3L[x(t)] = \frac{2}{s^2}$

$$L[x(t)]\{2s+3\} + L[y(t)]\{-s\} = \frac{2}{s^2} + 1$$

$$= \frac{2+s^2}{s^2}$$

(3)

$$y(t) = e^t - 2\int_0^t y(u)\cos(t-u)\,du$$

$$y(t) = e^t - 2\int_0^t y(u)\cos(t-u)\,du$$

$$y(t) = e^t - 2\{y(t) * \cos t\}$$

$$L[y(t)] = L[e^t] - 2\left[L[y(t)] \cdot L[\cos t]\right]$$

$$L[y(t)] = \frac{1}{s-1} - 2\left[L[y(t)] \cdot \frac{s}{s^2+1}\right]$$

$$L[y(t)] + 2L[y(t)]\frac{s}{s^2+1} = \frac{1}{s-1}$$

$$L[y(t)]\left[1 + \frac{2s}{s^2+1}\right] = \frac{1}{s-1}$$

$$L[y(t)]\left[\frac{s^2+1+2s}{s^2+1}\right] = \frac{1}{s-1}$$

$$L[y(t)]\left[\frac{(s+1)^2}{s^2+1}\right] = \frac{1}{s-1}$$

$$L[y(t)] = \frac{1}{s-1}\,\frac{s^2+1}{(s+1)^2}$$

$$L[y(t)] = \frac{s^2+1}{(s-1)(s+1)^2}$$

$$L[y(t)] = \frac{s^2+1}{(s^2+2s+1)(s-1)}$$

$$\frac{s^2+1}{(s^2+2s+1)(s-1)} = \frac{As+B}{s^2+2s+1} + \frac{C}{s-1}$$

$$s^2+1 = (As+B)(s-1) + C(s^2+2s+1)$$

$$s^2+1 = As^2 - As + Bs - B + Cs^2 + 2Cs + C$$

Put $s = 1$

$$2 = C(1+2+1) \Rightarrow 4C = 2 \Rightarrow C = 1/2$$

Scanned by CamScanner

Coeff of s^0

$A+C=1$

$A+\frac{1}{2}=1$

$A=\frac{1}{2}$

$$L^{-1}\left[\frac{\frac{1}{2}s+\frac{1}{2}}{s^2+2s+1}\right]+L^{-1}\left[\frac{\frac{1}{2}}{s-1}\right]$$

$$\frac{1}{2}L^{-1}\left[\frac{s-1}{s^2+2s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right]$$

$$\frac{1}{2}\left[L^{-1}\frac{s+1-1}{(s+1)^2}\right.$$

$$\frac{s^2+1}{(s+1)^2(s-1)}=\frac{A}{(s+1)}+\frac{B}{(s+1)^2}+\frac{C}{s-1}$$

$$s^2+1=A(s+1)(s-1)+B(s-1)+C(s+1)^2$$

$A(s^2-1)$ $\quad$ $C(s^2+2s+1)$

~~coeff~~ put $s=-1$

$2=-2B$

$B=-1$

put $s=1$

$2=C(2)^2$

$4C=2$

$C=1/2$

coeff of s^2

$A+C=1$

$A=1-\frac{1}{2}=1-\frac{1}{2}$

$$L^{-1}\left[\frac{1}{2(s+1)}\right]+L^{-1}\left[\frac{-1}{(s+1)^2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right]$$

$$\frac{1}{2}e^{-t}-e^{-t}t+\frac{1}{2}e^{t}$$

2) $y(t)=1-e^{t}\int_0^t \sin(t-u)\,y(u)\,du$

Sol: $L[y(t)]=\frac{1}{s}-\frac{1}{s-1}\left[L[\sin t]\,L[y(t)]\right]$

$L[y(t)]=\frac{1}{s}-\frac{1}{s-1}\left[L[\sin t]\,L[y(t)]\right]$

Scanned by CamScanner

$$L[y(t)] + \frac{1}{s-1}\left(\frac{1}{s^2+1} L[y(t)]\right) = \frac{1}{s}$$

$$L[y(t)]\left[1 + \frac{1}{s-1}\frac{1}{s^2+1}\right] = \frac{1}{s}$$

$(s-1)(s^2+1)$
s^3+s-s^2
s^3-s^2

$$L[y(t)]\left[\frac{s^3-s^2+s-1+1}{(s^2+1)(s-1)}\right] = \frac{1}{s}$$

$$L[y(t)]\left[\frac{s^3-s^2+s}{(s^2+1)(s-1)}\right] = \frac{1}{s}$$

$$L[y(t)] = \frac{1}{s}\frac{(s^2+1)(s-1)}{s^3-s^2+s}$$

$$L[y(t)] = \frac{1}{s^2}\frac{(s^2+1)(s-1)}{(s^2-s+1)}$$

$$\frac{s^3-s^2+s-1}{s^2(s^2-s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2-s+1}$$

$s^3-s^2+s-1 = A(s)(s^2-s+1) + B(s^2-s+1) + (Cs+D)(s^2)$

~~Put s=0~~ As^3-As^2+As $\quad Cs^3+Ds^2$

$A = 2/3$ $\quad$ Put $s=0$ $\quad$ Coeff of s^3 $\quad$ Coeff of s^2

$-1 = B1$ $\quad A+C=1$ $\quad -A+B+D=-1$

$O+C=1$ $\quad -A-1+D=-1$

$B=-1/3$ $\quad C=1/3$ $\quad -A+D=0$

$O+D=0$

$D=0$

coeff of s

$A-B=1$

$A+1=1$

$A=1-1$

$A=0$

$$\frac{1}{2}+\frac{1}{2}L^{-1}\left[\frac{s}{s^2-2s+3}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s^2-2s+3}\right]$$

$$L^{-1}\left[\cancel{-\frac{2}{s^2}}\right] + L^{-1}\left[\cancel{\frac{s}{s^2-s+1}}\right] = \frac{2}{3} + \frac{1}{2}L^{-1}\left[\frac{s-1}{(s-1)^2+2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s-1)^2+2}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{(s-1)^2+2}\right]$$

$$= \frac{2}{3} + \frac{1}{2}e^{t}\cos\sqrt{2}t + \frac{1}{2}\frac{1}{\sqrt{2}}\sin\sqrt{2}t - \frac{1}{3\sqrt{2}}\sin\sqrt{2}t$$

Scanned by CamScanner

$$= L^{-1}\left[\frac{1}{s^2}\right] + \cancel{L^{-1}\left[\frac{1}{s}\right]} \cdot$$

$$\cancel{t + L^{-1}\left[\frac{s}{s^2 - s + 1}\right]}$$

$$\cancel{t + L^{-1}}$$

$$\frac{s}{s^2+s+1} = \frac{As+B}{s^2+s+1}$$

$$s = As + B \qquad B = 0$$

$$A = 1$$

$$\frac{s}{\ }$$

* $y'(t) = \sin t + \int_0^t y(t-u)\cos u\,du \qquad y(0) = 0$

$$s[L[y(t)] - y(0)] = L[\sin t] + L[y(t)]\,L[\cos t]$$

$$s L[y(t)] = \frac{1}{s^2+1} + \frac{s}{s^2+1} L[y(t)]$$

$$L[y(t)]\left[s - \frac{s}{s^2+1}\right] = \frac{1}{s^2+1}$$

$$L[y(t)]\left[\frac{s^3+s-s}{s^2+1}\right] = \frac{1}{s^2+1}$$

$$L[y(t)] = \frac{1}{s^3}$$

$$y(t) = L^{-1}\left[\frac{1}{s^3}\right]$$

$$y(t) = \frac{t^2}{2}$$

8

* Solve the simultaneous differential eqns by using Laplace transform $\frac{dy}{dt} - 2x = \sin 2t$, $x(0) = 1$

$$\frac{dx}{dt} - 2y = \cos 2t, \quad y(0) = 0$$

Sol:- $y'(t) - 2x = \sin 2t$ — ①

$x' - 2y = \cos 2t$ — ②

Scanned by CamScanner

① $\Rightarrow L[y'(t)] + 2L[x(t)] = L[\sin 2t]$

$$sL[y(t)] - y(0) + 2L[x(t)] = \frac{2}{s^2+4}$$

$$\Rightarrow L[x(t)]\{2\} + L[y(t)]\{s\} = \frac{2}{s^2+4} \quad \text{--- ②}$$

② $L[x'(t)] - 2L[y(t)] = L[\cos 2t]$

$$s\{L[x(t)] - x(0)\} - 2L[y(t)] = \frac{s}{s^2+4}$$

$$L[x(t)]\{s\} + L[y(t)]\{-2\} = \frac{s}{s^2+4} + 1$$

$$L[x(t)]\{s\} + L[y(t)]\{-2\} = \frac{s^2+4+s}{s^2+4}$$

Solve for $L[x(t)]$ & $L[y(t)]$

$$\Delta = \begin{vmatrix} 2 & s \\ s & -2 \end{vmatrix} = -4 - s^2 = -(4+s^2)$$

$$\Delta x = \begin{vmatrix} \frac{2}{s^2+4} & s \\ \frac{s^2+4+s}{s^2+4} & -2 \end{vmatrix} = \frac{-4}{(s^2+4)} - \frac{s^3+4s+s^2}{s^2+4}$$

$$= \frac{-4 - s^3 - 4s - s^2}{s^2+4}$$

By cramers rule $L[x(t)] = \frac{\Delta x}{\Delta}$

$$= \frac{-(4+s^3+4s+s^2)}{-(s^2+4)(4+s^2)}$$

$$= \frac{s^3+s^2+4s+4}{(s^2+4)^2}$$

$$= \frac{s^2(s+1)+4(s+1)}{(s^2+4)^2}$$

Scanned by CamScanner

$$= \frac{(s^2+4)(s+1)}{(s^2+4)^2} = \frac{s+1}{s^2+4}$$

$$x(t) = L^{-1}\left[\frac{s+1}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+4}\right] + L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$x(t) = \cos 2t + \frac{1}{2}\sin 2t.$$

solve for $L[y(t)]$

$$\Delta y = \begin{vmatrix} 2 & \frac{2}{s^2+4} \\ s & \frac{s^2+4+s}{s^2+4} \end{vmatrix}.$$

$$= \frac{2s^2+8+2s}{s^2+4} - \frac{2s}{s^2+4}$$

$$= \frac{2s^2+8+2s-2s}{s^2+4} = \frac{2s^2+8}{s^2+4}$$

$$= \frac{2(s^2+4)}{s^2+4} = 2.$$

By cramer's rule $L[y(t)] = \frac{\Delta y}{\Delta}$

$$L[y(t)] = \frac{-2}{(s^2+4)}$$

$$y(t) = L^{-1}\left[\frac{-2}{s^2+4}\right]$$

$$= -\sin 2t.$$

* $2\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1, \quad x(0) = 0$

$\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0, \quad y(0) = 0.$

Scanned by CamScanner

$3x'(t) + y'(t) + 2x(t) = 1$ —— (1)

$x'(t) + 4y'(t) + 3y(t) = 0$ —— (2)

(1) $\Rightarrow$ $3[sL[x(t)] - x(0)] + [sL[y(t)] - y(0)] + 2L[x(t)] = 1$

$L[x(t)](3s+2) + sL[y(t)] = 1$

$L[x(t)](3s+2) + sL[y(t)] = \frac{1}{s}$ —— (3)

(2) $\Rightarrow$ $sL[x(t)] - x(0) + 4[sL[y(t)] - y(0)] + 3L[y(t)] = 0$

$L[y(t)][4s+3] + sL[x(t)] = 0$.

$sL[x(t)] + L[y(t)](4s+3) = 0$ —— (4)

By cramer's rule

$$\Delta = \begin{vmatrix} 3s+2 & s \\ s & 4s+3 \end{vmatrix} = 12s^2 + 9s + 8s + 6 - s^2$$

$$= 11s^2 + 17s + 6 = (s+1)(11s+6)$$

$$\Delta x = \begin{vmatrix} \frac{1}{s} & s \\ 0 & 4s+3 \end{vmatrix} = \frac{4s+3}{s}$$

$$\frac{\Delta x}{\Delta} = \frac{4s+3}{s(11s^2+17s+6)} \qquad (s+1)(11s+6)$$

$$= \frac{4s+3}{s(s+1)(11s+6)}$$

Scanned by CamScanner

$$\frac{4s+3}{s(s+1)(11s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{11s+6}$$

$$4s+3 = A(s+1)(11s+6) + B(s)(11s+6) + C(s)(s+1)$$

Put $s=0$

$$3 = A(6)$$

$$6A = 3$$

$$A = \frac{1}{2}$$

Put $s=-1$

$$-4+3 = -B(-11+6)$$

$$-1 = 5B$$

$$B = \frac{-1}{5}$$

Put $s = \frac{-6}{11}$

$$4\left(\frac{-6}{11}\right)+3 = C\left(\frac{-6}{11}\right)\left(\frac{-6}{11}+1\right)$$

$$-\frac{24}{11}+3 = -\frac{6C}{11}\left(\frac{5}{11}\right)$$

$$-24+33 = -\frac{30C}{11}$$

$$-30C = (9)(11)$$

$$C = \frac{99}{30} = -\frac{33}{10}$$

$$\begin{array}{r} 33 \\ -24 \\ \hline 9 \end{array}$$

$$x(t) = L^{-1}\left[\frac{1}{2}\left(\frac{1}{s}\right)\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{33}{10}L^{-1}\left[\frac{1}{11s+6}\right]$$

$$= \frac{1}{2}(1) - \frac{1}{5}e^{-t} - \frac{33}{10}\,\frac{1}{11}\left[\frac{1}{s+\frac{6}{11}}\right]$$

$$= \frac{1}{2} - \frac{1}{5}e^{-t} - \frac{33}{110}e^{-6t/11}$$

$$\Delta y = \begin{vmatrix} 3s+2 & \frac{1}{s} \\ s & 0 \end{vmatrix} = \frac{s}{s} = 1$$

$$\frac{\Delta y}{\Delta} = \frac{1}{(s+1)(11s+6)}$$

Scanned by CamScanner

$$\frac{1}{(s+1)(11s+6)} = \frac{A}{s+1} + \frac{B}{11s+6}$$

$$-1 = A(11s+6) + B(s+1)$$

Put $s = -1$

$$-1 = A(-11+6)$$

$$-5A = -1$$

$$A = \frac{+1}{5}$$

Put $s = \frac{-6}{11}$

$$1 = B\left(\frac{-6}{11}+1\right)$$

$$-1 = B\left(\frac{5}{11}\right)$$

$$B = \frac{-11}{5}$$

$$L[y(t)] = \frac{+1}{5}\left[\frac{1}{s+1}\right] - \frac{11}{5}\left[\frac{1}{11s+6}\right]$$

$$= \frac{+1}{5}e^{-t} - \frac{1}{5}e^{-6t/11}$$

$$= \frac{1}{5}\left(e^{-t} - e^{-6t/11}\right)$$

Hw,

① $x' - 2x + 3y = 0$ $\quad x(0) = 8,\ y(0) = 3$

$y' - y + 2x = 0$

② $\frac{dx}{dt} + y = \sin t$ $\quad x = 2,\ y = 0$ at $t = 0$

$\frac{dy}{dt} + x = \cos t$

* $Dx - y = e^t$ $\quad x(0) = 1$

$Dy + x = \sin t$ $\quad y(0) = 0$

Sol:

$$x'(t) - y(t) = e^t$$

$$sL[x(t)] - x(0) - L[y(t)] = L[e^t]$$

$$sL[x(t)] - L[y(t)] = \frac{1}{s-1} + 1$$

$$sL[x(t)] - L[y(t)] = \frac{1+s-1}{s-1} = \frac{s}{s-1}$$

Scanned by CamScanner

Unit-I Laplace Transforms

1) Problems on omitting exponential functions, 's' and $\frac{1}{s}$

Formulae:-

* $L^{-1}[e^{-as}F(s)] = L^{-1}[F(s)]\Big|_{t \to t-a}$.

* $L^{-1}[sF(s)] = \frac{d}{dt}\{L^{-1}[F(s)]\}$

* $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$

* Find $L^{-1}\left[\frac{e^{-s}}{(s+2)(s+3)}\right]$

Soln

$$L^{-1}\left[\frac{1}{(s+2)(s+3)}\right]_{t \to t-1}$$

$$\frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$1 = A(s+3) + B(s+2)$

put $s=-3$ put $s=-2$

$1 = -1B$ $1 = A$

$B = -1$

$$L^{-1}\left[\frac{1}{s+2}\right] + L^{-1}\left[\frac{-1}{s+3}\right]\Bigg|_{t \to t-1}$$

$$e^{-2t} - e^{-3t}\Big|_{t \to t-1}$$

$$e^{-2(t-1)} - e^{-3(t-1)}$$

$$e^{-2t+2} - e^{-3t+3}$$

Scanned by CamScanner

* $L^{-1}\left[\frac{e^{-3s}}{s^2(s^2+1)}\right]$

Soln: $L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]\Bigg|_{t\to t-3}$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+1}$$

$$1 = As(s^2+1)+B(s^2+1)+(Cs+D)(s^2)$$

Put $=0$

$B(1)=1$

$B=1$

Coeff of s

$A=0$

Coeff of s^3

$A+C=0$

$C=0$

Coeff of s^2

$B+D=0$

$1+D=0$

$D=-1$

$$L^{-1}\left[\frac{1}{s^2}\right]+L^{-1}\left[\frac{-1}{s^2+1}\right]\Bigg|_{t\to t-3}$$

$t - \sin t$

$(t-3)-\sin(t-3)$

Add in assign

① $y''-3y'+2y=4t+e^{3t}$ $\quad y(0)=1,\ y'(0)=-1$

* $L^{-1}\left[\frac{e^{-\pi s}(s+1)}{s^2+s+1}\right]$

* $L^{-1}\left[\frac{s+1}{s^2+s+1}\right]_{t\to(t-\pi)}$

Scanned by CamScanner

$$L^{-1}\left[\frac{s+1}{s^2+s+1}\right]_{t \to t-\pi}$$

Denominator: $s^2+s+1 = \left(s+\frac{1}{2}\right)^2 + \frac{3}{4} + 1$

$$L^{-1}\left[\frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]_{t \to t-\pi}$$

$$L^{-1}\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right] + L^{-1}\left[\frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]_{t \to t-\pi}$$

$$e^{-\frac{1}{2}t} L^{-1}\left[\frac{s}{s^2+\frac{3}{4}}\right] + L^{-1}\left[\frac{1/2}{s^2+3/4}\right]_{t \to t-\pi}$$

$a^2 = \frac{3}{4}$

$a = \frac{\sqrt{3}}{2}$

$$e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}\, e^{-\frac{1}{2}t}\, \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$e^{-\frac{1}{2}t}\left[\cos\left(\frac{\sqrt{3}t}{2}\right) + \sin\left(\frac{\sqrt{3}}{2}t\right)\right]_{t \to t-\pi}$$

$$e^{-\frac{1}{2}(t-\pi)}\left[\cos\left[\frac{\sqrt{3}(t-\pi)}{2}\right] + \sin\left[\frac{\sqrt{3}(t-\pi)}{2}\right]\right]$$

* $$L^{-1}\left[\frac{1+e^{-\pi s}}{s^2+1}\right]$$

Sol:- $$L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right]$$

$$\sin t + L^{-1}\left[\frac{1}{s^2+1}\right]_{t \to t-\pi}$$

$$\sin t + \sin(t-\pi) = 0$$

Scanned by CamScanner

* $L^{-1}\left[\frac{s}{(s+2)^4}\right]$

Sol: $\frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^4}\right]$

$\frac{d}{dt}\left[e^{-2t} L^{-1}\left[\frac{1}{s^4}\right]\right]$

$\frac{d}{dt}\left[e^{-2t}\frac{t^3}{6}\right]$

$\frac{1}{6}\frac{d}{dt}\left[e^{-2t} t^3\right]$

$\frac{1}{6}\left[-2e^{-2t} t^3 + e^{-2t} 3t^2\right]$

$\frac{e^{-2t} t^2}{6}\left[-2t+3\right]$

* $L^{-1}\left[\frac{s^2}{(s-2)^3}\right]$

Sol: $\frac{d^2}{dt^2}\left[\frac{1}{(s-2)^3}\right]$

$\frac{d^2}{dt^2}\left[e^{+2t}\frac{t^2}{2}\right]$

$\frac{1}{2}\frac{d^2}{dt^2}\left[e^{2t} t^2\right]$

$\frac{1}{2}\frac{d}{dt}\left[2t e^{2t} + 2e^{2t} t^2\right]$

$\frac{1}{2}\left\{\frac{d}{dt}(2te^{2t}) + \frac{d}{dt} e^{2t} t^2\right]$

Scanned by CamScanner

$$\frac{1}{2}\left\{2(e^{2t}+2te^{2t})+2e^{2t}t^2+e^{2t}2t\right\}$$

$$\frac{1}{2}\left\{2e^{2t}+2te^{2t}+2t^2e^{2t}+2te^{2t}\right\}$$

$$\frac{1}{2}\left\{2e^{2t}+2t^2e^{2t}+4te^{2t}\right\}$$

$$\frac{2e^{2t}}{2}\left\{1+t^2+2t\right\}$$

$$e^{2t}(t^2+2t+1)$$

HW:-

* $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ * $L^{-1}\left[\frac{s^2}{(5s-6)^4}\right]$

* Sol:- $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Sol: $\int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]ds$

$$\int_0^t \frac{1}{a}\sin at\,dt$$

$$\frac{1}{a}\left[-\frac{\cos at}{a}\right]_0^t = \frac{1}{a}\left[\frac{-\cos at}{a}+\frac{1}{a}\right]$$

$$= \frac{1}{a^2}\left[1-\cos at\right].$$

* $L^{-1}\left[\frac{1}{s^2(s+a)}\right]$

Sol: $\int_0^t\int_0^t L^{-1}\left[\frac{1}{s+a}\right]$

$$\int_0^t\int_0^t e^{-at}\,dt$$

$$\int_0^t\left[\frac{e^{-at}}{-a}\right]_0^t dt = \int_0^t\left[\frac{e^{-at}}{-a}+\frac{1}{a}\right]dt \quad \boxed{\frac{1}{a^2}(e^{-at}-1)+\frac{t}{a}}$$

$$\int_0^t \frac{e^{-at}}{-a}dt+\int_0^t\frac{1}{a}dt$$

$$\frac{1}{a^2}\left[e^{-at}\right]_0^t+\left[\frac{t}{a}\right]_0^t$$

Scanned by CamScanner

* $L^{-1}\left[\frac{1}{s(s+2)^3}\right]$

$$\int_0^t L^{-1}\left(\frac{1}{(s+2)^3}\right)$$

$$\int_0^t e^{-2t}\frac{t^2}{2}\,dt$$

$$\frac{1}{2}\int_0^t e^{-2t}t^2\,dt$$

$u = t^2$ $\quad v = e^{-2t}$

$u' = 2t$ $\quad v_1 = \frac{e^{-2t}}{-2}$

$u'' = 2$ $\quad v_2 = \frac{e^{-2t}}{4}$

$v_3 = \frac{e^{-2t}}{-8}$

$$= \left[\frac{t^2e^{-2t}}{-2} - (2t)\frac{e^{-2t}}{4} + (2)\frac{e^{-2t}}{-8}\right]_0^t$$

$$= \left[\frac{t^2e^{-2t}}{-2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]_0^t$$

$$= \frac{t^2e^{-2t}}{-2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} - \left[\frac{t^2}{-2} - \frac{t}{2} - \frac{1}{4}\right]$$

$$= \left[\frac{t^2e^{-2t}}{-2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{4}\right]$$

$$= \frac{e^{-2t}}{2}\left[-t^2 - t - \frac{1}{2}\right] + \frac{1}{2}\left[t^2 + t + \frac{1}{2}\right]$$

* $L^{-1}\left[\frac{s^2-s+2}{s(s+2)(s-3)}\right]$

Scanned by CamScanner

Sol: $\int_0^t L^{-1}\left[\frac{s^2-s+2}{(s+2)(s-3)}\right]$

$$\frac{s^2-s+2}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3}$$

$$s^2-s+2 = A(s-3)+B(s+2)$$

Put $s=-2$

$$4+2+2 = -5A$$

$$8 = -5A$$

$$A = \frac{-8}{5}$$

Put $s=3$

$$9-3+2 = 5A$$

$$8 = 5A$$

$$A = \frac{8}{5}$$

$$L^{-1}\left[\frac{s^2-s+2}{(s+2)(s-3)}\right] = -\frac{8}{5}e^{-2t} + \frac{8}{5}e^{3t}$$

$$\int_0^t -\frac{8}{5}e^{-2t}\,dt + \int_0^t \frac{8}{5}e^{3t}\,dt$$

$$-\frac{8}{5}\left[\frac{e^{-2t}}{-2}\right]_0^t + \frac{8}{5}\left[\frac{e^{3t}}{3}\right]_0^t$$

$$-\frac{8}{5}\left[\frac{e^{-2t}}{-2}+\frac{1}{2}\right] + \frac{8}{5}\left[\frac{e^{3t}}{3}-\frac{1}{3}\right]$$

$$\frac{-8e^{-2t}}{-10} - \frac{8}{10} + \frac{8e^{3t}}{15} - \frac{8}{15}$$

Applications of Laplace Transform

⇒ **Electric circuits:-**

$t \rightarrow$ time
$q \rightarrow$ charge
$i \rightarrow$ current $\Rightarrow \frac{dq}{dt}$

} Variables

Constants:
- $R \rightarrow$ resistance — Ω
- $L \rightarrow$ inductance — H
- $C \rightarrow$ capacitance — F

Scanned by CamScanner

Kirchoff's Law

Single circuit

The sum of voltage drop is equal to the applied voltage

(or)

The sum of current flowing into the junction is zero

Multiple circuit

The amount of current flowing into the junction is equal to amount of current flowing out of it.

* In a electric circuits when EMF $E(t)$, resistance (R) and inductance (L). If the switch is connected at $t=0$ and disconnected at $t=a$. Find the current i at any instant

Sol: The given problem is of

RL circuit:-

By kirchoff's law

$$\boxed{iR + L\cdot\frac{di}{dt} = E(t)} \Rightarrow iR + Li'(t) = E(t) \quad (1)$$

$$E(t) = \begin{cases} E, & 0<t<a \\ 0, & t<a \end{cases} \quad (2)$$

Taking L.T on both sides for (1)

Scanned by CamScanner

$$R\,L[i(t)] + L[sL\{i(t)-i(0)\}] = L[E(t)]$$

$$\Rightarrow L[i(t)]\{R+sL\} = \int_0^{\infty} e^{-st} E(t)\,dt$$

$$= E\int_0^{\infty} e^{-st}\,dt$$

$$= E\left[\frac{-e^{-st}}{s}\right]_0^{a}$$

$$L[i(t)]\{R+sL\} = \frac{E}{s}\left[1-e^{-sa}\right]$$

$$i(t) = L^{-1}\left[\frac{E(1-e^{-sa})}{s(R+sL)}\right]$$

$$= E\left\{L^{-1}\left(\frac{1}{s(R+sL)}\right) - L^{-1}\left[\frac{e^{-as}}{s(R+sL)}\right]\right\} \quad (B)$$

I II

(I) $\Rightarrow \dfrac{1}{s(R+sL)} = \dfrac{A}{s} + \dfrac{B}{R+sL}$

$1 = A(R+sL) + B(s)$

Put $s=0$

$A = 1/R$

s term

$0 = AL + B$

$0 = \frac{L}{R} + B$

$B = -\frac{L}{R}$

$$I \Rightarrow \frac{1}{R}L^{-1}\left[\frac{1}{s}\right] - \frac{L}{R}\frac{1}{L}\left[\frac{1}{\left(s+\frac{R}{L}\right)}\right]$$

$$\frac{1}{R}(1) - \frac{e^{-Rt/L}}{R}$$

(II) $\Rightarrow L^{-1}\left[\dfrac{e^{-as}}{s(R+sL)}\right] = L^{-1}\left[\dfrac{1}{s(R+sL)}\right]\Bigg|_{t\to t-a}$

$$= \frac{1}{R}\frac{d}{ds}\frac{1}{L}e^{-Rt/L}\Bigg|_{t\to t-a} \quad \frac{1}{R}\frac{e^{-R(t-a)/L}}{-R}$$

$$= \frac{1}{R}\frac{1}{L}\frac{e^{-Rt/L}}{(-R/L)}\Bigg|_{t\to t-a}$$

Scanned by CamScanner

$$i(t) = I - II$$

$$= E\left\{\frac{-e^{-Rt/L}}{R} + \frac{e^{-Rt/L + Ra/L}}{R}\right\}$$

$$= \frac{E}{R} \times e^{-Rt/L} \cdot \left[-1 + e^{Ra/L}\right]$$

* A voltage $E.e^{-at}$ is supplied at time $t=0$ to a circuit containing inductance L and resistance R. show that the current at any instance $i = \frac{E}{(R-aL)}\left[e^{-at} - e^{-Rt/L}\right]$ where $i=0$ at $t=0$.

Sol: The given prblm is of LR circuit.

$$iR + L\frac{di}{dt} = Ee^{-at} \Rightarrow iR + L\,i'(t) = Ee^{-at} \quad \text{--- (1)}$$

$$L[i(t)] + L[sL[i(t)] - i(0)] = L[Ee^{-at}]$$

$$L[i(t)]\{R+sL\} = E\frac{1}{s+a}$$

$$L[i(t)] = \frac{E}{s+a(R+sL)}$$

$$i(t) = L^{-1}\left[\frac{E}{(s+a)(sL+R)}\right]$$

$$i(t) = E\,L^{-1}\left[\frac{1}{(s+a)(sL+R)}\right]$$

$$\frac{1}{(s+a)(sL+R)} = \frac{A}{s+a} + \frac{B}{sL+R}$$

$$\Rightarrow 1 = A(sL+R) + B(s+a)$$

put $s=-a$

$1 = A(sL+R)$

$A = \frac{1}{sL+R}$

$A = \frac{1}{R-aL}$

By s term

$AL+B = 0$

$\frac{L}{R-aL} + B = 0$

$B = \frac{-L}{R-aL}$

$$E\left\{L^{-1}\left[\frac{1}{(R-aL)(s+a)}\right] - L^{-1}\left[\frac{L}{(R-aL)(sL+R)}\right]\right.$$

$$\frac{E}{R-aL}\left\{e^{-at} - \frac{L}{R}e^{-Rt/L}\right\}$$

$$\frac{E}{R-aL}\left\{e^{-at} - e^{-Rt/L}\right\}$$

* The current 'i' and charge 'q' are in a series circuit containing an inductance 'L', capacitance 'C' and EMF 'E'. Express 'i' and 'q' in terms of 't' given that LC and E are constants. Such that the initial value of i and q is zero.

Sol: Given problem is of LC circuit

$$\boxed{L\frac{di}{dt} + \frac{q}{C} = E} \qquad i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E$$

$$L\,q''(t) + \frac{q(t)}{C} = E \quad (1), \quad i(0) = q(0) = 0$$

$$L\left\{s^2 L[q(t)] - sq(0) - q'(0)\right\} + \frac{1}{C}L[q(t)] = L[E]$$

$$L[q(t)]\left\{Ls^2 + \frac{1}{C}\right\} = \frac{E}{s}$$

$$L[q(t)] = \frac{EC}{s(CLs^2+1)}$$

Scanned by CamScanner

$$L[q(t)] = \frac{\quad}{s\left[s^2 + \frac{1}{LC}\right]}$$

$$L[q(t)] = \frac{E}{sL\left[s^2 + \frac{1}{LC}\right]}$$

$$q(t) = \frac{E}{L} L^{-1}\left[\frac{1}{s\left[s^2 + \frac{1}{LC}\right]}\right]$$

$$= \frac{E}{L}\int_0^t \frac{1}{\left(s^2 + \frac{1}{LC}\right)}\, dt$$

$$a^2 = \frac{1}{LC} \quad a = \frac{1}{\sqrt{LC}}$$

$$\frac{E}{L}\frac{1}{(1/\sqrt{LC})}\int_0^t \sin\left(\frac{t}{\sqrt{LC}}\right) dt$$

$$\frac{E}{L}\sqrt{LC}\left[\frac{-\cos\left(\frac{t}{\sqrt{LC}}\right)}{(1/\sqrt{LC})}\right]_0^t$$

$$\frac{E}{L}(LC)\left[-\cos\left(\frac{t}{\sqrt{LC}}\right)\right]_0^t$$

$$EC\left(-\cos\left(t/\sqrt{LC}\right) + 1\right)$$

$$q(t) = EC\left[1 - \cos\left(t/\sqrt{LC}\right)\right]$$

$$i = \frac{dq}{dt} = \frac{d\, EC\left[1 - \cos\left(t/\sqrt{LC}\right)\right]}{dt}$$

$$= EC\left(+\sin\left(t/\sqrt{LC}\right)\right)\frac{1}{\sqrt{LC}}$$

$$q(t) = E\sqrt{C}\,\sin\left[\frac{t}{\sqrt{LC}}\right]$$

Scanned by CamScanner

* An alternating emf $E \sin at$ is applied to an inductance L, capacitance C in series. Show that the current at any instance in the circuit is $\frac{Ea}{L(n^2-a^2)}$ $[\because \cos at = \cos nt]$, $n^2 = \frac{1}{LC}$

Sol

LC circuit

$$L\frac{di}{dt} + \frac{q}{C} = E \sin at \qquad i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E \sin at$$

$$L q''(t) + \frac{q(t)}{C} = E \sin at$$

$$L[L[q''(t)]] + \frac{L[q(t)]}{C} = L[E \sin at]$$

$$L[s^2 L[q(t)] - s q'(0) - q'(0)] + \frac{1}{C} L[q(t)] = E \frac{a}{s^2+a^2}$$

$$L[q(t)] \left\{ Ls^2 + \frac{1}{C} \right\} = \frac{Ea}{s^2+a^2}$$

$$L[q(t)] = \frac{Ea}{s^2+a^2\left[s^2+\frac{1}{C}\right]}$$

$$L[q(t)] = \frac{Ea}{L[s^2+a^2]\left[s^2+\frac{1}{LC}\right]}$$

$$L[q(t)] = \frac{Ea}{L[s^2+a^2][s^2+n^2]}$$

$$\frac{1}{(s^2+a^2)(s^2+n^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+n^2}$$

$$1 = (As+B)(s^2+n^2) + (Cs+D)(s^2+a^2)$$

$$1 = As^3 + An^2 + Bs^2 + Bn^2 + Cs^3 + Csa^2 + Ds^2 + Da^2$$

Coeff of s^3: $A + C = 0$, $C = 0$

Coeff of s^2: $B + D = 0$, $D = \frac{-1}{n^2+a^2}$

Coeff of s: $An^2 + Ca^2 = 0$, $An^2 - Aa^2 = 0$, $A(n^2-a^2) = 0$, $A = 0$

$Bn^2 + Da^2 = 1$

$Bn^2 - Ba^2 = 1$ $\quad B(n^2-a^2) = 1 \Rightarrow B = \frac{1}{n^2-a^2}$

Scanned by CamScanner

$$L^{-1}\left[\frac{\frac{1}{n^2-a^2}}{s^2+a^2}\right] - L^{-1}\left[\frac{\frac{1}{n^2-a^2}}{s^2+n^2}\right]$$

$$= \frac{1}{(n^2-a^2)(s^2+a^2)} - \frac{1}{(n^2-a^2)(s^2+n^2)}$$

$$L(q(t)) = \frac{E_a}{L}\left[\frac{1}{n^2-a^2}\right] L^{-1}\left[\frac{1}{s^2+a^2} - L^{-1}\left[\frac{1}{s^2+n^2}\right]\right]$$

$$L[q(t)] = \frac{E_a}{L(n^2-a^2)}\left[\frac{1}{a}\sin at - \frac{1}{n}\sin nt\right]$$

$$i(t) = \frac{dq}{dt} = \frac{E_a}{L(n^2-a^2)}\left[\frac{\cancel{a}\cos at}{\cancel{a}} - \frac{\cancel{n}}{\cancel{n}}\cos nt\right]$$

$$i(t) = \frac{E_a}{L(n^2-a^2)}\left[\cos at - \cos nt\right]$$

* A periodic emf $E_0 \sin at$ is supplied at the time $t=0$ to a circuit consist of resistor and capacitance in series the initial value of the current and the charge 'q' in the condensor are 'zero'. Find the current at any time 't' using Laplace transform.

1) RC circuit

$$iR + \frac{q}{C} = E_0 \sin at \qquad i = \frac{dq}{dt}$$

$$R\frac{dq}{dt} + \frac{q}{C} = E_0 \sin at$$

Scanned by CamScanner

$$R q'(t) + \frac{q(t)}{C} = E_0 \sin at$$

$$R[L[q'(t)] + \frac{L[q(t)]}{C} = E_0 L[\sin at]$$

$$R[s L[q(t)] - q(0)] + \frac{L[q(t)]}{C} = E_0 \frac{a}{s^2+a^2}$$

$$L[q(t)] \left\{ sR + \frac{1}{C} \right\} = \frac{E_0 a}{s^2+a^2}$$

$$L[q(t)] = \frac{E_0 a}{(s^2+a^2)(sR+\frac{1}{C})}$$

$$q(t) = L^{-1}\left[\frac{Ea}{(s^2+a^2)(sR+\frac{1}{C})}\right]$$

$$q(t) = L^{-1}\left[\frac{Ea}{R(s^2+a^2)(s+\frac{1}{RC})}\right]$$

$$\frac{1}{(s^2+a^2)(s+\frac{1}{RC})} = \frac{A}{(s+\frac{1}{RC})} + \frac{Bs+C}{s^2+a^2}$$

$$1 = A(s^2+a^2) + (Bs+C)(s+\frac{1}{RC})$$

Put $s = -\frac{1}{RC}$

$$1 = A\left(\frac{1}{R^2C^2}+a^2\right)$$

$$A = \frac{1}{\left(\frac{1}{R^2C^2}+a^2\right)} = \frac{R^2C^2}{1+a^2R^2C^2}$$

coeff of s^2

$A+B=0$

$$B = \frac{-1}{\left(\frac{1}{R^2C^2}+a^2\right)}$$

$$B = \frac{-R^2C^2}{1+a^2R^2C^2}$$

~~coeff of~~ s Const

$$Aa^2 + C\frac{1}{RC} = 1$$

$$\frac{B}{RC} + C = 0$$

$$C = \frac{-RC}{1+a^2R^2C^2}$$

$$a^2\left(\frac{1}{R^2C^2}+a^2\right) + \frac{C}{RC} = 1$$

$$\frac{C}{RC} = 1 - \frac{a^2}{R^2C^2} + a^4 \Rightarrow \frac{C}{RC} = \frac{R^2C^2 - a^2 + a^4R^2C^2}{R^2C^2}$$

$$C = \frac{R^2C^2 + a^2 + a^4R^2C^2}{RC}$$

Scanned by CamScanner

$$\frac{Ea}{R}\left[C^{-1}\left(\frac{R^2C^2}{s^2+a^2R^2C^2}\right)+\left(\frac{R^2C^2-RC}{s^2+a^2R^2C^2}\right)\right]$$

$$\frac{Ea}{R}\times\frac{-RC}{1+a^2R^2C^2}\left[C^{-1}\left(\frac{RCs}{s^2+a^2}+\frac{1}{s^2+a^2}-\frac{RC}{s+1/RC}\right)\right]$$

$$\frac{EaC}{1+a^2R^2C^2}\left[RC\cos at+\frac{1}{a}\sin at-RC.e^{-t/RC}\right]$$

An inductor of 3H is in series with a resistance of 30 Ω and an EMF of 150 V. Assume that the current is zero (i=0) and time t=0. Find the current at any instant t. $i(t)=5(1-e^{-10t})$

RL circuit:-

$$iR+L\frac{di}{dt}=E(t)$$

$$iR+Li'(t)=E(t)$$

$$i(30)+3\,i'(t)=150$$

Adding LT on both sides

$$L[i]+L[L\,i'(t)]=L[E(t)]$$

$$RL[i(t)]+L[L[i'(t)]]=L[E(t)]$$

$$RL[i(t)]+L\{s\,L[i(t)]-i(t)\}=\int_0^\infty e^{-st}E(t)\,dt$$

$$=E\int_0^\infty e^{-st}\,dt$$

$$=E\left[\frac{-e^{-st}}{\ }\right]^\infty$$

Scanned by CamScanner

$$L[i(t)]\{R+sL\} - 0 = \frac{E}{s}\left[1-e^{-sa}\right]$$

$$L[i(t)] = \frac{E\left[1-e^{-sa}\right]}{s[R+sL]}$$

$$i(t) = L^{-1}\left[\frac{E(1-e^{-sa})}{s[R+sL]}\right]$$

$$= E\left[L^{-1}\left(\frac{1}{s(R+sL)}\right) - L^{-1}\left(\frac{e^{-as}}{s(R+sL)}\right)\right]$$

I $\qquad$ II

$$I \Rightarrow \frac{1}{s(R+sL)} = \frac{A}{s} + \frac{B}{R+sL}$$

$$1 = A(R+sL) + B(s)$$

put $s=0$

$$1 = A(R+0)+0$$

$$1 = AR$$

$$\boxed{A = 1/R}$$

comparing co-efficient of s

$$\Rightarrow 0 = AL + B$$

$$0 = \frac{L}{R} + B$$

$$\boxed{B = -\frac{L}{R}}$$

$$I \Rightarrow L^{-1}\left[\frac{1}{Rs}\right] + L^{-1}\left[\frac{-L/R}{R+sL}\right]$$

$$= \frac{1}{R}L^{-1}\left[\frac{1}{s}\right] - \frac{\cancel{L}}{R}L^{-1}\left[\frac{1}{\cancel{L}} \cdot \frac{1}{s+R/L}\right]$$

$$= \frac{1}{R}(1) - \frac{1}{R}L^{-1}\left[\frac{1}{s+\frac{R}{L}}\right]$$

$$= \frac{1}{R} - \frac{1}{R}e^{-R/L\,t} \quad \text{—— (1)}$$

Scanned by CamScanner

$$\text{II} \Rightarrow L^{-1}\left[\frac{e^{-as}}{s(R+sL)}\right] = L^{-1}\left[\frac{1}{s(R+sL)}\right]\Bigg|_{t \to t-a}$$

$$= \frac{1}{R} - \frac{e^{-\frac{Rt}{L}}}{R}\Bigg|_{t \to t-a}$$

$$= \frac{1}{R} - \frac{e^{-\frac{R(t-a)}{L}}}{R} \quad \text{—— (2)}$$

$i(t) = (1) - (2)$

$$= \cancel{\frac{1}{R}} - \cancel{\frac{e^{-\frac{Rt}{L}}}{R}} - \cancel{\frac{1}{R}} - \left[\cancel{\frac{e^{-\frac{Rt}{L}}}{R}} - \cancel{e^{-\frac{Ra}{L}}}\right]$$

$$= E\left[\frac{1}{R} - \frac{e^{-\frac{Rt}{L}}}{R} - \left[\frac{1}{R} - \frac{e^{-\frac{R(t-a)}{L}}}{R}\right]\right]$$

$$= E\left[-\frac{e^{-\frac{Rt}{L}}}{R} + \frac{e^{-\frac{R(t-a)}{L}}}{R}\right]$$

$$i(t) = \frac{E}{R}\, e^{-\frac{Rt}{L}}\left[-1 + e^{\frac{Ra}{L}}\right]$$

$$= \frac{150}{30}\, e^{-\frac{30t}{3}}\left[-1 + e^{-\frac{30a}{3}}\right]$$

$$= 5e^{-10t}\left[-1 + e^{-10a}\right]$$

Scanned by CamScanner

* Obtain the solution of the equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E, \text{ where } i(0)=0,\ i'(0)$$

L, R, C are constants also $a = \frac{R}{2L}$, $\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$

$k^2 = -\omega^2$

Soln

$i = \frac{dq}{dt} \Rightarrow i\,dt = dq$ so $e = \frac{q}{C}$

$\int_0^t i\,dt = q$

RLC circuit

$$L[Li'(t)] + RL[i(t)] + \frac{1}{C}L\left[\int_0^t i\,dt\right] = L[E]$$

$$L[s\,L[i(t)] - i(0)] + R\,L[i(t)] + \frac{1}{C}\frac{L[i(t)]}{s} = L[E]$$

$$\boxed{\because L\left[\int_0^t f(t)\,dt\right] = \frac{L[f(t)]}{s}}$$

$$L[i(t)]\left[sL + R + \frac{1}{Cs}\right] = \frac{E}{s}$$

$$L[i(t)]\left[Cs^2L + CsR + 1\right] = \frac{E \times Cs}{s}$$

$$L[i(t)]\left[Cs^2L + CsR + 1\right] = CE$$

$\div$ LC on both sides

$$L[i(t)]\left\{s^2 + \frac{sR}{L} + \frac{1}{LC}\right\} = \frac{E}{L}$$

$$L[i(t)] = \frac{E}{L} \times \frac{1}{\left[s^2 + \frac{sR}{L} + \frac{1}{LC}\right]}$$

$$\frac{E}{L}\frac{1}{\left(s^2 + \frac{sR}{L} + \frac{1}{LC}\right)} = \frac{E}{L}\frac{1}{\left(s^2 + \frac{sR}{L} + \frac{1}{LC} + \frac{R^2}{4L^2} - \frac{R^2}{4L^2}\right)}$$

$A^2 = s^2$ $\quad 2AB = \frac{Rs}{L}$

$A = s$ $\quad 2sB = \frac{Rs}{L}$

$B = \frac{R}{2L}$

$B^2 = \frac{R^2}{4L^2}$

$$= \frac{E}{L}\frac{1}{\left(\left[s + \frac{R}{2L}\right]^2 + \frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$= \frac{E}{L}\frac{1}{\left(s + \frac{R}{2L}\right)^2 + \omega^2}$$

Scanned by CamScanner

$$= \frac{E}{L} \times L^{-1}\left[\frac{1}{(s^2+a^2)^2+\omega^2}\right] \quad (*)$$

$$= \frac{E}{L} e^{-at} L^{-1}\left[\frac{1}{s^2+\omega^2}\right]$$

$$q(t) = \frac{E}{L\omega} e^{-at} \sin\omega t.$$

from ~~*~~ eq

Case(i) $\omega^2 > 0$

$$\frac{E}{L} \times L^{-1}\left[\frac{1}{(s^2+a^2)^2+\omega^2}\right]$$

$$= \frac{E}{L\omega} e^{-at} \sin\omega t$$

Case(ii) $\omega^2 < 0$

$$\frac{E}{L} L^{-1}\left[\frac{1}{(s^2+a^2)^2+\omega^2}\right]$$

$$\frac{E}{L} L^{-1}\left[\frac{1}{(s^2+a^2)^2-k^2}\right]$$

$$\frac{E}{Lk} e^{-at} \cancel{\sinh kt} \sinh kt$$

Case (i) $\omega^2 = 0$

$$\frac{E}{L} L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

$$\frac{E}{La} e^{-at} \cancel{\sin at}(t)$$

$$\frac{E(t)}{L} e^{-at}$$

A particle of mass 'm' can ~~be~~ oscillate above the position of equilibrium under the effect of restoring force 'mk^2' times the displacement. It started from the rest

Scanned by CamScanner

by a constant force F which acts for a time 'T' and then ceases. Find the amplitude of the subsequent oscillations.

Sol: Let mass 'm' be mass of particle.

$\frac{dx}{dt} \rightarrow$ velocity $\quad x \rightarrow$ displacement

$\frac{d^2x}{dt^2} \rightarrow$ acceleration.

By newton's second law of motion

$f = ma$ —— (1)

Define the constant force F in terms of unit step function,

$$u(t-T) = \begin{cases} 1, & 0 < t < T \\ 0, & T > t. \end{cases}$$

$L[u(t-T)] = \frac{e^{-sT}}{s}$

Therefore also the restoring force acts opposite to the direction of particle

$\therefore f = F[1-u(t-T)] - mk^2x$ —— (2)

$\therefore$ from (1) & (2)

$ma = F[1-u(t-T)] - mk^2x$

$m\frac{d^2x}{dt^2} = F[1-u(t-T)] - mk^2x$

$m\left(\frac{d^2x}{dt^2} + k^2x\right) = F[1-u(t-T)]$

$\div$ 'm' on both sides

$\frac{d^2x}{dt^2} + k^2x = \frac{F}{m}[1-u(t-T)]$ —— (3)

$x''(t) + k^2x(t) = \frac{F}{m}[1-u(t-T)]$

Taking L.T on both sides

$s^2L[x(t)] - sx(0) - x'(0) + k^2L[x(t)] = \frac{F}{m}L[1-u(t-T)]$

$= \frac{F}{m}\left[\frac{1}{s} - \frac{e^{-sT}}{s}\right]$

$L[x(t)][s^2+k^2] = \frac{F}{ms}\left[\frac{1}{s} - e^{-sT}\right]$

Scanned by CamScanner

$$L[x(t)] = \frac{F}{ms}\frac{(1-e^{-st})}{s^2+k^2}$$

$$x(t) = L^{-1}\left[\frac{F}{ms}\frac{(1-e^{-st})}{(s^2+k^2)}\right]$$

$$x(t) = \frac{F}{m}\left[L^{-1}\left(\frac{1-e^{-st}}{s(s^2+k^2)}\right)\right] \quad \text{—(4)}$$

$$\frac{1}{s(s^2+k^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+k^2}$$

$$1 = A(s^2+k^2) + (Bs+C)s$$

Put $s=0$

$Ak^2 = 1$

$A = \frac{1}{k^2}$

Coeff of s^2

$A+B=0$

$B = -\frac{1}{k^2}$

Coeff of s

$C=0$

$$\frac{1}{s} \quad \frac{1}{k^2(s)} + \left(\frac{-\frac{1}{k^2}s}{s^2+k^2}\right)$$

$$L^{-1}\left(\frac{1}{k^2(s)}\right) - L^{-1}\left(\frac{1}{k^2}\left(\frac{s}{s^2+k^2}\right)\right)$$

$$\frac{1}{k^2} - \frac{1}{k^2}\cos kt$$

Scanned by CamScanner

$$L^{-1}\left[\frac{1-e^{-st}}{s(s^2+k^2)}\right] = L^{-1}\left(\frac{1}{s(s^2+k^2)}\right) - L^{-1}\left[\frac{e^{-st}}{s(s^2+k^2)}\right]$$

$$= L^{-1}\left(\frac{1}{k^2} - \frac{1}{k^2}\cos kt\right) - L^{-1}\left[\frac{1}{s(s^2+k^2)}\right]\Big|_{t\to t-T}$$

$$= \frac{1}{k^2} - \frac{1}{k^2}\cos kt - L^{-1}\left[\frac{1}{k^2} - \frac{1}{k^2}\cos kt\right]_{t\to t-T}$$

$$= \frac{1}{k^2} - \frac{1}{k^2}\cos kt - \frac{1}{k^2} - \frac{1}{k^2}\cos k(t-T)$$

$$= \frac{-1}{k^2}\left[\cos kt - \cos k(t-T)\right]$$

$$x(t) = \frac{F}{m}\frac{-1}{k^2}\left[\cos kt - \cos k(t-T)\right]$$

$$x(t) = \frac{F}{mk^2}\left(\cos k(t-T) - \cos kt\right)$$

* A particle of mass 3g moves on the x-axis and is attracted towards a fixed point in its path with a force whose numerical value '12x'. Assume that the particle initially at rest is $x = 5$. Determine the position of the particle at any time 't'.

(i) when there is no other force $x(t) = 5\cos 2t$

(ii) when there is a damping force which is 6 times the velocity. $x(t) = 5e^{-2t}$

Scanned by CamScanner

Deflection of Beams.

The equation of motion is given by

$$\frac{d^4y}{dx^4} = \frac{w}{EI}$$

w = weight of particle

E = Young's modulus.

I = Moment of inertia.

The solution in terms of 'x'.

* A beam is clamped at its ends, at $s=0$, $s=L$ carries a uniform weight w per unit length. Show that the deflection at any point is given by $y(x) = \frac{w x^2 (l-x)^2}{24EI}$

Sol

$$\frac{dx^4}{dx^4} = \frac{w}{EI}$$

$y''''(x)$. $0 < x < l$

$y''''(x) = w(x) = \frac{w}{EI}$

Initial condition

$y(0) = 0 = y'(0)$

Boundary condition

$y(l) = 0$

$y'(l) = 0$

$$s^4 L[y(x)] - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = L\left[\frac{w}{EI}\right]$$

Let $y''(0) = c_1$

$y'''(0) = c_2$

$$s^4 L[y(x)] - sc_1 - c_2 = \frac{w}{EIs}$$

$$s^4 L[y(x)] = \frac{w}{EIs} + sc_1 + c_2$$

$$L[y(x)] = \frac{w}{EIs^5} + \frac{c_1 s}{s^4} + \frac{c_2}{s^4}$$

Scanned by CamScanner

$L[y(x)] =$

$y(x) = \frac{w}{EI}\left[\frac{1}{5^3}\right] + C_2\left[\frac{1}{5^4}\right]$

$$y(x) = \frac{w}{EI}\frac{x^4}{4!} + C_1\frac{x^2}{2!} + C_2\frac{x^3}{3!} \quad \text{—— (4)}$$

Using Boundary conditions

$y(l) = 0$

$$\frac{wl^4}{24EI} + C_1\frac{l^2}{2} + C_2\frac{l^3}{6} = 0 \quad \text{—— (1)}$$

Diff

$\Rightarrow y'(l) = 0$

$$y'(x) = \frac{w}{EI}\frac{4x^3}{24EI} + C_1\frac{2x}{2} + C_2\frac{3x^2}{6} = 0$$

$$\frac{w}{6EI}\times l^3 + C_1 l + C_2\frac{l^2}{2} = 0 \quad \text{—— (2)}$$

$$(1) \Rightarrow C_1\left(\frac{l^2}{2}\right) + C_2\left(\frac{l^3}{6}\right) = \frac{-wl^4}{24EI}$$

$$(2)\times\frac{l}{2} \Rightarrow C_1\left(\frac{l^2}{2}\right) + C_2\left(\frac{l^3}{4}\right) = \frac{-w}{6EI}$$

$(-)$ $(-)$ $(+)$

$$C_2\left(\frac{l^3}{6} - \frac{l^3}{4}\right) = \frac{-w}{6EI}\left(\frac{l^4}{4EI} + \frac{1}{EI}\right)$$

$$C_2 \cdot l^3\left(\frac{-1}{12}\right) = \frac{wl^4}{EI}\left(\frac{-1}{24} - \frac{12+24}{24\times 12}\right)$$

$$C_2 = \frac{wl}{2EI} \qquad C_2 = \frac{-wl}{2EI}$$

Scanned by CamScanner

Subs: C_2 in (1)

$$\frac{wl^4}{24EI} + \frac{Cl^2}{2} + \left(\frac{-wl}{2EI}\right)\frac{l^3}{6} = 0$$

$$\frac{wl^4}{24EI} + \frac{Cl^2}{2} - \frac{wl^4}{12EI} = 0$$

$$\frac{Cl^2}{2} = \frac{wl^4}{12EI} - \frac{wl^4}{24EI}$$

$$\frac{Cl^2}{2} = \frac{wl^4}{12EI}\left[1 - \frac{1}{2}\right]$$

$$\frac{Cl^2}{2} = \frac{wl^4}{12EI}\left(\frac{1}{2}\right)$$

$$C_1 = \frac{wl^2}{24EI}$$

$$C_1 = \frac{wl^2}{12EI}$$

Subsitute C_1, C_2 in (*)

$$y(x) = \frac{w}{EI} \times \frac{x^4}{24} + \frac{wl^2}{12EI}\left(\frac{x^2}{2}\right) - \frac{wl}{2EI}\frac{l^3}{6}$$

$$= \frac{wx^4}{24EI} + \frac{wl^2x^2}{24EI} - \frac{wl^3x^3}{12EI}\frac{x}{l}$$

$$= \cancel{\frac{wx^2}{12EI}\left(\frac{x^2}{12} + l^2 - \right.}$$

$$= \frac{wx^2}{24EI}\left[x^2 + l^2 - 2lx\right]$$

Scanned by CamScanner

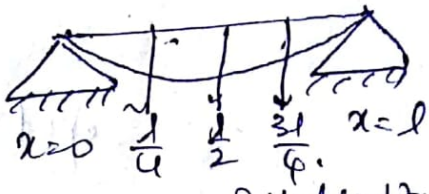
$$\frac{wx^2}{2uEI}(l-x)^2$$

$$y(x) = \frac{wx^2(l-x)^2}{2uEI}$$

* Calculate the maximum deflection of a encastre beam of length "l" feet carrying a uniformly distributed load w (lb/ft) on it central half ft

Sol: Given length of beam = l feet

load = w lb/ft.



Here

$$y''''(x) = \frac{w}{EI}(x), \quad \frac{l}{4} < x < \frac{3l}{4}$$

Initial condition
$y(0) = 0$
$y'(0)$
Boundary condition
$y(l) = 0$
$y'(l) = 0$

Here $w(x) = w\left[u\left(x-\frac{l}{4}\right) - u\left(x-\frac{3l}{4}\right)\right]$

Unit step func

$$L[u(t-a)] = \frac{e^{-sa}}{s}$$

Taking LT on both sides

$$s^4 L[y(x)] - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{w}{EI}\left[L\left[u\left(x-\frac{l}{4}\right)\right] - L\left[u-\frac{3l}{4}\right]\right]$$

$$s^4[L[y(x)] - c_1 s - c_2 = \frac{w}{EI}\left\{\frac{e^{-sl/4}}{s} - \frac{e^{-s3l/4}}{s}\right\}$$

$$s^4 L[y(x)] = \frac{w}{EI}\left\{\frac{e^{-sl/4}}{s} - \frac{e^{-3sl/4}}{s}\right\} + c_1 s + c_2$$

Scanned by CamScanner