Conformal Mapping
- simple transformation
- standard transformation
  - $w = \sin z$
  - $w = \cos z$
  - $w = \sinh z$
  - $w = e^z$
  - $w = z + \frac{1}{z}$

Pr: $w = \sin z$

$$u + iv = \sin z$$
$$= \sin(x+iy)$$
$$= \sin x \cos iy + \cos x \sin iy$$
$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

on comparing,

$$u = \sin x \cdot \cosh y \longrightarrow (1)$$
$$v = \cos x \cdot \sinh y \longrightarrow (2)$$

Elimination of y:-

$$\cosh y = \frac{u}{\sin x} \quad ; \quad \sinh y = \frac{v}{\cos x}$$

$$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

Take $x = c$,

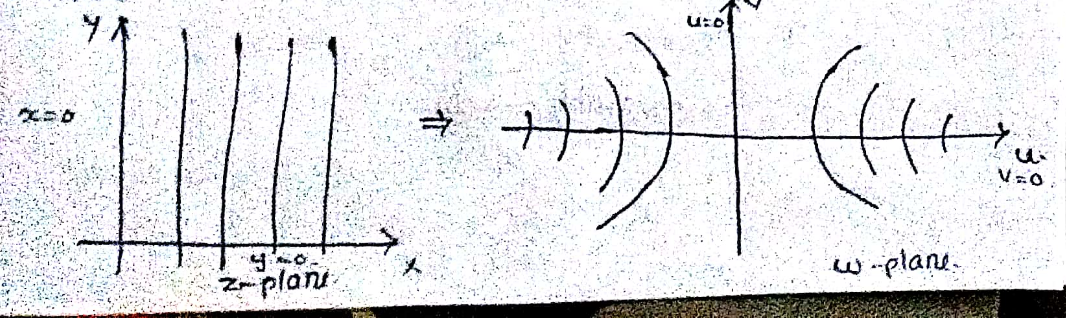
$$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$$

Take, $a^2 = \sin^2 c$; $b^2 = \cos^2 c$.

$$\left(\frac{u^2}{a^2} - \frac{v^2}{b^2}\right) = 1$$

$\therefore$ Equation of hyperbola.

The set of straight lines $(x=c)$ in z-plane is transformed into confocal hyperbola in w-plane.



Elimination of x:-

$$\sin x = \frac{u}{\cosh y}$$

$$\cos x = \frac{v}{\sinh y}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

Take $y = k$ (straight line)

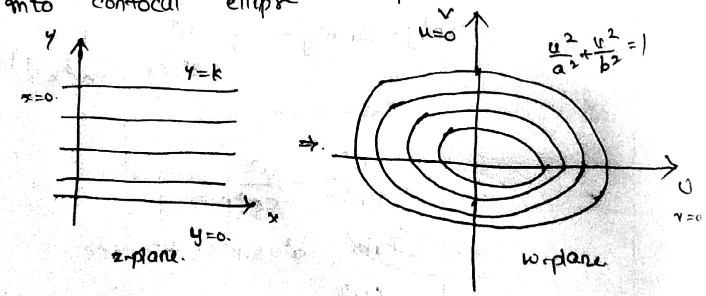
$$\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$$

Take $\cosh^2 c = a^2$, $\sinh^2 c = b^2$

$$\boxed{\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1}$$

$\therefore$ equation of ellipse

$\therefore$ set of straight lines ($y=k$) in z-plane is transformed into confocal ellipse in w-plane.



(8) $w = \cos z$

$u + iv = \cos(x+iy)$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos x \cosh y - i \sin x \sinh y$

$u = \cos x \cosh y \rightarrow (1)$

$v = -\sin x \sinh y \rightarrow (2)$

Elimination of y:-

$$\cosh y = \frac{u}{\cos x} \quad ; \quad \sinh y = \frac{-v}{\sin x}$$

$$\cosh^2 y - \sinh^2 y = 1$$

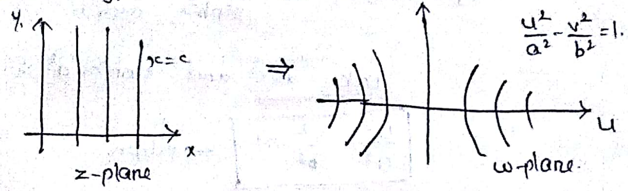
$$\Rightarrow \left(\frac{u}{\cos x}\right)^2 - \left(\frac{-v}{\sin x}\right)^2 = 1 \Rightarrow \frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1$$

Take $x = c$ (straight line).

$$a^2 = \cos^2 c \; ; \; b^2 = \sin^2 c$$

$$\boxed{\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1} \rightarrow \text{hyperbola.}$$

$\therefore$ set of straight lines $(x=c)$ in z-plane transformed into confocal hyperbola in w-plane.



Elimination of x:-

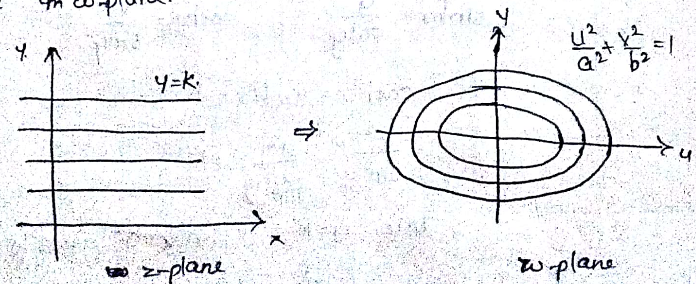
$$\cos x = \frac{u}{\cosh y} \quad ; \quad \sin x = \frac{-v}{\sinh y}$$

$$\cos^2 x + \sin^2 x = 1 \Rightarrow \frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1.$$

tak $y = k$ and $\cosh^2 k = a^2$, $\sinh^2 k = b^2$

$$\boxed{\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1} \rightarrow \text{ellipse.}$$

$\therefore$ set of st. Lines $y = k$ in z-plane transformed into confocal ellipse in w-plane.



(3) $w = \sinh z$ $\qquad$ $\sin(iz) = i\sinh z$

$\sinh z = \frac{1}{i}\sin(iz)$

$= -i\sin(iz)$

$= -i\sin(iz)$

$= -i\sin(i(x+iy))$

$= -i\sin(ix-y)$

$= -i[\sin ix\cos y - \cos ix\sin y]$

$= -i[i\sinh x\cos y - \cosh x\sin y]$.

$= \sinh x\cos y + i\cosh x\sin y$

$u = \sinh x\cos y$ $\qquad$ $v = \cosh x\sin y$.

Elimination of y:-

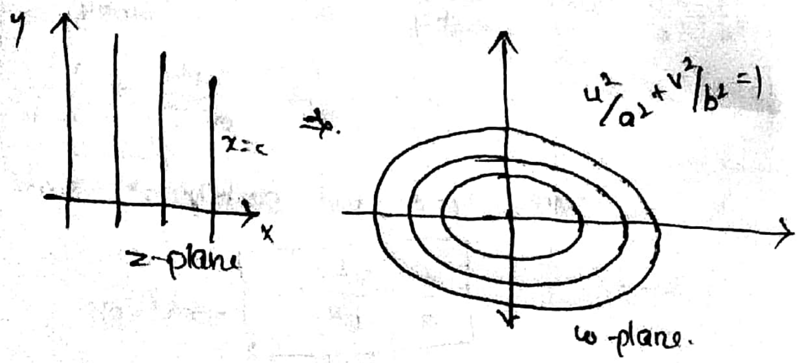
$\cos y = \frac{u}{\sinh x}$; $\sin y = \frac{v}{\cosh x}$

$\cos^2 y + \sin^2 y = 1 \rightarrow \frac{u^2}{\sinh^2 x} + \frac{v^2}{\cosh^2 x} = 1$

take $x = c$ and $\sinh^2 c = a^2$ & $\cosh^2 c = b^2$

$\boxed{\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1} \rightarrow$ ellipse

$\therefore$ set of st. line ($x=c$) in z-plane transformed into confocal ellipse in w-plane.



Elimination of x:-

$\sinh x = \frac{u}{\cos y}$; $\cosh x = \frac{v}{\sin y}$

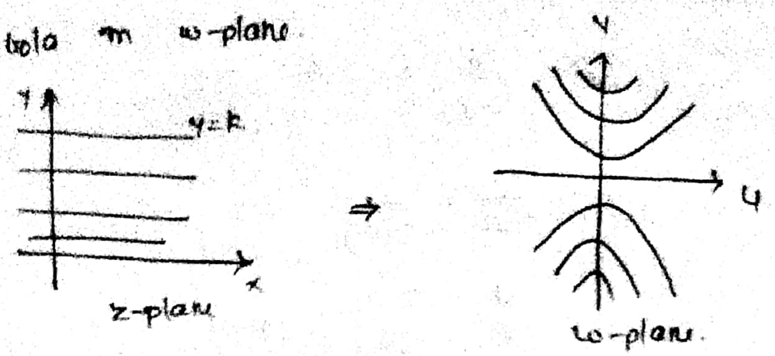
$\cosh^2 x - \sinh^2 x = 1$

$\frac{v^2}{\sin^2 y} - \frac{u^2}{\cos^2 y} = 1$

take $y = k$, $\sin^2 k = a^2$; $\cos^2 k = b^2$

$\left(\frac{v^2}{a^2} - \frac{u^2}{b^2}\right) = 1 \rightarrow$ hyperbola

$\therefore$ set of st. lines $(y=k)$ in z-plane converted into confocal hyperbola in w-plane.



(d) $w = e^z$

$u+iv = e^x \cdot e^{iy}$

$= e^x(\cos y + i\sin y)$

$= e^x\cos y + i e^x \sin y$

$u = e^x\cos y$

$v = e^x \sin y$

Eliminating y;

$\cos y = \frac{u}{e^x}$; $\sin y = \frac{v}{e^x}$

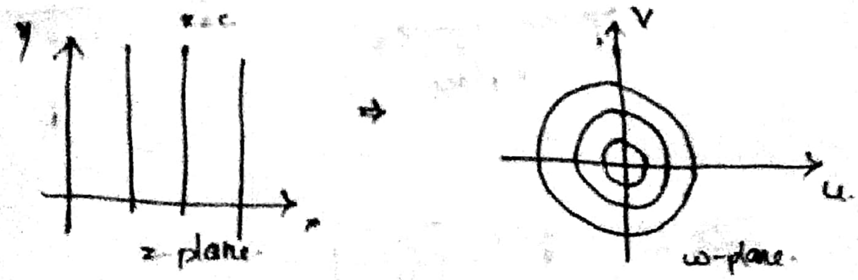
$\cos^2 y + \sin^2 y = 1 \Rightarrow \frac{u^2}{e^{2x}} + \frac{v^2}{e^{2x}} = 1$

$u^2+v^2 = e^{2x} \rightarrow$ circle

case (i):- $x = c$, $u^2+v^2 = e^{2c} \rightarrow$ circle

$u^2+v^2 = k^2 \rightarrow$ circle

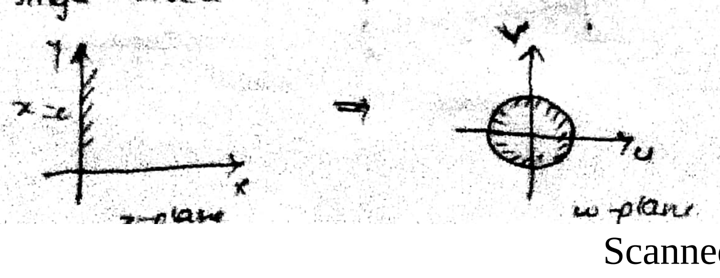
$\therefore$ A set of st. lines $(x=c)$ is in z-plane is transformed into concentric circles in w-plane.



case (ii):- $x=0$, (Y-axis) $u^2+v^2 = e^{2(0)} = 1$.

$u^2+v^2 = 1 \rightarrow$ eq. of circle with unit radius.

$\therefore$ $x=0$ line in z-plane is transformed into a single circle in w-plane



Elimination of x:-

$$e^x = \frac{u}{\cos y} \; ; \quad e^x = \frac{v}{\sin y}$$

$$\frac{u}{\cos y} = \frac{v}{\sin y}$$

$$\frac{u}{v} = \frac{\cos y}{\sin y} = \cot y$$

$$\boxed{\frac{v}{u} = \tan y.} \Rightarrow v = u \tan y$$

$$\boxed{y = \tan^{-1}\left[\frac{v}{u}\right]}$$

$y=k$ straight line.

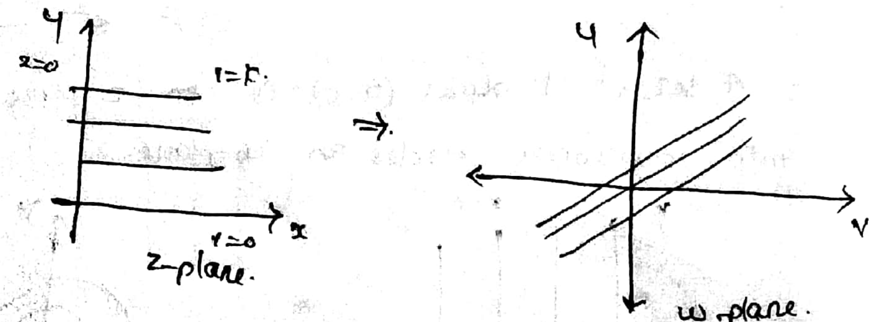
case (iii):-

$$\frac{v}{u} = \tan k.$$

$$v = u(\tan k) = u(m).$$

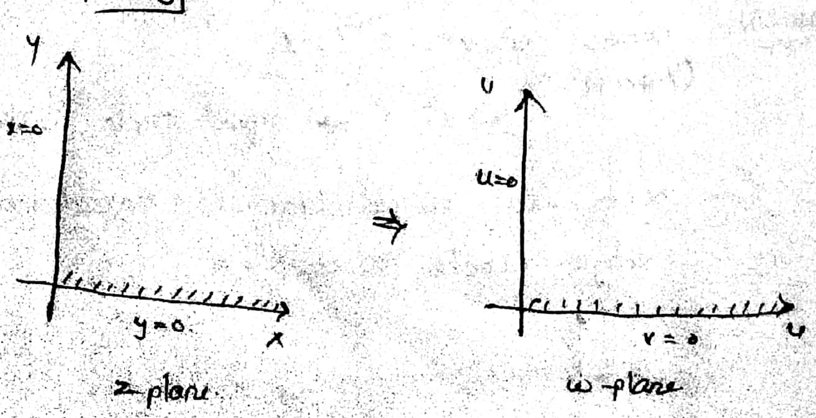
$$\boxed{v = m u}$$

$\therefore$ the st-line $y=k$ in z-plane transformed into set of lines in w-plane.



case (iv):- $y=0$, $\boxed{\frac{v}{u}=0}$

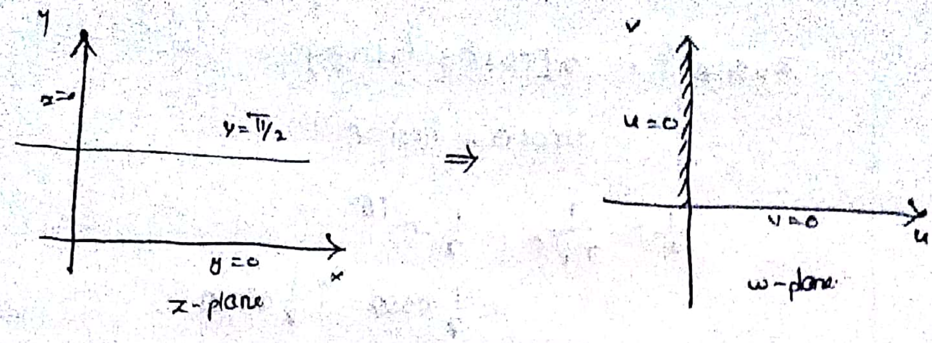
$\boxed{v=0}$



case (v):- $y = \pi/2$

$u = e^x \cdot \cos(\pi/2) = 0$

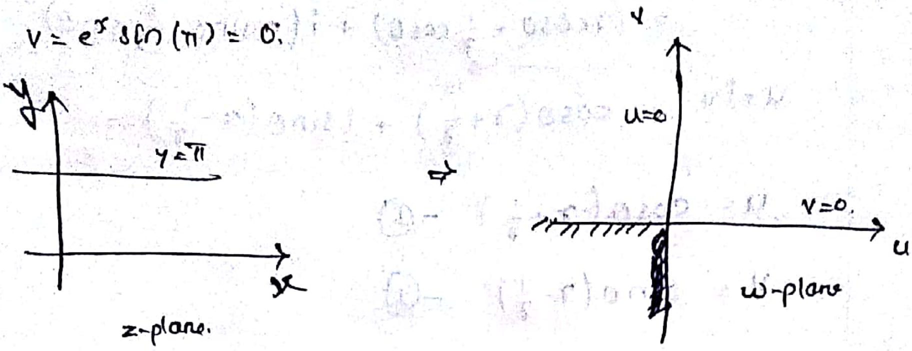
$v = e^x \cdot \sin(\pi/2) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots (>0)$



case (vi):- $y = \pi$

$u = e^x \cos(\pi) = -e^x$

$v = e^x \sin(\pi) = 0.$

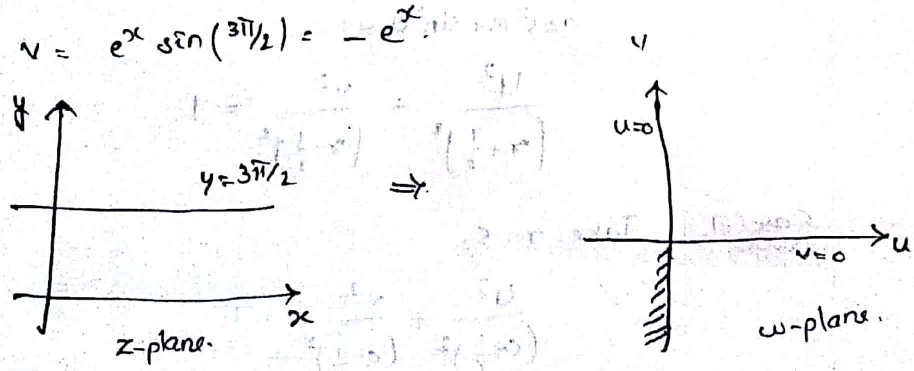


The line $y = \pi/2$ is transformed into the -ve u axis.

case (vii):- $y = 3\pi/2$.

$u = e^x \cos(3\pi/2) = 0$

$v = e^x \sin(3\pi/2) = -e^x$.



case (viii): $y = 2\pi$

$u = e^x \cos(2\pi) = e^x$

$v = e^x \sin(2\pi) = 0$.

(5) Discuss the transform of $w = z + \frac{1}{z}$ (in polar form) under the transfor $z+\frac{1}{z}$, what will be the image of $|z| = r$ if $r = c$ and $(c \neq 1)$, $(c=1)$

$$z = r\cdot e^{i\theta} = r[\cos\theta + i\sin\theta]$$
$$= r\cos\theta + ir\sin\theta$$

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}\cdot e^{-i\theta}$$
$$= \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

$$u+iv = r\cos\theta + ir\sin\theta + \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$
$$= \left(r\cos\theta + \frac{1}{r}\cos\theta\right) + i\left(r\sin\theta - \frac{1}{r}\sin\theta\right)$$

$$u+iv = \cos\theta\left(r+\frac{1}{r}\right) + i\sin\theta\left(r-\frac{1}{r}\right)$$

$$u = \cos\theta\left(r+\frac{1}{r}\right) \quad -(1)$$
$$v = \sin\theta\left(r-\frac{1}{r}\right) \quad -(2)$$

Elimination of θ:-

$$\cos\theta = \frac{u}{\left[r+\frac{1}{r}\right]} \;;\; \sin\theta = \frac{v}{\left[r-\frac{1}{r}\right]}$$

$$\cos^2\theta + \sin^2\theta = 1$$

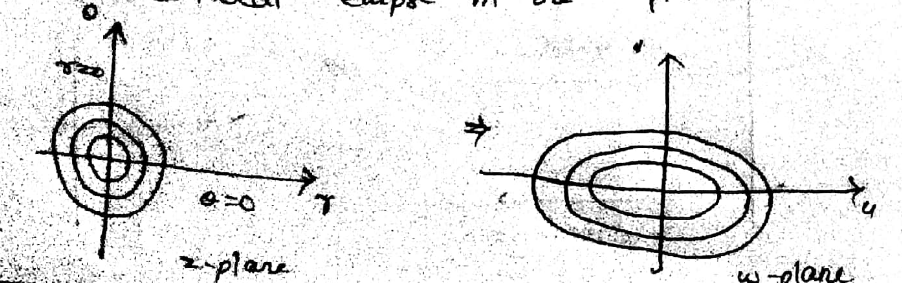
$$\frac{u^2}{\left[r+\frac{1}{r}\right]^2} + \frac{v^2}{\left[r-\frac{1}{r}\right]^2} = 1$$

Case (i):- Take $r = c$,

$$\underbrace{\frac{u^2}{(c+\frac{1}{c})^2}}_{a^2} + \underbrace{\frac{v^2}{(c-\frac{1}{c})^2}}_{b^2} = 1$$

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \Rightarrow \text{ellipse.}$$

$\therefore$ A set of concentric circles in z-plane is transformed into confocal ellipse in the w-plane.



Case (ii):- Take $r=1$, is a unit ...

$$\frac{u^2}{(r+\frac{1}{r})^2} + \frac{v^2}{(r-\frac{1}{r})^2} = 1$$

$$\frac{u^2}{(r+\frac{1}{r})^2} + \infty = 1$$

$$\infty \neq 1$$

$\therefore$ This case can not be explained.

Elimination of r:-

$$r+\frac{1}{r} = \frac{u}{\cos\theta} \; ; \quad r-\frac{1}{r} = \frac{v}{\sin\theta}$$

$$\left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta}$$

$$2\cdot r\cdot\frac{1}{r} + 2\cdot r\cdot\frac{1}{r} = \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$$

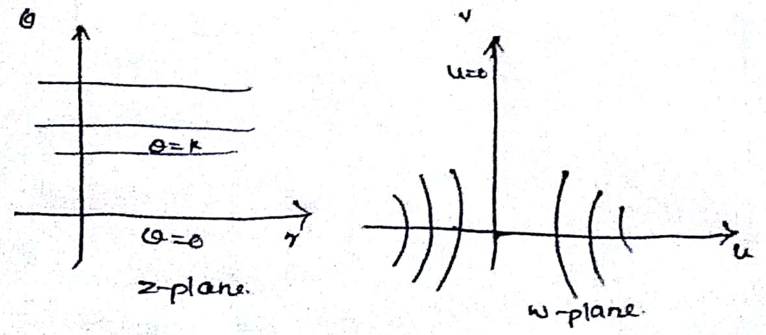
$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1$$

$\theta = k$,

$$\underbrace{\frac{u^2}{(2\cos\theta)^2}}_{a^2} - \underbrace{\frac{v^2}{(2\sin\theta)^2}}_{b^2} = 1$$

$\underbrace{2\cos k}_{a}$, $\underbrace{2\sin k}_{b}$.

$$\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1 \Rightarrow \text{hyperbola.}$$



Simple transformations:-

(a) Magnification

(b) translation.

(c) rotation.

(d) Inversion.

(e)