On inversion and noting that $L^{-1}\{\bar{f}'(s)\} = -t f(t)$, we get

$$-ty = -\sin t - \tfrac{1}{2} t \sin t \qquad \text{[See § 21.12 (11)]}$$

or $$y = \frac{1}{2}\left(1 + \frac{2}{t}\right) \sin t$$ which is the desired solution.

Example 21.33. *Solve* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0,\ y(0) = 2,\ y'(0) = 0.$

Solution. Taking Laplace transform of both sides of the equation, we get

$$L(xy'') + L(y') + L(xy) = 0$$

or $$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + [s\bar{y} - y(0)] - \frac{d\bar{y}}{ds} = 0 \quad \text{or} \quad (s^2+1)\frac{d\bar{y}}{ds} + s\bar{y} = 0$$

Separating the variables, $$\int \frac{d\bar{y}}{\bar{y}} + \int \frac{s\,ds}{s^2+1} = c$$

or $$\log \bar{y} + \tfrac{1}{2}\log(s^2+1) = \log c' \quad \text{or} \quad \bar{y} = \frac{c'}{\sqrt{(s^2+1)}}$$

Inversion gives $\quad y = c' J_0(x)$

To find c', we have $\quad y(0) = c' J_0(0),\quad$ *i.e.*, $\quad c' = 2$

Hence $\quad y = 2J_0(x).$

Example 21.34. *An alternating e.m.f. E sin ωt is applied to an inductance L and a capacitance C in series. Show by transform method, that the current in the circuit is* $\dfrac{E\omega}{(p^2-\omega^2)L}(\cos \omega t - \cos pt)$, *where* $p^2 = 1/LC$.

Solution. If i be a current and q the charge at time t in the circuit, then its differential equation is

$$L\frac{di}{dt} + \frac{q}{c} = E\sin\omega t \qquad [\because\ R = 0]$$

Taking Laplace transform of both sides, we get

$$L[s\bar{i}(s) - i(0)] + \frac{1}{C}L(q) = E\,.\,\frac{\omega}{s^2+\omega^2}$$

Since $i = 0$ and $q = 0$ at $t = 0$

$$\therefore \quad L s\bar{i}(s) + \frac{1}{C}L(q) = \frac{E\omega}{s^2+\omega^2} \qquad \ldots(i)$$

Also taking Laplace transform of $i = dq/dt$, we get

$$\bar{i}(s) = L(dq/dt) = sL(q) - q(0)$$

i.e. $$L(q) = \bar{i}(s)/s \qquad [\because\ q(0) = 0]$$

$\therefore\ (i)$ becomes $$L s\bar{i}(s) + \frac{1}{C}[\bar{i}(s)/s] = \frac{E\omega}{s^2+\omega^2}$$

or $$\left(Ls + \frac{1}{Cs}\right)\bar{i}(s) = \frac{E\omega}{s+\omega^2} \quad \text{or} \quad \bar{i}(s) = \frac{E\omega s}{L(s^2+1/LC)(s^2+\omega^2)}$$

or $$\bar{i}(s) = \frac{E\omega}{L}\cdot\frac{s}{(s^2+p^2)(s^2+\omega^2)} \quad \text{where } p^2 = 1/LC$$

$$\bar{i}(s) = \frac{E\omega}{L(p^2-\omega^2)}\left\{\frac{s}{s^2+\omega^2} - \frac{s}{s^2+p^2}\right\}$$

Now taking inverse Laplace transform of both sides, we get

$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} L^{-1}\left\{\frac{s}{s^2 + \omega^2} - \frac{s}{(s^2 + p^2)}\right\}$$

or
$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} (\cos \omega t - \cos pt).$$

PROBLEMS 21.6

Solve the following equations by the transform method :

1. $y'' + 4y' + 3y = e^{-t}, y(0) = y'(0) = 1.$ (*V.T.U., 2008 S ; Kurukshetra, 2005*)
2. $(D^2 - 1)x = a \cosh t, x(0) = x'(0) = 0.$
3. $y'' + y = t, y(0) = 1, y'(0) = 0.$ (*Mumbai, 2009*)
4. $y'' - 3y' + 2y = e^{3t}$, when $y(0) = 1$ and $y'(0) = 0$. (*V.T.U., 2010*)
5. $(D^2 - 3D + 2)y = 4e^{2t}$ with $y(0) = -3, y(0) = 5$. (*Mumbai, 2008*)
6. $y'' + 25y = 10 \cos 5t$ given that $y(0) = 2, y'(0) = 0$. (*S.V.T.U., 2008*)
7. $(D^2 + \omega^2)y = \cos \omega t, t > 0$, given that $y = 0$ and $Dy = 0$ at $t = 0$.
8. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t, y = \frac{dy}{dt} = 0$ when $t = 0$. (*Kurukshetra, 2005 ; Madras, 2003*)
9. $\frac{d^4y}{dt^4} - k^4y = 0$, where $y(0) = 1, y'(0) = y''(0) = y'''(0) = 0$.
10. $y''''(t) + 2y''(t) + y(t) = \sin t$, when $y(0) = y'(0) = y''(0) = y'''(0) = 0$.
11. $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 5y = e^{-t} \sin t$, where $y(0) = 0$ and $y'(0) = 1$. (*P.T.U., 2010*)
12. $y'' + 2y' + 5y = 5y = 5(t - 2), y(0) = 0, y'(0) = 0.$ (*P.T.U., 2005 S*)
13. $\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2 e^{2t}$, where $y = 1, \frac{dy}{dt} = 0, \frac{d^2y}{dt^2} = -2$ at $t = 0$.
14. $(D^2 + 1)x = t \cos 2t, x = Dx = 0$ at $t = 0$. (*Raipur, 2005 ; U.P.T.U., 2005*)
15. $ty'' + 2y' + ty = \sin t$, when $y(0) = 1$.
16. $ty'' + (1 - 2t)y' - 2y = 0$, when $y(0) = 1, y'(0) = 2$. (*P.T.U., 2002*)
17. $y'' + 2ty' - y = t$, when $y(0) = 0, y'(0) = 1$. (*U.P.T.U., 2003*)
18. $ty'' + y' + 4ty = 0$ when $y(0) = 3, y'(0) = 0$.
19. A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistance R. Show (by the transform method) that the current at time t is $\frac{E}{R - aL}(e^{-at} - e^{-Rt/L})$. (*V.T.U., 2000*)
20. Workout example 12.17, p. 465 by the transform method.
21. Obtain the equation for the forced oscillation of a mass m attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is k, when the driving force is $F_0 \sin at$. Solve this equation (using the Laplace transforms) when $a^2 \neq k/m$, given that initial velocity and displacement (from equilibrium position) are zero.

$\left[\textbf{Hint :} \text{ The equation of motion is } \frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_0}{m} \sin at \text{ and } x = \frac{dx}{dt} = 0 \text{ when } t = 0.\right]$

21.16 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform method can also be applied with advantage to the solution of simultaneous linear differential equations.

Example 21.35. *Solve the simultaneous equations* $\frac{dx}{dt} + 5x - 2y = t, \frac{dy}{dt} + 2x + y = 0$ *being given* $x = y = 0$ *when* $t = 0$. [Ex. 13.38] and $$L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\,.\,u(t-a) \qquad ...(\lambda)$$

$$\therefore \quad L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\,.\,\frac{s}{s^2+\pi^2}\right\} + L^{-1}\left\{e^{-s}\,.\,\frac{\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi\,(t-1/2)\,.\,u\,(t-1/2) + \sin\pi\,(t-1)\,.\,u\,(t-1)$$

$$= \sin\pi t\,.\,u\,(t-1/2) - \sin\pi t\,.\,u\,(t-1) = \{u\,(t-1/2) - u\,(t-1)\}\sin\pi t$$

(*ii*) $$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left\{e^{-cs}\left(-\frac{1}{a^2}\,.\,\frac{1}{s}+\frac{1}{a}\,.\,\frac{1}{s^2}+\frac{1}{a^2}\,.\,\frac{1}{s+a}\right)\right\}$$

Using (λ) above, we have

$$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = -\frac{1}{a^2}\{1\,.\,u\,(t-c)\} + \frac{1}{a}\{(t-c)\,.\,u(t-c)\} + \frac{1}{a^2}\,\{e^{-a\,(t-c)}\,.\,u(t-c)\}$$

$$= \frac{1}{a^2}\{a\,(t-c) - 1 + e^{-a\,(t-c)}\}\,u(t-c).$$

Example 21.44. *A particle of mass m can oscillate about the position of equilibrium under the effect of a restoring force mk^2 times the displacement. It started from rest by a constant force F which acts for time T and then ceases. Find the amplitude of the subsequent oscillation.*

Solution. The constant force F acting from $t = 0$ to $t = T$ can be expressed as

$$F\,[1 - u(t-T)],\ 0 < t < T$$

$\therefore$ equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = F\,[1 - u\,(t-T)] - mk^2x \quad \text{or} \quad \frac{d^2x}{dt^2} + k^2x = \frac{F}{m}[1 - u(t-T)]$$

Taking Laplace transform of both sides, we get

$$(s^2+k^2)\,\bar{x} = \frac{F}{ms}\,(1 - e^{-sT}) \qquad [\because \ x = 0,\ \dot{x} = 0 \text{ at } t = 0]$$

or $$\bar{x} = \frac{F}{m}\,.\,\frac{1-e^{-sT}}{s(s^2+k^2)} = \frac{F}{m}(1-e^{-sT})\,.\,\frac{1}{k^2}\left(\frac{1}{s} - \frac{s}{s^2+k^2}\right)$$

$$= \frac{F}{mk^2}\left\{(1-e^{-sT})\frac{1}{s} - (1-e^{-sT})\,.\,\frac{s}{s^2+k^2}\right\}$$

Taking inverse Laplace transform, we obtain

$$x = \frac{F}{mk^2}[(1-\cos kt) - \{1 - \cos k\,(t-T)\}]\,u\,(t-T)$$

i.e., $$x = \frac{F}{mk^2}(1-\cos kt) \text{ for } 0 < t < T$$

and $$x = \frac{F}{mk^2}(1-\cos kt) - \{1 - \cos k\,(t-T)\} \text{ for } t > T$$

$$= \frac{F}{mk^2}\{\cos k\,(t-T) - \cos kT\} \text{ for } t > T$$

or $$x = \frac{2F}{mk^2}\sin\frac{kT}{2}\,.\,\sin k\,(t-T/2) \text{ for } t > T$$

Hence the amplitude of subsequent oscillation (*i.e.*, for $t > T$) $= \dfrac{2F}{mk^2}\sin\dfrac{kT}{2}$.

Example 21.45. *In an electrical circuit with e.m.f. E(t), resistance R and inductance L, the current i builds up at the rate given by*

$$L\,di/dt + Ri = E(t). \qquad ...(i)$$

If the switch is connected at t = 0 and disconnected at t = a, find the current i at any instant.

Solution. We have $i = 0$ at $t = 0$ and $E(t) = \begin{cases} E \text{ for } 0 < t < a \\ 0 \text{ for } t > a \end{cases}$

$\therefore$ taking the Laplace transform of both sides, (*i*) becomes

$$(Ls + R)\bar{i} = \int_0^\infty e^{-st}\, E(t)\, dt = \int_0^a e^{-st}\, E dt = \frac{E}{s}(1 - e^{-as})$$

or
$$\bar{i} = \frac{E}{s(Ls+R)} - \frac{Ee^{-as}}{s(Ls+R)}$$

On inversion, we get
$$i = L^{-1}\left\{\frac{E}{s(Ls+R)}\right\} - L^{-1}\left\{\frac{Ee^{-as}}{s(Ls+R)}\right\} \qquad ...(ii)$$

Now
$$L^{-1}\left\{\frac{E}{s(Ls+R)}\right\} = \frac{E}{R}\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+R/L}\right)\right\} = \frac{E}{R}(1 - e^{-Rt/L})$$

and
$$L^{-1}\left\{\frac{Ee^{-as}}{s(Ls+R)}\right\} = \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\, u(t-a) \qquad \text{[By the second shifting property]}$$

Thus (*ii*) becomes
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} - \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\, u(t-a)$$

Hence
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} \text{ for } 0 < t < a$$

and
$$i = \frac{E}{R}[\{1 - e^{-Rt/L}\} - \{1 - e^{-R(t-a)/L}\}] = \frac{E}{R}e^{-Rt/L}(e^{-Ra/L} - 1) \text{ for } t > a.$$

Example 21.46. *Calculate the maximum deflection of an encastre beam 1 ft. long carrying a uniformly distributed load w lb./ft. on its central half length.*

Solution. Taking the origin at the end A, we have

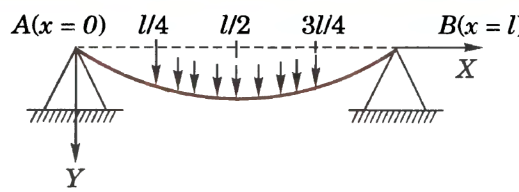

Fig. 21.6

$$EI\frac{d^4y}{dx^4} = w(x)$$

where $w(x) = w\{u(x - l/4) - u(x - 3l/4)\}$

Taking the Laplace transform of both sides, (Fig. 21.6), we get

$$EI[s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)]$$
$$= w\left(\frac{e^{-ls/4}}{s} - \frac{e^{-3ls/4}}{s}\right)$$

Using the conditions $y(0) = y'(0) = 0$ and taking $y''(0) = c_1$ and $y'''(0) = c_2$, we have

$$EI\,\bar{y} = w\left(\frac{e^{-ls/4}}{s^5} - \frac{e^{-3ls/4}}{s^5}\right) + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

On inversion, we get $EIy = \dfrac{w}{24}[(x - l/4)^4 u(x - l/4) - (x - 3l/4)^4 u(x - 3l/4)] + \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3$ ...(*i*)

For $x > 3l/4$,
$$EIy = \frac{w}{24}[(x - l/4)^2 - (x - 3l/4)^2] + \tfrac{1}{2}c_1x^2 + \tfrac{1}{6}c_2x^3$$

and
$$EIy' = \frac{w}{6}[(x - l/4)^3 - (x - 3l/4)^3] + c_1x + \tfrac{1}{2}c_2x^2$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get $0 = \dfrac{w}{24}\left\{\left(\dfrac{3l}{4}\right)^4 - \left(\dfrac{l}{4}\right)^4\right\} + \frac{1}{2}c_1l^2 + \frac{1}{6}c_2l^3$

and
$$0 = \frac{w}{6}\left\{\left(\frac{3l}{4}\right)^3 - \left(\frac{l}{4}\right)^3\right\} + c_1l + \tfrac{1}{2}c_2l^2$$

whence $c_1 = 11\,wl^2/192$; $c_2 = -\,wl/4$.

Thus for $l/4 < x < 3l/4$, (*i*) gives $EIy = \frac{w}{24}\left(x + \frac{1}{4}\right)^4 + \frac{11wl^2}{384}x^2 - \frac{wl}{24}x^3$

Hence the maximum deflection $= y(l/2) = \frac{13wl^4}{6144EI}$.

21.18 (1) UNIT IMPULSE FUNCTION

The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the *unit impulse function* (also called *Dirac delta function**).

Thus unit impulse function is considered as the limiting form of the function (Fig. 21.7) :

$$\delta_\varepsilon(t-a) = 1/\varepsilon, \quad a \le t \le a+\varepsilon$$
$$= 0, \quad \text{otherwise}$$

as $\varepsilon \to 0$. It is clear from Fig. 21.7 that as $\varepsilon \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

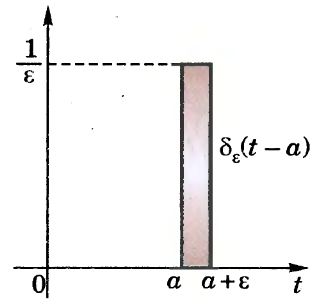

Fig. 21.7

Thus the unit impulse function $\delta(t-a)$ *is defined as follows :*

$$\delta(t-a) = \infty \text{ for } t = a \;; = 0 \text{ for } t \ne a,$$

such that $\int_0^\infty \delta(t-a)\,dt = 1. \qquad (a \ge 0)$

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ε per unit length over the portion of the beam between $x = a$ and $x = a + \varepsilon$. Thus

$$w(x) = w_0/\varepsilon \qquad a < x < a+\varepsilon,$$
$$= 0, \qquad \text{otherwise}$$

i.e., $w(x) = w_0\delta(x+a)$.

(2) Transform of unit impulse function. If $f(t)$ be a function of t continuous at $t = a$, then

$$\int_0^\infty f(t)\,\delta_\varepsilon(t-a)\,.\,dt = \int_0^{a+\varepsilon} f(t)\,.\,\frac{1}{\varepsilon}\,dt$$

$$= (a+\varepsilon-a)\,f(\eta)\,.\,\frac{1}{\varepsilon} = f(\eta), \qquad \text{where } a < \eta < a+\varepsilon.$$

by Mean value theorem for integrals.

As $\varepsilon \to 0$, we get $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$.

In particular, when $f(t) = e^{-st}$, we have $\mathbf{L\{\delta(t-a)\} = e^{-as}}$.

Example 21.47. *Evaluate* (*i*) $\int_0^\infty \sin 2t\,\delta(t-\pi/4)\,dt$ (*ii*) $L\frac{1}{t}\,\delta(t-a)$.

Solution. (*i*) We know that $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

$$\therefore \quad \int_0^\infty \sin 2t\,\delta(t-\pi/4)\,dt = \sin(2\,.\,\pi/4) = 1$$

(*ii*) We know that $L\{\delta(t-a)\} = e^{-as}$

$$\therefore \quad L\left[\frac{1}{t}\delta(t-a)\right] = \int_s^\infty L[\delta(t-a)]\,ds = \int_s^\infty e^{-as}\,ds$$

$$= \left|\frac{e^{-as}}{-a}\right|_s^\infty = \frac{1}{a}e^{-as}.$$

* After the English physicist *Paul Dirac* (1902-84) who was awarded the Nobel prize in 1933 for his work in Quantum mechanics.