Ex. 90: Solve, by Laplace Transform method, the equation $\frac{d^4y}{dt^4} - k^4 y = 0$ **with the initial conditions** $y(0) = 1$; $y'(0) = y''(0) = y'''(0) = 0$. **(Nov. 87 Bharathiar Uty.)**

Solution: The given equation is

$$\frac{d^4y}{dt^4} - \lambda^4 y = 0 \qquad \text{.... (1)}$$

Taking the Laplace Transformation of both sides, we get

$$s^4 L(y) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) - k^4 L(y) = L(0) = 0.$$

i.e. $(s^4 - k^4) L(y) - s^3 = 0$ using the given initial conditions

$$L(y) = \frac{s^3}{s^4 - k^4} = \frac{s^3}{(s^2 + k^2)(s^2 - k^2)}$$

$$= \frac{1}{2}\left[\frac{s}{s^2 + k^2} + \frac{s}{s^2 - k^2}\right],$$

Hence $y = \frac{1}{2} L^{-1}\left(\frac{s}{s^2 + k^2}\right) + \frac{1}{2} L^{-1}\left(\frac{s}{s^2 - k^2}\right)$

$$= \frac{1}{2}(\cos kt + \cosh kt)$$

Ex. 91: The current i **and the charge** q **in a series circuit containing an inductance** L**, a capacitance** C **and an e.m.f.** L **satisfy the equations** $L\frac{di}{dt} + \frac{q}{C} = E$, $i = \frac{dq}{dt}$. **Express** i **and** q **in terms of time** t**, given that** L, C, E **are constants and that the initial values of i and q are both zero.**

.... (1)

Solution: $L\frac{di}{dt} + \frac{q}{C} = E$

.... (2)

Using $i = \frac{dq}{dt}$, (1) becomes $L\frac{d^2q}{dt^2} + \frac{q}{C} = E$

Taking Laplace Transform of both sides of (2), we get

$$L \cdot \alpha\left(\frac{d^2q}{dt^2}\right) + \frac{1}{C}\alpha(q) = \alpha(E).$$

(Here we have used α to denote Laplace Transform as L is inductance) i.e. $L\left[s^2\alpha(q) - sq(0) - q'(0)\right] + \frac{1}{C}\alpha(q) = \frac{E}{s}$

i.e. $\left(Ls^2 + \frac{1}{C}\right)\alpha(q) = \frac{E}{s}$ since $q(0) = 0$ and $q'(0) = i(0) = 0$.

Hence $\alpha(q) = \dfrac{E}{s\left(Ls^2 + \frac{1}{C}\right)}$

$$q = \alpha^{-1}\frac{E}{s\left(Ls^2 + \frac{1}{C}\right)} = \alpha^{-1}\frac{\left(\frac{E}{L}\right)}{s\left(s^2 + \frac{1}{CL}\right)}$$

Neglecting the factor $\frac{1}{s}$,

$$\text{inverse of } \frac{\frac{E}{L}}{s^2 + \frac{1}{CL}} = \frac{E}{L}\cdot\sqrt{CL}\,\sin\left(\frac{1}{\sqrt{CL}}t\right).$$

$$\text{Hence } q = \frac{E}{L}\cdot\sqrt{CL}\int_0^t \sin\left(\frac{1}{\sqrt{CL}}t\right)dt$$

$$= \frac{E\sqrt{C}}{\sqrt{L}}\left[-\cos\frac{1}{\sqrt{CL}}t\right]_0^t \sqrt{CL}$$

$$= -EC\cdot\left[\cos\frac{1}{\sqrt{CL}}t - 1\right] = EC\left[1 - \cos\frac{t}{\sqrt{CL}}\right]$$

$$i = \frac{dq}{dt} = EC\cdot\frac{1}{\sqrt{CL}}\sin\frac{1}{\sqrt{CL}}t$$

$$= E\sqrt{\frac{C}{L}}\,\sin\left(\frac{1}{\sqrt{CL}}t\right)$$

Alternatively a solution for i can be found directly by transforming equation (1).

We have $L\frac{di}{dt} + \frac{q}{C} = E$ where $i = \frac{dq}{dt}$

Hence $q = \int i\,dt$ and equation (1) becomes

$$L\frac{di}{dt} + \frac{1}{C}\int i\,dt = E \qquad \text{.... (2)}$$

Taking the Laplace Transform of both sides of (2), we get

$$L\,[\,s\,\alpha\,(i) - i\,(0)\,] + \frac{1}{C}\cdot\frac{1}{s}\,\alpha\,(i) = \alpha\,(E) = \frac{E}{s}\,.$$

i.e. $$\left(Ls + \frac{1}{Cs}\right)\alpha\,(i) = \frac{E}{s} \text{ since } i\,(0) = 0.$$

$$\alpha\,(i) = \frac{E}{s\left(Ls + \frac{1}{Cs}\right)} = \frac{E}{Ls^2 + \frac{1}{C}} = \frac{E}{L\left(s^2 + \frac{1}{CL}\right)}$$

$$= \frac{E}{L}\cdot\sqrt{CL}\cdot\frac{\left(\frac{1}{\sqrt{CL}}\right)}{s^2 + \frac{1}{CL}}$$

Hence $$i = \frac{E}{L}\sqrt{CL}\;\alpha^{-1}\frac{\left(\frac{1}{\sqrt{CL}}\right)}{s^2 + \frac{1}{CL}} = E\sqrt{\frac{C}{L}}\,\sin\left(\frac{t}{\sqrt{CL}}\right).$$

Exercises 1.13

Solve the following differential equations by Laplace Transform methods (1 to 16).

1. $\dfrac{dx}{dt} + 3x = e^{-2t}$ given that $x = 4$ when $t = 0$.
2. $y'' + 4y = 4;\ y\,(0) = 0 = y'\,(0)$ **(Nov. 95 M.S. Uty)**
3. $(D^2 - 2D - 8)\,y = 4;\ y\,(0) = 0, y'\,(0) = 1$ **(Nov. 89 Bharathi dasan Uty)**
4. $y'' - 3y' + 2y = 2e^{3x}$ given $y\,(0) = 2$ and $y'\,(0) = 3.$ **(Nov. 88 Bharathi dasan Uty)**
5. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = \sin t$ if $\dfrac{dy}{dt} = 0$ and $y = 2$ when $t = 0$ **(Nov. 88 Bharathi dasan Uty)**
6. $y'' + 4y' + 3y = e^{-t}$ given $y\,(0) = 1$ and $y'\,(0) = 0.$ **(Nov. 95 M.S. Uty)**
7. $x'' - 2x' + x = e^{-t}$ with $x\,(0) = 2, x'\,(0) = -1.$ **(Nov. 95 M.S. Uty)**

§30. Simultaneous Differential Equations: Laplace transform methods can also be applied to the solution of simultaneous linear differential equations, as illustrated in the following worked examples:

Ex. 92: Solve the equations

$$3\frac{dx}{dt}+\frac{dy}{dt}+2x=1$$

$$\frac{dx}{dt}+4\frac{dy}{dt}+3y=0$$

given that $x=0$ and $y=0$ when $t=0$

(Bharathiar Uty Nov. 88; M.U. Ap. 93; M.S. Uty Ap. 95)

Taking the Laplace Transformation of the two equations, we get

$$3\left[\, sL(x)-x(0)\,\right]+sL(y)-y(0)+2L(x)=L(1)=\frac{1}{s}$$

i.e. $(3s+2)\,L(x)+sL(y)=\dfrac{1}{s}$ .... (1) $(\because x(0)=0=y(0))$

and $sL(x)-x(0)+4\,[sL(y)-y(0)]+3L(y)=0$

i.e. $sL(x)+(4s+3)\,L(y)=0$ .... (2)

We solve the (algebraic) equations (1) and (2) to get $L(x)$ and $L(y)$.

$$L(x)=\frac{\begin{vmatrix}\frac{1}{s} & s\\ 0 & 4s+3\end{vmatrix}}{\begin{vmatrix}3s+2 & s\\ s & 4s+3\end{vmatrix}}=\frac{\frac{1}{s}(4s+3)}{(3s+2)(4s+3)-s^2}$$

$$=\frac{4s+3}{s(11s^2+17s+6)}=\frac{4s+3}{s(11s+6)(s+1)}$$

From (2), $L(y)=-\dfrac{s}{4s+3}\cdot\dfrac{4s+3}{s(11s+6)(s+1)}$

$$=-\frac{1}{(11s+6)(s+1)}$$

Hence $x = L^{-1}\dfrac{4s+3}{s\,(11s+6)\,(s+1)} = L^{-1}\left[\dfrac{\left(\frac{1}{2}\right)}{s} + \dfrac{-\frac{33}{10}}{11s+6} + \dfrac{-\frac{1}{5}}{s+1}\right]$

$$= \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - \frac{33}{10} \times \frac{1}{11} L^{-1}\frac{1}{\left(s+\frac{6}{11}\right)} - \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{2} - \frac{3}{10}e^{-\frac{6t}{11}} - \frac{1}{5}e^{-t}$$

$$y = -L^{-1}\frac{1}{(11s+6)\,(s+1)}$$

$$= -L^{-1}\left[\frac{\left(\frac{11}{5}\right)}{11s+6} + \frac{-\frac{1}{5}}{s+1}\right]$$

$$= -\frac{11}{5} \times \frac{1}{11} L^{-1}\frac{1}{\left(s+\frac{6}{11}\right)} + \frac{1}{5}L^{-1}\frac{1}{(s+1)} = -\frac{1}{5}e^{-\frac{6t}{11}} + \frac{1}{5}e^{-t}$$

Ex. 93: Solve the equations $\dfrac{dx}{dt} - y = e^t,\ \dfrac{dy}{dt} + x = \sin t$ given that $x(0) = 1$ and $y(0) = 0$.

(Nov. 88 Bharathier Uty; Nov. 88, Nov. 89, Ap. 95 Bharathidasan Uty; Ap. 95, Nov. 95 M.S. Uty.)

Taking Laplace Transforms of the two equations, we get

$$sL(x) - x(0) - L(y) = L(e^t) = \frac{1}{s-1}$$

$$\text{i.e. } sL(x) - L(y) = 1 + \frac{1}{s-1} = \frac{s}{s-1} \qquad \text{.... (1) } (\because x(0) = 1)$$

$$\text{and } sL(y) - y(0) + L(x) = L(\sin t) = \frac{1}{s^2+1}$$

$$\text{i.e. } L(x) + sL(y) = \frac{1}{s^2+1} \qquad \text{... (2) } (\because y(0) = 0)$$

From (1) and (2), $L(x) \cdot (s^2+1) = \dfrac{s^2}{s-1} + \dfrac{1}{s^2+1}$

Hence $L(x) = \dfrac{s^2}{(s-1)(s^2+1)} + \dfrac{1}{(s^2+1)^2}$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + \frac{1}{(s^2+1)^2}$$

So $x = \dfrac{1}{2}\left[L^{-1}\left(\dfrac{1}{s-1}\right) + L^{-1}\left(\dfrac{s}{s^2+1}\right) + L^{-1}\left(\dfrac{1}{s^2+1}\right) + 2L^{-1}\dfrac{1}{(s^2+1)^2}\right]$

$$= \frac{1}{2}\left[e^t + \cos t + \sin t + \sin t - t\cos t\right]$$

Referring to the table on page 734.

$$= \frac{1}{2}\left(e^t + \cos t + 2\sin t - t\cos t\right)$$

Also $(s^2+1)\,L(y) = \dfrac{s}{s^2+1} - \dfrac{s}{s-1}$

$$L(y) = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)}$$

$$= \frac{s}{(s^2+1)^2} - \frac{1}{2}\left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$

Hence $y = \dfrac{1}{2}\left[L^{-1}\dfrac{2s}{(s^2+1)^2} - L^{-1}\dfrac{1}{s-1} + L^{-1}\dfrac{s}{s^2+1} - L^{-1}\dfrac{1}{s^2+1}\right]$

$$= \frac{1}{2}\left(t\sin t - e^t + \cos t - \sin t\right)$$

Ex. 94: Solve : $x' + y' + x - y = 2$

$$x'' + x' - y' = \cos t$$

given $x(0) = 0,\ x'(0) = 2,\ y(0) = 1$. **(Nov. 95 M.S. Uty)**

Taking Laplace Transforms of the two equations, we get

$$sL(x) - x(0) + sL(y) - y(0) + L(x) - L(y) = L(2) = \frac{2}{s}$$

i.e. $(s+1)\,L(x) + (s-1)\,L(y) = \dfrac{2}{s} + 1 = \dfrac{s+2}{s}$ .... (1)

$(\because x(0) = 0,\ y(0) = 1)$

$$s^2L(x) - sx(0) - x'(0) + sL(x) - x(0) - [\,sL(y) - y(0)\,]$$

$$= L(\cos t) = \frac{s}{s^2+1}$$

i.e. $(s^2 + s)\, L(x) - 2 - sL(y) + 1 = \dfrac{s}{s^2+1}$

$$(s^2 + s)\, L(x) - s\, L(y) = \frac{s}{s^2+1} + 1 = \frac{s^2+s+1}{s^2+1} \qquad \text{.... (2)}$$

Solving (1) and (2) by Cramer's rule, we get

$$L(x) = \frac{\begin{vmatrix} \dfrac{s+2}{s} & s-1 \\ \dfrac{s^2+s+1}{s^2+1} & -s \end{vmatrix}}{\begin{vmatrix} s+1 & s-1 \\ s^2+s & -s \end{vmatrix}}$$

$$= \frac{-(s+2) - \dfrac{(s-1)(s^2+s+1)}{s^2+1}}{-s(s+1) - (s^2+s)(s-1)}$$

$$= \frac{-(s+2)(s^2+1) - (s-1)(s^2+s+1)}{(s^2+1) \cdot -s(s+1)(1+s-1)}$$

$$= \frac{2s^3 + 2s^2 + s + 1}{s^2(s+1)(s^2+1)} = \frac{(s+1)(2s^2+1)}{s^2(s+1)(s^2+1)} = \frac{2s^2+1}{s^2(s^2+1)}$$

$$= \frac{1}{s^2} + \frac{1}{s^2+1} \text{ on splitting into partial fractions}$$

So $x = L^{-1}\left(\dfrac{1}{s^2}\right) + L^{-1}\left(\dfrac{1}{s^2+1}\right) = t + \sin t$

$$L(y) = \frac{\begin{vmatrix} s+1 & \dfrac{s+2}{s} \\ s^2+s & \dfrac{s^2+s+1}{s^2+1} \end{vmatrix}}{\begin{vmatrix} s+1 & s-1 \\ s^2+1 & -s \end{vmatrix}}$$

$$= \frac{\dfrac{(s+1)(s^2+s+1)}{s^2+1} - \dfrac{(s^2+s)(s+2)}{s}}{-s(s+1) - (s^2+s)(s-1)}$$

$$= \frac{(s+1)(s^2+s+1) - (s^2+1)(s+1)(s+2)}{(s^2+1) \cdot -s(s+1)(1+s-1)}$$

$$= \frac{s^2+s+1-(s^2+1)(s+2)}{-s^2(s^2+1)} = \frac{s^3+s^2+1}{s^2(s^2+1)}$$

$$= \frac{1}{s^2} + \frac{s}{s^2+1} \text{ on splitting into partial fractions}$$

Hence $y = L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{s}{s^2+1}\right) = t + \cos t$

Exercises 1.14

Solve the following simultaneous differential equations by Laplace Transform method.

1. $\frac{dx}{dt} + y = 2\cos t$; $\frac{dy}{dt} - x = 1$ given that $x(0) = -1$ and $y(0) = 1$. **(Nov. 89 M.U.)**

2. $\frac{dx}{dt} + y = \sin t$; $\frac{dy}{dt} + x = \cos t$ given that $x = 2$ and $y = 0$ when $t = 0$. **(Ap. 91, Ap. 92 M.U.; M.S. Uty Nov. 94; Ap 96 M.K.U.)**

3. $\frac{dx}{dt} + \frac{dy}{dt} + x + y = 1$; $\frac{dy}{dt} = 2x + y$ given that $x = 0, y = 1$ at $t = 0$.

4. $\dot{x} + \dot{y} + x = -e^{-t}$, $\dot{x} + 2\dot{y} + 2x + 2y = 0$ given that $x(0) = -1, y(0) = 1$

5. $2\frac{dx}{dt} - \frac{dy}{dt} + 3x = 2t$

   $\frac{dx}{dt} + 2\frac{dy}{dt} - 2x - y = t^2 - t$

   given that $x(0) = y(0) = 1$. **(Ap. 92 M.U.)**

6. $\frac{dx}{dt} - \frac{dy}{dt} - 2x + 2y = 1 - 2t$

   $\frac{d^2x}{dt^2} + 2\frac{dy}{dt} + x = 0$ subject to the conditions that $x = 0$, $y = 0$, $\frac{dx}{dt} = 0$ when $t = 0$.

Remark. The following rule may be adopted for solving differential equations. In the given differential equation replace $D^n x$ by $s^n L\{x\}$, $D^{n-1}x$ by $s^{n-1} L\{x\}$, ... $D^2 x$ by $L^2(x)$, Dx by $s L\{x\}$ and x by $L\{x\}$. Then for $D^n x$ in the left hand side add $s^{n-1}x_0 + s^{n-2}x_1 + \ldots\ldots x_{n-1}$ in right hand side where

$$x_0 = (x)_{t=0} \text{ and } x_r = (D^r x)_{t=0} = \left(\frac{d^r x}{dx^r}\right)_{t=0}$$

and replace the function, if any on right hand side by its Laplace transform, then express $L\{x\}$ in the form $\left[\frac{G(s)}{H(s)}\right]$ and finally take the inverse Laplace transform to get the solution.

Ex. 44. *Solve the differential equation* $\frac{dx}{dt} + \alpha x = 0$ *by Laplace transform method subject to the initial condition that* $x = x_0$ *at* $t = 0$.

(*MS Univ. 2003*)

Solution. Given equation is

$$\frac{dx}{dt} + \alpha x = 0 \text{ or } x' + \alpha x = 0 \text{ where } x' = \frac{dx}{dt}$$

$$L(x') + \alpha L(x) = L(0)$$

$$\Rightarrow \quad [sL(x) - (x)_{t=0}] + \alpha L(x) = 0$$

$$\Rightarrow \quad (s + \alpha) L(x) = (x)_{t=0}$$

As $\quad x = x_0$ at $t = 0$

$$\therefore \quad L(x) = \frac{x_0}{s + \alpha}$$

Taking inverse Laplace transform we get

$$L^{-1} Lx = L^{-1}\left(\frac{x_0}{s+\alpha}\right) \text{ or } x = x_0 L^{-1}\left(\frac{1}{s+\alpha}\right)$$

$$\Rightarrow \quad x = x_0 e^{-\alpha t}$$

This is required solution.

Ex. 45. *Using Laplace transform, solve*

$$\frac{dx}{dt} + y = 0 \text{ and } \frac{dy}{dt} - x = 0$$

(*Delhi, 2000*)

under the condition $x(0) = 1$, $y(0) = 0$.

Solution. Given equations are

$$\frac{dx}{dt} + y = 0 \quad \text{or} \quad x' + y = 0 \quad \text{...(1)}$$

and $$\frac{dy}{dt} - x = 0 \quad \text{or} \quad y' - x = 0 \quad \text{...(2)}$$

Taking Laplace transform of (1) and (2), we get

$$[s L(x) - x(0)] + L(y) = 0 \quad \text{...(3)}$$

and $$[s L(y) - y(0)] - L(x) = 0 \quad \text{...(4)}$$

As $x(0) = 1$ and $y(0) = 0$, we have

$$[s L(x) - 1)] + L(y) = 0 \quad \text{...(5)}$$

and $$s L(y) - L(x) = 0 \quad \text{...(6)}$$

Solving (5) and (6)

$$L(x) = \frac{s}{s^2+1} \text{ and } L(y) = \frac{1}{s^2+1}$$

Taking inverse Laplace transforms

$$x = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t \text{ and } y = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

Thus the solution is $x = \cos t$ and $y = \sin t$.

Ex. 46. *Using Laplace transformation method solve the differential equation* $y'' + 9y = 0$; *satisfying the initial conditions* $y(0) = 0$ *and* $y'(0) = 2$

Given that $\quad L^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin 3t.$ (Meerut 1980)

Solution. Given differential equation is $y'' + 9y = 0$. ...(1)

Taking Laplace transform of both the sides ; we get

$$L\{y''\} + 9\,L\{y\} = L\{0\}$$

i.e., $\quad s^2 L\{y\} - s(y)_{t=0} - \left(\frac{dy}{dt}\right)_{t=0} + 9\,L\{y\} = L\{0\}$

i.e., $\quad (s^2+9)\,L\{y\} = L(0) + s(y)_{t=0} + \left(\frac{dy}{dt}\right)_{t=0}$

$$= L(0) + sy(0) + y'(0)$$

$$= 0 + s.0 + 2 = 2$$

i.e., $\quad L\{y\} = \frac{2}{s^2+9} = \frac{2}{3}\left(\frac{3}{s^2+9}\right)$

Taking inverse *Laplace transform* ; we get

$$y(t) = \frac{2}{3}\,L^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{2}{3}\sin 3t$$

Ex. 47. *Solve the differential equation*

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0\,;\ x_0 = x_1 = 1$$ (Bhartidasan 1989, Meerut 1992)

Solution. Given equation is

$$D^2x - 2Dx + 2x = 0 \text{ or } x'' - 2x' + 2x = 0$$

Taking Laplace transform of both sides ; we get

$$L\{x''\} - 2\,L(x') + 2\,L\{x\} = L(0)$$

i.e., $\quad s^2 L\{x\} - s(x)_{t=0} - (x')_{t=0} - 2[s\,L(x) - (x)_{t=0}] + 2\,L\{x\} = L(0)$

i.e., $(s^2 - 2s + 2)\,L\{x\} = L(0) + (s-2)(x)_{t=0} + 1\cdot(x')_{t=0} = 0 + (s-2)\cdot 1 + 1$

$$\left[\begin{array}{l}\text{since } (x)_{t=0} = x_0 = 1 \\ \text{and } (x')_{t=0} = x_1 = 1\end{array}\right]$$

$\therefore$ $\quad L\{x\} = \frac{s-1}{s^2-2s+2}$

Taking inverse Laplace transform, we get

$$x(t) = L^{-1}\left\{\frac{s-1}{s^2-2s+2}\right\} = e^t\cos t$$

Ex. 48. *Solve the differential equation*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x} \sin x$$

where $g(0) = 0$ *and* $y'(0) = 1.$

Solution. Given equation is

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x} \sin x$$

Taking Laplace transform of this equation

$$L(y'') + 2L(y') + 5L(y) = L(e^{-x} \sin x)$$

$$[s^2 L(y) - s\,y(0) - y'(0)] + 2[S\,L(y) - y(0)] + 5L(y) = L(e^{-x} \sin x]$$

$$\Rightarrow \quad (s^2 + 2s + 5)\,L(y) - (s+2)\,y_0 - y'(0) = L(e^{-x} \sin x)$$

Substituting $y_0 = 0$ and $y'(0) = 1$, we get

$$(s^2 + 2s + 5)(y) - 0 - 1 = \sqrt{\frac{1}{(s+1)^2} + 1}$$

$$\Rightarrow \quad (s^2 + 2s + 5)\,L(y) = 1 + \frac{1}{(s+1)^2 + 1} = \frac{(s+1)^2 + 1 + 1}{(s+1)^2 + 1}$$

$$\Rightarrow \quad L(y) = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

On resolving RHS into partial fraction

$$L(y) = \frac{2}{3}\left\{\frac{1}{(s^2 + 2s + 5)}\right\} + \frac{1}{3}\left\{\frac{1}{(s^2 + 2s + 2)}\right\}$$

Taking inverse of Laplace transform

$$y(x) = \frac{2}{3}L^{-1}\left\{\frac{1}{(s^2 + 2s + 5)}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s^2 + 2s + 2}\right\}$$

$$= \frac{1}{3}L^{-1}\frac{2}{(s+1)^2 + (2)^2} + \frac{1}{3}L^{-1}\frac{1}{(s+1)^2 + 1^2}$$

$$= \frac{1}{3}e^{-x} \sin 2x + \frac{1}{3}e^{-x} \sin x$$

$$\Rightarrow \quad y(x) = \frac{1}{3}e^{-x}(\sin 2x + \sin x)$$

This is required solution.

(*b*) Ordinary differential equation with variable coefficients

Ex. 49. *Solve* $t\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + 4tx = 0$ *(Bharatidasan 2001, 1989)*

when $x(0) = 3$ and $x_1(0) = 0$

Solution. Taking Laplace transform of given equation

$$L\{tx''\} + L\{x'\} + 4L\{tx\} = L(0) = 0$$

or $$-\frac{d}{ds}[s^2 L\{x\} - s(x)_{t=0} - (x')_{t=0}] + [sL\{x\} - (x)_{t=0}] - 4\frac{d}{ds}[L\{x\}] = 0$$

$$\left[\begin{aligned}&\text{since } (x)_{t=0} = x(0) = 3\\ &\text{and } (x')_{t=0} = x_1(0) = 0\end{aligned}\right]$$

or $$\frac{d}{ds}[s^2 L\{x\} - 3s - [s\,L\{x\} - 3] + 4\frac{d}{ds}[L\{x\}] = 0$$

or $$(s^2+4)\frac{d}{ds}[L\{x\}] + s\,L\{x\} = 0$$

or $$\frac{d\,[L\{x\}]}{L\{x\}} + \frac{s\,ds}{s^2+4} = 0$$

Integrating, we get

$$\log_e [L\{x\}] + \frac{1}{2}\log_e (s^2+4) = \log_e k,\ k \text{ being a constant}$$

or $$L\{x\} = \frac{k}{\sqrt{(s^2+4)}}$$

Taking inverse Laplace transform

$$x(t) = L^{-1}\left\{\frac{k}{\sqrt{(s^2+4)}}\right\} = k\,J_0(2t)$$

Now at $t = 0$, $x = 3$; thereby giving $3 = k\,J_0(0) = k$

$\therefore$ $$x(t) = 3\,J_0(2t)$$

(c) Solution of Integral Equations

An integral equation is defined as the equation which contains a dependent variable under an integral sign.

Ex. 50. *Solve the following integral equation*

$$F(t) = 1 + 2\int_0^t F(t-x)\,e^{-2x}\,dx$$

Solution. Taking Laplace transform of both the sides, we get

$$L\{F(t)\} = L\{1\} + 2L\left\{\int_0^t F(t-x)\,e^{-2x}\,dx\right\} \qquad \ldots(1$$

Assuming $L\{F(t)\} = f(s)$; so that $L(1) = 1/s$

and $L\left\{\int_0^t F(t-x)\,e^{-2x}\,dx\right\} = \frac{f(s)}{s+2}$, equation (1) gives

$$f(s) = \frac{1}{s} + \frac{2f(s)}{s+2} \text{ i.e., } f(s) = \frac{1}{s} + \frac{2}{s^2}$$

Hence $F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s} + \frac{2}{s^2}\right\} = 1 + 2t.$

Ex. 51. *Solve the following equation for F(t) with the condition that F(0) = 0 :*

$$F'(t) = \sin t + \int_0^t F(t-x)\cos x\,dx$$

Solution. Taking Laplace transform of both the sides, we get

$$L\{F'(t)\} = L\{\sin t\} + L\left\{\int_0^t F(t-x)\cos x\,dx\right\}$$

Using $L\{F(t)\} = f(s)$; we get

$$sf(s) - F(0) = \frac{1}{s^2+1} + \frac{f(s).s}{s^2+1}$$

As $F(0) = 0$, we get

$$s f(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s^2+1}$$

which yields $f(s) = \frac{1}{s^3}$

Hence $L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s^3}\right\}$

$\therefore$ $F(t) = \frac{1}{2}t^2$

(d) Solution of Boundary Value Problems

Ex. 52. *A string is stretched between two fixed points, (0, 0) and (l, 0) and is released from position $u = u_0 \sin(\pi x/l)$ then show that the expression for its subsequent displacement $u(x, t)$ is*

$$u(x, t) = u_0 \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{h\pi t}{l}\right)$$

where h^2 is diffusivity.

Solution. The equation of vibration of string is given by

$$\frac{\partial^2 u}{\partial t^2} = h^2\frac{\partial^2 u}{\partial x^2} \quad \ldots(1)$$

The boundary conditions are

$$u(0, t) = 0, \quad u(l, t) = 0 \quad \ldots(2)$$

and initial conditions are

$$u(x, 0) = u_0 \sin\left(\frac{\pi x}{l}\right) \text{ and } \left(\frac{\partial u}{\partial t}\right)_{t=0} = u_t(x, 0) = 0 \quad \ldots(3)$$

The Laplace transform of equation (1) gives

$$s^2 f(x, s) - su(x, 0) - u_t(x, 0) = h^2\frac{\partial^2 f}{\partial x^2} \quad \ldots(4)$$

where $f = f(x, s) = L\{u(x, t)\}$,

Using (3) equation (4) gives

$$\frac{\partial^2 f}{\partial x^2} - \frac{s^2}{h^2}f = -\frac{u_0 s}{h^2}\sin\frac{\pi x}{l} \quad \ldots(5)$$

The Laplace transform of (2) yields

$$f(0, s) = 0\,;\, f(l, s) = 0 \quad \ldots(6)$$

The solution of equation (4) is

$$f = Ae^{+sx/h} + Be^{-sx/h} + \frac{u_0 s \sin\frac{\pi x}{l}}{s^2 + \frac{\pi^2 h^2}{l^2}} \quad \ldots(7)$$

Now $\quad f(0, s) = 0$; this implies $A + B = 0$

and $\quad f(l, s) = 0$ implies $Ae^{sl/h} + Be^{-sl/h} = 0$

Solving these equations, we get $A = B = 0$

Thus equation (7) becomes

$$f(x, s) = \frac{u_0 s \sin\frac{\pi x}{l}}{s^2 + \frac{\pi^2 h^2}{l^2}} \qquad \ldots (8)$$

Taking inverse Laplace transform of (8) ; we get

$$u(x, t) = L^{-1}\{f(x, s)\} = u_0 \sin\frac{\pi x}{l} L^{-1}\left\{\frac{s}{s^2 + \frac{\pi^2 h^2}{l^2}}\right\}$$

$$= u_0 \sin\frac{\pi x}{l} \cos\frac{\pi h t}{l} \qquad \ldots (9)$$

Ex. 53. *An inductor of 3 henry is in series with a resistance of 30 ohms and an e.m.f. of 150 volts. Assuming that the current is zero at $t = 0$, find the current at time $t > 0$.*

Solution. Given $L = 3$ henry, $R = 30$ ohms, E = 150 volt

and $\quad I = \frac{dq}{dt} = 0$ at $t = 0$.

The differential equation of given circuit is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} = E \qquad \text{(since capacity } C = 0)$$

Substituting given values ; we get

$$3\frac{d^2q}{dt^2} + 30\frac{dq}{dt} = 150 \text{ or } 3q'' + 30q' = 150 \qquad \ldots (1)$$

Taking Laplace transform of (1) ; we get

$$3L\{q''\} + 30L\{q'\} = 150L\{1\}$$

or $$3[s^2 L\{q\} - sq(0) - q'(0)] + 30[s L\{q\} - q(0)] = \frac{150}{s} \qquad \ldots (2)$$

At $t = 0$, $I = dq/dt = q' = 0$ i.e., $q'(0) = 0$ and $q(0) = 0$

Therefore equation (2) becomes

$$3s^2 L\{q\} + 30s L\{q\} = \frac{150}{s}$$

or $$L\{q\} = \frac{50}{s^2(s + 10)}$$

Taking inverse Laplace transform, we get

$$q(t) = 50 L^{-1}\left\{\frac{1}{s^2(s + 10)}\right\} = 50 \times \frac{1}{10}\left[t + \frac{e^{-10t} - 1}{10}\right] = 5\left(t + \frac{e^{-10t} - 1}{10}\right)$$

Hence current $I(t) = \frac{dq}{dt} = 5\left[1 + \frac{-10e^{-10t} - 0}{10}\right] = 5(1 - e^{-10t})$