Formulae for B.V.P

* $F_s\left[\frac{\partial u}{\partial x}\right] = -\frac{p\pi}{l} F_c(u)$

* $F_c\left[\frac{\partial u}{\partial x}\right] = \frac{p\pi}{l} F_s(u) - u(0,t) + (-1)^p u(l,t)$

* $F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{-p^2\pi^2}{l^2} F_s(u) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(l,t)\right]$

* $F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\frac{p^2\pi^2}{l^2} F_c(u) + \frac{\partial u}{\partial x}(l,t)\cos p\pi - \frac{\partial u}{\partial x}(0,t)$

Problems solving Boundary value problems using Finite F.S.T & F.C.T (B.V.P)

① Using finite F.T, Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ given $u(0,t)=0$ & $u(4,t)=$ [illegible] $u(x,0)=2x$, where $0<x<4$, $t>0$ Nov 16

Sol:

Given that $u(0,t)$, → Take Finite F.S.T, x by $\sin\left(\frac{p\pi x}{l}\right)$, $l=4$ we have,

$$\therefore \int_0^4 \frac{\partial u}{\partial t}\sin\left(\frac{p\pi x}{4}\right)dx = \int_0^4 \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{p\pi x}{4}\right)dx.$$

$$\Rightarrow F_s\left[\frac{\partial u}{\partial t}\right] = F_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{d}{dt}\left[F_s(u)\right] = \frac{-p^2\pi^2}{l^2} F_s(u) + \frac{p\pi}{l}\left[u(\underbrace{l,t}_{0}) - (-1)^p u(\underbrace{4,t}_{0})\right]$$

$$= \frac{-p^2\pi^2}{16}\left[F_s(u)\right] + 0$$

$$\Rightarrow \frac{d[F_s(u)]}{[F_s(u)]} = \frac{-p^2\pi^2}{16}\, dt$$

Taking $\int$ w.r.to 't'

$$\log\left[F_s(u)\right] = \frac{-p^2\pi^2}{16} t + c$$

$$\Rightarrow F_s(u) = e^{-p^2\pi^2 t/16} \cdot \underbrace{e^c}_{A}$$

$$\Rightarrow \int_0^l u(x,t)\sin\left(\frac{p\pi x}{l}\right)dx = A\cdot e^{-p^2\pi^2 t/16} \longrightarrow ①$$

Given: $u(x,0) = 2x$.

$$\Rightarrow \int_0^4 u(x,0)\sin\left(\frac{p\pi x}{4}\right)dx = A$$

$$\Rightarrow \int_0^4 2x\sin\left(\frac{p\pi x}{4}\right)dx = A$$

$u = x \qquad v = \sin\left(\frac{p\pi x}{4}\right)$

$u' = 1 \qquad v_1 = -\cos\left(\frac{p\pi x}{4}\right) \times \frac{4}{p\pi}$

$v_2 = -\sin\left(\frac{p\pi x}{4}\right) \times \frac{16}{p^2\pi^2}$

$$A = 2\left[-x\cos\left(\frac{p\pi x}{4}\right)\times\frac{4}{p\pi} + \frac{16}{p^2\pi^2}\sin\left(\frac{p\pi x}{4}\right)\right]_{x=0}^{x=4}$$

U.L $(x=4)$ L.L $(x=0)$

$$A = 2\left[-4\cos\left(\frac{p\pi\times 4}{4}\right)\frac{4}{p\pi} + \frac{16}{p^2\pi^2}\cancel{\sin\left(\frac{4p\pi}{4}\right)}\right] - \left[0+0\right]$$

$$\boxed{A = \frac{-32}{p\pi}\cos p\pi \to (-1)^p}$$ sub in ①

$\therefore F_s(u) = \frac{-32}{p\pi}\times(-1)^p \times e^{-p^2\pi^2 t/16}$

(or $F_s[u(x,t)]$)

By finite Fourier inversion for sine,

$$f(x) = \sum_{p=1}^{\infty} F_s[p]\sin\left(\frac{p\pi x}{l}\right)$$

Here, $u(x,t) = \frac{2}{l}\sum_{p=1}^{\infty} F_s[u(x,t)]\sin\left(\frac{p\pi x}{4}\right)$

$$= -\frac{\cancel{32}\,8}{\pi}\times\frac{2}{\cancel{4}}\times\sum_{p=1}^{\infty}(-1)^p\times e^{-p^2\pi^2 t/16}\times\sin\left(\frac{p\pi x}{4}\right)$$

$$= \frac{-16}{\pi}\sum_{n=1}^{\infty}(-1)^p\times e^{-p^2\pi^2/16}\times\sin\left(\frac{p\pi x}{4}\right)$$

② Solve: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0<x<6$, $t>0$. Nov 17, May 18

$\left(\frac{\partial u}{\partial x}\right)(0,t) = 0$, $\left(\frac{\partial u}{\partial x}\right)(6,t) = 0$ & $u(x,0) = 2x$.

Sol: $\frac{\partial u}{\partial x}$ is given at $(0,t)$ & $(6,t)$, take finite F.C.T

$\times$ by $\cos\left(\frac{p\pi x}{l}\right)$, $l=6$, we have.

$$\int_0^6\left(\frac{\partial u}{\partial t}\right)\cos\left(\frac{p\pi x}{6}\right)dx = \int_0^6\left(\frac{\partial^2 u}{\partial x^2}\right)\cos\left(\frac{p\pi x}{6}\right)dx$$

$$F_c\left[\frac{\partial u}{\partial t}\right] = F_c\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\frac{d}{dt}F_c[u] = -\frac{p^2\pi^2}{l^2}F_c(u) + \frac{\partial u}{\partial x}(l,t)\cos p\pi - \frac{\partial u}{\partial x}(0,t)$$

$$= -\frac{p^2\pi^2}{36}F_c(u) + \frac{\partial u}{\partial x}\cancel{(6,t)}^{0}\cos p\pi - \frac{\partial u}{\partial x}\cancel{(0,t)}^{0}$$

$$\frac{d[F_c(u)]}{[F_c(u)]} = -\frac{p^2\pi^2}{36}dt$$

Taking int.

$$\log F_c[u] = -\frac{p^2\pi^2 t}{36} + C$$

Taking Exp;

$$\Rightarrow F_c[u] = e^C e^{-p^2\pi^2 t/36}$$

$$= A\cdot e^{-p^2\pi^2 t/36} \to ①$$

$$\Rightarrow \int_0^l u(x,t)\cos\left(\frac{p\pi x}{l}\right)dx = Ae^{-p^2\pi^2 t/36}$$

Using, $t=0$, $u(x,0) = 2x$, $l=6$

$$\Rightarrow \int_0^6 u(x,0)\cos\left(\frac{p\pi x}{6}\right)dx = A.$$

$$\Rightarrow \int_0^6 2x\cdot\cos\left(\frac{p\pi x}{6}\right)dx = A$$

$u = x$ $\quad v = \cos\left(\frac{p\pi x}{6}\right)$

$u' = 1$ $\quad v_1 = \sin\left(\frac{p\pi x}{6}\right) \times \frac{6}{p\pi}$

$v_2 = -\frac{36}{p^2\pi^2}\cos\left(\frac{p\pi x}{6}\right)$

$$A = 2\left[\frac{6}{p\pi} \times x \times \sin\left(\frac{p\pi x}{6}\right) + \frac{36}{p^2\pi^2} \times \cos\left(\frac{p\pi x}{6}\right)\right]_{x=0}^{x=6}$$

U.L $(x=6)$ $\quad$ L.L $(x=0)$

$$A = 2\left\{\left[0 + \frac{36(-1)^p}{p^2\pi^2}\right] - \left[0 + \frac{36}{p^2\pi^2}\right]\right\}$$

$$\boxed{A = \frac{72}{p^2\pi^2}\left[(-1)^p - 1\right]}$$ sub in (1)

$$F_c(u) = \frac{72}{p^2\pi^2}\left[(-1)^p - 1\right] e^{-p^2\pi^2 t/36}$$

By inversion formula for cosine

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l}\sum_{p=1}^{\infty} F_c(s) \cos\left(\frac{p\pi x}{l}\right) dx$$

Here

$$u(x,t) = \frac{1}{6} \times F_c(0) + \frac{\cancel{2}^1}{\cancel{6}_3}\sum_{p=1}^{\infty} \frac{72}{p^2\pi^2}\left[(-1)^p - 1\right] e^{-p^2\pi^2 t/36}$$

$$= \frac{1}{6}\int_0^6 (2x)\,dx + 24\sum_{p=1}^{\infty} \frac{\left[(-1)^p - 1\right]}{p^2\pi^2} e^{-p^2\pi^2 t/36}$$

$$= \frac{\cancel{2}}{6}\left[\frac{x^2}{\cancel{2}}\right]_0^6 + 24\sum_{p=1}^{\infty} \frac{\left[(-1)^p - 1\right]}{p^2\pi^2} e^{-p^2\pi^2 t/36}$$

$$= 6 + \frac{24}{\pi^2}\sum_{p=1}^{\infty} \frac{\left[(-1)^p - 1\right]}{p^2} e^{-p^2\pi^2 t/36}.$$

(5) Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $0 < x < 4$, $t > 0$, given $u(0,t) = 0$, $u(4,t) = 0$, $u(x,0) = 3\sin\pi x - 2\sin 5x$.

Sol:

$u(0,t)$ is given, take Finite Fourier sine transform, $\times$ by $\sin\left(\frac{p\pi x}{l}\right)$, $l = 4$, we have

$$\int_0^4 \left(\frac{\partial u}{\partial t}\right)\sin\left(\frac{p\pi x}{4}\right) dx = 2\int_0^4 \left(\frac{\partial^2 u}{\partial x^2}\right)\sin\left(\frac{p\pi x}{4}\right) dx$$

$$\Rightarrow F_s\left[\frac{\partial u}{\partial t}\right] = 2 \cdot F_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{d}{dt}\left[F_s(u)\right] = 2\left[-\frac{p^2\pi^2}{l^2} F_s(u) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(l,t)\right]\right]$$

$$= -\frac{\cancel{2}p^2\pi^2}{\cancel{16}\,8} F_s(u)$$

$$\Rightarrow \frac{d\left[F_s(u)\right]}{F_s(u)} = -\frac{p^2\pi^2}{8} dt$$

Taking $\int$,

$$\Rightarrow \log F_s(u) = -\frac{p^2\pi^2 t}{8} + c$$

Taking Exp,

$$\Rightarrow F_s(u) = e^{-p^2\pi^2 t} \cdot e^c = A \cdot e^{-p^2\pi^2 t/8} \rightarrow (1)$$

Using $t = 0$, $l = 4$, $u(x,0) = 3\sin\pi x - 2\sin 5x$.

$$\Rightarrow \int_0^4 u(x,0) \cdot \sin\left(\frac{p\pi x}{4}\right) dx = A\, \cancel{e^{-p^2\pi^2 t/8}}\ 1$$

$$\Rightarrow A = \int_0^4 (3\sin\pi x - 2\sin 5x)\sin\left(\frac{p\pi x}{4}\right) dx.$$

$= 0$, $p \neq 4$, $p \neq 5 \times 4$
$\neq 20$

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When $p=4$, $A=\int_0^4 [3\sin\pi x - 2\sin 5\pi x]\sin\pi x\,dx$

$$= 3\int_0^4 \sin^2\pi x\,dx - 2\int_0^4 \sin(\underset{A}{5\pi x})\sin(\underset{B}{\pi x})\,dx$$

$$= 3\int_0^4 \left[\frac{1-\cos 2\pi x}{2}\right]dx - \frac{2}{2}\int_0^4 \frac{\cos[\overset{A-B}{5\pi x-\pi x}]}{4\pi x} - \frac{\cos[\overset{A+B}{5\pi x+\pi x}]}{6\pi x}$$

$$= \frac{3}{2}\left[x - \cancel{\frac{\sin 2\pi x}{2\pi}}_{0}\right]_{x=0}^{x=4} - \left[\cancel{\frac{\sin(4\pi x)}{4\pi}} - \cancel{\frac{\sin(6\pi x)}{6\pi}}_{0}\right]_{x=0}^{x=4}$$

$$= \frac{3}{2}\times 4$$

$$= 6.$$

When $p=20$, $A=\int_0^4[3\sin\pi x - 2\sin 5\pi x]\sin\left(\frac{20\pi x}{4}\right)dx$

$$=3\int_0^4 \cancel{\sin\pi x\sin 5\pi x\,dx}_{0} - 2\int_0^4 \sin^2 5\pi x\,dx$$

similar to $p=4$ case

$$= -2\int_0^4 \left[\frac{1-\cos 10\pi x}{2}\right]dx$$

$$= -\frac{2}{2}\left[x - \cancel{\frac{\sin 10\pi x}{10\pi}}\right]_{x=0}^{x=4}$$

$= -4.$ sub in ①

$$F_s(u) = \underset{(p=4)}{6\cdot e^{-p^2\pi^2 t/8}} - \underset{(p=20)}{4\cdot e^{p^2\pi^2 t/8}}$$

By inversion formula for sine

$$f(x)=\frac{2}{l}\sum_{p=1}^{\infty} F_s(u)\sin\left(\frac{p\pi x}{l}\right)$$

$$=\frac{2}{4}\left[6e^{-p^2\pi^2 t/8}\sin\left(\frac{4\pi x}{4}\right) - 4e^{-p^2\pi^2 t/8}\sin\left(\frac{20\pi x}{4}\right)\right]$$

$$=\frac{1}{2}\left[6e^{-16\pi^2 t/8}\sin(\pi x) - 4e^{-400\pi^2 t/8}\sin(5\pi x)\right]$$

$$= 3e^{-2\pi^2 t}\sin(\pi x) - 4e^{-50\pi^2 t}\sin(5\pi x)$$

4- Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0<x<\pi$, $t>0$, given $u(0,t)=u(\pi,t)=0$, $t>0$, $u(x,0)=\sin^3 x$.

Sol: Given that $u(0,t)$ & $u(\pi,t)$, take Finite F.S.T x by $\sin\left(\frac{p\pi x}{l}\right)$; $l=\pi$, we have,

$$\Rightarrow \int_0^\pi \left(\frac{\partial u}{\partial t}\right)\sin\left(\frac{p\pi x}{\pi}\right)dx = \int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{p\pi x}{\pi}\right)dx$$

$$\Rightarrow F_s\left[\frac{\partial u}{\partial t}\right] = F_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{d}{dt}[F_s(u)] = -\frac{p^2\pi^2}{l^2}F_s(u) + \frac{p\pi}{l}[u(0,t) - (-1)^p u(l,t)]$$

$$= -\frac{p^2\pi^2}{\pi^2}F_s(u) + \frac{p\pi}{l}[\underset{0}{u(0,t)} - (-1)^p \underset{0}{u(\pi,t)}]$$

$$= -p^2 F_s(u)$$

$$\Rightarrow \frac{d[F_s(u)]}{[F_s(u)]} = -p^2\,dt.$$

Taking int,

$$\log F_s(u) = -p^2 t + c$$

Taking Exp.,

$$F_s(u) = e^{-p^2 t}\cdot e^{c} = A\cdot e^{-p^2 t} \rightarrow ①$$

Using, $t=0$, $u(x,0)=\sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$

$$\Rightarrow \int_0^l u(x,t)_0 \sin\left(\frac{p\pi x}{l}\right)dx = A\cdot \cancel{e^{-p^2 t}}^{0}$$

(i.e) $A \Rightarrow \frac{1}{4}\int_0^\pi (3\sin x - \sin 3x)\left(\sin\frac{p\pi x}{\pi}\right) = 0$, $p\neq 1$, $p\neq 3$.

When $p=1$, $A = \frac{1}{4}\int_0^{\pi}(3\sin x - \sin 3x)\sin x\, dx$

$$= \frac{1}{4}\times\int_0^{\pi} 3\sin^2 x\, dx - \int_0^{\pi} \underset{A}{\sin(3x)}\,\underset{B}{\sin(x)}\, dx$$

$$= \frac{1}{4}\left[\frac{3}{2}\int_0^{\pi}(1-\cos 2x)dx - \frac{1}{2}\int_0^{\pi}\cos(2x) - \cos(4x)\, d\right.$$

$$= \frac{1}{4}\times\left[\frac{3}{2}\left[x - \left(\frac{\cancel{\sin 2x}}{2}\right)\right]_0^{\pi} - \frac{1}{2}\left[\frac{\cancel{\sin 2x}}{2} - \left(\frac{\cancel{\sin 4x}}{4}\right)\right]_0\right]$$

$$= \frac{3\pi}{8}.$$

When $p=3$, $A = \frac{1}{4}\int_0^{\pi}(3\sin x - \sin 3x)\sin 3x\, dx$

$$= \frac{1}{4}\times\int_0^{\pi}\cancel{3\sin x\sin 3x\,dx}_{\to 0} - \int_0^{\pi}\sin^2 3x\, dx$$

similar to prev. case

$$= -\frac{1}{4}\times\frac{1}{2}\int_0^{\pi}\left[1 - \cos 6x\right]dx$$

$$= -\frac{1}{8}\left[x - \frac{\cancel{\sin 6x}}{6}\right]_{x=0}^{x=\pi}$$

$$= -\pi/8 \cdot \quad \text{sub in (1)}.$$

$$\therefore F_s(u) = \underset{(p=1)}{\frac{3\pi}{8}e^{-p^2t}} - \underset{(p=3)}{\frac{\pi}{8}e^{-p^2t}}$$

By inversion formula for sine,

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty} F_s(u)\cdot\sin\left(\frac{p\pi x}{l}\right)$$

$$= \frac{2}{\pi}\left[\underset{(p=1)}{\frac{3\pi}{8}e^{-p^2t}}\sin\left(\frac{\cancel{p\pi} x}{\cancel{\pi}}\right) - \underset{(p=3)}{\frac{\pi}{8}e^{-p^2t}\sin\left(\frac{3\cancel{\pi}x}{\cancel{\pi}}\right)}\right.$$

$$= \frac{2}{\pi}\left[\frac{3\pi}{8}e^{-t}\sin x - \frac{\pi}{8}e^{-9t}\sin 3x\right]$$

$$= \frac{\cancel{2}}{\cancel{\pi}}\times\frac{3\cancel{\pi}}{\cancel{8}_4}\times e^{-t}\sin x - \frac{\cancel{2}}{\cancel{\pi}}\times\frac{\cancel{\pi}}{\cancel{8}_4}\times e^{-9t}\sin 3x$$

$$= \frac{3}{4}\times e^{-t}\sin x - \frac{e^{-9t}}{4}\sin 3x \,//.$$

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(5) Solve: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 10$, given $u(0,t) = u(10,t) = 0$ for $t > 0$ & $u(x,0) = 10x - x^2$, $0 < x < 10$.

Sol: $u(0,t)$ & $u(10,t)$ is given, take finite F.S.T

$\times$ by $\sin\left(\frac{p\pi x}{l}\right)$, $l = 10$, we have

$$\int_0^{10} \left(\frac{\partial u}{\partial t}\right) \sin\left(\frac{p\pi x}{10}\right) dx = \int_0^{10} \frac{\partial^2 u}{\partial x^2} \sin\left(\frac{p\pi x}{10}\right) dx$$

$$\Rightarrow F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{d}{dt}\left[F_s(u)\right] = \frac{-p^2\pi^2}{l^2} F_s(u) + \frac{p\pi}{l}\left[u(0,t) - (-1)^p u(l,t)\right]$$

$$= \frac{-p^2\pi^2}{100} F_s(u) + \frac{p\pi}{10}\left[u(0,t) - (-1)^p u(10,t)\right]$$

$$\Rightarrow \frac{d[F_s(u)]}{F_s(u)} = \frac{-p^2\pi^2}{100} dt$$

Taking $\int$,

$$\Rightarrow \log F_s(u) = \frac{-p^2\pi^2 t}{100} + c$$

Taking Exp.,

$$\Rightarrow F_s(u) = e^{-p^2\pi^2 t/100} \cdot e^c = e^{-p^2\pi^2 t/100} \cdot A \rightarrow (1)$$

Using $t = 0$, $l = 10$, $u(x,0) = 10x - x^2$, $0 < x < 10$, we have

$$\Rightarrow \int_0^{10} u(x,0) \sin\left(\frac{p\pi x}{10}\right) dx = \cancel{e^{-p^2\pi^2 t/100}}\, 1\; A$$

$$\Rightarrow A = \int_0^{10} (10x - x^2) \sin\left(\frac{p\pi x}{10}\right) dx.$$

$u = 10x - x^2$ $\qquad v = \sin\frac{p\pi x}{10}$

$u' = 10 - 2x$ $\qquad v_1 = -\frac{10}{p\pi}\cos\left(\frac{p\pi x}{10}\right)$

$u'' = -2$ $\qquad v_2 = -\frac{100}{p^2\pi^2}\sin\left(\frac{p\pi x}{10}\right)$

$\qquad v_3 = +\frac{1000}{p^3\pi^3}\cos\left(\frac{p\pi x}{10}\right)$

$$A = \left[(10x-x^2)\left(\frac{-10}{p\pi}\right)\cos\left(\frac{p\pi x}{10}\right) + (10-2x)\frac{100}{p^2\pi^2}\sin\left(\frac{p\pi x}{10}\right) + \frac{2000}{p^3\pi^3}\cos\right]_{x=0}^{x=10}$$

U.L $(x=10)$ $\qquad$ L.L $[x=0]$

$$= \left[0 - 0 - \frac{2000}{p^3\pi^3}\cos\left(\frac{p\pi \times 10}{10}\right)\right] - \left[0 + 0 - \frac{2000}{p^3\pi^3}(1)\right]$$

$$= \frac{-2000}{p^3\pi^3}\left[(-1)^p - 1\right]$$

$$= \begin{cases} 0, & p\ \text{even} \\ \frac{4000}{p^3\pi^3}, & \text{odd} \end{cases}$$ sub in (1)

$$\therefore F_s(u) = \frac{4000}{p^3\pi^3} e^{-p^2\pi^2 t/100} \quad (p \to \text{odd})$$

Using inversion formula for sine, we have,

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty} F_s(p)\sin\left(\frac{p\pi x}{l}\right)$$

$$\Rightarrow u(x,t) = \frac{2}{10}\sum_{p=\text{odd}}^{\infty}\frac{4000}{p^3\pi^3} e^{-p^2\pi^2 t/100}\sin\left(\frac{p\pi x}{l}\right)$$

$$= \frac{800}{\pi^3}\sum_{p=\text{odd}}^{\infty}\frac{1}{p^3}\times e^{-p^2\pi^2 t/100}\times\sin\left(\frac{p\pi x}{10}\right)$$