Unit-I

Laplace Transforms

* Transform is a equivalance form of mathematical operation.

Def:- A function $f(t)$ can be defined and continous on some intervals, for some positive values of t. then the laplace transform of t is given by $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$, $t > 0$.

where s is a function associated with $f(t)$.

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$$\Gamma n = \int_0^\infty e^{-x} x^n\,dx$$

$$\Gamma n = n\,\Gamma n-1$$
$$= (n-1)!$$
$$\Gamma 1/2 = \sqrt{\pi}/2$$

Piece-wise continous:-

* The function $f(t)$ is said to be piece-wise continous (or) sectionally continous on a closed interval $[a,b]$, if the interval can be subdivided into smaller sub-intervals in each of which the $f(t)$ should be continous.

Exponential Order:-

* A function $f(t)$ is said to be of exponential order if it satisfy the condition

$$\lim_{t\to\infty} e^{-st} f(t) = 0.$$

① Ex: $f(t) \cdot t$

$$\lim_{t\to\infty} e^{-st} t^2 = \lim_{t\to\infty} \frac{t^2}{e^{st}}$$

$$= \lim_{t\to\infty} \frac{2t}{s.e^{st}} \quad (\text{L-Hospital's})$$

$$= \lim_{t\to\infty} \frac{2}{s^2 . e^{st}}$$

$$= 0$$

② Ex = $f(x) = x^n$

$$\lim_{x\to\infty} e^{-sx} x^n = \lim_{x\to\infty} \frac{x^n}{e^{sx}}$$

$$= \lim_{x\to\infty} \frac{n.x^{n-1}}{s.e^{sx}}$$ the power of x goes on decreares.

$$= \lim_{x\to\infty} \frac{n(n-1)\ldots 2.1}{s^n . e^{sx}}$$

$$= \lim_{x\to\infty} \frac{n!}{s^n . e^{sx}}.$$

* P-form:-

$$\int_0^\infty p.e^{-pt} f(t) . dt$$

Condition for Laplace transform:-

* $f(t)$ should be continous (or) piecewise continous on interval $[a,b]$.

* $f(t)$ should be of exponential order:

(5) $e^{i\theta} = \cos\theta + i\sin\theta$

$$L[e^{iat}] = \frac{1}{s-ia}$$

$$L[\cos at + i\sin at] = \frac{1}{(s-ia)}$$

$$L[\cos at] + i\,L[\sin at] = \frac{1}{s-ia} \times \frac{s+ia}{s+ia}$$

$$L[\cos at] + i\,L[\sin at] = \frac{(s+ia)}{s^2+a^2}$$

Real part : $L[\cos at] = \dfrac{s}{s^2+a^2}$

Imag. part : $L[\sin at] = \dfrac{a}{s^2+a^2}$

Note:-

Additive property: $L[f_1(t) + f_2(t)] = L[f_1(t)] + L[f_2(t)]$.

Linearity Property:-

$$L[k.f(t)] = k.\,L[f(t)].$$

(6) $L[\cosh at]$ & $L[\sinh at]$

$$\cosh at = \frac{e^{at}+e^{-at}}{2}$$

$$= L\left[\frac{e^{at}+e^{-at}}{2}\right] = \frac{1}{2}\left\{L[e^{at}] + L[e^{-at}]\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2} \times \left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

$||^{ly}$. $L[\sinh at] = \dfrac{a}{s^2-a^2}$.

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$f(t)$	$L(t)$
1	$1/s$
k	k/s
t^n	$\frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cosh at$	s/s^2-a^2
$\sinh at$	a/s^2-a^2

Problems:-

1. Find the L.T of the following:

(i) $L[t^2+7+\cos^2 t]$

$= L[t^2] + L[7] + L[\cos^2 t]$

$L[t^2] = \frac{n!}{s^{n+1}} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

$L[7] = \frac{K}{s} = 7/s$

$L[\cos^2 t] = L\left[\frac{1+\cos 2t}{2}\right] = \frac{1}{2}\{L[1] + L[\cos 2t]\}$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+2^2}\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right].$

$\therefore L[t^2+7+\cos^2 t] = \frac{2}{s^3} + \frac{7}{s} + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right].$

(ii) $L[t^{3/2} + \cosh 3t + \cos 3t \cdot \cos 5t]$

$= L[t^{3/2}] + L[\cosh 3t] + L[\cos 3t \cdot \cos 5t]$

$L[t^{3/2}] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\Gamma(\frac{3}{2}+1)}{s^{n+1}} = \frac{\frac{3}{2}\Gamma(3/2)}{s^{5/2}} = \frac{\frac{3}{4}\sqrt{\pi}}{s^{5/2}}$

$L[\cosh 3t] = \frac{s}{s^2-a^2} = \frac{s}{s^2-9}$

$L[\cos 3t \cdot \cos 5t] = \frac{1}{2}\{L[\cos(8t) + \cos(-2t)]\}$

$= \frac{1}{2}\left[\frac{s}{s^2+64} + \frac{s}{s^2+4}\right].$

$\therefore L[t^{3/2} + \cosh 3t + \cos 3t \cdot \cos 5t] = \frac{3\sqrt{\pi}}{4 s^{5/2}} + \frac{s}{s^2-9} + \frac{1}{2}\left[\frac{s}{s^2+64} + \frac{s}{s^2+4}\right]$

(iii) $L[\sin^3 2t + \cos^3 t]$

$= L[\sin^3 2t] + L[\cos^3 t].$

$\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$

$\cos^3\theta = \frac{1}{4}[3\cos\theta + \cos 3\theta]$

$= \frac{1}{4}\{L[3\sin\theta - \sin 3\theta]\} + \frac{1}{4}\{L[3\cos\theta] + L[\cos 3\theta]\}$

$$\frac{1}{4}\left[3L[\sin\overset{2t}{\theta}] - L[\sin 3\overset{2t}{\theta}]\right] + \frac{1}{4}\left[3L[\cos\overset{t}{\theta}] + L[\cos 3\overset{t}{\theta}]\right]$$

$$= \frac{1}{4}\left[3\,\frac{2}{s^2+4} - \frac{6}{s^2+36}\right] + \frac{1}{4}\left[3\cdot\frac{s}{s^2+1} + \frac{s}{s^2+9}\right]$$

$$= \cancel{\frac{1}{4}\left[\frac{12}{s^2+1} - \frac{36}{s^2+9}\right] + \frac{1}{4}\left[\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right]}.$$

$$= \cancel{\frac{1}{4}\left[\frac{12+3s}{s^2+1} + \frac{s-36}{s^2+9}\right]} \;//$$

$$= \frac{1}{4}\left(\frac{6}{s^2+4} - \frac{6}{s^2+36}\right) + \frac{1}{4}\left[\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right].$$

(iv) $L[\sin t\cdot\sin 2t\cdot\sin 3t]$ $\qquad \sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$L\left[\frac{1}{2}[\cos(t-2t) - \cos(3t)]\sin 3t\right].$$

$$= L\left[\frac{1}{2}[\cos t - \cos 3t]\sin 3t\right].$$

$$= \frac{1}{2}\left[L[\cos t\sin 3t - \cos 3t\sin 3t]\right].$$

$$\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)].$$

$$= \frac{1}{2}\left[L\left[\frac{1}{2}(\sin 4t + \sin(2t))\right]\right] - \frac{1}{2}\left[L\left[\frac{1}{2}\sin(6t) - \sin 0\right]\right].$$

$$= \frac{1}{4}\left[L[\sin 4t] + L[\sin 2t]\right] - \frac{1}{4}L[\sin 6t]. \qquad *$$

$$= \frac{1}{4}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right] - \frac{1}{4}\left[\frac{6}{s^2+36}\right].$$

$$= \frac{1}{4}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4} - \frac{6}{s^2+36}\right].$$

(v)

find the Laplace transform of $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ 0, & \text{otherwise.} \end{cases}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore L[f(t)] = \int_0^\pi e^{-st} \cos t\, dt$$

$\because \sin n\pi = 0$
$\cos n\pi = (-1)^n$

$$\because \int e^{at} \cos bt\, dt = \frac{e^{at}}{a^2+b^2}[a\cos bt + b\sin bt]$$

$a = -s,$

$$= \left[\frac{e^{-st}}{s^2+1}[-s\cos t + \sin t]\right]_{t=0}^{\pi}$$

$$= \frac{e^{-\pi s}}{s^2+1}[-s\cos(\pi) + \sin(\pi)] - \frac{1}{s^2+1}[-s\cos(0) + \sin(0)].$$

$$= \frac{e^{-\pi s}}{s^2+1}[-s\cdot(-1)^1 + 0] - \frac{1}{s^2+1}[-s+0].$$

$$= \frac{e^{-\pi s}}{s^2+1}[s] + \frac{1}{s^2+1}(s) = \frac{1}{s^2+1}[se^{-\pi s} + s] = \frac{s}{s^2+1}[e^{-\pi s}+1].$$

$f(t) = \begin{cases} \sin 2t, & (0,\pi) \\ 0, & \text{otherwise.} \end{cases}$

$$\therefore L[f(t)] = \int_0^\pi e^{-st} \sin 2t\, dt.$$

$$\because \int e^{at} \sin bt\, dt = \frac{e^{at}}{a^2+b^2}[a\sin bt - b\cos bt].$$

$$= \left[\frac{e^{-st}}{s^2+4}[-s\sin 2t - 2\cos 2t]\right]_{t=0}^{\pi}$$

$$= \frac{e^{-s\pi}}{s^2+4}\left[-s\sin 2\pi - 2\cos 2\pi\right] - \frac{1}{s^2+4}\left[-s\sin 0 - 2\cos 0\right]$$

$$= \frac{e^{-s\pi}}{s^2+4}\left[-2\cdot(-1)^2\right] - \frac{1}{s^2+4}\left[-2\right].$$

$$= \frac{1}{s^2+4}\left[-2\cdot e^{-s\pi} + 2\right]$$

* $f(t) = \begin{cases} e^{-3t}, & 0<t<4 \\ 0, & t>4 \end{cases}$

$$L[f(t)] = \int_0^4 e^{-st}\cdot e^{-3t}\,dt = \int_0^4 e^{-t[s+3]}\,dt$$

$$= \left[\frac{e^{-t[3+s]}}{s+3}\right]_0^4$$

$$= \frac{e^{-4(3+s)}}{s+3} - \frac{1}{s+3}.$$

$$= \frac{e^{-4(3+s)} - 1}{s+3}$$

First Shifting Theorm:-

If $L[f(t)] = F[s]$, then

(i) $L[e^{at}.f(t)] = F[s-a]$ [omit e^{at} term and replace s by s-a]

(ii) $L[e^{-at} f(t)] = F[s+a]$ [omit e^{at} term and replace s by s+a]

(i) $L[e^{at}.f(t)] = \int_0^\infty e^{-st}.e^{at}.f(t)\,dt$

$= \int_0^\infty e^{-st+at} f(t)\,dt$

$= \int_0^\infty e^{-[s-a]t} f(t)\,dt$

$= F[s-a].$

Homework:-

(i) $L[\cos^4 t]$

$\cos^4 t = [\cos^2 t]^2 = \left[\frac{1+\cos 2t}{2}\right]^2 = \frac{1}{4}[1+\cos^2 2t + 2\cos 2t]$

$= L\left[\frac{1}{4}[1+\cos^2 2t + 2\cos 2t]\right] = \frac{1}{4}.\left[L[1] + L[\cos^2 2t] + L[2\cos 2t]\right]$

$= \frac{1}{4}\left[\frac{1}{s} + L\left[\frac{1+\cos 4t}{2}\right] + L[2\cos 2t]\right]$

$= \frac{1}{4}\left[\frac{1}{s} + \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right) + 2.\frac{s}{s^2+4}\right]$

$= \frac{1}{4}\left[\frac{1}{s} + \frac{1}{2s} + \frac{s}{2(s^2+16)} + \frac{2s}{s^2+4}\right]$

(ii) $L[\sin^3 5t + \cosh 2t + e^{-5t}]$.

$= L[\sin^3 5t] + L[\cosh 2t] + L[e^{-5t}]$.

$L[\sin^3 5t] = \frac{1}{4}.L[35\text{in} 5t - \sin 15t]$

$= \frac{1}{4}\left[L[3\sin 5t] - L[\sin 15t]\right].$

$= \frac{1}{4}\left[3.\frac{5}{s^2+25} - \frac{15}{s^2+225}\right] = \frac{1}{4}\left[\frac{15}{s^2+25} - \frac{15}{s^2+225}\right]$

$L[\cosh 2t] = \frac{s}{s^2-a^2}$

$= \frac{s}{s^2-4}$

$L[e^{-5t}] = \frac{1}{s+a} = \frac{1}{s+5}$

$= \frac{1}{4}\left[\frac{15}{s^2+25} + \frac{15}{s^2+225}\right] + \frac{s}{s^2-4} + \frac{1}{s+5}$ (i)

(iii) $L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{\pi}} \cdot L\left[\frac{1}{\sqrt{t}}\right] = \frac{1}{\sqrt{\pi}} L[t^{-1/2}]$

$\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-1/2+1}} = \frac{1}{\sqrt{\pi}} \times \frac{\Gamma\frac{1}{2}}{s^{1/2}} = \frac{1}{\cancel{\sqrt{\pi}}} \times \frac{\cancel{\sqrt{\pi}}}{s^{1/2}}$

$= \frac{1}{s^{1/2}} = \frac{1}{\sqrt{s}}$

Second shifting theorem:-

If $L[f(t)] = F[s]$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0, & \text{otherwise} \end{cases}$ ii)

$$L[G(t)] = e^{-as} \cdot F[s]$$

$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$\therefore L[G(t)] = \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$

$= \int_a^\infty f(t-a)\, e^{-st}\,dt$

Put $u = t-a$

$du = dt$.

t	a	∞.
$u=t-a$	0	∞

$= \int_0^\infty f(u) \cdot e^{-s[u+a]}\,du$

$= \int_0^\infty f(u) \cdot e^{-su} \cdot e^{-sa}\,du = e^{-sa} \int_0^\infty e^{-su} \cdot f(u)\,du$.

$= e^{-sa} L[f(t)] = e^{-sa} F[s]$

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Derivative of transforms:-

$$L[t.f(t)] = \frac{-d}{ds}[L[f(t)]]$$

$$L[t^n.f(t)] = (-1)^n \frac{d^n}{ds^n}[L[f(t)]].$$

Find the laplace transform of following:

(i) $e^{-3t}.\cosh 3t$

$$= L[e^{-3t}.\cosh 3t]$$

$$L[e^{at}.f(t)] = L[f(t)] \Big|_{s \to s-a}.$$

$$L[\cosh 3t] \Big|_{s \to s+3}$$

$$= \left[\frac{s}{s^2-9}\right]_{s \to s+3}$$

$$= \frac{s+3}{(s+3)^2-9} = \frac{s+3}{s^2+\cancel{9}+6s-\cancel{9}} = \frac{s+3}{s^2+6s} = \frac{s+3}{s(s+6)}.$$

ii) $e^{-2t}.\sin 3t.\cos 5t.$

$$= L[e^{-2t}.\sin 3t.\cos 5t].$$

$$L[\sin 3t.\cos 5t].$$

$$= \frac{1}{2}.L[\sin(8t)+\sin(-2t)].$$

$$= \frac{1}{2}.[L[\sin 8t] - L[\sin 2t]].$$

$$= \frac{1}{2}\left[\frac{8}{s^2+64} - \frac{2}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{8}{s^2+4+4s+64} - \frac{2}{s^2+4+4s+4}\right].$$

$$= \frac{1}{2}\left[\frac{8}{s^2+4s+68} - \frac{2}{s^2+4s+8}\right]$$

$$= \left[\frac{4}{s^2+4s+68} - \frac{1}{s^2+4s+8}\right]$$

iii) $e^{-2t}\cdot \sin(2t+3)$

$$L[e^{2t}\cdot \sin(2t+3)] = L[\sin(2t+3)]\Big|_{s\to s+2}.$$

$$L[\sin 2t\cos 3 + \cos 2t \sin 3]$$

$$= \cos 3\cdot \frac{2}{s^2+4} + \sin 3\cdot \frac{s}{s^2+4}$$

$$= \cos 3\cdot \frac{2}{s^2+4s+8} + \sin 3\cdot \frac{s+2}{s^2+4s+8}.$$

$$= \frac{2\cos 3}{s^2+4s+8} + \frac{(s+2)\sin 3}{s^2+4s+8}$$

$$= \frac{1}{s^2+4s+8}\left(2\cos 3 + (s+2)\sin 3\right).$$

* Find the laplace transform of $[t\cdot \sinh 3t]$

(i) $L[t\cdot \sinh 3t]$

$$= -\frac{d}{ds}\left[L[\sinh 3t]\right]$$

$$= -\frac{d}{ds}\left[\frac{3}{s^2-9}\right].$$

$$= \frac{3\times 2s}{(s^2-9)^2} = \frac{6s}{s^4-18s^2+81}.$$

(ii) $L[t^2\cdot \cos 2t]$.

$$= (-1)^2\cdot \frac{d^2}{ds^2} L[\cos 2t].$$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right].$$

$$= \frac{d}{ds}\left[\frac{(s^2+4)(1)-(s)(2s)}{s^4+16+8s^2}\right] = \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)^2(-2s) - (4-s^2)[\dots]}{}$$

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$$= \frac{-2s\,(s^2+4)^2 - (4-s^2)\left(4s\,(s^2+4)\right)}{(s^2+4)^4}$$

$$= \frac{(s^2+4)\left[-2s(s^2+4) - (4s)(4-s^2)\right]}{(s^2+4)^3}$$

$$= \frac{-2s^3 - 8s - 16s + 4s^3}{(s^2+4)^3} \qquad \frac{d}{d}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

$$= \frac{2s^3 - 24s}{(s^2+4)^3} = \frac{2s(s^2-12)}{(s^2+4)^3}$$

* Find the laplace transform of

(i) $e^t.\ t^2.\sin 2t$

$$L[e^t.\ t^2.\sin 2t] = L[t^2.\sin 2t]\Big|_{s\to s-1}$$

$$= (-1)^2.\ \frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right] = \frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right]\Big|_{s\to s-1}$$

$$= \frac{d}{ds}\left[\frac{-2\times 2s}{(s^2+4)^2}\right]\Big|_{s\to s-1} = -4\cdot\frac{d}{ds}\left[\frac{s}{(s^2+4)^2}\right]$$

$$= -4.\left[\frac{(s^2+4)^2(1) - s\left(2(s^2+4).2s\right)}{(s^2+4)^4}\right]$$

$$= -4\left[\frac{\cancel{(s^2+4)}\,(s^2+4-4s^2)}{(s^2+4)^{4}}\right] = \frac{-4\,(-3s^2+4)}{(s^2+4)^3}$$

$$= \cancel{\frac{-4\,(1-4(s-1)^2)}{((s-1)^2+4)^3}} = \frac{-4\,(-3(s-1)^2+4)}{((s-1)^2+4)^3}$$

$$= \frac{-4\,(-3(s^2+1-2s)+4)}{(s^2-2s+1+4)^3} = \frac{-4\,(-3s^2-3+6s+4)}{(s^2-2s+5)^3}$$

$$= \frac{-4\,(-3s^2+6s+1)}{(s^2-2s+5)^3} = \frac{12s^2-24s-4}{(s^2-2s+5)^3}$$

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Homework:

* Find the laplace transform of following:

(i) $e^{at} \cdot \sinh bt$

(ii) $e^{t} \cdot t^{-1/2}$

(iii) $e^{-2t} \cdot t \cdot \cos 3t$

(iv) $t \cdot \sin 3t \cdot \sin 5t$

(v) $t^2 \cdot e^{3t} \cdot \sin 4t$

Unique step function:

* The unique step function are also known as Heaviside function. denoted by

$$U(t-a) / H(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$

$$L[U(t-a)] = \int_0^\infty e^{-st} \cdot U(t-a) \cdot dt$$

$$= \cancel{\int_0^a e^{-st}(0)}\, dt + \int_a^\infty e^{-st}(1)\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= 0 - \frac{e^{-sa}}{-s} = \frac{e^{-sa}}{s} \cdot //$$

$= L[\sinh bt] \Big| s \text{ by } s-a.$

$= \frac{b}{s^2-b^2} \Big|_{s \to s-a}$

$= \frac{b}{(s-a)^2-b^2} = \frac{b}{s^2+\cancel{a^2}_{a^2}-2as-\cancel{b^2}} = \frac{b}{s^2-2as+a^2-b^2}$

(ii) $L[e^t . t^{-1/2}]$

$= L[t^{-1/2}] \Big|_{s \text{ by } s-1}$

$= \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s-1}} = \sqrt{\frac{\pi}{s-1}}$

(iii) $L[e^{-2t}.t.\cos 3t].$

$= L[t \cdot \cos 3t] \Big|_{s \to s+2.}$

$= -\frac{d}{ds}\left[L[\cos 3t]\right]$

$= -\frac{d}{ds}\left[\frac{s}{s^2+9}\right]$

$= \frac{(s^2-9)(1) - s(2s)}{(s^2+9)^2} = \frac{s^2-9-2s^2}{(s^2+9)^2} = \frac{-9-s^2}{(s^2+9)^2} = -\frac{(s^2+9)}{(s^2+9)^2}$

$s \to s+2.$

$= \frac{-(s^2+4+4s+9)}{(s^2+4+4s+9)^2} = -\frac{(s^2+4s+13)}{(s^2+4s+\cancel{9}^{+13})^2}.$ //

(iv) $L[t. \sin 3t \cdot \sin 5t]$

$= -\frac{d}{ds}.\left[L[\sin 3t.\sin 5t]\right] = -\frac{d}{ds}.\left[L\left[\frac{1}{2}[\cos 2t - \cos 8t]\right]\right]$

$= \frac{1}{2} \cdot -\frac{d}{ds}\left[L[\cos 2t] - L[\cos 8t]\right] = \frac{1}{2} \cdot -\frac{d}{ds}\left[\frac{s}{s^2+4} - \frac{s}{s^2+64}\right]$

$= -\frac{1}{2}\left[\frac{(s^2+4)(1)-(s)(2s)}{(s^2+4)^2}\right] + \frac{1}{2}\left[\frac{(s^2+64)(1)-s(2s)}{(s^2+64)^2}\right]$

(v) $L[e^{3t}.t^2.\sin 4t]$.

$L[t^2 \sin 4t] \;\Big|_{s \to s-3}$

$$= \frac{d^2}{ds^2}\left[L[\sin 4t]\right] = \frac{d^2}{ds^2}\left[\frac{4}{s^2+16}\right] = \frac{d}{ds}\left[\frac{4\times 2s}{(s^2+16)^2}\right] = \frac{d}{ds}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$= \frac{(s^2-16)^2(8) - (8s)(2(s^2-16))(2s)}{(s^2+16)^4}$$

$$= \frac{(s^2-16)(8s^2-128-32s^2)}{(s^2+16)^4} = \frac{(-128-24s^2)}{(s^2+16)^3}$$

$$= \frac{(-128-24(s^2+9-6s))}{((s^2+9-6s)+16)^3} = \frac{-128-24s^2-216+144s}{(s^2-6s+25)^3}$$

$$= \frac{-24s^2+144s-344}{(s^2-6s+25)^3} \;//$$

$*$ $\int_0^\infty e^{-2t}.t\cos t.dt$

$L[f(t)] = \int_0^\infty e^{-st} f(t).dt$

on comparision $f(t) = t\cos t$, $s = 2$.

$= L[t\cos t]_{s=2}$

$= L[\cos t]_{s=2} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s=2}$

$= \left[\frac{(s^2+1)1 - s(2s)}{(s^2+1)^2}\right]_{s=2} = \left[\frac{-s^2+1}{(s^2+1)^2}\right]_{s=2}$

$= -\frac{(-3)}{25} = 3/25.$

Integral of transforms:-

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds$$

$$L\left[\frac{f(t)}{t^n}\right] = \underbrace{\int_s^\infty\int_s^\infty\int_s^\infty \cdots \int_s^\infty}_{n \text{ times}} L[f(t)]\,ds$$

1. $L\left[\frac{1-e^{-2t}}{t}\right]$

Apply L'optial

$$\underset{t\to 0}{Lt}\ \frac{1-e^{-2t}}{t} = \frac{1}{0} = \infty \text{ (indeter)}$$

$$= \underset{t\to 0}{Lt}\ \frac{2\cdot e^{-2t}}{1} = 2.$$

$$L\left[\frac{1-e^{-2t}}{t}\right] = \int_s^\infty L[1-e^{-2t}]\,ds.$$

$$= \int_s^\infty \left[L[1] - L[e^{-2t}]\right] ds$$

$$= \int_s^\infty \left[\left[\frac{1}{s}\right] - \left[\frac{1}{s+2}\right]\right] ds$$

$$= \int_s^\infty \frac{ds}{s} - \int_s^\infty \frac{ds}{s+2} = \cancel{\log}\left[= \left[\log s - \log s+2\right]_s^\infty\right.$$

$$= \cancel{\left[\log s\right]_s^\infty} - \cancel{\left[\log(s+2)\right]_s^\infty} = \left[\log\frac{s}{s+2}\right]_s^\infty$$

$$= \left[\log\frac{\cancel{s}}{\cancel{s}\left(1+\frac{2}{s}\right)}\right]_s^\infty$$

$$= \log 1 - \log\frac{1}{1+\frac{2}{s}}.$$

$$= -\log\left[\frac{s}{s+2}\right] = \log\left[\frac{s+2}{s}\right].$$

2. $L\left[\frac{e^{-at}-e^{-bt}}{t}\right].$

~~Apply~~ L'optial rule,

$$\underset{t\to 0}{Lt}\ \frac{e^{-at}-e^{-bt}}{t} = \frac{1-1}{0} = \frac{0}{0} \text{ (indeter)}.$$

$$\underset{t\to 0}{Lt} = \frac{-a\cdot e^{-at}+b\cdot e^{-bt}}{1} = -a+b = b-a.$$

f(t) exist.

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty L[e^{-at}-e^{-bt}]\,ds.$$

$$= \int_s^\infty \left[L[e^{-at}] - L[e^{-bt}]\right] ds$$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds.$$

$$= \int_s^\infty \frac{ds}{s+a} - \frac{ds}{s+b}$$

$$= \Big[\log[s+a] - \log[s+b]\Big]_s^\infty$$

$$= \left[\log\left[\frac{s+a}{s+b}\right]\right]_s^\infty = \left[\log\left[\frac{1+a/s}{1+b/s}\right]\right]_s^\infty$$

$$= \log[1] - \log\left[\frac{1+a/s}{1+b/s}\right]$$

$$= -\log\left[\frac{s+a}{s+b}\right] = \log\left[\frac{s+b}{s+a}\right]$$

* Find $L\left[\frac{\sin at}{t}\right]$ and hence evaluate $\int_0^\infty \frac{\sin t}{t}\, dt = \pi/2$.

$$\lim_{t\to 0} \frac{\sin at}{t} = \frac{0}{0}$$

$$\lim_{t\to 0} \frac{a\cos at}{1} = \frac{a}{1} = a.$$

$f(t)$. exist.

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\, ds.$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\, ds = a\int_s^\infty \frac{ds}{s^2+a^2} = \cancel{a}\left[\frac{\tan^{-1}\left[\frac{s}{a}\right]}{\cancel{a}}\right]_s^\infty$$

$$= \left[\tan^{-1}\left[\frac{s}{a}\right]\right]_s^\infty = \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \pi/2 - \tan^{-1}\left(\frac{s}{a}\right). = L\left[\frac{\sin at}{t}\right]$$

$$= \int_0^\infty e^{-st} \cdot \frac{\sin at}{t}\, dt$$

$s=0,\ a=1$

$$\therefore \int_0^\infty \frac{\sin t}{t}\, dt = \left[\frac{\pi}{2} - \tan^{-1}\left[\frac{s}{a}\right]\right]_0^\infty = \frac{\pi}{2} - \cancel{\tan^{-1}\left(\frac{0}{1}\right)}$$

$$= \pi/2.$$

* Find the LT of $L\left[\frac{e^{-3t}\cdot \sin 2t}{t}\right]$

$$\lim_{t\to 0} = \frac{0}{0}$$

$$\lim_{t\to 0} \frac{\cancel{-3e^{-3t}\,2\cos 2t}}{\cancel{1}} = \frac{-2}{1} = \frac{e^{-3t}(2\cos 2t) + \sin 2t\,(-3e^{-3t})}{1}$$

$\therefore f(t)$ exist. $= 2.$

$$L\left[\frac{e^{-3t}\cdot \sin 2t}{t}\right] = \int_s^\infty L[\sin 2t]\,ds \Big|_{s\to s+3}$$

$\tan^{-1}(s)+\cot^{-1}(s)=\pi/2$

$$= \int_s^\infty \frac{2}{s^2+4}\,ds \Big|_{s\to s+3}$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\left[\tan^{-1}\left[\frac{s}{2}\right]\right]_s^\infty \Big|_{s\to s+3}$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{2}\right) \Big|_{s\to s+3}$$

$$= \pi/2 - \tan^{-1}\left[\frac{s+3}{2}\right].//$$

$$= \cot^{-1}\left[\frac{s+3}{2}\right].$$

* $L\left[\frac{\sin^2 2t}{t}\right]$

$$\lim_{t\to 0} = \frac{0}{0}$$

$$\lim_{t\to 0} \frac{2\sin 2t\,(2\cos 2t)}{1} = 0 \qquad f(t) \text{ exist}$$

$$L\left[\frac{\sin^2 2t}{t}\right] = \int_s^\infty L[\sin^2 t]\,ds = \int_s^\infty L\left[\frac{1-\cos 4t}{2}\right] ds$$

$$= \frac{1}{2}\cdot\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+16}\right] ds$$

$$= \frac{1}{2}\left[\left[\log s\right]_s^\infty\right] - \frac{1}{4}\int_s^\infty \frac{2s}{s^2+16}\,ds$$

$$= \frac{1}{2}\left[\log s\right]_s^\infty - \frac{1}{4}\left[\log(s^2+16)\right]_s^\infty$$

* Prove that $\int_0^\infty \frac{1-\cos 2t}{t^2}.dt$

$$\underset{t\to 0}{Lt} \frac{1-\cos 2t}{t^2} = \frac{0}{0}.$$

$$\underset{t\to 0}{Lt} \frac{1+2\sin 2t}{2t} = \infty.$$

$$\underset{t\to 0}{Lt} \frac{4\cos 2t}{2} = \frac{4\times 1}{2} = 2. \quad \therefore f(t) \text{ exist}$$

$$= \int_s^\infty \int_s^\infty \left[L[1] - L[\cos 2t] \right] ds.$$

$$= \int_s^\infty \left[\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+2} \right) ds \right] ds.$$

$$= \int_s^\infty \left[\log s - \frac{1}{2} \log(s^2+4) \right]_s^\infty ds.$$

$$= \int_s^\infty \left[\log \frac{s}{\sqrt{s^2+4}} \right]_s^\infty ds. = \int_s^\infty \left[\log \frac{\cancel{s}}{\cancel{s}\sqrt{1+\frac{4}{s^2}}} \right]_s^\infty ds.$$

$$= \int_s^\infty \log \frac{\sqrt{s^2+4}}{s} ds. = \int_s^\infty \log \sqrt{\frac{s^2+4}{s^2}} ds.$$

$$= \frac{1}{2} \cdot \int_s^\infty \log \left(1 + \frac{4}{s^2} \right) ds.$$

$$u = \log \left[1 + \frac{4}{s^2} \right] \quad dv = ds \Rightarrow v = s.$$

$$du = \frac{1}{\left(1+\frac{4}{s^2}\right)} \times \left[0 + \frac{4\times -2}{s^3} \right] ds.$$

$$= \frac{\cancel{s^2}}{s^2+4} \times \frac{-8}{s^{\cancel{3}}} ds.$$

$$\int u dv = uv - \int v du.$$

$$= \left[s \log \left[1 + \frac{4}{s^2} \right] \right]_s^\infty - \int_s^\infty \cancel{s} \times \frac{-8}{(s^2+4)\cancel{s}} ds.$$

$$= 0 - s \log \left[\frac{1+4}{s^2} \right] + \left[8 \times \frac{1}{2} \tan^{-1}\left(\frac{s}{2} \right) \right]_s^\infty.$$

$$= -s \log \left[1 + \frac{4}{s^2} \right] + 4 \left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{2} \right) \right]$$

Homework:-

(1) $\int_0^\infty e^{-t}\, t\cos t \, dt.$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$f(t) = t\cos t \quad , s = 1$$

$$= L[t\cos t]_{s=1}$$

$$= L[\cos t]_{s=1} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s=1}$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right]_{s=1} = \left[\frac{1-s^2}{(s^2+1)^2}\right]_{s=1} = \frac{1-1}{(1+1)^2} = 0.$$

(2) $\int_0^\infty e^{-t}\, t\sin t \, dt.$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$f(t) = t\sin t, \quad s = 1$$

$$L[t\sin t]_{s=1} = L[\sin t]_{s=1} = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]_{s=1}$$

$$= -\left[(-1)(s^2+1)^{-2}(2s)\right]_{s=1}$$

$$= +\left[\frac{1}{(1+1)^2}(2\times 1)\right] = \frac{1}{4}\times 2 = \frac{1}{2}.$$

(3) $L\left[\frac{1-\cos at}{t}\right]$

$$\lim_{t\to 0} \frac{1-\cos at}{t} = \frac{0}{0} \Rightarrow \lim_{t\to 0} \frac{a\sin at}{1} = \frac{a.0}{1} = 0 \quad \therefore f(t) \text{ exist}$$

$$= \int_s^\infty \left[L[1] - L[\cos at]\right] ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] ds = \int_s^\infty \frac{1}{s}\, ds - \int_s^\infty \frac{s}{s^2+a^2}\, ds = \left[\log s\right]_s^\infty - \frac{1}{2}\left(\log(s^2+a^2)\right)_s^\infty$$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right]_s^\infty = \left[\log\left[\frac{1}{\sqrt{1+\frac{a^2}{s^2}}}\right]\right]_s^\infty$$

$$= -\log\left[\frac{1\times s}{\sqrt{s^2+a^2}}\right] = \log\left[\frac{\sqrt{s^2+a^2}}{s}\right].$$

4. $L\left[\frac{\sin 3t \cdot \cos 2t}{t}\right]$ $\quad \sin 3t \cos 2t = \frac{1}{2}\left[\sin(5t) + \sin(t)\right]$

$$\frac{1}{2}\left[\int_s^\infty L[\sin 5t] + L[\sin t]\, ds\right] = \frac{1}{2}\left[\int_s^\infty \left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right] ds\right]$$

$$= \left[\frac{5}{2} \cdot \frac{1}{5} \tan^{-1}\left[\frac{s}{5}\right] + \frac{1}{2} \cdot \tan^{-1}\left[\frac{s}{1}\right]\right]_s^\infty$$

$$= \left(\frac{1}{2}\tan^{-1}\left[\frac{s}{5}\right]\right]_s^\infty + \left[\frac{1}{2}\tan^{-1}\left[\frac{s}{1}\right]\right]_s^\infty$$

$$= \left[\frac{1}{2} \times \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left[\frac{s}{5}\right]\right] + \left[\frac{1}{2} \times \frac{\pi}{2} - \frac{1}{2}\tan^{-1}(s)\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{s}{5}\right] + \frac{\pi}{4} - \frac{1}{2}\tan^{-1}(s)$$

$$= \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left[\frac{s}{5}\right] - \frac{1}{2}\tan^{-1}(s).$$

5. P.T. $L\left[\frac{\cos at - \cos bt}{t}\right]$

$$\lim_{t\to 0} \frac{\cos at - \cos bt}{t} = \frac{0}{0} \Rightarrow \lim_{t\to 0} \frac{-a\sin at + b\sin bt}{1} = 0 \quad \therefore f(t) \text{ exist}$$

$$= \int_s^\infty \left[L[\cos at] - L[\cos bt]\right] ds = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$$

$$= \left[\log(s^2+a^2)^{1/2} - \log(s^2+b^2)^{1/2}\right]_s^\infty$$

$$= \left[\log\left[\frac{s\sqrt{1+a^2/s^2}}{s\sqrt{1+b^2/s^2}}\right]\right]_s^\infty = \log 1 - \log\left[\frac{\sqrt{1+\frac{a^2}{s^2}}}{\sqrt{1+\frac{b^2}{s^2}}}\right] = -\log\left[\frac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}}\right]$$

$$= \log\frac{\sqrt{s^2+b^2}}{\ldots} = \frac{1}{\ldots}\log\left(\frac{s^2+b^2}{\ldots}\right]$$

Solve $\int_0^\infty \frac{e^{-\sqrt{2t}} \sin\sqrt{2t}\, dt}{t}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

Put $u = \sqrt{2t}$

t	0	∞
$u=\sqrt{2t}$	0	∞

$$du = \sqrt{2} \cdot \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{\sqrt{2t}} = \frac{dt}{u}$$

$$I = \int_0^\infty \frac{e^{-u} \sin u}{\left(\frac{u^2}{2}\right)} \cancel{u} \cdot du = \int_0^\infty \frac{2e^{-u} \sin u\, du}{u}$$

$$= 2 \cdot \int_0^\infty \frac{e^{-t} \sin t}{t}\, dt = 2 \cdot L\left[\frac{\sin t}{t}\right]_{s=1}$$

$$= 2 \int_s^\infty L[\sin t]\, ds \Big|_{s=1} = 2 \int_s^\infty \frac{1}{s^2+1}\, ds \Big|_{s=1} = 2\left[\frac{1}{1} \cdot \tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty \Big|_{s=1}$$

$$= 2\left[\tan^{-1}(\infty) - \tan^{-1}(s)\right]_{s=1} = 2\left[\frac{\pi}{2} - \tan^{-1}(1)\right] = 2\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{2}$$

Formulae:-

* $L[f(t)] = F[s]$
* $L[t \cdot f(t)] = -\frac{d}{ds}\left[L[f(t)]\right] = -\frac{d}{ds}\left[F(s)\right]$.
* $\int_0^\infty e^{-st} f(t)\, dt = F(s)$

$$F(s) = L[f(t)]$$

$$\frac{d}{ds}\left(F(s)\right) = \frac{d}{ds}\left[L(f(t))\right].$$

$$F'(s) = \int_0^\infty \frac{d}{ds}\left(e^{-st} \cdot f(t)\right) dt$$

$$F'(s) = \int_0^\infty f(t) \cdot e^{-s} \times (-t)\, dt$$

$$= -\int_0^\infty f(t) \cdot t \cdot e^{-st}\, dt$$

$$= -L[t \cdot f(t)].$$

(2) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \{L[f(t)]\}$

$= (-1)^n \frac{d^n}{ds^n} \{F(s)\}$

$n=2,$

$L[t^2 f(t)] = L[t \cdot t \cdot f(t)] = -\frac{d}{ds}[L[t\, f(t)]]$

$= -\frac{d}{ds} \times -\frac{d}{ds}\left(L[f(t)]\right)$

$L[t^2 \cdot f(t)] = \frac{d^2}{ds^2} L[f(t)]$

$\vdots$

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \cdot L[f(t)]$

(3) $L[f(t)] = F(s)$ & $\underset{t\to 0}{Lt} \frac{f(t)}{t}$ exist then

$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)] \cdot ds$

$F[s] = L[f(t)]$

$\Rightarrow \int_s^\infty L[f(t)]\, ds$

$= \int_s^\infty \int_0^\infty e^{-st} f(t)\, dt\, ds$ (change the order of integration)

$= \int_0^\infty \int_s^\infty e^{-st} f(t)\, ds\, dt = \int_s^\infty f(t) \cdot \left[\frac{e^{-st}}{-t}\right]_s^\infty dt$

$= \int_s^\infty \frac{f(t)}{t}\left[e^{-\infty} - e^{-st}\right] dt = \int_s^\infty \frac{e^{-st} f(t)}{t}\, dt = L\left[\frac{f(t)}{t}\right]$

By applying limit at some point it will be a Laplace transform.

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Change of scape property:-

* If $a>0$, $L[f(at)] = \frac{1}{a}\cdot F\left[\frac{s}{a}\right]$.

$L[f(t)] = F[s]$

$$L[f(at)] = \int_0^\infty e^{-st}\cdot f(at)\,dt$$

t	0	∞
u=at	0	∞

$u = at$

$du = a\cdot dt$

$$= \int_0^\infty e^{-s\left[\frac{u}{a}\right]}\cdot f(u)\cdot\frac{du}{a}$$

$$= \frac{1}{a}\cdot\int_0^\infty e^{\frac{-su}{a}}\cdot f(u)\cdot du$$

$$= \frac{1}{a}\cdot\int_0^\infty e^{-st/a}\cdot f(t)\,dt$$

$$= \frac{1}{a}\cdot F\left(\frac{s}{a}\right]$$

* $L[f'(t)] = s\cdot L[f(t)] - f(0)$.

$$L[f'(t)] = \int_0^\infty e^{-st}\cdot f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\cdot d[f(t)]$$

$u = e^{-st}$ $\quad dv = d[f(t)]$

$du = -s\cdot e^{-st}dt$ $\quad v = f(t)$.

$$\int u\,dv = uv - \int v\,du$$

$$= \left[e^{-st}\cdot f(t)\right]_0^\infty - \int_0^\infty f(t)\cdot -s\cdot e^{-st}dt$$

$$= [0 - f(0)] + s\cdot\int_0^\infty e^{-st}f(t)\cdot dt$$

$$= s\cdot L[f(t)] - f(0).$$

* $$L[f''(t)] = \int_0^\infty e^{-st}\cdot[f''(t)]\,dt$$

$$= \int_0^\infty e^{-st}\cdot d[f'(t)]\,dt.$$

$u = e^{-st}$ $\quad dv = d[f'(t)]$.

$du = -s\cdot e^{-st}dt$ $\quad v = f'(t)$

$$[e^{-st}f'(t)]^\infty - \int^\infty f'(t)\cdot -s\cdot e^{-st}dt$$

$$= [0 - f'(0)] + s \cdot \int_0^\infty f'(t) \cdot e^{-st} dt$$

$$= -f'(0) + s \cdot L[f'(t)].$$

$$= -f'(0) + s\,[s \cdot L[f(t)] - f(0)].$$

$$= s^2 \cdot L[f(t)] - s \cdot f(0) - f'(0).$$

* $L\left[\int_0^t f(t)dt\right] = \frac{1}{s} \cdot L[f(t)]$

$$F(t) = \int_0^t f(t)dt \quad ; \quad F(0) = 0$$

$$F'(t) = f(t)$$

$$L[F'(t)] = s \cdot L[F(t)] - F(0)$$

$$= s \cdot L[F(t)]$$

$$\frac{1}{s} \cdot L[F'(t)] = L[F(t)].$$

$$\frac{1}{s} \cdot L[f(t)] = L\left[\int_0^t f(t)dt\right].$$

Intial Value Theorm (IVT):-

IVT: $\underset{t \to 0}{Lt} f(t) = \underset{s \to \infty}{Lt} s \cdot L[f(t)]$

FVT:- $\underset{t \to \infty}{lt} f(t) = \underset{s \to 0}{lt} s \cdot L[f(t)].$

$$f(t) = e^{-t}[\cos t + \sin t] + 1$$

Homework:-

(1) $f(t) = e^{-t}(t+2)^2$

IVT:-

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s\cdot L[f(t)]$$

$$L.H.S = \underset{t\to 0}{Lt}\ f(t) = \underset{t\to 0}{Lt}\ e^{-t}(t+2)^2$$

$$= e^{-(0)}(0+2)^2$$

$$= 1(0+2)^2 = 4$$

$$R.H.S = \underset{s\to\infty}{Lt}\ s\cdot L[f(t)].$$

$$L[f(t)] = L[e^{-t}(t+2)^2]$$

$$= L[t^2+4t+4]\Big|_{s\to s+1}$$

$$= L[t^2]+4L[t]+L[4] = \frac{2!}{s^{2+1}} + \frac{4\cdot 1!}{s^{1+1}} + \frac{4}{s}$$

$$= \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} = \frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{(s+1)}$$

$$= \frac{2+4(s+1)+4(s+1)^2}{(s+1)^3}$$

$$= \frac{2+4s+4+4s^2+4+8s}{(s^3+1+3s^2+3s)}$$

$$= \frac{4s^2+12s+10}{s^3+3s^2+3s+1}$$

$$s\cdot L[f(t)] = \frac{\cancel{s^3}\left(4+\frac{12}{s}+\frac{10}{s^2}\right)}{\cancel{s^3}\left(1+\frac{3}{s}+\frac{3}{s^2}+\frac{1}{s^3}\right)}$$

$$\underset{s\to\infty}{Lt}\ s\cdot L[f(t)] = \underset{s\to\infty}{Lt}\ \frac{4+\frac{12}{s}+\frac{10}{s^2}}{1+\frac{3}{s}+\frac{3}{s^2}+\frac{1}{s^3}}$$

$$= \frac{4}{1}$$

$$= 4$$

$\therefore L.H.S = R.H.S$

F.V.T:-

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\cdot L[f(t)].$$

$f(t) = e^{-t}(t+2)^2$

$$L.H.S = \underset{t\to\infty}{Lt}\ e^{-t}(t+2)^2$$

$$= 0$$

$$R.H.S = \underset{s\to 0}{Lt}\ sL[f(t)]$$

$$s\cdot L[f(t)] = \frac{4s^3+12s^2+10s}{s^3+3s^2+3s+1}$$

$$\underset{s\to 0}{Lt}\ s\cdot L[f(t)] = \frac{4(0)+12(0)+10(0)}{(0)^3+3(0)^2+3(0)+1} = \frac{0}{1} = 0.$$

③ $f(t) = 5\cdot e^{-3t}$

I.V.T:-

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s\cdot L[f(t)].$$

$$LHS: \underset{t\to 0}{Lt}\ 5\cdot e^{-3t} = 5\cdot e^{-3(0)}$$

$$= 5\cdot e^{0}$$

$$= 5$$

$$R.H.S: s\cdot L[f(t)] = 5s\cdot L[e^{-3t}]$$

$$= 5s\cdot\frac{1}{s+3}$$

$$\underset{s\to\infty}{Lt}\ s\cdot L[f(t)] = \underset{s\to\infty}{Lt}\ \frac{5s}{s+3} = \underset{s\to\infty}{Lt}\ \frac{\cancel{s}[5]}{\cancel{s}[1+\frac{3}{s}]}$$

$$= 5.$$

F.V.T:-

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\cdot L[f(t)].$$

$$L.H.S: \underset{t\to\infty}{Lt}\ f(t) = 5\underset{t\to\infty}{Lt}\ e^{-3t} = 5(0) = 0$$

$$R.H.S: \underset{s\to 0}{Lt}\ s\cdot L[f(t)] = \underset{s\to 0}{Lt}\ \frac{5s}{s+3} = \frac{5(0)}{(0+3)} = 0.$$

③ $f(t) = 1 - e^{-at}$

IVT:-

$$\underset{t\to 0}{Lt} f(t) = \underset{s\to\infty}{Lt} s \cdot L[f(t)]$$

L.H.S: $\underset{t\to 0}{Lt} f(t) = \underset{t\to 0}{Lt} (1-e^{-at})$

$$= 1 - e^{-a(0)} = 1 - e^{0} = 1-1 = 0$$

R.H.S:

$$\underset{s\to\infty}{Lt} s\cdot L[f(t)]$$

$$s\cdot L[f(t)] = s\cdot L[1-e^{-at}] = s\cdot\left[\frac{1}{s} - \frac{1}{s+a}\right]$$

$$= 1 - \frac{s}{s+a}$$

$$\underset{s\to\infty}{Lt}\ 1 - \frac{s}{s+a} = \underset{s\to\infty}{Lt}\ 1 - \frac{\cancel{s}(1)}{\cancel{s}(1+\frac{a}{s})} = 1 - \underset{s\to\infty}{Lt} \frac{1}{1+\frac{a}{s}} = 1-1 = 0.$$

F.V.T:-

$$\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt} s\cdot L[f(t)].$$

L.H.S: $\underset{t\to\infty}{Lt} f(t) = \underset{t\to\infty}{Lt} (1-e^{-at}) = 1 - \underset{t\to\infty}{Lt} e^{-at} = 1 - 0 = 1$

R.H.S: $\underset{s\to 0}{Lt} \left(1 - \frac{s}{s+a}\right) = 1 - \underset{s\to 0}{Lt} \frac{s}{s+a} = 1 - \frac{0}{0+a} = 1 - 0 = 1.$

Periodic functions:-

* A function $f(t)$ is said to have a period p if $f(t+p) = f(t)$ in some interval. (p is the smallest positive integer).

$$f(t+p) = f(t)$$

$$f(t) = \sin t$$

$$f(t+\pi) = \sin(t+\pi)$$

$$= \sin t\cos\pi + \cos t\sin\pi$$

$$= -\sin t.$$

π is not a period.

$$f(t+2\pi) = \sin(t+2\pi) = \sin t\cos 2\pi + \cos t \sin 2\pi$$

$$= \sin t$$

$$= f(t)$$

Similarly, $f(t) = \cos t$ the period ...

$f(t) = \tan t$ the period is π.

Laplace transform of periodic function:-

$$\frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\,dt = L[f(t)]$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^p e^{-st} f(t)\,dt + \int_p^\infty e^{-st} f(t)\,dt$$

Put $t = u+p$

$dt = du$

$u = t-p$

t	p	∞
$u = t-p$	0	∞

e^{-su}
e^{-sp}

$$= \int_0^p e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+p)} \underbrace{f(u+p)}_{f(u)}\,du \quad [\because f(t+p) = f(t)]$$

$$= \int_0^p e^{-st} f(t)\,dt + e^{-ps} \int_0^\infty e^{-su} f(u)\,du$$

u = dummy.

$$L[f(t)] = \int_0^p e^{-st} f(t)\,dt + e^{-ps}\, L[f(t)]$$

$$L[f(t)][1 - e^{-ps}] = \int_0^p e^{-st} f(t)\,dt$$

$$\boxed{L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt.}$$

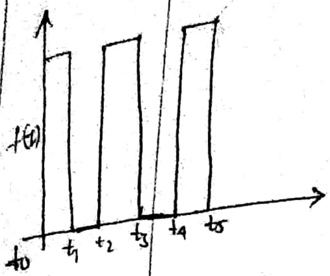
* Find the laplace transform of the rectangular wavefunction

$$f(t) = \begin{cases} +k, & 0<t<b \\ -k, & b<t<2b. \end{cases}$$

$$\frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2bs}} \left\{ \int_0^b e^{-st}(k)\,dt + \int_b^{2b} e^{-st}(-k)\,dt \right\}$$

$$= \frac{k}{(1-e^{-2bs})} \left\{ \int_0^b e^{-st}\,dt - \int_b^{2b} e^{-st}\,dt \right\}$$

$$= \frac{k}{1-e^{-2bs}} \left\{ \left[\frac{e^{-st}}{-s}\right]_0^b - \left[\frac{e^{-st}}{-s}\right]_b^{2b} \right\}$$



$$= \frac{k}{s(1-e^{-2bs})}\left\{[-e^{-bs}-(-1)] - [(-e^{-2bs})-(-e^{-bs})]\right\}$$

$$= \frac{k}{s(1-e^{-2bs})}[-e^{-bs}+1+e^{-2bs}-e^{-bs}].$$

$$= \frac{k}{s(1-e^{-2bs})}[1-2e^{-bs}+e^{-2bs}]$$

$A^2-2AB+B^2$

$$= \frac{k}{s(1-e^{-2bs})}[(1-e^{-bs})]^2$$

$$= \frac{k}{s[\cancel{1-e^{-bs}}][1+e^{-bs}]}[1-e^{-bs}]^{\cancel{2}}$$

$$= \frac{k}{s} \times \frac{[1-e^{-bs}]}{[1+e^{-bs}]}$$

$$= \frac{k}{s} \times \tanh\left[\frac{bs}{2}\right]$$

$2\theta = bs$

$\theta = \frac{bs}{2}$.

$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$

$\tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$

$\frac{e^{-\theta}}{e^{-\theta}}$

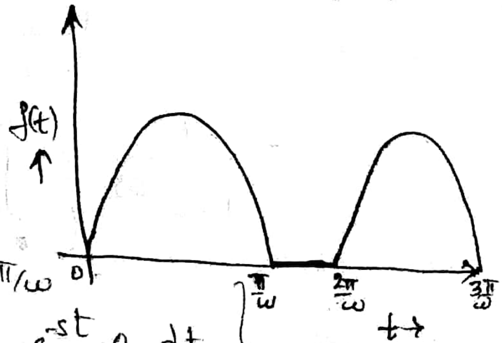
$= \frac{1-e^{-2\theta}}{1+e^{-2\theta}}$

H.W.

* $f(t) = \begin{cases} +E, & 0<t<a/2 \\ -E, & a/2<t<a. \end{cases} = \frac{E}{s} \times \tanh\left[\frac{bs}{4}\right].$

* find the laplace transform of half wave sine recitifier.

$$f(t) = \begin{cases} \sin\omega t, & 0<t<\pi/\omega \\ 0, & \frac{\pi}{\omega}<t<\frac{2\pi}{\omega}. \end{cases}$$



$P = +2\pi/\omega$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left\{ \int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \underbrace{\int_{\pi/\omega}^{2\pi/\omega} e^{-st}.0.dt}_{(0)} \right\}$$

$$\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}\left[a\sin bt - b\cos bt\right]$$

here $a=-s$, $b=\omega$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left[\frac{e^{-st}}{s^2+\omega^2} \left[-s\sin\omega t - \omega\cos\omega t \right] \right]_0^{\pi/\omega}$$

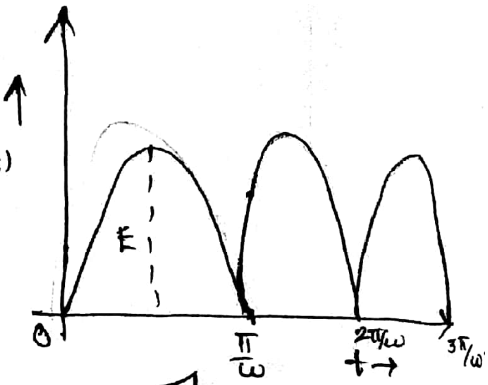
$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left\{ \left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2} \left[-s\underset{(0)}{\sin \omega \times \frac{\pi}{\omega}} - \omega \underset{(-1)}{\cos \omega \times \frac{\pi}{\omega}} \right] \right] - \left[\frac{1}{s^2+\omega^2} \left[-s\underset{(0)}{\sin 0} - \omega \underset{1}{\cos 0} \right] \right] \right\}$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}} \left[\frac{\omega \cdot e^{-s\pi/\omega} + \omega}{s^2+\omega^2} \right]$$

$\cancel{f(t) = \begin{cases} \sin\omega t, & 0 < t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}}$

$$= \frac{\omega}{1-e^{-\frac{2\pi}{\omega}s}} \frac{(1+e^{-s\pi/\omega})}{s^2+\omega^2}$$

$$= \frac{\omega}{(1-e^{-\pi/\omega s})} \times \frac{1}{s^2+\omega^2} //$$



* $f(t) = \begin{cases} \sin 10t, & 0 < t < \frac{\pi}{10} \\ 0, & \frac{\pi}{10} < t < \frac{\pi}{5} \end{cases}$

Ass * full wave sine rectifier $f(t) = \sin t$, $p = \pi$

Ass * full wave cos rectifier $f(t) = \cos t$, $p = \pi$

* Find the Laplace transform of triangular waveform $f(t) = \begin{cases} t, & 0 < t < a \\ (2a-t), & a < t < 2a \end{cases}$

$$= \frac{1}{1-e^{-2as}} \left\{ \underbrace{\int_0^a e^{-st}\, t\, dt}_{I_1} + \underbrace{\int_a^{2a} e^{-st} (2a-t)\, dt}_{I_2} \right\} \rightarrow (1)$$

$$\int uv\, dx = uv_1 - u'v_1 + \cdots$$

I_1:

$u = t \qquad v = e^{-st}$

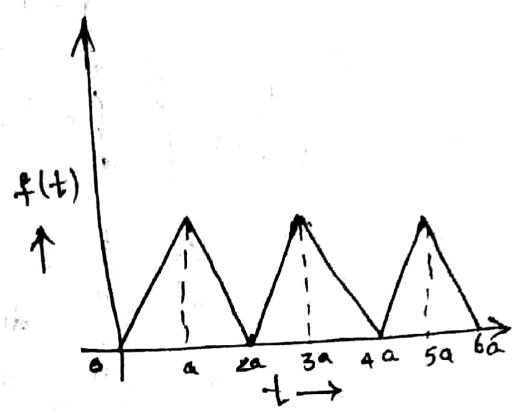
$u' = 1 \qquad v_1 = \frac{e^{-st}}{-s}$

$u'' = 0 \qquad v_2 = \frac{ae^{-st}}{} - \frac{e^{-st}}{}$

$$\therefore I_1 = \left[\frac{t \cdot e^{-st}}{-s}\right]_{t=0}^{a} - \left[\frac{1 \cdot e^{-st}}{s^2}\right]_{t=0}^{a}$$

$$= \left\{\frac{a\, e^{-sa}}{-s} - (0)\right\} - \left[\frac{e^{-as}}{s^2} - \frac{1}{s^2}\right].$$

$$I_1 = -\frac{a e^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}$$



I_2 :-

$u = 2a - t \qquad v = e^{-st}$

$u' = -1 \qquad v_1 = \frac{e^{-st}}{-s}$

$u'' = 0. \qquad v_2 = \frac{e^{-st}}{s^2}$

$$I_2 = \left[\frac{(2a-t)\, e^{-st}}{-s} - \frac{(-1) e^{-st}}{s^2}\right]_{t=a}^{2a}.$$

$$= \left\{\left[0 + \frac{e^{-2as}}{s^2}\right] - \left[\frac{a \cdot e^{-as}}{-s} + \frac{e^{-as}}{s^2}\right]\right\}$$

$$I_2 = \frac{e^{-2as}}{s^2} + \frac{a \cdot e^{-as}}{s} - \frac{e^{-as}}{s^2}$$

sub I_1, I_2 in (1)

$$(1) \rightarrow \frac{1}{1-e^{-2as}}\left[\frac{-2 \cdot e^{-sa}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2}\right]$$

$$\Rightarrow \frac{1}{s^2[1-(e^{-as})^2]}\left[-2e^{-sa} + 1 + (e^{-as})^2\right].$$

$$= \frac{1}{s^2(\cancel{1-e^{-as}})(1+e^{-as})} \times [1-e^{-as}]^2$$

$$= \frac{1}{s^2}\left\{\frac{[1-e^{-as}]}{[1+e^{-as}]}\right\}$$

$$= \frac{1}{s^2}\tanh\left[\frac{as}{2}\right].$$

* $f(t) = \begin{cases} t, & 0<t<1 \\ 2-t, & 1<t<2 \end{cases} \quad = \frac{1}{s^1} \tanh\left[\frac{s}{a}\right]$

$$\Rightarrow \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

$p = 2.$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st}\, t\, dt + \int_1^2 e^{-st}\, 2t\, dt. \right.$$

$v = e^{-st}$ $\quad u = t$

$v_1 = \frac{e^{-st}}{-s}$ $\quad u' = 1$

$v_2 = \frac{e^{-st}}{s^2}$ $\quad u'' = 0.$

$v = e^{-st}$ $\quad u = 2-t$

$v_1 = \frac{e^{-st}}{s}$ $\quad u' = -1$

$v_2 = \frac{e^{-st}}{s^2}$ $\quad u'' = 0.$

$$I_1 = \left[\frac{t \cdot e^{-st}}{-s}\right]_0^1 - \left[\frac{(1)\, e^{-st}}{s^2}\right]_0^1$$

$$= \left[\frac{-e^{-s}}{s} - 0\right] - \left[\frac{e^{-s}}{s^2} \times \frac{1}{s^2}\right].$$

$$= \frac{e^{-s} - s \cdot e^{-s} + 1}{s^2}$$

$$I_2 = \left[\frac{(2-t)\, e^{-st}}{-s}\right]_1^2 - \left[\frac{(-1)\, e^{-st}}{s^2}\right]_1^2$$

$$= \left[0 - \frac{e^{-s}}{-s}\right] + \left[\frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}\right]$$

$$= \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s^2}$$

$=$

$$= \frac{1}{s^2} \tanh\left[\frac{s}{a}\right].$$

$$\tanh\theta = \frac{1-e^{-2\theta}}{1+e^{-2\theta}}$$

$$\coth\theta = \frac{1+e^{-2\theta}}{1-e^{-2\theta}}.$$

Inverse Laplace Transform:-

If $L[f(t)] = F[s]$, then $f(t) = L^{-1}[F(s)]$

where L^{-1} is the inverse Laplace operator.

Linearity Property:- $L^{-1}[c_1 \cdot F_1(s) + c_2 \cdot F_2(s)]$

$$= c_1 \cdot L^{-1}[F_1(s)] + c_2 \cdot L^{-1}[F_2(s)]$$

Proof:-

$$L[f_1(t)] = F_1(s) \Rightarrow f_1(t) = L^{-1}[F_1(s)]$$

$$L[f_2(t)] = F_2(s) \Rightarrow f_2(t) = L^{-1}[F_2(s)]$$

W.K.T, Linearity property of L.T,

$$L[c_1 \cdot f_1(t) + c_2 \cdot f_2(t)] = c_1 \cdot L[f_1(t)] + c_2 \cdot L[f_2(t)]$$

$$= c_1 \cdot F_1(s) + c_2 \cdot F_2(s)$$

$$e_1 \cdot f_1(t) + c_2 \cdot f_2(t) = L^{-1}[c_1 \cdot F_1(s) + c_2 \cdot F_2(s)].$$

$$c_1 \cdot L^{-1}[F_1(s)] + c_2 \cdot L^{-1}[F_2(s)] = L^{-1}[c_1 \cdot F_1(s) + c_2 \cdot F_2(s)]$$ //

Formulae:-

* $L^{-1}\left[\frac{1}{s}\right] = 1$; $L^{-1}\left[\frac{k}{s}\right] = k$
* $L[t^n] = \frac{n!}{s^{n+1}}$; $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$ $L[t] = \frac{1}{s^2} \Rightarrow t = L^{-1}\left[\frac{1}{s^2}\right]$
* $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
* $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
* $L^{-1}\left[\frac{s}{s^2 7 a^2}\right] = \cos at$
* $L^{-1}\left[\frac{a}{s^2 7 a^2}\right] = \frac{1}{a}\sin at$
* $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
* $L^{-1}\left[\frac{1}{a}\cdot\frac{a}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

* $L^{-1}[F(s-a)] = e^{at}.F(s)$ $\quad | s-a \to s$

* $L^{-1}[F(s+a)] = e^{-at}F(s)$ $\quad | s+a \to s$

Pr: Find $L^{-1}\left[\frac{s+2}{s^2+4s+8}\right]$

$s^2+4s+4+4$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+4}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+2^2}\right].$$ By shifting property.

$s+2 \to s$

$$= L^{-1}\left[\frac{s}{s^2+2^2}\right] \cdot e^{-2t}$$

$$= e^{-2t}. \cos 2t.$$

Pr: Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

$$= L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right].$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}\right] = L^{-1}\left[\frac{\cancel{s+2}}{(s+2)^{\cancel{2}}}\right] - L^{-1}\left[\frac{2}{(s+2)^2}\right].$$

$$= L^{-1}\left[\frac{1}{s+2}\right] - 2. L^{-1}\left[\frac{1}{(s+2)^2}\right].$$ Replace ~~s+2~~ s+2 by s

$$= e^{-2t} - 2.e^{-2t}.t$$

$$= e^{-2t}(1-2t)$$

Method Of Partial Fraction:-

Degree of Nr < Degree of Dr

* $$\frac{F(s)}{(s-a)(s-b)(s-c)} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c}$$

* $$\frac{F(s)}{(s-a)^2(s-b)} = \frac{A}{(s-a)} + \frac{B}{(s-b)^2} + \frac{C}{(s-b)}$$

* $$\frac{F(s)}{(s^2+a^2)(s+b)} = \frac{AS+B}{(s^2+a^2)} + \frac{C}{(s+b)}$$

① Find $L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right]$

$$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

Put $s=1$,

$1 = A(-1)(-2)$

$2A = 1$; $A = 1/2$.

Put $s=2$;

$1 = B(1)(-1)$

$B = -1$

Put $s=3$;

$5 = C(2)(1)$

$2C = 5 \Rightarrow C = 5/2$.

$$\Rightarrow L^{-1}\left[\frac{1}{2(s-1)} - \frac{1}{(s-2)} + \frac{5}{2(s-3)}\right]$$

$$\Rightarrow \frac{1}{2}\cdot L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s-2}\right] + \frac{5}{2}\cdot L^{-1}\left[\frac{1}{s-3}\right].$$

$$\Rightarrow \frac{1}{2}\cdot e^{t} - 1\, e^{2t} + \frac{5}{2}\cdot e^{3t}$$

$$= \frac{1}{2}e^{t} + \frac{5}{2}e^{3t} - e^{2t}.$$

Homework:-

① $L^{-1}\left[\frac{1}{(s+2)(s+3)}\right]$

$$\frac{1}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)}$$

$$1 = A(s+3) + B(s+2)$$

Put $s=-3$, $1 = B(-1) \Rightarrow B = -1$

Put $s=-2$, $1 = A(1) \Rightarrow A = 1$

$$\rightarrow L^{-1}\left[\frac{1}{(s+2)} - \frac{1}{(s+3)}\right] = L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{s+3}\right].$$

$$= e^{-2t} - e^{-3t}$$

② $L^{-1}\left[\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right]$

$$s^2-3s+2 = s^2-2s-1s+2$$
$$= s(s-2)-1(s-2) \Rightarrow (s-1)(s-2)$$

$$= L^{-1}\left[\frac{4s^2-3s+5}{(s+1)(s-1)(s-2)}\right].$$

$$\frac{4s^2-3s+5}{(s+1)(s-1)(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$$

$$4s^2-3s+5 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1).$$

Let $s=-1$, $4+3+5 = A(-2)(-3) \Rightarrow 12 = 6A \Rightarrow A = 2$

$s=1$, $4-3+5 = B(2)(-1) \Rightarrow 6 = -2B \Rightarrow B = -3$

$s=2$, $16-6+5 = C(3)(2) \Rightarrow 15 = 6C \Rightarrow C = 5/2$

$$L^{-1}\left[\frac{2}{s+1} - \frac{3}{s-1} + \frac{5}{2(s-2)}\right] = 2\cdot L^{-1}\left[\frac{1}{s+1}\right] - 3\cdot L^{-1}\left[\frac{1}{s-1}\right] + \frac{5}{2}\cdot L^{-1}\left[\frac{1}{s-2}\right]$$

$$= 2\cdot e^{-t} - 3\cdot e^{t} + \frac{5}{2}\cdot e^{2t}$$

* find the inverse laplace transform of $L^{-1}\left[\frac{2s-1}{s^2(s-1)^2}\right]$

$$\frac{2s-1}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$$

$$(2s-1) = A(s)(s-1)^2 + B(s-1)^2 + C(s)^2(s-1) + D(s^2)$$

Put $s=0$, $-1 = B(0-1)^2 \Rightarrow \boxed{B=-1.}$

$s=1$, $1 = D(1)^2 \Rightarrow \boxed{D=1}$

coeff of s^3, $A + C = 0$

coeff of s^2, $-2A + B - C + D = 0$

$-2A - C = 0$

$2A + \cancel{C} = 0$

$A + \cancel{C} = 0$

$$\boxed{A = 0 \Rightarrow C = 0.}$$

$$\frac{2s-1}{s^2(s-1)^2} = \left[0 - \frac{1}{s^2} + 0 + \frac{1}{(s-1)^2}\right]$$

$$L^{-1}\left[\frac{2s-1}{s^2(s-1)^2}\right] = L^{-1}\left[\frac{1}{(s-1)^2} - \frac{1}{s^2}\right] = L^{-1}\left[\frac{1}{(s-1)^2}\right] - L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{t}\cdot L^{-1}\left[\frac{1}{s^2}\right] - \frac{t^{2-1}}{(2-1)!}$$ replace $s-1$ by s

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$= e^{t}\cdot \frac{t}{1!} - \frac{t}{1!}$

$= t(e^{t} - 1)$ //

* find $L^{-1}\left[\frac{1}{s(s+1)^3}\right]$

$$\frac{1}{s(s+1)^3} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}.$$

$$1 = A(s+1)^3 + B(s)(s+1)^2 + C(s)(s+1) + D(s)$$

Put $s=-1$, $\quad 1 = D(-1) \Rightarrow D = -1.$

on comparing coeff, of s^3,

$$0 = A + B \quad \Rightarrow \boxed{A = -B}$$

coeff of s^2,

$$0 = 3A + 2B + C$$

coeff of s,

$$0 = 3A + B + C + D.$$

$$3A - 2A + C = 0 \Rightarrow \quad A + \cancel{C} = 0.$$

$$2A + C - 1 = 0. \quad\quad 2A + \cancel{C} = 1$$

$$-A = -1$$

$$\boxed{A = 1}$$

$$1 + C = 0 \quad\quad \boxed{B = -1}$$

$$\boxed{C = -1}$$

$$L^{-1}\left[\frac{1}{s(s+1)^3}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}\right].$$

$$= L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2}\right] - L^{-1}\left[\frac{1}{(s+1)^3}\right].$$

$$= 1 - e^{-t} - e^{-t}\cdot L^{-1}\left[\frac{1}{s^2}\right] - e^{-t}\cdot L^{-1}\left[\frac{1}{s^3}\right].$$

$$= 1 - e^{-t} - e^{-t}\cdot\frac{t}{1!} - e^{-t}\cdot\frac{t^2}{2!}.$$

$$= 1 - e^{-t} - e^{-t}\cdot t - \frac{e^{-t}\cdot t^2}{2} = \frac{2 - 2e^{-t} - 2t\cdot e^{-t} - t^2 e^{-t}}{2}.$$

$$= 1 - e^{-t}\left[1 - t - \frac{t^2}{2}\right] //.$$

* find the $L^{-1}\left[\frac{3s+1}{(s-2)(s^2+1)}\right]$

$$\frac{3s+1}{(s-2)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

Put $s=1$, $4 = A(1+1)+0$

$2A = 4 \Rightarrow \boxed{A=2.}$

Compare s^2, $0 = A+B$

$2+B=0 \Rightarrow \boxed{B=-2}$

s, $3 = C-B$

$3 = C+2$

$C = 3-2 \Rightarrow \boxed{C=1}$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = L^{-1}\left[\frac{2}{s-1} + \frac{(-2)s+1}{s^2+1}\right]$$

$$= 2\cdot L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1-2s}{s^2+1}\right].$$

$$= 2\cdot e^{+t}\cdot 1 + L^{-1}\left[\frac{-2s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2\cdot e^{t} - 2\cdot L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2\cdot e^{t} - 2\cos t + \sin t //$$

* find the $L^{-1}\left[\frac{(1-s)}{(s+1)(s^2+4s+13)}\right]$.

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{(s+1)} + \frac{Bs+C}{s^2+4s+13}$$

$$1-s = A(s^2+4s+13) + (Bs+C)(s+1).$$

Put $s=-1$.

$2 = A(1-4+13) \Rightarrow 10A = 2 \Rightarrow A = 1/5.$

$A+B = 0 \Rightarrow \frac{1}{5}+B = 0 \quad B = -1/5.$

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$-1 = 4A + B + C$

$4\left(\frac{1}{5}\right) - \frac{1}{5} + C = -1$

$\frac{4}{5} - \frac{1}{5} + C = -1$

$\frac{3}{5} + C = -1$

$C = -\frac{1}{1} - \frac{3}{5} = \frac{-5-3}{5} = -8/5.$

$$L^{-1}\left[\frac{1}{5(s+1)} + \frac{-1(s)+(-8)}{5(s^2+4s+13)}\right].$$

$$= \frac{1}{5}\cdot L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4s+13}\right] - \frac{8}{5}L^{-1}\left[\frac{1}{s^2+4s+13}\right].$$

$$= \frac{1}{5}\cdot e^{-t}\cdot 1 - \frac{1}{5}\cdot L^{-1}\left[\frac{s+2-2}{(s+2)^2+9}\right] - \frac{8}{5}\cdot L^{-1}\left[\frac{1}{(s+2)^2+9}\right].$$

$$= \frac{1}{5}\cdot e^{-t} - \frac{1}{5}\cdot e^{-2t}\cdot \cos 3t - \frac{8}{5}\cdot e^{-2t}\cdot \frac{1}{3}\cdot \sin 3t.$$

$$\cancel{= \frac{1}{5}\left[e^{-t} - e^{-2t}\cos 3t - \frac{8}{3}e^{-2t}\sin 3t\right]}$$

$$= -\frac{1}{5}\cdot L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$= \frac{1}{5}\cdot e^{-t} - \frac{1}{5}\cdot e^{-2t}\cdot\cos 3t - \frac{2}{5}\cdot e^{-2t}\cos 3t - \frac{8}{15}e^{-2t}\sin 3t.$$

$$= \frac{1}{5}\cdot e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}\cdot e^{-2t}\cos 3t - \frac{8}{15}e^{-2t}\sin 3t.$$

* find $L^{-1}\left[\frac{3s^2+2s-1}{(s^2+4)(s^2-2s+1)}\right].$

$$\frac{3s^2+2s-1}{(s^2+4)(s-1)^2} = \frac{As+B}{(s^2+4)} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}.$$

$$3s^2+2s-1 = (As+B)\overset{(s^2-2s+1)}{(s-1)^2} + C(s^2+4)(s-1) + D(s^2+4)$$

Put $s=1$, $4 = 5D \Rightarrow \boxed{D = 4/5}$

Coeff of s^3

3, $2 = A - 2B + 4C$

$2 = A - 2B + 4(-A)$

$2 = -3A - 2B \Rightarrow$ $3A - 2B = 2$

$-A + B = 3 - \frac{4}{5} = \frac{11}{5}$

$+3A - 2\cancel{B} = 2$

$-2A + \cancel{2}B = \frac{22}{5}$

$A = 2 + \frac{22}{5} = \frac{-12}{5}$

$C = \frac{12}{5}$

HOMEWORK

1) $L^{-1}\left[\frac{5s^2-15-11}{(s+1)(s-2)^3}\right] \Rightarrow \frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$

$5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$

Put $s=2$,

$-21 = 3D$

$D = -7$

Put $s=-1$

$9 = -27A$

$A = -\frac{1}{3}$

comparing coeff'n $\Rightarrow B = \frac{1}{3}$

$C = 5+6A+3B$

$= 5+6A-3A$

$= 5+3A = 5-\frac{1}{3}(3) = 4$

$= L^{-1}\left[-\frac{1}{3}\left(\frac{1}{s+1}\right)\right] + L^{-1}\left[\frac{1}{3}\left[\frac{1}{s-2}\right]\right] + L^{-1}\left[\frac{5}{(s-2)^2}\right] + L^{-1}\left(\frac{-7}{(s-2)^3}\right)$

$= -\frac{1}{3}e^{t} + \frac{1}{3}e^{2t} + 5e^{2t}\left(L^{-1}\left(\frac{1}{s^2}\right)\right] - 7.e^{2t}\, L^{-1}\left[\frac{1}{s^3}\right]$

$= -\frac{1}{3}.e^{t} + \frac{1}{3}.e^{2t} + 5e^{2t}(t) - 7e^{2t}.\frac{t^2}{2}$

(2) $L^{-1}\left[\frac{1}{(s^2+4s+4)(s-3)}\right] = \frac{1}{(s+2)^2(s-3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s-3)}$

$1 = A(s-3)(s+2) + B(s-3) + C(s+2)^2$

Put $s=3$, $C(25)=1$

$C = \frac{1}{25}$

Put $s=-2$,

$1 = B(-5)$

$+B = -\frac{1}{5}$

coeff'n,

$0 = A+C$

$A = -C = -\frac{1}{25}$

$= L^{-1}\left[\frac{-1}{25}\left(\frac{1}{s+2}\right)\right] - \frac{1}{5}L^{-1}\left[\frac{1}{(s+2)^2}\right] + \frac{1}{25}\left\{L^{-1}\left[\frac{1}{s-3}\right]\right\}$

$= -\frac{1}{25}e^{-2t} - \frac{1}{5}e^{-2t} + \frac{1}{25}e^{3t}$

(3) $L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$

$2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$

Put $s=3$,

$5 = C(2)(1)$

$C = 5/2$

Put $s=2$,

$1 = B(1)(-1)$

$B = -1$

Put $s=1$,

$1 = A(-1)(-2)$

$A = 1/2$

$$L^{-1}\left[\frac{1}{2}\cdot\frac{1}{(s-1)}\right] - L^{-1}\left[\frac{1}{s-2}\right] + \frac{5}{2}\, L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \frac{1}{2}\cdot e^{t} - e^{2t} + \frac{5}{2}\cdot e^{3t}$$

(4) $$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$1 = A(s^2+2s+2) + (Bs+C)(s)$$

coeff terms.

$0 = A+B$; $0 = 2A + C$; $1 = 2A$

$B = -1/2$; $C = -2A$; $A = 1/2$

$= -2\left(\frac{1}{2}\right)$

$= -1$

$$= L^{-1}\left[\frac{1}{2}\cdot\frac{1}{s}\right) + L^{-1}\left[\frac{-\frac{1}{2}s - 1}{s^2+2s+2}\right]$$

$$= \frac{1}{2}\cdot L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}\cdot L^{-1}\left[\frac{s+1-1}{(s+1)^2+1^2}\right] - L^{-1}\left[\frac{1}{(s+1)^2+(1)^2}\right].$$

$$= \frac{1}{2}(1) - \frac{1}{2}\cdot e^{-t}\cos t + \frac{1}{2}\overset{e^{-t}}{\sin t} - \sin t\cdot e^{-t}.$$

$$= \frac{1}{2} - \frac{1}{2}\cdot e^{-t}\cos t + \frac{1}{2}\cdot e^{-t}\sin t - e^{-t}\sin t$$

(4)

$$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}. \quad = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{Cs+D}{(s^2+1)}$$

$$s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2$$

$$s = A(s^3+s^2+s+1) + B(s^2+1) + (Cs+D(s^2+2s+1)$$

s^3 $0 = A+C$; $s^2, 0 = A+B+2C+D$; $s, 1 = A+C+2D.$

$\boxed{A = -C}$ $\qquad B+C+D=0$ $\qquad 2D=1$

$B+\cancel{C} = -1/2$ $\qquad \boxed{D = 1/2}$

$B-\cancel{C} = -1/2$

$0 = A+B+D$ $\qquad 2B = -1$ $\qquad B+C = -1/2$

$A+B = -D$ $\qquad \boxed{B = -1/2}$ $\qquad -\frac{1}{2} + C = -\frac{1}{2}$

$A+B = -1/2$ $\qquad \boxed{C=0 \quad A=0}$

$B-C = -1/2$

$$L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right] = L^{-1}\left[\frac{0}{\quad}\right] + -\frac{1}{2}\cdot L^{-1}\left[\frac{1}{(s+1)^2}\right].$$

$$+ \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right].$$

$$= -\frac{1}{2}\cdot e^{-t}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2}\cdot \sin t$$

$$= -\frac{1}{2}\cdot e^{-t}\cdot t + \frac{1}{2}.\sin t \;.$$

Convolution:-

The convolution of two functions $f(t)$ and $g(t)$ where $t \geq 0$ is given by $f(t)*g(t)$ (or) $f*g = \int_0^t f(u)\cdot g(t-u)\,du$

$t \to u$ $\qquad t \to t-u$

where $*$ denotes convolution operator.

Convolution theorm:-

Note:- $f*g = g*f$ i.e., convolutional operator is commutative

$$f*g = f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$** \quad \int^a f(t)dt = \int^a f(a-t)dt$$

$$= \int_0^t f(t-u) \cdot g(t-(t-u))\, du$$

$$= \int_0^t f(t-u) \cdot g(u)\, du$$

$$= \int_0^t g(u) \cdot f(t-u)\, du = g(t) * f(t)$$

$$= g * f$$

Statement for convolution theorem:-

* If $f(t)$, $g(t)$ be two functions defined for $t \geq 0$ then, $L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$. It is also called as faltung theorm (or) faltung integral.

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$\Rightarrow$ L.H.S,

$$L[f(t) * g(t)] = \int_0^\infty e^{-st} [f(t) * g(t)]\, dt$$

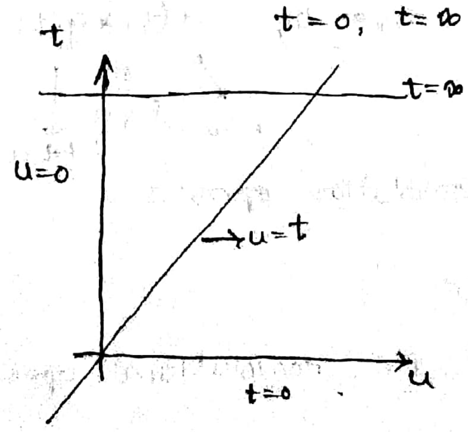
$$= \int_0^\infty e^{-st} \left[\int_0^t f(u) \cdot g(t-u)\, du \right] dt.$$

$$= \int_0^\infty \int_0^t e^{-st} \cdot f(u) \cdot g(t-u)\, du\, dt \quad (3)$$

By change the order of integration,

$u = 0$, $u = t$

$t = 0$, $t = \infty$



$du\,dt \rightarrow dt\,du$.

constant $u = 0 \rightarrow u = \infty$.

variable $t = u \rightarrow t = \infty$

③ $\Rightarrow \displaystyle\int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)\cdot g(t-u)\, dt\, du$

$u+v=t \Leftarrow v = t-u$

$dv = dt$

t	u	∞
$v=t-u$	0	∞

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-s[u+v]}\cdot f(u)\cdot g(v)\, dv\, du$$

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-su}\cdot e^{-sv}\, f(u)\cdot g(v)\cdot du\, dv$$

$$= \left[\int_{u=0}^{\infty} e^{-su} f(u)\, du\right]\left[\int_{v=0}^{\infty} e^{-sv} g(v)\, dv\right]$$

$= L[f(t)]\cdot L[g(t)]$. = R.H.S.

∴ Hence proved.

$* \quad L^{-1}[F(s)\cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

1. $L^{-1}\left[\dfrac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\dfrac{1}{(s+a)}\times\dfrac{1}{(s+b)}\right]$

where a, b are some constants

$$= L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$= L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au}\cdot e^{-b(t-u)}\, du$$

$$= e^{-bt}\int_0^t e^{-au}\cdot e^{bu}\, du$$

$$= e^{-bt}\int_0^t e^{-u(a-b)}\, du$$

$$= e^{-bt}\cdot\left[\frac{e^{-(a-b)u}}{(b-a)}\right]_{u=0}^{t} = \frac{e^{-bt}}{b-a}\left[e^{-(a-b)u}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{-(a-b)t} - 1\right] = \frac{e^{-bt}}{b-a}\left[e^{-at}\cdot e^{bt} - 1\right]$$

$$= \frac{1}{b-a}\left(e^{-at}-1\right)\times e^{-bt}) = -\left(e^{-at}-e^{-bt}\right)/(b-a)$$

② find $L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right] = 4 \cdot L^{-1}\left[\frac{1}{(s^2+2s+5)^2}\right]$

$= 4 \cdot \left[L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] * L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]\right]$

$= 4 \cdot e^{-t} \cdot L^{-1}\left(\frac{1}{s^2+2^2}\right) * e^{-t} \cdot L^{-1}\left[\frac{1}{s^2+2^2}\right].$

$= 4 \cdot e^{-t} \cdot \frac{1}{2}\sin 2t * e^{-t} \cdot \frac{1}{2}\sin 2t$

$= 2\left\{e^{-t}\sin 2t * e^{-t}\sin 2t\right\}$

$t \to u \qquad t \to t-u.$

$= 2\int_0^t e^{-u}\sin 2u \cdot e^{-(t-u)} \cdot \sin(2t-2u)\,du.$

$= 2 e^{-t}\int_0^t \sin 2u \cdot \sin(2t-2u)\,du.$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$= 2 \cdot e^{-t} \cdot \int_0^t \frac{1}{2}\left[\cos(4u-2t) - \cos(2t)\right]du.$

$= e^{-t}\left[\frac{\sin(4u-2t)}{4} - u\cos 2t\right]_{u=0}^{t}$

$= e^{-t}\left[\frac{\sin 2t}{4} - \frac{\sin(-2t)}{4}\right] - \left\{(t-0)\cos 2t\right\}$

$= e^{-t}\left\{\frac{\cancel{2}\sin 2t}{\cancel{4}\,2} - t\cos 2t\right\}.$

③ $L^{-1}\left[\frac{1}{s^2(s+5)}\right] = L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s+5}\right].$

$= t * e^{-5t} = t * e^{-5t}$

$= \int_0^t u \cdot e^{-5(t-u)}\,du = e^{-5t}\int_0^t u \cdot e^{5u}\,du.$

$u\int dv = uv - \int v\,du. \qquad u = u\,;\ dv = e^{5u}du$

$u' = 1 \qquad v = \frac{e^{5u}}{5}$

$= \left[\frac{e^{5u}}{5}\right]_0^t - \int_0^t \frac{e^{5u}}{5}. \qquad v_1 = \frac{e^{5u}}{25}$

$\left[\ 5t \quad ? \quad e^{5u}\right.$

Homework:-

* $L^{-1}\left[\frac{1}{(s+5)(s-6)}\right]$.

$= L^{-1}\left[\frac{1}{s+5}\right] * L^{-1}\left[\frac{1}{s-6}\right]$.

$= e^{-5t} * e^{6t}$

$= \int_0^t e^{-5u} \cdot e^{6(t-u)}\, du$

$= \int_0^t e^{-5u} \cdot e^{6t} \cdot e^{-6u}\, du$.

$= e^{6t} \int_0^t e^{-5u-6u}\, du = e^{6t} \cdot \int_0^t e^{-11u}\, du = e^{6t} \cdot \left[\frac{e^{-11u}}{-11}\right]_0^t$

$= e^{6t}\left[\frac{e^{-11t}-1}{-11}\right] = \frac{e^{-5t} - e^{6t}}{-11} = \frac{e^{6t} - e^{-5t}}{11}$. //

* $L^{-1}\left[\frac{10}{(s+1)\cdot(s^2+4)}\right]$.

$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+4}\right]$.

$= e^{-t} * \frac{1}{2} \cdot \sin 2t$.

$= \frac{10}{2}\int_0^t e^{-u} * \sin 2(t-u)\, du$.

$= t-u=v$

$-du = dv$.

$= \frac{1}{2}\int_0^t e^{-(t-v)} \sin 2v\, dv$.

$= \frac{1}{2} e^{-t} \int_0^t e^{v} \sin 2v\, dv$.

$\frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx$.

$= \frac{e^{v}}{v^2+4v^2}(v \sin 2v - 2v \cos 2v)$.

$= \frac{1}{2} \cdot e^{-t} \cdot \frac{e^{v}}{5v^2} (v \sin 2v - 2v \cos 2v)$.

$=$

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* find the inverse Laplace transform of $L^{-1}\left[\frac{1}{s^2(s+1)^3}\right]$ by convolution method:-

$$= L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{(s+1)^3}\right].$$

$$= \frac{t^1}{1!} * e^{-t}.\frac{t^2}{2}$$

$$= t * e^{-t}.\frac{t^2}{2} = e^{-t}\frac{t^2}{2} * t$$

$$= \int_0^t e^{-u}\frac{u^2}{2}.(t-u)\,du = \frac{1}{2}\int_0^t e^{-u}(tu^2-u^3)du$$

$u = u^2t-u^3$ $\quad v = e^{-u}$

$u' = 2ut-3u^2$ $\quad v_1 = -e^{-u}$

$u'' = 2t-6u$ $\quad v_2 = e^{-u}$

$u''' = -6$ $\quad v_3 = -e^{-u}$

$\quad v_4 = e^{-u}$

$$= \left[(u^2t-u^3)(-e^{-u}) - (2ut-3u^2)(e^{-u}) + (2t-6u)(-e^{-u}) - (-6)e^{-u}\right]_0^t$$

$$= \left[\left\{0-(-t^2)(e^{-t}) + (-4t)(-e^{-t}) - (6)e^{-t}\right\} - \left\{0-0+2t(-1) - (-6)(1)\right.\right.$$

$$= \left[\frac{1}{2}\left\{(e^{-t}.t^2 + e^{-t}.4t + 6.e^{-t}) + \{2t-6\}\right\}\right].$$

$$= \frac{1}{2}.e^{-t}(t^2+4t+6) + \left\{\frac{1}{2}(2t-6)\right\}$$

$$= \frac{1}{2}.e^{-t}(t^2+4t+6) + (t-3).$$

Some standard formulae:-

$$* = L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos(au).\cos(bt-bu)\,du$$

$$\frac{1}{2}\int_0^t \cos(au+bt-bu) + \cos(au-bt+bu)\}\,du.$$

$$= \frac{1}{2}\int_0 [\cos[u(a-b)+bt]$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{(a-b)} + \frac{\sin(a+b)u-bt}{(a+b)} \right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin(a-b)t+bt}{(a-b)} \right) + \left[\frac{\sin(a+b)t-bt}{(a+b)} \right] \right] - \left[\frac{\sin bt}{(a-b)} + \frac{\sin(-bt)}{(a+b)} \right]$$

$$= \frac{1}{2}\left\{ \frac{\sin at}{(a-b)} + \frac{\sin at}{(a+b)} - \frac{\sin bt}{(a-b)} + \frac{\sin bt}{(a+b)} \right\}$$

$$= \frac{1}{2}\left\{ \sin at\left(\frac{1}{(a-b)} + \frac{1}{(a+b)} \right) + \sin bt\left[\frac{1}{(a+b)} - \frac{1}{(a-b)} \right] \right\}$$

$$= \frac{1}{2}\left\{ \sin at \frac{2a}{(a^2-b^2)} \right\} + \left\{ \frac{1}{2} \sin bt \times \frac{-2b}{a^2-b^2} \right\}$$

$$= \left(\frac{a\sin at - b\sin bt}{a^2-b^2} \right)$$

(7) $L^{-1}\left[\frac{-s^2}{(s^2+a^2)^2} \right]$

$$= L^{-1}\left[\frac{s}{s^2+a^2} \right] * L^{-1}\left[\frac{s}{s^2+a^2} \right]$$

$$= \cos at * \cos at$$

$$= \int_0^t \cos(au) \cdot \cos(at-au)\, du.$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+at-au) + \cos(au-at+au) \right] du.$$

$$= \frac{1}{2}\int_0^t \left(\cos(at) + \cos(2au) \right] du.$$

$$= \frac{1}{2}\left\{ \left[u\cos at \right]_0^t + \left(\left[\frac{\sin 2au}{2a} \right]_0^t \right\} \right.$$

$$= \frac{1}{2}\left\{ \left[t\cos at \right] + \left[\frac{\sin 2at}{2a} \right] \right\}. \quad = \frac{t\cos at}{2} + \frac{\sin at}{2a}$$

$$* \quad L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{a}\sin at * \frac{1}{a}\sin at$$

$$= \frac{1}{a^2}\int_0^t \sin au \sin(at-au)\,du.$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(au-at+au) - \cos(\cancel{au}+at-\cancel{au})\right].$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(2au-at) - \cos at\right]du.$$

$$= \frac{1}{2a^2}\left[\left[\frac{\sin(2au-at)}{2a}\right]_0^t - \left[t\cos at\right]_0^t\right].$$

$$= \frac{1}{2a^2}\left[\left[\frac{(\sin at)+\sin at}{2a}\right] - t\cos at\right].$$

$$= \frac{1}{2a^2}\left[\frac{\cancel{2}\sin at}{\cancel{2}a} - t\cos at\right].$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at.\right].$$

Homework:-

$$① \quad L^{-1}\left[\frac{1}{(s^2+25)(s^2+36)}\right] = L^{-1}\left[\frac{1}{s^2+25}\right] * L^{-1}\left[\frac{1}{s^2+36}\right].$$

$$= \frac{1}{5}\sin 5t * \frac{1}{6}\sin 6t$$

$$= \frac{1}{30}\int_0^t \sin 5t \sin 6t\,du = \frac{1}{30}\int_0^t \sin 5u\cdot\sin(6t-6u)\,du$$

$$= \frac{1}{60}\int_0^t \cos(11u-6t) - \cos(6t-u)\;du.$$

$$= \frac{1}{60}\left\{\left[\frac{\sin(11u-6t)}{11}\right]_0^t + \left[\frac{\sin(6t-u)}{1}\right]_0^t\right\}$$

$$= \frac{1}{60}\left[\frac{\sin(5t)+\sin 6t}{11} + \frac{\sin 5t+\sin 6t}{1}\right].$$

$$= \frac{1}{60}(\sin 5t+\sin 6t)\left\{\frac{1}{11}+1\right\} = \frac{1}{60}\sin 5t+\sin 6t\left[\frac{12}{11}\right]$$

$$= \frac{1}{55}\left(\sin 5t+\sin 6t\right).$$

2) $L^{-1}\left[\dfrac{s^2}{(s^2+9)(s^2+81)}\right] = L^{-1}\left(\dfrac{s}{s^2+9}\right) * L^{-1}\left(\dfrac{s}{s^2+81}\right)$

$= \cos 3t * \cos 9t$

$= \int_0^t \cos 3u \cos(9t-9u)\,du$

$= \frac{1}{2}\int_0^t \cos(9t-6u) + \cos(12u-9t)\,du$

$= \frac{1}{2}\left\{\left[\dfrac{\sin(9t-6u)}{-6}\right]_0^t + \left[\dfrac{\sin(12u-9t)}{12}\right]_0^t\right\}$

$= \frac{1}{2}\left\{\left[\dfrac{\sin 3t - \sin 9t}{-6} + \dfrac{\sin 3t + \sin 9t}{12}\right]\right\}$

$= \frac{1}{2}\left\{\left[\dfrac{\sin 9t - \sin 3t}{6} + \dfrac{\sin 3t + \sin 9t}{12}\right]\right\}$

$= \dfrac{3\sin 3t - 9\sin 9t}{(9-81)}$

$= \dfrac{9\sin 9t - 3\sin 3t}{72} = \dfrac{3\sin 9t - \sin 3t}{24}$

3) $L^{-1}\left[\dfrac{s^2}{(s^2+49)^2}\right] = L^{-1}\left[\dfrac{s}{s^2+49}\right] * L^{-1}\left[\dfrac{s}{s^2+49}\right]$

$= \cos 7t * \cos 7t$

$= \int_0^t \cos 7u \cdot \cos(7t-7u)\,du = \frac{1}{2}\int_0^t \cos(7t) + \cos(14u-7t)\,du$

$= \frac{1}{2}\left\{\left[u\cos 7t\right]_0^t + \left(\dfrac{\sin(14u-7t)}{14}\right)_0^t\right\}$

$= \frac{1}{2}\left[\left[t\cos 7t\right] + \left[\dfrac{\sin 7t + \sin 7t}{14}\right]\right]$

$= \frac{1}{2}\left\{\left[t\cos 7t\right] + \left[\dfrac{2\sin 7t}{14}\right]\right\}$

$= \frac{1}{2}\left\{t\cos 7t + \dfrac{\sin 7t}{7}\right\}$

4) $\mathcal{L}^{-1}\left[\frac{1}{(s^2+64)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+64}\right] * \mathcal{L}^{-1}\left[\frac{1}{s^2+64}\right]$

$$= \frac{1}{8}\sin 8t * \frac{1}{8}\sin 8t$$

$$= \frac{1}{64}\int_0^t \sin 8u \cdot \sin(8t-8u)\,du$$

$$= \frac{1}{64}\int_0^t \cos(16u-8t) - \cos(8t)\,du$$

$$= \frac{1}{2(64)}\left\{\left[\frac{\sin(16u-8t)}{16}\right]_0^t - \left[u\cos 8t\right]_0^t\right\}$$

$$= \frac{1}{128}\left\{\left[\frac{\sin 8t + \sin 8t}{16} - t\cos 8t\right]\right\}$$

$$= \frac{1}{128}\left\{\left[\frac{\sin 8t}{2} - t\cos 8t\right]\right\}$$

$$= \frac{1}{128}\left[\frac{\sin 8t}{2} - t\cos 8t\right]$$

$= \frac{1}{2\cdot 8}$ sin

Solution of Ordinary Differential Equation:-

To find, $L[y(t)] = ?$

$$L[y'(t)] = s\cdot L[y(t)] - y(0)$$

$$L[y''(t)] = s^2 \cdot L[y(t)] - sy(0) - y'(0)$$

$$L[y'''(t)] = s^3 L[y(t)] - s^2 y(0) - sy'(0) - y''(0).$$

① Solve $(D^2 - 3D + 2)y = 4$, $y(0) = 2$, $y'(0) = -3$.

$$= y''(t) - 3y'(t) + 2y(t) = 4$$

$$L[y''(t)] - 3\cdot L[y'(t)] + 2L[y(t)] = L[4]$$

$$\left[s^2 L[y(t)] - s\cdot y(0) - y'(0)\right] - 3\left[s\cdot L[y(t)] - y(0)\right] + 2\cdot L[y(t)] = 4/s.$$

$$L[y(t)]\left[s^2 - 3s + 2\right] - 2s + 3 + 6 = \frac{4}{s}.$$

$$L[y(t)]\left[s^2 - 3s + 2\right] = \frac{4}{s} + 2s - 9$$

$$= \frac{2s^2 - 9s + 4}{s}$$

$$y(t) = L^{-1}\left[\frac{2s^2-9s+4}{s(s^2-3s+2)}\right]$$

$$y(t) = L^{-1}\left[\frac{2s^2-9s+4}{s(s-1)(s-2)}\right]$$

$$\frac{2s^2-9s+4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$$

$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2s^2-9s+4$$

Put $s=1$, $B(-1) = 2-9+4 = -3$

$$\boxed{B=3}$$

Put $s=2$, $2C(1) = 8-18+4$

$$2C = -6$$

$$\boxed{C=-3}$$

Put $s=0$,

$$A(-1)(-2) = 4$$

$$2A = 4$$

$$\boxed{A=2}$$

$$= L^{-1}\left[\frac{2}{s} + \frac{3}{s-1} - \frac{3}{s-2}\right].$$

$$= 2L^{-1}\left[\frac{1}{s}\right] + 3L^{-1}\left[\frac{1}{s-1}\right] - 3L^{-1}\left[\frac{1}{s-2}\right].$$

$$= 2(1) + 3 \cdot e^{+t} - 3 \cdot e^{2t}$$

$$= 2 + 3e^{t} - 3e^{2t} //$$

(8) $(D+2)^2 \cdot y = 4 \cdot e^{-2t}$, $y(0) = -1$; $y'(0) = 4$

$$(D^2+4D+4)y = 4 \cdot e^{-2t}$$

$$y''(t) + 4 \cdot y'(t) + 4 \cdot y(t) = 4 \cdot e^{-2t}$$

$$L[y''(t)] + 4 \cdot L[y'(t)] + 4 \cdot L[y(t)] = 4 \cdot L[e^{-2t}].$$

$$\left[s^2 \cdot L[y(t)] - s(y(0)) - y'(0)\right] + 4\left[s \cdot L[y(t)] - y(0)\right] + 4 \cdot L[y(t)] = \frac{4}{s+2}.$$

$$= L[y(t)]\{s^2+4s+4\} + s \cancel{+4} \cancel{+4} = \frac{4}{s+2}$$

$$L[y(t)] = \frac{4}{s+2} - s$$

$$(s^2+4s+4)\, L(y(t)) = \frac{4 - s(s+2)}{(s+2)}$$

$$L[y(t)] = \frac{4-s^2-2s}{(s+2)^3} = \frac{-s^2-2s+4}{(s+2)^3}.$$

$$\frac{-s^2-2s+4}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}.$$

$$-s^2-2s+4 = A(s+2)^2 + B(s+2) + C(1)$$

$$-s^2-2s+4 = A(s^2+4s+4) + B(s+2) + C$$

compare s^2, $A = -1$

s, $4A+B = -2$

$-4+B = -2$

$\boxed{B = 2}$

$4 = 4A+2B+C$

$4 = -4 \cancel{+4} + C$ $\boxed{C=4}$.

$$= L^{-1}\left[\frac{-1}{s+2} + \frac{2}{(s+2)^2} + \frac{4}{(s+2)^3}\right].$$

$$= -1 \cdot L^{-1}\left[\frac{1}{s+2}\right] + 2 \cdot L^{-1}\left[\frac{1}{(s+2)^2}\right] + 4 \cdot L^{-1}\left[\frac{1}{(s+2)^3}\right].$$

$$= -1 \cdot e^{-2t} + 2 \cdot e^{-2t} \cdot (t) + 4 \cdot e^{-2t} \cdot \frac{t^2}{2}$$

$$= e^{-2t}\left(\frac{\cancel{4}^2 t^2}{\cancel{2}} + 2t - 1\right)$$

$$= e^{-2t}(2t^2+2t-1)$$

(3) Solve $(\ddot{x} + 4\dot{x} + 3x) = 10\sin t$, $x(0) = x'(0) = 0$.

$y''(t) + 4y'(t) + 3 \cdot y(t) = 10 \sin t.$ $\quad y(0) = y'(0) = 0.$

$$[s^2 L[y(t)] - s\,y(0) - y'(0)] + 4[s\,L[y(t)] - y(0)] + 3 \cdot L[y(t)]$$
$$= 10 \cdot L[\sin t].$$

$$L[y(t)]\left\{s^2+4s+3\right\} = 10\cdot\frac{1}{s^2+1^2}$$

$$= \frac{10}{s^2+1^2}$$

$s^2+3s+1s+3$

$s(s+3)$

$$L[y(t)] = \frac{10}{(s^2+1)(s+3)(s+1)}$$

$$y(t) = L^{-1}\left[\frac{10}{(s^2+1)(s+3)(s+1)}\right]$$

$$10 = \frac{As+B}{s^2+1} + \frac{C}{s+3} + \frac{D}{s+1}$$

$$10 = (As+B)(s+3)(s+1) + C(s+1)(s^2+1) + D(s+3)(s^2+1)$$

Put $s = -3$,

$$10 = B(-2)(10) \Rightarrow C = -1/2$$

Put $s = -1$,

$$10 = C(2)(2) \Rightarrow D = 5/2$$

coeff,

$$0 = A+B+C$$

$$0 = A + \frac{-1}{2} + \frac{5}{2} \Rightarrow A = \frac{1}{2} - \frac{5}{2} = \frac{-4}{2} = -2.$$

$$y(t) = \frac{-2s - \frac{1}{2}}{(s^2+1)} - \frac{1}{2(s+3)} + \frac{5}{2(s+1)}$$

$$= \frac{-4s-1}{2(s^2+1)} - \frac{1}{2(s+3)} + \frac{5}{2(s+1)}$$

$=$

$$\frac{10}{(s^2+1)(s+1)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{(s+1)} + \frac{D}{(s+3)}$$

$$10 = (As+B)(s+1)(s+3) + C(s+3)(s^2+1) + D(s+1)(s^2+1).$$

Put $s=-3$, $\quad D(-2)\cancel{(10)} = \cancel{10}$

$$D = -\tfrac{1}{2}$$

Put $s=-1$, $\quad \overset{5}{\cancel{10}} = C(+2)(\cancel{2}).$

$$C = \tfrac{5}{2}.$$

s^3, $\quad 0 = A + C + D.$

$$0 = A + \frac{5}{2} - \frac{1}{2} \Rightarrow A + \frac{4}{2} = 0 \Rightarrow \boxed{A=-2}$$

s^2, $\quad 0 = 4A + B + 3C + D.$

$$0 = -8 + B + \frac{15}{2} - \frac{1}{2}$$

$$0 = -8 + B + \frac{14}{2}\,7 \qquad -1+B=0 \quad \boxed{B=1}.$$

$$= \frac{-2s+1}{s^2+1} + \frac{5}{2(s+1)} - \frac{1}{2(s+3)}$$

$$= L^{-1}\left(\frac{1-2s}{s^2+1}\right) + \frac{5}{2}\cdot L^{-1}\left[\frac{1}{s+1}\right] - 2\cdot L^{-1}\left[\frac{1}{s+3}\right].$$

$$= L^{-1}\left[\frac{1}{s^2+1}\right] - 2\cdot L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{5}{2}\cdot e^{-t} - 2\cdot e^{-3t}$$

$$= \sin t - 2\cos t + \frac{5}{2}\cdot e^{-t} - 2e^{-3t} \,//$$

4) $D^2y + 2Dy + 2y = 5\sin t.$

$$= y''(t) + 2y'(t) + 2y(t) = 5\sin t$$

$$= \left[s^2 L(y(t)) - sy(0) - y'(0)\right] + 2\left[sL(y(t)) - y(0)\right] + 2L(y(t)) = \frac{5}{s^2+1}$$

$$= L(y(t))\left[s^2+2s+2\right] = \frac{5}{s^2+1}$$

$$= y(t) = L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$$

$\cancel{L^{-1}\left[\frac{5}{\ }\right.}$

$$\frac{5}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+2s+2)}$$

$$5 = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1).$$

s^3, $0 = A+C \Rightarrow \boxed{A=-C}$ $\quad A+0+C+0=0$. — ①

s^2, $0 = 2A+B+D \Rightarrow$ $\quad 2A+B+D=0$. — ②

s, $0 = 2A+2B+C \Rightarrow$ $\quad 2A+2B+C=0$. — ③ $\cancel{2B+C=0}$

cm, $5 = 2B+D \Rightarrow$ $\quad 2B+D=5$. — ④

$\cancel{B+D-2C}=0$.

$\cancel{2B+D+C}$

③ $\rightarrow$ $2B+C-2C=0$

$2B=C$

$\boxed{B=\frac{C}{2}}$

② $\rightarrow$ $-2C+\frac{C}{2}+D=0$.

$-4C+C+2D=0$.

$-3C+2D=0$

$D=\frac{3C}{2}$.

④ $\rightarrow$ $C+D=5$

$C+\frac{3C}{2}=5$ $\quad \cancel{\frac{3C}{2}=5}$

$\frac{5C}{2}=5$.

$5C=10$ $\quad \boxed{C=2}$

$\boxed{\cancel{C=4}}$

$B=1$

$D=\frac{3\times\cancel{2}}{\cancel{2}}=3$. $\quad \boxed{A=-2}$

$$= L^{-1}\left[\frac{-2s+1}{s^2+1}\right] + L^{-1}\left[\frac{2s+3}{(s^2+2s+2)}\right].$$

$$= -2L^{-1}\left[\frac{s}{s^2+1}\right] + 1L^{-1}\left[\frac{1}{s^2+1}\right] + 2\cdot L^{-1}\left[\frac{s}{(s+1)^2+1}\right] + 3\cdot L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -2\cdot\cos t + 1\cdot\sin t + 2\cdot e^{-t}\cdot\cos t + 3\cdot e^{-t}\cdot\sin t.$$

$$= 2\cos t\,(e^{-t}-2) + \sin t\,(1+3e^{-t}).$$

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HOMEWORK:-

① $(D^2+2D+5)y = 0, \quad y(0)=2, \quad y'(0) = -4$

$= y''(t) + 2\cdot y'(t) + 5y(t) = 0.$

$= [s^2 L[y(t)] - sy(0) - y'(0)] + 2[s\cdot L(y(t)) - y(0)] + 5\cdot L[y(t)] = 0$

$= L[y(t)][s^2+2s+5] - s(2) \cancel{+4} - 2\cancel{(2)} = 0.$

$= L(y(t)) = \frac{2s}{s^2+2s+5}$

$y(t) = L^{-1}\left[\frac{2s}{s^2+2s+5}\right] = 2L^{-1}\left[\frac{s}{(s+1)^2+4}\right]$

$\frac{2s}{s^2+2s+5} = \frac{As+B}{s^2+2s+5}$

$= 2\cdot L^{-1}\left[\frac{s}{(s+1)^2+2^2}\right] = 2\cdot e^{-t}\cdot L^{-1}\left[\frac{s}{s^2+2^2}\right]$

$= 2e^{-t}\cos 2t.$

② $(D^2-2D+1)y = e^t, \quad y(0)=2, \quad y'(0)=1$

$= y''(t) - 2y'(t) + y(t) = e^t$

$= [s^2 L[y(t)] - sy(0) - y'(0)] - 2(s\cdot L(y(t)) - y(0)] + L[y(t)] = \cancel{e^t} L[e^t]$

$= L(y(t))[s^2-2s+1] - 2s - 1 - 2(2) = L[e^t] = \frac{1}{s-1}$

$= L[y(t)][s^2-2s+1] - 2s - 5 = \frac{1}{(s-1)}$

$= L(y(t)) = \frac{1}{(s-1)\cancel{(s^2-2s+1)}} + 2s+5. = \frac{1+2s(s-1)+5(s-1)}{(s^2-2s+1)(s-1)}$

$= \frac{2s^2-2s+5s-5+1}{(s^2-2s+1)(s-1)}$

$= \frac{2s^2+3s-4}{(s^2-2s+1)(s-1)}$

$\frac{2s^2-3s-4}{(s-1)^3} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} = \cancel{\frac{-5}{s-1} - \frac{9}{(s-1)^2}}$

~~$2s^2 - 3s - 4 = A(s-1) + B.$~~

~~Put $s=1$, $B = 2-3-4 = 2-7 = -5$~~

~~$-4 = -A + B.$~~

~~$-4 = 5 + B \Rightarrow B = -9$~~

$= 2s^2-3s-4 = A(s^2-2s+1) + B(s-1) + C.$

$\cancel{2 = A}$

$s=1, \quad C = 2-3-4 = 2-7 = -5$

$\boxed{2 = A}; \quad -3 = -2A + B$

$-3 = -4 + B$

$\boxed{B=1}$

$$= \mathcal{L}^{-1}\left[\frac{2}{(s-1)} + \frac{1}{(s-1)^2} - \frac{5}{(s-1)^3}\right]$$

$$= 2\times e^t + e^t(t) - 5\cdot e^t\left(\frac{t^2}{2}\right)$$

$$= e^t\left(\frac{5t^2}{2} + e^t + 2\right).$$

4). $y''(t) + 4y'(t) + 13y(t) = 26$; $y(0) = 3$; $y'(0) = 4$

$$\left(s^2 L[y(t)] - sy(0) - y'(0)\right) + 4\left(sL[y(t)] - y(0)\right) + 13L[y(t)] = L(26)$$

$$= L[y(t)](s^2+4s+13) - 3s - 4 + 12 = \frac{26}{s}$$

$$L[y(t)](s^2+4s+13) = \frac{26}{s} + 3s - 8$$

$$L[y(t)] = \frac{26+3s^2-8s}{s(s^2+4s+13)}$$

$$3s^2-8s+26 = \frac{A}{s} + \frac{Bs+C}{(s^2+4s+13)}$$

$$3s^2-8s+26. = A(s^2+4s+13) + (Bs+C)s$$

$3 = A+B$; $\quad -8 = 4A+C$; $\quad 26 = 13A$

$B = 1$. $\quad -8 = 8+C \quad$ $A = 2$

$\boxed{C = -16}$

$$= \frac{2}{s} + \frac{1s-16}{s^2+4s+13}$$

$$= 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{s}{(s+2)^2+9}\right] - 16L^{-1}\left[\frac{1}{(s+2)^2+9}\right].$$

$$= 2(1) + e^{-2t}\cos 3t - 16e^{-2t} \times \frac{1}{3}\sin 3t$$

$$= 2 + e^{-2t}\cos 3t - \frac{16}{3}e^{-2t}\sin 3t$$

* $(D^2+2D+5)y = e^{-t}\sin t \quad y(0) = 0$; $y'(0) = 1$

$$y''(t) + 2y'(t) + 5y(t) = L\left[e^{-t}\sin t\right]$$

$$\left[s^2 L[y(t)] - s\cdot y(0) - y'(0)\right] + 2\cdot\left[s\cdot L[y(t)] - y(0)\right] + 5\cdot L[y(t)]$$

$$= L[e^{-t}\sin t].$$

$$L[y(t)](s^2+2s+5) - 1 = L[e^{-t}\sin t].$$

$$= L[\sin t]\,\big|_{s\to s+1}$$

$$= \frac{1}{s^2+1} = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2} + 1$$

$$L[y(t)](s^2+2s+5) = \frac{s^2+2s+2+1}{(s^2+2s+2)} = \frac{s^2+2s+3}{(s^2+2s+2)}$$

$$L[y(t)] = \frac{s^2+2s+3}{\ldots(s^2+2s+5)}$$

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2+2s+5)}$$

$$s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

$$= (As^3+2As^2+5As+Bs^2+2Bs+5B) +$$
$$(Cs^3+2Cs^2+2Cs+Ds^2+2Ds+2D)$$

coeff, s^3, $0 = A + C \rightarrow (1)$

s^2, $1 = 2A+B+2C+D \quad -(2)$

s, $2 = 5A+2B+2C+2D \quad -(3)$

Con, $3 = 5B+2D \quad -(4)$

$\boxed{A = -C}$

in (2) $\quad -\cancel{2C}+B+\cancel{2C}+D = 1$

$B + D = 1$

$5B + 2D = 3$

$2B + 2D = 2$

$3B = 1$

$\boxed{B = 1/3}$

$\frac{5}{3} + 2D = 3$

$2D = 3 - \frac{5}{3} = \frac{4}{3}$

$\boxed{D = 2/3}$

$5A + \frac{2}{3} + 2C + \frac{4}{3} = 2.$

$-3C = 2 - \frac{2}{3} - \frac{4}{3}$

$-3C = 2 + \left(\frac{-2-4}{3}\right) = 2 - \frac{6}{3} = 0.$

$3C = 0$

$\boxed{C=0}$

$\boxed{A=0}.$

$$= L^{-1}\left[\frac{1/3}{s^2+2s+2}\right] + L^{-1}\left[\frac{2/3}{s^2+2s+5}\right]$$

$$= \frac{1}{3} \cdot L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3} \cdot L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= \frac{1}{3} \cdot e^{-t} \cdot \sin t + \frac{2}{3} \times \frac{1}{2} \cdot e^{-t} \sin 2t$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{e^{-t} \sin 2t}{3} = \frac{e^{-t}}{3}(\sin t + \sin 2t)$$

$$\frac{4s^4+2s^3+2s\ldots}{s^4(s^2+\ldots)}$$

* $y'' + y' = t^2 + 2t \quad y(0) = 4 \quad y'(0) = -2$

$y''(t) + y'(t) = t^2 + 2t$

$\left[s^2 L[y(t)] - s y(0) - y'(0)\right] + \left(s \cdot L[y(t)] - y(0)\right) = L[t^2 + 2t]$

$L[y(t)]\ (s^2+s) - 4s + 2 - 4 = \frac{2!}{s^{2+1}} + 2 \cdot L[t]$

$= \frac{2}{s^3} + 2 \cdot \frac{1}{s^{1+1}} = \frac{2}{s^3} + \frac{2}{s^2}$

$L[y(t)]\ (s^2+s) - 4s - 2 = \frac{2s^2 + 2s^3}{s^5} = \frac{\cancel{s^2}(2+2s)}{s^{\cancel{5}3}}$

$= \frac{2+2s}{s^3}$

$L[y(t)] = (s^2+s) = \frac{2+2s}{s^3} + 4s + 2$

$= \frac{2+2s+4s^4+2s^3}{s^3(s^2+s)} = \frac{4s^4+2s^3+2s+2}{s^3(s^2+s)}$

$= \frac{4s^4+2s^3+2s+2}{s^5+s^4}$

$\frac{4s^4+2s^3+2s+2}{s^4(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \cancel{\frac{Ds+E}{s^2+s}} + \frac{D}{s^4} + \frac{E}{s+1}$

~~$= A(s^2+s) \cdot s^2 + Bs(s^2+s) + C(s^2+s) + (Ds+E)(s^3)$~~

~~$= As^4 + As^3 + Bs^3 + Bs^2 + Cs^2 + Cs + Ds^4 + Es^3$~~

~~$s^4,\ 4 = A + D$ — 1~~

~~$s^3,\ 2 = A + B + E$ — 2~~

~~$s^2,\ 0 = B + C$~~

~~$s,\ 2 = C$~~

~~Con, $2 =$~~

$= A(s^3)(s+1) + B(s^2)(s+1) + C(s)(s+1) + D(s+1) + E(s^4)$

Put $s=0$, $\boxed{2 = D}$

$s=-1$, $\cancel{4} - \cancel{2} - \cancel{2} + 2 = E$

$\boxed{E=2}$

s^4, $4 = A + C$ $\quad$ s^3, $2 = A + B$ $\quad$ s^2, $0 = B + C$

$4 = A + 2$ $\quad$ $\cancel{2} = \cancel{A} + B$ $\quad$ $0 = 0 + C$

$\boxed{A=2}$ $\quad$ $\boxed{B=0}$ $\quad$ $\boxed{C=0}$

$$= L^{-1}\left[\frac{2}{s}\right] + L^{-1}\left[\cancel{0}\right] + \cancel{L^{-1}[\;]} + L^{-1}\left[\frac{}{s^4}\right] + L^{-1}\left[\frac{2}{s+1}\right].$$

$$= 2\cdot L^{-1}\left[\frac{1}{s}\right] + 2\cdot L^{-1}\left[\frac{1}{s^4}\right] + 2\cdot L^{-1}\left[\frac{1}{s+1}\right].$$

$$= 2(1) + 2\left(\frac{t^3}{3!}\right) + 2\cdot e^{-t}$$

$$= 2(1) + \frac{\cancel{2}t^3}{3\times\cancel{2}} + 2e^{-t}$$

$$= 2e^{-t} + \frac{t^3}{3} + 2.$$

* $(D^2+4)y = \sin 2t, \quad y(0)=3, \quad y'(0)=4.$

$$(s^2+4)\,L[y(t)] - 3s - 4 = \frac{2}{s^2+4}.$$

$$(s^2+4)\,L[y(t)] = \frac{2}{s^2+4} + 3s + 4$$

$$L[y(t)] = \frac{2}{(s^2+4)^2} + \frac{3s}{(s^2+4)} + \frac{4}{(s^2+4)}$$

$$y(t) = 2\cdot L^{-1}\left[\frac{2}{(s^2+4)^2}\right] + 3\cdot L^{-1}\left(\frac{s}{s^2+4}\right) + 4\cdot L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= 2L^{-1}\left(\frac{2}{(s^2+2^2)^2}\right) + 3\cdot\cos 2t + 2\sin 2t$$

$$= \frac{1}{8}\left(\frac{\sin 2t}{2} - t\cos 2t\right] + 3\cos 2t + 2\sin 2t \;\textit{ll.}$$

Homework:

(1) $(D^2+D-2)y = 3\cos 3t - 11\sin 3t \quad y(0)=0, \; y'(0)=6.$

$$y''(t) + y'(t) - 2y(t) = L(3\cos 3t - 11\sin 3t).$$

$$\left[s^2L[y(t)] - sy(0) - y'(0)\right] + \left[L[y(t)] - s\,y(0)\right] - 2\,L[y(t)] = 3\cdot\frac{s}{s^2+9} - \frac{11\times 3}{s^2+9}$$

$$L[y(t)]\left[s^2+s-2\right] \cancel{-6} = \frac{3s-33}{s^2+9} + 6 = \frac{3s-33+6s^2+54}{s^2+9}$$

$$= \frac{6s^2+3s+21}{(s^2+9)}$$

$$L[y(t)] = \frac{6s^2+3s+21}{(s^2+9)(s^2+s-2)}$$

(2) $y'''(t) - 3y''(t) + 3y'(t) - y(t) = t^2 e^t$;
$y(0) = 1$; $y'(0) = 0$; $y''(0) = -2$

$$\left[s^3 L[y(t)] - s^2 y(0) - s y'(0) - y''(0)\right] - 3\left[s^2 L[y(t)] - s y(0) - y'(0)\right]$$

$$+3\left[s L[y(t)] - y(0)\right] - L[y(t)] = L[t^2 \cdot e^t]$$

$$= L[t^2] \,|\, s \text{ by } s-1 = \frac{2!}{s^{2+1}}$$

$$= \frac{2}{(s-1)^3}$$

$$L[y(t)](s^3 - 3s^2 + 3s - 1) - s^2 + 2 - s - 1 = \frac{2}{(s-1)^3}$$

$$L[y(t)](s^3 - 3s^2 + 3s - 1) = \frac{2}{(s-1)^3} + s^2 + s - 1$$

$$= \frac{2}{(s-1)^3(s-1)^3} + \frac{s^2}{(s-1)^3} + \frac{s}{(s-1)^3} - \frac{1}{(s-1)^3}.$$

$$L[y(t)] = \frac{2}{(s-1)^6} + \frac{s^2 + s - 1}{(s-1)^3}$$

$$\frac{s^2 + s - 1}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}.$$

$$s^2 + s - 1 = A(s-1)^2 + B(s-1) + C(1).$$

Put $s = 1$, $1 + 1 - 1 = C$

$\boxed{C = 1}$

$\cancel{s^2,\ 1 = A + B}$

$\cancel{s,\ 1 = -2A - 2B \Rightarrow 2A + 2B = -1}$

$\cancel{2A + 2B = 2}$

$\cancel{A + B = -1/2}$

$\cancel{A + B = 1}$

s^2, $\boxed{1 = A}$

s, $1 = -2A - 2B$.

$1 = -2 - 2B$

$-2B = 3$

$B = -3/2$

$$= L^{-1}\left[\frac{1}{s-1}\right] + \frac{-3}{2} L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{1}{(s-1)^6}\right]$$

$$= e^t - \frac{3}{2} \cdot e^t \cdot \frac{t}{1!} + e^t \cdot \frac{t^2}{2} + 2 \cdot e^t \cdot \frac{t^5}{5!}$$

$$= e^t\left[1 - \frac{3t}{2} + \frac{t^2}{2} + \frac{t^5}{60}\right]$$

Solution of Integral equation:-

* An integral equation is of the form

$$y = f(t) + \int_0^t F(u) \cdot G(t-u)\,du.$$

$$= f(t) + F(t) * G(t)$$

* Solve the integral equation $y'(t) = \sin t + \int_0^t y(t-u)\cos u\,du$, $y(0)=1$

$$y'(t) = \sin t + \int_0^t y(t-u)\cos u\,du.$$

$$y'(t) = \sin t + y(t) * \cos t$$

$$L(y'(t)] = L[\sin t] + L[y(t)] \cdot L[\cos t]$$

$$s \cdot L[y(t)] - y(0) = \frac{1}{s^2+1} + L[y(t)] \cdot \frac{s}{s^2+1}$$

$$s \cdot L[y(t)] - \frac{s}{s^2+1} \cdot L[y(t)] = \frac{1}{s^2+1}$$

$$L(y(t)] \left[s - \frac{s}{s^2+1} \right] = \frac{1}{s^2+1}$$

$$L(y(t)] = \left[\frac{s^3 + \cancel{s} - \cancel{s}}{(\cancel{s^2+1})} \right] = \frac{1}{(\cancel{s^2+1})}$$

$$L[y(t)] = \frac{1}{s^3}.$$

$$y(t) = L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{(3-1)!} = \frac{t^2}{2!} = \frac{t^2}{2}.$$

* $F'(t) = t + \int_0^t F(t-u)\cos u\,du \qquad F(0) = 4.$

$$F'(t) = t + F(t) * \cos t$$

$$L[F'(t)] = L[t] + L[F(t)] \cdot L[\cos t].$$

$$s \cdot L[F(t)] - 4 = \frac{1}{s^2} + L[F(t)] \cdot \frac{s}{s^2+1}$$

$$L[F(t)] \left\{ s - \frac{s}{s^2+1} \right] = \frac{1}{(s^2)} + 4$$

$$L[F(t)] \left[\frac{s^3 + \cancel{s} - \cancel{s}}{} \right] \quad \frac{1+4s^2}{}$$

$$L[F(t)] = \frac{(1+4s^2)}{s^2} \times \frac{(s^2+1)}{s^3} = \frac{(1+4s^2)(s^2+1)}{s^5}$$

$$= \frac{s^2+1+4s^4+4s^2}{s^5} = \frac{4s^4+5s^2+1}{s^5}$$

$$L[F(t)] = \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}$$

$$F(t) = L^{-1}\left[\frac{4}{s}\right] + L^{-1}\left(\frac{5}{s^3}\right) + L^{-1}\left(\frac{1}{s^5}\right)$$

$$= 4 \times L^{-1}\left[\frac{1}{s}\right] + 5\,L^{-1}\left[\frac{1}{s^3}\right] + 1\,L^{-1}\left[\frac{1}{s^5}\right]$$

$$= 4(1) + 5\left(\frac{t^2}{2!}\right) + \frac{t^4}{4!}$$

$$= 4 + \frac{5t^2}{2} + \frac{t^4}{4!}.$$

* $y(t) = at + \int_0^t y(u) \cdot \sin(t-u)\, du.$

$y(t) = at + \sin t * y(t)$

$L[y(t)] = L[at] + L[\sin t] \cdot L[y(t)].$

$L[y(t)] = a \cdot L[t] + \frac{1}{s^2+1} \cdot L[y(t)].$

$L[y(t)]\left[1 - \frac{1}{s^2+1}\right] = \frac{a}{s^2}.$

$L[y(t)]\left[\frac{s^2+\cancel{1}-\cancel{1}}{s^2+1}\right] = \frac{a}{s^2} \Rightarrow L[y(t)] = \frac{a}{s^2} \times \frac{s^2+1}{s^2}$

$L[y(t)] = \frac{as^2+a}{s^4} = \frac{a}{s^2} + \frac{a}{s^4}.$

$y(t) = a\,L^{-1}\left[\frac{1}{s^2}\right] + a\,L^{-1}\left[\frac{1}{s^4}\right].$

$= a \times t + a \cdot \frac{t^3}{3!}$

$= at + \frac{at^3}{6}.$

* $y(t) = t + \frac{1}{6}\int_0^t (t-u)^3\, y(u)\, du.$

$y(t) = t + \frac{1}{6} \cdot [t^3 * y(t)].$

$L[y(t)] = L(t) + \frac{1}{6} L[t^3] \cdot L[y(t)].$

$= \frac{1}{s^2} + \cancel{\frac{1}{6}} \frac{\cancel{3!}}{s^4} L[y(t)]$

$$L[y(t)]\left[1-\frac{1}{s^4}\right]=\frac{1}{s^2}$$

$$L[y(t)]\left[\frac{s^4-1}{\cancel{s^4}}\right]=\frac{1}{\cancel{s^2}}$$

$$L[y(t)]=\frac{s^2}{s^4-1}=\frac{s^2}{((s^2)^2-1^2)}$$

$$=\frac{s^2}{(s^2+1)(s^2-1)}$$

$$\cancel{=\frac{s^2+1-1}{(s^2+1)(s^2-1)}}$$

$$\frac{s^2}{(s^2+1)(s^2-1)}=\frac{As+B}{(s^2+1)}+\frac{Cs+D}{(s^2-1)}$$

$$s^2=(As+B)(s^2-1)+(Cs+D)(s^2+1)$$

$$s^2=(As^3-As+Bs^2-B)+(Cs^3+Cs+Ds^2+D)$$

$0=A+C$; $1=B+D$; $0=-A+C$ $\boxed{A=C}$

$0=-B+D$

$1=2D$

$\boxed{D=1/2}$

$B+\frac{1}{2}=1$

$\boxed{B=1/2}$

$$s^2=\frac{As+B}{s^2+1}+\frac{C}{s+1}+\frac{D}{s-1}$$

$$s^2=(As+B)(s+1)(s-1)+C(s^2+1)(s-1)+D(s+1)(s^2+1).$$

Put $s=1$, $1=D(2)(2)$

$\boxed{D=1/4}$

Put $s=-1$, $1=C(-2)(2)$

$1=C(-4)$

$\boxed{C=-1/4}$

$0=A+C$

$\boxed{A=-C}$

$\boxed{A=1/4}$

$1=B+C+D$

$1=B-\frac{1}{4}+\frac{1}{4}$

$\boxed{B=1}$

$$= \frac{1S+4}{4(s^2+1)} - \frac{1}{4(s+1)} + \frac{1}{4(s-1)}$$

$$= \frac{1}{4} \cdot L^{-1}\left[\frac{s}{s^2+1}\right] + 1 \cdot L^{-1}\left[\frac{1}{s^2+1}\right] - \frac{1}{4} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{4} \cdot L^{-1}\left[\frac{1}{s-1}\right]$$

$$= \frac{1}{4} \cdot \cos t + \sin t - \frac{1}{4} \cdot e^{-t} + \frac{1}{4} \cdot e^{t}.$$

$$= \frac{1}{4}\left[e^{-t} + e^{t} + \cos t\right] + \sin t$$

Solution of simultaneous ODE:-

* $2x' - y' + 3x = 2t \quad , x(0) = 1$

$x' + 2y' - 2x - y = t^2 - t \quad , y(0) = 1$

$2 \cdot L[x'(t)] - L[y'(t)] + 3 \cdot L[x(t)] = 2 \cdot L[t].$

$2 \cdot [s \cdot L[x(t)] - x(0)] - [s \cdot L[y(t)] - y(0)] + 3 \cdot L[x(t)] = 2 \times \frac{1}{s^2}.$

$L[x(t)]\{2s+3\} + L[y(t)]\{-s\} = \frac{2}{s^2} + 1 = \frac{2+s^2}{s^2}$ —— (1).

$L[x'(t)] + 2 \cdot L[y'(t)] - 2L[x(t)] - L[y(t)] = L[t^2] - L[t].$

$[s \cdot L[x(t)] - x(0)] + 2 \cdot [sL[y(t)] - y(0)] - 2L[x(t)] - L[y(t)] = \frac{2}{s^3} - \frac{1}{s^2}$

$L[x(t)]\{s-2\} + L[y(t)]\{2s-1\} = \frac{2}{s^3} - \frac{1}{s^2} + 3$

$$= \frac{2 - s + 3s^3}{s^3}$$

By Cramer's Rule,

$$x = \frac{\Delta x}{x}, \quad y = \frac{\Delta y}{y}$$

$$\Delta = \begin{vmatrix} 2s+3 & -s \\ (s-2) & (2s-1) \end{vmatrix}$$

$$= (2s+3)(2s-1) + s(s-2)$$

$$= 4s^2 - 2s + 6s - 3 + s^2 - 2s$$

$$= 5s^2 + 2s - 3.$$

$$= 5s^2 + 5s - 3s - 3$$

$$= 5s(s+1) - 3(s+1)$$

$$\Rightarrow (5s-3)(s+1)$$

$$s = 3/5, \quad s = -1.$$

$$\Delta x = \begin{vmatrix} \frac{2+s^2}{s^2} & -s \\ \frac{2-s+3s^3}{s^3} & 2s-1 \end{vmatrix}$$

$$= \frac{(2+s^2)(2s-1)}{s^2} = \frac{4s-2+2s^3-s^2}{s^2} = \frac{2s^3-s^2+4s-2}{s^2}$$

$$+ s\left[\frac{2-s+3s^3}{s^3}\right] = \frac{2-s+3s^3}{s^2}$$

$$= \frac{2s^3-s^2+4s-2+2-s+3s^3}{s^2}$$

$$= \frac{5s^3-s^2+3s}{s^2} = \frac{5s^2-s+3}{s}$$

$$L(x(t)) = \frac{\Delta x}{\Delta} = \frac{5s^2-s+3}{s(s+1)(5s-3)}$$

$$\frac{5s^2-s+3}{s(s+1)(5s-3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{5s-3}$$

$$= A(s+1)(5s-3) + B(s)(5s-3) + Cs(s+1)$$

Homework:-

(1) $y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)\,du$

$y(t) = e^{-t} - 2\,[\cos t * y(t)]$

$L[y(t)] = L(e^{-t}) - 2\{L[\cos t]\cdot L[y(t)]\}$

$L[y(t)] = \frac{1}{s+1} - 2\left\{\frac{s}{s^2+1}\cdot L[y(t)]\right\}$

$L[y(t)]\left[1 + \frac{2s}{s^2+1}\right] = \frac{1}{s+1}$

$L[y(t)] = \frac{1}{s+1} \times \frac{s^2+1}{(s^2+2s+1)} = \frac{s^2+1}{(s+1)^3}$.

$\frac{s^2+1}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$.

$s^2+1 = A(s^2+2s+1) + B(s+1) + C$

Put $s=-1$, $\boxed{2 = C}$ coeff s^2, $\boxed{1 = A}$

s, $0 = 2A+B$

$0 = 2+B$

$\boxed{B=-2}$

$y(t) = L^{-1}\left[\frac{1}{s+1}\right] - 2L^{-1}\left[\frac{1}{(s+1)^2}\right] + 2L^{-1}\left[\frac{1}{(s+1)^3}\right]$.

$= e^{-t} - 2\cdot e^{-t}\cdot(t) + \cancel{2}\cdot e^{-t}\cdot\frac{t^2}{\cancel{2}}$

$= e^{-t}(1-2t+1) = 2e^{-t}(1-t)$

(3) $y'(t) = \sin t + \int_0^t y(t-u)\cos 2u\,du$, $y(0)=0$.

$y'(t) = \sin t + [\cos 2t * y(t)]$

$[sL[y(t)] - y(0)] = \frac{1}{s^2+1} + \frac{s}{s^2+4}L[y(t)]$.

$L[y(t)]\left[s - \frac{s}{s^2+4}\right] = \frac{1}{s^2+1} \Rightarrow L[y(t)]\left(\frac{s^3+3s}{s^2+4}\right) = \frac{1}{s^2+1}$

$L[y(t)] = \frac{(s^2+4)}{(s^2+1)(s^3+3s)}$.

$\frac{s^2+4}{(s^2+1)(s^3+3s)} = \frac{As+B}{s^2+1} + \frac{Cs^2+Ds+E}{s^3+3s}$.

$s^2+4 = (As+B)(s^3+3s) + (Cs^2+Ds+E)(s^2+1)$

$0 = A+C \Rightarrow \boxed{A=-C}$

s^3, $0 = B + D$ $\quad s^2$, $1 = 3A + E + C$ $\quad 0 = 3B + \cancel{D}$

$3A + E - A = 0$ $\quad 0 = B + \cancel{D}$

$2A + E = 0.$ $\quad 2B = 0$

$\boxed{E = 4}$ $\quad \boxed{B = 0}$

$\boxed{D = -0}$

$3A + 4 - A = 1$ $\quad \cancel{2A + 4 = 0}$

$2A = -3$ $\quad \cancel{2A = -4}$

$\boxed{A = -3/2}$ $\quad \cancel{\boxed{A = -2}}$

$\boxed{C = 3/2}$

$$y(t) = L^{-1}\left[\frac{-\frac{3}{2}s}{s^2+1}\right] + L^{-1}\left[\frac{\frac{3}{2}s^2+4}{s^3+3s}\right].$$

$$= -\frac{3}{2} \cdot L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{3}{2} \cdot L^{-1}\left[\frac{s^2}{s^3+3s}\right] + 4\, L^{-1}\left[\frac{1}{s^3+3s}\right]$$

$$= -\frac{3}{2} \cdot \cos t + \frac{3}{2} \cdot L^{-1}\left[\frac{s}{s^2+3}\right] + 4\, L^{-1}\left[\frac{1}{s^3}\right] + \frac{4}{3} \cdot L^{-1}\left[\frac{1}{s}\right]$$

$$= -\frac{3}{2}\cos t + \frac{3}{2}\cos(\sqrt{3})t + 4 \cdot \frac{t^2}{2} + \frac{4}{3} \; (1).$$

$$= -\frac{3}{2}\cos t + \frac{3}{2}\cos(\sqrt{3})t + 2t^2 + \frac{4}{3}.$$

* Simultaneous differential equations :-

$\frac{dy}{dt} + 2x = \sin 2t \quad , x(0) = 1 \quad \Rightarrow y' + 2x = \sin 2t \rightarrow (1)$

$\frac{dx}{dt} - 2y = \cos 2t , \quad y(0) = 0 \quad \Rightarrow x' - 2y = \cos 2t \rightarrow (2)$

$(1) \rightarrow L[y'(t)] + 2L[x(t)] = L[\sin 2t]$.

$s \cdot L[y(t)] - y(0) + 2 \cdot L[x(t)] = \frac{2}{s^2+4}$

$L[x(t)]\{2\} + L[y(t)]\{s\} = \frac{2}{s^2+4} \rightarrow (3)$

$(2) \Rightarrow L[x'(t)] - 2L[y(t)] = L[\cos 2t]$

$s \cdot L[x(t)] - x(0) - 2L[y(t)] = \frac{s}{s^2+4}$

$L[x(t)]\{s\} + L[y(t)]\{-2\} = \frac{s}{s^2+4} + 1$

Solve for $L[x(t)]$ & $L[y(t)]$

$$\Delta = \begin{vmatrix} 2 & s \\ s & -2 \end{vmatrix} = -4 - s^2 = -(4+s^2).$$

$$\Delta x = \begin{vmatrix} \frac{2}{s^2+4} & s \\ \frac{s^2+s+4}{s^2+4} & -2 \end{vmatrix}$$

$$\Rightarrow = \frac{-4}{s^2+4} - \frac{s^3+s^2+4s}{s^2+4} = \frac{-s^3-s^2-4s-4}{s^2+4}$$

By cramer's rule, $L(x(t)) = \frac{\Delta x}{\Delta} = \frac{\cancel{-}(s^3+s^2+4s+4)}{\cancel{-}(s^2+4)(s^2+4)}$

$$= \frac{s^2(s+1) + 4(s+1)}{(s^2+4)^2}$$

$$= \frac{\cancel{(s^2+4)}(s+1)}{(s^2+4)^{\cancel{2}}} = \frac{s+1}{s^2+4}$$

$$x(t) = L^{-1}\left[\frac{s+1}{s^2+4}\right] = L^{-1}\left[\frac{s}{s^2+4}\right] + L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \cos 2t + \frac{1}{2}\sin 2t.$$

$$\Delta y = \begin{vmatrix} 2 & \frac{2}{s^2+4} \\ s & \frac{s^2+s+4}{s^2+4} \end{vmatrix}$$

$$= \frac{2s^2+2s+8}{s^2+4} - \frac{2s}{s^2+4} = \frac{2s^2+8}{s^2+4} = \frac{2\cancel{(s^2+4)}}{\cancel{(s^2+4)}}$$

$$= 2.$$

$$L[y(t)] = \frac{\Delta y}{\Delta} = \frac{2}{-(s^2+4)} = \frac{-2}{s^2+4}$$

$$= y(t) = L^{-1}\left[\frac{-2}{s^2+4}\right] = -2 \cdot L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= -\cancel{2} \times \frac{1}{\cancel{2}} \sin 2t$$

$$= -\sin 2t.$$

2. $3 \cdot \frac{dx}{dt} + \frac{dy}{dt} + 2x = 1, \quad x(0) = 0$

$\frac{dx}{dt} + 4 \cdot \frac{dy}{dt} + 3y = 0, \quad y(0) = 0.$

$3x' + y' + 2x = 1$

$3 \cdot L[x'(t)] + y'(t) + 2L[x(t)] = L[1].$

$3[s \cdot L[x(t)] - x(0)] + [s \cdot L[y(t)] - y(0)] + 2 \cdot L[x(t)] = \frac{1}{s}.$

$L[x(t)][3s+2] + s \cdot L[y(t)] = \frac{1}{s}. \quad — 1$

$x' + 4y' + 3y = 0.$

$L[x'(t)] + 4y'(t) + 3y(t) = 0.$

$[s \cdot L[x(t)] - x(0)] + 4[s \cdot L[y(t)] - y(0)] + 3 \cdot L[y(t)] = 0$

$L[y(t)][4s+3] + L[x(t)](s) = 0. \quad — 2$

$$\Delta = \begin{vmatrix} 3s+2 & s \\ s & 4s+3 \end{vmatrix} = (3s+2)(4s+3) - s^2$$

$$= 12s^2 + 9s + 8s + 6 - s^2$$

$$= 11s^2 + 17s + 6.$$

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$$L[x(t)] = \frac{\Delta x}{\Delta}$$

$$\Delta x = \begin{vmatrix} \frac{1}{s} & s \\ 0 & 4s+3 \end{vmatrix}$$

$11s^2+11s+6s+6$

$11(s+1)+6(s+1)$

$$= \frac{4s+3}{s}$$

$$L[x(t)] = \frac{\Delta x}{\Delta} = \frac{(4s+3)/s}{11s^2+17s+6} = \frac{4s+3}{11s^3+7s^2+6s}$$

$$= \frac{(4s+3)}{s(s+1)(11s+6)}$$

$$\frac{4s+3}{s(s+1)(11s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{11s+6}$$

$$4s+3 = A(s+1)(11s+6) + B(s)(11s+6) + C(s)(s+1)$$

Put $s=-1$, $\quad -1 = B(-1)(-11+6)$ $\quad$ Put $s=0$, $\quad 3 = A(1)(6)$

$-1 = B(-1)(-5)$ $\quad\quad 3 = A(6)$

$\boxed{B = -1/5}$ $\quad\quad \boxed{A = 1/2}$

Put $s = \frac{-6}{11}$, $\quad -\frac{24}{11}+3 = C\left(\frac{-6}{11}\right)\left(\frac{-6}{11}+1\right)$

$$\frac{9}{\cancel{11}} = C\left(\frac{-6}{\cancel{11}}\right)\left(\frac{5}{11}\right)$$

$$\cancel{9}^{3} = C(\cancel{-6}_{2})\left(\frac{5}{11}\right)$$

$$\boxed{C = \frac{-33}{10}}$$

$$x(t) = L^{-1}\left[\frac{1}{2s}\right] + \frac{-1}{5} \cdot L^{-1}\left(\frac{1}{s+1}\right) - \frac{33}{10} \cdot L^{-1}\left[\frac{1}{11s+6}\right]$$

$$= \frac{1}{2}(1) - \frac{1}{5} \cdot e^{-t} - \frac{33}{10}\left\{\frac{1}{11} \times L^{-1}\left[\frac{1}{s+\frac{6}{11}}\right]\right\}$$

$$= \frac{1}{2} - \frac{1}{5}e^{-t} - \frac{3}{10} \times e^{-6/11\, t}$$

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$$\Delta y = \begin{vmatrix} 3s+2 & \frac{1}{s} \\ s & 0 \end{vmatrix}$$

$$= 0 - s \times \frac{1}{s} = -1.$$

$$L[y(t)] = \frac{\Delta y}{\Delta} = \frac{-1}{(s+1)(11s+6)}$$

$$L[y(t)] = \frac{-1}{(s+1)(11s+6)}$$

$$\frac{-1}{(s+1)(11s+6)} = \frac{A}{s+1} + \frac{B}{(11s+6)}$$

$-1 = A(11s+6) + B(s+1)$ $\quad s = \frac{-6}{11}$ $\quad -1 = A(0) + B\left(\frac{-6}{11}+1\right)$

put $s = -1$, $\quad -1 = A(-11+6)$ $\qquad -1 = B\left(\frac{5}{11}\right)$

$\cancel{-}1 = \cancel{-}5A.$ $\qquad B = -11/5.$

$A = 1/5.$

$$y(t) = L^{-1}\left[\frac{1}{5(s+1)}\right] - \frac{11}{5} L^{-1}\left[\frac{1}{11s+6}\right]$$

$$= \frac{1}{5} \times e^{-t} - \frac{11}{5} \times \frac{1}{11} \times e^{-6/11 t}.$$

$$= \frac{1}{5} e^{-t} - \frac{1}{5} e^{-6/11 t}$$

$$= \frac{1}{5}\left(e^{-t} - e^{-6/11 t}\right).$$

(3) $Dx - y = e^t$ $\quad x(0) = 1$

$Dy + x = \sin t$ $\quad y(0) = 0.$

$$x' - y = e^t$$

$$x'(t) - y(t) = e^t.$$

$$s \cdot L[x(t)] - L[y(t)] = \frac{1}{s-1} + 1 = \frac{1+s-1}{s-1} = \frac{s}{s-1}.$$

$$y' + x = \sin t$$

$$s \cdot L[y(t)] + L[x(t)] = \frac{1}{s^2+1}$$

$$\Delta = \begin{vmatrix} s & -1 \\ 1 & s \end{vmatrix}$$

$$= s^2 - (-1) = s^2 + 1.$$

$$\Delta x = \begin{vmatrix} \frac{s}{s-1} & -1 \\ \frac{1}{s^2+1} & s \end{vmatrix} = \frac{s^2}{s-1} + \frac{1}{s^2+1} = \frac{s^4+s^2+s-1}{(s-1)(s^2+1)}$$

$$L[x(t)] = \frac{s^4+s^2+s-1}{(s^2+1)^2(s-1)} = \frac{\Delta x}{\Delta}$$

$$= \frac{s^2\,\cancel{(s^2+1)}}{(s^2+1)^2(s-1)} + \frac{\cancel{(s-1)}}{(s^2+1)^2\cancel{(s-1)}}$$

$$x(t) = L^{-1}\left[\frac{s^2}{(s^2+1)(s-1)}\right] + L^{-1}\left[\frac{1}{(s^2+1)^2}\right].$$

$$\frac{1}{2a^2}[\sin at - t\cos at].$$

$$s^2 = (As+B)(s-1) + C(s^2+1).$$

put $s = +1$, $\quad 1 = 2C$

$$C = 1/2$$

coeff s^2, $\quad 1 = A + C$

$$A = 1 - 1/2 = 1/2$$

s, $0 = -A + B$

$$0 = -\tfrac{1}{2} + B, \quad \boxed{B = 1/2}.$$

$$= L^{-1}\left[\frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1}\right] + L^{-1}\left[\frac{1/2}{s-1}\right].$$

$$= \frac{1}{2}L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right)$$

$$= \frac{1}{2}\cos t + \frac{1}{2}e^t + \frac{1}{2(1)}[\sin t - t\cos t].$$

$$= \frac{1}{2}[\cos t + e^t + \sin t - t\cos t].$$

Homework:-

* $x' - 2x + 3y = 0$, $x(0) = 8$

$y' - y + 2x = 0$, $y(0) = 3$.

$$x'(t) - 2x(t) + 3y = 0.$$

$$s\cdot L[x(t)] - 8 - 2L[x(t)] + 3L[y(t)] = 0.$$

$$L[x(t)]\{s-2\} + 3L[y(t)] = 8 \quad \text{--- (1)}$$

$$y'(t) - y(t) + 2x(t) = 0.$$

$$s\cdot L[y(t)] - L[y(t)] + 2\cdot L[x(t)] - 3 = 0.$$

$$L[y(t)]\{s-1\} - L[y(t)]\,[?] = 3$$

$$\Delta = \begin{vmatrix} s-2 & 3 \\ -2 & s-1 \end{vmatrix} = (s-2)(s-1)+6 = s^2-3s+2+6$$
$$= s^2-3s+8$$

$$\Delta x = \begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix} = 8s-8-9 = 8s-17$$

$$\Delta y = \begin{vmatrix} s-2 & 8 \\ -2 & 3 \end{vmatrix} = 3s-6+16 = 3s+10.$$

$$L[x(t)] = \frac{\Delta x}{\Delta} = \frac{8s-17}{s^2-3s+8}$$

$$= \frac{8s-17}{s^2-3s+4} = \frac{8s-17}{(s+1)(s-4)}$$

$$= \frac{A}{s+1} + \frac{B}{s-4}$$

$$8s-17 = A(s-4)+B(s+1)$$

Put $s=4$,

$32-17 = B(5)$

$B = 3$

Put $s=-1$

$-25 = A(-5)$

$A = 5$

$$L^{-1}\left[\frac{5}{s+1}\right] + L^{-1}\left[\frac{3}{s-4}\right] = 5e^{-t}+3e^{4t}$$

(2)

$$3y = 2x - x' \qquad \frac{d}{dt}(5e^{-t}+3e^{4t})$$

$$= 10e^{-t} + 6e^{4t} + 5e^{-t} - 12e^{4t}$$

$$3y = 15e^{-t} - 6e^{4t}$$

$$y(t) = 5e^{-t} - 2e^{4t}$$

(1)

$\frac{dx}{dt} + y = \sin t, \qquad x = 2$

$\frac{dy}{dt} + x = \cos t \qquad y=0$ ~~at~~ ~~t=0~~.

$\therefore x'(t) + y(t) = \sin t$

$y'(t) + x(t) = \cos t.$

$$L[x'(t)] + L[y(t)] = \frac{1}{s^2+1}$$

$$\{s \cdot L[x(t)] - x(0)\} + L[y(t)] = \frac{1}{s^2+1}$$

$L[x(t)]\,[s] + \ldots$

$$= 1+2s^2+2/s^2+1 = \frac{2s^2+3}{s^2+1} \quad (1)$$

$$L[y'(t)] + L[x(t)] = \frac{s}{s^2+1}$$

$$s \cdot L[y(t)] - y(0) + L[x(t)] = \frac{s}{s^2+1}$$

$$L[x(t)]\{1\} + L[y(t)]\{s\} = \frac{s}{s^2+1}$$

$$\begin{vmatrix} s & 1 \\ 1 & s \end{vmatrix} = s^2-1 \qquad \begin{vmatrix} 2s^2+3/s^2+1 & 2 \\ s/s^2+1 & s \end{vmatrix} = \frac{2s^3+3s}{s^2+1} - \frac{s}{s^2+1}$$

$$= \frac{2s^3+3s-s}{s^2+1}$$

$$L[x(t)] = \frac{2s^2+2s}{(s^2+1)(s^2-1)} = \frac{2s}{s^2-1}$$

$$x(t) = 2L^{-1}\left[\frac{s}{s^2-1}\right] = 2\cosh t$$

$$y(t) = \sin t - 2\sinh t.$$

Problems on omitting exponential function on s & 1/s :-

* $L^{-1}[e^{-as} \cdot F(s)] = L^{-1}[F(s)]_{t \to t-a}$.

* $L^{-1}[s \cdot F(s)] = \frac{d}{dt}\left\{L^{-1}[F(s)]\right\}$

$L^{-1}[s^2 \cdot F(s)] = \frac{d^2}{dt^2}\left\{L^{-1}[F(s)]\right\}$

* $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$.

*

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* $L^{-1}\left[\dfrac{e^{-s}}{(s+2)(s+3)}\right] = L^{-1}\left[\dfrac{1}{(s+2)(s+3)}\right]_{t \to t-1}$

$\dfrac{1}{(s+2)(s+3)} = \dfrac{A}{s+2} + \dfrac{B}{s+3}$

$1 = A(s+3) + B(s+2)$

$s = -3, \quad 1 = B(-1) \quad \boxed{B=-1} \qquad s=-2, \quad 1 = A(-2+3) \quad \boxed{A=1}$

$= L^{-1}\left[\dfrac{1}{s+2}\right] - L^{-1}\left[\dfrac{1}{s+3}\right]$

$= e^{-2t} - e^{-3t}$

$= e^{-2(t-1)} - e^{-3(t-1)} = e^{-2t+2} - e^{-3t+3}.$

* $L^{-1}\left[\dfrac{e^{-3s}}{s^2(s^2+1)}\right] = L^{-1}\left(\dfrac{1}{s^2(s^2+1)}\right)_{t \to t-3}.$

$\dfrac{1}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+1}$

$1 = As(s^2+1) + B(s^2+1) + (Cs+D)(s^2)$

$= As^3 + As + Bs^2 + B + Cs^3 + Ds^2$

Coeff, $s^3, \quad 0 = A + C$

$s^2, \quad 0 = B + D.$

$s, \quad \boxed{0 = A} \quad \boxed{C=0}$

$cn, \quad \boxed{1 = B} \quad \boxed{D=-1}$

$= L^{-1}\left[0 + \dfrac{1}{s^2} - \dfrac{1}{s^2+1}\right].$

$= L^{-1}(0) + L^{-1}\left[\dfrac{1}{s^2}\right] - L^{-1}\left[\dfrac{1}{s^2+1}\right] = \dfrac{t^{2-1}}{1!} - \sin t$

$= t - \sin t$

$= (t-3) - \sin(t-3).$

* $L^{-1}\left[\dfrac{e^{-\pi s}(s+1)}{s^2+s+1}\right] = L^{-1}\left(\dfrac{s+1}{s^2+s+1}\right)_{t \to (t-\pi)}$

$A^2 \quad 2AB$

$A = s, \; B = 1/2$

$= L^{-1}\left[\dfrac{(s+1)}{(s+\frac{1}{2})^2 - \frac{1}{4} + 1}\right] = L^{-1}\left[\dfrac{s + 1/2 + 1/2}{(s+\frac{1}{2})^2 + \frac{3}{4}}\right] = L^{-1}\left[\dfrac{s+1/2}{(s+\frac{1}{2})^2 + \frac{3}{4}}\right] + L^{-1}\left[\dfrac{1/2}{(s+\frac{1}{2})^2+\frac{3}{4}}\right]$

$$= e^{-\frac{1}{2}t} L^{-1}\left[\frac{}{s^2+\frac{3}{4}}\right] + \frac{}{2}$$

$$= e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}\cdot e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)\times\frac{2}{\sqrt{3}}$$

$$= e^{-\frac{1}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \sin\left(\frac{\sqrt{3}}{2}t\right)\right] \quad t \to t-\pi$$

$$= e^{-\frac{1}{2}(t-\pi)}\left[\cos\left(\frac{\sqrt{3}}{2}(t-\pi)\right) + \sin\left[\frac{\sqrt{3}}{2}(t-\pi)\right]\right].$$

$*$ $$L^{-1}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] = L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right].$$

$$= \sin t + L^{-1}\left[\frac{1}{s^2+1}\right] \quad t \to t-\pi$$

$$= \sin t + \sin(t)$$

$$= \sin t + \sin(t-\pi).$$

$*$ $$L^{-1}\left[\frac{s}{(s+2)^4}\right].$$

$$= \frac{d}{dt}\left[L^{-1}\left[\frac{1}{(s+2)^4}\right]\right].$$

$$= \frac{d}{dt}\left[e^{-2t}\, L^{-1}\left[\frac{1}{s^4}\right]\right]$$

$$= \frac{d}{dt}\left[e^{-2t}\times\frac{t^3}{6}\right]. = \frac{1}{6}\left[e^{-2t}\times 3t^2 + t^3(-2)e^{-2t}\right]$$

$$= \frac{e^{-2t}t^2}{6}\left[3-2t\right]$$

$*$ $$L^{-1}\left[\frac{s^2}{(s-2)^3}\right] = \frac{d^2}{dt^2}\left[L^{-1}\left[\frac{1}{(s-2)^3}\right]\right]$$

$$= \frac{d^2}{dt^2}\left[e^{+2t} L^{-1}\left[\frac{1}{s^3}\right]\right] = \frac{d^2}{dt^2}\left[e^{2t}\cdot\frac{t^2}{2}\right]$$

$$= \frac{1}{2}\frac{d}{dt}\left[e^{2t}(2t) + t^2(2e^{2t})\right].$$

$$= \frac{1}{2}\left[e^{2t}(2) + 2t(e^{2t})(2) + t^2(4e^{2t}) + (2e\right.$$

$$= \frac{1}{2}\left[2e^{2t} + 4t\cdot e^{2t} + 4t^2\cdot e^{2t} + 4e^{2t}t\right].$$

$$= \frac{1}{2}e^{2t}\left[\right.$$

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* $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]. = \frac{1}{2}\left\{t\cos at + \left[\frac{\sin 2at}{2a}\right]\right\}.$

$= \cancel{\frac{d^2}{dt^2}\left[L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]\right]}.$

* $L^{-1}\left[\frac{s^2}{(5s-6)^4}\right]. = \frac{d^2}{dt^2}\left[L^{-1}\left[\frac{1}{(5s-6)^4}\right]\right].$

$= \frac{d^2}{dt^2}\left[\frac{1}{5^4}L^{-1}\left[\frac{1}{(s-\frac{6}{5})^4}\right]\right]$

$= \frac{d^2}{dt^2}\left[\frac{1}{5^4}e^{+\frac{6}{5}t}L^{-1}\left[\frac{1}{s^4}\right]\right].$

* $L^{-1}\left[\frac{1}{s^2(s+a)}\right].$

$= \int_0^t\int_0^t e^{-at}\,dt = \int_0^t \left(\frac{e^{-at}}{(-a)}\right]_0^t dt = \frac{1}{-a}\int_0^t \left[e^{-at}-1\right] dt$

$= \int_0^t e^{-at}dt - \frac{1}{a}\int_0^t 1\,dt$

$= \frac{1}{a} \times \frac{1}{-a}\left(\left[e^{-at}\right]_0^t\right) - \frac{1}{a}\left(1\right]_0^t$

$= \frac{1}{a^2}\left(e^{-at}-1\right] - \frac{1}{a}(t].$

* $L^{-1}\left[\frac{1}{s(s+2)^3}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+2)^3}\right]dt = \int_0^t e^{-2t}.\,L^{-1}\left[\frac{1}{s^3}\right]dt$

$= \int_0^t e^{-t} \times \frac{t^2}{2}\,dt = \frac{1}{2}\int_0^t e^{-t} \times t^2\,dt$

$u = t^2, \quad x = e^{2t}$

$u' = 2t, \quad v_1 = e^{-2t}/-2$

$u'' = 2, \quad v_2 = e^{-2t}/4$

$v_3 = e^{-2t}/-8$

$= \left[\frac{t^2 e^{-2t}}{-2} - (2t)\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right]_0^t\right.$

$= \frac{t^2.e^{-2t}}{-2} - 2t\left(\frac{e^{-2t}}{4}\right) + \frac{2e^{-2t}}{-8} - 1/4.$

*

Applications of Laplace ...

Electric circuits:-

R, L, C are some constants.

Kirchoff Laws:-

* The sum of voltage drop is equal to the applied voltage (or)
* The sum of currents flowing into the junction is zero.
* For two (or) more circuits, The amount of current flowing into the junction is equal to the amount of current flowing out of the junction.

Problems:-

* In a electric circuit with emf $E(t)$, Resistance (R) and inductance (L). If the switch is connected at $t=0$, and disconnected at $t=a$, find the current i, at any instant.

Sol: given problem is of RL circuit

$$i \cdot R + L \cdot \frac{di}{dt} = E(t) \Rightarrow iR + L \cdot i'(t) = E(t) \rightarrow (1)$$

$$E(t) = \begin{cases} E, & 0<t<a \\ 0, & t>a \end{cases} \quad (2)$$

Taking L.T on both sides,

$$R \cdot L[i(t)] + L\left\{ s \cdot L[i(t)] - i(0) \right\} = L[E(t)]$$

$$= \int_0^a e^{-st} E(t) dt$$

$$= E \int_0^a e^{-st} dt$$

$$= E \left[\frac{e^{-st}}{-s} \right]_0^a$$

$$= \frac{E}{...} \left[e^{-sa} - 1 \right]$$

$$L[i(t)]\{R+sL\} = \frac{E}{s}[1-e^{-as}]$$

$$i(t) = L^{-1}\left[\frac{E[1-e^{-as}]}{s[R+s\cdot L]}\right] = E\cdot L^{-1}\left[\frac{1}{s[R+s\cdot L]}\right] - L^{-1}\left[\frac{e^{-as}}{s(R+sL)}\right] \quad -③$$

$$\text{I} \qquad\qquad\qquad\qquad \text{II}$$

$$\text{I} \Rightarrow \frac{1}{s[R+s\cdot L]} = \frac{A}{s} + \frac{B}{R+s\cdot L}$$

$$1 = A(R+sL) + Bs$$

s term, $\quad 0 = AL + B$

Put $s = 0$ $\quad 0 = \frac{L}{R} + B$

$$1 = AR$$

$$\boxed{A = 1/R} \qquad \boxed{B = -L/R}$$

$$\therefore \text{I} \Rightarrow \frac{1}{R}\cdot L^{-1}\left[\frac{1}{s}\right] - \frac{L}{R}\times\frac{1}{L}\, L^{-1}\left[\frac{1}{(s+\frac{R}{L})}\right]$$

$$\boxed{\text{I} = \frac{1}{R}(1) - \frac{e^{-Rt/L}}{R}}$$

$$\text{II} \Rightarrow \frac{1}{R} - \frac{e^{-R(t-a)/L}}{R}$$

$$\Rightarrow E\left\{-\frac{e^{Rt/L}}{R} + \frac{e^{-\frac{Rt}{L}+\frac{Ra}{L}}}{R}\right\}.$$

$$= \frac{E}{R}\times e^{-Rt/L}\left[-1+e^{Ra/L}\right].$$

* A voltage $E\cdot e^{-at}$ is applied at time $t=0$ with inductance L and Resistance R. Show that current at any instant is given by $i = \frac{E}{(R-aL)}\left[e^{-at} - e^{-Rt/L}\right]$ where $i=0$ at $t=0$.

$$iR + L\cdot\frac{di}{dt} = E\cdot e^{-at}$$

$$iR + L\cdot i'(t) = E\cdot e^{-at}$$

$$R\cdot L[i(t)] + L\{s\cdot L[i(t)] - i(0)\} = E\cdot L[e^{-at}]$$

$$L[i(t)][R+sL] = \frac{E}{s+a}$$

$$i(t) = E\cdot L^{-1}\left[\frac{1}{(s+a)(R+sL)}\right].$$

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$$\frac{1}{(s+a)(R+sL)} = \frac{A}{(s+a)} + \frac{B}{sL+R}$$

$$1 = A(sL+R) + B(s+a)$$

Put $s = -a$, $1 = A(-aL+R)$

$$A = \frac{1}{R-aL}$$

coeff s: $0 = AL + B$

$$0 = \frac{L}{R-aL} + B$$

$$\boxed{B = \frac{-L}{R-aL}}$$

Sub in ②

$$i(t) = E \cdot L^{-1}\left\{ \frac{1}{(R-aL)(s+a)} - \frac{L}{(R-aL)(sL+R)} \rightarrow L(s+R/L) \right\}$$

$$= \frac{E}{(R-aL)}\left[e^{-at} - e^{-Rt/L} \right]$$

* The current i and charge q are in a series circuit containing an inductance 'L', capacitance 'c' and emf E. Express i & q in term of 't' given that L, C, E are constant.

given problem is of LC circuit,

$$L \cdot \frac{di}{dt} + \frac{q}{c} = E$$

$$L \cdot \frac{d^2q}{dt^2} + \frac{q}{c} = E \quad (or) \quad Lq''(t) + \frac{q(t)}{c} = E \quad -①$$

$$L\left\{ s^2 \cdot L[q(t)] - s \cdot q(0) - q'(0) \right] + \frac{1}{c} \cdot L[q(t)] = L[E] = \frac{E}{s}$$

$$L[q(t)]\left\{ s^2 \cdot L + \frac{1}{c} \right\} = \frac{E}{s}$$

$$L[q(t)] = \frac{E}{s[s^2L + \frac{1}{c}]}$$

$$q(t) = L^{-1}\left[\frac{E}{s(s^2L + \frac{1}{c})} \right]$$

$$= \frac{E}{s.L\left[s^2+\frac{1}{LC}\right]} \Rightarrow q(t) = \frac{E}{L}\cdot L^{-1}\left[\frac{1}{s\left[s^2+\frac{1}{LC}\right]}\right].$$

$$q(t) = \frac{E}{L}\int_0^t L^{-1}\left[\frac{1}{s^2+\frac{1}{LC}}\right]dt.$$

$$a^2 = \frac{1}{LC} \Rightarrow a = \frac{1}{\sqrt{LC}}$$

$$= \frac{E}{L}\times\frac{1}{\left(\frac{1}{\sqrt{LC}}\right)}\int_0^t \sin\left(\frac{t}{\sqrt{LC}}\right)dt$$

$$= \frac{E}{L}\times\sqrt{LC}\left[\frac{-\cos\left(\frac{t}{\sqrt{LC}}\right)}{\frac{1}{\sqrt{LC}}}\right]_{0=t}^{t=t}$$

$$= \frac{E}{\cancel{\sqrt{L}}}\sqrt{C}\,\cancel{\sqrt{LC}}$$

$$= EC\left[-\cos\left(\frac{t}{\sqrt{LC}}\right) - (-1)\right]$$

$$\boxed{q = EC\left[1-\cos\left[\frac{t}{\sqrt{LC}}\right]\right].}$$

$$i = \frac{dq}{dt}$$

$$= E\sqrt{C}\times\sqrt{C}\sin\left(\frac{t}{\sqrt{LC}}\right)\times\frac{1}{\sqrt{L}\cancel{C}}$$

$$= \frac{E}{\sqrt{L}}\times\sqrt{C}\cdot\sin\left[\frac{t}{\sqrt{LC}}\right].$$

* An alternating emf $E\sin at$ is applied to an inductance 'L' and capacitance 'C' are in series. Show that

By kirchoff's law, $L\cdot\frac{di}{dt} + \frac{q}{c} = E\sin at \Rightarrow L\frac{d^2q}{dt^2} + \frac{q}{c} = E\sin at$

$$\Rightarrow L\,q''(t) + \frac{q(t)}{c} = E\sin at \rightarrow ①$$

Taking L.T on both sides.

$$L\left[s^2 L[q(t)] - s.q(0) - q'(0)\right] + \frac{L[q(t)]}{c} = E\cdot L[\sin at].$$

$$L[q(t)]\left\{Ls^2+\frac{1}{c}\right\} = \frac{E\times a}{s^2+a^2}$$

$$L[q(t)] = \frac{Ea}{(s^2+a^2)\left(s^2L+\frac{1}{c}\right)} = \frac{Ea}{L(s^2+a^2)(s^2+n^2)}$$

$$\frac{Ea}{L(s^2+a^2)(s^2+n^2)}$$

$$\frac{1}{(s^2+a^2)(s^2+n^2)} = \frac{(As+B)}{(s^2+a^2)} + \frac{(Cs+D)}{(s^2+n^2)}$$

$$= (As+B)(s^2+n^2) + (Cs+D)(s^2+a^2)$$

$$1 = As^3 + n^2As + Bs^2 + Bn^2 + Cs^3 + Csa^2 + Ds^2 + Da^2$$

coeff, s^3, $0 = A + C$

s^2, $0 = B + D$

s, $0 = An^2 + Ca^2$

con, $1 = B + D$

$A = 0$; $C = 0$.

$B = \frac{+1}{n^2-a^2}$; $D = \frac{-1}{n^2-a^2}$

$$② \Rightarrow \frac{Ea}{L}\left\{L^{-1}\left[\frac{+1}{(n^2-a^2)(s^2+a^2)}\right] - L^{-1}\left[\frac{1}{(n^2-a^2)(s^2+n^2)}\right]\right\}.$$

$$\frac{Ea}{L(n^2-a^2)}\left\{+\frac{1}{a}\sin at - \frac{1}{n}\sin nt\right\} = q(t).$$

$$i = \frac{dq}{dt} = \frac{Ea}{L(n^2-a^2)}\left\{\cancel{\frac{+1}{a}}\,\cancel{\times a} \times \cos at - \cancel{\frac{1}{n}}\,\cancel{\times n}\cos nt\right\}.$$

$$= \frac{Ea}{L(n^2-a^2)}\left\{-\cos nt + \cos at\right\}.$$

* A periodic emf $E_0 \sin at$ is applied at the time $(t=0)$, to a circuit consisting of resistor & capacitance in series. The initial value of current (i) & charge (q) in the condensor are zero. find the current at any time t using Laplace transform. *

$$iR + \frac{q}{C} = E_0 \sin at \quad \text{(or)} \quad R\frac{dq}{dt} + \frac{q}{C} = E_0 \sin at$$

$$Rq'(t) + \frac{q(t)}{C} = E_0 \sin at$$

$$R[sL[q(t)] - q(0)] + \frac{L[q(t)]}{C} = E_0 \times \frac{a}{s^2+a^2}$$

$$L[q(t)]\left\{Rs + \frac{1}{C}\right\} = \frac{E_0\, a}{s^2+a^2}$$

$$L[q(t)] = \frac{E_0 a}{(s^2+a^2)(sR+\frac{1}{c})} = \frac{E_0 a}{R(s^2+a^2)(s+\frac{1}{Rc})}$$

$$\frac{1}{(s^2+a^2)(s+\frac{1}{Rc})} = \frac{As+B}{(s^2+a^2)} + \frac{c}{(s+\frac{1}{Rc})}$$

$$1 = (As+B)(s+\frac{1}{Rc}) + c(s^2+a^2)$$

Put $s = -1/Rc$

$$1 = c\left(\frac{1}{R^2c^2}+a^2\right)$$

$$C = \frac{1}{\left(\frac{1}{R^2c^2}+a^2\right)} = \frac{1}{\frac{1+a^2R^2c^2}{R^2c^2}} = \frac{R^2c^2}{1+a^2R^2c^2}$$

$$0 = \frac{A}{Rc} + B.$$

$$B = -\frac{A}{Rc} = -\frac{R^2c^2/1+a^2R^2c^2}{Rc}$$

$$\boxed{B = -\frac{Rc}{1+a^2R^2c^2}} \quad \boxed{A = \frac{-R^2c^2}{1+a^2R^2c^2}}$$

$$= \frac{E_0 a}{R} \times \frac{-Rc}{1+R^2c^2a^2}\left[\frac{(Rc)s}{s^2+a^2} + \frac{1}{s^2+a^2} - \frac{Rc}{(s+1/Rc)}\right].$$

$$= \frac{E_0 a}{R} \times \left\{\frac{-R^2c^2}{1+R^2c^2a^2} \times \cos at - \frac{Rc}{1+R^2c^2a^2} \times \frac{1}{a}\sin at + \frac{R^2c^2}{1+R^2c^2a^2} \times e^{-t/Rc}\right\}$$

* A inductor of 3 Henry is in series with resistor of 30 Ω an emf of 150V. Assume that the current is 0 at time t=0. find the current at any instant "t".

$$I(t) = 5\cdot(1 - e^{-10t}).$$

$$y^{iv}(x) = \frac{w}{EI}(x)$$

Taking L.T on both sides,

$$s^4 L[y(x)] - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = L\left[\frac{w}{EI}\right]$$

$$y''(0) = c_1 \qquad\qquad = \frac{w}{EI} \times \frac{1}{s}$$

$$y'''(0) = c_2$$

$$s^4 L[y(x)] - s c_1 - c_2 = \frac{w}{EI \cdot s}$$

$$s^4 [L[y(x)]] = \frac{w}{EI \cdot s} + c_1 s + c_2$$

$$L[y(x)] = \frac{w}{EI\, s^5} + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

$$= \frac{w}{EI} L^{-1}\left[\frac{1}{s^5}\right] + c_1 \left[\frac{1}{s^3}\right] L^{-1} + c_2 L^{-1}\left[\frac{1}{s^4}\right]$$

$$= \frac{w}{EI} \times \frac{x^4}{4!} + c_1 \cdot \frac{x^2}{2!} + c_2 \cdot \frac{x^3}{3!}$$

$$= \frac{w}{EI} \frac{x^4}{24} + c_1 \cdot \frac{x^2}{2} + c_2 \cdot \frac{x^3}{6}$$

to find c_1 & c_2, Bon, condition, $y(l) = 0$,

$$\frac{wl^4}{24EI} + c_1 \cdot \frac{l^2}{2} + \frac{c_2 \cdot l^3}{6} = 0 \longrightarrow (1)$$

$y'(l) = 0$,

$$y'(x) = \frac{w}{EI} \times \frac{4x^3}{24_6} + c_1 \times \frac{2x}{2} + c_2 \times \frac{3x^2}{6_2}$$

$$= \frac{wx^3}{EI \times 6} + c_1 x + c_2 \frac{x^2}{2} = 0$$

$$y'(l) = \frac{wl^3}{EI \times 6} + c_1 \times l + c_2 \times \frac{l^2}{2} = 0 \rightarrow (2)$$

$$① \Rightarrow \quad c_1\left(\frac{l^2}{2}\right) + c_2\left(\frac{l^3}{6}\right) = \frac{-wl^4}{24EI}$$

$$②\times\frac{l}{2} \Rightarrow \quad c_1\left(\frac{l^2}{2}\right) + c_2\left(\frac{l^3}{4}\right) = \frac{-wl^4}{6EI\times 2}$$

$$l^3 c_2\left(\frac{1}{6}-\frac{1}{4}\right) = \frac{wl^4}{EI}\left[-\frac{1}{24}+\frac{1}{12}\right]$$

$$l^3 c_2\left[\frac{-2}{24}\right] = \frac{wl^4}{EI}\left[\frac{-12+24}{24\times 12}\right]$$

$$-c_2 = \frac{+wl}{2EI}$$

$$\boxed{c_2 = -\frac{wl}{2EI}}$$

c_2 in ①,

$$\frac{wl^4}{24EI} + c_1\cdot\frac{l^2}{2} - \frac{wl}{2EI}\times\frac{l^3}{6} = 0$$

$$\frac{wl^4}{24EI} - \frac{wl^4}{12EI} = -c_1\times\frac{l^2}{2}$$

$$\frac{wl^4}{EI}\left[\frac{1}{24}-\frac{1}{12}\right] = -c_1\times\frac{l^2}{2}$$

$$\left[\frac{12-24}{12\times 24}\right] = -c_1\times\frac{l^2}{2}$$

$$\left[\frac{+12}{12\times 24}\right] = -c_1\times\frac{l^2}{2}$$

$$\frac{wl^2}{EI}\left[\frac{1}{12}\right] = c_1$$

$$y(x) = \frac{w}{EI}\times\frac{x^4}{24} + \frac{wl^2}{EI}\times\frac{x^2}{12} + \frac{-wL}{2EI}\times\frac{x^3}{6}$$

$$= \frac{w}{24EI}\left[x^4 + l^2 x^2 2 - lx^3 \times 2\right] = \frac{wx^2}{24EI}\left[x^2 + l^2 - 2lx\right]$$

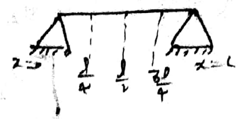
$$= \frac{w}{24EI}\left[\cancel{x^2(x^2 - 2x) + 2l^2}\right] = \frac{wx^2}{24EI}\left[(l-x)^2\right]$$

$$= \frac{wx^2(l-x)^2}{24EI}$$

* calculate the maximum deflection of a beam of length 'l' feet carrying a uniformly distributed load, $w(x) = w$ lb/feet on its central half length.

given that length of beam = l feet.
weight $w(x) = w$, lb/feet

$$y''''(x) = \frac{w(x)}{E \cdot I}$$



$$y''''(x) = \frac{w}{E \cdot I}$$

here $w(x) = w\left[u\left(x-\frac{l}{4}\right) - u\left(x-\frac{3l}{4}\right)\right]$ $\left\{\begin{matrix}1\\0\end{matrix}\right.$ $\left\{\begin{matrix}1\\0\end{matrix}\right.$

unitstep function.

$$L[u(t-a)] = \frac{e^{-sa}}{s}$$

Taking L.T on both sides,

$$s^4 L[y(x)] - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{w}{EI}\left[L\left[u\left(x-\frac{l}{4}\right)\right] - L\left(u\left(x-\frac{3l}{4}\right)\right)\right]$$

$$s^4 L[y(x)] = \frac{w}{EI}\left[\frac{e^{-sl/4}}{s} - \frac{e^{-\frac{3l}{4}s}}{s}\right] + c_1 s + c_2$$

$$L[y(x)] = \frac{w}{EI}\left\{\frac{e^{-\frac{l}{4}s}}{s^5} - \frac{e^{-\frac{3l}{4}s}}{s^5}\right\} + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

$$y(x) = \frac{w}{EI}\left\{\left[\frac{x^4}{4!}\right]_{x \to x-\frac{l}{4}} - \left(\frac{x^4}{4!}\right)_{x \to x-\frac{3l}{4}}\right\} + c_1 \cdot \frac{x^2}{2!} + c_2 \cdot \frac{x^3}{3!}$$

$$y(x) = \frac{w}{EI}\left\{\frac{(x-l/4)^4}{24} - \frac{(x-\frac{3l}{4})^4}{24}\right\} + c_1\frac{x^2}{2} + c_2\frac{x^3}{6} \longrightarrow (*)$$

Boundary conditions,

$$y(l) = 0, = \frac{w}{EI}\left\{\frac{(3l/4)^4}{24} - \frac{(l/4)^4}{24}\right\} + c_1\frac{l^2}{2} + c_2\frac{l^3}{6}$$

$$y'(x) = \frac{w}{EI}\left\{\frac{4\left(x-\frac{l}{4}\right)^3}{24} - \frac{4\left(x-\frac{3l}{4}\right)^3}{24}\right\} + c_1 \times \frac{2x}{2} + c_2 \times \frac{3x^2}{6}$$

$$= \frac{w}{EI}\left\{\frac{(x-l/4)^3}{6} - \frac{(x-3l/4)^3}{6}\right\} + c_1 x + c_2 \times \frac{x^2}{2}.$$

$$y'(l) = \frac{w}{EI}\left\{\frac{(3l/4)^3}{6} - \frac{(l/4)^3}{6}\right\} + c_1 l + c_2 \frac{l^2}{2}.$$

$$① \Rightarrow -\frac{w}{EI}\left[\frac{(3l/4)^4}{24} - \frac{(l/4)^4}{24}\right] = \cancel{c_1 \frac{l^2}{2}} + c_2 \frac{l^3}{6}.$$

$$② \times \frac{l}{2} \Rightarrow -\frac{w}{EI}\frac{l}{2}\left\{\frac{(3l/4)^3}{6} - \frac{(l/4)^3}{6}\right\} = \cancel{c_1 \frac{l^2}{2}} + \frac{c_2 l^3}{4}$$

$$c_2\left(\frac{l^3}{6} - \frac{l^3}{4}\right) =$$

$$c_1 = \frac{-wl}{4}, \quad c_2 = \frac{11wl^2}{192} \rightarrow \text{sub on } (*)$$

$$y\left(\frac{l}{2}\right) = \frac{13wl^4}{6144EI}.$$