69. $\dfrac{1}{s^2(s+1)^3}$ 70. $\dfrac{1}{s^4+4}$

3.11 SOLUTIONS OF DIFFERENTIAL AND INTEGRAL EQUATIONS

As mentioned in the beginning, Laplace transform technique can be used to solve differential (both ordinary and partial) and integral equations. We shall apply this method to solve only ordinary linear differential equations with constant coefficients and a few integral and intergo-differential equations. The advantage of this method is that it gives the particular solution directly. This means that there is no need to first find the general solution and then evaluate the arbitrary constants as in the classical approach.

3.11.1 Procedure

1. We take the Laplace transforms of both sides of the given differential equation in $y(t)$, simultaneously using the given initial conditions. This gives an algebraic equation in $\bar{y}(s) = L\{y(t)\}$.

Note $L\{y^{(n)}(t)\} = s^n\,\bar{y}(s) - s^{n-1}y(0) - s^{n-2}y'(0) \ldots\ldots y^{(n-1)}(0)$.

2. We solve the algebraic equation to get $\bar{y}(s)$ as a function of s.
3. Finally we take $L^{-1}\{\bar{y}(s)\}$ to get $y(t)$. The various methods we have discussed in the previous sections will enable us to find $L^{-1}\{\bar{y}(s)\}$.

The procedure is illustrated in the worked examples given below:

WORKED EXAMPLE 3(d)

Example 3.1 Using Laplace transform, solve the following equation $L\dfrac{di}{dt} + Ri = E\,e^{-at}$; $i(0) = 0$, where L, R, E and a are constants.

Taking Laplace transforms of both sides of the given equation, we get,

$$L \cdot L\{i'(t)\} + RL\{i(t)\} = EL\{e^{-at}\}$$

i.e., $$L\{s\,\bar{i}(s) - i(0)\} + R\bar{i}(s) = \frac{E}{s+a}, \text{ where } \bar{i}(s) = L\{i(t)\}$$

i.e., $$(Ls + R)\,\bar{i}(s) = \frac{E}{s+a}$$

$\therefore$ $$\bar{i}(s) = \frac{E}{(s+a)(Ls+R)}$$

$$= E\left\{\frac{\left(\frac{1}{R-aL}\right)}{s+a}+\frac{\left(\frac{1}{aL-R}\right)}{s+R/L}\right\}$$

Taking inverse Laplace transforms

$$i(t)=\frac{E}{R-aL}\left[L^{-1}\left\{\frac{1}{s+a}\right\}-L^{-1}\left\{\frac{1}{s+R/L}\right\}\right]$$

$$=\frac{E}{R-aL}(e^{-at}-e^{-Rt/L})$$

Example 3.2 Solve $y''-4y'+8y=e^{2t}$, $y(0)=2$ and $y'(0)=-2$.
Taking Laplace transforms of both sides of the given equation, we get

$$[s^2\bar{y}(s)-sy(0)-y'(0)]-4[s\bar{y}(s)-y(0)]+8\bar{y}(s)=\frac{1}{s-2}$$

i.e., $$(s^2-4s+8)\bar{y}(s)=\frac{1}{s-2}+(2s-10)$$

$\therefore$ $$\bar{y}(s)=\frac{1}{(s-2)(s^2-4s+8)}+\frac{2s-10}{s^2-4s+8}$$

$$=\frac{A}{s-2}+\frac{Bs+C}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$

$$=\frac{\frac{1}{4}}{s-2}+\frac{-\frac{1}{4}s+\frac{1}{2}}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}s-\frac{19}{2}}{s^2-4s+8}$$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$$

$$y=\frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right)+e^{2t}L^{-1}\left\{\frac{\frac{7}{4}s-6}{s^2+4}\right\}$$

$$=\frac{1}{4}e^{2t}+e^{2t}\left(\frac{7}{4}\cos 2t-3\sin 2t\right)$$

$$=\frac{1}{4}e^{2t}(1+7\cos 2t-12\sin 2t)$$

and $$\bar{y}(s)=\frac{1}{s^2+1}-\frac{2}{\left(s^2+1\right)^2}$$

$$\therefore \quad x=\cos t+2\int_0^t \sin t\,dt+\int_0^t\left\{L^{-1}\left(\frac{1}{s^2+1}\right)-2L^{-1}\frac{1}{\left(s^2+1\right)^2}\right\}dt$$

$$=\cos t+2(1-\cos t)+\int_0^t(\sin t-\sin t+t\cos t)\,dt$$

$$=1+t\sin t$$

and $$y=\sin t-2\times\frac{1}{2}(\sin t-t\cos t)$$

$$=t\cos t.$$

Example 3.16 Show that the solution of the equation

$L\dfrac{di}{dt}+Ri+\dfrac{1}{C}\displaystyle\int_0^t i\,dt=E, i(0)=0$ [where L, R, E are constants] is given by

$$i=\begin{cases}\dfrac{E}{\omega L}e^{-at}\sin\omega t, \text{if } \omega^2>0\\ \dfrac{E}{L}te^{-at}, \text{if } \omega=0\\ \dfrac{E}{kL}e^{-at}\sinh kt, \text{if } \omega^2<0\end{cases}$$

where $$a=\frac{R}{2L},\ \omega^2=\frac{1}{LC}-\frac{R^2}{4L^2} \text{ and } k^2=-\omega^2.$$

Note ☑ The given equation is an integro-differential equation, as the unknown (dependent variable) i occurs within the integral and differential operations.]

Taking Laplace transforms of the given equation, we get

$$Ls\bar{i}(s)+R\bar{i}(s)+\frac{1}{Cs}\bar{i}(s)=\frac{E}{s}$$

i.e. $$\left(LCs^2+RCs+1\right)\bar{i}(s)=EC$$

$$\therefore \quad \bar{i}(s)=\frac{EC}{LCs^2+RCs+1}$$

$$=\frac{E}{L}\cdot\frac{1}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

$$= \frac{E}{L} \cdot \frac{1}{\left(s+\frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$= \frac{E}{L} \cdot \frac{1}{(s+a)^2 + \omega^2}, \text{if } \omega^2 > 0$$

$\therefore$
$$i(t) = \frac{E}{L\omega} \cdot e^{-at} \sin \omega t$$

If
$$\omega = 0,$$

$$\bar{i}(s) = \frac{E}{L} \cdot \frac{1}{(s+a)^2}$$

$\therefore$
$$i(t) = \frac{E}{L} t e^{-at}$$

If $\omega^2 < 0$ and $\omega^2 = -k^2$,

$$\bar{i}(s) = \frac{E}{L} \cdot \frac{1}{(s+a)^2 - k^2}$$

$\therefore$
$$i(t) = \frac{E}{Lk} e^{-at} \sinh kt.$$

Example 3.17 Solve the simultaneous equations

$3x' + 2y' + 6x = 0$ and

$$y' + y + 3\int_0^t x\,dt = \cos t + 3\sin t, \quad x(0) = 2 \quad \text{and} \quad y(0) = -3.$$

Taking Laplace transforms of both the equations, we get

$$3[s\bar{x}(s) - 2] + 2[s\bar{y}(s) + 3] + 6\bar{x}(s) = 0 \text{ and}$$

$$s\bar{y}(s) + 3 + \bar{y}(s) + \frac{3}{s}\bar{x}(s) = \frac{s}{s^2+1} + \frac{3}{s^2+1}$$

i.e.
$$(3s+6)\bar{x}(s) + 2s\bar{y}(s) = 0 \tag{1}$$

and
$$\frac{3}{s}\bar{x}(s) + (s+1)\bar{y}(s) = \frac{s+3}{s^2+1} - 3 \tag{2}$$

Solving (1) and (2) for $\bar{x}(s)$, we have

$$[3(s+2)(s+1) - 6]\bar{x}(s) = -\frac{2s(s+3)}{s^2+1} + 6s$$

i.e.
$$\bar{x}(s) = -\frac{2}{3} \cdot \frac{1}{s^2+1} + \frac{2}{s+3}$$