26.06.'17

Sets:-

A set is a collection of well-defined objects.

Representation of sets:-

1) Enumeration

The elements of a set are enumerated (listed) within brases.

2) Set builder Notation

Describing a set in a cons manner.

eg: $Q = \{a/b : a, b \in Z, b \neq 0\}$

Here 1) a/b is a typical element of the set, which is a function.

2) The colon is to be read as such that or where and | is interchangeably used.

3) $a, b \in Z$ is an abbrivation for a and b are integers.

4) , in this notation represents 'and'

3) Standard Symbols

Frequently used sets are given symbols that are reserved for them alone,

eg: 1) $\mathbb{P}$ = positive integers $= \{1, 2, 3, \cdots\}$

3) $\mathbb{Z}$ = Integers $= \{\cdots -1, 0, 1, \cdots\}$

2) $\mathbb{N}$ = Natural numbers $= \{0, 1, 2, 3, \cdots\}$

4) $\mathbb{Q}$ = Rational numbers

5) $\mathbb{R}$ = Real Numbers

6) $\mathbb{C}$ = Complex Numbers

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Finite Set :- A set is a finite set if it has a finite no. of elements. Any set that is not finite is an Infinite Set.

Cardinality :- Let A be a finite set. the no. of different elements in A is called its cardinality and is denoted by |A|.

Subsets :- Let A and B be sets. We say that A is a subset of B (in notation $A \subseteq B$) if and only if every element of A is an element of B.

27.06.17 A proposition is a sentence to which one and only one of the terms true or false can be meaningfully applied.

A introgative and Exclamatory sentences are not proposition.

eg: New Delhi is the Capital of India

Proposition Over Universe :-

Let U be an non-empty set. A proposition over U is a sentence that contains a variable that can take any value on U and that has a definite truth value as a result of any such substitution.

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Truth set : If P is a proposition over U the truth set of P is $T_p = \{a \in U \mid p(a) \text{ is true}\}$.

eg: In the universe of $\mathbb{Z}$ the truth set of $x^2+x=0$ is $\{0\}$, when the universe is extended from $\mathbb{Z}$ to $\mathbb{C}$, the truth set becomes $\{0, \pm i\}$ $z = \{x : x^3+x=0, x \in \mathbb{C}\}$

Note : The term solution set is often used for the truth set of an equation.

Mathematical Induction :-

A technique for proving proposition over the $\mathbb{P}$ is Mathematic Induction & also known as Finite Induction.

Principle of MI :-

Let $P(n)$ be a proposition over $\mathbb{P}$, then $P(n)$ is a tautology (universal truth) if

1) $P(1)$ is true and
2) for all $n \geq 1$, $P(n) \Rightarrow P(n+1)$

Basis step => $P(1)$ is true

Assumption step => $P(n)$ is true

Induction step => $P(n+1)$ is true.

$P(n)$: $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

$P(1)$: $1 = \frac{1(1+1)}{2} = 1$

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Example 1:

for all $n \geq 1$, prove by Mathematical Induction that n^3+2n is a multiple of 3.

Solution :

Let $P(n)$: n^3+2n is a multiple of 3.

Basis step : $P(1)$: $n^3+2n \Rightarrow 1^3+2 = 3$ is a multiple of 3

Assumption step : $P(n)$ is true, ie.,

n^3+2n is a multiple of 3.

Induction step : To prove $P(n+1)$ is true,

ie., to prove $(n+1)^3+2(n+1)$ is a multiple of 3.

$$(n+1)^3+2(n+1) = n^3+3n^2+3n+1+2n+2$$
$$= n^3+2n+3n^2+3n+2+1$$
$$= (n^3+2n)+3(n^2+n+1)$$

By assumption step n^3+2n is a multiple of 3 and $3(n^2+n+1)$ is also a multiple of 3. Thus $(n+1)^3+2(n+1)$ is also a multiple of 3. Thus $P(n+1)$ is true. Thus by Principle of MI n^3+2n is a multiple of 3, for all $n \geq 1$.

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2) Prove by the principle of MI that $a^n - b^n$ is divisible by $(a-b)$ for all +ve Z-n.

Let P(1) is true

$a^1 - b^1 = a - b$

$\therefore$ P(1) is divisible by $(a-b)$

Assumption step :-

$a^n - b^n = k(a-b)$

$\Rightarrow a^n = k(a-b) + b^n$

Induction Step :-

$a^{n+1} - b^{n+1} = a.a^n - b.b^n$

$= a(b^n + k(a-b)) - b.b^n$

$= b^n(a-b) + ak(a-b)$

$= (a-b)(b^n + ak)$

$\therefore$ P(n+1) is true.

$\therefore$ $(a^n - b^n)$ is divisible by $(a-b)$

for all +ve integer

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Principle of MI Generalized Version:-

If $p(n)$ is a proposition over $\{k_0, k_0+1, k_0+2, \ldots\}$ where k_0 is any integer then $p(n)$ is a tautology if

i) $p(k_0)$ is true and

ii) for all $n \geq k_0$, $p(n) \rightarrow p(n+1)$

1) State and prove Generalized demorgan's law using generalized principle of MI.

Statement: Generalized demorgan's law is given by

$$(A_1 \cup A_2 \cup A_3 \ldots \cup A_n)' = A_1' \cap A_2' \cap A_3' \ldots \cap A_n'$$

and

$$(A_1 \cap A_2 \cap A_3 \ldots \cap A_n)' = A_1' \cup A_2' \cup A_3' \ldots \cup A_n'$$

Proof:-

Basis Step: Here $k_0 = 2$,

ie., $p(2)$: $(A_1 \cup A_2)' = A_1' \cap A_2'$

As

$x \in (A_1 \cup A_2)'$

$x \notin (A_1 \cup A_2)$

$x \notin A_1$ & $x \notin A_2$

$x \in A_1'$ & $x \in A_2'$

$x \in (A_1' \cap A_2')$

$A_1 = (1, 2, 3)$

$A_2 = (2, 4)$

$A_1 \cap A_2 = 2$

$A_1 \cup A_2 = 1, 2, 3, 4$

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Assumption Step :-

Let us assume $p(n)$ is true. i.e.,

$(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n)' = A_1' \cap A_2' \cdots \cap A_n'$

Induction Step :-

$(A_1 \cup A_2 \cup \dots \cup A_n \cup A_{n+1})' = ((A_1 \cup A_2 \cup \dots \cup A_n) \cup A_{n+1})'$

$= (A_1 \cup A_2 \cup \dots \cup A_n)' \cap A_{n+1}'$

$= A_1' \cap A_2' \cap \dots \cap A_n' \cap A_{n+1}'$ (by assumption)

$\therefore$ $p(n+1)$ is true.

Thus by generalized principle of MI, $p(n)$ is true for all n.

2) $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

Basis step : $k_0 = 0$

$p(0) = 1 = 2^{0+1} - 1 = 1$

$\therefore$ $p(0)$ is true.

Assumption step :-

$p(n)$ is true

$1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

Induction step :-

$p(n+1) : 1 + 2 + 2^2 + \dots + 2^n + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1}$

$$= 2 \cdot 2^{n+1} - 1$$

$$= 2^{n+2} - 1$$

$\therefore$ $P(n+1)$ is true.

30.06.17

Relations

Let A and B be sets, a relation from A into B is any subset of $A \times B$.

$A \times B = \{(x, y) : x \in A, y \in B\}$ $|A| = n$, $|B| = m$ $\therefore |A \times B| = n \times m$

↳ Order Pair

eg:- $A = \{1, 2, 3\}$; $B = \{a, b\}$ => $A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$|A \times B| = 6$

Relation On a Set:

A relation from a set A into A is called a relation on A.

eg:- Let $A = \{1, 2, 3, 6, 12\}$ define a relation r on A as $r = \{(1,1), (1,2), (1,3), (1,6), (1,12), (2,2), (2,6), (2,12), (3,3), (3,6), (3,12), (6,6), (6,12), (12,12)\}$

D_{12} => divisors of 12.

r can be expressed as is divisor of or is a multiple of.

Divides :- (a/b)

Let $a, b \in \mathbb{Z}$ if a/b if and only if there exist an integer k such that $ak = b$.

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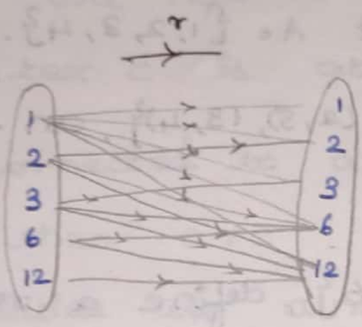
then a is divisor of b or b is a multiple of a.

Note:-

In general a relation is denoted by r, R, $\leq$

Graphical Representation of Relation :-

A picture of any relation r can be drawn as follows, where the arrows indicates the relation



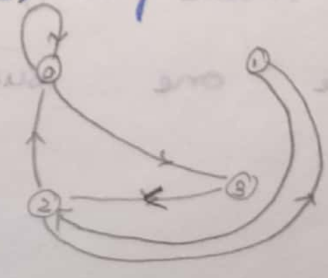
Notation :-

x related to y can be written as xRy, or $x\,ry$ or $x \leq y$. ($\leq$ just a symbol of relation)

Graph of Relation:-

consider $A = \{0, 1, 2, 3\}$ and Let $r = \{(0,0), (0,3), (1,2), (2,1), (3,2), (2,0)\}$. The elements of A are called the vertices of the graph. They are represented by points or small circles. connect any two vertices with an arrow. ie, if A and B are vertices then if A related to B, they are connected by an arrow.

The graph of relation r is given by $\Rightarrow$



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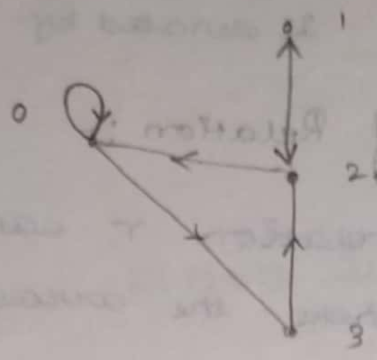
Which can be simplifed as follows.



7.7.'17

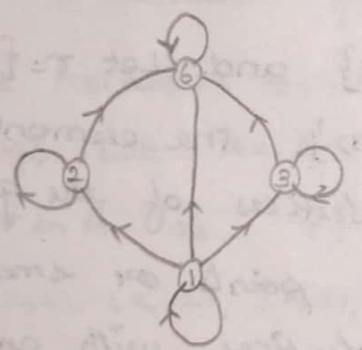
consider the set $A = \{1, 2, 3, 4\}$. Define a relation $S = \{(1,2), (2,3), (3,4)\}$, an S is known as Succesor Relation.

Let $D = \{1, 2, 3, 6\}$, define a relation

$r = \{(1,1), (1,2), (1,3), (1,6), (3,6)$
$(2,2), (2,6), (3,3), (6,6)\}$ $\Rightarrow D_6$

$a \, r \, b$ if and only if a divides b.

The graph of r is given by,



Hasse Diagram:

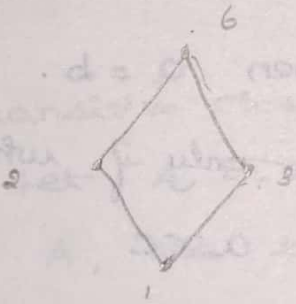
A Hasse diagram (or) Order diagram is the one such that

* each vertex of A must be related to itself. In this case the arrows from a vertex to itself is not necessary.

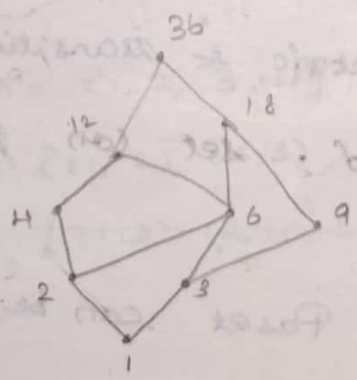
* If vertex b appers above vertex a and if vertex a is connected to vertex b by an edge then a r b so direction arrows are not necessary.

* If vertex c is above a vertex a & if c is connected to a by a sequence of edges, then a r c

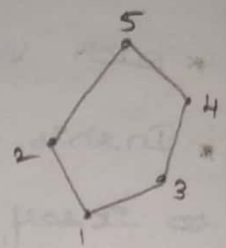
* The vertices are denoted by points rather than by circles. Therefore the Hasse diagram of r is



D36 ⇒ D = {1, 2, 3, 4, 6, 9, 12, 18, 36}



consider the Hasse diagram



the relation represented by this Hasse diagram is

$\{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,5), (3,3), (3,4), (3,5), (4,4), (4,5), (5,5)\}$

Reflexive:- Let A be a set & Let r be a relation on A, then r is reflexive if and only if a r a for all $a \in A$

Anti Symmetric :-

r is antisymmetric if & only if a r b & $a \neq b$ then b r a is false

Equivalently: (for antisymmetric)

a r b & b r a then $a = b$.

Transitive :

r is transitive if & only if whenever a r b & b r c then a r c

Poset :-

A relation on a set 'A' that is reflexive, antisymmetric & transitive is called a partially ordered set (or) po-set

Note:

Any (or) Every Poset can be represented by a Hasse diagram

Symmetric :-

Let r be a relation on a set A. r is symmetric if & only if whenever a r b it follows that b r a.

Equivalance Relation :-

A relation r on a set A is called an Equivalance relation if & only if it is reflexive, symmetric & transitive.

ex: A = {1, 2, 3}

then e = {(1,1), (2,2), (3, 3), (1,3), (3,1)}

Ex : 6.3.3

Transitive closure :

Let A be a set and r be a relation on A, the transitive closure of r is denoted by r^+, which is the smallest transitive relation containg r as a subset.

eg:- A = {1, 2, 3, 4}

S = {(1, 2), (2,3), (3,4)}

~~S^+ = {(1,3), (2,4), (1,4)~~

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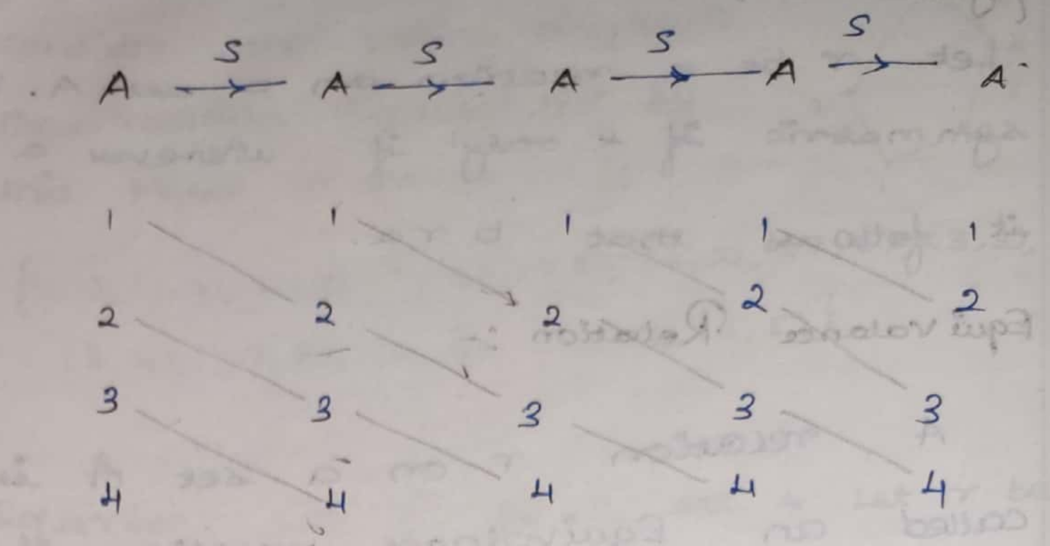

$S^2 = \{(1,3), (2,4)\}$

$S \cup S^2 = \{(1,2), (1,3), (2,3), (2,4), (3,4)\}$

$S^3 = \{(1,4)\}$

$S^+ = S \cup S^2 \cup S^3 = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

Theorem :-

If r is a relation on a set A & cardinality of A $(|A|) = n$ then the transitive closure of r is the union of first n powers of r. i.e.,

$r^+ = r \cup r^2 \cup r^3 \cup \ldots \cup r^n$

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Adjuscency matrix :-

Let $A = \{a_1, a_2, a_3, \ldots, a_m\}$ & $B = \{b_1, b_2, \ldots b_n\}$ be finite sets of cardinality m & n respectively. Let r be a relation from A into B, then r can be represented by the $m \times n$ matrix R defined by

$$R_{ij} = \begin{cases} 1 & \text{if } a_i \, r \, b_j \\ 0 & \text{otherwise} \end{cases}$$

The matrix R is called the adjuscency matrix of r.

Let $A = \{2, 5, 6\}$ and r be the relation defined by $r = \{(2,2), (2,5), (5,6), (6,6)\}$ on A.

$$R = \begin{array}{c c} & \begin{array}{ccc} 2 & 5 & 6 \end{array} \\ \begin{array}{c} 2 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

is the adjuscency matrix of r.

Bolean arithmetics :-

$0 + 0 = 0$	$0 \cdot 0 = 0$
$0 + 1 = 1$	$1 \cdot 0 = 0$
$1 + 0 = 1$	$0 \cdot 1 = 0$
$1 + 1 = 1$	$1 \cdot 1 = 1$

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$$R^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot1+1\cdot0+0\cdot0 & 1\cdot1+1\cdot0+0\cdot0 & 1\cdot0+1\cdot1+0\cdot1 \\ 0\cdot1+0\cdot0+1\cdot0 & 0\cdot1+0\cdot0+0\cdot1 & 0\cdot0+0\cdot1+1\cdot1 \\ 0\cdot1+0\cdot0 & & \end{bmatrix}$$

$$R^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

If $R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$RS = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$SR = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(RS)^T = SR$

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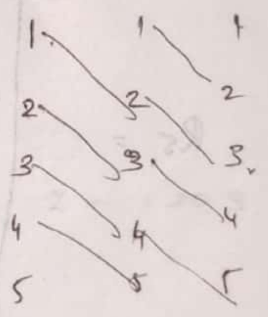
Let $r = \{(1,4),(2,1),(2,2)(2,3)(3,2),(4,3),(4,5),(5,1)\}$

$$R \quad \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



$S = \{(1,2),(2,3),(3,4)\}$

$$S = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

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$$S^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \{(1,3), (2,4)\}$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R^3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow R^4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$R_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

6.5.2

Theorem :-

Let r be a relation on a finite set and let R^+ be the matrix of r^+, the transitive closure of r then

$R^+ = R + R^2 + \dots + R^n$ using Boolean arithmetic

Warshall's Algorithm :- pg (118)

1. $T = R$
2. For $k = 1$ to n Do
3. For $i = 1$ to n Do
4. For $j = 1$ to n Do
5. $T[i,j] = T[i,j] + T[i,k] \cdot T[k,j]$
6. Terminate with $T = R^+$ $\quad ij = ij \quad k \cdot kj$

$T(i,j) = T(i,j) + (T[i,k) \cdot T$

$$R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$n = 3 \Rightarrow k = 1$ to 3, $j = 1$ to 3, $i = 1$ to 3

$\underline{k=1}$

$\underline{i=1}$

$\underline{j=1}$

$T[1,1] = T[1,1] + T[1,1] \cdot T[1,1]$

$= 1 + 1 \cdot 1 = 1$

$\underline{j=2}$

$T[1,2] = T[1,2] + T[1,1] \cdot T[1,2]$

$= 1 + 1 \cdot 1 = 1$

$\underline{j=3}$

$T[1,3] = T[1,3] + T[1,1] \cdot T[1,3]$

$= 0 + 1 \cdot 0 = 0$

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i = 2

j = 1

$T[2,1] = T[2,1] + T[2,1] \cdot T[1,1]$

$= 0 + 0 \cdot 1 = 0$

j = 2

$T[2,2] = T[2,2] + T[2,1] \cdot T[1,2]$

$= 0 + 0 \cdot 1 = 0$

j = 3

$T[2,3] = T[2,3] + T[2,1] \cdot T[1,3]$

$= 1 + 0 \cdot 0 = 1$

i = 3

j = 1

$T[3,1] = T[3,1] + T[3,1] \cdot T[1,1]$

$= 0 + 0 \cdot 1 = 0$

j = 2

$T[3,2] = T[3,2] + T[3,1] \cdot T[1,2]$

$= 0 + 0 \cdot 1 = 0$

j = 3

$T[3,3] = T[3,3] + T[3,1] \cdot T[1,3]$

$= 1 + 0 \cdot 0 = 1$

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$\underline{K=2}$

$\underline{i=1}$

$\underline{j=1}$

$T[1,1] = T[1,1] + T[1,2] \cdot T[2,1]$

$= 1 + 1 \cdot 0 = 1$

$\underline{j=2}$

$T[1,2] = T[1,2] + T[1,2] \cdot T[2,2]$

$= 1 + 1 \cdot 0 = 1$

$\underline{j=3}$

$T[1,3] = T[1,3] + T[1,2] \cdot T[2,3]$

$= 0 + 1 \cdot 1 = 1$

$\underline{i=2}$

$\underline{j=1}$

$T[2,1] = T[2,1] + T[2,2] \cdot T[2,1]$

$= 0 + 0 \cdot 0 = 0$

$\underline{j=2}$

$T[2,2] = T[2,2] + T[2,2] \cdot T[2,2]$

$= 0 + 0 \cdot 0 = 0$

$\underline{j=3}$

$T[2,3] = T[2,3] + T[2,2] \cdot T[2,3]$

$= 1 + 0 \cdot 1 = 1$

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i=3

j=1

$T[3,1] = T[3,1] + T[3,2] \cdot T[2,1]$

$= 0 + 0 \cdot 0 = 0$

j=2

$T[3,2] = T[3,2] + T[3,2] \cdot T[2,2]$

$= 0 + 0 \cdot 0 = 0$

j=3

$T[3,3] = T[3,3] + T[3,2] \cdot T[2,3]$

$= 1 + 0 \cdot 1 = 1$

K=3

i=1

j=1

$T[1,1] = T[1,1] + T[1,3] \cdot T[3,1]$

$= 1 + 0 \cdot 0 = 1$

j=2

$T[1,2] = T[1,2] + T[1,3] \cdot T[3,2]$

$= 1 + 0 \cdot 0 = 1$

j=3

$T[1,3] = T[1,3] + T[1,3] \cdot T[3,2]$

$= 0 + 0 \cdot 0 = 0$

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$i = 2$

$j = 1$

$T[2,1] = T[2,1] + T[2,3] \cdot T[3,1]$

$= 0 + 1 \cdot 0 = 0$

$j = 2$

$T[2,2] = T[2,2] + T[2,3] \cdot T[3,2]$

$= 0 + 1 \cdot 0 = 0$

$j = 3$

$T[2,3] = T[2,3] + T[2,3] \cdot T[3,3]$

$= 1 + 1 \cdot 1 = 1$

$i = 3$

$j = 1$

$T[3,1] = T[3,1] + T[3,3] \cdot T[3,1]$

$= 0 + 1 \cdot 0 = 0$

$j = 2$

$T[3,2] = T[3,2] + T[3,3] \cdot T[3,2]$

$= 0 + 1 \cdot 0 = 0$

$j = 3$

$T[3,3] = T[3,3] + T[3,3] \cdot T[3,3]$

$= 1 + 1 \cdot 1 = 1$

Matrix Refer book

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Warshall's Algorithm:-

Let r be a relation on the set $\{1, 2, \ldots n\}$ with relation matrix R. The matrix of the transitive closure R^+, can be computed by the equation $R^+ = R + R^2 + \ldots + R^n$. By using ordinary polynomial evaluation methods, you can compute R^+ with n-1 matrix multiplications:

$$R^+ \; R\,(I + R\,(I + (\cdots R\,(I + R)\cdots)))$$

For example, if $n = 3$, $R = R\,(I + R\,(I + R))$

We can make use of the fact that if T is a relation matrix, $T + T = T$ due to the fact that $1 + 1 = 1$ in Boolean arithmetic. Let

$S_k = R + R^2 + \ldots + R^k$, then

$R = S_1$

$$S_1\,(I + S_1) = R\,(I + R) = R + R^2 = S_2$$

$$S_2\,(I + S_2) = (R + R^2)\,(I + R + R^2)$$

$$= (R + R^2) + (R^2 + R^3) + (R^3 + R^4)$$

$$= R + R^2 + R^3 + R^4 = S_4$$

Similarly

$$S_4\,(I + S_4) = S_8$$

etc.,

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Notice how each matrix multiplication doubles the number of terms that have been added to the sum that you currently have computed. In algorithmic form we compute R^2 as follows.

Algorithm : Transitive closure Algorithm :

Let R be a known relation matrix and let R^+ be its transitive closure matrix, which is to be computed.

1. $T := R$
2. Repeat

    2.1 $S := T$

    2.2 $T := S(I + S)$ // using Boolean arithmetic

    Until $T = S$

3. Terminate with $T = R^+$.

Notes :

a) often the higher power terms in S_n do not contribute anything to R^+. When the condition $T = S$ becomes true in step. 2, this is an indication that no higher- powered terms are needed.

b) To compute R^+ using this algorithm, you need to perform no more than $\lceil \log_2 n \rceil$ matrix multiplications, where $\lceil x \rceil$ is the least integer

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that is greater than or equal to x. For example, if r is a relation on 25 elements, no more than $\lceil \log_2 25 \rceil = 5$ matrix multiplications are needed.

A second algorithm, Warshall's Algorithm, reduces computation time to the time that it takes to perform one matrix multiplication.

Algorithm 6.5.2; Warshall's Algorithm:

Let R be a known relation matrix and let R^+ be its transitive closure matrix, which is to be computed.

1.0 $T := R$

2.0 For $k := 1$ to n Do

For $i := 1$ to n Do

For $j := 1$ to n Do

$T[i,j] = T[i,j] + T[i,k] \cdot T[k,j]$

3.0 Terminate with $T = R^+$

Definition Function :- A function from a set A into a set B is a relation from A into B such that each element of A is related to exactly one element of the set B. The set A is called the domain of the function and the set B is called the codomain.

Image of an Element :- Let $f: A \to B$, If $a \in A$, then $f(a)$ is used to denote that element of B to which a is related. $f(a)$ is called the image of a, or, more precisely, the image of a under f. We write $f(a) = b$ to indicate that the image of a is b.

Range of a Function : The range of a function is the set of images of its domain. If $f: X \to Y$, then the range of f is denoted $f(X)$ and

$$f(X) = \{ f(a) \mid a \in X \}$$
$$= \{ b \in Y \mid \exists a \in X \text{ such that } f(a) = b \}$$

Note that the range of a function is a subset of its codomain. $f(X)$ is also read as "the image of the set X under the function f" or simply "the image of f"

Definition Injective function:- A function $f: A \to B$ is injective if

$a, b \in A$, $a \neq b \Rightarrow f(a) \neq f(b)$

Notice that the condition for a injective function is equivalent to

$a, b \in A$, $f(a) = f(b) \Rightarrow a = b$

Injective functions are also called injections, or one-to-one functions.

Definition Surjective Function:- A function $f: A \to B$ is surjective if its range, $f(A)$, is equal to its codomain B.

Notice that the condition for a surjective function is equivalent to

For all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Surjective functions are also called surjections, or onto functions.

Definition Bijective Function:- A function $f: A \to B$ is bijective if it is both injective and surjective.

Bijective functions are also called one-to-one, onto functions.

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Ex. 7.1.2

Let R be the real nos. Then $L = \{(x, 3x) \mid x \in R\}$ is a fn from R into R, or more simply L is a fun. of R.

We will use a different system of notation for fns. than the one we used for relations. If f is a fn. from set A into B, we will write: $f : A \to B$.

for eg: $y = 1/x$ (or) $f(x) = 1/x$ both define the fn $\{(x, 1/x) \mid x \in R, x \neq 0\}$ here the domain was assumed to be those elements of R whose substitution for x makes sense, the non-zero real nos and the co-domain was assumed to be R.

11.07.17

Equal cardinality:

2 sets are said to have the same cardinality if there exist a bijection b/w them.

Countable: If a set is finite (or) has the same cardinality as the set of +ve integers, then it is called a countable set.

Eg: The set $2\mathbb{P}$ of even integers has the same cardinality as the set $\mathbb{P}$ of +ve integers.

Define a fn. $f : \mathbb{P} \to 2\mathbb{P}$ by $f(a) = 2a$ for all $a \in \mathbb{P}$.

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To prove f is injective,

Let $a, b \in P$, assume $f(a) = f(b)$

$$f(a) = f(b)$$

$$\Rightarrow 2a = 2b$$

$$\Rightarrow a = b$$

$\therefore f$ is injective.

To prove f is onto,

Let $b \in 2P$, if $b \in 2P$ then $b = 2k$ for some $k \in P$

$$\Rightarrow f(k) = 2k = b$$

Thus each element of $2P$ is the image of some element of P. i.e., there is a bijection from P to $2P$.

$\Rightarrow P$ & $2P$ has same Cardinality.

Pigeon ~~W~~hole Principle :-

Let f be a fn. from a finite set X onto a finite set Y. If $n \geq 1$ and (cardinality of x) $|X| \geq n\,|Y|$ then there exist an element of Y, that is the

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image of atleast n+1 elements of X.

Theorem:
7.3.2

Statement: If $f: A \to B$ & $g: B \to C$ are injections then $g \circ f: A \to C$ is an injection.

compose

Proof:- assume that $(g \circ f)(x_1) = (g \circ f)x_2$, where $x_1, x_2 \in A$

Composition: One of the most important operations on functions is that of composition.

Composition of functions: Let $f: A \to B$ and $g: B \to C$. Then the composition of f followed by g, written $g \circ f$ is a function from A into C defined by $(g \circ f)(x) = g(f(x))$, which is read "g of f of x."

~~The~~ It is to be noted that it is traditional to write the composition of functions from right to left. Thus, in the above definition, the first function performed in computing $g \circ f$, which is f. On the other hand, for relations, the composition r s is read from left to right, so that the first relation is r.

Theorem : 7.3.1

Function composition is associative

That is, if $f: A \to B$, $g: B \to C$ and $h: C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof Technique : In order to prove that two functions are equal, we must use the definition of equality of functions: Assuming that the functions have the same domain, they are equal if, for each domain element, the images of that element under the two functions are equal.

Proof : We wish to prove that

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x) \text{ for all } x \in A,$$

Which is the domain of both functions.

$(h \circ (g \circ f))(x) = h((g \circ f)(x))$ by the definition of composition of h with $g \circ f$

$= h(g(f(x)))$ by the definition of the composition of g with f.

Similarly

$((h \circ g) \circ f)(x) = (h \circ g)(f(x))$ by the definition of composition of $h \circ g$ with f.

$= h(g(f(x)))$ by the definition of composition of h with g.

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Notice that no matter how the functions the expression $h \circ g \circ f$ is grouped, the final image of any element of $x \in A$ is $h(g(f(x)))$ and so $h \circ (g \circ f) = (h \circ g) \circ f$.

Definition : Powers of Functions.

Let $f : A \to A$

1) $f^1 = f$; that is, $f^1(a) = f(a)$, for $a \in A$.

2) For $n \geq 1$, $f^{n+1} = f \circ f^n$; that is, $f^{n+1}(a) = f(f^n(a))$ and $a \in A$.

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If $A \subseteq B$ & $B \subseteq A$ then $A = B$.

Theorem 7.3.2 :

Proof :- Let $x_1, x_2 \in A$, assume $(g \circ f) x_1 = (g \circ f) x_2$

By definition of composition $(f \circ g)(x) = f[g(x)]$

$(g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow f(x_1) = f(x_2)$ [$\because$ g is injection]

$\Rightarrow x_1 = x_2$ [$\because$ f is injection].

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Theorem 7.3.3.

If $f: A \to B$ & $g: B \to C$ are surjection, then $(g \circ f): A \to C$ is a surjection.

Proof: Let $c \in C$, given g is surjective, $\therefore$ there exist $b \in B$ such that $g(b) = c$ also f is onto $\Rightarrow$ there exist $a \in A$ such that $f(a) = b$.

Now $(g \circ f)(a) = g[f(a)]$
$= g[b]$
$= c$

Thus for an element $c \in C$ we have an element $a \in A$, such that $(g \circ f)(a) = c$.

$\therefore (g \circ f)$ is surjective.

Identity function :- For any set A, the identity fn. on A is a fn. from A onto A such that $i(a) = a$. i.e., Identity fn. is denoted by i. If $f: A \to A$ is a fn. then $f \circ i = i \circ f = f$.

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Inverse function :- Let $f : A \to A$, if there exist a fn. $g : A \to A$ such that $g \circ f = f \circ g = i$ then g is called the inverse of f and it is denoted by f^{-1}.

Example : Let $g : R \to R$ be defined by $g(x) = x^3$, then g^{-1} can be defined as follows.

$g^{-1} : R \to R \Rightarrow g^{-1}(x) = x^{1/3} = \sqrt[3]{x}$

Theorem : 7.3.4

Let $f : A \to A$. f^{-1} exist if & only if f is a bijection

Case 1: assume that f^{-1} exist.

To prove f is Binjection.

assume that $f(a) = f(b) = c$.

We know that $f(s) = t$ is equivalent to

$(s, t) \in f \therefore f(a) = c$ is

$(a, c) \in f$ & $f(b) = c$ is $(b, c) \in f$.

$\Rightarrow (c, a) \in f'$ & $(c, b) \in f'$

by hypothesis f' is a fn. & c cannot have two images. Thus f is injective. (since $a = b$)

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f is surjective.

f^{-1} is a fn. $\Rightarrow$ It must use all its domain. ie., A. Let b be any element of A. Then b has an image under f^{-1}. ie, $f^{-1}B$.

$\Rightarrow (b, f^{-1}(b)) \in f^{-1}$.

By definition of the inverse this is equivalent to $(f^{-1}(b), b) \in f^{-1}$.

$\Rightarrow$ b is in the range of f.

Since b is arbitary, the range of f must be all of A.

Logic :-

conjunction: (And). If P and q are propositions, their conjunction, p and q (denoted as $p \wedge q$), is defined by the truth table.

P	q	$P \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

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Disjunction : (Or). If p and q are propositions, their disjunction is p or q, denoted $p \vee q$, and is defined by the truth table.

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Negation ; (Not). If p is a proposition, its negation, not p, is denoted $\sim p$ and is defined by the following Truth table.

p	$\neg p$
0	1
1	0

Conditional operator :- The conditional statement if p then q, denoted $p \rightarrow q$, is defined by the following truth table.

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

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Bi conditional Operator :- (if and only if)

If p and q are propositions, the biconditional statement "p if and only if q" denoted $p \leftrightarrow q$, is defined by the following truth table.

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

14.07.17

Tautology:-

An expression involving logical variables that is true in all cases is called a tautology

eg:- $(p \wedge q) \rightarrow q$, $p \vee \neg p$

Contradiction:-

An expression involving logical variables that is false for all cases is called a contradiction.

eg: $p \wedge \neg p$, $(p \wedge q) \wedge (\neg p \vee \neg q)$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \vee \neg q$	$(p \wedge q) \wedge (\neg p \vee \neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

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Equivalance :-

Let S be a set of propositions & let r & s be proposition generated by S. r and s are equivalent if and only if $r \leftrightarrow s$ is a tautology.

It is denoted by $r \Leftrightarrow s$

eg :- $p \vee q \Leftrightarrow \neg p \wedge \neg q$

p	q	¬p	¬q	p∨q	¬p∨¬q	(5) ↔ (6)
0	0	1	1	1	1	1
0	1	1	0	0	0	1
1	0	0	1	0	0	1
1	1	0	0	0	0	

$(p \leftrightarrow q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	p→q	q→p	③∧④	p↔q	⑤↔⑥
0	0	1	1	1	1	1
0	1	1	0	0	0	1
1	0	0	1	0	0	1
1	1	1	1	1	1	1

Implication:-

Let S be a set of prosition & r & s be propositions generated by S. We say that r implies s, $r \Rightarrow s$ if $r \rightarrow s$ is a tautology.

eg:- $p \Rightarrow (p \vee q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

Laws of Logic:-

Commutative Laws:-

* $a \wedge b \Leftrightarrow b \wedge a$
* $a \vee b \Leftrightarrow b \vee a$

Associative Laws:-

* $a \wedge (b \wedge c) \Leftrightarrow (a \wedge b) \wedge c$
* $a \vee (b \vee c) \Leftrightarrow (a \vee b) \vee c$

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Distributive Laws:-

* $a \wedge (b \vee c) \Leftrightarrow (a \wedge b) \vee (b \wedge c)$
* $a \vee (b \wedge c) \Leftrightarrow (a \vee b) \wedge (a \vee c)$

Identity Laws:-

* $p \vee 0 \Leftrightarrow p$
* $p \wedge 1 \Leftrightarrow p$

Negation Laws:-

* $p \wedge \neg p \Leftrightarrow 0$
* $p \vee \neg p \Leftrightarrow 1$

Idempotent Laws:-

* $p \wedge p \Leftrightarrow p$
* $p \vee p \Leftrightarrow p$

Null Laws:-

* $p \wedge 0 \Leftrightarrow 0$
* $p \vee 1 \Leftrightarrow 1$

Absorption Laws:-

* $p \wedge (p \vee q) \Leftrightarrow p$
* $p \vee (p \wedge q) \Leftrightarrow p$

De-Morgan's laws:-

* $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$
* $\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Involution laws:-

* $\neg (\neg p) \Leftrightarrow p$

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Rule T :- We may introduce a formula S in a derivation if S is tautologically implied by any one or more of the preceding formulae in the derivation.

Common Implications & Equivalences :-

Detachement :

$(P \to q) \wedge P \Rightarrow q$

Disjunctive Simplification:

$(P \vee q) \wedge \neg P \Rightarrow q$ &

$(P \vee q) \wedge \neg q \Rightarrow P$

Indirect Reasoning:

$(P \to q) \wedge \neg q \Rightarrow \neg P$

$P \Rightarrow P \vee Q$

Disjunctive addition:-

$P \Rightarrow (P \vee q)$

P, Q

$P \wedge Q$

Conjunctive simplification:-

$P \wedge q \Rightarrow P$ & $P \wedge q \Rightarrow q$

$J \vee S \Rightarrow J$

$S \vee J \Rightarrow S$

Chain rule :-

$(P \to q) \wedge (q \to r) \Rightarrow (P \to r)$

$J \Rightarrow J \vee S$

conditional equivalance :-

$(P \to q) \Leftrightarrow (\neg P \vee q)$

Bicondition equivalance :-

$(P \leftrightarrow q) \Leftrightarrow (P \to q) \wedge (q \to P)$

$\Leftrightarrow (\neg P \vee q) \wedge (\neg q \vee P)$

$\Leftrightarrow (P \wedge q) \vee (\neg P \wedge \neg q)$

contrapositive :-

$(P \to q) \Leftrightarrow (\neg q \to \neg P)$

17.07.'19

Rule P => Introducing Proposition at any point of time in the derivation.

Mathematic theorem proving:-

Propositional Calculus:

Direct proving => Method ①.

1) Derive S from $P \to \neg Q$, $Q \vee R$, $\neg S \to P$ & $\neg R$.

Step	proposition	Justification/ Reasoning
1)	$Q \vee R$	proposition
2)	$\neg R$	proposition
3)	Q	Detachment
4)	$P \to \neg Q$	proposition
5)	$Q \to \neg P$	contrapositive
6)	$\neg P$	Detachment (3, 5)
7)	$\neg S \to P$	proposition
8)	$\neg P \to S$	contrapositive
9)	S	Detachment (6, 8)

$(Q \vee R \Leftrightarrow R \vee Q \Leftrightarrow \neg R \to Q)$

2) Show that $R \vee S$ is a valid conclusion from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ & $(A \wedge \neg B) \to (R \vee S)$

Step	proposition	Reasoning.
(1)	$C \vee D$	proposition
(2)	$(C \vee D) \to \neg H$	proposition
(3)	$\neg H$	Detachment
(4)	$\neg H \to (A \wedge \neg B)$	proposition
(5)	$A \wedge \neg B$	Detachment
(6)	$(A \wedge \neg B) \to R \vee S$	proposition
(7)	$R \vee S$	Detachment.

3) Show by direct proof that U is the consulsion from $S \to T$, $Q \to R$, $P \to Q$, $T \to U$, $R \to S$ & P

Step	proposition	Reasoning
(1)	P	proposition
(2)	$P \to Q$	proposition
(3)	Q	Detachment

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(4)	$Q \to R$	proposition
(5)	R	Detachment
(6)	$R \to S$	proposition
(7)	S	Detachment
(8)	$S \to T$	proposition
(9)	T	Detachment
(10)	$T \to U$	proposition
(11)	U	Detachment.

Conditional proof:-

1) Derive $P \to (Q \to S)$ using conditional proof if necessary from $P \to (Q \to R)$, $Q \to (R \to S)$

Step	proposition	Reasoning
(1)	$P \to (Q \to R)$	Premise
(2)	P	additional premise
(3)	$Q \to R$	detachment
	$\neg Q \vee R$	conditional equivalance.
(4)		
(5)	$Q \to (R \to S)$	Premise
(6)	$\neg Q \vee (R \to S)$	conditional equivalance.
7)	$\neg Q \vee (R \wedge (R \to S))$	

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8) $\neg Q \vee S$
9) $Q \to S$
10) $P \to (Q \to S)$ by conditional proof.

Theorem: A true proposition derived from axioms of mathematical system is called a theorem.

Proof:- A proof of a theorem is a finite sequence of logically valid steps that demonstrate that the permises of a theorem imply the conclusion.

Axioms: assertion about the properties of the universe and rules for creating and justifying more assertions. These rules always include the system of logic that we have developed to this point.

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Conditional proof:

If we can derive S from r and a set of premises then, we can derive $r \to S$ from the set of premises alone.

Indirect Method of Proof:-
(Definition - book)
(Inconsistence)
(Proof of contradiction.

here → the additional premise is the negation of the conclusion.

1) Prove that D follows from $A \to B$, $A \to C$, $\neg(B \wedge C)$ and $D \vee A$ by indirect method of proof.

$\neg(B \wedge C) \Leftrightarrow \neg B \vee \neg C \Leftrightarrow B \to \neg C$ (or) $C \to \neg B$

$D \vee A \Leftrightarrow \neg D \to A$ (or) $\neg A \to D$

Steps	Proposition	Reasoning.
(1)	$A \to B$	premise
(2)	$\neg(B \wedge C)$	premise
(3)	$B \to \neg C$	premise.
(4)	$A \to \neg C$	chain rule
(5)	$A \to C$	premise
(6)	$A \to (\neg C \wedge C)$	Disjunctive addition
(7)	$\neg D$	additional premise
(8)	$D \vee A$	premise
(9)	$\neg D \to A$	
(10)	A	
(11)	$\neg C \wedge C$	
(12)	0	

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2) prove by indirect method that

$\neg q, p \to q, p \vee t \Rightarrow t$

Step	proposition	reasoning
(1)	$p \to q$	premise
(2)	$\neg q$	premise
(3)	$\neg p$	Indirect reasoning
(4)	$p \vee t$	premise
(5)	$\neg p \to t$	conditional equivalance
(6)	t	detachment (3, 5)
(7)	$\neg t$	additional premise.
(8)	$t \wedge \neg t$	disjunctive addition (7, 8)
(9)	0	Negation Law

3) using Indirect method of proof derive $P \to \neg S$ from $P \to Q \vee R$, $Q \to \neg P$, $S \to \neg R$, P

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$P \rightarrow \neg S \Rightarrow \neg P \vee \neg S \Rightarrow \neg(P \wedge S)$

Steps	proposition	reasoning.
(1)	$P \rightarrow Q \vee R$	premise
(2)	P	premise
(3)	$Q \vee R$	detachment.
(4)	$R \vee Q$	commutative
(5)	$\neg R \rightarrow Q$	conditional equivalance.
(6)	$Q \rightarrow \neg P$	premise
(7)	$\neg R \rightarrow \neg P$	chain Rule
(8)	$S \rightarrow \neg R$	Premise
(9)	$S \rightarrow \neg P$	chain Rule
(10)	$\neg(P \wedge S)$	conditional equivalance
(11)	$P \wedge S$	additional premise
(12)	0	Negative Law

Mathematical System :

A mathematical system consists of

1) A set or universe, U

2) Definition : Sentence that explain the meaning of concepts that relate to the universe. Any term used in describing the universe itself is said to be undefined.

3) Axioms - assertions

4) Theorem - The additional assertion.

Rules for formal proof:

1) A proof must end in a finite no. of steps.

2) Each steps must be either a premise or a previous steps using any Valid equivalence or implication.

3) For a direct proof, the last step must be the conclusion of the theorem. for an indirect proof, the last step must be a contradiction.

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4) Justification column: The column labelled. Justification is analogs to the comments that appear in most good computer programs. They simply make the proof more readable.

Conditional Conclusion :

The conclusion of a theorem is often a conditional proposition. The condition of the conclusion can be included as a premise in the proof of the theorem. The object of the proof is then to prove the consequence of the conclusion. This rule is indefined of the logical law:

$$p \rightarrow (h \rightarrow c) \Leftrightarrow (p \wedge h) \rightarrow c$$

Indirect proof / proof by contradiction:-

consider a theorem $p \Rightarrow c$, where p represents $p_1, p_2, \ldots \& p_n$ the premise. The method of indirect proof is based on the equivalance.

$$p \rightarrow c \Leftrightarrow \sim (p \wedge \sim c)$$

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Normal form:

The formula is said to be in disjunctive normal form if it is a sum of elementary products. eg: $(a \wedge b) \vee (\sim a \wedge b)$

Note:

Disjunction of variables & their negation is called an elementary sum. And Conjunction of variables & their negation is called an elementary product.

Elementary sum	Elementary product.
$a \vee b$	$a \wedge b$
$a \vee \sim b$	$a \wedge \sim b$
$\sim a \vee b$	$\sim a \wedge b$
$\sim a \vee \sim b$	$\sim a \wedge \sim b$

The formula is said to be in conjunctive normal form if it is a product of elementary sums. eg: $(a \vee b) \wedge (\sim a \vee b)$

Principle dnf:

Principle disjunctive normal form is a dnf but in every conjunction a variable or its negation but not both will appear exactly ones.

Principle Cnf:-

Pcnf is a normal form but in every disjunction a variable or its negation but not both will exactly ones.

Example:

1) obtain dnf of $P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$

$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$

$P \to Q$

$\Leftrightarrow \neg P \vee Q$

$\Leftrightarrow P \to ((\neg P \vee Q) \wedge \neg\neg(Q \wedge P))$

$(\neg P \vee Q) \wedge A$

$\Leftrightarrow (\neg P \wedge A) \vee$

$\Leftrightarrow (Q \wedge A)$

$\Leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge \underbrace{(P \wedge Q)}_{A})$

$\Leftrightarrow \neg P \vee (\neg P \wedge P \wedge Q) \vee (Q \wedge P \wedge Q)$

$\Leftrightarrow \neg P \vee 0 \vee (Q \wedge P)$

$\Leftrightarrow \neg P \vee (Q \wedge P)$

This is required dnf.

21.07.17

2) Obtain cnf of $P \to (P \to Q) \wedge (\neg(\neg Q \vee \neg P))$

$\Leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (P \wedge Q))$

$\Leftrightarrow (\neg P \vee (\neg P \vee Q)) \wedge (\neg P \vee (P \vee Q))$

$\Leftrightarrow (\neg P \vee \neg P \vee Q) \wedge (\neg P \vee P) \wedge (\neg P \vee Q)$

$\Leftrightarrow (\neg P \vee Q) \wedge 1 \wedge (\neg P \vee Q)$

$\Leftrightarrow (\neg P \vee Q)$

This is pcnf

3) Obtain the principle normal form of
and or

$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge 1) \vee (\neg P \wedge 1 \wedge R) \vee (1 \wedge Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge (\neg Q \vee Q) \wedge R) \vee ((P \vee \neg P) \wedge Q \wedge R)$.

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\cancel{P \wedge Q \wedge R}) \vee (\neg P \wedge Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R)$

This is pdnf form.

Let $F \equiv (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\neg F \equiv (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

$\therefore \neg(\neg P) \equiv \neg\left[(P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)\right]$

$\equiv \left[(\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R)\right]$

$\equiv (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

This is pcnf form.

4) Obtain the normal form of $(\neg P \to R) \wedge (P \leftrightarrow Q)$

Wrong

$(\neg P \to R) \wedge (P \leftrightarrow Q)$

$\Leftrightarrow (\neg P \to R) \wedge (P \to Q) \wedge (Q \to P)$

$\Leftrightarrow (P \vee R) \wedge (\neg P \vee Q) \wedge (\neg Q \vee R)$

$\Leftrightarrow (P \vee R \vee O) \wedge (\neg P \vee Q \vee O) \wedge (\neg Q \vee R \vee O)$

$\Leftrightarrow (P \vee (Q \wedge \neg Q) \vee R) \wedge (\neg P \vee Q \vee (R \wedge \neg R)) \wedge ((\neg P \wedge P) \vee \neg Q \vee R)$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge \cancel{(P \vee \neg Q \vee R)} \wedge (\neg P \vee \neg Q \vee R)$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R)$

This is pcnf form.

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Let $F \equiv (\sim P \rightarrow R) \wedge (P \leftrightarrow Q)$

$\sim F \equiv \sim[(P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R)]$

$\equiv (\sim P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge Q \wedge \sim R) \vee (P \wedge \sim Q \wedge \sim R) \vee (P \wedge \sim Q \wedge R) \vee (P \wedge Q \wedge \sim R)$.

$\equiv (P \wedge Q \wedge \sim R) \vee (P \wedge \sim Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge \sim R)$

$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

P	Q	R	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	Ans
0	0	0	0	0	0	0
0	0	1	0	0	0	1
0	1	0	0	0	0	0
0	1	1	0	1	1	1
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	0	0	1	1
1	1	1	1	1	1	1

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24.7.17

Predicate Calculus

$P(x)$: x is a prime

$T(x, y)$: x is taller than y

Symbolizing the Statement:

=> You can take the flight if and only if you buy a ticket.

F : you can take the flight

T : you by a ticket

$F \leftrightarrow T$

=> The home team win's whenever it is raining

T : The home team wins

f : it trains

$f \Rightarrow T$

=> The automated replay cannot be send when the fill system is full

f : The file System is full

$\sim A$: The automated replay can be send

$f \leftrightarrow \sim A$

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→ You cannot ride the rollar couster if you are under 4 feet tall unless you are older than 16 years old.

¬p : You can ride the rollar couster

A : you are under 4 feet tall.

Q : & you are older than 16 years old.

1) If the directory database is open then the monitor is put in a closed state if the system not in its initial state.

2) Determine the validity of the following argument.

i) My father praises me only if I can proud of myself either I do well in sports or I cannot be proud of myself if I study hard then I cannot do well in sports. Therefore if father praises me then I do not study well.

ii) x is a prime number. The variable x is a subject of the statement is a prime number is the predicate.

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This can be denoted by $p(x)$. which is the value of the propositional function P, at x, If the value is assigned to variable x. The statement $p(x)$ is become the proposition and has a truth value.

Definition:

Universal quantification of $p(x)$:

It is the "$p(x)$ is true $\forall$ values of x in the universe of discourse (or domain."

In Notation: $(\forall x)(p(x))$ denote the universal quantification of $p(x)$. Here $\forall$ is called the Universal quantifier.

Existential quantifier:

It is the proposition "There exist an element x in universe of discourse such that $p(x)$ is true."

In Notation: $(\exists x)(p(x))$ for the existential quantification of $p(x)$. Here $\exists$ is called the existential quantifier.

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Example :-

=> Let $p(x)$ is the statement $x+1>x$, where the universe of discourse consist of all real numbers.

It is clear that $\forall$ $p(x)$ is true were the universe of discourse is the set of all real numbers.

=> Let $q(x)$ be the statement $x^2+5x+6=0$, where the universe of discourse is the set of all integers.

$\therefore$ There exist x $Q(x)$ is true. Since the integers -2 & -3 Satisfy the given equation.

Negating Qualifiers:

$\neg(\forall x)(p(x))$ is true

$\Leftrightarrow (\exists x)(\neg p(x))$

$\neg(\exists x)(p(x)) \Leftrightarrow (\forall x)(\neg p(x))$

Symbolize the following Statement:

1) All men are gaints.

Solution :

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M(x) : x is a man

G(x) : x is a gaint

the given statement can be refrased

as $\forall x$: x is a man then x is a gaint.

$\therefore \quad \forall x\,(\,M(x) \rightarrow (G(x)))$

2) Some men are not gaints.

25.7.17

$A \rightarrow n$ elements

$(P(A)) = 2^n$

No. of 0 elem Sub = 1

1 $= {}^nC_1$ or

2 $= {}^nC_2$

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27.7.'17

Equivalences and Implications for Predicate calculus

Implications :

1) $(\forall x)\ (P(x)) \Leftrightarrow \neg (\exists(x))\ (\neg P(x))$

2) $(\forall x)\ (\neg P(x)) \Leftrightarrow \neg (\exists(x))\ (P(x))$

3) $\neg (\forall x)(P(x)) \Leftrightarrow (\exists x)(\neg P(x))$

4) $\neg (\forall x)\ (\neg P(x)) \Leftrightarrow (\exists x)\ (P(x))$

5) $(\exists x)\ (A(x) \vee B(x)) \Leftrightarrow (\exists x)\ A(x) \vee (\exists x)\ B(x)$

6) $(\forall x)\ (A(x) \wedge B(x)) \Leftrightarrow (\forall x)\, A(x) \wedge (\forall (x))\ (B(x))$

7) $\neg (\exists x)\ (A(x)) \Leftrightarrow (\forall x)\ (\neg A(x))$

8) $\neg (\forall x)\ (A(x)) \Leftrightarrow (\exists x)(\neg A(x))$

9) $(\forall x)\ (A \vee B(x)) \Leftrightarrow A \vee (\forall x)\ (B(x))$

10) $(\exists x)\ (A \wedge B(x)) \Leftrightarrow A \wedge (\exists x)\ B(x)$

11) $(\forall x)\ A(x) \rightarrow B \Leftrightarrow (\exists x)\ (A(x) \rightarrow B)$

12) $A \rightarrow (\forall x)(B(x)) \Leftrightarrow (\forall x)(A \rightarrow B(x))$

13) $A \rightarrow (\exists x)\ (B(x)) \Leftrightarrow (\exists x)\ (A \rightarrow B(x))$

14) $(\forall x)\ (A(x)) \vee (\forall x)\ (B(x)) \Rightarrow \forall (x)\ (A(x) \vee B(x))$

15) $(\exists x)\ (A(x) \wedge B(x)) \Rightarrow (\exists x)\ (A(x)) \wedge (\exists x)(B(x))$

Theory of infrance for Predicate Calculus:

Rule US (universal specification): removing $\forall$

If a statement of the form $(\forall x)(p(x))$ is assumed to be true (when $p(x)$ may or maynot contain other symbols in addition to x) then the universal specifier may be droped from the statement to obtain $P(x)$, which is true for each object in the universe (or) to obtain $P(b)$ which is true for a specific object of the universe. US

Rule UG (Universal Generalisation): Insert $\forall$

If a statement $p(x)$ is true for a arbitrary x of the universe (where $p(x)$ may or maynot contain other symbols in addition to x) then the US may be prefixed to obtain $(\forall(x))(P(x))$ provided that every existential object in $p(x)$ which depends on x is covered by a quantifier.

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Rule ES (Existential Specification): (remove $\exists$)

If $\exists(x)\,(p(x))$ is assumed to be true, (where $p(x)$ may or maynot contain the other symbols in add. to x) then the existential object b can be used to represent the obj. for which $p(x)$ is true provide b has not already been used to represent some obj. in the discussion or in $p(x)$.

Rule EG (Existential Generalisation): (insert $\exists$)

Let c be an obj. having a given property & Let $p(c)$ be a symbolic Expression of this Property possed by c which may or maynot contain other symbols in addition to c. If x is not one of the symbols occuring in $p(c)$ then the existence of x can be expressed by replacing c by x & prefixing the $\exists$ to obtain $(\exists x)\,(p(x))$

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23.7.19

Verify the validity of the following argument.

1) The lions are dangerous animals. There are lions. Therefore there are dangerous animals.

Proof:-

$L(x)$: x is a lion.

$D(x)$: x is a dangerous animal.

$\forall x \, (L(x) \rightarrow D(x))$ => Proposition

$(\exists x) \, L(x)$

$(\exists x) \, D(x)$ => conclusion

Arguments:

Step	Proposition	Justification.
(1)	$(\exists x) L(x)$	P- propositional
2	$L(P)$	ES, (1)
3	$(\forall x) \, (L(x) \rightarrow D(x))$	P- Propositions.
4	$L(P) \rightarrow D(P)$	US (3)
5	$D(P)$	T- Detachment (2), (4)
6.	$(\exists x) \, (D(x))$	EG (5)

2) Sockratie's arguments

Verify the validity of the following argument.

All men are mortal. Sockratles is a Man. therefore He is mortal.

$L(x)$: x is man

$D(x)$: x is mortal

S : Socrates.

$$\forall x\ (L(x) \rightarrow D(x))$$

$$\frac{L(S)}{D(S)}$$

Step	Proposition	Justification
1	$\forall x(L(x) \rightarrow D(x))$	Proposition
2	$L(S) \rightarrow D(S)$	US (1)
3	$L(S)$	proposition
4	$D(S)$	Detachement (2), (3)

3) Show that from a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (\forall y)(M(y) \rightarrow W(y))$ and b) $(\exists y)(M(y) \wedge \neg W(y))$ the conclusion $(\forall x)(F(x) \rightarrow \neg S(x))$ follows.

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Step	Proposition	Justification
1	$\exists(y)\ (M(y) \wedge \rightarrow \neg W(y))$	proposition
2	$M(z) \wedge \rightarrow \neg W(z)$	ES, (1)
3	$\neg (M(z) \rightarrow W(z))$	conditional equivalence

$M(z) \wedge \neg (W(z))$

$\Leftrightarrow \neg(\neg M(z)) \wedge \neg (W(z))$

$\neg (\neg M(z) \vee W(z))$

$\Leftrightarrow \neg (M(z) \rightarrow W(z))$

$p \wedge \neg q$

$(\neg p \vee q)$

29.7.19

* Recurrence Relation :-

A Recurrence Relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous terms of the sequence, namely $a_0, a_1, \ldots, a_{n-1}, \forall \mathbb{Z}\ n$, with $n \geq n_0$, where n_0 is a non-negative integer.

soln:-

Eg => fibonnaccii => $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$$\underset{a_0}{\uparrow}\ \underset{a_1}{\uparrow}\ \underset{a_2}{\uparrow}\ \underset{a_3}{\uparrow}\ \underset{a_4}{\uparrow}$$

if $\boxed{a_n = a_{n-1} + a_{n-2}, \quad n \geq 2}$ recurrence relation,

we cannot find for a_0 & a_1.

$\therefore a_2 = a_1 + a_0$ Hence $n \geq 2$ $\therefore n_0 = 2$

Sequence :-

A Sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

eg:

Let $\{a_n\}$ be a sequence that satisfy the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for $n = 2, 3, 4, \ldots$

Let $a_0 = 0$, & $a_1 = 1$. Therefore $a_2 = a_1 + a_0 = 0 + 1 = 1$

then $a_3 = a_2 + a_1 = 1 + 1 = 2$ etc.,

$\therefore$ The solution is the sequence

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

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1) Consider D defined by $D(k) = 5 \cdot 2^k$, $k \geq 0$, find the recurrence relation on D.

Solution:
For $k \geq 0$, it is given $D(k) = 5 \cdot 2^k$.

$\Rightarrow D(k-1) = 5 \cdot 2^{k-1}$

$\therefore D(k)/D(k-1) = 5 \cdot 2^k / 5 \cdot 2^{k-1} = 2$

$\Rightarrow D(k) - 2D(k-1) = 0, \; k \geq 0$

Order:

The recurrence relation on a sequence S is of order k if S(n) is expressed as a fn. of $S(n-1), S(n-2), \ldots, (S(n-k))$

eg: Highest index - least index = order k.

$S(n) = \_ S(n-1) + \_ S(n-2) + \ldots + \_ S(n-k)$

$\therefore n - (n-k) = k$

In above eg. ex: ①, the order is $k - (k-1) = 1$

Linear recurrence relation with constant coefficient:

The recurrence relation on a sequence S is a linear " " with constant coefficients, if it is of the form

$S(k) + C_1 S(k-1) + C_2 S(k-2) + \ldots + C_n S(k-n) = f(k)$,

$k \geq n$, where $C_1, C_2, \ldots C_n$ are numbers and f is a fn. defined for $k \geq n$. Also if $C_n \neq 0$ then the relation is said to be of order n.

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Note:

A Linear Recurrence Relation with constant coefficient is simply called as Linear Relation.

Homogeneous Relation:

An n^{th} order linear relation is a homogeneous relation if $f(k) = 0 \ \forall k$

Associated Homogeneous Relation:

For a recurrence relation

$S(k) + C_1 S(k-1) + C_2 S(k-2) + \dots + C_n S(k-n) = f(k)$

the associated homogeneous relation is

$S(k) + C_1 S(k-1) + C_2 S(k-2) + \dots + C_n S(k-n) = 0$

30/7/17

1) Show that $\neg 2$, $p \rightarrow 2 \Rightarrow \neg p$

Step	Proposition	Reason
1	$p \rightarrow 2$	P
2	$\neg 2 \rightarrow \neg p$	T (1) & E18
3	$\neg 2$	P
4	$\neg p$	T (2) (3) & I_{11}

2) show that

$\{(p \to q) \wedge (r \to s),\ (q \to t) \wedge (s \to u),\ \neg(t \wedge u),\ p \to r\}$

$\Rightarrow \neg p$

step	proposition	Reason
1	$(p \to q) \wedge (r \to s)$	P
2	$p \to q$	T (1) conjunctive simplification
3	$r \to s$	T (1) " "
4	$(q \to t) \wedge (s \to u)$	P
5	$q \to t$	T, (4) Simplification
6	$s \to u$	T (4) "
7)	$p \to t$	T (2), (5) chain rule hypothetical Sylogism
8	$\neg t \to \neg p$	contrapositive T (7) & E, C
9	$r \to u$	T(3), (6), HS
10	$p \to r$	P
11	$p \to u$	T (10)(9) HS
12	$\neg u \to \neg p$	T (11) & E18
13	$(\neg t \vee \neg u) \to \neg p$	T(8) & 12) & I
14	$\neg(t \wedge u) \to \neg p$	T(13) Dem Law
15	$\neg(t \wedge u)$	P
16	$\neg p$	T (14), (15)

3) show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ and Q

Step	Proposition	Reason
1	$\neg R \vee P$ $R \to P$ condition for equivalence	P
2	R	P (additional premise)
3	P	T(1), (2) & I_{10}
4	$P \to (Q \to S)$	P
5	$Q \to S$	T, (3)(4) & I_{11}
6	Q	P
7	S	T(5)(6) & I_{11}
8	$R \to S$	CP

4) Show that $P \to S$ can be derived from the premises $\neg P \vee Q$, $\neg Q \vee R$, $R \to S$

Step	proposition	Reason
1	$\neg P \vee Q$ $P \to Q$	P
2	P	P (addition of premise)
3	Q	T(1), (2) & I_{10}
4	$\neg Q \vee R$ $Q \to R$	P
5	R	T(3), (4) & I_{10}

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6	$R \rightarrow S$	P
7	S	T(5),(6) & I_{11}
8	$P \rightarrow S$	CP

5) Show that if $p \rightarrow q$, $q \rightarrow r$, $\neg(p \wedge r)$ & $p \vee r$ then r

step	proposition	reason
1	$p \rightarrow q$	P
2	$q \rightarrow r$	P
3	$p \rightarrow r$	T(1), (2) & I_{13}
4	$\neg p \vee r$	T, (3) & E_{16}
5	$p \vee r$	P
6	$(p \wedge \neg p) \vee r$	T(4) (5) distributive law
7	r	T(6) $p \wedge \neg p = F$
8	$\neg(p \wedge r)$	P
9	$\neg p \vee \neg r$	T(8) Dem. La
10	$r \wedge (\neg p \vee \neg r)$	T(7), (9)
11	$(r \wedge \neg p) \vee (r \wedge \neg r)$	T, (10) Dis. Law
12	$r \wedge \neg p$	T(11) $F = r \wedge \neg r$
13	r	T(12) simplification.

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6) Prove by indirect method that

$(\neg q)$, $p \to q$, $p \vee t \Rightarrow t$

step	proposition	Reason
1	$p \vee t$	P
2	$\neg t$	P (additional)
3	p	T (1) (2)
4	$p \to q$	P
5	q	T (3), (4)
6	$\neg q$	P
7	$q \wedge \neg q$	T (5), (6) contradiction

7) using indirect proof show that $p \to q$, $q \to r$, $\neg(p \wedge r)$, $p \vee r \Rightarrow r$

Step	Proposition	Reason
1	$q \to r$	P
2	$\neg p r$	P (additional)
3	$\neg q$	T (1), (2)
4	$p \to q$	P
5	$\neg p$	T (3), (4)

6 $p \vee r$ $\neg p \to r$ $\quad$ P

7 r $\quad$ T (5) (6)

8 $r \wedge \neg r$ $\quad$ T (2) (7)

Theory of inference for predicate calculus.

1) verify the validity of the following arguments.

Every living thing is a plant / an animal. John's gold fish is alive & it is not a plant. All animals have hearts. therefore John's gold fish has a heart.

$P(x)$: x is a plant.

$A(x)$: x is an animal

$H(x)$: x has a heard.

g : John's gold fish.

$(\forall x)\ (P(x) \vee A(x))$

$\neg P(g)$

$(\forall x)\ (A(x) \to H(x))$

$H(g)$

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Step	proposition	Reason
1	$(\forall x)(P(x) \vee A(x))$	P
2	$\neg P(g)$	P
3	$P(g) \vee A(g)$	US(1)
	$p \vee q \Rightarrow \neg p \rightarrow q$	
4	$A(g)$	T,(2)(3)
5	$(\forall x)(A(x) \rightarrow H(x))$	P
6	$A(g) \rightarrow H(g)$	US(5)
7	$H(g)$	T,(4),(6)

2) Give an argument which will establish the validity

All integers are rational num.

Some " are powers of 2.

Therefore, Some rational num. are Powers of 2.

$P(x)$: x is an integer

$R(x)$: x is a rational no.

$S(x)$: x is a power of 2

$(\forall x) \quad (P(x) \rightarrow R(x))$

$(\exists x) \quad (P(x) \wedge S(x))$

$\exists(x) \quad (R(x) \wedge S(x))$

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Step	Proposition	Reason
1	$(\exists x)(P(x) \wedge S(x))$	P
2	$P(b) \wedge S(b)$	ES (1)
3	$P(b)$	T, (2)
4	$S(b)$	T, (2)
5	$\forall (P(x) \rightarrow R(x))$	P
6	$P(b) \rightarrow R(b)$	US (5)
7	$R(b)$	T(3)(6)
8	$R(b) \wedge S(b)$	T (7)(4)
9	$(\exists x)(R(x) \wedge S(x))$	EG, (8)

3) a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (\forall y)(M(y) \rightarrow W(y))$

b) $(\exists y)(M(y) \wedge \sim W(y))$

$P \wedge \sim Q = \sim(P \rightarrow Q)$

Step	Proposition	Reason
1	$(\exists y)(M(y) \wedge \sim W(y))$	P
	$M(z) \wedge \sim W(z)$	ES (1)
2	$\sim(M(z) \rightarrow W(z))$	T, (2), E17
3	$(\exists y) \sim [M(y) \rightarrow W(y)]$	EG, (3)
4	$\sim (\forall y)(M(y) \rightarrow W(y))$	(4), E26
5		
6	$(\exists x)(M(y) \rightarrow W(y)) \rightarrow (\forall y)(M(y) \rightarrow W(y))$	P
7	$\sim(\exists x)(F(x) \wedge (S(x))$	T(5),(6), I12

8	$(\forall x) \sim (F(x) \wedge S(x))$	T, (7), E_{25}
9	$\sim [F(x) \wedge S(x)]$	US (8)
10	$F(x) \rightarrow \sim S(x)$	T, (9) E_9, E_{16}
11	$(\forall x)(F(x) \rightarrow \sim S(x))$	UG, (10)

4) If $(\forall x)(P(x) \rightarrow Q(x))$: $(\exists y)(P(y))$ then,

$(\exists z)\ Q(z)$

Step	Proposition	Reason
1	$\sim [\exists z] Q(z)$	P (additional premise)
2	$(\forall z) \sim Q(z)$	T (1)
3	$(\exists y)\ P(y)$	P
4	$P(a)$	ES, (3)
5	$\sim Q(a)$	US, (2)
6	$P(a) \wedge \sim Q(a)$	T, (4)(5)
7	$\sim (P(a) \rightarrow Q(a))$	T, (6)
8	$(\forall x)(P(x) \rightarrow Q(x))$	P
9	$P(a) \rightarrow Q(a)$	US (8)
10	$(P(a) \rightarrow Q(a)) \wedge \sim (P(a) \rightarrow Q(a))$	T (7) (9), contradiction.

$P \wedge \sim q$ $\sim(\sim p \vee q)$ $\neg(\neg p \vee q)$ $p \rightarrow q$

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5) using CP (or) otherwise obtain the following implementation

$(\forall x)(P(x) \to Q(x)), (\forall x)(R(x) \to \neg Q(x))$

$\Downarrow$

$(\forall x)(R(x) \to \neg P(x))$

Step	Proposition	Reason
1	$(\forall x)(P(x) \to Q(x))$	P
2	$(\forall x)(R(x) \to \neg Q(x))$	P
3	$R(x) \to \neg Q(x)$	US, (2)
4	$R(x)$	P (add premise)
5	$\neg Q(x)$	T (3), (4)
6	$P(x) \to Q(x)$	US, (1)
7	$\neg P(x)$	T (5)(6)
8	$R(x) \to \neg P(x)$	CP, (4)(7)
9	$(\forall x)(R(x) \to \neg P(x))$	UG (9)

6) There is a mistake in the following derivaties find it. Is the conclusion valid ? If so, obtain the correct derivation

1	$(\forall x)(P(x) \to Q(x))$	P
2	$P(y) \to Q(y)$	US (1)

3	$(\exists x)(P(x))$	P
4	$P(y)$	ES (3)
5	$Q(y)$	T(2)(4), I_{11}
6	$(\exists x) Q(x)$	EG (5)

(a)

1	$(\exists x) P(x)$	P_1
2	$P(y)$	ES (1)
3	$(\forall x)(P(x) \rightarrow Q(x))$	P
4	$P(y) \rightarrow Q(y)$	US (3)
5	$Q(y)$	T (2)(4), I_{11}
6	$(\exists x) Q(x)$	EG, (5)

Statements Involving more than the Quantifier.

1) In the universe of all integers, let $Q(x, y) : x + y = 10$,
which of the following statements are true in this universe

(i) $(\exists x) (\forall y) Q(x, y)$

(ii) $(\forall x) (\forall y) Q(x, y)$

(iii) $(\exists x) (\exists y) Q(x, y)$

(iv) $(\forall x) (\exists y) Q(x, y)$

(i) $(\exists x)(\forall y)\ Q(x,y)$ means "there exists an x such that $x+y=10$, for all integer y".

It is not true as there is no common x satisfy this relation for all y. If there is an integer x, such that $x_1 + y_1 = 10$ for some y_1 then $x_1 + 2y_1 = 10$ is not true.

(ii) $(\forall x)(\forall y)\ Q(x,y)$ means for every integer x and for every integer y, we have $x+y=10$.

This is not true. when $x=3$, & $y=5$ then $x+y$ is $\neq 10$.

(iii) The statement is true for $(\exists x)(\exists y)\ Q(x,y)$ means there is an integer x and there is an integer y such that $x+y=10$.

It is true when $x=3$ & $y=7$, we have $3+7=10$ (There are many pairs (x,y) such that $x+y=10$)

(iv) $(\forall x)((\exists y)\ Q(x,y))$ means to each integer x, there is atleast one integer y such that $x+y=10$. This statement is true since to each integer x, which is an integer take $y=10-x$, then $x+y=10$.

2) Write the following statement in the symbolic form.

"Everyone who likes fun will enjoy each of these plays"

$L(x)$: x likes fun

$P(y)$: x is a play

$E(x,y)$: x will Enjoy y

Then $(\forall x)\ [L(x) \rightarrow (\forall y)(P(y) \rightarrow E(x,y))]$

represents, the given statement. The statement can also be represented as 'for each x, if x likes fun and for each y, if y is a play, then x enjoys y'. So it could be symbolized as

$(\forall x)(\forall y)\ [L(x) \wedge P(y) \rightarrow E(x,y)]$

31.7.17

1) Find the recurrence relation satisfying $y_n = A(3)^n + B(-4)^n$.

$12y_n + 3y_{n+1} - 4y_{n+1} + y_{n+2} = 0$

solution:

Given $y_n = A(3)^n + B(-4)^n \rightarrow ①$

$y_{n+1} = A(3)^{n+1} + B(-4)^{n+1}$

$\Rightarrow y_{n+1} = 3A3^n - 4B(-4)^n \longrightarrow (2)$

$y_{n+2} = 9A3^n + 16B(-4)^n \longrightarrow (3)$

eliminating A & B from (1), (2), & (3), we have.

$$y_{n+1} = 3A3^n - 4B(-4)^n \longrightarrow (2)$$

$$3y_n = 3A3^n + 3B(-4)^n \longrightarrow (1)\times 3$$

$(-)\quad (-)\quad (-)$

$$y_{n+1} - 3y_n = -7B(-4)^n$$

$$-7B(-4)^n = y_{n+1} - 3y_n \longrightarrow (4)$$

$$3y_{n+1} = 9A3^n - 12B(-4)^n \longrightarrow (2)\times 3$$

$$y_{n+2} = 9A3^n + 16B(-4)^n \longrightarrow (3)$$

$(-)\quad (-)\quad (-)$

$$3(y_{n+1}) - y_{n+2} = 4B(-4)^n \longrightarrow (5)$$

$(4)\times 4 + (5)\times 7$

$$\Rightarrow -28B(-4)^n = 4y_{n+1} - 12y_n$$

$$-28B(-4)^n = 21y_{n+1} - 7y_{n+2}$$

$$\Rightarrow 25y_{n+1} - 7y_{n+2} - 12y_n = 0$$

2) $y_n = (A + Bn)4^n$

$= A(4^n) + Bn(4^n) \rightarrow ①$

$y_{n+1} = (A + B(n+1))4^{n+1}$

$= A4^{n+1} + Bn4^{n+1} + B4^{n+1}$

$= 4A4^n + 4B4^n n + 4B4^n \rightarrow ②$

$y_{n+2} = (A + B(n+2))4^{n+2}$

$= A4^{n+2} + Bn4^{n+2} + 2B4^{n+2}$

$= 16A4^n + 16Bn4^n + 32B4^n \rightarrow ③$

② => $y_{n+1} = \cancel{4A4^n} + \cancel{4B4^n n} + 4B4^n$

①×4 => $4y_n = \cancel{4A4^n} + \cancel{4Bn4^n}$

(-) (-) (-)

$y_{n+1} - 4y_n = 4B4^n \rightarrow ④$

③ => $y_{n+2} = \cancel{16A4^n} + \cancel{16Bn4^n} + 32B4^n$

① ×16 => $16y_n = \cancel{16A4^n} + \cancel{16Bn4^n}$

(-) (-)

$y_{n+2} - 16y_n = 32B4^n \rightarrow ⑤$

~~(5) => $y_{n+2} - 16y_n = 32\ B4^n$~~

~~(4)×8 => $8y_{n+1} - 32y_n = 34\ B4^n$~~

$$y_{n+2} - 16y_n = 8(y_{n+1} - 4y_n)$$

$$y_{n+2} - 8y_{n+1} + 16y_n = 0$$

~~$4y_{n+1} + y_{n+2} = 21A(3)^n \to (5)$~~

~~3(3)×(4) − (5)~~

~~$12y_n + 3y_{n+1} - 4y_{n+1} - y_{n+2} = 0$~~

~~$y_{n+2} + y_n$~~

Algorithm for solving n^{th} order Linear homogeneous recurrence relation:

Step-1: Write the characteristic eqn. of the given homogeneous relation.

Step-2: Find all the roots of the ch. equ. These roots are known as ch. roots.

Step-3:
3.1- If the roots $a_1, a_2, \ldots, a_n$ of the ch. eqn are distinct then the general soln. of the recurrence relation $s(k)$ is $s(k) = b_1 a_1^k + b_2 a_2^k + \ldots + b_n a_n^k \to (1)$

3.2 - If the root a_j is repeated 'p' - times thn $b_j a_j^k$ must be replaced by

$(c_0 + c_1 k + c_2 k^2 + \dots + c_p k^{p-1}) a_j^k$

Step - 4: If n initial conditions are given, thn obtain n linear eqn in n-unknown $b_1, b_2, \dots, b_n$ and replace the values in equ ①

1) Solve $D(k) - 8D(k-1) + 16 D(k-2) = 0$ where $D(2) = 16$, $D(3) = 80$

Solution:-

The order of given recurrence relation is 2.

∴ The characteristic eqn is

$a^2 - 8a + 16 = 0$

$\therefore a = 4, 4$

Since the roots are repeated, We have

$D(k) = (C_0 + C_1 k) 4^k$

Now, given => $D(2) = 16$

=> $16 = D(2) = (C_0 + C_1(2)) 4^2$

$16 = 16 C_0 + 32 C_1 \rightarrow ①$

$64 C_0 + 192 C_1 = 64 \rightarrow ②$

$D(3) = 80$

$$80 = (c_0 + c_1 (3)) 4^3$$

$$80 = 64 c_0 + 192 c_1$$

$c_0 = 1/2$, $c_1 = 1/4$

$$D(k) = (1/2 + 1/4 k) 4^k$$

7.8.'17

2) Solve the recurrence relation $S(k) - 4 S(k-1)$ $- 11 (S(k-2)) + 30 (S(k-3)) = 0$, $S(0) = 0$, $S(1) = -35$, $S(2) = -85$.

Solution:

The ch. equ is

$$a^3 - 4a^2 - 11a + 30 = 0$$

$a = 2$

$$\begin{array}{c|cccc} 2 & 1 & -4 & -11 & 30 \\ & \downarrow & 2 & -4 & -30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$a^2 - 2a - 15 = 0$$

$-\cancel{2}a = +15$

$a = \frac{15}{2}$

$\therefore (a-5)(a+3) = 0$

$a = 5, -3$

$\therefore$ The roots of a of $2, 5, -3$

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$$S(k) = b_1 a_1^k + b_2 a_2^k + \dots + b_n a_n^k$$

$$= b_1 2^k + b_2 5^k + b_3 (-3)^k$$

$$S(0) = 1b_1 + b_2 + b_3 = 0$$

$$S(1) = 2b_1 + 5b_2 - 3b_3 = -35$$

$$S(2) = 4b_1 + 25b_2 + 9b_3 = -85$$

1, 4, -5

$$\cancel{2b_1} + 2b_2 + 2b_3 = 0$$
$$\cancel{2b_1} + 5b_2 - 3b_3 = -35$$

$$-3b_2 + 5b_3 = 35$$

$$\cancel{4b_1} + 4b_2 + 4b_3 = 0$$
$$\cancel{4b_1} + 25b_2 + 9b_3 = -85$$

$$-21b_2 - \cancel{5b_3} = 85$$
$$-3b_2 + \cancel{5b_3} = 35$$

$$-24b_2 = 120$$

$$b_2 = 120/-24$$

$$\boxed{b_2 = -5}$$

$$b_1 - 5 + 4 = 0$$

$$-3(-5) + 5b_3 = 35$$

$$\boxed{\therefore b_1 = 1}$$

$$5b_3 = 20$$

$$\boxed{b_3 = 4}$$

$\therefore S(k) = 2^k - 5(5^k) + 4(-3)^k$

H.W

3) Write the recurrence relation for fibonacci number and solve.

$F_{n+3} = F_{n+2} + F_{n+1}$, $n = 0, 1, 2, \ldots$ where

$F_1 = 0$ & $F_2 = 1$ (or)

$F(n+3) - F(n+2) - F(n+1) = 0$ where

$F(1) = 0$ & $F(2) = 1$

Solution of Non-Homogeneous Recurrence Relation:

When the given recurrence relation is non-homogeneous the general solution is the sum of

1) solution for the corresponding Homogeneous relation

2) particular soln, depending on the RHS of the given recurrence relation.

Procedure for finding the particular solution:

case - 1:

RHS of the relation is $a_0 + a_1 k + \ldots + a_m k^m$ (this is a polynomial in K)

substitute $D_0 + D_1 K + \ldots + D_n K^n$ in place of $T(K)$, $a_0 + a_1(K-1) + \ldots + a_m (K-1)^n$ in place of $T(K-1)$ and so on in the given recurrence relation.

If RHS is ca^k, then substitute d_0a^k in place of ~~a~~ $T(K)$, d_0a^{k-1} in place of $T(K-1)$, ... in the given relation.

Step-2:

At the end of step 1, we get a polynomial in k with co-efficient ~~d_0~~, d_1, etc on LHS which is equal to the RHS of the given r.r.

equate the co.eff of power of k on both sides to get the value of d_0, d_1, etc.,

Step-3:

Use initial conditions for getting the values of the unknowns of the general sln. which is the sum of the sln. for the homogeneous relation & the particular solution.

Case: 2:-

If RHS in a given relation is a constant, replace $T(K)$, $T(K-1)$, ... by d_0 and follow steps 2, 3.

case : 3

When RHS is Ca^k and a coincides with the characteristic root,

when a is a simple root of the ch. equ, replace $d_0 k a^k$ etc.,

when a is a double root of the ch equ, replace $d_0 k^2 a^k$ etc,

Example :-

1) Solve $S(K) - S(K-1) - 6(S(K-2)) = -30$

where $S(0) = 20$, $S(1) = -5$.

Solution :-

I) It is homogeneous solution,

The ch. equ is

$$a^2 - a - 6 = 0$$

$$a = -2, +3$$

$\therefore$ The homogeneous solution is

$$c_1(-2)^k + c_2 3^k.$$

II) Particular solution.

The RHS of the given recurrence relation is a constant,

$\therefore$ take d and replace S^k, S^{k-1}, S

$S(k)$, $S(k-1)$, $(S(k-2))$ by d.

$d - d - 6d = -30$

$\Rightarrow \boxed{d = 5}$

i.e., the particular solution is 5

$\therefore$ General sln is $S(K) = C_1(-2)^K + C_2 3^K + 5$

$S(0) = 20 = C_1 + C_2 + 5 \rightarrow (1)$

$S(1) = -5 = -2C_1 + 3C_2 + 5 \rightarrow (2)$

$$-2C_1 + 3C_2 + 5 = -5$$
$$2C_1 + 2C_2 + 10 = 40$$

$$5C_2 + 15 = 35$$

$$5C_2 = 20$$

$$\boxed{C_2 = 4}$$

$$C_1 = 20 - 4 - 5$$

$$\boxed{\therefore C_1 = 11}$$

$\therefore S(K) = 11(-2)^K + 4(3)^K + 5$

10.8.17

2) Solve $S(K) - 3S(K-1) - 4S(K-2) = 4^K$

Solution :-

a) Homogeneous solution :-

$a^2 - 3a - 4 = 0$

$a = -1, 4$

∴ Homogeneous sln is

$$b_1(-1)^K + b_2(4)^K.$$

b) Particular solution:-

The RHS is 4^K. ie., the base is 4, which is one function of the characteristic simple root.

∴ Take $dK4^K$.

ie., replace $S(K)$ by $dK4^k$, $S(K-1)$ by $d(K-1)4^{K-1}$

and $S(K-2)$ by $d(K-2)4^{K-2}$

∴ Given recurrence relation becomes,

$$dK4^K - 3d(K-1)4^{K-1} - 4d(K-2)4^{K-2} = 4^K$$

$$\Rightarrow 4^K\left[dK - \frac{3}{4}d(K-1) - \frac{4}{4^2}d(K-2)\right] = 4^K$$

$$\Rightarrow 16dK - 12dK + 12d - 4dK + 8d = 1$$

$$\Rightarrow d = 0.8$$

∴ particular solution is $0.8K4^K$.

∴ The general solution is $S(K) = b_1(-1)^K + b_2(4)^K + 0.8K4^K$

3) solve $S(K) - 4S(K-1) + 4S(K-2) = 3K + 2^K$, $S(0) = 1$, $S(1) = 1$

solution:-

(a) Homogeneous solution is,

$$a^2 - 4a + 4 = 0 \Rightarrow a = 2, 2$$

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$\therefore$ Homogeneous solution is $(b_1 + b_2 K)2^K$

b) Particular solution; corresponds to $3K$:

$3K$ is of the form $c_1 + c_2 K$.

$\therefore$ Take $d_0 + d_1 K$.

i.e., replace $S(K)$ by $d_0 + d_1 K$, $S(K-1)$ by $d_0 + d_1(K-1)$, $S(K-2)$ by $d_0 + d_1(K-2)$

Thus we have,

$$d_0 + d_1 K - 4d_0 - 4d_1 K + 4d_1 + 4d_0 + 4d_1 K - 8d_1 = 3K$$

$$d_0 + \underline{d_1 K} - 4d_1 = \underline{3K} \quad \text{equating coeff of } K \text{ \& constant}$$

$$\Rightarrow \boxed{d_1 = 3,\ d_0 = 12} \quad \Rightarrow d_0 = 12$$

$\therefore$ particular soln. corresponds to $3K$ is $12 + 3K$.

c) particular integral corresponds to 2^K:

The base 2 is a double root of the characteristic polynomial.

$\therefore$ Take $dK^2 2^K$ i.e., replace $S(K)$ by $dK^2 2^K$, $S(K-1)$ by $d(K-1)^2 2^{K-1}$ & $S(K-2)$ by $d(K-2)^2 2^{K-2}$.

Thus

$$dK^2 2^K - 4d(K-1)^2 2^{K-1} + 4d(K-2)^2 2^{K-2} = 2^K$$

$$\Rightarrow 2^K\left[dK^2 - 2d(K^2 + 1 - 2K) + d(K^2 + 4 - 4K)\right] = 2^K$$

$$\Rightarrow \cancel{dK^2} - 2dK^2 - 2d + 4dK + dK^2 + 4d - 4dK = 1$$

$2d = 1$

$\therefore d = 0.5$

$\therefore$ particular solution corresponds to 2^K is $0.5K^2 2^K$

$\therefore$ The general solution is $S(K) = (b_1 + b_2 K)2^K + 12 + 3K + 0.5K^2 2^K$

given => $S(0) = 1$

$b_1 + 12 = 1 \Rightarrow b_1 = -11$

$S(1) = 1$

$2b_1 + 2b_2 + 12 + 3 + 1 = 1$

$2(-11) + 2b_2 + 18 = 1$

$2b_2 = 7$

$b_2 = 7/2 = 3.5$

$\therefore S(K) = (-11 + 3.5K)2^K + 12 + 3K + 0.5K^2 2^K$

4) solve $S(K) - 4S(K-1) + 4S(K-2) = (K+1)2^K$

solution:

a) Homogeneous sln :-

$a^2 - 4a + 4 = 0$

$a = 2, 2$

$\therefore$ Homogeneous sln is $(b_1 + b_2 K)2^K$

b) Particular solution:

The base for 2^K in the RHS is 2 which is the characteristic double root, we can take $K^2(d_0 + d_1 K)2^K$.

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i.e., replace $S(K)$ by $K^2(d_0 + d_1 K)2^K$, $S(K-1)$ by

$(K-1)^2(d_0 + d_1(K-1))2^{K-1}$ & $S(K-2) = (K-2)^2(d_0 + d_1(K-2))2^{K-2}$

Thus.

$$K^2(d_0 + d_1 K)2^K - 4\left[(K-1)^2(d_0 + d_1(K-1))2^{K-1}\right] +$$

$$4\left[(K-2)^2(d_0 + d_1(K-2))2^{K-2}\right] = (K+1)2^K$$

$$2^K\left[d_0K^2 + d_1K^3 - 2(K^2-2K+1)(d_0 + d_1K - d_1)\right.$$

$$\left. + (K^2-4K+4)(d_0 + d_1K - 2d_1)\right] = (K+1)2^K$$

$$\Rightarrow d_0K^2 + d_1K^3 - 2K^2d_0 - 2K^3d_1 + 2d_1K^2 + 4d_0K + 4d_1K^2$$

$$- 4d_1K - 2d_0 - 2d_1K + 2d_1 + d_0K^2 + d_1K^3 - 2d_1K^2$$

$$- 4d_0K - 4d_1K^2 + 8d_1K + 4d_0 + 4d_1K - 8d_1 = K+1$$

$$2d_0 + 6d_1K - d_1K^3 - 6d_1 = K+1$$

$$d_1K^3 - d_1K^2 + 2d_1 + K = -1$$

$$\cancel{d_1K^3 - d_1K^2 + K + 2d_1 + 1 = 0}$$

$$6d_1K + 2d_0 - 6d_1 = K+1$$

$6d_1 = 1$ $\qquad 2d_0 - 6d_1 = 1$

$\boxed{d_1 = 1/6}$ $\qquad 2d_0 = 2$

$\boxed{d_0 = 1}$

∴ The particular s(n is $k^2(1+1/6\,k)2^k$

∴ The general s(n is $(b_1+b_2k)2^k+k^2(1+1/6\,k)2^k$ //

Generating function:-

Defination:- The generating function of a sequence $s_0, s_1, s_2, \ldots$ is the power series.

$$G(S;Z) = S_0 + S_1 Z + S_2 Z^2 + \ldots$$

11.08.'17

consider the recurrence relation,

$$S(K) = 2S(K-1), \quad S(0)=1$$

$$S(1) = 2S(0)$$
$$S(2) = 2S(1) = 2\times2\, S(0) = 2^2 S(0)$$

Let the generating function be

$$G(S;Z) = S(0) + S(1)Z + S(2)Z^2 + \ldots$$

$$= S(0) + 2S(0)Z + 2^2S(0)Z^2 + \ldots$$

$$\textcircled{1} \leftarrow \quad = 1 + 2Z + (2Z)^2 + (2Z)^3 + \ldots$$

$$= 1/1-2Z \;//$$

(otherwise

$= S(0)Z^0 + S(1)Z^n + S(2)Z^2 + \ldots$

$= \sum_{n=0}^{\infty} S(n) Z^n$)

Procedure:

1) Write the given recurrence relation as an equ with zero on RHS.

2) Multiply the LHS of the equ. got in ① by Z^n and take the sum over all n.

The resulting power series is rewritten in terms of $G(S;Z)$.

3) write $G(S;Z)$ as a function of Z, which is the required generating function.

Example:

1) Find the generating function of P if P is given by $P(K) - 6P(K-1) + 5P(K-2) = 0$,

$P(0) = 2, \quad P(1) = 2$

Solution:-

Let $G(P;Z)$ be the generating function for P, then $G(P;Z) = \sum_{n=0}^{\infty} P(n) Z^n$ (summation notation)

1) Replacing the index k of (P) by n, the given recurrence relation becomes,

$$P(n) - 6P(n-1) + 5P(n-2) = 0, \quad n \geq 2 \rightarrow ①$$

2) Multiplying the equation ① by Z^n,

$$P(n)Z^n - 6P(n-1)Z^n + 5P(n-2)Z^n = 0, \quad n \geq 2$$

$$\sum_{n=2}^{\infty} \left[P(n)Z^n - 6P(n-1)Z^n + 5P(n-2)Z^n \right] = 0$$

$P(0)+P(1)z+P(2)z^2+P(3)z^3$

$$\Rightarrow \sum_{n=2}^{\infty} P(n)Z^n - 6Z\sum_{n=2}^{\infty} P(n-1)Z^{n-1} + 5Z^2 \sum_{n=2}^{\infty} P(n-2)Z^{n-2} = 0$$

$$\Rightarrow \left[(P(0) + P(1)Z + P(2)Z^2 + \dots) - (P(0) + P(1)Z) \right] - 6Z\left[\sum_{n=0}^{\infty} P(n)Z^n - P(0) \right] + 5Z^2 G(P;Z) = 0$$

$\Rightarrow G(P;z) - 2 - 2z - 6z\{G(P;z) - 2\} + 5z^2 G(P;z) = 0$

$\Rightarrow G(P;z)\,[1 - 6z + 5z^2] - 2 + 10z = 0$

$$\Rightarrow G(P;z) = \frac{2-10z}{1-6z+5z^2}$$

$$= \frac{2(1-5z)}{(5z-1)(z-1)} = \frac{-2}{(z-1)}$$

$$= 2/_{1-z} = 2(1-z)^{-1}$$

$$= 2[1 + z + z^2 + \dots + z^n + \dots]$$

2) Using generating fn. solve the difference equation $y_{n+2} - y_{n+1} - 6y_n = 0$ given $y_1 = 1$, $y_0 = 2$.

Solution:-

Let $G(y;z)$ be the generating fn. of the sequence $\{y_n\}$, then $G(y;z) = \sum_{n=0}^{\infty} y_n z^n$.

The given recurrence relation is

$y_{n+2} - y_{n+1} - 6y_n = 0$, Multiplying by z^n and summing over n from 0 to ∞, we have

$$\sum_{n=0}^{\infty} [y_{n+2} - y_{n+1} - 6y_n]\, z^n = 0$$

$$\sum_{n=0}^{\infty} y_{n+2} z^n - \sum_{n=0}^{\infty} y_{n+1} z^n - 6\sum_{n=0}^{\infty} y_n z^n = 0$$

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$$\frac{1}{z^2}\sum_{n=0}^{\infty} y_{n+2} z^{n+2} - \frac{1}{z}\sum_{n=0}^{\infty} y_{n+1} z^{n+1} - 6\sum_{n=0}^{\infty} y_n z^n = 0$$

$$\Rightarrow \frac{1}{z^2}\left[G(y;z) - y(0) - y(1)z\right] - \frac{1}{z}\left[G(y;z) - y(0)\right]$$

$$-6\, G(y;z) = 0$$

$$G(y;z)\left[\frac{1}{z^2} - \frac{1}{z} - 6\right] - \frac{2}{z^2} - \frac{1}{z} + \frac{2}{z} = 0$$

$$G(y;z)\left[\frac{1-z-6z^2}{z^2}\right] - \left[\frac{2+z-2z}{z^2}\right] = 0$$

$$G(y;z)\,\frac{1-z-6z^2}{z^2} - \frac{2-z}{z^2} = 0$$

$$G(y;z) = \frac{2-z}{1-z-6z^2}$$

$$= \frac{2-z}{(1-3z)(1+2z)}$$

$$= \frac{A}{1-3z} + \frac{B}{1+2z}$$

$$(2-z) = A(1+2z) + B(1-3z)$$

$$= A + 2zA + B - 3zB$$

$$A + B = 2 \qquad\qquad 2A + 2B = 4$$

$$2A - 3B = -1 \qquad\qquad 2A - 3B = -1$$

$$5B = 5$$

$$\boxed{B=1} \qquad \boxed{A=1}$$

$$= \frac{1}{1-3z} + \frac{1}{1+2z}$$

$$= (1-3z)^{-1} + (1+2z)^{-1}$$

$$= \{1+3z+(3z)^2+\dots+(3z)^n+\dots\}$$
$$+ \{1-2z+(2z)^2+\dots+(-1)^n(2z)^n+\dots\}$$

$\therefore y_n =$ co.efficient of z^n

$$= 3^n + (-2)^n$$

3) solve the recurrence relation $s(n) = s(n-1) + 2(n-1)$, with $s(0) = 3$, $s(1) = 1$ by finding its generating function.

solution:-

The given recurrence relation is $s(n) = s(n-1) + 2(n-1)$

$\Rightarrow s(n) - s(n-1) - 2n + 2 = 0$

Let $G(S; z)$ be the G.Fn of z, then

$$G(S; z) = \sum_{n=0}^{\infty} s(n) z^n.$$

Multiplying z^n & summing $n = 1$ to ∞, we have

$$\sum_{n=1}^{\infty} [s(n) - s(n-1) - 2n + 2] z^n = 0$$

$$\sum_{n=1}^{\infty} s(n) z^n - \sum_{n=1}^{\infty} s(n-1) z^n - 2 \sum_{n=1}^{\infty} n z^n + 2 \sum_{n=1}^{\infty} z^n = 0$$

$$G(S; z) - s(0) - \frac{z}{z} G(S; z) - 2(z + 2z^2 + 3z^3 + \dots)$$
$$+ 2(z + z^2 + z^3 + \dots) = 0.$$

$$G(S; z)(1-z) - 3 - 2z(1-z)^{-2} + 2(z)(1-z)^{-1} = 0$$

$$G(S; z) = \frac{3}{1-z} + \frac{2z}{(1-z)^3} + \frac{2z}{(1-z)^2}$$

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Graph Theory.

Graph: A graph G is an ordered triple $(V(G), E(G), \psi)$ consisting of

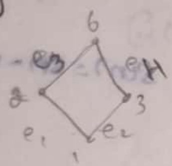
i) a non empty finite set $V(G)$ vertex set (or) vertices set

ii) a finite set $E(G)$ Edge (or) Edges set.

Which is disjoint from $V(G)$, may be empty

iii) an incidence function ψ that associates with each element of $E(G)$ and $V(G)$ an unordered pair of elements of $V(G)$.

eg:



G_1

Consider this Hasse diagram, here

$V(G) = \{1, 2, 3, 6\}$

$E(G) = \{e_1, e_2, e_3, e_4\}$

$\psi(e_1) \Rightarrow (1, 2), \psi(e_2) \Rightarrow (1, 3), \psi(e_3) \Rightarrow (2, 6), \psi(e_4) \Rightarrow (3, 6)$

If unordered pairs $\to$ graph or undirected graph

" ordered " $\to$ digraph or directed graph.

Note: The elements of $V(G)$ are called vertices of G & the elements of $E(G)$ are called the edges of G. If e is an edge and $\psi(e) = (u, v)$ then we say that e is an edge joining u and v, and the vertices

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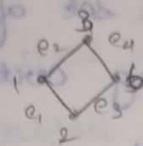
u and v are called the end vertices of e.

Notation:

Graphs are usually represented by diagram using a point for each vertex & a line for each edge.

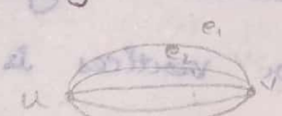
example: G_1 $V(G) = \{1, 2, 3, 6\}$

$E(G_1) = \{e_1, e_2, e_3, e_4\}$



If e is an edge in a graph G such that $\varphi(e) = (u, u)$ for some $u \in V(G)$ then e is said to be a loop in G.

If $e_1, e_2, \ldots, e_m$, $m \geq 2$ are edges in G, such that $\varphi(e_1) = \varphi(e_2) = \cdots = \varphi(e_m) = (u, v)$ then $e_1, e_2, \ldots, e_m$ are said to be parallel edges joining u & v, where $u, v \in G$.

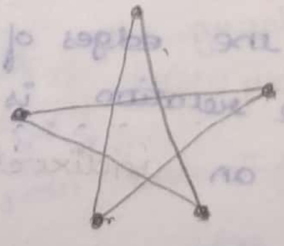
eg: 

[not a simple graph]

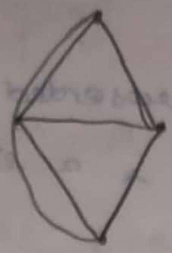
class of Graph:

A graph which has no loop and no parallel edge is said to be a simple graph.

example:



This is a simple graph of 5 vertices and 5 edges.

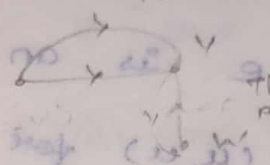


This was the 1st graph in graph theory drawn by Euler

18-8-17 Directed Graph:

A Directed Graph consist of a set of Vertices V and a set of edges E, connecting certain elements of V in an ordered pair from V. → this is parallel.

Simple Directed Graph:



This is not parallel (not a simple directed graph)

A simple Graph is one for which there is not more than one vertex directed from any vertex to another vertex.

Since U, V has parallel edges.

Multigraph:

The graph with atleast 2 edges from one vertex to some other vertices is called Multigraph, The above eg: u,v is a Multigraph.

example: A network of computers can be described using the graphs.

An edge b/w any 2 computer (vertices) denotes the 2 way communication is possible b/w the 2 computers. Hence the edges of this graph are symmetric not directed. Also the relation is symmetric. Thus this graph is an undirected Graph.

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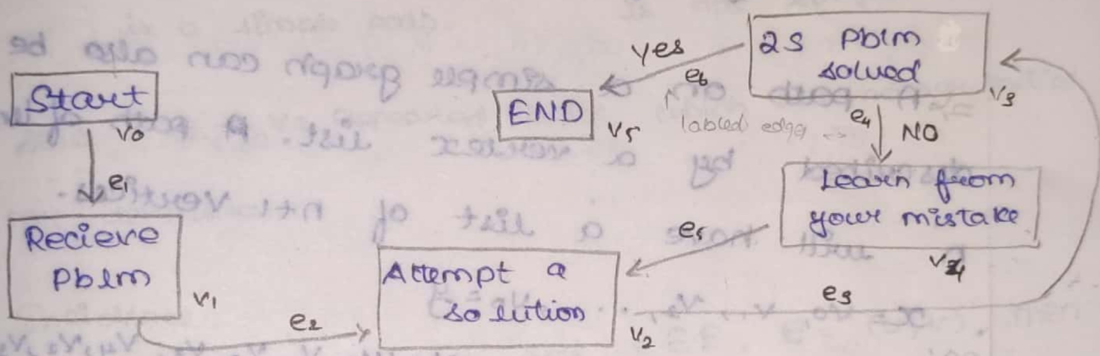
Complete Graph:

A complete undirected graph of n vertices, is an undirected graph with the property that each pair of distinct vertices are connected to one another & it is denoted by K_n.

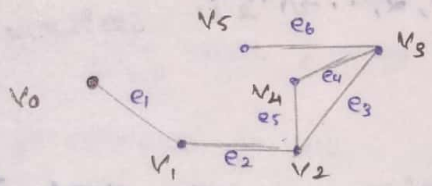
eg: K_2 : ; K_3 ; K_4 : ; K_5 :

Example:- A flow chart is a common eg. of a simple graph that requires lables for its vertices & some of its edges.

The following flow chart is an eg. that illustrates how many pblms are solved.



Graph theoritical Notation:



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If x and y are any 2 vertices of a graph then a path b/w x & y describes a motion from x & y along the edges of the graph. Vertex x is called Initial Vertex of the path & y is called Terminal Vertex. A path b/w x & y can away be describes by its edge list. i.e., the list of edges used : $(e_1, e_2, \ldots e_m)$.

Here

1) the initial vertex of e_1 is x

2) the terminal vertex of e_i is initial vertex of e_{i+1}, $i = 0, 1, 2, \ldots n-1$

3) The terminal vertex of e_m is y.

for eg: In the above graph there are 2 paths

i) $(e_1\ e_2\ e_3\ e_6)$; ii) $(e_1\ e_2\ e_3\ e_4\ e_5\ e_3\ e_6)$

⇒ The no. of edges in the edge list of a path is the path length.

The length of the 1st path is 4 & 2nd is 7.

⇒ A path on a simple graph can also be described by a vertex list. A path of length n will have a list of $n+1$ vertices.

$x = v_0, v_1, v_2, \ldots, v_n = y$

eg:

i) $(v_0, v_1, v_2, v_3, v_5)$ ii) $(v_0, v_1, v_2, v_3, v_4, v_2, v_3, v_5)$

where $k = 0, 1, 2, \ldots, n-1$ with (v_k, v_{k+1}) is an edge.

⇒ A circuit is a path that terminates at its initial vertex. It is otherwise known as cycle

eg: In the above graph (v_2, v_3, v_4, v_2), (or) (v_3, v_4, v_2, v_3) (or) (v_4, v_2, v_3, v_4) (or) in edge series (e_3, e_4, e_5)

⇒ A sub path of the graph is any portion of the path described by one or more consecutive edges in the edge list.

eg: e_1, $e_1 e_2$, $e_1 e_2 e_3$, $e_3 e_6$ etc.,

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=> Any path is its own subpath is an improper subpath of itself. other subpaths are called proper subpaths.

=> A path or circuit is simple if it contains no proper subpaths i.e., A path or circuit is simple if it does not visit any vertex more than once except for the common initial & terminal vertex in the circuit.

eg: i) $(v_0, v_1, v_2, v_3, v_5)$ is a simple path.

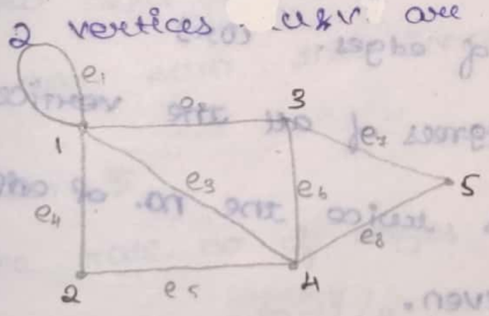
ii) $(v_0, v_1, v_2, v_3, v_4, v_2, v_3, v_5)$ is not simple.

since v_2 & v_3 appeared twice which are neither initial nor terminal.

Incidence:

Let G be a graph and $e \in E$, $e = (u, v)$. Then e is said to be incident with the vertices u & v. Also the vertices u & v are said to be incident with e.

2 vertices u & v are said to be adjacent if (u, v) is an edge of the graph.



vertex 1 is adjacent to 1, 2, 3, 4

" 2 " " 1, 4

" 3 " " 1, 4, 5

" 4 " " 1, 2, 5, 3

" 5 " " 3, 4

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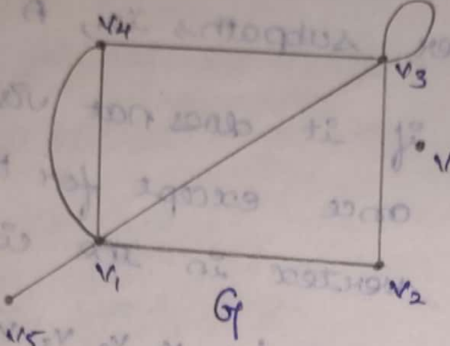
Let v be a vertex in a graph G, then the degree $d_G(v)$ of the vertex v in G is the no. of edges of G that are incident with v where each loop is counted twice.



$d_G(v_1) = 5$; $d_G(v_2) = 2$; $d_G(v_3) = 5$; (self loop counted twice)

$d_G(v_4) = 3$; $d_G(v_5) = 1$; $d_G(v_6) = 0$;

Theorem -1

Hand- Shaking theorem:

Statement: Let G be an undirected graph with ε (epsilon) no. of edges, then $\sum_{v \in V} d(v) = 2\varepsilon$. (or)

$\sum_{i=1}^{e} d(v_i) = 2e$. ie., sum of degrees of all vertices is twice the no. of edges. (or)

The sum of degrees of all the vertices of an undirected graph is twice the no. of edges of the graph and hence even.

Proof: Let G be an undirected graph with ε edges & n vertices $v_1, v_2, \ldots, v_n$.

Since each edge contributes two degrees, to the sum of degrees of all vertices in G, we have $\sum_{i=1}^{n} d(v_i) = 2e$.

Theorem - 2

In an undirected graph the no. of odd degree vertices are even.

Proof: Let V_1, and V_2 be the set of odd degree and even degree vertices in G.

$V_1 = \{v_1, v_2, \dots v_k\}$; $V_2 = \{v_l, v_m, \dots, v_s\}$

It is clear that $V_1 \cup V_2 = V$ & $V_1 \cap V_2 = \phi$. Also

$$\sum_{v \in V} d(v) = \sum_{u \in V_1} d(u) + \sum_{w \in V_2} d(w) \rightarrow (1)$$

By theorem - 1 we have $\sum_{v \in V} d(v) = 2E$

an even number, ie., LHS of equ (1) is an even integer. Also, V_2 consist of all vertices of even degree $\Rightarrow d(w)$ is even for $w \in V_2$.

This implies $\sum_{w \in V_2} d(w)$ is even. This implies $\sum_{u \in V_1} d(u)$ must be even. $\rightarrow (2)$. ie, $\sum_{u \in V_1} d(u) = 2\delta$

Now LHS of (2) is the sum of odd numbers with even sum this implies $|V_1|$ is even.
cardinality.

Theorem - 3

The max. no of edges in a simple graph with 'n' vertices is $n(n-1)/2$

Proof: The proof of this thm is by the Principle of Mathematical Induction.

Basis step: $n=1$

a Graph with 1 vertex has no edges.

$\Rightarrow$ The result is true for $n=1$.

For $n=2 \Rightarrow$ 1 may have atmost 1 edge; $\frac{2(2-1)}{2}=1$.

Assumption step:

Assume that the result is true for $n=k$ ie., A Graph with k vertices has atmost $k(k-1)/2$ edges.

Induction step:

Take $n=k+1$; ie., the graph G has k+1 vertices.

Let G' be the graph obtained from G by deleting one vertex say $v \in V$. Since G' has k vertices by hypothesis G' has atmost $k(k-1)/2$ edges.

Now add the vertex v to G' such that v may be adjacent to all the k vertices of G'.

$\therefore$ The total no. of edges in G are $\frac{k(k-1)}{2}+k$

$$= \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2},$$

Thus the result is true for k+1.

Hence the max. no. of edges in a simple graph with 'n' vertices is $n(n-1)/2$.

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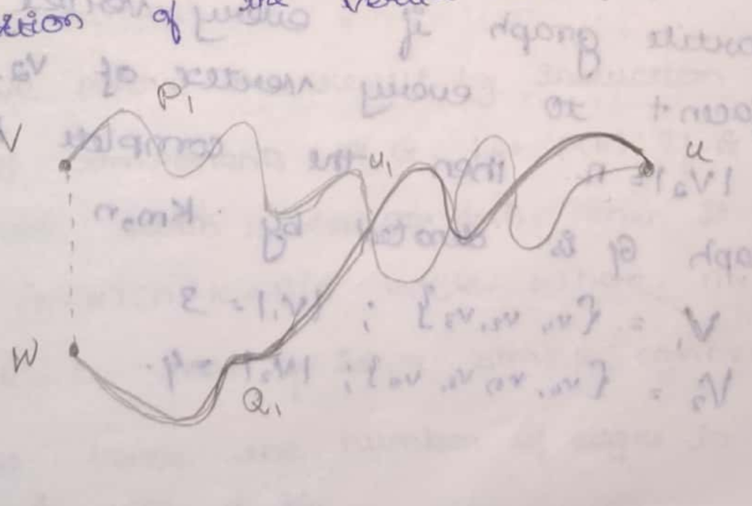
Sufficiency:

It souffices to prove the converse for connected graphs. Let G be a connected graph that contain no odd cycle. choose an arbitary vertex u and define a portion (X, Y) of V by defining

$X = \{v \in V \mid d(u, v) \text{ is even}\}$

$Y = \{v \in V \mid d(u, v) \text{ is odd}\}$

We claim that G is a bipartite ~~bi part~~ graph with bi partition $\{X, Y\}$. Let v and $w \in X$. Let P be a shortes u-v path and Q be a shortest u-w path in G. Let u_1 be the last vertex common to P and Q. Since P & Q are shortes paths (u, u_1) sections of P & Q are shortest u-u_1 paths and therefore have the same length say K. As the length of the path P & Q are even. The length of (u_1, v). Section P_1 of P and (u_1, w). section Q_1 of Q are both either even or odd. If vw were an edge in G, then the cycle $P_1 Q_1^{-1} v$ is a cycle of odd length, contrary to the hypothesis. Hence no two vertices in X are adjacent. My no two vertices in Y are adjacent. Thus (X, Y) is a bipartition of the vertex set and G is bipartite.



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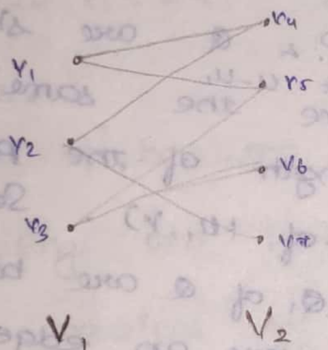
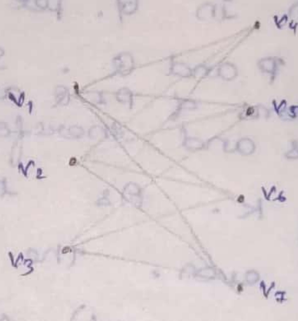
Bipartite Graph:

An undirected Graph G is said to be bipartite if its vertex set V can be partitioned into subsets V_1 & V_2 i.e., $V_1 \cup V_2 = V$ & $V_1 \cap V_2$ = (empty) ϕ such that every edge E(G) has one end in V_1 & the other end in V_2. Such a partition (V_1, V_2) is called a bipartition of G.

Example: Complete binary Graph. $K_{3,4}$





A simple bipartite Graph G with bipartition (V_1, V_2) is said to be a complete bipartite graph if every vertex of V_1 is adjacent to every vertex of V_2. If $|V_1| = m$ & $|V_2| = n$ then the complete bipartite Graph G is denoted by $K_{m,n}$

$V_1 = \{v_1, v_2, v_3\}$; $|V_1| = 3$

$V_2 = \{v_4, v_5, v_6, v_7\}$; $|V_2| = 4$.

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Necessary Part.

Let G be bipartite with bipartition (X, Y).

Let $C = v_0 v_1 \dots v_{k-1} v_k$ where $v_k = v_0$ be a cycle in G. Assume the $v_0 \in X$. Then as $v_0 v_1$ is an edge & G is bipartite $v_1 \in Y$. As $v_1 \in Y$ & $v_1 v_2$ is as edge we have $v_2 \in X$.

Proceeding like this, we have $v_{2i} \in X$ & $v_{2i+1} \in Y$. $\Rightarrow v_k \in X$ gives C is an even cycle. Thus G contains no odd cycles.

Let u & v be distinct vertices in a Graph G. If there is a $u-v$ path in G then a $u-v$ path of minimum length is called Geodesic. The length of a geodesic b/w u & v is called the distance b/w u & v & it is denoted by $d(u, v)$

Theorem - 5

A simple graph with n vertices and k components can have atmost $(n-k)(n-k+1)/2$ edges.

Proof: We prove the result by induction on the number of components of G. Let $P(k)$: If G is a simple graph with k components, then it can have atmost $(n-k)(n-k+1)/2$ edges. Where $n = |V(G)|$.

If $k = 1$, then G is a simple connected graph and hence the number of edges in $G \leq$ number of edges of K_n

$\leq (n-1)n/2$

where $n = |V(G)|$ and K_n is the complete graph of n vertices.

Thus $p(1)$ is true $\longrightarrow$ ①

Assume that $p(m)$ is true, for some m. $\longrightarrow$ ②.

Let G be a simple graph with n vertices and $(m+1)$ components. Let H_1 be a component of G. Let $|V(H_1)| = n_1$. As G has m remaining components and each component has atleast one vertex, we have $n_1 \leq n-m$. Let H_2 be the subgraph of G induced by $V(G) - V(H_1)$. Then H_2 is a simple graph with $n-n_1$ vertices and m components, and by (2), H_2 can have atmost $(n-n_1-m)(n-n_1-m+1)/2$ edges. As H_1 is a connected simple graph with n vertices, by (1), it can have atmost $(n_1-1)n_1/2$ edges. Thus the number of edges in $G \leq \frac{n_1(n_1-1)}{2} + (n-m-n_1)(n-m+1-n_1)/2$

$= \frac{1}{2}[n_1^2 - n_1 + (n-m)(n-m+1) - n_1(n-m+1+n-m) + n_1^2]$

$= \frac{1}{2}[(n-m)(n-m-1+2) - 2n_1(n-m) - 2n_1 + 2n_1^2]$

$= \frac{1}{2}[(n-m)(n-m+1) - 2(n-m)(n_1-1) + 2n_1(n_1-1)]$

$\leq \frac{1}{2}(n-m)(n-m-1)$ as $n_1 \leq n-m$.

Thus from ① & ②, $P(m+1)$ is also true.

By induction principal, $P(K)$ is true for all positive integers K.

Worked Examples:

1) Let G be a graph and u and v be two distinct vertices of G. Show that if there is a u-v path walk in G, then there is also a u-v path in G.

Solution:

It is given that there is a u-v walk in G. Among all u-v walks in G, find one walk with least length. Let $ux_1x_2\ldots x_mv$ be a u-v walk with least length. We claim that the vertices $u, x_1, x_2, \ldots, x_m$ and v are all distinct. It is given that $u \neq v$. If $u = x_i$ for some i, $1 \leq i \leq m$, then $u = x_i x_{i+1} \ldots x_m v$ is a u-v walk of length $\leq m$ which is a contradiction. So $u \neq x_i$, for all $i = 1, 2, \ldots, m$. If $x_i = x_j$ for some $1 \leq j \leq i \leq m$, then $ux_1 \ldots x_i x_{j+1} \ldots x_m v$ is a u-v walk with length $\leq m$, again it is a contradiction. If $x_i = v$ for some i, then $ux_1 \ldots x_i$ is a u-v walk, leading to a contradiction. Thus the vertices $u, x_1, \ldots, x_m, v$ are all distinct and hence $ux_1x_2\ldots x_mv$ is a u-v path.

Remark: The following result can be proved similarly:

If u and v are two distinct vertices of a digraph G, and if there is a u-v directed walk in G, then there is a u-v directed path in G.

5) If a graph has n vertices and a vertex v is connected to a vertex w, then there exists a path from v to w of length no more than $(n-1)$.

Solution:

Let $vu_1u_2\ldots u_{m-1}w$ be a path in G from v to w. By the definition of the path, the vertices $v, u_1, u_2, \ldots, u_{m-1}$ and w are all distinct. As G

contains only n vertices, it follows that.
$m+1 \leq n$. i.e., $m \leq n-1$.

6) show that in a simple diagram $G = \langle V, E \rangle$, every node of the digraph lies in exactly one strong component.

Solution :

Let G be a simple digraph. Define a relation 'disconnection' on $V(G)$ as follows:

$u, v \in V$ then u and v are disconnected if each is reachable from the other.

This relation is an equivalence relation on V and we get a partition $V_1, V_2, \dots V_m$ of V. The subdigraphs $G[V_1], G[V_2], \dots, G[V_m]$ are the discomponents (strong components) of G. If V is a vertex (node) in G, then $v \in V_i$ for exactly one i and $v \in G[V_i]$. Thus every vertex (node) of G lies exactly one strong component.

7) Prove that a simple graph with n vertices must be connected if it has more than $(n-1)(n-2)/2$ edges

Solution : (solution is similar to proof of Theorem 5)

Let G be a simple graph with more than $(n-1)(n-2)/2$ edges. Assume that G is not connected. Select any one of the connected component of G. Let V_1 be the vertex set of that component. Take $V_2 = V(G) - V_1$ and $m = |V_1|$.

Then

(i) $1 \leq m \leq n-1$

(ii) There is no edge joining a vertex of V_1 and a vertex of V_2 and

(iii) $|V_2| = n - m \geq 1$.

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So $|E(G)| = |E(G[V_1])| + |E(G[V_2])|$

$$\le \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$

$$= \frac{1}{2}[m(m-1) + (n-m)(n-m-1)]$$

$$= 1/2\,[(n-m)(n-m-1) + m^2 - m]$$

$$= 1/2\,[n(n-1) - nm - m(n-m-1) + m^2 - m]$$

$$= 1/2\,[(n-1)(n-2) + 2(n-1) - 2nm + m^2 + m + m^2 - m]$$

$$= 1/2\,[(n-1)(n-2) + 2n - 2 - 2nm + 2m^2]$$

$$= 1/2\,[(n-1)(n-2) + 2n(1-m) + 2(m^2-1)]$$

$$= 1/2\,[(n-1)(n-2) + 2(1-m)\{n-1-m\}]$$

$$= 1/2\,[(n-1)(n-2) - 2(m-1)(n-m-1)]$$

$\le 1/2\,(n-1)(n-2)$ since $(m-1)(n-m-1) \ge 0$ for $1 \le m \le n-1$

which is a contradiction as G has more than $(n-1)(n-2)/2$ edges. Hence G is connected.

8) Let G be a simple graph and the minimum degree $\delta(G) \ge 2$. Then G contains a cycle of length $\ge \delta + 1$

solution :

Let G be a simple graph and $\delta(G) \ge 2$.

Let $P: u_0 u_1 \dots u_m$ be a longest path in G. We claim that the length of this path $= m \ge \delta(G)$. Suppose $m < \delta(G)$. As $\deg(u_m) \ge \delta(G)$, there is a vertex $v \in V(G)$ such that $u_m v$ is an edge and $v \ne u_i$, for all $i = 0, 1, 2, \dots, m-1$. Now $u_0 u_1 \dots u_m v$ is a path of length $m+1$, which is a contradiction. Thus $m \ge \delta(G)$. Now as P is a longest path, u_0 is not adjacent to any vertex in $V(G) - \{u_1, u_2, \ldots, u_m\}$. (if u_0 is adjacent to a vertex w, where $w \neq u_i$ $i = 1, 2, \ldots, m$, then we get a new path $w u_0 u_1 \ldots u_m$ of length $> m$, which is a contradiction). As $\deg(u_0) \geq \delta(G) \geq 2$ and as $u_0 v$ is an edge implies $v = \{u_1, u_2, \ldots, u_m\}$, it follows that $u_0 u_k$ is an edge for some $k \geq \delta(G)$. Now $u_0 u_1 \ldots u_k u_0$ is a cycle of length $k+1 \geq \delta(G)+1$.

9) Let G be a simple graph with n vertices. Show that if $\delta(G) \geq [n/2]$, then G is connected.

Proof:

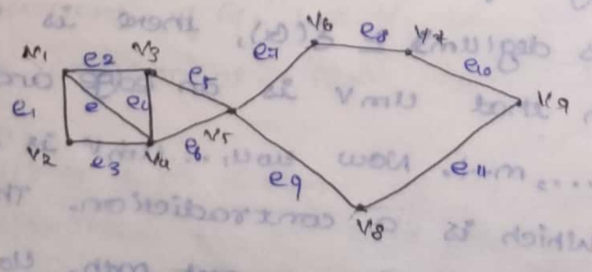
Let u and v be two distinct vertices in G, we claim that there is a u-v path in G. If uv is an edge in G, then it is a u-v path. Assume that uv is not an edge in G. Let A be the set of all vertices which are adjacent to u and B be the set of all vertices which are adjacent to v. Then $u, v \notin A \cup B$, and hence $|A \cup B| \leq n-2$.

As now $|A| = \deg(u) \geq \delta(G) \geq [n/2]$. Similarly $|B| \geq [n/2]$.

Hence we get $|A| + |B| \geq n-1$. Now from $|A \cup B| + |A \cap B| = |A| + |B|$, it follows that $|A \cap B| \geq 1$.

So $A \cap B \neq \phi$. Take a vertex $w \in A \cap B$. Then uwv is a u-v path in G. Thus for every pair of distinct vertices u, v, there is a u-v path in G. In otherwords, G is connected.

24.08.'17



This contains odd cycles hence it is not bipartite

Geodesic:

$d(v_1, v_2) = 1$ $d(v_1, v_6) = 3$

$d(v_1, v_3) = 1$ $d(v_1, v_7) = 4$

$d(v_1, v_4) = 1$ $d(v_1, v_8) = 3$

$d(v_1, v_5) = 2$ $d(v_1, v_9) = 4$

Theorem: 4

Let G be an undirected graph, G is bipartite if and only if it contains no odd cycles.

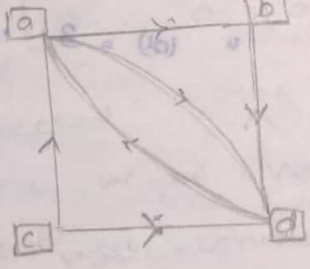
[Necessary part & sufficiency]

28.8.17

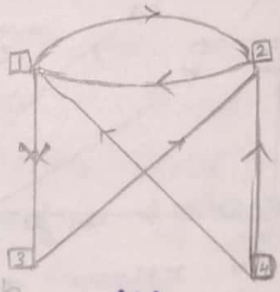
Isomorphic Graphs:

Two graph (V, E) and (V', E') are isomorphic if there exist a Bijunction $f: V \to V'$ such that $(v_i, v_j) \in E$ implies $(f(v_i), f(v_j)) \in E'$. For multigraphs the no. of edges connecting v_i to v_j must equal to the no. of edges from $f(v_i)$ to $f(v_j)$.

Example:



(a)



(b)

$f(a) = 1$

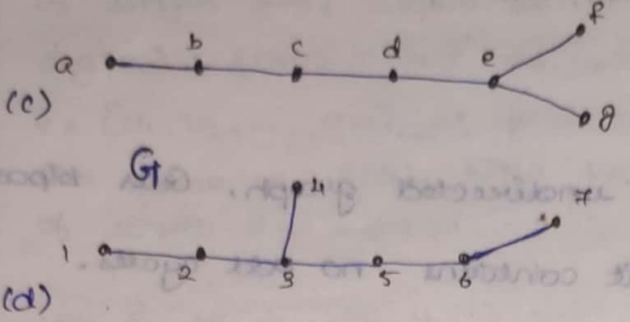
$f(b) = 3$

$f(c) = 4$

$f(d) = 2$

(we will consider adjacent, indegree & outdegree)

NOTE: The position of vertices are not essential.



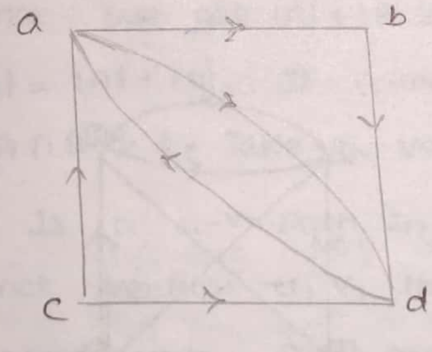
In this graph adjacency is not preserved, Bcoz for e there is 2 - onedegree f, g & 1 - 2degree d, But for 3, we have only one - onedegree 4 & 2 - two degree 2, 5.

Hence this is not isomorphic

Outdegree & Indegree:

If v is a vertex of a directed graph then the outdegree of v denoted outdeg(v) is the no. of edges of the graph that initiate at v.

The Indegree of v denoted indeg(v) is the no. of edges that terminate at v.



indeg (a) = 2 outdeg(a) = 2
" (b) = 1 " (b) = 1
" (c) = 0 " (c) = 2
" (d) = 3 " (d) = 1

Graph Sequence:

A finite non - increasing sequence of integers $d_1, d_2, \dots, d_n$ is a graphic if there exist a simple graph with n vertices having the sequence as its degree sequence.

Example.

Degree sequence of fig (c) & (d)

3, 2, 2, 2, 1, 1, 1. is graphic.

This is the Graph sequence of the graph (c)(d).

Bcoz it matches with the graph (c)&(d).

Adjacency Matrix:

The adjacency matrix of a graph G is $a_{ij} = 1$ if and only if vertex i is connected to vertex j in the graph.

adjacency matrix for graph (c)

$$\begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ c & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ e & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ g & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Sum of these columns is the degree of that particular vertex.

29.8.17 Connected

Let V & W be vertices of a directed Graph. vertex V is connected to vertex W if there is a path from V to W.

2 vertices are strongly connected if they are connected in both directions to one another.

→ A Graph is connected if for each pair of distinct vertices V & W either V is connected to W or W is connected to V.

=> A Graph is strongly connected if every pair of its vertices is strongly connected.

Eg: Graph (a) is connected. It is not strongly connected.

a & b (and) a & d are strong connected vertices.

c is connected to a, b, c but a is not connect to c, b is not connected to c.

Theorem 9.3.1:

If a graph has 'n' vertices and vertex u is connected to vertex w, then there exist a path from u to w of length no more than n.

Proof: suppose u is connected to w, but the shortest path from u to w has length 'm' where m > n. A vertex list for a path of length m will have m+1 vertices. This path can be represented as $u = v_0, v_1, v_2, \ldots, v_m$, ~~$v_{m+1}$~~ $= w$. Since there are only n vertices in the graph, and m vertices are listed in the path after v_0, by pigeon hole principle there must be some duplication in m vertices of the vertex list, implies there is a circuit in the path. Hence the minimum length can be reduced, which is a contradiction to our assumption, where the path from u - w is shortest.

Hence the theorem.

Methods for testing connectivity:-

1) adjacency matrix method
2) broadcasting method.

Adjacency matrix method: suppose there the information about edges in a graph is in the form of adjacency matrix G. Define a relation 'r' on G as $v \, r \, w$ if there is an edge connecting v to w. The composition of r with itself is r^2 & it is defined by $v r^2 w$ if there exist a vertex y such that $v r y$ & $y r w$ i.e., v is connected to w by a path of length 2. By induction r^k, $k \geq 1$ is defined by $v r^k w$ if & only if there is a path of length k from v to w.

194 - 9.4.1

195 - 9.4.2

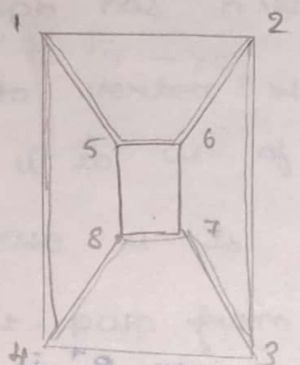
Since the transitive closure R^+ is the union of $R, R^2, R^3, \ldots$ it is evident that the transitive closure of r ascertains the existence of the path from v to w.

NOTE: The advantage of the adjacency matrix method is that the transitive closure matrix gives the existence of path b/w all the vertices. The disadvantage is that this method tells us whether a path exist but not what the path is.

Hamiltonian Graph:

A Hamiltonian path through a graph is a path whose vertex list contains each vertex of the graph exactly ones, except if the path is a circuit in which case the initial vertex appears a 2^{nd} time as the terminal vertex. If the path is a circuit then it is called a Hamiltonian circuit. A HG is a graph that posses a Hamiltonian path.

It is of exponential order



It includes all vertices but not all edges.

1 5 6 2 3 7 8 4 1 is a Hamiltonian circuit.

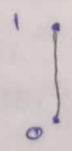
1 5 6 2 3 7 8 4 is a H-Path.

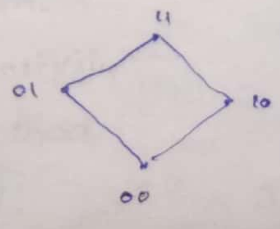
n-cube:

Let $n \geq 1$ and Let B^n be the set of strings of zero's & one's with length n. The n-cube is the undirected graph with a vertex for each string in B^n & an edge connecting each pair of strings that differ in exactly one position.

Example: $\{0, 1\} \Rightarrow$ 1-cube

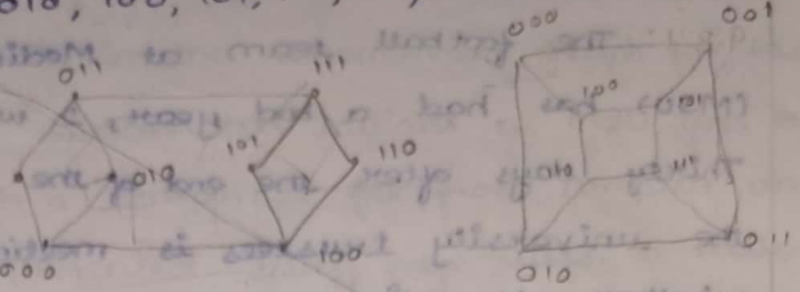
$\{00, 01, 10, 11\}$
2-cube





3 - cube

{000, 001, 010, 100, 101, 110, 111, 011}



Gray code :

A Hamaltonian circuit of the n-cube can be described recursively. The circuit is called the Gray code which is the Hamatonian circuit of the n-cube.

The standard way to write the gray code is as a column of strings were the last string is followed by the first string to compute the circuit.

Basis : $n = 1$

The gray code for 1-cube is $G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad G_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1100 \\ 1101 \\ 1111 \\ 1110 \\ 1010 \\ 1011 \\ 1001 \\ 1000 \end{bmatrix}$$

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Broadcasting:

9.3.1: The football team at Mediocre State University (MSU) has had a bad year, 2 wins and 9 losses. Thirty days after the end of the football season, the university trustees is meeting to decide whether to rehire the head coach; things look bad for him. However, on the day of the meeting, the coach releases the following list of results from the past year:

Mediocre State defeated Local A&M.

Local A&M defeated City College.

City College defeated Corn State U.

... (25 results later)

Tough Tech defeated Enormous State University (ESU)

And ESU went on to win the national championship!

The trustees were so impressed that they rehired the coach with a raise in pay! How did the coach come up with such a list?

In reality, such lists exist occasionally and appear in newspapers from time to time. of course they really don't prove anything since each team that defeated MSU in our example above can produce a similar chain of results. Since College football records are readily available, the coach could have found this list by trail & error. All that he needed to start with was that his team won at least one game. Since ESU lost one game, there was some hope of producing the chain.

The problem of finding this list is equivalent to finding a path in the tournament graph for last year's football season that initiates at MSU and ends at ESU. Such a graph is far from complete and would be represented using edge lists. To make the coach's problem interesting, let's imagine that only the winner of a game members the result of the game. The coach's problem has now taken on the flavor of a maze. To reach ESU, he must communicate with the various teams along the path. One way that the coach could have discovered his list in time is by sending the emails to the coaches of the two teams that MSU defeated during the season.

Observations: From the conditions of this message, it should be clear that if everyone cooperates and if coaches participate within a day of receiving the message:

Algorithm 9.3.1 A broadcasting algorithm for finding a path b/w vertex i and j of a graph having n vertices. The each item V_k of a list $V = \{V_1, V_2, \ldots V_n\}$, consist of a Boolean field V_k. found and an integer field V_k. from. The sets $D_1, D_2, \ldots,$ called depth sets, have the property that if k is in D_r, then the shortest path from vertex i to vertex k is of length r. In step 5, a stack is used to put the vertex list for the path from the vertex i to j in proper order.

1. Set the value V_k. found equal to false, $k = 1, 2, \ldots, n$
2. $r = 0$
3. $D_0 = \{i\}$
4. While $(\neg V_j.\text{found})$ and $(D_r \neq \emptyset)$
   - 4.1 $D_{r+1} = \emptyset$
   - 4.2 For each k in D_r do
     - For each edge (k, t) do
       - If V_t. found = = False
         - then V_t. found = True

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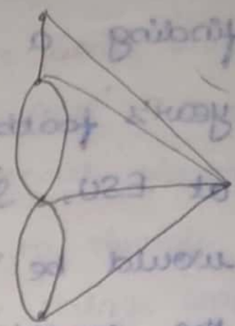
$V_t.from = k$

$D_{r+1} = D_{r+1} \cup \{t\}$



4.3 $r = r+1$

5. If $V_j.found$ = True then

5.1 S = Empty Stack

5.2 $k = j$

5.3 While $V_k.from \neq i$

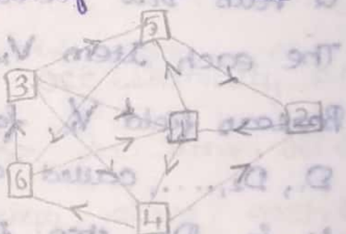
5.3.1 push k onto S

5.3.2 $k = V_k.from$

Example 9.3.2

Consider the graph in fig. The existence of a path from vertex 2 to 3 is not difficult to determine by examination. After a few sec, you should be able to find two paths of length 4.

Suppose that the edges from each vertex are stored in ascending order by terminal vertex. For eg, the edges from vertex 3 would be in the order (3,1), (3,4), (3,5). In addition, assume that in the body of step 4 of the algorithm, the elements of D_r are used in ascending order. Then at the end of step 4, the value of V will be



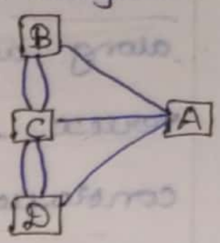
k	1	2	3	4	5	6
$V_k.found$	T	T	T	T	T	T
$V_k.from$	2	4	6	1	1	4
Depth set	1	3	4	2	2	3

(value of r for which $k \in D_r$)

Therefore, the path (2, 1, 4, 6, 3) is produced by the algorithm. Note that if we wanted a path from 2 to 5, the information in V produces the path (2, 1, 5) since $V_k.from = 1$ & $V_1.from = 2$. A shortest circuit that initiates at vertex 2 is also available by noting that $V_2.from = 4$, $V_4.from = 1$ & $V_1.from = 2$; thus the circuit (2, 1, 4, 2) is obtained.

Theorem 9.4.1: Euler's Theorem. Konigsberg case:

No walking tour of Konigsberg can be designed so that each bridge is used exactly once.



Proof: The map of Konigsberg can be represented as an undirected multigraph. The four land masses are the vertices and each edge represents a bridge. The desired tour is then a path that uses each edge once and only once. Since the path can start and end at 2 different vertices, there are two remaining vertices that must be intermediate vertices in the path. If x is an intermediate vertex, then every time that you visit x, you must use two of its incident edges, one to enter and one to exit. Therefore, there must be an even number of edges connecting x to the other vertices. Since every vertex in the konigsberg graph has an odd no. of edges, no tour of the type that is desired is possible.

As, is typical of most mathematicians, Euler wasn't satisfied.

Definitions: Eulerian Paths, Circuits, Graphs: A Eulerian path through a graph is a path whose edge list contains each edge of the graph exactly once. If the path is a Circuit, then it is called as Eulerian Circuit. A Eulerian graph is a graph that possesses a Eulerian Path.

Theorem 9.4.2: Euler's theorem - General Case.

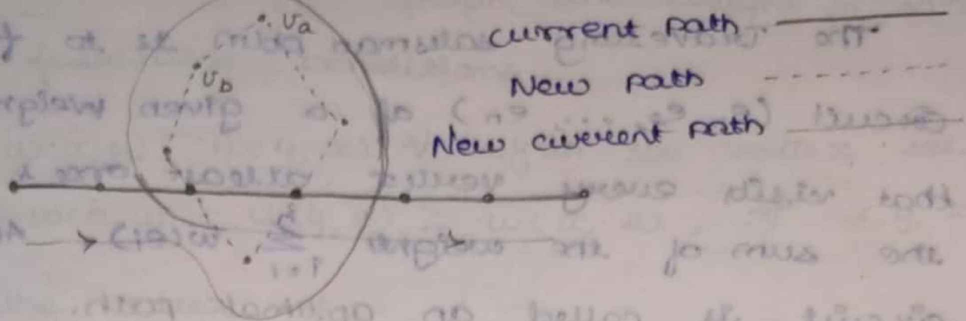
An undirected graph is Eulerian if and only if it is connected and has either zero or 2 vertices with an odd degree. If no vertex has an odd degree, then the graph has a Eulerian Circuit.

Proof: It can be proven by induction that the no. of vertices in an undirected graph that have an odd degree must be even. We will leave the proof of

Let K be the no. of vertices with odd degree.

Phase 1. If $k=0$, start at any vertex, v_0 and travel along any path, not using any edge twice. Since each vertex has an even degree, this path can always be continued past each vertex that you reach except v_0. The result is a circuit that includes v_0. If $k=2$, let v_0 be either one of the vertices of odd degree. Trace any path starting at v_0 using up edges until you can go no further, as in the $k=0$ case. This time, the path that you obtain must end at the other vertex of odd degree that we will call v_1. At the end of Phase 1, we have an initial path that may or may not be Eulerian. If it is not Eulerian, Phase 2 can be repeated until all of the edges have been used. Since the no. of unused edges is decreased in any use of phase 2, a Eulerian path must be obtained in a finite no. of steps.

Phase 2. As we enter this phase, we have constructed a path that uses a proper subset of the edges in our graph. We will refer to this path as the current path. Let V be the vertices of our graph, E the edges, & E_u the edges that have been used in the current path. Consider the graph $G' = (V, E - E_u)$. Note that every vertex in G' has an even degree. Select any edge, e, from G'. Let v_a & v_b be the vertices that e connects. Once we complete the circuit that is the new path, we resume the traversal of the current path.



Augmenting the current path in proof of Th. 9.4.2

Gray code:

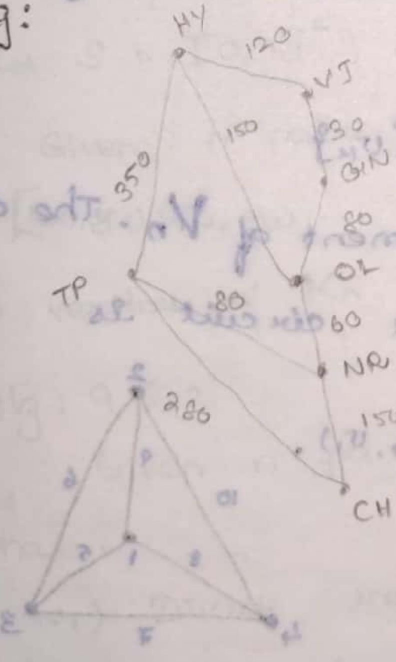
Recursively G_{n+1} can be obtained for the n^3, $n \geq 1$ by
1) Listing G_n with each string prefix with 0.
2) Reversing the list of strings in G_n with each string prefixed with 1.

ie, $G_{n+1} = \begin{pmatrix} 0 & G_n \\ 1 & G_n^r \end{pmatrix}$ where G_n^r is the reverse of the list G_n

Graph optimization:-

weighted Graph: A weighted graph (V, E, w) is a graph (V, E) together with a weight function w from E to $\mathbb{R}$ ie, $w: E \to \mathbb{R}$. If $e \in E$, $w(e)$ is the weight on edge e.

eg:



	HY	VJ	GIN	OL	NR	TP	CH
HY	-	120	150	150	210	350	360
VJ	120	-	30	110	170	250	320
GN	150	30	-	80	140	220	290
OL	150	110	80	-	60	140	210
NR	210	170	140	60	-	80	150
TP	350	250	220	140	80	-	280
CH	360	320	290	210	150	280	-

Travelling Salesman Problem:

The Travelling salesman pblm is to find a circuit $(e_1, e_2, \ldots, e_n)$ of a given weighted graph that visits every vertex atleast once & minimizes the sum of the weights $\sum_{i=1}^{n} w(e_i)$. Any such circuit is called an optimal path.

Algorithm 9.5.1

The closest Neighbour alg. $G = (V, E, w)$ be a complete weighted graph with $|V| = n$. The closest neighbour circuit through G starting at vertex v_1 is $(v_1, v_2, \ldots, v_n)$ defined by the following steps

1. $V_1 = V - \{v_1\}$
2. For $k = 2$ to $n-1$

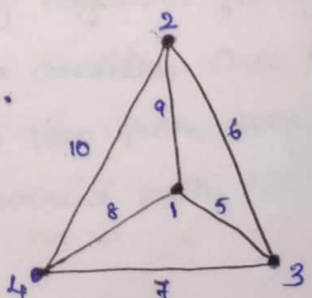
   2.1 v_k = the closest vertex in V_{k-1} to v_{k-1}, i.e., $w(v_{k-1}, v_k) = \min\{(v_{k-1}, v) : v \in V\}$

   In case of a Tie for closest v_k may be chosen arbitarily.

   2.2 $V_k = V_{k-1} - \{v_k\}$

3. v_n = the only element of V_n. The cost of the closest neighbour circuit is

$$\sum_{k=1}^{n-1} w(v_k, v_{k+1}) + w(v_n, v_1).$$



4.9.17

For the complete graph, the weight fn. satisfies the following conditions.

1) $w(x,y) = w(y,x)$, $\forall$ x,y in the vertex set.

2) $w(x,y) + w(y,z) \geq w(x,z)$, $\forall$ x,y,z in the vertex set.

The 1st condition is called the symmetric condition
" 2nd " " " the triangle inequality

proof:- 105-109 LTU

Theorem 9.5.1:

If (V,E,w) is the complete weighted graph that satisfies the symmetry and triangle inequality conditions then $\frac{C_{cn}}{C_{opt}} \leq \frac{\lceil \log_2(2n) \rceil}{2}$, where C_{opt} is the cause of optimal circuit in a graph & C_{cn} is the closest neighbour circuit in a graph.

Algorithm 9.5.2: The Strip Algorithm :-

Unit Square - Let $[0,1] = \{x : 0 \leq x \leq 1\}$, and Let $S = [0,1]^2$, the unit square.

Given : n pairs of real numbers $[(x_1,y_1)(x_2,y_2)\ldots(x_n,y_n)]$ in S that represent the n vertices of K_n. Find a circuit of the graph

Alg: 9.5.2

Given n pts in the unit square.

Phase - 1 :

1.1) Divide the square into $\lceil \sqrt{n/2} \rceil$ vertical strips. Let d be the width of each strip.

If a pt lines on a boundary b/w 2 strips consider it as a part of the left hand strip.

1.2) starting from the left find the 1^{st} strip that contains one of the pts. Locate the starting pt by selecting the 1^{st} pt, that is encountered in that strip as you travel from bottom to top. Assume that the 1^{st} pt is (x_1, y_1)

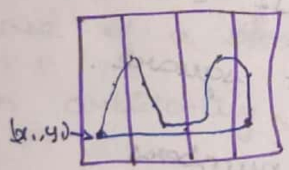
1.3) Alternate traveling up & down the strips that contain vertices until all of the vertices have been reached

1.4) Return to the starting point.

Phase 2:

2.1) shift all strips $d/2$ units to the right creating a small strip on the left.

2.2) Repeat steps 1.2 - 1.4 of phase 1, with the new strips.



Phase- I

$d/2$

phase - II

When 2 phases are complete choose the shorter of the 2 circuits obtained

NOTE: Within a strip the order in which we visit the pts is determined as follows.

(x_i, y_i) preceeds (x_j, y_j) if $y_i < y_j$ or if $y_i = y_j$ & $x_i < x_j$, In traveling down a strip replace $y_i < y_j$ with $y_i > y_j$

Networks And Maximum Flow Pblm:

Network: It is a simple weighted directed graph that contains 2 distinguished vertices called the source and the sink with the property that the indegree of the source & the outdegree of sink are both 0.

The max. rate of flow through a pipe or a wire/cable or any medium is called its capacity, and is the information that the weight fn. of a network contains

The max. flow of pblm is derived from the objective of moving the material or information from the source to the signal.

Define a flow as a fn. f from ie, $f: E \to R$ such that

1) flow of material through any edge is non negative & no larger than its capacity $0 \le f(e) \le w(e), \forall e \in E$.

2) for each vertex other than source & sink the total amount of material that is directed into a vertex is equal to the total amount that is directed out. ie, flow into V = flow out of V.

$$\sum_{(x,v)\in E} f(x,v) = \sum_{(v,y)\in E} f(v,y) \to ①$$

$$\sum_{(x,v)\in E} f(x,v) = \sum_{(v,y)\in E} f(v-y)$$

5.9.17 Theorem 9.5.2

statement: If f is a flow then $\sum_{(source,\, v)\in E} f(source, v)$

$$= \sum_{(v,\, sink)\in E} f(v, sink)$$

Proof: For any flow fn. f from E to R we have $0 \le f(e) \le w(e)$ $\forall e \in E$ & for each vertex other than the source & sink $\sum_{(x,v)\in E} f(x,v) = \sum_{(v,y)\in E} f(v,y) \to ①$

This imply flow into v = flow out of v.

$\Rightarrow$ flow into v − flow out of $v = 0$

Summing up these differences for each vertex in $V' = V - \{source, sink\}$ we have

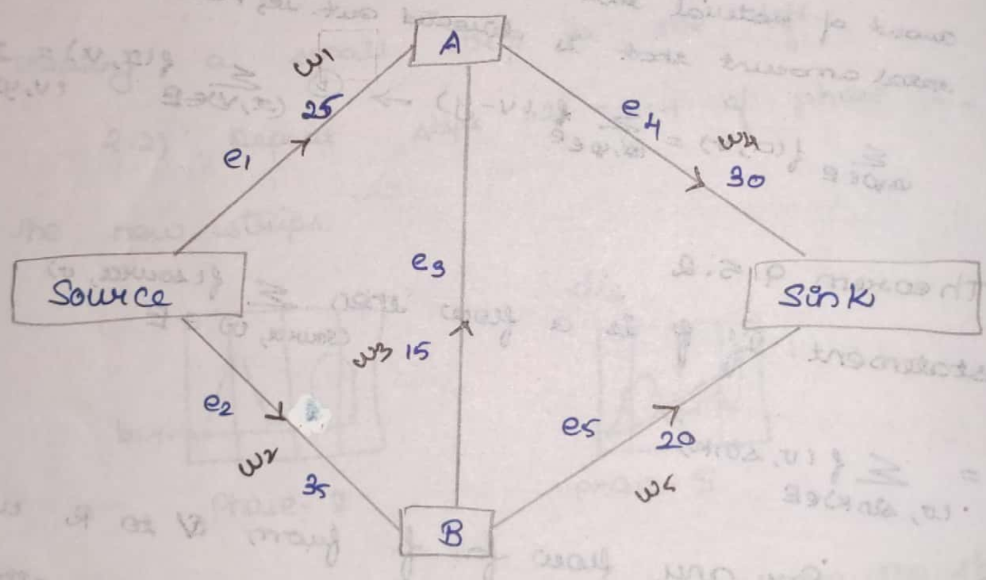
$$\sum_{v \in V'} \left(\sum_{(x,v) \in E} f(x,v) - \sum_{(v,y) \in E} f(v,y) \right) = 0 \quad \to (2)$$

Now if an edge connects 2 vertices in V its flow appears as both a +ve & a -ve term in (2). The only +ve terms that are not cancelled out are the flows into sink. The only -ve terms that remain are the flows out of the source. Therefore

$$\sum_{(source, v) \in E} f(source, v) - \sum_{(v, sink) \in E} f(v, sink) = 0$$

Maximal Flows:

Example: Flow Diagram:



For the above network consider a flow f_1 defined by $f_1(e_1) = 25$, $f_1(e_2) = 20$, $f_1(e_3) = 0$, $f_1(e_4) = 25$, $f_1(e_5) = 20$

Flow Augmenting Path: Given a flow f in a network (V, E), a flow augmenting path with respect to f is a simple path from the source to the sink using edges both in their forward & their reverse directions such that for each edge e in the path $w(e)-f(e)>0$ if e is used in its forward direction and $f(e)>0$ if e is used in the reverse direction. were $w(e)$ is the capacity of w.

$w(e_1)-f_1(e_1)=0$; $w(e_2)-f_1(e_2)=15>0$;

$w(e_3)-f_1(e_3)=15>0$; $w(e_4)-f_1(e_4)=5>0$;

$w(e_5)-f_1(e_5)=0$ ∴ augmenting path is (e_2, e_3, e_4).

These the differences represent unused capacities. The smallest value represent (difference) the amount of flow that can be added to each edge in the path. ie., $f_2(e_2)=f_1(e_1)+5=25$; $f_2(e_3)=f_1(e_3)+5=5$; $f_2(e_4)=25+5=30$. and $f_2(e_1)$ & $f_2(e_5)$ are left as it is as 25, 20 respectively.

Maximum flow = 50 ie., total inflow at the sink = 30 + 20 = 50.

9.5.3 $f_2=f_1+$ min. distance for edges in the path &

206 $f_2=f_1$ for other edges.

Other Graph Optimization Problems.

1) The minimum spanning tree pblm.

Given a weighted graph (V, E, w) find a subset E' of E with the properties that (V, E') is connected & the sum of the weights of edges in E' are as small as possible.

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Algorithm 9.5.3:

The Ford & Fulkerson Algorithm.

1) Define the flow function f_0 by $f_0(e) = 0$ for each edge $e \in E$

2) $i = 0$

3) Repeat:

3.1) If possible, find a flow augmenting path with respect to f_i

3.2) If a flow augmenting path exists, then:

3.2.1) Determine

$$d = \min\{\{w(e) - f_i(e) \mid e \text{ is used in forward direction}\}$$
$$\{f_i(e) \mid e \text{ is used in reverse direction}\}\}$$

3.2.2) Define f_{i+1} by

$f_{i+1}(e) = f_i(e)$ if e is not part of the flow augmenting path.

$f_{i+1}(e) = f_i(e) + d$ if e is used in forward direction

$f_{i+1}(e) = f_i(e) - d$ if e is used in reverse direction

3.2.3) $i = i + 1$

until no flow augmenting path exists.

4) Terminate with a maximal flow f_i.

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Depth-First Search for the Sink Initiating at the Source:

Let E' be the set of directed edges that can be used in producing a flow augmenting path. Add to the network a vertex called start & the edge.

1) S = vertex set of the network.

2) P = start.

3) P = source (* Move P along the edge (start, source)*)

4) While P is not equal to start or sink do

If an edge E' exists, that takes you from P to another vertex in S

then set P to be that next vertex & delete the edge from E'.

else reassign P to be the vertex that P was reached from (ie, backtracks).

5) If P = start,

then no flow augmenting path exists.

else P = sink, you have found a flow augmenting path.

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7.9.'17

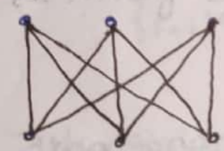
Other Graph Optimization pblm :-

2) The minimum matching pblm: Given an undirected weighted graph (V, E, w) with an even no. of vertices where up the vertices so that each pair is connected by an edge & ~~some~~ sum of these edges is as small as possible

3) Graph Centre pblm: Given a connected undirected weighted graph. Find a vertex called the centre in the graph, with the property that the dis. from the centre to every other vertex is as small as possible (ie, either as minimizing the sum of the distances to each vertex or as " the max. dis. from the centre to a vertex.)

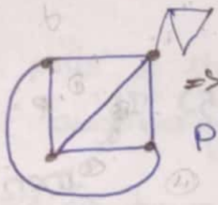
Planarity & Colouring:

A graph is planar if it can be drawn in a plane so that no edges cross.



=> eg. for Non-planar

3-utility Puzzle (pg-210, 211)



=> eg. for Planar graph.

4 regions $r = 4$ $v = 4$ $e = 6$

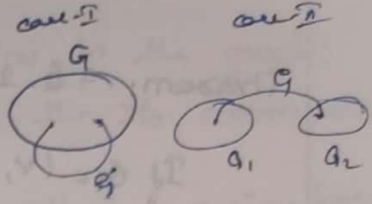
(211) Theorem : 9.6.1 Euler's Pblm :

If $G = (V, E)$ is a connected planar graph with r regions, v vertices & e edges then $(v + r - e) = 2$ for any such graph.

Proof: Euler's formula is proved by induction on e for $e \geq 0$

Basis : If $e = 0$ then G is a graph with 1 vertex & 1 region => ie, there is 1 infinite region $(r = 1)$ $(v = 1)$

$\therefore v + r - e = 1 + 1 - 0 = 2.$



Hence the Basis.

Assumption:

Induction: suppose there G has k edges, $k \geq 1$ & all connected planar graphs with < k edges satisfies $v + r - e = 2$.

Select any edge ie, part of the boundary of the infinite region & call it e_1. Let G' be the graph obtained from G by deleting e_1. Then we have 2 diff possibilities.

i) G' is connected. (ii) G' has 2 connected components G_1 & G_2.

case - I: if G' is connected the induction hypothesis can be applied to it. If G' has v'-vertices, r'-regions & e'-edges then by assumption step we have

$v' + r' - e' = 2$. $\therefore$ For G $v' = v$ (since no vertices were removed from G'). $r' = r - 1$ (1 region of G merged with the infinite region when e_1 is removed) & $e' = k - 1$ since G had k edges & e_1 is removed.

$\therefore v + r - e = v + r - k = v' + r' + 1 - k' - 1$

$\Rightarrow v' + r' - e' = 2$ (by assumption step).

case : 2 $\Rightarrow$ If G' is not connected it must consist of two connected components G_1 & G_2. Let G_1 has v_1 vertices, e_1 edges & r_1 regions with $v_1 + r_1 - e_1 = 2$ & Let G_2 has v_2-vertices, r_2-regions & e_2-edges with $v_2 + r_2 - e_2 = 2$; $v = v_1 + v_2$; $e = e_1 + e_2 + 1$

$r = r_1 + r_2 - 1$.

$v + r - e = v_1 + v_2 + r_1 + r_2 - 1 - e_1 - e_2 - 1$

$= (v_1 + r_1 - e_1) + (v_2 + r_2 - e_2) - 1 - 1$

$= 2 + 2 - 2$

$\therefore v + r - e = 2$ //

Theorem: 9.6.2:

If $G = (V, E)$ is a connected planar graph with v vertices; $v \geq 3$ and e edges then

$e \leq 3v - 6$.

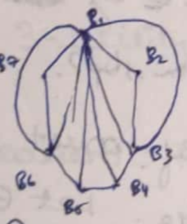
Theorem: 9.6.3:

If G is a connected planar graph then it has a vertex with degree 5 or less.

Graph colouring pblm: Given an undirected graph of $G = (V, E)$ we can find a colouring fn. f from V into a set of colours H such that (v_i, v_j) belongs to $E \Rightarrow f(v_i) \neq f(v_j)$ & H has the smallest possible cardinality. The cardinality of H is called the chromatic number of G. & it is denoted by $\chi(G)$ (chi)

adjacent vertices should not have same colour

eg:



R

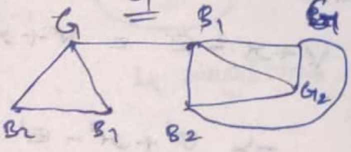
$(v_i, v_j) \in E$

$f(v_i) \neq f(v_j)$

$|H| = 7$

$\chi(R) = 7$

G



$\chi(G) = 4$

Note: A colouring function onto an n element set is called an n-colouring.

Theorem: 9.6.4: The 4-colour thm:

If G is a planar graph then $\chi(G) = 4$

Theorem: 9.6.5 5 colour Thm:

If G is a planar graph then $\chi(G) \leq 5$

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Proof by Induction on no. of vertices in the graph:

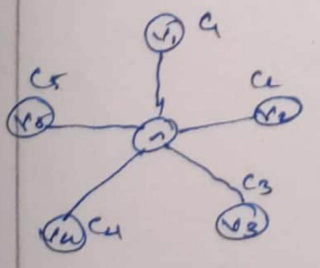
Basis: If the graph has 1 vertex then its chromatic no. is 1.

Assum: Assume that all planar graphs with n-1 vertices have a cromatic no of 5 or less.

Induction: ~~Assume that~~ let G be a planar graph with n vertices. by Thm 9.6.2 there exist a vertex v with ~~deg~~ deg $v \le 5$.

Let $G-v$ be a planar graph obtained by deleting v & all edges that connect v to other vertices in G. Now by Induction $G-v$ has a 5 colouring. Assume ~~that~~ the colours used are c_1, c_2, c_3, c_4, c_5. If $\deg(v) < 5$ then we can produce a five colouring of G by selecting a colour that is not used in colouring the vertices that are connected to v with an edge in G.

If degree of $v = 5$ & if 5 vertices that are adjacent to v are not all coloured differently. ∴ we are now left with the possibility that v_1, v_2, v_3, v_4 & v_5 are all connected to v by an edge & they are all coloured differently. Assume that they are coloured with c_1, c_2, c_3, c_4, c_5 respectively.



(*) New Note

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