Unit - I

Solution of Equations and Eigen Value Problems

Solution of algebraic and transcendental equations: Regula-Falsi method – Fixed point theorem (statement only) – Fixed point iteration method – Newton Raphson method – Solution of linear system of equations: Cholesky decomposition method – Eigen values of a matrix: Power method.

Solution of algebraic and transcendental equations :

Algebraic equations :

If $f(x)$ is purely a polynomial in x then its called by an algebraic equations .

Ex :

$x^3 + 5x + 6 = 0$, $4x^4 - 2x^3 + x^2 - 7x + 12 = 0$ are algebraic equations .

Transcendental equations :

If $f(x)$ contains some other functions (transcendental) such as trigonometric , logarithmic or exponential etc .

Ex :

$3x - \cos x - 1 = 0$, $3x + \sin x - e^x = 0$, $x \log_{10} x - 1.2 = 0$ are transcendental equations .

Solution of an equation :

The values of x which makes $f(x)$ zero are known as zero's or roots of the function $f(x)$.

Newton - Raphson Method or Newton's Method :

Let $f(x) = 0$ be the given equation then Newton's Method is given by

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\}$$

Condition for convergence of Newton's Method :

Condition for convergence of Newton's method is $|f(x)f''(x)| < |f'(x)|^2$

Note :

1. Order of convergence of Newton's method is 2 .
2. Newton's method is also called method of tangents .
3. If $f(a)$ and $f(b)$ have opposite signs then one root of $f(x) = 0$ lies between a and b .
4. If the root lies between a and b then choose an initial value as $x_0 = \frac{a+b}{2}$.

Problems :

1. Find the positive root of $x^3 = 6x - 4$ by Newton's method correct to four decimal places.

Solution :

Given $x^3 = 6x - 4$

$\Rightarrow x^3 - 6x + 4 = 0$

Let $f(x) = x^3 - 6x + 4$ $\qquad ---------(1)$

Differentiate with respect to x ,

$$f'(x) = 3x^2 - 6 \qquad ---------(2)$$

$$\left\{\because \frac{d}{dx}\, x^n = n\, x^{n-1}\,,\ \frac{d}{dx}\, constant = 0\right.$$

$Subs\ x = 0 \Rightarrow f(0) = 0^3 - (6 * 0) + 4 = 4 = +ve$

$Subs\ x = 1 \Rightarrow f(1) = 1^3 - (6 * 1) + 4 = -1 = -ve$

Here $f(0)$ and $f(1)$ are opposite in sign

$\therefore$ *The root lies between 0 and 1*

Then the initial value is $x_0 = \frac{0+1}{2} = 0.5$

Newton's Method is given by

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\} \qquad ---------(3)$$

1^{st} *Iteration :*

Subs $n = 0$ *in* (3),

$$x_1 = x_0 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_0 - \left\{\frac{x_0^3 - 6x_0 + 4}{3x_0^2 - 6}\right\}$$

$$= 0.5 - \left\{\frac{(0.5)^3 - (6*0.5) + 4}{\left(3*(0.5^2)\right) - 6}\right\}$$

$$x_1 = 0.7143$$

2^{nd} *Iteration :*

Subs $n = 1$ *in* (3),

$$x_2 = x_1 - \left\{\frac{f(x_1)}{f'(x_1)}\right\}$$

$$= x_1 - \left\{\frac{x_1^3 - 6x_1 + 4}{3x_1^2 - 6}\right\}$$

$$= 0.7143 - \left\{\frac{(0.7143)^3 - (6*0.7143) + 4}{\left(3*(0.7143^2)\right) - 6}\right\}$$

$$x_2 = 0.7319$$

3^{rd} *Iteration :*

Subs $n = 2$ *in* (3),

$$x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\}$$

$$= x_2 - \left\{\frac{x_2^3 - 6x_2 + 4}{3x_2^2 - 6}\right\}$$

$$= 0.7319 - \left\{\frac{(0.7319)^3-(6*0.7319)+4}{\left(3*(0.7319^2)\right)-6}\right\}$$

$$x_3 = 0.7321$$

4^{th} Iteration :

Subs $n = 3$ *in* (3) ,

$$x_4 = x_3 - \left\{\frac{f(x_3)}{f'(x_3)}\right\}$$

$$= x_3 - \left\{\frac{x_3^3-6x_3+4}{3x_3^2-6}\right\}$$

$$= 0.7321 - \left\{\frac{(0.7321)^3-(6*0.7321)+4}{\left(3*(0.7321^2)\right)-6}\right\}$$

$$x_4 = 0.7321$$

Since in 3^{rd} *and* 4^{th} *iterations the values of are same, we stop the process here.*

$\therefore$ *The root of the given equation is* 0.7321 .

2. Find the positive root of $x^3 - 2x + 0.5 = 0$ ***by Newton's method correct to four decimal places.***

Solution :

Given $x^3 - 2x + 0.5 = 0$

Let $f(x) = x^3 - 2x + 0.5 \qquad \text{---------} (1)$

Differentiate with respect to x ,

$$f'(x) = 3x^2 - 2 \qquad \text{---------} (2)$$

Subs $x = 0 \Rightarrow f(0) = 0^3 - (2*0) + 0.5 = 0.5 = +ve$

Subs $x = 1 \Rightarrow f(1) = 1^3 - (2*1) + 0.5 = -0.5 = -ve$

Here $f(0)$ *and* $f(1)$ *are opposite in sign*

$\therefore$ *The root lies between* 0 *and* 1

Then the initial value is $x_0 = \frac{0+1}{2} = 0.5$

Newton's Method is given by

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\} \qquad \text{---------} (3)$$

1^{st} Iteration :

Subs $n = 0$ *in* (3) ,

$$x_1 = x_0 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_0 - \left\{\frac{x_0^3-2x_0+0.5}{3x_0^2-2}\right\}$$

$$= 0.5 - \left\{\frac{(0.5)^3-(2*0.5)+0.5}{\left(3*(0.5^2)\right)-2}\right\}$$

$$x_1 = 0.2$$

2^{nd} Iteration :

Subs $n = 1$ in (3) ,

$$x_2 = x_1 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_1 - \left\{\frac{x_1^3-2x_1+0.5}{3x_1^2-2}\right\}$$

$$= - \left\{\frac{(0.5)^3-(2*0.5)+0.5}{\left(3*(0.5^2)\right)-2}\right\}$$

$$x_2 = 0.2574$$

3^{rd} Iteration :

Subs $n = 2$ in (3) ,

$$x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\}$$

$$= x_2 - \left\{\frac{x_2^3-2x_2+4}{3x_2^2-2}\right\}$$

$$= 0.2574 - \left\{\frac{(0.2574)^3-(2*0.2574)+0.5}{\left(3*(0.2574^2)\right)-2}\right\}$$

$$x_3 = 0.2587$$

4^{th} Iteration :

Subs $n = 3$ in (3) ,

$$x_4 = x_3 - \left\{\frac{f(x_3)}{f'(x_3)}\right\}$$

$$= x_3 - \left\{\frac{x_3^3-2x_3+0.5}{3x_3^2-2}\right\}$$

$$= 0.2587 - \left\{\frac{(0.2587)^3-(2*0.2587)+0.5}{\left(3*(0.2587^2)\right)-2}\right\}$$

$$x_4 = 0.2587$$

Since in 3^{rd} and 4^{th} iterations the values of are same, we stop the process here.

$\therefore$ The root of the given equation is 0.2587 .

3. Find the positive root of $x^4 - x - 10 = 0$ by Newton's method correct to four decimal places.

Solution :

Given $x^4 - x - 10 = 0$

Let $f(x) = x^4 - x - 10$ $\quad ---------(1)$

Differentiate with respect to x ,

$f'(x) = 4x^3 - 1$ $\quad ---------(2)$

Subs $x = 0 \Rightarrow f(0) = 0^4 - 0 - 10 = -10 = -ve$

Subs $x = 1 \Rightarrow f(1) = 1^4 - 1 - 10 = -10 = -ve$

Subs $x = 2 \Rightarrow f(2) = 2^4 - 2 - 10 = 4 = +ve$

Here $f(1)$ *and* $f(2)$ *are opposite in sign*

$\therefore$ *The root lies between* 1 *and* 2

Then the initial value is $x_0 = \frac{1+2}{2} = 1.5$

Newton's Method is given by

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\} \quad ---------(3)$$

1^{st} *Iteration :*

Subs $n = 0$ *in* (3) ,

$$x_1 = x_0 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_0 - \left\{\frac{x_0^4 - x_0 - 10}{4x_0^3 - 1}\right\}$$

$$= 0.5 - \left\{\frac{(1.5)^4 - 1.5 - 10}{\left(4*(0.5^3)\right) - 1}\right\}$$

$$x_1 = 2.015$$

2^{nd} *Iteration :*

Subs $n = 1$ *in* (3) ,

$$x_2 = x_1 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_1 - \left\{\frac{x_0^4 - x_0 - 10}{4x_0^3 - 1}\right\}$$

$$= 2.015 - \left\{\frac{(2.015)^4 - 2.015 - 10}{\left(4*(2.0155^2)\right) - 1}\right\}$$

$$x_2 = 1.8741$$

3^{rd} *Iteration :*

Subs $n = 2$ *in* (3) ,

$$x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\}$$

$$= x_2 - \left\{\frac{x_2^4-6x_2-10}{4x_2^3-1}\right\}$$

$$= 1.8741 - \left\{\frac{(1.8741)^4-1.8741-10}{\left(4*(1.8741^3)\right)-1}\right\}$$

$$x_3 = 1.8559$$

4^{th} Iteration :

Subs $n = 3$ in (3) ,

$$x_4 = x_3 - \left\{\frac{f(x_3)}{f'(x_3)}\right\}$$

$$= x_3 - \left\{\frac{x_3^4-x_3-10}{4x_3^3-1}\right\}$$

$$= 1.8559 - \left\{\frac{(1.8559)^3- 1.8559 -10}{\left(4*(1.8559^3)\right)-1}\right\}$$

$$x_4 = 1.8556$$

5^{th} Iteration :

Subs $n = 4$ in (3) ,

$$x_5 = x_4 - \left\{\frac{f(x_4)}{f'(x_4)}\right\}$$

$$= x_4 - \left\{\frac{x_4^4 - x_4-10}{4x_4^3-1}\right\}$$

$$= 1.8556 - \left\{\frac{(1.8556)^3-1.8556-10}{\left(4*(1.8556^3)\right)-1}\right\}$$

$$x_5 = 1.8556$$

Since in 4^{th} and 5^{th} iterations the values of are same, we stop the process here.

$\therefore$ The root of the given equation is 1.8556 .

4. Find the positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to four decimal places.

Solution :

Given $3x - \cos x - 1 = 0$

Let $f(x) = 3x - \cos x - 1$ $\qquad ---------(1)$

Differentiate with respect to x ,

$$f'(x) = 3 - (-\sin x)$$

$$f'(x) = 3 + \sin x \qquad ---------(2)$$

$$\left\{\because \frac{d}{dx} \cos x = -\sin x\right.$$

Subs $x = 0 \Rightarrow f(0) = (3 * 0) - \cos 0 - 1 = -2 = -ve$

$Subs\ x = 1 \Rightarrow f(1) = (3*1) - \cos 1 - 1 = 1.4597 = +ve$

$Here\ f(0)\ and\ f(1)\ are\ opposite\ in\ sign$

$\therefore\ The\ root\ lies\ between\ 0\ and\ 1$

$Then\ the\ initial\ value\ is\ x_0 = \frac{0+1}{2} = 0.5$

$Newton's\ Method\ is\ given\ by$

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\} \qquad ---------(3)$$

1^{st} *Iteration :*

$Subs\ n = 0\ in\ (3),$

$$x_1 = x_0 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_0 - \left\{\frac{3x_0 - \cos x_0 - 1}{3 + \sin x_0}\right\}$$

$$= 0.5 - \left\{\frac{(3*0.5) - \cos 0.5 - 1}{3 + \sin 0.5}\right\}$$

$$x_1 = 0.6085$$

2^{nd} *Iteration :*

$Subs\ n = 1\ in\ (3),$

$$x_2 = x_1 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_1 - \left\{\frac{3x_1 - \cos x_1 - 1}{3 + \sin x_1}\right\}$$

$$= 0.6085 - \left\{\frac{(3*0.6085) - \cos 0.6085 - 1}{3 + \sin 0.6085}\right\}$$

$$x_2 = 0.6071$$

3^{rd} *Iteration :*

$Subs\ n = 2\ in\ (3),$

$$x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\}$$

$$= x_2 - \left\{\frac{3x_2 - \cos x_2 - 1}{3 + \sin x_2}\right\}$$

$$= 0.6071 - \left\{\frac{(3*0.6071) - \cos 0.6071 - 1}{3 + \sin 0.6071}\right\}$$

$$x_3 = 0.6071$$

Since in 2^{nd} and 3^{rd} iterations the values of are same, we stop the process here.

$\therefore$ *The root of the given equation is* 0.6071.

5. *Find the positive root of $\log_{10} x - 1.2 = 0$ by Newton's method correct to four decimal places with*

an initial value $x_0 = 15.5$

Solution :

Given $\log_{10} x - 1.2 = 0$

Let $f(x) = \log_{10} x - 1.2$ $\quad$ ---------(1)

Differentiate with respect to x ,

$$f'(x) = \frac{1}{x} \log_{10} e \qquad \text{---------}(2)$$

$$\left\{\because \frac{d}{dx} \log_a x = \frac{1}{x} \log_a e\right.$$

The given initial value is $x_0 = 15.5$

Newton's Method is given by

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\} \qquad \text{---------}(3)$$

1^{st} ***Iteration :***

Subs $n = 0$ *in* (3) ,

$$x_1 = x_0 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_0 - \left\{\frac{\log_{10} x_0 - 1.2}{\frac{1}{x_0} \log_{10} e}\right\}$$

$$= 15.5 - \left\{\frac{\log_{10}(15.5) - 1.2}{\frac{1}{15.5} \log_{10} e}\right\}$$

$$x_1 = 15.8451$$

2^{nd} ***Iteration :***

Subs $n = 1$ *in* (3) ,

$$x_2 = x_1 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_1 - \left\{\frac{\log_{10} x_1 - 1.2}{\frac{1}{x_1} \log_{10} e}\right\}$$

$$= 15.8451 - \left\{\frac{\log_{10}(15.8451) - 1.2}{\frac{1}{15.8451} \log_{10} e}\right\}$$

$$x_2 = 15.8489$$

3^{rd} ***Iteration :***

Subs $n = 2$ *in* (3) ,

$$x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\}$$

$$= x_2 - \left\{ \frac{\log_{10} x_2 - 1.2}{\frac{1}{x_2} \log_{10} e} \right\}$$

$$= 15.8489 - \left\{ \frac{\log_{10}(15.8489) - 1.2}{\frac{1}{15.8489} \log_{10} e} \right\}$$

$$x_3 = 15.8489$$

Since in 2^{nd} and 3^{rd} iterations the values of are same, we stop the process here.

$\therefore$ *The root of the given equation is* 15.8489 .

6. Find the positive root of $x \log_{10} x - 1.2 = 0$ by Newton's method correct to three decimal places.

Solution :

Given $x \log_{10} x - 1.2 = 0$

Let $f(x) = x \log_{10} x - 1.2 \qquad ----------(1)$

Differentiate with respect to x ,

$$f'(x) = x * \frac{1}{x} \log_{10} e + \log_{10} x * 1$$

$$f'(x) = \log_{10} e + \log_{10} x \qquad ----------(2)$$

$$\left\{ \because \frac{d}{dx} uv = udv + vdu \right.$$

$$u = x \Rightarrow du = 1$$

$$v = \log_{10} x \Rightarrow dv = \frac{1}{x} \log_{10} e$$

Subs $x = 1 \Rightarrow f(1) = 1 * \log_{10} 1 - 1.2 = -0.2 = -ve$

Subs $x = 2 \Rightarrow f(2) = 2 * \log_{10} 2 - 1.2 = 0.4990 = +ve$

Here $f(1)$ *and* $f(2)$ *are opposite in sign*

$\therefore$ *The root lies between* 1 *and* 2

Then the initial value is $x_0 = \frac{1+2}{2} = 1.5$

Newton's Method is given by

$$x_{n+1} = x_n - \left\{ \frac{f(x_n)}{f'(x_n)} \right\} \qquad ----------(3)$$

1^{st} Iteration :

Subs $n = 0$ *in* (3) ,

$$x_1 = x_0 - \left\{ \frac{f(x_0)}{f'(x_0)} \right\}$$

$$= x_0 - \left\{ \frac{x_0 \log_{10} x_0 - 1.2}{\log_{10} e + \log_{10} x_0} \right\}$$

$$= 1.5 - \left\{ \frac{1.5 \log_{10} 1.5 - 1.2}{\log_{10} e + \log_{10} 1.5} \right\}$$

$$x_1 = 3.033$$

2^{nd} ***Iteration :***

Subs $n = 1$ ***in*** (3) ,

$$x_2 = x_1 - \left\{\frac{f(x_0)}{f'(x_0)}\right\}$$

$$= x_1 - \left\{\frac{x_1 \log_{10} x_1 - 1.2}{\log_{10} e + \log_{10} x_1}\right\}$$

$$= 3.033 - \left\{\frac{3.033 \log_{10} 3.033 - 1.2}{\log_{10} e + \log_{10} 3.033}\right\}$$

$$x_2 = 2.748$$

3^{rd} ***Iteration :***

Subs $n = 2$ ***in*** (3) ,

$$x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\}$$

$$= x_2 - \left\{\frac{x_2 \log_{10} x_2 - 1.2}{\log_{10} e + \log_{10} x_2}\right\}$$

$$= 2.748 - \left\{\frac{2.748 \log_{10} 2.748 - 1.2}{\log_{10} e + \log_{10} 2.748}\right\}$$

$$x_3 = 2.741$$

4^{th} ***Iteration :***

Subs $n = 3$ ***in*** (3) ,

$$x_4 = x_3 - \left\{\frac{f(x_3)}{f'(x_3)}\right\}$$

$$= x_3 - \left\{\frac{x_3 \log_{10} x_3 - 1.2}{\log_{10} e + \log_{10} x_3}\right\}$$

$$= 2.741 - \left\{\frac{2.741 \log_{10} 2.741 - 1.2}{\log_{10} e + \log_{10} 2.741}\right\}$$

$$x_4 = 2.741$$

Since in 3^{rd} ***and*** 4^{th} ***iterations the values of are same, we stop the process here.***

$\therefore$ ***The root of the given equation is*** 2.741 ***.***

7. Find an iterative formula to find $\sqrt{N}$ ***, where*** N ***is a positive number and hence find the*** $\sqrt{17}$ ***by Newton's method .***

Solution :

Given $x = \sqrt{N}$

$\Rightarrow x^2 = N$

$$x^2 - N = 0$$

$$\text{Let } f(x) = x^2 - N \qquad ---------(1)$$

Differentiate with respect to x ,

$$f'(x) = 2x \qquad ---------(2)$$

Newton's Method is given by

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\}$$

$$= x_n - \left\{\frac{x_n^2 - N}{2x_n}\right\}$$

$$= x_n - \left\{\frac{x_n^2}{2x_n} - \frac{N}{2x_n}\right\}$$

$$= x_n - \left\{\frac{x_n}{2} - \frac{N}{2x_n}\right\}$$

$$= x_n - \frac{x_n}{2} + \frac{N}{2x_n}$$

$$= \frac{x_n}{2} + \frac{N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \qquad ---------(3)$$

To find $\sqrt{17}$ *:*

Here $N = 17$

Choose $x_0 = 4.5$

1^{st} *Iteration :*

Subs $n = 0$ *in* (3) ,

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right)$$

$$= \frac{1}{2}\left(4.5 + \frac{17}{4.5}\right)$$

$$x_1 = 4.1389$$

2^{nd} *Iteration :*

Subs $n = 1$ *in* (3) ,

$$x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right)$$

$$= \frac{1}{2}\left(4.1389 + \frac{17}{4.1389}\right)$$

$$x_2 = 4.1231$$

3^{rd} Iteration :

Subs $n = 2$ in (3) ,

$$x_3 = \frac{1}{2}\left(x_2 + \frac{N}{x_2}\right)$$

$$= \frac{1}{2}\left(4.1231 + \frac{17}{4.1231}\right)$$

$$x_3 = 4.1231$$

Since in 2^{nd} and 3^{rd} iterations the values of are same, we stop the process here.

$\therefore$ *The value of $\sqrt{17}$ is 4.1231 .*

8. Find an iterative formula to find the reciprocal of a given number N , where N is a positive number and hence find the $\frac{1}{12}$ by Newton's method .

Solution :

Given $x = \frac{1}{N}$

$$\Rightarrow \frac{1}{x} = N$$

$$\frac{1}{x} - N = 0$$

$$\text{Let } f(x) = \frac{1}{x} - N \qquad ---------(1)$$

Differentiate with respect to x ,

$$f'(x) = \frac{-1}{x^2} \qquad ---------(2)$$

Newton's Method is given by

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\}$$

$$= x_n - \left\{\frac{\frac{1}{x_n} - N}{\frac{-1}{x_n^2}}\right\}$$

$$= x_n - \left\{\left(\frac{1}{x_n} - N\right) * (-x_n^2)\right\}$$

$$= x_n - \left\{\frac{-x_n^2}{x_n} + Nx_n^2\right\}$$

$$= x_n - (-x_n + Nx_n^2)$$

$$= x_n + x_n - Nx_n^2$$

$$= 2x_n - Nx_n^2$$

$$x_{n+1} = x_n(2 - Nx_n) \qquad ---------(3)$$

To find $\frac{1}{12}$:

$Here\ N = 12$

$Choose\ x_0 = 0.05$

1^{st} ***Iteration :***

$Subs\ n = 0\ in\ (3)$,

$$x_1 = x_0\,(2 - Nx_0)$$

$$= 0.05\,\big(2 - (12 * 0.05)\big)$$

$$x_1 = 0.07$$

2^{nd} ***Iteration :***

$Subs\ n = 1\ in\ (3)$,

$$x_2 = x_1\,(2 - Nx_1)$$

$$= 0.07\,\big(2 - (12 * 0.07)\big)$$

$$x_2 = 0.0812$$

3^{rd} ***Iteration :***

$Subs\ n = 2\ in\ (3)$,

$$x_3 = x_2\,(2 - Nx_2)$$

$$= 0.0812\,\big(2 - (12 * 0.0812)\big)$$

$$x_3 = 0.0833$$

4^{th} ***Iteration :***

$Subs\ n = 3\ in\ (3)$,

$$x_4 = x_3\,(2 - Nx_3)$$

$$= 0.0833\,\big(2 - (12 * 0.0833)\big)$$

$$x_4 = 0.0833$$

Since in 3^{rd} and 4^{th} iterations the values of are same, we stop the process here.

$\therefore$ *The value of* $\frac{1}{12}$ *is* 0.0833 .

9. Find an iterative formula to find the p^{th} root of positive number , where N is a positive number and hence find the cube root of 17 by Newton's method .

Solution :

$Given\ x = \sqrt[p]{N}$

$\Rightarrow x^p = N$

$x^p - N = 0$

$Let\ f(x) = x^p - N \qquad --------- (1)$

Differentiate with respect to x ,

$$f'(x) = px^{p-1} \qquad ---------(2)$$

Newton's Method is given by

$$x_{n+1} = x_n - \left\{\frac{f(x_n)}{f'(x_n)}\right\}$$

$$= x_n - \left\{\frac{x_n^p - N}{px_n^{p-1}}\right\}$$

$$= x_n - \left\{\frac{x_n^p}{px_n^{p-1}} - \frac{N}{px_n^{p-1}}\right\}$$

$$= \frac{x_n px_n^{p-1} - x_n^p + N}{px_n^{p-1}}$$

$$= \frac{x_n px_n^p x_n^{-1} - x_n^p + N}{px_n^{p-1}}$$

$$= \frac{px_n^p - x_n^p + N}{px_n^{p-1}}$$

$$= \frac{(p-1)\, x_n^p + N}{px_n^{p-1}}$$

$$x_{n+1} = \frac{(p-1)\, x_n^p + N}{px_n^{p-1}} \qquad ---------(3)$$

To find $\sqrt[3]{\sqrt{17}}$:

Here $N = 17$, $p = 3$

Choose $x_0 = 2$

1^{st} Iteration :

Subs $n = 0$, $p = 3$, $N = 17$ in (3) ,

$$x_1 = \frac{(3-1)\, x_0^3 + 17}{3x_0^{3-1}}$$

$$= \frac{2(2^3)+17}{3(2^2)}$$

$$x_1 = 2.75$$

2^{nd} Iteration :

Subs $n = 1$, $p = 3$, $N = 17$ in (3) ,

$$x_2 = \frac{(3-1)\, x_1^3 + 17}{3x_1^{3-1}}$$

$$= \frac{2(2.75^3)+17}{3(2.75^2)}$$

$$x_2 = 2.5826$$

3^{rd} ***Iteration :***

Subs $n = 2$, $p = 3, N = 17$ ***in*** (3) ,

$$x_2 = \frac{(3-1)\,x_1^3+17}{3x_1^{3-1}}$$

$$= \frac{2(2.5826^3)+17}{3(2.5826^2)}$$

$$x_3 = 2.5733$$

4^{th} ***Iteration :***

Subs $n = 3$, $p = 3, N = 17$ ***in*** (3) ,

$$x_4 = \frac{(3-1)\,x_3^3+17}{3x_3^{3-1}}$$

$$= \frac{2(2.5733^3)+17}{3(2.5733^2)}$$

$$x_4 = 2.5713$$

5^{th} ***Iteration :***

Subs $n = 4$, $p = 3, N = 17$ ***in*** (3) ,

$$x_5 = \frac{(3-1)\,x_4^3+17}{3x_4^{3-1}}$$

$$= \frac{2(2.5713^3)+17}{3(2.5713^2)}$$

$$x_5 = 2.5713$$

Since in 4^{th} ***and*** 5^{th} ***iterations the values of are same, we stop the process here.***

$\therefore$ ***The value of*** $\sqrt[3]{17}$ ***is*** 2.5713 .