Sorting Techniques

Design and Analysis of Algorithms
Lecture Set 2

Outline

- The problem
- Importance and Application
- Bubble Sort
- Insertion Sort
- Selection Sort
- Heap Sort

The problem

- Given a set of n numbers that may be in any order, the goal is to output the numbers in sorted order
  - A set of elements S are sorted in ascending order when $S_i < S_{i+1}$ for all $1 <= i < n$
  - A set of elements S are sorted in descending order when $S_i > S_{i+1}$ for all $1 <= i < n$
- Can sort just the key or entire records

Importance of Sorting

- Basic block around which many other algorithms are built on
- Interesting ideas in design of algorithms appear in the context of sorting
  - Divide and conquer, data structures, randomized algorithms
- Remains the most ubiquitous combinatorial algorithmic problem in practice
- Loads of study on this problem

Applications of Sorting

- Searching
  - Whether linear or binary, it is easier to search for an element if the list is sorted
- Closest Pair
  - Given a set of n numbers, find the pair of numbers that have the smallest difference between them
  - Closest pair lie next to each other somewhere in the sorted order

Applications of Sorting

- Finding duplicates among n elements in a list
  - In sorted list, it is easy to find this by scanning adjacent elements
- Frequency Distribution
  - Find the number of occurrences of different elements in a list
  - Identical items together in a sorted list
    - Just scan list once
- Selection
  - Find the kth largest element in a list

Sorting Algorithms

- Comparison Sort
- Bucket Sort
- Counting Sort
- Radix Sort
- Heap Sort

Bubble Sort

- A basic sorting algorithm where corresponding elements are checked and swapped
  - The smallest elements bubble up the way

http://t2.gstatic.com/images?q=tbn:ANd9GcRnvLV-rVerCKzEZ7sN7WwBIUL09M_FqIkgX52w5mSLElDgtFVRX-FgHtSsWg

Pseudocode

- BUBBLESORT(A)

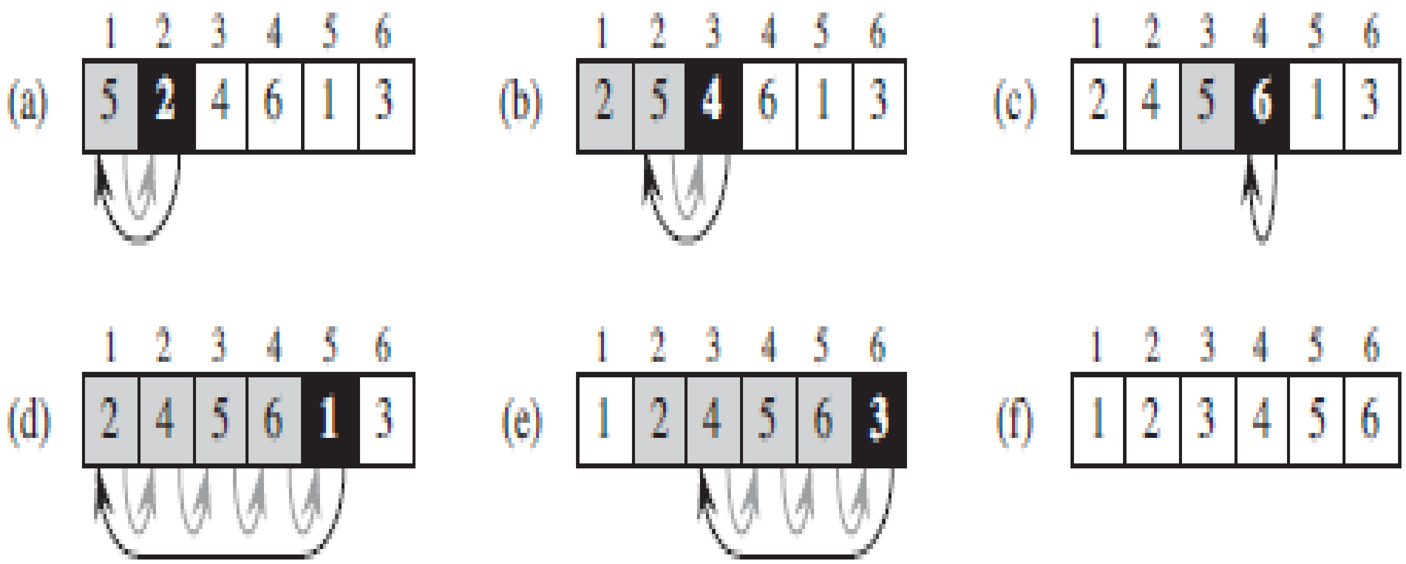
```
for i = 1 to A.length-1
  for j = A:length downto i – 1
     if A[j] < A[j-1]
        exchange A[j] with A[j -1]
```

Bubble Sort: Analysis

- Best Case
  - O(n)
  - List already sorted, hence zero swaps after first iteration
- Worst case
  - $O(n^2)$
- On average
  - (n-1)*n/2 comparisons
  - $O(n^2)$

Insertion Sort

- At each iteration an element is removed from the list and inserted into the right place
  - An any iteration the first i items are in place



Src: CLR – Ch2- Pg22

Insertion Sort: Pseudocode

- INSERTION-SORT(A)

  1. ***for*** j = 2 ***to*** A.length

  2. key = A[j] // Insert A[j] into the sorted sequence A[1 : : j-1]

  3. i = j-1

  4. ***while*** i > 0 ***and*** A[i] > key

  5. A[i+1] = A[i]

  6 i = i-1

  7 A[i-1] = key

Src: CLR – Ch2

Insertion Sort : Analysis

- At each iteration
  - Element compared and/or swapped with atmost i elements
    - i varies from 1 to n
- n such elements inserted
- Average case and worst case- $O(n^2)$
  - Worst case when list sorted in reverse order
- Best Case – O(n)
  - When list is already sorted
    - No swaps needed

Selection Sort

- At each iteration the minimum element is chosen and inserted in the top of the list

```
S E L E C T I O N S O R T
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C E E I L N O O R S S T T
```

Src: Steven S. Skiena, “The Algorithm Design Manual”, Second Edition

Selection Sort: Pseudocode

```
selection_sort(int s[], int n){
        int i,j; /* counters */
        for (i=0; i<n; i++) {
                min=i;    /* index of minimum */
                for (j=i+1; j<n; j++)
                        if (s[j] < s[min]) min=j;
                swap(s[i],s[min]);
        }
}
```

Src: Steven S. Skiena, “The Algorithm Design Manual”, Second Edition

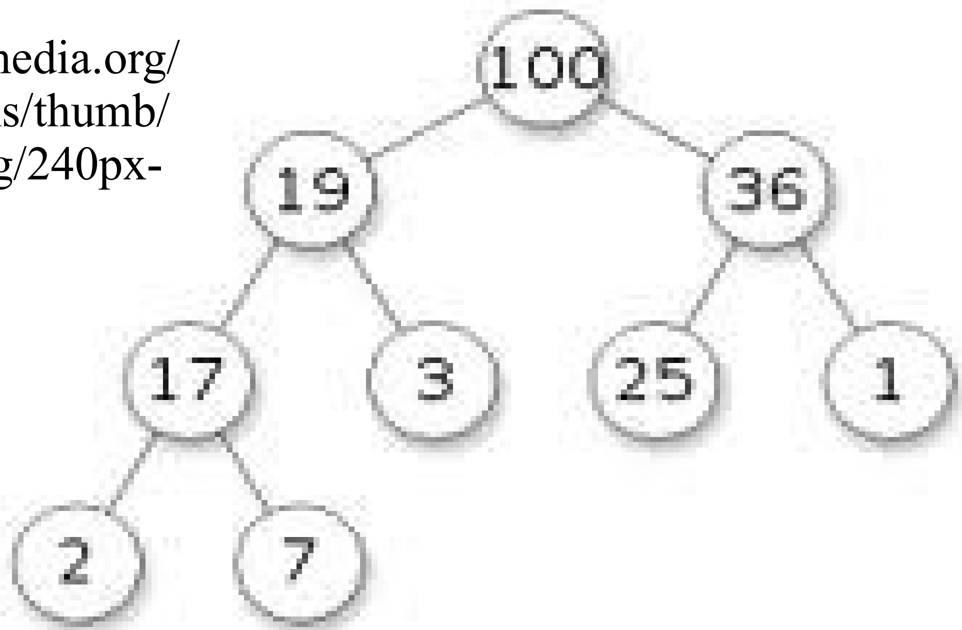
Analysis of Selection Sort

- Complexity depends on cost of finding minimum element in remaining list
- Best Case – O(n^2)
  - Almost sorted still requires cost of finding min
- Average Case – $O(n^2)$
  - Cost of finding minimum element is i per iteration = n(n-1)/2
- Worst Case – $O(n^2)$
  - When list is in reverse sorted order
  - The minimum is always at the end

Review : Heaps

- Priority queue
  - Elements in sorted order
- Stores elements in a binary tree
  - insertions and deletions logarithmic time

http://upload.wikimedia.org/wikipedia/commons/thumb/3/38/Max-Heap.svg/240px-Max-Heap.svg.png

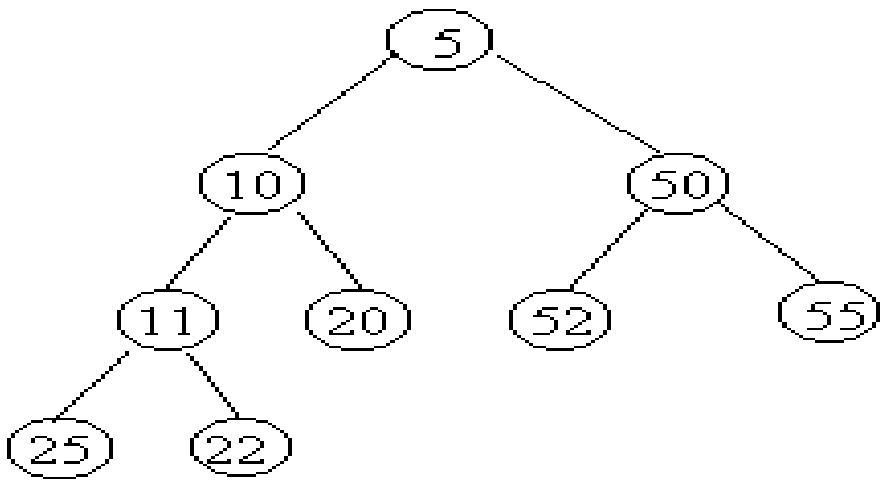


Heap: Properties

- Heap-Order Property
  - For every node v other than the root, the key stored at v is greater than or equal to the key stored at v's parent
- Complete Binary tree
  - A binary tree with height h is complete if the levels 0,1,2,...h-1 have the maximum number of nodes possible and
  - All internal nodes are to the left of the external nodes
  - Helps keep the height of the heap small

Heap Implementation

- Implemented using vector representation
- The last node is the rightmost node in the last level



5	10	50	11	20	52	55	25	22
0	1	2	3	4	5	6	7	8

Heap Sort

- A priority queue based sort
- Step1: Create the heap
  - Inserting elements takes O(log k) where k is the number of elements in the heap at that time
  - Using bottom-up approach cost is O(n)
- Step 2: Sorting
  - Remove the minimum element in each iteration and store in external array
  - Complexity: O(log k) where k is the number of elements in the heap at that time
- Total Cost: O(nlogn)

In-Place Heap Sort

- Uses the vector representation
  - Use the left portion of the vector S to store elements in heap upto i-1 rank
    - These elements are sorted
  - Right portion of S to store the other elements in sequence
- Start with empty heap and move the boundary between heap and sequence from left to right
  - In step i, expand heap by adding element at i-1
- Start with empty sequence and move the boundary
  - In step i, remove maximum from heap and store at rank n-i
- We rearrange instead of using extra memory

Heap-Sort

BUILD-MAX-HEAP(A)

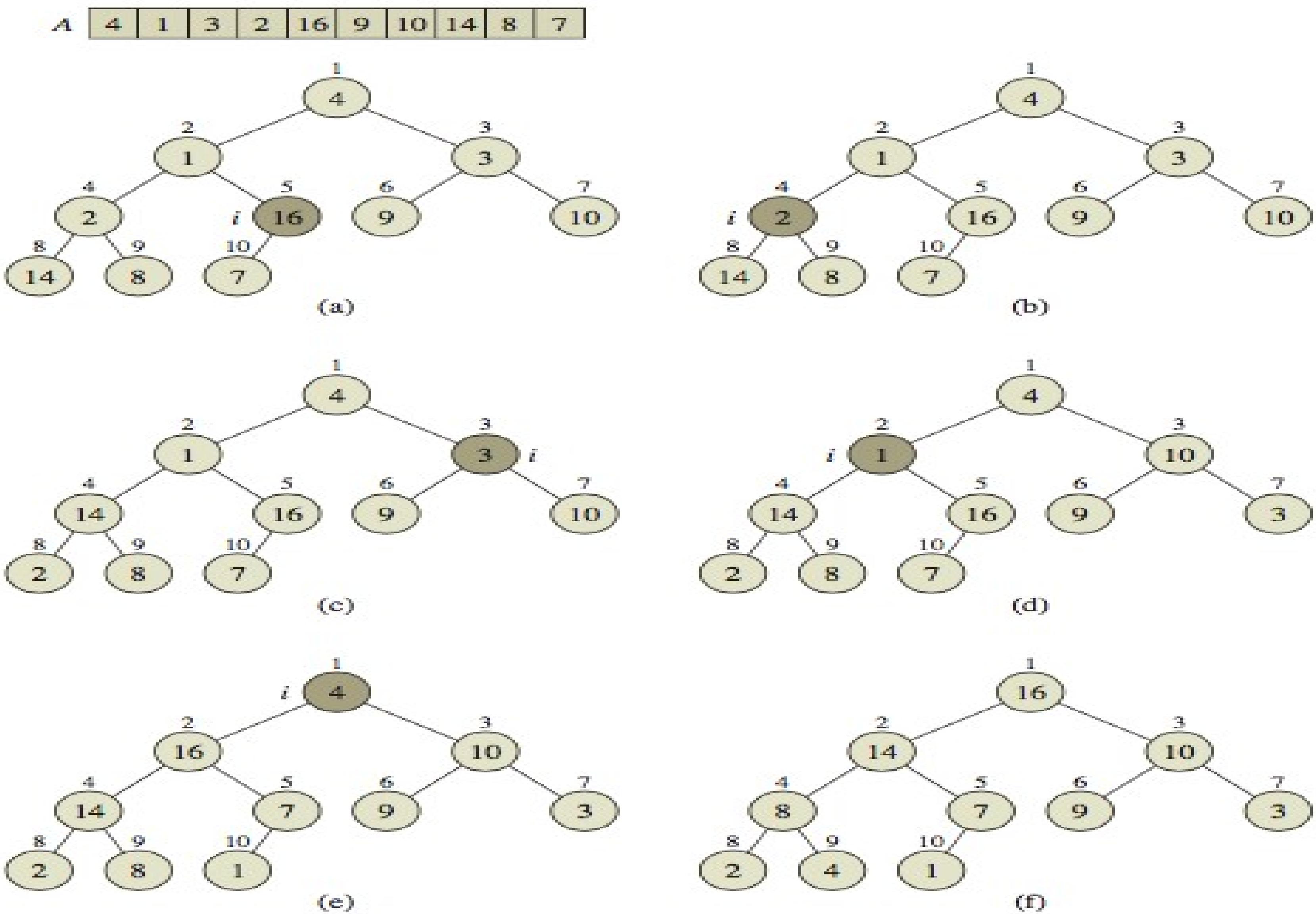
for i = A.length downto 2

exchange A[1] with A[i]

A[heap-size] = A[heap-size-]

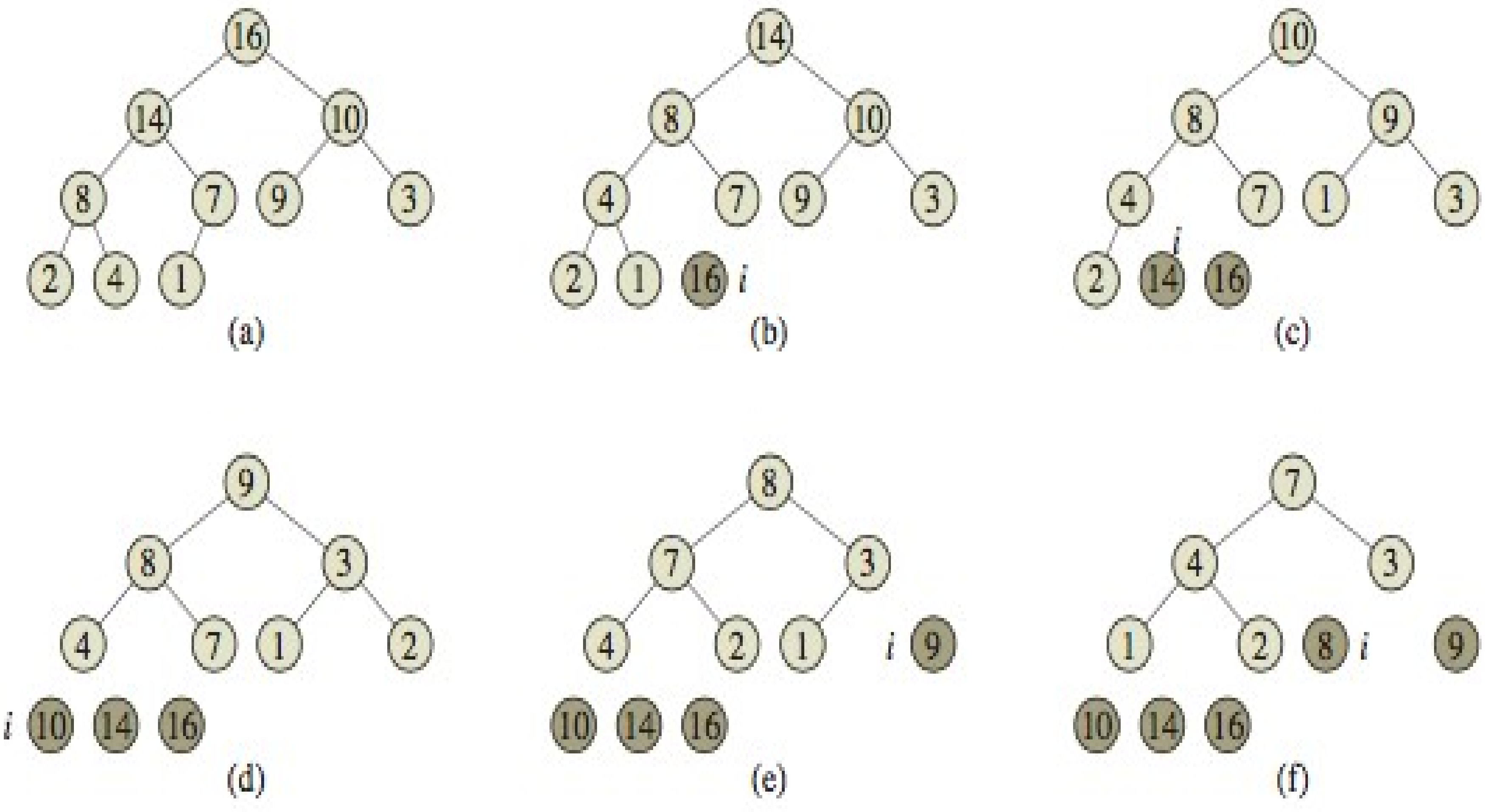
MAX-HEAPIFY(A,1)

In-place Heap Sort



Src:Introduction to Algorithms: CLRS

In-place Heap Sort



Src:Introduction to Algorithms: CLRS

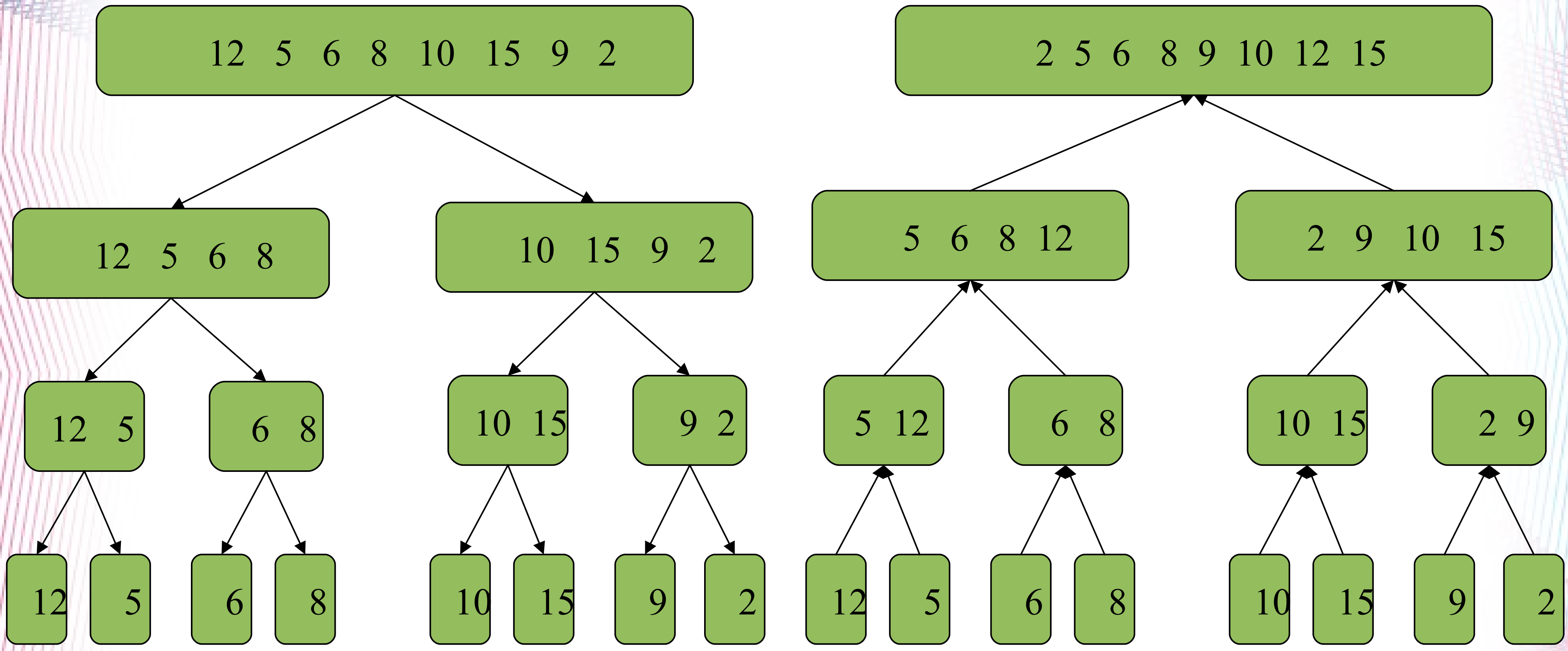
Merge-Sort

- Uses divide and conquer strategy to sort a set of numbers
- Divide:
  - If S has zero or one element, return S
  - Divide S into two sequences S_1 and S_2 each containing half of the elements of S
- Recur
  - Recursively apply merge sort to S_1 and S_2
- Conquer
  - Merge S_1 and S_2 into a sorted sequence

Merging two sorted sequences

- Iteratively remove smallest element from one of the two sequences S_1 and S_2 and add it to end of output sequence S

Merge Sort


12 5 6 8 10 15 9 2
2 5 6 8 9 10 12 15
12 5 6 8
10 15 9 2
5 6 8 12
2 9 10 15
12 5
6 8
10 15
9 2
5 12
6 8
10 15
2 9
12
5
6
8
10
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9
2
12
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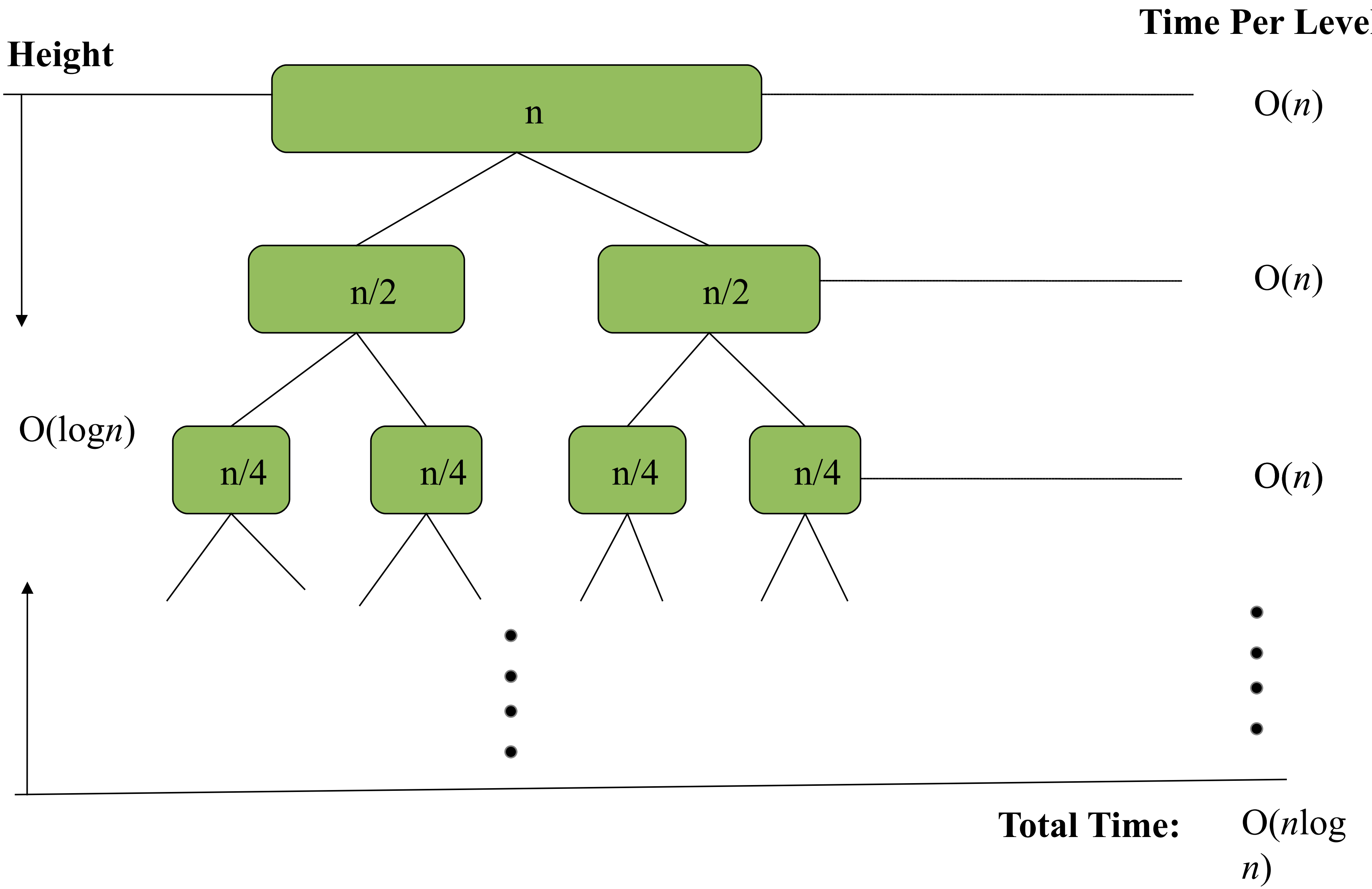
10
15
9
2

Merge Sort

```
mergesort(item_type s[], int low, int high) {
        int i; /* counter */
        int middle; /* index of middle element */
            if (low < high) {
                    middle = (low+high)/2;
                    mergesort(s,low,middle);
                    mergesort(s,middle+1,high);
                    merge(s, low, middle, high);
            }
}
```

Analysis of Merge Sort



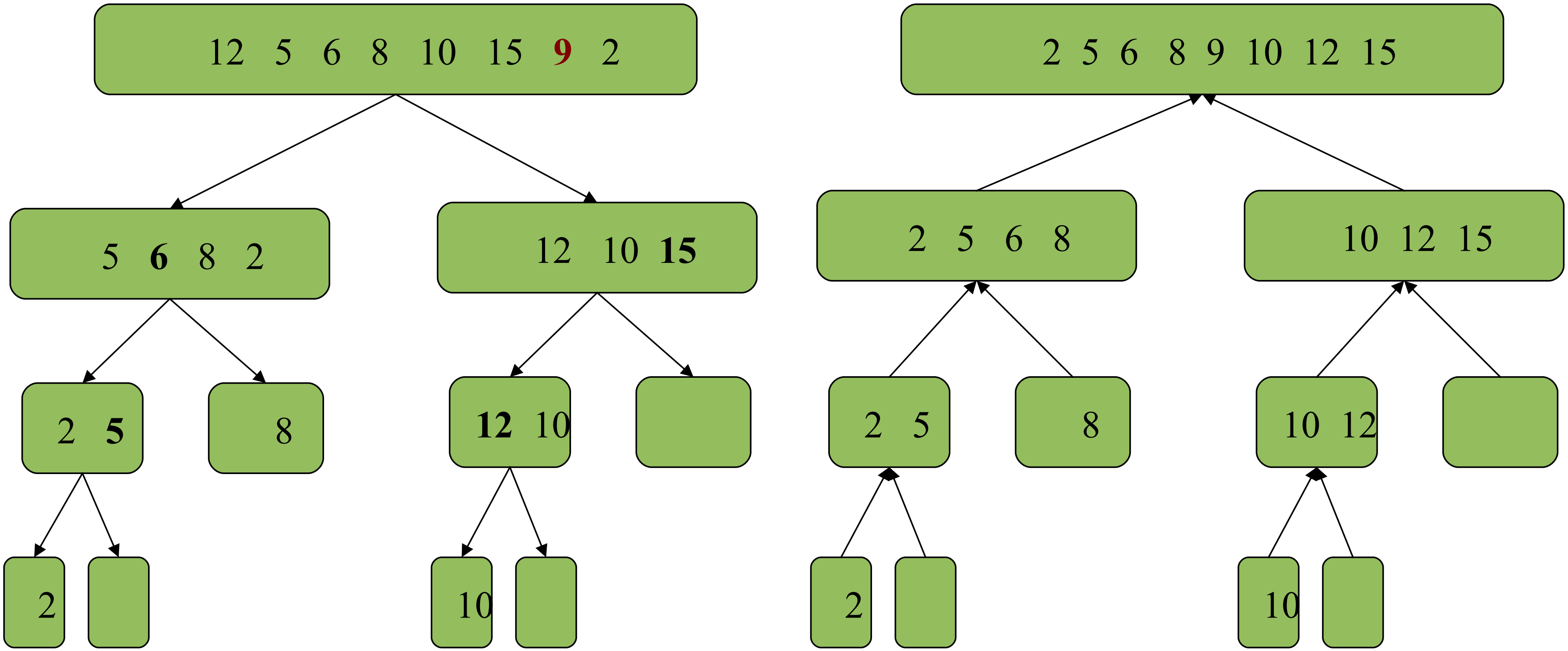
Merge Sort: Analysis

- work done on the kth level involves merging 2^k pairs sorted list, each of size $n/2^{k+1}$
  - A total of atmost n--2^k comparisons
- Linear time for merging at each level
  - Each of the n elements appear exactly in one of the subproblems at each level
- Requires extra memory

Quick Sort

- A divide and conquer strategy which also uses randomization
- Divide
  - Select a random pivot p. Divide S into two subarrays, where one contains elements < p and the other which are > p
- Recur
  - Sort subarrays by recursively applying quicksort on each subarray
- Conquer
  - Since the subarrays are already sorted, no work is needed in merge part

Quick Sort



12 5 6 8 10 15 9 2
5 6 8 2
12 10 15
2 5
8
12 10
2
10
2 5 6 8 9 10 12 15
2 5 6 8
10 12 15
2 5
8
10 12
2
10

Quick Sort

- quickSort(Element[] E, **int** first, **int** last)

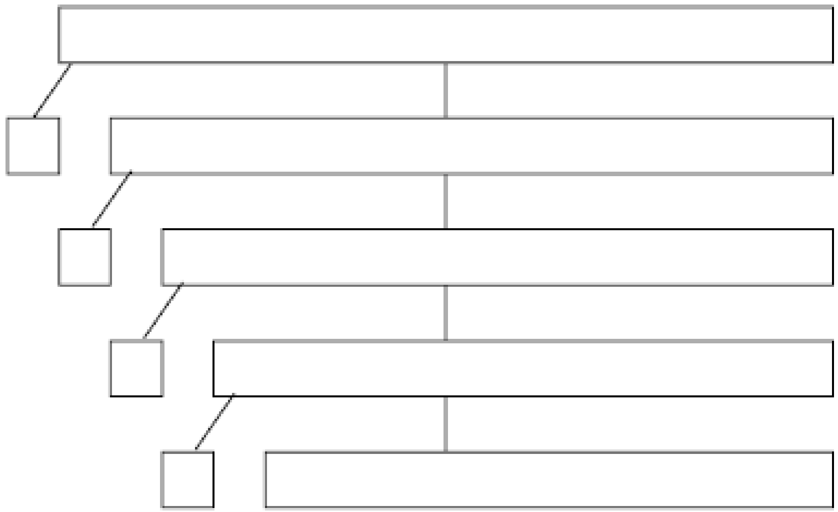
```
        if (first < last)
            Element pivot = E[first];
            int splitPoint = partition(E,pivot,first,last);
            E[splitPoint] = pivot;
            quickSort(E, first,splitpoint-1);
            quickSort(E, splitpoint+1, last);
```

Quick Sort

- Selection of pivot
  - Can be first or last element (here)
  - Median element
  - Random element

Quick Sort Analysis

- Worst Case Running Time
  - Occurs when pivot is always the largest element
  - Selecting the first element or last element as pivot causes this problem when list is already sorted
  - Running time proportional to n+(n-1)+(n-2)+....1
  - $O(n^2)$



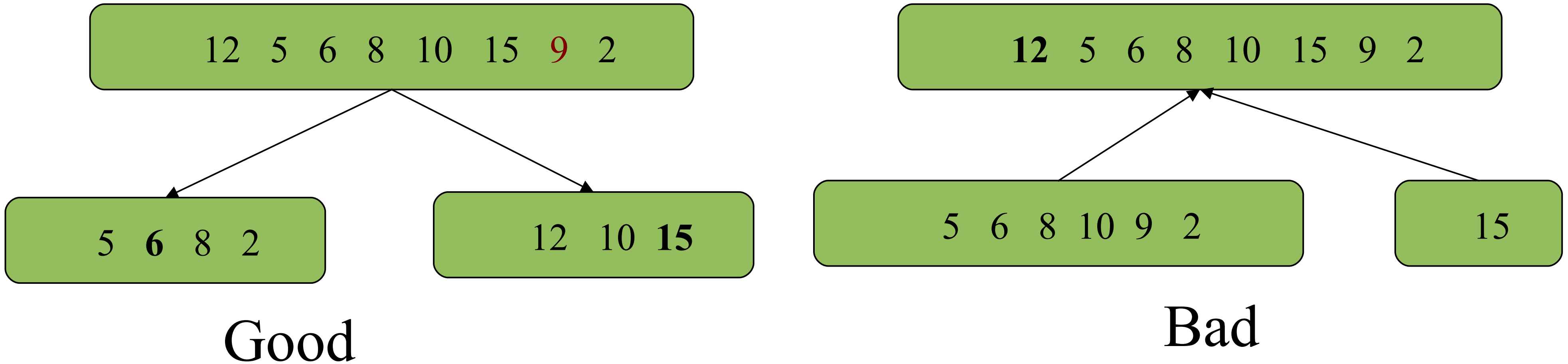
Src: Skiena, Algorithm Design Manual, Chapter 4.6

Average Case Analysis

- The partition in a quick sort of size s can be
  - Good : if the sizes of partitions are each less than 3s/4
    - A partition is good with the probability ½
    - ½ the pivots cause good partitions

2 5 6 8 9 10 **12** 15

  - Bad : if one of the partitions has size greater than 3s/4



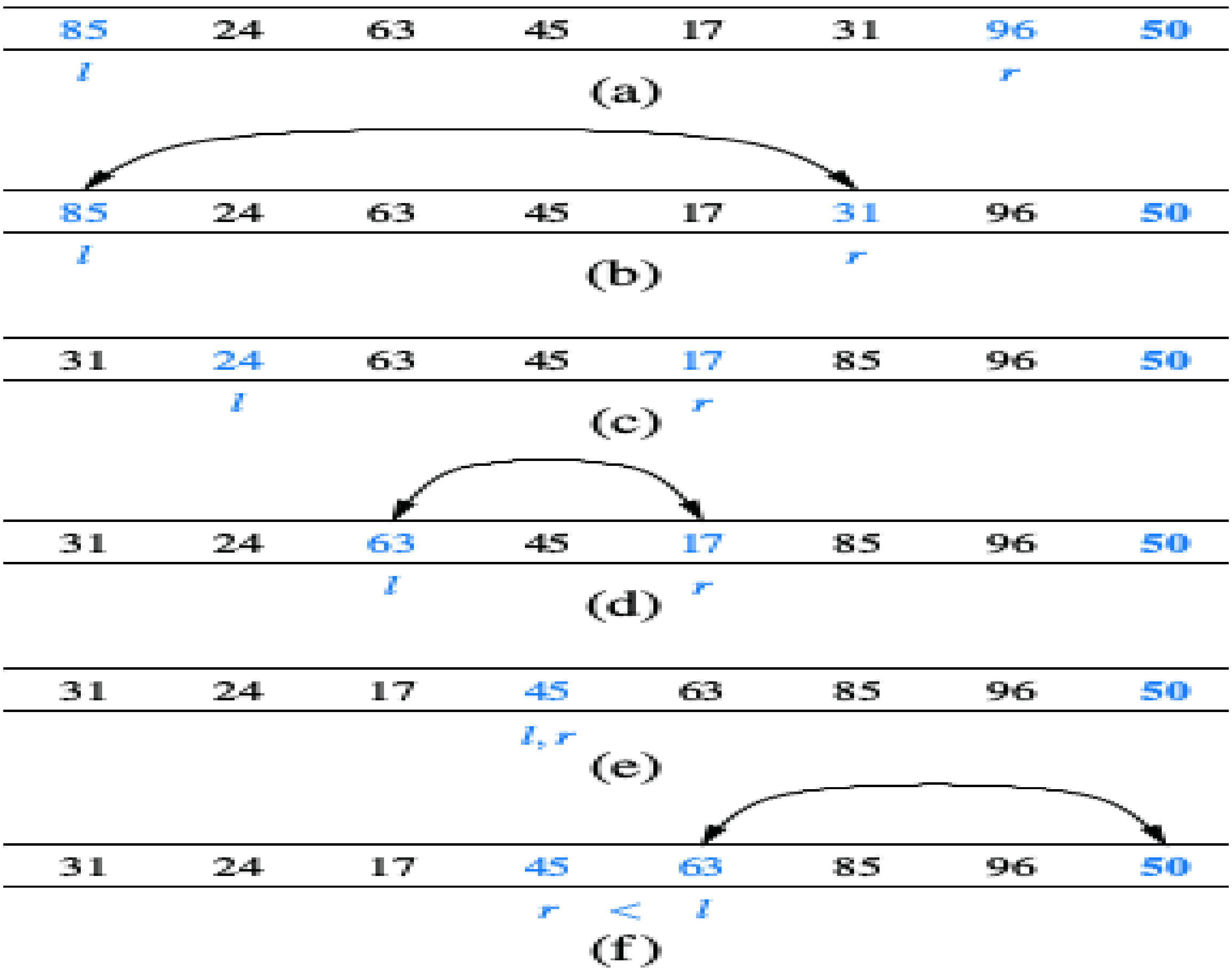
Average Case Analysis

- For a node of depth i, we expect
  - i/2 ancestors are good calls , they are result of good pivots
  - The size of the input sequence for the current call is at most $(3/4)^{i/2}n$
- For a node of depth $2\log_{4/3}n$,
  - the expected input size is one
  - The expected height of the quick-sort tree is O(log n)
  - The amount or work done at the nodes of the same depth is O(n)
- Expected running time of quicksort is O(nlogn)

In-Place Quick Sort

- Quick sort can be implemented without extra memory
- Replace operation used in partition step
  - Elements less than pivot have rank less than p of pivot
  - Elements greater than pivot have rank greater than p

In-Place Quick Sort



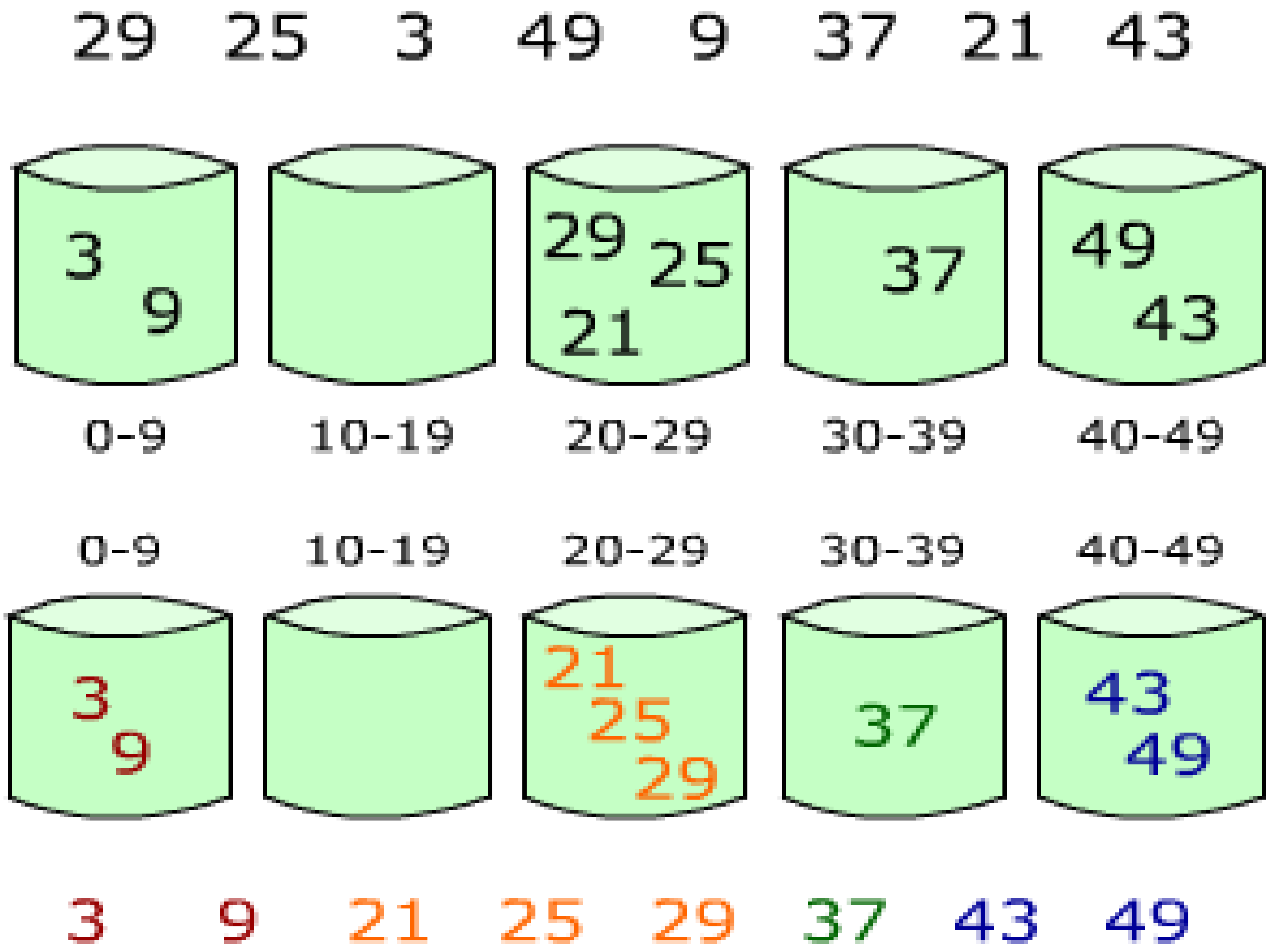
Quick Sort: Partitioning

- PARTITION(A,p, r) //inplace partitioning
  - x = A[r]
  - i = p-1
  - ***for*** j = p ***to*** r-1
    - ***if*** A[j] <= x
      - i = i + 1
      - exchange A[i] with A[j]
  - exchange A[i+1] with A[r]
- return i+1

External Sort: Bucket Sort

- Assumes elements are uniformly distributed
- Distribution sort
  - Array is partitioned into n equal sized subintervals/buckets
  - Each interval is a bucket
    - e.g each bucket can contain element starting with a particular alphabet
  - Each bucket sorted separately
    - Can use bucket sort again
    - Use other sorting techniques
- Elements from the buckets merged to form the sorted list

Bucket Sort



http://en.wikipedia.org/wiki/File:Bucket_sort_2.png

Bucket Sort: Properties

- Let be S be a sequence of n (key, element) items with keys in the range [0, N − 1]
  - Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)
  - Key-type Property
    - The keys are used as indices into an array and cannot be arbitrary objects
    - No external comparator
- Stable Sort Property
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

Methods for Bucketing

- Integer keys in the range [a, b]
  - Put item (k, o) into bucket B[k − a]
- String keys from a set D of possible strings, where D has constant size (e.g., names of the Indian states)
  - Sort D and compute the rank r(k) of each string k of D in the sorted sequence
  - Put item (k, o) into bucket B[r(k)]

Bucket Sort: Pseudocode

BUCKET-SORT(A)

let B[0: : n-1] be a new array

n = A:length

for i = 0 **to** n - 1

make B[i] an empty list

for i = 1 **to** n

insert A[i] into list B[floor(n A[i])]

for i = 0 **to** n 1

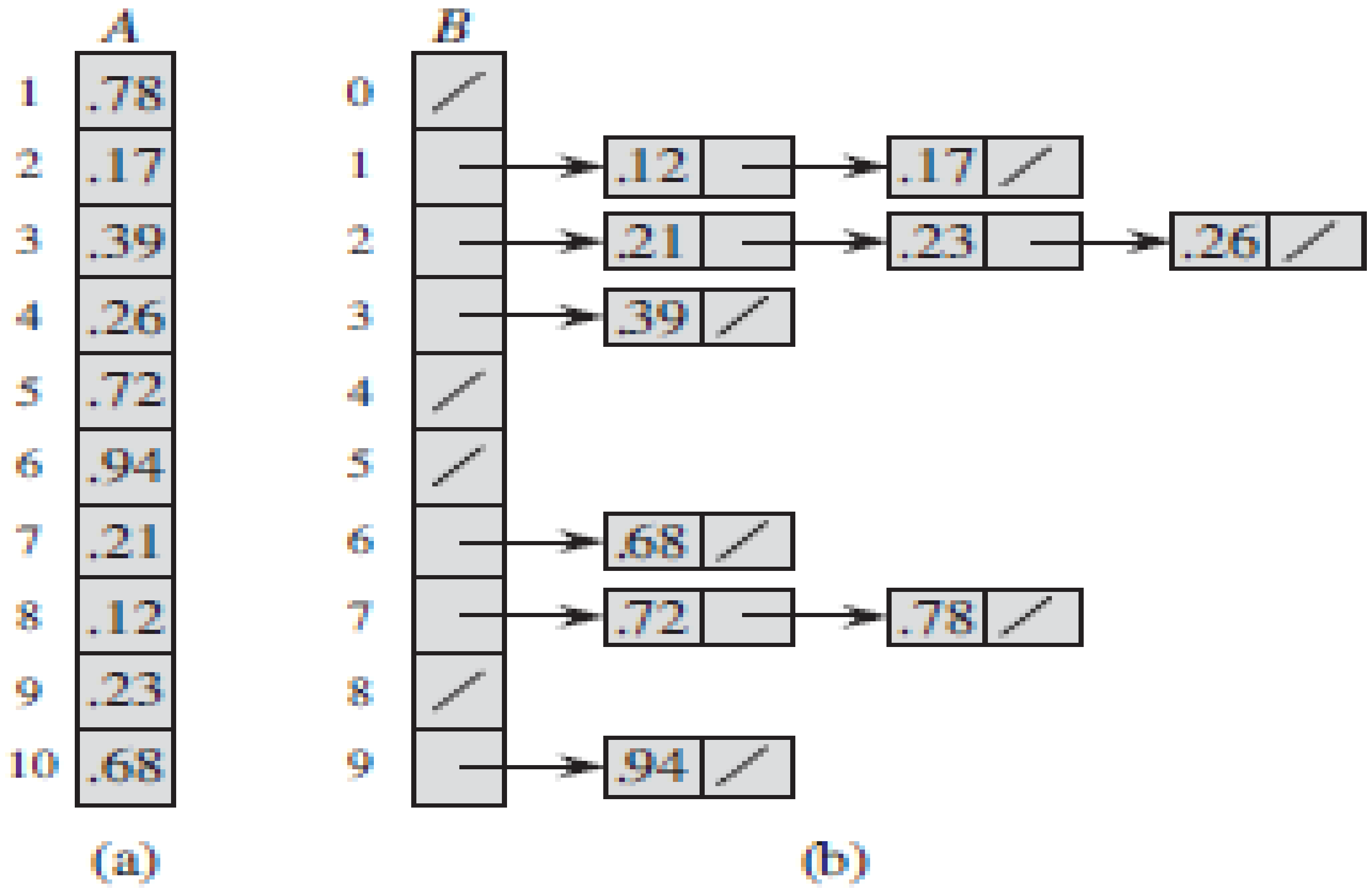
sort list B[i] with insertion sort

concatenate the lists B[0],B[1],....,B[n] together in order

Analysis of Bucket Sort

- Expected time is O(n)
  - Each step takes O(n) time except for sorting each bucket
  - Expected time to sort bucket Bi is $E[O(n_i^2)] = O(E[n_i^2]]$
    - Depends on distribution of each random variable n
  - $\sum_{i=1}^{n} O\left(E\left(n_i^2\right)\right)$
  - Given n elements and n buckets, probability that a given element falls in a bucket B[i] is 1/n
    - Probability follows Binomial distribution
      - Mean $E[n_i] = np = 1$, Variance $Var[n_i] = np(1-p) = 1-1/n$
    - $E[n_i^2] = Var[n_i] + (E[n_i])^2 = 1 - 1/n + 1^2 = $ (1)
    - Substituting in above equation expected time is O(n)
- Worst case complexity – O(n^2) (skewed distribution)

Bucket Sort: Example



Src: Corman, Rivest etal, “Introduction to Algorithms”, 8.4

Problem

- Using the previous example as a model, illustrate the operation of BUCKET-SORT on the array A
(.79, .13, .16, .64, .39, .20, .89, .53, .71, .42).
- Use bucket sort to sort the following elements
  - (233, 456, 567, 777, 554, 343, 267, 466, 786, 900, 600, 545, 531, 879, 821, 289, 192, 945)
  - How are the buckets chosen?

Lexicographic Order

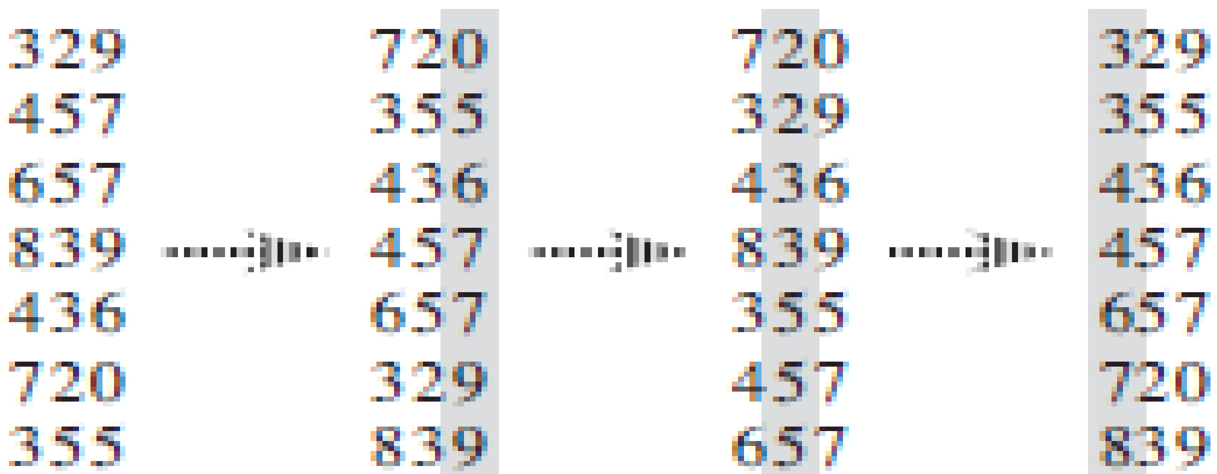
- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
  - Example: coordinates in 3d space
- The lexicographic order of two d-tuples is recursively defined as follows
  - $(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$ if
  - $x_1 < y_1 \vee x_1 = y_1 \wedge (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$
    - i.e., the tuples are compared by the first dimension, then by the second dimension etc

Radix Sort

- Specialization of lexicographic sort
  - Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing a stable sort algorithm d times
  - Radix sort uses bucket sort for each dimension
  - Applicable where keys in each dimension are integers
- Example:
  - (7, 4, 6) (5, 1, 5) (2, 4, 6) (2, 1, 4) (3, 2, 4)
  - (2, 1, 4) (3, 2, 4) (5,1, 5) (7, 4, 6) (2, 4, 6)
  - (2, 1, 4) (5,1,5) (3, 2, 4) (7, 4, 6) (2, 4, 6)
  - (2, 1, 4) (2, 4, 6) (3, 2, 4) (5, 1, 5) (7, 4, 6)

Radix Sort

- RADIX-SORT(A,d)
  - for i = 1 to d
    - use a stable sort to sort array A on digit i
- Given n d-digit numbers in which each digit can take on up to k possible values, RADIX-SORT correctly sorts these numbers in Θ(d.(n+k)) time if the stable sort it uses takes Θ(n+k) time.



Src: Corman, Rivest etal, “Introduction to Algorithms”, 8.3