19CSE433 Computer Graphics & Visualization

Professional Elective 1

5^{th} Semester,2021-22 Odd

2019-22 Batch, BTech CSE

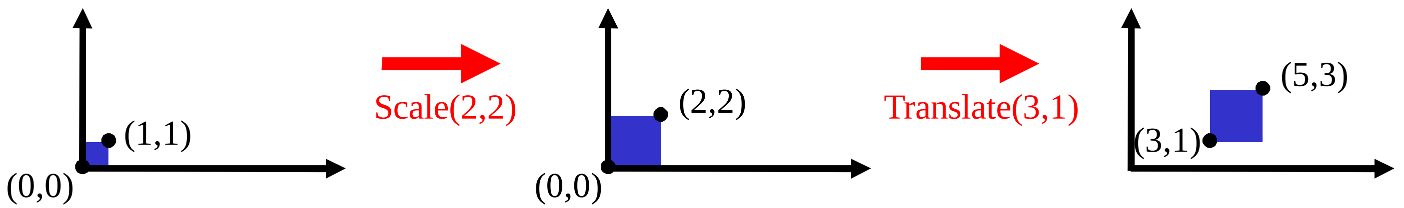
Dr.S.Padmavathi,

Department of Computer Science and Engineering,

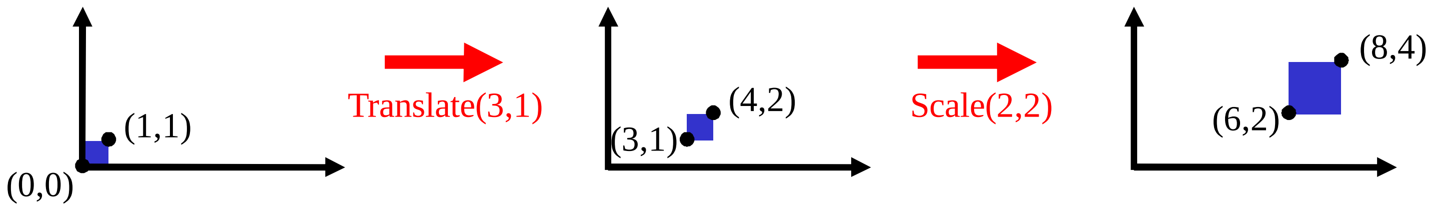
Amrita School of Engineering, Coimbatore

Non-commutative Composition

Scale then Translate: p' = T (S p) = TS p



Translate then Scale: p' = S (T p) = ST p



Two Transform Paths

Scale then Translate

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

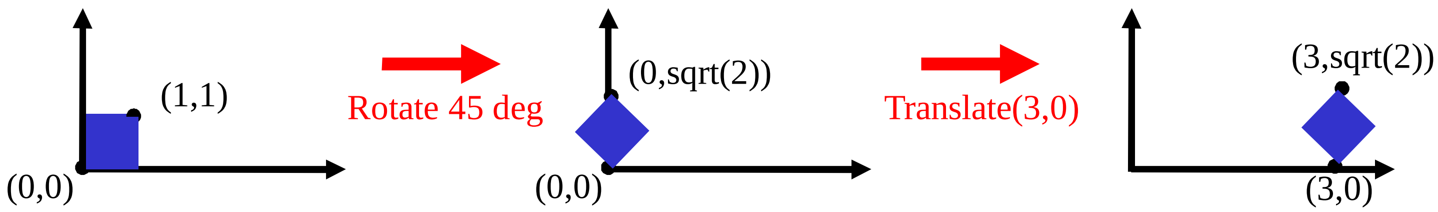
Translate then Scale

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

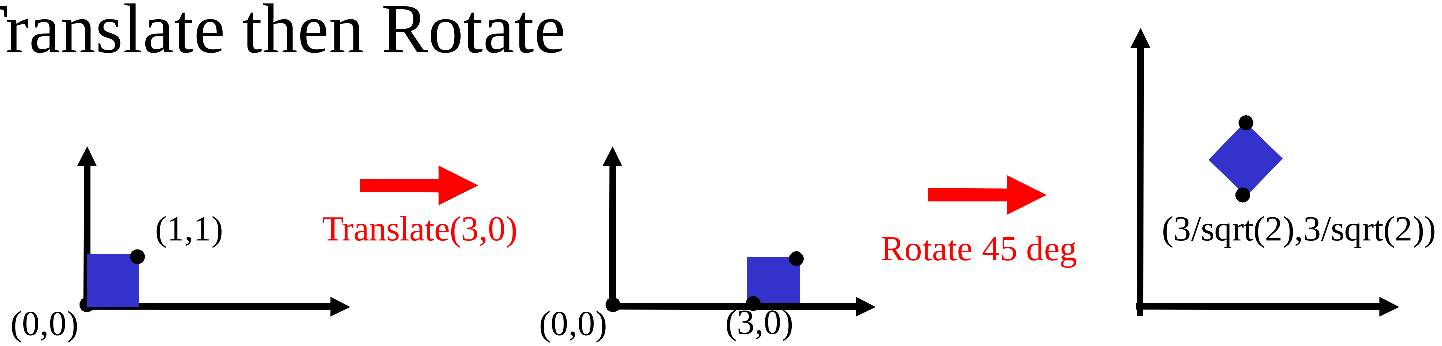
$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

How are transforms combined?

Rotate then Translate



Translate then Rotate



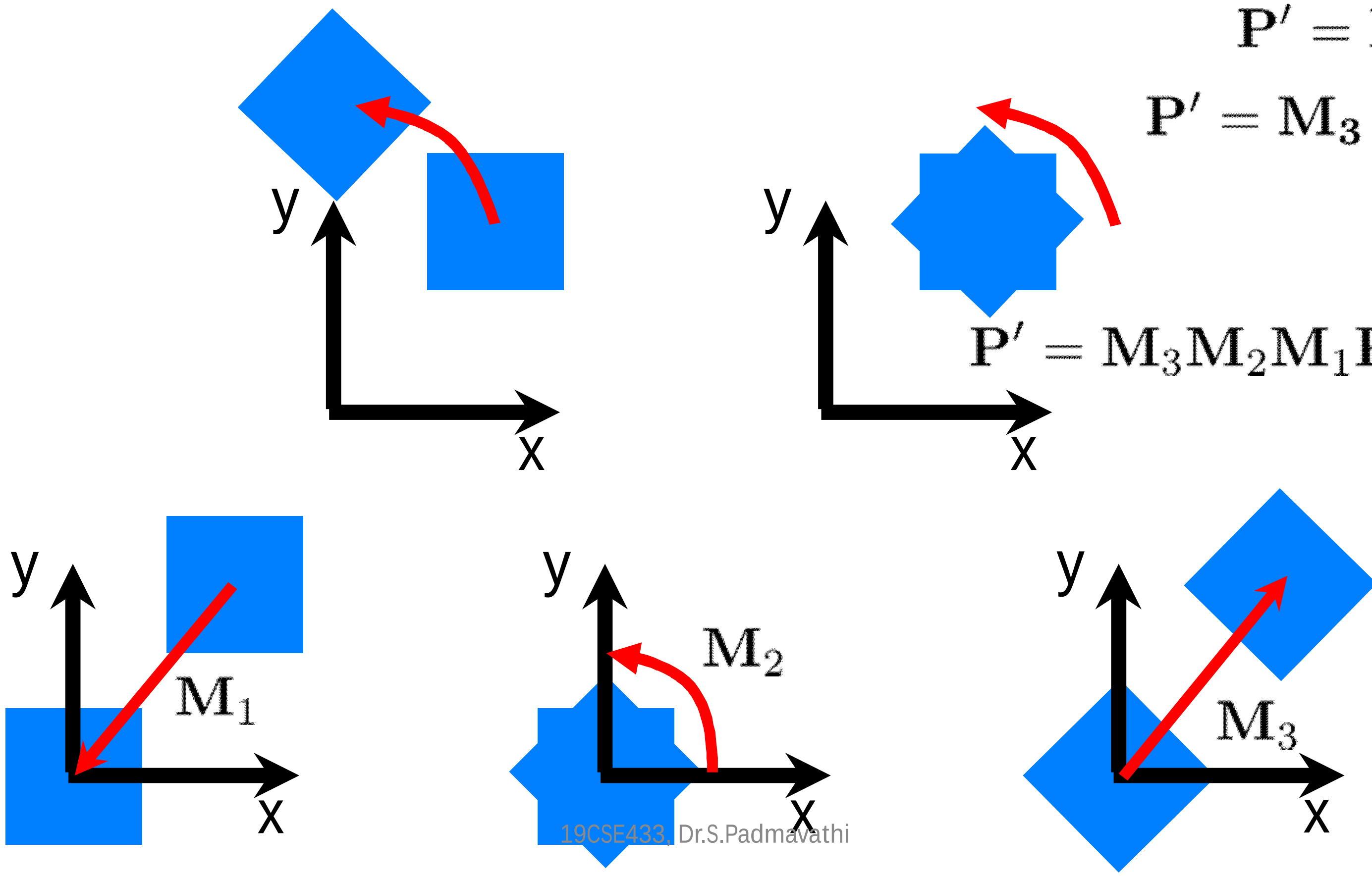
Caution: matrix multiplication is NOT commutative!

Application: Rotation about a non-origin point

- Always compose from right to left
  - Here, transform M_1 is applied first
  - Transform M_3 is applied last

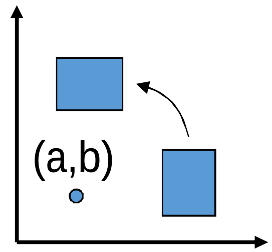
$$\mathbf{P'} = \mathbf{M_3 M_2 M_1 P}$$

$$\mathbf{P'} = \mathbf{M_3}\left(\mathbf{M_2}\left(\mathbf{M_1}\left(\mathbf{P}\right)\right)\right)$$

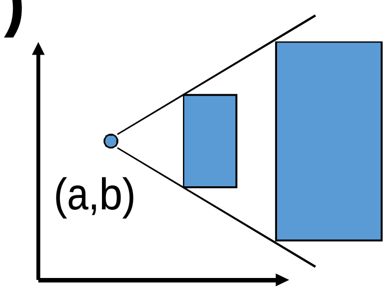


Matrix Composition

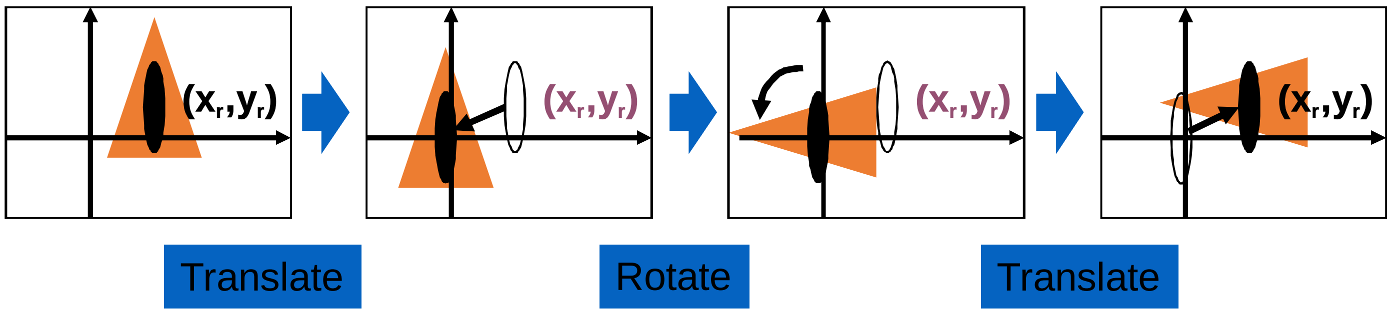
- **Rotate by θ around arbitrary point (xr,yr)**
  - $$M = T(xr, yr) \times R(\theta) \times T(\text{-}xr, \text{-}yr)$$



- **Scale by sx, sy around arbitrary point (xf,yf)**
  - $$M = T(xf, yf) \times S(sx, sy) \times T(\text{-}xf, \text{-}yf)$$



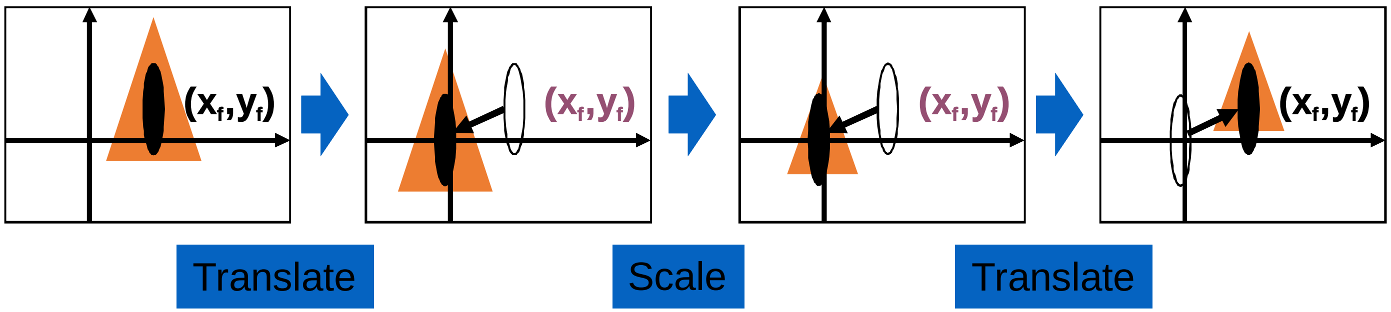
Pivot-Point Rotation



$$T(x_r, y_r)\cdot R(\theta)\cdot T(-x_r, -y_r) = R(x_r, y_r, \theta)$$

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r \sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

General Fixed-Point Scaling



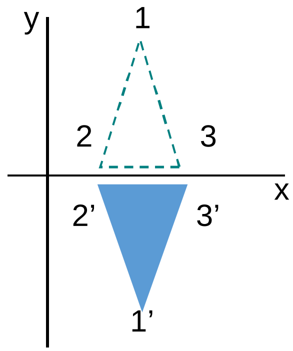
$$T(x_f, y_f) \cdot S(s_x, s_y) \cdot T(-x_f, -y_f) = S(x_f, y_f, s_x, s_y)$$

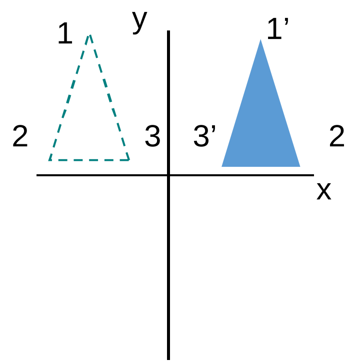
$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

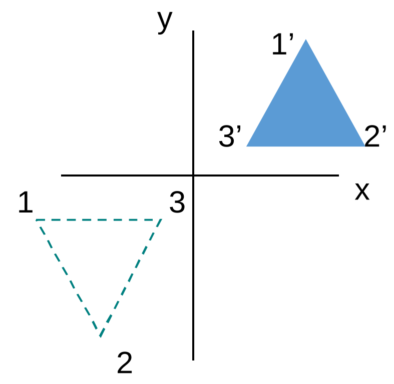
Reflection

- **Reflection with respect to the axis**
  - Reflections on x-axis
  - Reflections on y-axis
  - Xy axis (origin)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



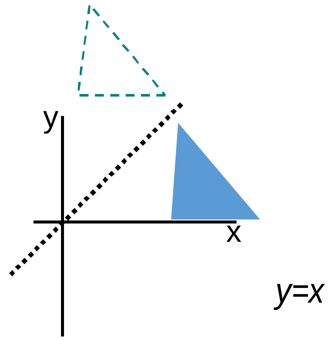




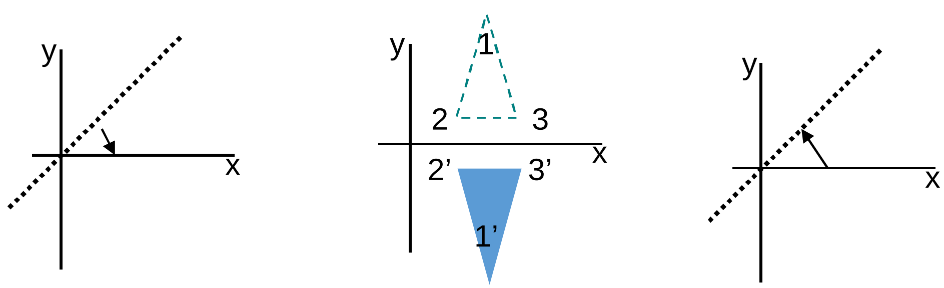
Reflection

- **Reflection with respect to a Line**

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- **Clockwise rotation of 45° → Reflection about the x axis → Counterclockwise rotation of 45°**

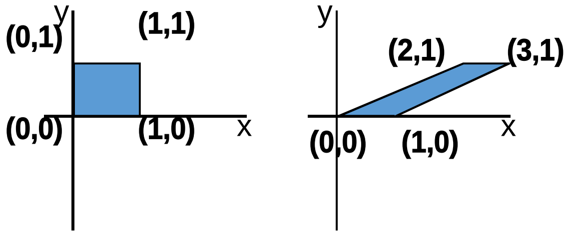


Shear

- **Converted to a parallelogram**

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x' = x + sh_x \cdot y, \quad y' = y$

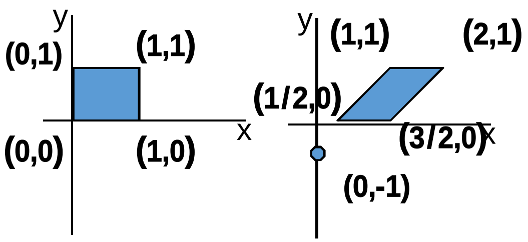


$(Sh_x=2)$

- **Transformed to a shifted parallelogram**
  (Y = Yref)

$$\begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x' = x + sh_x \cdot (y - y_{ref}), \quad y' = y$



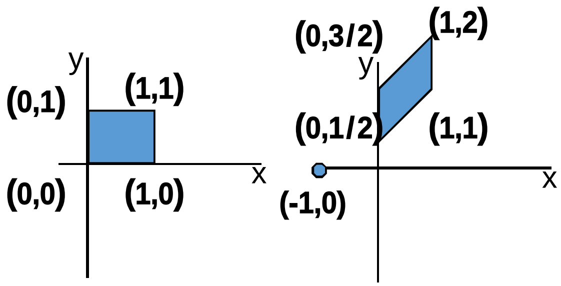
$(Sh_x=1/2,\ y_{ref}=-1)$

Shear

- **Transformed to a shifted parallelogram**

  (X = Xref)

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$



$x' = x, \quad y' = sh_y \cdot (x - x_{ref}) + y$

Translations in homogenised coordinates

- Transformation matrices for 2D translation are now 3x3.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + d_x$$
$$y' = y + d_y$$
$$1 = 1$$

Concatenation.

- We perform 2 translations on the same point:

$$P' = T(d_{x1}, d_{y1}) \cdot P$$

$$P'' = T(d_{x2}, d_{y2}) \cdot P'$$

$$P'' = T(d_{x1}, d_{y1}) \cdot T(d_{x2}, d_{y2}) \cdot P = T(d_{x1} + d_{x2}, d_{y1} + d_{y2}) \cdot P$$

So we expect :

$$T(d_{x1}, d_{y1}) \cdot T(d_{x2}, d_{y2}) = T(d_{x1} + d_{x2}, d_{y1} + d_{y2})$$

Concatenation.

The matrix product $T(d_{x1}, d_{y1}) \cdot T(d_{x2}, d_{y2})$ is:

$$\begin{bmatrix} 1 & 0 & d_{x1} \\ 0 & 1 & d_{y1} \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & d_{x2} \\ 0 & 1 & d_{y2} \\ 0 & 0 & 1 \end{bmatrix} = ?$$

Concatenation.

The matrix product $T(d_{x1}, d_{y1}) \cdot T(d_{x2}, d_{y2})$ is:

$$\begin{bmatrix} 1 & 0 & d_{x1} \\ 0 & 1 & d_{y1} \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & d_{x2} \\ 0 & 1 & d_{y2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_{x1} + d_{x2} \\ 0 & 1 & d_{y1} + d_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of translations.

1. $T(0,0) = I$
2. $T(s_x, s_y) \cdot T(t_x, t_y) = T(s_x + t_x, s_y + t_y)$
3. $T(s_x, s_y) \cdot T(t_x, t_y) = T(t_x, t_y) \cdot T(s_x, s_y)$
4. $\mathrm{T}^{-1}(s_x, s_y) = T(-s_x, -s_y)$

Homogeneous form of scale.

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Concatenation of scales.

The matrix product $S(s_{x1}, s_{y1}) \cdot S(s_{x2}, s_{y2})$ is:

$$\begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Only diagonal elements in the matrix - easy to multiply!

Homogeneous form of rotation.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For rotation matrices,

$$R^{-1}(\theta) = R(-\theta).$$

Rotation matrices are orthogonal, i.e :

$$R^{-1}(\theta) = R^{T}(\theta)$$

Orthogonality of rotation matrices.

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R^T(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(-\theta) = \begin{bmatrix} \cos-\theta & -\sin-\theta & 0 \\ \sin-\theta & \cos-\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Other properties of rotation.

$$R(0) = I$$

$$R(\theta) \cdot R(\phi) = R(\theta + \phi)$$

and

$$R(\theta) \cdot R(\phi) = R(\phi) \cdot R(\theta)$$

But this is only because the axis of rotation is the same

For 3D rotations, need to be more careful