Design and Analysis of Algorithms

Algorithm Analysis

Odd Semester- 2020-21

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Course Details

- Lecture Notes – on Teams
- Text Book
  - Michael T Goodrich, Roberto Tamassia, "Algorithm Design: Foundations, Analysis and Internet Examples", John Wiley and Sons, 2001
  - Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, "Introduction to Algorithms, Second Edition", The MIT Press, 2001

Evaluation

- Grade Policy
  - Final – 35% -> 15 + 20 (Viva)
  - Midterm – 10 Online, 10 Viva
  - Continuous Theory – 15
    - Problem solving assignments given regularly and in class
    - Both written submissions and class participation to be evaluated
  - Continuous Lab
    - 15: 3 long contests on HPOJ
    - 15: Continuous assessment consisting of group exercises and HPOJ contests

Lecture Schedule (Tentative)

Term I

- Algorithm Analysis
- Sorting Algorithms
- Graph Algorithms

Term II

- Recurrence Analysis
- Divide and Conquer
- Greedy Strategy
- Dynamic Programming

Lecture Schedule

Term III

- Backtracking and Branch and Bound
- String Algorithms
- Network Flow
- Introduction to NP Completeness

May be modified over time

Problem Solving

1

Identify problems in real world solvable by computers

2

Understand the problem

- Understand the inputs
- Output requirements
- Constraints under which the problem must operate

3

Identify potential solutions

4

Select best solution

- Fastest
- Most accurate

Pseudocode

- High level description of an algorithm
- More structured than English prose
- Less detailed than an actual program
  - Hides program design details

```
Algorithm arrayMax(A, n)
        Input array A of n integers
        Output maximum element of A

        currentMax ← A[0]
        for i ← 1 to n - 1 do
                if A[i] < currentMax then
                        currentMax ← A[i]
        return currentMax
```

Pseudocode

Expressions

- ← assignment, like = in Java
- = Equality testing, like == in Java
- Superscripts and other mathematical formatting allowed

Method Declaration

- Algorithm *method*(arg1…)
  - Input..
  - Ouput..

Indentation replaces braces

Control Flow

if … then .. [else…]

while .. do ..

repeat … until ..

for … do…

Analyzing Algorithms

- Correctness
- Amount of Work done
- Space used
- Simplicity, clarity
- Optimality

Correctness

Understand	Understand what correctness means • Define the characteristics of the input an algorithm is expected to work on • The results that each input must produce
Prove	Prove the statement about the relationship between input and output
Prove	Prove Correctness of algorithm

Proof of Correctness

- Simple Techniques
  - By example
  - By contrapositives and contradiction
  - Induction
  - Loop Invariants

Analysis of Amount of Work done

Algorithm

- Set of simple instructions to be followed to solve a problem

Algorithm Analysis

- Determine resources, time and space the algorithms requires
- Helps choose among different algorithms to a solution

Goal

- Estimate time required to execute the algorithm
- Reduce the running time of the program
- Understand results of careless use of recursion

Issues in calculating running time

- Running time grows with input size
- Varies with different inputs
- Actual running time can be calculated in seconds or milliseconds
  - The system setup must be same for all inputs
    - Same hardware and software must be used
  - Actual time may be affected by other programs running on the same machine
- A theoretical analysis is usually preferred

Average Case and Worst Case

- Running time of an algorithm is not constant
  - Depends on input
    - Can run fast for certain inputs and slow for others
    - e.g linear search
- Average Case Cost
  - Cost of the algorithm on average
  - Difficult to calculate
- Worst Case
  - Gives an upper limit for the running time
  - Easier to analyze

What we need

Model of Computation

- Mathematical Framework
- Asymptotic Notation

What to Analyze

- Running Time Calculations

Checking the analysis

Random Access Machine Model

- Model of Computation to analyze algorithms
- Primitive Operations
  - Assigning a value to a variable
  - Performing an arithmetic operation
  - Calling a method
  - Comparing two numbers
  - Indexing into an array
  - Following an object reference
  - Returning from a method
- Count primitives to give high level estimate

Counting Primitives: Recap

```
Algorithm FindMax(S, n)
Input : An array S storing n numbers, n>=1
Output: Max Element in S
curMax <-- S[0] (2 operations)
i ← 0 (1 operations)
while i< n-1 do (n comparison operations)
    if curMax < A[i] then (2(n-1) operations)
        curMax <-- A[i] (2(n-1) operations)
    i ← i+1; (2 (n-1) operations)
return curmax (1 operations)
```

Complexity between 5n and 7n-2

Problems

- Calculate running time:

-
  ```
  sum = 0;
  for( i=1; i<n; i*=2 )
      sum++;
  ```

-
  ```
  sum = 0;
    for( i=0; i<n; i++ )
        for( j=1; j<n; j*=2 )
           sum++;
  ```

-
  ```
  sum = 0;
  for( i=0; i<n; i++ )
      for( j=0; j<n*n; j++ )
          sum++;
  ```

Problem continued

-
  ```
  sum = 0;
  for( i=1; i<2n; i++ )
      for( j=1; j<=i; j++ )
          sum++;
  ```

-
  ```
  for (i = 1; i <= n; i++) {
    for (j = 1; j <= n; j += i)
        x = x + 1;
  }
  ```

Problems

Consider the task of finding the missing element in a sequence of n elements

Consider the task of finding the frequency of occurrence of each element in a set

Prefix averages

The i-th prefix average of an array X is average of the first (i □ 1) elements of X:

A[i] =X[0] + X[1] + … + X[i])/(i+1)

Two algorithms

Problems

```
Algorithm prefixAverage1($X$,$n$)
    Input array $X$ of integers
    Output array $A$ of prefix averages of $X$
    $A \leftarrow$ new array of $n$ integers
    for $i \leftarrow 0$ to $n$-1 do
        $s \leftarrow X[0]$
        for $j \leftarrow 1$ to $i$ do
            $s \leftarrow s + X[j]$
        $A[i] \leftarrow s/(i+1)$
    return $A$
```

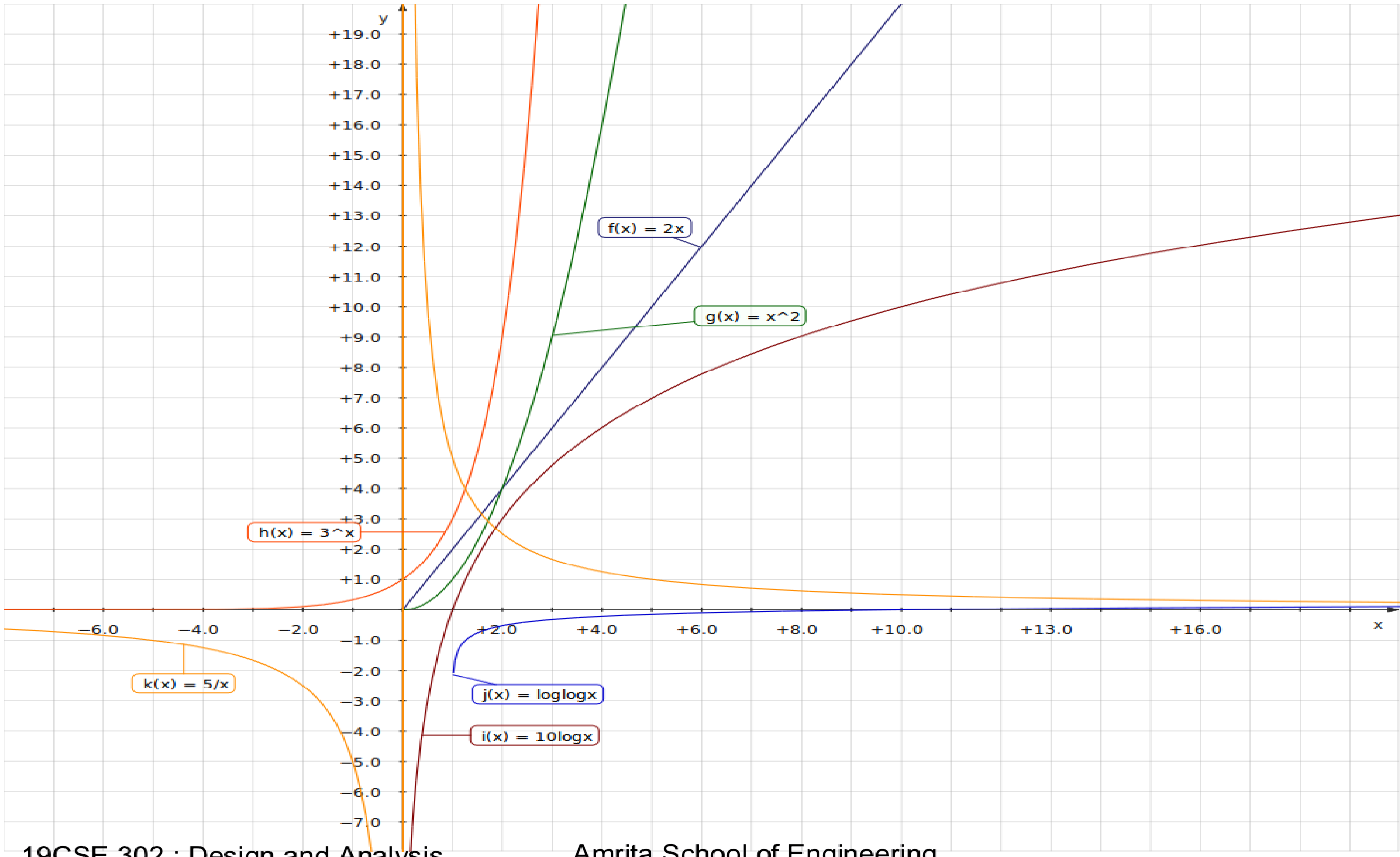
Problems

```
Algorithm prefixAverage2($X$,$n$)
    Input array $X$ of integers
    Output array $A$ of prefix averages of $X$
    $A \leftarrow$ new array of $n$ integers
    for $i \leftarrow 0$ to $n-1$ do
        $s \leftarrow s + X[i]$
        $A[i] \leftarrow s/(i+1)$
    return $A$
```

Growth Rates of Running Time

- Important factor to be considered when estimating running time
- When experimental setup (hardware/software) changes
  - Running time is affected by a constant factor
    - 2n or 3n or 100n is still linear
  - Growth rate of the running time is not affected
- Growth rates of functions
  - Linear
  - Quadratic
  - Exponential

Some Function Plots



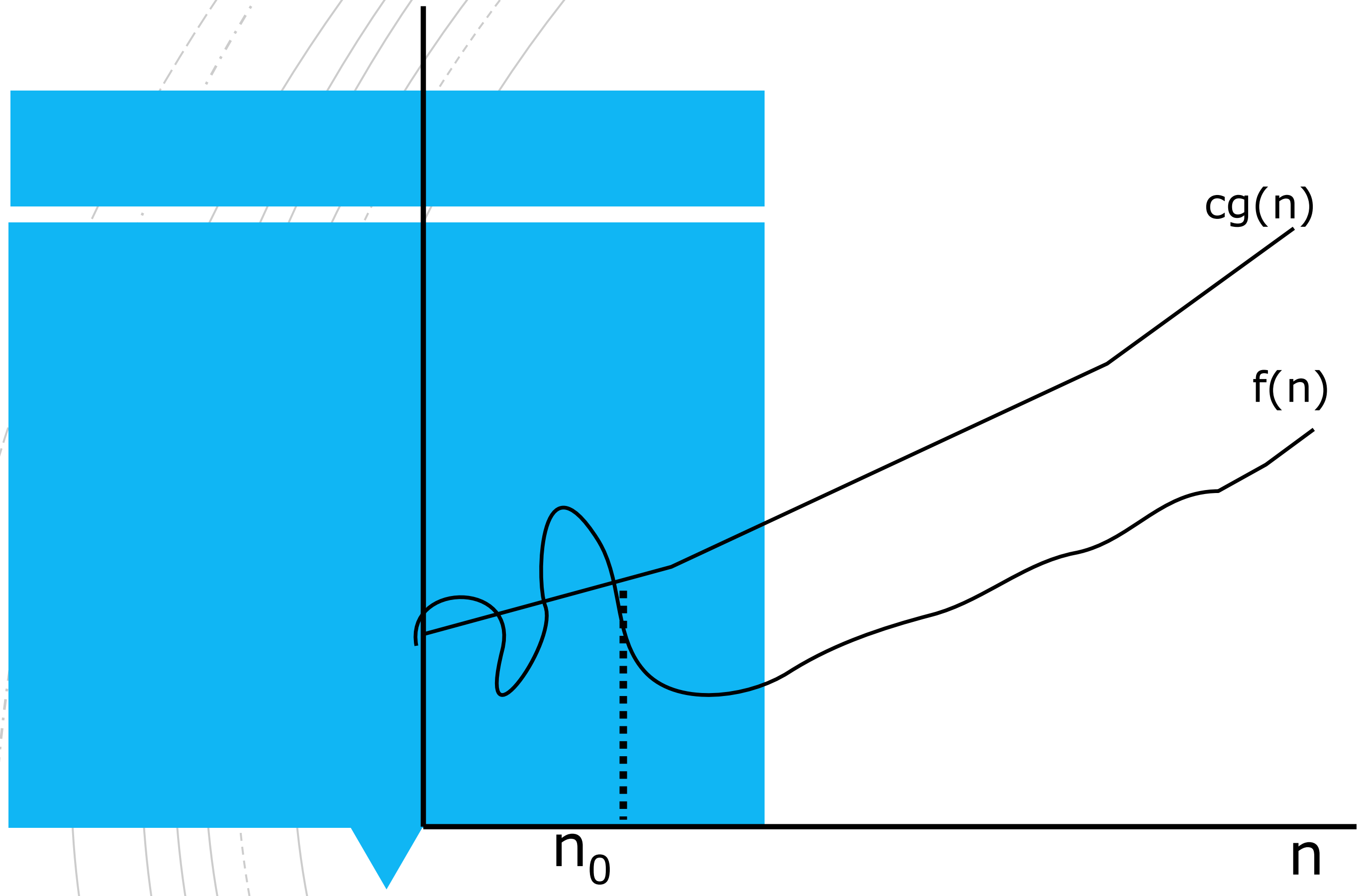
Asymptotic Analysis

- Can be defined as a method of describing limiting behavior
- Used for determining the computational complexity of algorithms
- A way of expressing the main component of the cost of an algorithm using the most determining factor
    - e.g if the running time is $5n^2+5n+3$, the most dominating factor is $5n^2+5n+3$
    - Capturing this dominating factor is the purpose of asymptotic notations

Big Oh Notation

- Given a function f(n) we say, f(n) = O(g(n)) if there are positive constants c and n_0 such that f(n)<= cg (n) when n>= n_0
- Example
  - 2n + 8 is O(n)
  - 2n+8 <= cn
  - (c-2)n >= 8
  - n >= 8/(c-2)
  - Choose c = 3, and n_0 as 8, then the rule holds

O(n) – growth function



$$f(n) = O(g(n))$$

Example

- Example: the function n^2 is not $O(n)$
  - Must prove $n^2 <= cn$
  - $n <= c$
  - The above inequality cannot be satisfied since c must be a constant
  - Hence proof by contradiction

More Examples

Show

Show 7n-2 is O(n)

- need c > 0 and n0 >= 1 s.t 7n-2 <= cn for n >= n0
- this is true for c = 7 and n0 = 1

Show

Show 3n^3 + 20n^2 + 5 is O(n^3)

- find c, n0 s.t 3n^3 + 20n^2 + 5 <= cn^3 for n >= n0
- this is true for c = 4 and n0 = 21

Show

Show 3 log n + log log n is O(log n)

- need c > 0 and n0 >= 1 such that 3 log n + log log n <= c log n for n >= n0
- this is true for c = 4 and n0 = 2

Problems

- Show that $6n^2 + 20n$ is $O(n^3)$
- When does the time taken to calculate the circumference of a circle run faster than the time taken to find the area of a circle, considering n to be the radius.

Some problems

- Order the following functions by the big-Oh notation
- $6n\log n$, 2^{100}, $\log^2 n$, $1/n$, n^3, $n^2\log n$
- What is the total running time of counting from 1 to n in binary if the time needed to add 1 to the current number i is proportional to the number of bits in the binary expansion of i that must change in going from i to i+1
- Is 2^{n+1} $O(2^n)$?
- Is 2^{2n} $O(2^n)$?

Big Oh Significance

- The big-Oh notation gives an upper bound on the growth rate of a function
- "f(n) is O(g(n))" means that the growth rate of f(n) is no more than the growth rate of g(n)
  - Both can grow at the same rate
- Though 1000n is larger than n^2, n^2 grows at a faster rate
  - n^2 will be larger function after n = 1000
  - Hence $1000n = O(n^2)$
- The big-Oh notation can be used to rank functions according to their growth rate

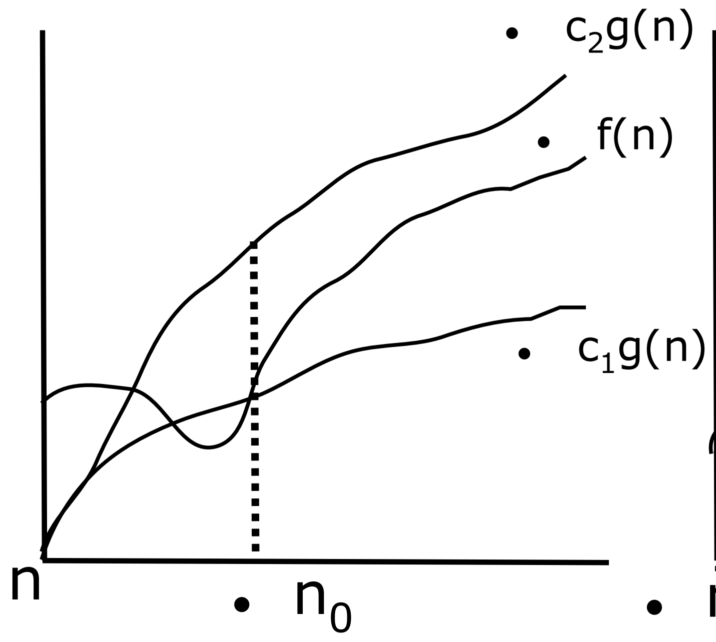
Some common rules

- If is f(n) a polynomial of degree d, then f(n) is $O(n^d)$, i.e.,
  - Drop lower-order terms
  - Drop constant factors
- Use the smallest possible class of functions to represent in big Oh
  - "2n is O(n)" instead of "2n is $O(n^2)$"
- Use the simplest expression of the class
  - "3n+ 5 is O(n)" instead of "3n + 5 is O(3n)"

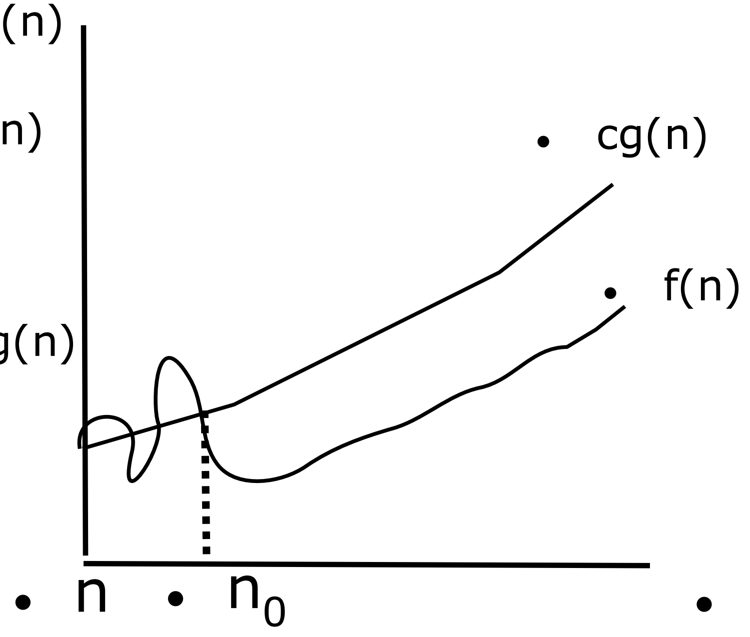
Asymptotic Notations

- $f(n) = O(g(n))$ if there are constants c and n_0 such that $f(n) <= cg(n)$ when $n >= n_0$
- $f(n) = \Omega\, g(n))$ if there are constants c and n_0 such that $f(n) >= cg(n)$ when $n >= n_0$
- $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. $f(n) <= c_1 g(n)$ and $>= c_2 g(n)$
- $f(n) = o(g(n))$ if $f(n) = O(g(n))$ and $f(n) != \Theta(g(n))$
  - $f(n) < cg(n)$
  - Goal
    - Establish relative order among functions!!

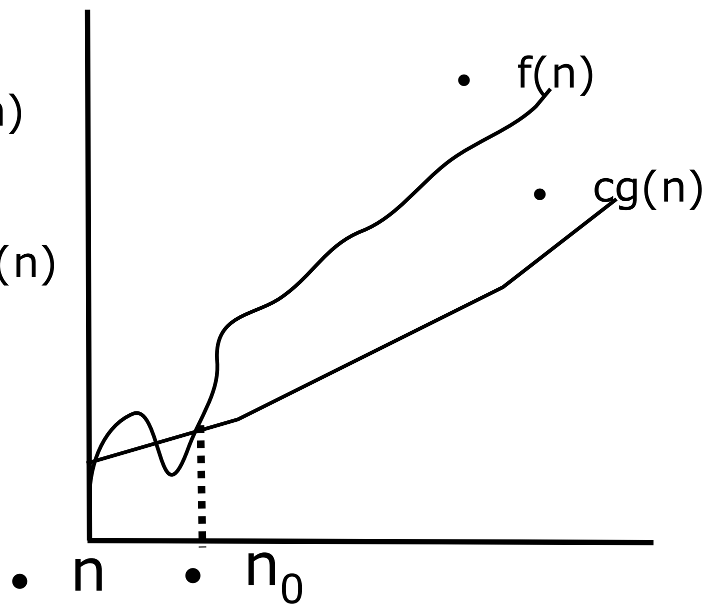
Growth of Functions



- f(n) = Θ(g(n))



- f(n) = O(g(n))



- f(n) = Ω(g(n))

Problems

- $n^3 - 3n^2 - n + 1 = \Theta(n^3)$.
- For each of the following pairs of functions, either f(n) is in O(g(n)), f(n) is in Ω(g(n)), or f(n) = Θ(g(n)). Determine which relationship is correct and briefly explain why.
    - $f(n) = \log n^2$; $g(n) = \log n + 5$
    - $f(n) = (n^2 - n)/2$, $g(n) = 6n$

Importance of Asymptotic Notation

- Though 1000n is larger than n^2, n^2 grows at a faster rate
  - n^2 will be larger function after n = 1000
  - 1000n = O($n^{2)}$
- If f(n) is O(g(n)), we are guaranteeing that f(n) grows at a rate no faster than g(n)
- f(n) is Ω(g(n)), then g(n) is lower bound

Importance of Asymptotics

•Table of max-size of a problem that can be solved in one second, one minute and one hour for various running times measures in microseconds [Goodrich]

Running Time	Maximum Problem Size (n)		
	1sec	1 min	1 hour
400n	2500	150000	9000000
20nlogn	4096	166666	7826087
$2n^2$	707	5477	42426
n^4	31	88	244
2^n	19	25	31

Asymptotic Rules

- If d(n) is O(f(n)), ad(n) is O(f(n)), for any a>0
  - $d(n) <= cf(n)$
  - $ad(n) <= acf(n)$ // ac is still a constant, hence proved
- If d(n) is O(f(n)), and e(n) is O(g(n)), then d(n)+e(n) is O(f(n)+g(n))
  - $d(n) <= c_1 f(n)$ and $e(n) <= c_2 g(n)$
  - $d(n)+e(n) <= c_1 f(n)+c_2 g(n)$
  - Choose a constant c_3 which is max of (c_1,c_2). Then $d(n)+e(n) <= c_3(f(n)+g(n))$

Asymptotic Rules

- 3. If d(n) is O(f(n)), and e(n) is O(g(n)), then d(n)e(n) is O(f(n)g(n))
  - $d(n) \le c_1 f(n)$ and $e(n) \le c_2 g(n)$
  - $d(n)e(n) \le c_1 f(n) c_2 g(n)$
  - $d(n)+e(n) \le c_3(f(n)+g(n))$ // $c_3 = c_1 c_2$
- 4. If d(n) is O(f(n)), and f(n) is O(g(n)), then d(n) is O(g(n))
  - $d(n) \le c_1 f(n)$ and $f(n) \le c_2 g(n)$
  - $\Longrightarrow d(n) \le c_1 c_2 g(n) \le c_3 g(n)$ // $c_3 = c_1 c_2$

Asymptotic Rules

- 5. If d(n) is O(f(n)), and e(n) is O(g(n)), then d(n)+e(n) is Max(O(f(n)), O(g(n)))
- 6. n^x is $O(a^n)$ for any fixed x>0, a>1
  - $n^x <= ca^n => \log n^x <= c \log a^n$
  - $x \log n <= cn \log a$
- 7. $\log n^x$ is O(log n) for any fixed x>0
  - $\log n^x <= c \log n => x\log n <= c\log n$
- 8. $\log^x n$ is $O(n^y)$ for some constant x>0, y>0
  - $(\log n)^x <= cn^y$

Example

- Show $2n^3 + 4n^2 \log n$ is $O(n^3)$
  - $\log n$ is $O(n)$ (rule 8)
  - $4n^2 \log n$ is $O(4n^3)$ (rule 3)
  - $2n^3 + 4n^2 \log n$ is $O(2n^3 + 4n^3)$ (rule 2)
  - $2n^3 + 4n^3$ is $O(n^3)$ (rule 1 or polynomial rule)
  - $2n^3 + 4n^2 \log n$ is $O(n^3)$ (rule 4)