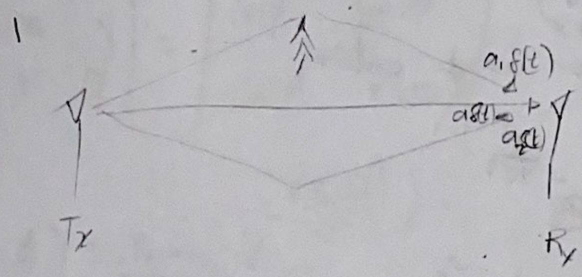
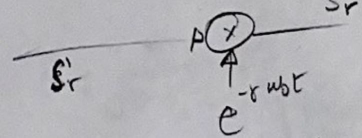
1° Trabalho de TP545

a) x e y são VA Gaussinas i.i.d com $\mu = 0$, σ e K



$$S_r' = a\, e^{j w_0 t} + a_1 e^{j(w_0 t + \phi_1)} + a_2 e^{j(w_0 t + \phi_2)} + \cdots$$

Em banda base



$$S_r = a + a_1 e^{j\phi_1} + a_2 e^{j\phi_2} + \cdots$$

$$= a + \sum_{k=1}^{N} a_k e^{j\phi_k}$$

$$= a + \sum_{k=1}^{N} a_k (\cos\phi_k + j \sin\phi_k)$$

$$= a + \sum_{k=1}^{N} a_k \cos\phi_k + j \sum_{k=1}^{N} \sin\phi_k$$

Se N é suficentemente grande então o $\sum_{k=1}^{N} a_k \cos\phi_k$ e $\sum_{k=1}^{N} a_k \sin\phi_k$ pode ser considera um VA

$S_r = (a + x) + j y$ Is asumindo que x e y son i.i.d

$$S_r = r e^{j\phi} = (a + x) + j y$$

Sendo $r = \sqrt{(a+x)^2 + y^2}$ $\quad \theta = tg^{-1}\left(\frac{y}{a+x}\right)$

A pdf de x e y $\quad p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$ $\quad p(y) = \frac{1}{2\pi\sigma} e^{-\frac{y^2}{2\sigma^2}}$

c) Proximo objetivo: Determina a pdf de (r, θ)

$$p(r,\phi) = p(x,y)\left(J\begin{bmatrix} x & y \\ x & \phi \end{bmatrix}\right)$$

sendo verdadeiro que x e y são iid então

$$p(x,y) = p(x)\,p(y)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$$J\begin{vmatrix} x & y \\ r & \phi \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{vmatrix}$$

$$r^2 = (a+x)^2 + y^2 \Rightarrow (a+x)^2 = r^2 - y^2$$

$$tg\phi = \frac{y}{(a+x)} \Leftrightarrow y = (a+x)\,tg\,\phi$$

$$(a+x)^2 = r^2 - (a+x)^2 tg^2\phi$$

$$r^2 = (a+x)^2 + (a+x)^2 tg^2\phi$$

$$= (a+x)^2\left[1 - tg^2\phi\right]$$

$$= (a+x)^2\left(1 - \frac{\sin^2\phi}{\cos^2\phi}\right)$$

$$= (a+x)^2\left(\frac{\cos^2\phi + \sin\phi}{\cos^2\phi}\right)$$

$$r^2 = (a+x)^2\left(\frac{1}{\cos^2\phi}\right)$$

$$r = (a+x)\frac{1}{\cos\phi} \Rightarrow a + x = r\cos\phi$$

$$\boxed{x = r\cos\phi - a}$$

$$y = (\cancel{a} + r\cancel{\cos\phi} - \cancel{a})\frac{\sin\phi}{\cancel{\cos\phi}}$$

$$\boxed{y = r\sin\phi}$$

$$\frac{\partial x}{\partial r} = \cos\phi \quad , \quad \frac{\partial y}{\partial r} = \sin\phi \quad , \quad \frac{\partial x}{\partial \phi} = -r\sin\phi \quad , \quad \frac{\partial y}{\partial \phi} = r\cos\phi$$

$$J\begin{vmatrix} x & y \\ r & \phi \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{vmatrix} = r\cos^2\phi + r\sin^2\phi = r(\cos^2\phi + \sin^2\phi) = r$$

$$p(r,\phi) = \frac{r}{2\pi\sigma_r^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$

sendo $x = r\cos\phi - a$ e $y = r\sin\phi$

$$= \frac{r}{2\pi\sigma_r^2} e^{-\frac{\left[(r\cos\phi - a)^2 + (r\sin\phi)^2\right]}{2\sigma^2}}$$

$$= \frac{r}{2\pi\sigma_r^2} e^{-\frac{\left[(r\cos\phi)^2 - 2ra\cos\phi + a^2 + (r\sin\phi)^2\right]}{2\sigma_r^2}}$$

$$= \frac{r}{2\pi\sigma_r^2} e^{-\frac{\left[r^2\left[\cos^2\phi + \sin^2\phi\right] + a^2 - 2ra\cos\phi\right]}{2\sigma^2}}$$

$$= \frac{r}{2\pi\sigma_r^2} e^{-\left(\frac{r^2 + a^2 - 2ra\cos\phi}{2\sigma_r^2}\right)}$$

$$p(r,\phi) = \frac{r}{2\pi\sigma^2} e^{-\left(\frac{r^2 + a^2 - 2ra\cos\phi}{2\sigma^2}\right)}$$

$$p(r) = \int_{-\infty}^{\infty} p(r,\phi)\, d\phi = \int_{-\infty}^{\infty} \frac{r}{2\pi\sigma^2} e^{-\left(\frac{r^2+a^2-2ra\cos\phi}{2\sigma^2}\right)} d\phi$$

$$= \int_{-\pi}^{\pi} \frac{r}{2\pi\sigma^2} e^{-\frac{(r^2+a^2)}{2\sigma^2}} e^{\frac{2ra\cos\phi}{2\sigma^2}} d\phi$$

Sendo $\phi \in [-\pi, \pi]$

$$\rightarrow \frac{r}{2\pi\sigma^2} e^{-\frac{(r^2+a^2)}{2\sigma^2}} \int_{-\pi}^{\pi} e^{+\frac{\cancel{2}ra\cos\phi}{\cancel{2}\sigma^2}} d\phi = \frac{r}{2\pi\sigma^2} \int_{-\pi}^{\pi} e^{\frac{ra}{\sigma}\cos\phi} d\phi$$

a integral de $\int_{-\pi}^{\pi} e^{a\cos b}\, db = 2\pi I_0(a)$ por tanto

$$p(r) = \frac{r}{\cancel{2\pi}\sigma_r^2} e^{-\frac{(r^2+a^2)}{2\sigma_r^2}} \cancel{2\pi} I_0\left(\frac{a r}{\sigma_r^2}\right)$$

$$\boxed{p(r) = \frac{r}{\sigma_r^2} e^{-\left(\frac{r^2+a^2}{2\sigma_r^2}\right)} I_0\left(\frac{a r}{\sigma_r^2}\right)} \quad (1)$$

Para : $K = \frac{a^2}{2\sigma^2} \Rightarrow a = \sqrt{2K\sigma^2}$ sendo $a > 0$

Podemos realizar alguma transformação com (1)

$$p(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} e^{-K} I_0\left(\frac{r \cdot \sqrt{2K \cdot \sigma^2}}{\sigma^2}\right)$$

Para realizar a simulação

b) Média $E(x) = \int_{-\infty}^{\infty} x f(x)$

$$E(r) = \int^{\infty} r \frac{r}{\sigma^2} e^{-\frac{(r^2+a^2)}{2\sigma^2}} I_0\left(\frac{r\sqrt{2K \cdot \sigma^2}}{\sigma^2}\right) dx$$

$C = -\frac{1}{\sigma^2}$

b) Media de $p(r)$

$$M_r = \int_{-\infty}^{\infty} r f(r)\, dr$$

$$= \int_{0}^{\infty} r f(r)\, dr$$

$$= \int_{0}^{\infty} r \cdot \frac{r}{\sigma^2} e^{-\left(\frac{r^2+a^2}{2\sigma_r}\right)} I_0\left(\frac{ar}{\sigma_r^2}\right) dr$$

$$= \int_{0}^{\infty} \frac{r^2}{\sigma_r^2} e^{-\frac{a^2}{2\sigma_r}} e^{-\frac{r^2}{2\sigma^2}} I_0\left(\frac{ar}{\sigma_r^2}\right) dr$$

$$= \frac{e^{-\frac{a^2}{2\sigma}}}{\sigma_r^2} \int_{0}^{\infty} e^{-\frac{r^2}{2\sigma^2}} I_0\left(\frac{ar}{\sigma_r^2}\right) dr$$

Seja $d = \frac{1}{2\sigma_r^2}$

$2d = \frac{1}{\sigma_r^2}$

$$= \frac{e^{-\frac{a^2}{d}}}{\sigma_r^2} \int_{0}^{\infty} e^{-dr^2} I_0(2d \cdot a \cdot r)\, dr$$

$$\int_{0}^{\infty} e^{-dr^2} I_0(2d \cdot ar) = \frac{e^{\frac{d^2}{2 \cdot a \cdot \log e}} \sqrt{\pi} \cdot \left(d\, I_1\left(\frac{d^2}{2a \log e}\right) + I_0\left(\frac{d^2}{2a \log_2}\right)\right)}{4a(\log e)^{5/2}}$$

$(d^2 + a \log e)$

$$= \frac{\frac{1/2\sigma_r^2}{e^{2 \cdot e \log_2}} \cdot \sqrt{\pi} \left(1/2\sigma_r^2\, I_1\left(-\frac{1/2\sigma_r^2}{2a \log e}\right) + I_0\left(-\frac{1/2\sigma_r^2}{2a \log_2}\right)\right) +}{4(a \log e)^{5/5}}$$

$\frac{1}{\sigma_r^2} + a \log e$

$\Rightarrow$

Sabendo que

$$e^{\frac{a}{\sigma_r^2}}\left[\left(1-\frac{a^2}{\sigma_r^2}\right)I_0\left(-\frac{a^2}{2\sigma_r^2}\right)-\frac{a^2}{\sigma_r^2}I_1\left(\frac{a^2}{\sigma_r^2}\right)\right]$$

$$e^{\frac{a}{\sigma_r^2}}\left[\left(1-\frac{a}{\sigma_r^2}\right)I_0\left(\frac{a^2}{2\sigma_r^2}\right)-\frac{a}{\sigma_r^2}I_1\left(\frac{a^2}{\sigma_r^2}\right)\right]$$

$$\mu=\sigma\sqrt{\pi/2}\;\underbrace{e^{-\frac{a^2}{2\sigma_r^2}}\left[\left(1-\frac{a^2}{\sigma^2}\right)I_0\left(\frac{a^2}{2\sigma^2}\right)-\frac{a^2}{\sigma_r^2}\left(I_1\left(-\frac{a^2}{2\sigma^2}\right)\right)\right]}$$

$$=\sigma\sqrt{\pi/2}\;e^{\frac{a^2}{2\sigma^2}}\qquad L_{1/2}\left(-a^2/2\sigma^2\right)$$

$$\mu_r=\sigma\sqrt{\pi/2}\;L_{1/2}\left(-\frac{a^2}{2\sigma^2}\right)$$

sendo $k=\frac{a^2}{2\sigma^2}$

Para $\sigma=1$ e

$$\mu_r=\sigma\sqrt{\frac{\pi}{2}}\;L_{1/2}\left(-k\right)$$

$$k=\frac{a^2}{2\sigma^2}\Rightarrow a^2=2\sigma^2k$$
$$a=\sqrt{2\sigma^2k}$$

Para $\sigma=1$ e $k=0$ então $a=0$ ⇒ implica distinção rayleigh

$$\mu_0=1\cdot\sqrt{\frac{\pi}{2}}\;L_{1/2}(0)$$

$$=0{,}88\,L_{1/2}(0)$$

$\to 1$

$$\mu=1{,}25$$

Para $\sigma=1$ e $k=1$ ≈ $a=\sqrt{2\cdot1\cdot1}=\sqrt{2}\neq 0$ distribuição Rice

$$\mu=1\cdot\sqrt{\frac{\pi}{2}}\;L_{1/2}(-1)=1{,}25\times1{,}44=1{,}8$$

Para σ_r, $K=1$ $\mu = 1{,}8$

c) Calcular a variancia

$$\sigma_r^2 = \int_0^\infty (r-\mu)^2 f(r)\, dr$$

$$= \int_0^\infty (r^2 - 2r\mu + \mu^2) f(r)\, dr$$

$$= \int_0^\infty r^2 f(r) - \int_0^\infty 2r\mu\, f(r)\, dr + \mu^2 \int_0^\infty f(r)\, dr$$

$$= \int_0^\infty \frac{r^3}{\sigma_r^2} e^{-\left(\frac{a^2+r^2}{2\sigma_r}\right)} I_0\left(\frac{ar}{\sigma^2}\right) dr - \int_0^\infty 2r \left[\sigma \sqrt{\frac{\pi}{2}} L_{1/2}\left(\frac{-a^2}{2\sigma^2}\right)\right]^2 \frac{r}{\sigma_r^2} e^{-\left(\frac{a^2+r^2}{2\sigma^2}\right)} I_0\left(\frac{ar}{\sigma_r}\right)$$

$$+ \left[\sigma \sqrt{\frac{\pi}{2}} L_{1/2}\left(-\frac{a^2}{\sigma_r^2}\right)\right]^2 \int_0^\infty \frac{r}{\sigma_r^2} e^{-\left(\frac{a^2+r}{2\sigma_r^2}\right)} I_0\left(\frac{ar}{\sigma^2}\right)$$

$$= \int_0^\infty \frac{r^3}{\sigma_r} e^{-\left(\frac{a^2+r^2}{2\sigma_r}\right)} I_0\left(\frac{ar}{\sigma_r^2}\right) - \left(\sigma \sqrt{\frac{\pi}{2}} L_{1/2}\left(\frac{a^2}{2\sigma^2}\right)\right) \cdot 2 \underbrace{\int_0^\infty \frac{r^2}{\sigma_2^2} e^{-\left(\frac{a^2+r}{2r^2}\right)} I_0\left(\frac{ar}{\sigma_2}\right)}_{\mu} +$$

$$\sigma^2 \frac{\pi}{2} L_{1/2}^2\left(-\frac{a^2}{2\sigma_r^2}\right) \int_0^\infty \frac{r}{\sigma_r} e^{-\left(\frac{a^2+r}{2\sigma_r}\right)} I_0\left(\frac{ar}{\sigma_r^2}\right)$$

$$= \int_0^\infty \frac{r^3}{\sigma_r} e^{-\left(\frac{a+r^2}{2\sigma}\right)} I_0\left(\frac{ar}{\sigma_r}\right) - \left[2\sigma \cdot \sqrt{\frac{\pi}{2}} L_{1/2}\left(\frac{-a^2}{2\sigma^2}\right) \cdot \sigma \sqrt{\frac{\pi}{2}} L_{1/2}\left(-\frac{a^2}{2\sigma^2}\right)\right] +$$

$$\sigma^2 \frac{\pi}{2} L_{1/2}^2\left(-\frac{a^2}{2\sigma^2}\right) \int_0^\infty \frac{r}{\sigma_r} e^{-\left(\frac{a^2+r}{2\sigma_r}\right)} I_0\left(a\frac{r}{\sigma_r^2}\right)$$

$$= \int_0^\infty \frac{r^3}{\sigma_r} e^{-\left(\frac{a^2+r^2}{2\sigma}\right)} I_0\left(\frac{ar}{\sigma_r}\right) - 2\sigma^2 \frac{\pi}{2} L_{1/2}^2\left(-\frac{a}{2\sigma^2}\right) \left[1 - \frac{1}{2} \int \frac{r}{\sigma_r} e^{-\left(\frac{a^2+r}{2\sigma_r}\right)} I_0\left(a\frac{r}{\sigma_r^2}\right)\right]$$

Aplicando a referencia ① e usando wolfram

$$\sigma^2 = 2\sigma^2 + a^2 - \frac{\pi\sigma^2}{2} L_{1/2}^2\left(-\frac{a^2}{2\sigma^2}\right)$$

Para $K = \frac{a^2}{2\sigma_r^2} \Rightarrow a^2 = 2\sigma_r^2 K$

$$\sigma_r^2 = 2\sigma_r^2 + 2\sigma_r^2 K - \frac{\pi\sigma^2}{2} L_{1/2}^2(-K)$$

Para $\sigma_1 = 1$ e $K=0$

$$\sigma_r^2 = 2 + 2.1.0 - \frac{\pi}{2} L_{1/2}^2(0)$$

$$= 2 - \frac{\pi}{2}(1) =$$

$\sigma_r^2 = 0{,}42$ equivalente a distribuição de Rayleigh.

Para $\sigma = 1$ e $K = 1$

$$\sigma_r^2 = 2 + 2 - \frac{\pi}{2} L_{1/2}^2(-1)$$

$$= 4 - \frac{\pi}{2}.(1{,}44)^2$$

$$= 4 - 3{,}25 = 0{,}75$$