# COMP9020 12s2 Final Exam Solutions

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**Question 1 (2 marks)** How many integers that are not divided by 5 are there between 345 and 12345?

There are $(12345 - (345 - 1)) = 12001$ integers between 345 and 12345.

$\lfloor 12345/5 \rfloor - \lfloor (345-1)/5 \rfloor = 2469 - 68 = 2401$ are divided by 5, so the answer is $12001 - 2400 = \mathbf{9600}$.

**Question 2 (2 marks)** What is $\gcd(343434345, 343434343)$?

$\gcd(343434345, 343434343) = \gcd(2, 343434343) = gcd(2, 1) = gcd(1, 1) = \mathbf{1}$

**Question 3 (4 marks)** Prove or disprove that $(S \cup T) \times (U \cup V) = (S \times U) \cup (T \times V)$ holds for all sets $S$, $T$, $U$, and $V$.

The claim is *not* true. $(a, a) \in \text{RHS} \setminus \text{LHS}$ for $S = V = \emptyset$ and $T = U = \{a\}$.

**Question 4 (3 marks)** Let $\phi_1 = (p \Rightarrow (q \wedge r))$, $\phi_2 = (s \Rightarrow (q \wedge p))$, and $\phi = \phi_1 \vee \phi_2$.

1. Draw Karnaugh maps for the three formulae, $\phi_1$, $\phi_2$, and $\phi$.
2. Read off a minimal DNF for $\phi$.
3. Give a minimal CNF for $\neg\phi_1$.

1.

$\phi_1$ :

| | $p$ | $p$ | $\bar{p}$ | $\bar{p}$ | |
|---|---|---|---|---|---|
| $q$ | 0 | | | | $\bar{s}$ |
| $q$ | 0 | | | | $s$ |
| $\bar{q}$ | 0 | 0 | | | $s$ |
| $\bar{q}$ | 0 | 0 | | | $\bar{s}$ |
| | $\bar{r}$ | $r$ | $r$ | $\bar{r}$ | |

$\phi_2$ :

| | $p$ | $p$ | $\bar{p}$ | $\bar{p}$ | |
|---|---|---|---|---|---|
| $q$ | | | | | $\bar{s}$ |
| $q$ | | | 0 | 0 | $s$ |
| $\bar{q}$ | 0 | 0 | 0 | 0 | $s$ |
| $\bar{q}$ | | | | | $\bar{s}$ |
| | $\bar{r}$ | $r$ | $r$ | $\bar{r}$ | |

$\phi$ :

| | $p$ | $p$ | $\bar{p}$ | $\bar{p}$ | |
|---|---|---|---|---|---|
| $q$ | | | | | $\bar{s}$ |
| $q$ | | | | | $s$ |
| $\bar{q}$ | 0 | 0 | | | $s$ |
| $\bar{q}$ | | | | | $\bar{s}$ |
| | $\bar{r}$ | $r$ | $r$ | $\bar{r}$ | |

2. A minimal DNF for $\phi$ is $\bar{\mathbf{p}} \vee \bar{\mathbf{s}} \vee \mathbf{q}$.
3. A minimal CNF for $\neg\phi_1$ is $\mathbf{p} \wedge (\bar{\mathbf{q}} \vee \bar{\mathbf{r}})$.

**Question 5 (3 marks)** Suppose Portia puts a portrait of herself in one of two caskets and places the following inscriptions on the caskets:

**Gold casket:** The portrait is not in this casket.

**Silver casket:** Exactly one of these inscriptions is true.

Portia tells her suitor to pick a casket that contains the portrait.
(1 mark) Which casket should the suitor choose?
(2 marks) Formalise the problem in propositional logic and justify your previous answer with a proof.

The suitor should choose the *gold casket*.

We formalise using four propositions:

$g$ is true iff the portrait is in the gold casket,

$s$ is true iff the portrait is in the silver casket,

$G$ is true iff the inscription on the gold casket is true,

$S$ is true iff the inscription on the silver casket is true, and

We need to model that there's a portrait in precisely one of the caskets.

$$g \Leftrightarrow \bar{s} \tag{1}$$

The inscriptions are modeled as follows.

$$G \Leftrightarrow \bar{g} \tag{2}$$
$$S \Leftrightarrow (S \Leftrightarrow \bar{G}) \tag{3}$$

$g$ (and $\bar{s}$) is consistent with (1)–(3): we have that $G$ must be false and $S$ can be anything. On the other hand, $s$ cannot be true because we then have that $G$ is true and (3) boils down to $S \Leftrightarrow \bar{S}$, which is unsatisfiable. Hence $s$ leads to a contradiction and $g$ must be true.

**Question 6 (2 marks)** In $\mathbb{B}^5$, what is the value of $(0, 0, 1, 1, 1) \vee (0, 1, 0, 1, 0)$?

$(\mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$

**Question 7 (4 marks)** Call a binary tree *complete* if every node has either 0 or 2 children. Prove that every nonempty complete binary tree has an odd number of nodes.

*Hint:* use induction on trees.

**Base case:** the smallest nonempty complete binary tree is (isomorphic to) $(\{v\}, \emptyset)$. It has 1 node.

**Inductive case:** The root node has 2 successors, each of which is a complete binary tree; the number $n$ of nodes is 1 (for the root) + $\ell$ (nodes in the left subtree) + $r$ (nodes in the right subtree). By the induction hypothesis both $\ell$ and $r$ must be odd. Hence $n$ is odd, too.

**Question 8 (2 marks)** How many Boolean algebra isomorphisms of $\mathcal{P}(\{a, b, c, d\})$ onto $\mathbb{B}^4$ are there? Explain your answer briefly.

Such an isomorphism is completely determined by how we map the atoms, which in this case are the singleton sets $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$. They must be mapped onto the atoms of $\mathbb{B}^4$, which are 0001, 0010, 0100, and 1000. There are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ onto functions between sets of size 4, so our answer is **24**.

**Question 9 (3 marks)** Consider the partial order $D_{50} = (\{\, x \in \mathbb{P} : x|50 \,\}, \sqsubseteq)$ defined on the positive divisors of 50 by $x \sqsubseteq y$ iff $x|y$. Give 3 different topological sorts of $D_{50}$.

There are at least 5 such toplogical sorts:

$$1, 2, 5, 10, 25, 50$$
$$1, 5, 2, 10, 25, 50$$
$$1, 2, 5, 25, 10, 50$$
$$1, 5, 2, 25, 10, 50$$
$$1, 5, 25, 2, 10, 50$$

**Question 10 (4 marks)** Let $S$, $T$, and $U$ be sets. Let $r_1, r_2 \subseteq S \times T$ and $r \subseteq T \times U$.

1. (3 marks) Prove that $(r_1 \cap r_2); r \subseteq (r_1; r) \cap (r_2; r)$.
2. (1 mark) Prove that $(r_1 \cap r_2); r = (r_1; r) \cap (r_2; r)$ does not hold in general.

1.

$$\begin{aligned}(s,u) \in (r_1 \cap r_2); r &\Leftrightarrow \exists t \in T\,((s,t) \in r_1 \cap r_2 \wedge (t,u) \in r)\\ &\Leftrightarrow \exists t \in T\,((s,t) \in r_1 \wedge (s,t) \in r_2 \wedge (t,u) \in r)\\ &\Rightarrow (s,u) \in r_1; r \wedge (s,u) \in r_2; r\\ &\Leftrightarrow (s,u) \in r_1; r \cap r_2; r\end{aligned}$$

2. Consider $r_1 = \{(s,t)\}$, $r_2 = \{(s,t')\}$, $r = \{(t,u), (t',u)\}$ where $t \neq t'$. Then LHS $= \emptyset$ whereas RHS $= \{(s,u)\}$

**Question 11 (3 marks)** Suppose $f : \mathbb{N} \longrightarrow \mathbb{R}$ is $\mathcal{O}(n)$. Prove that $g : \mathbb{N} \longrightarrow \mathbb{R}$ defined inductively by $g(0) = f(0)$ and $g(n+1) = g(n) + f(n+1)$ is $\mathcal{O}(n^2)$.

*Hint:* use the definition of $\mathcal{O}$.

**Proof:** Let $c_f \in \mathbb{R}_{>0}$ and $n_f \in \mathbb{P}$ be such that for all $n > n_f$ we have that $|f(n)| < c_f \cdot n$. (These constants exist by definition of $\mathcal{O}$. The definition of $g$ implies that $g(n) = \sum_{i=0}^{n} f(i)$. Let $c = \max\left(c_f, \frac{2|g(n_f)|}{n_f(n_f+1)}\right)$.

It follows that, for all $n > n_f$ we have that

$$\begin{aligned}g(n) = \sum_{i=0}^{n} f(n) = g(n_f) + \sum_{i=n_f+1}^{n} f(n)\\ &< g(n_f) + c_f \sum_{i=n_f+1}^{n} n\\ &\leq |g(n_f)| + c \sum_{i=n_f+1}^{n} n\\ &\leq c\frac{n_f(n_f+1)}{2} + c\sum_{i=n_f+1}^{n} n\\ &= c\frac{n(n+1)}{2}\\ &\leq c \cdot n^2 && \text{since } n > n_f > 1\end{aligned}$$

We conclude that $n_f$ and $c$ are constants that witness $g = \mathcal{O}(n^2)$.

**Question 12 (3 marks)** Consider a recurrence with $T(n) = 3T(\frac{n}{3}) + 17$. What is its order of growth?

By the master theorem: $T(n) = \mathcal{O}(\mathbf{n})$.

**Question 13 (3 marks)** Consider a call moveStack(n,0,1,2) given the C code snippet below. Characterise precisely the number of calls to moveDisk as a function of the value of n.

```
void moveStack (int n, int source, int aux, int dest)
{
  if (n == 1)
    moveDisk (source, dest);
  else {
```

```
    moveStack (n−1, source, dest, aux);
    moveDisk (source, dest);
    moveStack (n−1, aux, source, dest);
  }
}
```

$T(1) = 1$ and $T(n) = 2T(n-1) + 1$. So $\mathbf{T(n) = 2^n - 1}$.

**Question 14 (3 marks)** Consider a call cheapExp(3.1415,n) given the C code snippet below. Use $\mathcal{O}$ notation to characterise the time complexity as a function of the value of n.

```
double cheapExp(double x, unsigned int n) {
  double h;
  if (n == 0) return 1;
  if (n == 1) return x;

  h = cheapExp(x, n >> 1); // 2nd arg equals "n / 2"
  return (n & 1) ? x*h*h : h*h; // test equals "1 == n % 2"
}
```

$\mathbf{O(logn)}$

**Question 15 (3 marks)** The student council consists of six women and four men. Calculate how many different four-person committees can be formed when either of the following constraints is in place:

1. (1 mark) A committee must contain two women and two men.
2. (1 mark) A committee must contain at least one woman.
3. (1 mark) A committee must contain at least one man and one woman.

1. $\binom{6}{2} \cdot \binom{4}{2} = \frac{30}{2} \cdot \frac{12}{2} = 15 \cdot 6 = 90$
2. $\binom{10}{4} - 1 = 209$
3. $\binom{10}{4} - \binom{6}{4} - \binom{4}{4} = 194$

**Question 16 (3 marks)** Team $\alpha$ faces team $\beta$ in a 7-match series. Matches are either won or lost, i.e., there are no draws. It takes 4 wins to win the series. Team $\alpha$ has probability $p \in (0,1)$ of winning a match. They have lost the first two matches of the series already. What is the probability that they will lose the whole series?

We have two equally simple way to calculate the probablity:

1. $1 - P(\beta$ loses all 5 remaining matches$) - P(\beta$ loses 4 of the remaining matches$)$
2. $P(\alpha$ loses 5 more$) + P(\alpha$ loses 4 more$) + P(\alpha$ loses 3 more$) + P(\alpha$ loses 2 more$)$

1 looks simpler and we arrive at $1 - p^5 - \binom{5}{1}(1-p)p^4 = \mathbf{1 - p^5 - 5(1-p)p^4}$.

2 would be $\binom{5}{0}(1-p)^5 + \binom{5}{1}p(1-p)^4 + \binom{5}{2}p^2(1-p)^3 + \binom{5}{2}p^3(1-p)^2 = (1-p)^5 + 5p(1-p)^4 + 10p^2(1-p)^3 + 10p^3(1-p)^2$

**Question 17 (3 marks)** Suppose you are taking an exam consisting of 100 multiple choice questions. Each question has three possible answers exactly one of which is correct. A correct answer scores 1, an incorrect answer scores $-1/3$ and blank scores 0. You did not study at all, and decide to randomly guess all the answers and leave no blanks. What should you expect to score in the exam? Derive the correct answer to this question mathematically.

$\frac{1}{3}100 - \frac{2}{3}100\frac{1}{3} = \mathbf{\frac{100}{9}} \simeq 11.11\ldots$

**Question 18 (3 marks)** Calculate a weakest proposition $\phi$ that validates the Hoare triple

$$\{\phi\}\; y := y+3; x := f(x,y)\; \{x > 5 \wedge y < x\} \quad .$$

$$\begin{aligned}((x > 5 \wedge y < x)[{}^{f(x,y)}/_{x}])[{}^{y+3}/_{y}] &= (f(x,y) > 5 \wedge y < f(x,y))[{}^{y+3}/_{y}] \\ &= \mathbf{f(x,y+3) > 5 \wedge y+3 < f(x,y+3)}\end{aligned}$$

**Question 19** Find expressions $e_1$ and $e_2$ such that

$$\left\{\begin{array}{l} x_0 \in \mathbb{P} \wedge x = x_0 \wedge \\ y_0 \in \mathbb{P} \wedge y = y_0 \wedge \\ x \neq y \end{array}\right\} \begin{array}{l} \textbf{if } x > y \\ \textbf{then } x := e_1 \\ \textbf{else } y = e_2 \textbf{ fi} \end{array} \left\{\begin{array}{l} x + y < x_0 + y_0 \wedge \\ \forall p \in \mathbb{P}\,((p|x_0 \wedge p|y_0) \Leftrightarrow (p|x \wedge p|y)) \end{array}\right\}$$

is valid. Prove that your choices work. (Recall that $x_0$ and $y_0$ are so-called *ghost* variables that cannot occur in program code.)

$e_1$ could be $\mathbf{x - y}$ (or $x \bmod y$) and $e_2 = \mathbf{y - x}$ or $y \bmod x$.

**Question 20 (4 marks)** Use Hoare logic to prove the validity of

$$\begin{array}{l} \{n \in \mathbb{N}\} \\ x := 0; \\ k := 1; \\ \textbf{while } k \leq n \textbf{ do } x := x + b[k]; k := k+1 \textbf{ od} \\ \left\{x = \sum_{i=1}^{n} b[i]\right\} \end{array}$$

that is, fully annotate the code with intermediate assertions such that all resulting Hoare triples are valid.

$$\begin{array}{l} \{n \in \mathbb{N}\} \\ x := 0; \\ \{x = 0 \wedge 0 \leq n\} \\ k := 1; \\ \left\{x = 0 \wedge k = 1 \wedge x = \sum_{i=1}^{k-1} b[i] \wedge k \leq n+1\right\} \\ \left\{x = \sum_{i=1}^{k} b[i] \wedge 0 \leq k \leq n+1\right\} \\ \textbf{while } k \leq n \textbf{ do} \\ \quad \left\{x = \sum_{i=1}^{k-1} b[i] \wedge 0 < k \leq n\right\} \\ \quad \left\{x + b[k] = \sum_{i=1}^{k} b[i] \wedge 0 < k \leq n\right\} \\ \quad x := x + b[k]; \end{array}$$

$$\left\{x = \sum_{i=1}^{k} b[i] \wedge 0 < k \leq n\right\}$$

$k := k + 1$

**od**

$$\left\{x = \sum_{i=1}^{k-1} b[i] \wedge 0 < k = n + 1\right\}$$

$$\left\{x = \sum_{i=1}^{n} b[i]\right\}$$