COMP9020 Week 2
Binary Relations

- Textbook (R & W) - Ch. 3., Sec. 3.1, 3.4; Ch. 11, Sec. 11.1

Summary of topics

- Defining binary relations
- Properties of binary relations
- Equivalence relations, classes, and partitions
- Orderings

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Binary relations

A **binary relation between** S **and** T is a subset of $S \times T$: i.e. a set of ordered pairs.
Also: over S and T; from S to T; on S (if $S = T$).

Example (Special (Trivial) Relations)

Identity (diagonal, equality) $E = \{\, (x, x) : x \in S \,\}$

Empty $\emptyset$

Universal $U = S \times S$

Defining binary relations: Set-based definitions

Defining a relation $R \subseteq S \times T$:

- Explicitly listing tuples: e.g. $\{(1,1),(2,3),(3,2)\}$
- Set comprehension: $\{(x,y) \in [1,3] \times [1,3] : 5|xy-1\}$
- Construction from other relations: $\{(1,1)\} \cup \{(2,3)\} \cup \{(2,3)\}^{\leftarrow}$

Defining binary relations: Matrix representation

Defining a relation $R \subseteq S \times T$:

Rows enumerated by elements of S, columns by elements of T:

Examples

- The relation $\{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:

$$\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \\ \circ & \bullet & \circ \end{bmatrix}$$

- The relation $\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:

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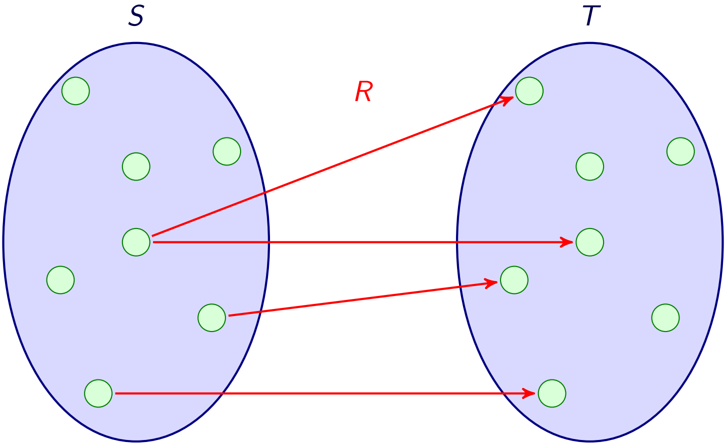
Defining binary relations: Graphical representation

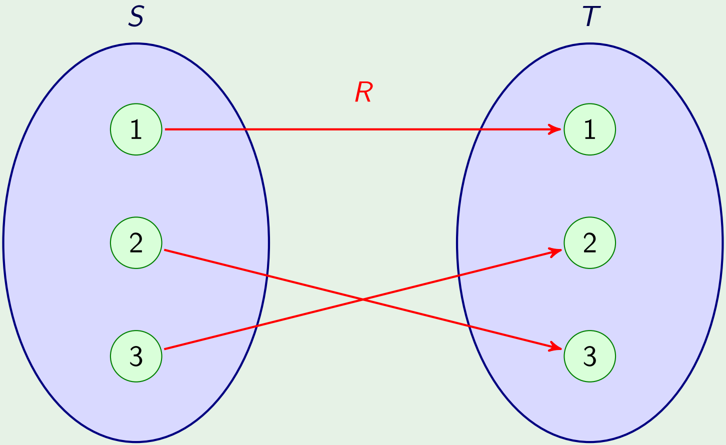
Defining a relation $R \subseteq S \times T$:



Defining binary relations: Graphical representation

Example

$R = \{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:



Defining binary relations: Graphical representation

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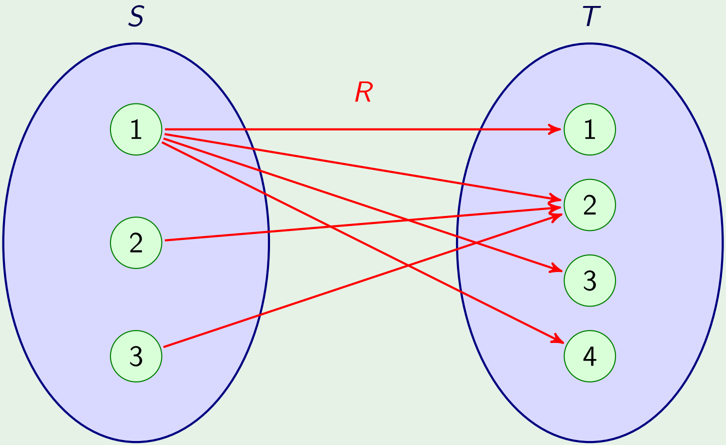
$\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:



Defining binary relations: Graph representation

If $S = T$ we can define $R \subseteq S \times S$ as a **directed graph** (week 5).

- Nodes: Elements of S
- Edges: Elements of R

Example

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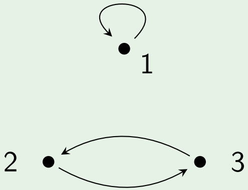

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Properties of Binary Relations $R \subseteq S \times S$

Definition

(R)	reflexive	For all $x \in S$: $(x, x) \in R$
(AR)	antireflexive	For all $x \in S$: $(x, x) \notin R$
(S)	symmetric	For all $x, y \in S$: If $(x, y) \in R$ then $(y, x) \in R$
(AS)	antisymmetric	For all $x, y \in S$: If (x, y) and $(y, x) \in R$ then $x = y$
(T)	transitive	For all $x, y, z \in S$: If (x, y) and $(y, z) \in R$ then $(x, z) \in R$

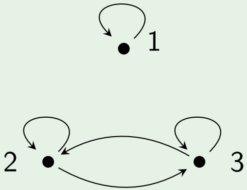
NB

- *Properties have to hold for all elements*
- *(S), (AS), (T) are conditional statements – they will hold if there is nothing which satisfies the 'if' part*

Relation properties: Examples

Examples

(R) Reflexivity: $(x, x) \in R$ for all x



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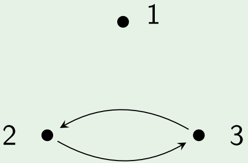

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Relation properties: Examples

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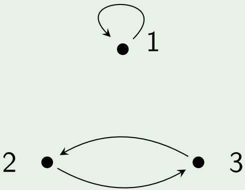
(R) Reflexivity: $(x, x) \in R$ for all x

(AR) Antireflexivity: $(x, x) \notin R$ for all x

(S) Symmetry: If $(x, y) \in R$ then $(y, x) \in R$ for all x, y



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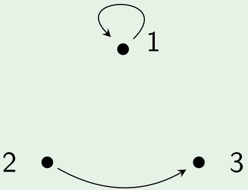
(AS) Antisymmetry: $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all x, y



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Relation properties: Examples

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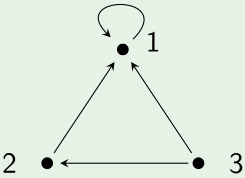
- **(AR)** Antireflexivity: $(x, x) \notin R$ for all x
- **(S)** Symmetry: If $(x, y) \in R$ then $(y, x) \in R$ for all x, y
- **(AS)** Antisymmetry: $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all x, y
- **(T)** Transitivity: $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all x, y, z.



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Interaction of Properties

A relation *can* be both symmetric and antisymmetric. Namely, when R consists only of some pairs $(x, x), x \in S$.
A relation *cannot* be simultaneously reflexive and antireflexive (unless $S = \emptyset$).

NB

$$\left.\begin{array}{r} \textit{nonreflexive} \\ \textit{nonsymmetric} \end{array}\right\} \textit{ is not the same as } \left\{\begin{array}{l} \textit{antireflexive/irreflexive} \\ \textit{antisymmetric} \end{array}\right.$$

Exercises

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$\boxed{3.1.1}$ The following relations are on $S = \{1, 2, 3\}$.
Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in R$ if $m + n = 3$?

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$?

$\boxed{3.1.2(b)}$ $(m, n) \in R$ if $m < n$?

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Complete the following table of common relations (over $\mathbb{Z}$) and their properties:

	(R)	(AR)	(S)	(AS)	(T)
$=$					
$\leq$					
$<$					
$\emptyset$					
$\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$					
$\mid$					
$\equiv \pmod 3$					

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$= \pmod 3$	?				

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$\boxed{3.1.10(a)}$ Give examples of relations with specified properties.

(AS), (T), not (R).

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Equivalence relations

Equivalence relations capture a general notion of "equality". They are relations which are:

- Reflexive (R): Every object should be "equal" to itself
- Symmetric (S): If x is "equal" to y, then y should be "equal" to x
- Transitive (T): If x is "equal" to y and y is "equal" to z, then x should be "equal" to z.

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Definition

A binary relation $R \subseteq S \times S$ is *equivalence relation* if it satisfies (R), (S), (T).

Example

Partition of $\mathbb{Z}$ into classes of numbers with the same remainder $\pmod{p}$; it is particularly important for p prime

$$\mathbb{Z}(p) = \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$$

One can define all four arithmetic operations (with the usual properties) on $\mathbb{Z}_p$ for a prime p; division has to be restricted when p is not prime.

Standard notation:
$m = n \pmod{p}$ stands for: $m \bmod p = n \bmod p$

NB

$(\mathbb{Z}_p, +, \cdot, 0, 1)$ are fundamental algebraic structures known as **rings**. *These structures are very important in coding theory and cryptography.*

Equivalence Classes and Partitions

Suppose $R \subseteq S \times S$ is an equivalence relation
The **equivalence class** $[s]$ (w.r.t. R) of an element $s \in S$ is

$$[s] = \{t : t \in S \text{ and } sRt\}$$

Collection of all equivalence classes $[S]_R = \{\ [s] : s \in S\ \}$ is a partition of S

$$S = \bigcup_{s \in S} [s]$$

Thus the equivalence classes are disjoint and jointly cover the entire domain. It means that every element belongs to one (and only one) equivalence class.

We call $s_1, s_2, \ldots$ *representatives* of (different) equivalence classes

For $s, t \in S$ either $[s] = [t]$, when sRt, or $[s] \cap [t] = \emptyset$, when $s \not R t$.

We commonly write $s \sim_R t$ when s, t are in the same equivalence class.

In the opposite direction, a partition of a set defines the equivalence relation on that set. If $S = S_1 \dot\cup \ldots \dot\cup S_k$, then we specify $s \sim t$ exactly when s and t belong to the same S_i.