Week 03a: Graph Algorithms

Directed Graphs

Directed Graphs (Digraphs)

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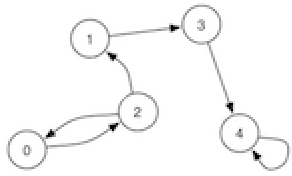
In our previous discussion of graphs:

- an edge indicates a relationship between two vertices
- an edge indicates nothing more than a relationship

In many real-world applications of graphs:

- edges are directional $(v \rightarrow w \neq w \rightarrow v)$



- edges have a *weight* (cost to go from $v \rightarrow w$)

... Directed Graphs (Digraphs)

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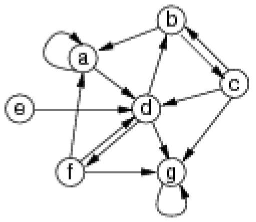
Example digraph and adjacency matrix representation:



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Undirectional $\Rightarrow$ symmetric matrix
Directional $\Rightarrow$ non-symmetric matrix

Maximum #edges in a digraph with V vertices: V^2

... Directed Graphs (Digraphs)

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Terminology for digraphs …

Directed path: sequence of $n \geq 2$ vertices $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n$

- where $(v_i,v_{i+1}) \in edges(G)$ for all v_i,v_{i+1} in sequence
- if $v_1 = v_n$, we have a *directed cycle*

Reachability: w is reachable from v if $\exists$ directed path $v,\ldots,w$

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Digraph Applications

Potential application areas:

Domain	**Vertex**	**Edge**
Web	web page	hyperlink
scheduling	task	precedence
chess	board position	legal move
science	journal article	citation
dynamic data	malloc'd object	pointer
programs	function	function call
`make`	file	dependency

... Digraph Applications

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Problems to solve on digraphs:

- is there a directed path from *s* to *t*? (transitive closure)
- what is the shortest path from *s* to *t*? (shortest path)
- are all vertices mutually reachable? (strong connectivity)
- how to organise a set of tasks? (topological sort)
- which web pages are "important"? (PageRank)
- how to build a web crawler? (graph traversal)

Digraph Representation

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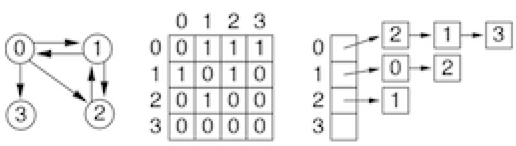
Similar set of choices as for undirectional graphs:

- array of edges (directed)
- vertex-indexed adjacency matrix (non-symmetric)
- vertex-indexed adjacency lists

V vertices identified by *0* .. *V-1*



digraph *adj matrix* *adj lists*

Reachability

Transitive Closure

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Given a digraph G it is potentially useful to know

- is vertex *t* reachable from vertex *s*?

Example applications:

- can I complete a schedule from the current state?
- is a malloc'd object being referenced by any pointer?

How to compute transitive closure?

... Transitive Closure

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One possibility:

- implement it via `hasPath(G,s,t)` (itself implemented by DFS or BFS algorithm)
- feasible if *reachable(G,s,t)* is infrequent operation

What if we have an algorithm that frequently needs to check reachability?

Would be very convenient/efficient to have:

```
reachable(G,s,t):
|  return G.tc[s][t]   // transitive closure matrix
```

Of course, if *V* is *very* large, then this is not feasible.

Exercise #1: Transitive Closure Matrix

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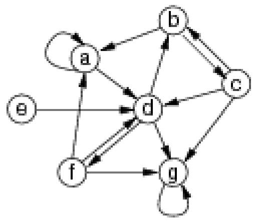
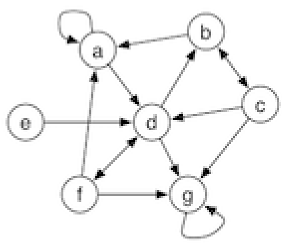
Which reachable *s* .. *t* exist in the following graph?



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Transitive closure of example graph:



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

adjacency matrix

	a	b	c	d	e	f	g
a	1	1	1	1	0	1	1
b	1	1	1	1	0	1	1
c	1	1	1	1	0	1	1
d	1	1	1	1	0	1	1
e	1	1	1	1	0	1	1
f	1	1	1	1	0	1	1
g	0	0	0	0	0	0	1

reachability matrix

... Transitive Closure

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Goal: produce a matrix of reachability values

- if *tc[s][t]* is 1, then *t* is reachable from *s*
- if *tc[s][t]* is 0, then *t* is not reachable from *s*

So, how to create this matrix?

Observation:

$\forall i,s,t \in$ vertices(G):
$(s,i) \in$ edges(G) and $(i,t) \in$ edges(G) $\Rightarrow$ $tc[s][t] = 1$

`tc[s][t]=1` if there is a path from s to t of length 2 $(s \rightarrow i \rightarrow t)$

... Transitive Closure

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If we implement the above as:

```
make tc[][] a copy of edges[][]
for all i∈vertices(G) do
   for all s∈vertices(G) do
      for all t∈vertices(G) do
         if tc[s][i]=1 and tc[i][t]=1 then
            tc[s][t]=1
         end if
      end for
   end for
end for
```

then we get an algorithm to convert `edges` into a *tc*

This is known as *Warshall's algorithm*

... Transitive Closure

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How it works …

After iteration 1, `tc[s][t]` is 1 if

- either $s \rightarrow t$ exists or $s \rightarrow 0 \rightarrow t$ exists

After iteration 2, `tc[s][t]` is 1 if any of the following exist

- $s \rightarrow t$ or $s \rightarrow 0 \rightarrow t$ or $s \rightarrow 1 \rightarrow t$ or $s \rightarrow 0 \rightarrow 1 \rightarrow t$ or $s \rightarrow 1 \rightarrow 0 \rightarrow t$

Etc. … so after the V^{th} iteration, `tc[s][t]` is 1 if

- there is any directed path in the graph from s to t

Exercise #2: Transitive Closure

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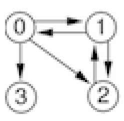
Trace Warshall's algorithm on the following graph:



digraph

	0	1	2	3
0	0	1	1	1
1	1	0	1	0
2	0	1	0	0
3	0	0	0	0

adj matrix

1st iteration `i=0`:

tc	[0]	[1]	[2]	[3]
[0]	0	1	1	1
[1]	1	1	1	1
[2]	0	1	0	0
[3]	0	0	0	0

2nd iteration `i=1`:

tc	[0]	[1]	[2]	[3]
[0]	1	1	1	1
[1]	1	1	1	1
[2]	1	1	1	1
[3]	0	0	0	0

3rd iteration `i=2`: unchanged

4th iteration `i=3`: unchanged

... Transitive Closure

Cost analysis:

- storage: additional V^2 items (each item may be 1 bit)
- computation of transitive closure: $O(V^3)$
- computation of `reachable()`: $O(1)$ after having generated `tc[][]`

Amortisation: would need many calls to `reachable()` to justify other costs

Alternative: use DFS in each call to `reachable()`
Cost analysis:

- storage: cost of queue and set during reachable
- computation of `reachable()`: cost of DFS = $O(V^2)$ (for adjacency matrix)

Digraph Traversal

Same algorithms as for undirected graphs:

`depthFirst(v):`

1. mark `v` as visited
2. for each `(v,w)`∈*edges*(G) do
   if `w` has not been visited then
   `depthFirst(w)`

`breadth-first(v):`

1. enqueue `v`

2. while queue not empty do
   dequeue `v`
   if `v` not already visited then
     mark `v` as visited
     enqueue each vertex `w` adjacent to `v`

Example: Web Crawling

Goal: visit every page on the web

Solution: breadth-first search with "implicit" graph

```
webCrawl(startingURL):
|  mark startingURL as alreadySeen
|  enqueue(Q,startingURL)
|  while Q is not empty do
|  |  nextPage=dequeue(Q)
|  |  visit nextPage
|  |  for each hyperLink on nextPage do
|  |  |  if hyperLink not alreadySeen then
|  |  |     mark hyperLink as alreadySeen
|  |  |     enqueue(Q,hyperLink)
|  |  |  end if
|  |  end for
|  end while
```

`visit` scans page and collects e.g. keywords and links

Weighted Graphs

Weighted Graphs

Graphs so far have considered

- edge = an association between two vertices/nodes
- may be a precedence in the association (directed)

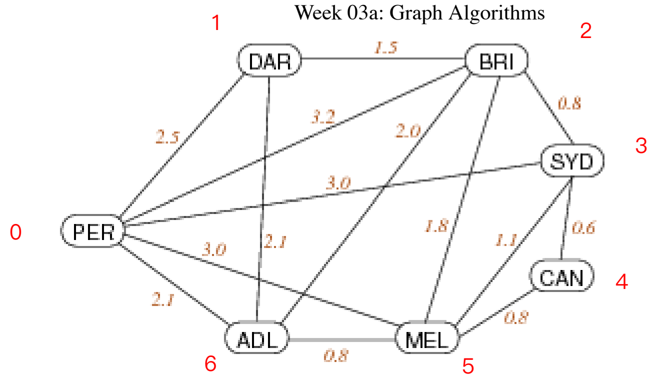
Some applications require us to consider

- a *cost* or *weight* of an association
- modelled by assigning values to edges (e.g. positive reals)

Weights can be used in both directed and undirected graphs.

... Weighted Graphs

Example: major airline flight routes in Australia



Representation: edge = direct flight; weight = approx flying time (hours)

... Weighted Graphs

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Weights lead to minimisation-type questions, e.g.

1. Cheapest way to connect all vertices?
   - a.k.a. *minimum spanning tree* problem
   - assumes: edges are weighted and undirected
2. Cheapest way to get from *A* to *B*?
   - a.k.a *shortest path* problem
   - assumes: edge weights positive, directed or undirected

Exercise #3: Implementing a Route Finder

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If we represent a street map as a graph

- what are the vertices?
- what are the edges?
- are edges directional?
- what are the weights?
- are the weights fixed?

Weighted Graph Representation

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Weights can easily be added to:

- adjacency matrix representation (0/1 → int or float)
- adjacency lists representation (add int/float to list node)

Both representations work whether edges are directed or not.

... Weighted Graph Representation

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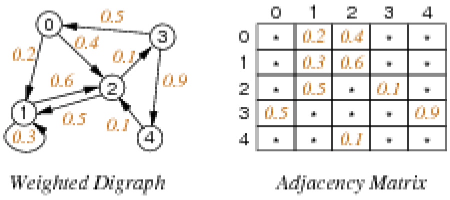
Adjacency matrix representation with weights:



Note: need distinguished value to indicate "no edge".

... Weighted Graph Representation

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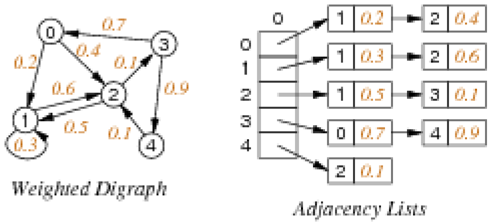
Adjacency lists representation with weights:



Note: if undirected, each edge appears twice with same weight

... Weighted Graph Representation

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Sample adjacency matrix implementation in C requires minimal changes to previous Graph ADT:

WGraph.h

```
// edges are pairs of vertices (end-points) plus positive weight
typedef struct Edge {
   Vertex v;
   Vertex w;
   int    weight;
} Edge;

// returns weight, or 0 if vertices not adjacent
int adjacent(Graph, Vertex, Vertex);
```

... Weighted Graph Representation

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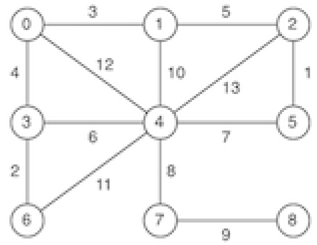
WGraph.c

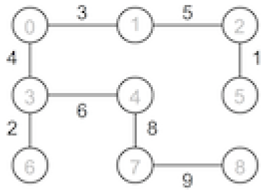
```
typedef struct GraphRep {
   int **edges;   // adjacency matrix storing positive weights
                  // 0 if nodes not adjacent
   int   nV;      // #vertices
   int   nE;      // #edges
} GraphRep;


void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));
   if (g->edges[e.v][e.w] == 0) {  // edge e not in graph
      g->edges[e.v][e.w] = e.weight;
      g->nE++;
   }
```

... Minimum Spanning Trees

Example:



An MST ...



... Minimum Spanning Trees

Brute force solution:

```
findMST(G):
|  Input  graph G
|  Output a minimum spanning tree of G
|
|  bestCost=∞
|  for all spanning trees t of G do
|  |  if cost(t)<bestCost then
|  |     bestTree=t
|  |     bestCost=cost(t)
|  |  end if
|  end for
|  return bestTree
```

Example of *generate-and-test* algorithm.

Not useful because #spanning trees is potentially large (e.g. n^{n-2} for a complete graph with n vertices)

... Minimum Spanning Trees

Simplifying assumption:

- edges in *G* are not directed (MST for digraphs is harder)

Kruskal's Algorithm

One approach to computing MST for graph *G* with *V* nodes:

1. start with empty MST
2. consider edges in increasing weight order

   - add edge if it does not form a cycle in MST
3. repeat until *V-1* edges are added

Critical operations:

- iterating over edges in weight order
- checking for cycles in a graph

... Kruskal's Algorithm

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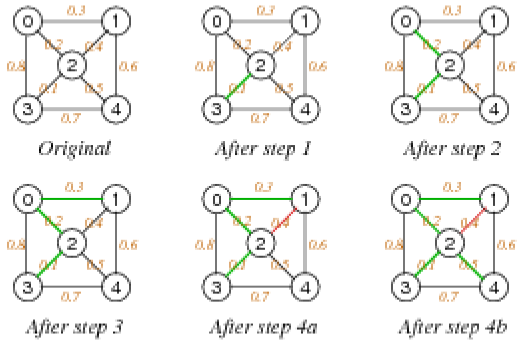
Execution trace of Kruskal's algorithm:



Exercise #5: Kruskal's Algorithm

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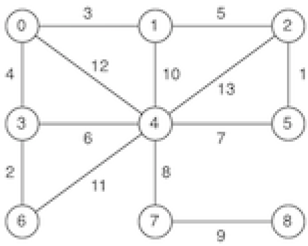
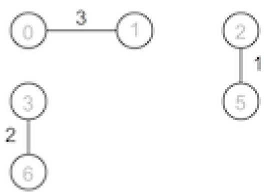
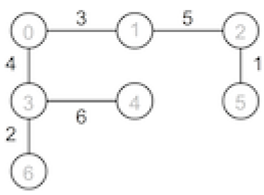
Show how Kruskal's algorithm produces an MST on:



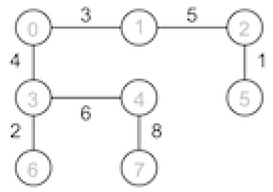
After 3rd iteration:



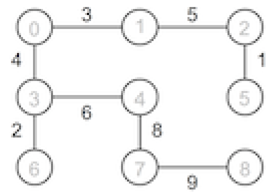
After 6th iteration:



After 7th iteration:



After 8th iteration (V-1=8 edges added):



... Kruskal's Algorithm

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Pseudocode:

```
KruskalMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
|  sort edges(G) by weight
|  for each e∈sortedEdgeList do
|  |  MST = MST ∪ {e}
|  |  if MST has a cyle then
|  |     MST = MST \ {e}
|  |  end if
|  |  if MST has n-1 edges then
|  |     return MST
|  |  end if
|  end for
```

... Kruskal's Algorithm

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Rough time complexity analysis …

- sorting edge list is $O(E \cdot log\ E)$
- at least V iterations over sorted edges
- on each iteration …
  - getting next lowest cost edge is $O(1)$
  - checking whether adding it forms a cycle: cost = ??

Possibilities for cycle checking:

- use DFS … too expensive?
- could use *Union-Find data structure* (see Sedgewick Ch.1)

Prim's Algorithm

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Another approach to computing MST for graph $G=(V,E)$:

1. start from any vertex v and empty MST

2. choose edge not already in MST to add to MST
    - must be incident on a vertex *s* already connected to *v* in MST
    - must be incident on a vertex *t* not already connected to *v* in MST
    - must have minimal weight of all such edges
3. repeat until MST covers all vertices

Critical operations:

- checking for vertex being connected in a graph
- finding min weight edge in a set of edges

... Prim's Algorithm

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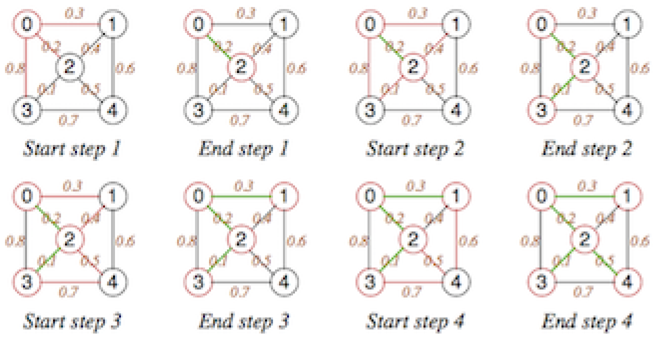
Execution trace of Prim's algorithm (starting at *s*=0):



Exercise #6: Prim's Algorithm

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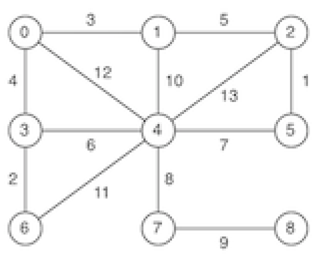
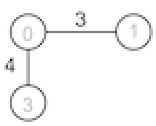
Show how Prim's algorithm produces an MST on:



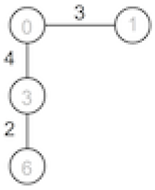
Start from vertex 0

After 1st iteration:

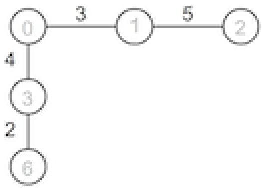
After 2nd iteration:



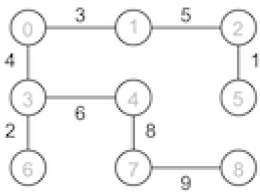
After 3rd iteration:



After 4th iteration:



After 8th iteration (all vertices covered):



... Prim's Algorithm

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Pseudocode:

```
PrimMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
|  usedV={0}
|  unusedE=edges(g)
|  while |usedV|<n do
|  |  find e=(s,t,w)∈unusedE such that {
|  |     s∈usedV, t∉usedV and w is min weight of all such edges
|  |  }
|  |  MST = MST ∪ {e}
|  |  usedV = usedV ∪ {t}
|  |  unusedE = unusedE \ {e}
|  end while
|  return MST
```

Critical operation: finding best edge

... Prim's Algorithm

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Rough time complexity analysis …

- *V* iterations of outer loop
- in each iteration …
  - find min edge with set of edges is $O(E) \Rightarrow O(V \cdot E)$ overall
  - find min edge with *priority queue* is $O(\log E) \Rightarrow O(V \cdot \log E)$ overall

---

Sidetrack: Priority Queues

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Some applications of queues require

- items processed in order of "priority"
- rather than in order of entry (FIFO — first in, first out)

Priority Queues (*PQueues*) provide this via:

- `join`: insert item into PQueue with an associated priority (replacing `enqueue`)
- `leave`: remove item with highest priority (replacing `dequeue`)

Time complexity for naive implementation of a PQueue containing N items …

- $O(1)$ for `join` $O(N)$ for `leave`

Most efficient implementation ("heap") …

- $O(\log N)$ for `join`, `leave`

Other MST Algorithms

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Boruvka's algorithm … complexity $O(E \cdot \log V)$

- the oldest MST algorithm
- start with V separate components
- join components using min cost links
- continue until only a single component

Karger, Klein, and Tarjan … complexity $O(E)$

- based on Boruvka, but non-deterministic
- randomly selects subset of edges to consider
- for the keen, here's the paper describing the algorithm

Shortest Path

Shortest Path

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Path = sequence of edges in graph G $p = (v_0,v_1), (v_1,v_2), \ldots, (v_{m-1},v_m)$

cost(path) = sum of edge weights along path

Shortest path between vertices *s* and *t*

- a simple path $p(s,t)$ where $s = first(p)$, $t = last(p)$
- no other simple path $q(s,t)$ has $cost(q) < cost(p)$

Assumptions: weighted digraph, no negative weights.

Finding shortest path between two given nodes known as *source-target* SP problem

Variations: *single-source* SP, *all-pairs* SP

Applications: navigation, routing in data networks, …

Single-source Shortest Path (SSSP)

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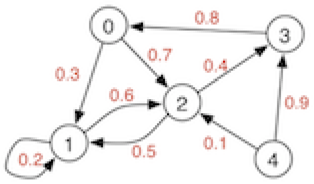
Given: weighted digraph G, source vertex s

Result: shortest paths from s to all other vertices

- `dist[]` V-indexed array of cost of shortest path from s
- `pred[]` V-indexed array of predecessor in shortest path from s

Example:



	0	1	2	3	4
dist	0	0.3	0.7	1.1	inf
pred	-	0	0	2	-

Shortest paths from s=0

Edge Relaxation

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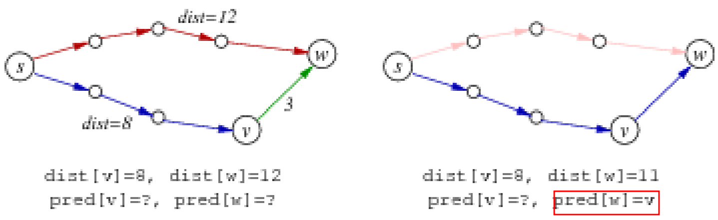
Assume: `dist[]` and `pred[]` as above (but containing data for shortest paths *discovered so far*)

`dist[v]` is length of shortest known path from s to v
`dist[w]` is length of shortest known path from s to w

Relaxation updates data for w if we find a shorter path from s to w:



Relaxation along edge e=`(v,w,weight)`:

The selection is shorter than the previous known path from s – w, so it can be updated

- if **`dist[v]+weight < dist[w]`** then
  update `dist[w]:=dist[v]+weight` and `pred[w]:=v`

Dijkstra's Algorithm

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One approach to solving single-source shortest path problem …

Data: G, s, `dist[]`, `pred[]` and

- *vSet*: set of vertices whose shortest path from s is unknown

Algorithm:

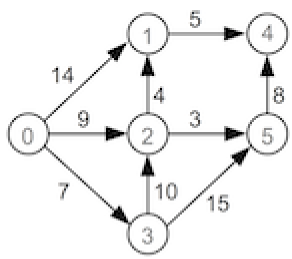
```
dist[]  // array of cost of shortest path from s
pred[]  // array of predecessor in shortest path from s

dijkstraSSSP(G,source):
```

```
|  Input graph G, source node
|
|  initialise dist[] to all ∞, except dist[source]=0
|  initialise pred[] to all -1
|  vSet=all vertices of G
|  while vSet≠∅ do
|  |  find s∈vSet with minimum dist[s]
|  |  for each (s,t,w)∈edges(G) do
|  |     relax along (s,t,w)
|  |  end for
|  |  vSet=vSet\{s}
|  end while
```

Exercise #7: Dijkstra's Algorithm

Show how Dijkstra's algorithm runs on (source node = 0):



	[0]	[1]	[2]	[3]	[4]	[5]
dist	**0**	∞	∞	∞	∞	∞
pred	–	–	–	–	–	–

---

	[0]	[1]	[2]	[3]	[4]	[5]
dist	**0**	∞	∞	∞	∞	∞
pred	–	–	–	–	–	–

dist	**0**	**14**	**9**	**7**	∞	∞
pred	–	**0**	**0**	**0**	–	–

dist	**0**	**14**	**9**	**7**	∞	**22**
pred	–	**0**	**0**	**0**	–	**3**

0 – 3

dist	**0**	**13**	**9**	**7**	∞	**12**
pred	–	**2**	**0**	**0**	–	**2**

0 – 3 – 2

dist	**0**	**13**	**9**	**7**	**20**	**12**
pred	–	**2**	**0**	**0**	**5**	**2**

0 – 3 – 2 – 5

dist	**0**	**13**	**9**	**7**	**18**	**12**

pred	–	2	0	0	1	2

... Dijkstra's Algorithm

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Why Dijkstra's algorithm is correct:

Hypothesis.
(a) For visited *s* … *dist[s]* is shortest distance from source
(b) For unvisited *t* … *dist[t]* is shortest distance from source *via visited nodes*

Proof.
Base case: no visited nodes, *dist[source]=0*, *dist[s]=∞* for all other nodes

Induction step:

1. If *s* is unvisited node with minimum *dist[s]*, then *dist[s]* is shortest distance from source to *s*:
   - if ∃ shorter path via only visited nodes, then *dist[s]* would have been updated when processing the predecessor of *s* on this path
   - if ∃ shorter path via an unvisited node *u*, then *dist[u]<dist[s]*, which is impossible if *s* has min distance of all unvisited nodes
2. This implies that (a) holds for *s* after processing *s*
3. (b) still holds for all unvisited nodes *t* after processing *s*:
   - if ∃ shorter path via *s* we would have just updated *dist[t]*
   - if ∃ shorter path without *s* we would have found it previously

... Dijkstra's Algorithm

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Time complexity analysis …

Each edge needs to be considered once ⇒ *O(E)*.

Outer loop has *O(V)* iterations.

Implementing "`find s∈vSet with minimum dist[s]`"

1. try all `s∈vSet` ⇒ cost = *O(V)* ⇒ overall cost = $O(E + V^2) = O(V^2)$
2. using a PQueue to implement extracting minimum
   - can improve overall cost to *O(E + V·log V)* (for best-known implementation)

All-pair Shortest Path (APSP)

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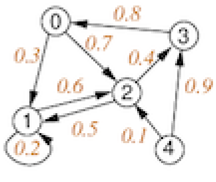
Given: weighted digraph *G*

Result: shortest paths between all pairs of vertices

- `dist[][]` *V×V*-indexed matrix of cost of shortest path from v_{row} to v_{col}
- `path[][]` *V×V*-indexed matrix of next node in shortest path from v_{row} to v_{col}

Example:



Weighted Digraph

dist

V	0	1	2	3	4
0	0	0.3	0.7	1.1	inf
1	1.8	0	0.6	1.0	inf
2	1.2	0.5	0	0.4	inf
3	0.8	1.1	1.5	0	inf
4	1.3	0.6	0.1	0.5	0

path

V	0	1	2	3	4
0	–	1	2	2	–
1	2	–	2	2	–
2	3	1	–	3	–
3	0	0	0	–	–
4	2	2	2	2	–

Shortest paths between all vertices

Digraph Applications

PageRank

Goal: determine which Web pages are "important"

Approach: ignore page contents; focus on hyperlinks

- treat Web as graph: page = vertex, hyperlink = directed edge
- pages with many incoming hyperlinks are important
- need to compute "incoming degree" for vertices

Problem: the Web is a *very* large graph

- approx. 10^{14} pages, 10^{15} hyperlinks

Assume for the moment that we could build a graph …

Most frequent operation in algorithm "Does edge (v,w) exist?"

... PageRank

Simple PageRank algorithm:

```
PageRank(myPage):
|  rank=0
|  for each page in the Web do
|  |  if linkExists(page,myPage) then
|  |     rank=rank+1
|  |  end if
|  end for
```

Note: requires *inbound* link check

... PageRank

V = # pages in Web, E = # hyperlinks in Web

Costs for computing PageRank for each representation:

Representation	**linkExists(v,w)**	**Cost**
Adjacency *matrix*	`edge[v][w]`	*1*
Adjacency *lists*	`inLL(list[v],w)`	≅ *E/V*

Not feasible …

- adjacency matrix … $V \cong 10^{14}$ ⇒ matrix has 10^{28} cells
- adjacency list … *V* lists, each with ≅10 hyperlinks ⇒ 10^{15} list nodes

So how to really do it?

... PageRank

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Approach: the random web surfer

- if we randomly follow links in the web …
- … more likely to re-discover pages with many inbound links

```
curr=random page, prev=null
for a long time do
|   if curr not in array ranked[] then
|      rank[curr]=0
|   end if
|   rank[curr]=rank[curr]+1
|   if random(0,100)<85 then            // with 85% chance ...
|      prev=curr
|      curr=choose hyperlink from curr  // ... crawl on
|   else
|      curr=random page                 // avoid getting stuck
|      prev=null
|   end if
end for
```

Could be accomplished while we crawl web to build search index

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Exercise #8: Implementing Facebook

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Facebook could be considered as a giant "social graph"

- what are the vertices?
- what are the edges?
- are edges directional?

What kind of algorithm would …

- help us find people that you might like to "befriend"?

Summary

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- Digraphs, weighted graphs: representations, applications
- Reachability
    - Warshall

- Minimum Spanning Tree (MST)
    - Kruskal, Prim
- Shortest path problems
    - Dijkstra (single source SPP)
    - Floyd (all-pair SSP)
- Flow networks
    - Edmonds-Karp (maximum flow)

- Suggested reading (Sedgewick):
    - digraphs … Ch. 19.1-19.3
    - weighted graphs … Ch. 20-20.1
    - MST … Ch. 20.2-20.4
    - SSP … Ch. 21-21.3
    - network flows … Ch. 22.1-22.2

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