The University of New South Wales

Final Exam

Session 2, 2012

COMP9020

Foundations of Computer Science

Time allowed: **2 hour**
Total number of questions: **20**
Maximum number of marks: **60**

Not all questions are worth the same.

Answer all questions.

Textbooks, lecture notes, etc. are not permitted, except for up to 2 double-sided A4 sheets containing handwritten notes.

Calculators may not be used.

Answers must be written in ink. Use a pencil or the back of the booklet for rough work. Your rough work will not be marked.

You can answer the questions in any order.

You may take this question paper out of the exam.

Write your answers into the answer booklet provided.

Question 1 (2 marks)

How many integers that are not divided by 5 are there between 345 and 12345?

Question 2 (2 marks)

What is $\gcd(343434345, 343434343)$?

Question 3 (4 marks)

Prove or disprove that $(S \cup T) \times (U \cup V) = (S \times U) \cup (T \times V)$ holds for all sets S, T, U, and V.

Question 4 (3 marks)

Let $\phi_1 = (p \Rightarrow (q \wedge r))$, $\phi_2 = (s \Rightarrow (q \wedge p))$, and $\phi = \phi_1 \vee \phi_2$.

(a) Draw Karnaugh maps for the three formulae, ϕ_1, ϕ_2, and ϕ.

(b) Read off a minimal DNF for ϕ.

(c) Give a minimal CNF for $\neg\phi_1$.

Question 5 (3 marks)

Suppose Portia puts a portrait of herself in one of two caskets and places the following inscriptions on the caskets:

Gold casket: The portrait is not in this casket.

Silver casket: Exactly one of these inscriptions is true.

Portia tells her suitor to pick a casket that contains the portrait.
(1 mark) Which casket should the suitor choose?
(2 marks) Formalise the problem in propositional logic and justify your previous answer with a proof.

Question 6 (2 marks)

In $\mathbb{B}^5$, what is the value of $(0, 0, 1, 1, 1) \vee (0, 1, 0, 1, 0)$?

Question 7 (4 marks)

Call a binary tree *complete* if every node has either 0 or 2 children. Prove that every nonempty complete binary tree has an odd number of nodes.

Hint: use induction on trees.

Question 8 (2 marks)

How many Boolean algebra isomorphisms of $\mathcal{P}(\{a, b, c, d\})$ onto $\mathbb{B}^4$ are there? Explain your answer briefly.

Question 9 (3 marks)

Consider the partial order $D_{50} = (\{\ x \in \mathbb{P} : x|50\ \}, \sqsubseteq)$ defined on the positive divisors of 50 by $x \sqsubseteq y$ iff $x|y$. Give 3 different topological sorts of D_{50}.

Question 10 (4 marks)

Let S, T, and U be sets. Let $r_1, r_2 \subseteq S \times T$ and $r \subseteq T \times U$.

(a) (3 marks) Prove that $(r_1 \cap r_2); r \subseteq (r_1; r) \cap (r_2; r)$.

(b) (1 mark) Prove that $(r_1 \cap r_2); r = (r_1; r) \cap (r_2; r)$ does not hold in general.

Question 11 (3 marks)

Suppose $f : \mathbb{N} \longrightarrow \mathbb{R}$ is $\mathcal{O}(n)$. Prove that $g : \mathbb{N} \longrightarrow \mathbb{R}$ defined inductively by $g(0) = f(0)$ and $g(n + 1) = g(n) + f(n + 1)$ is $\mathcal{O}(n^2)$.

Hint: use the definition of $\mathcal{O}$.

Question 12 (3 marks)

Consider a recurrence with $T(n) = 3T(\frac{n}{3}) + 17$. What is its order of growth?

Question 13 (3 marks)

Consider a call moveStack(n,0,1,2) given the C code snippet below. Characterise precisely the number of calls to moveDisk as a function of the value of n.

```
void moveStack (int n, int source, int aux, int dest)
{
  if (n == 1)
    moveDisk (source, dest);
  else {
    moveStack (n−1, source, dest, aux);
    moveDisk (source, dest);
    moveStack (n−1, aux, source, dest);
  }
}
```

Question 14 (3 marks)

Consider a call cheapExp(3.1415,n) given the C code snippet below. Use $\mathcal{O}$ notation to characterise the time complexity as a function of the value of n.

```
double cheapExp(double x, unsigned int n) {
  double h;
  if (n == 0) return 1;
  if (n == 1) return x;

  h = cheapExp(x, n >> 1); // 2nd arg equals "n / 2"
  return (n & 1) ? x*h*h : h*h; // test equals "1 == n % 2"
}
```

Question 15 (3 marks)

The student council consists of six women and four men. Calculate how many different four-person committees can be formed when either of the following constraints is in place:

(a) (1 mark) A committee must contain two women and two men.

(b) (1 mark) A committee must contain at least one woman.

(c) (1 mark) A committee must contain at least one man and one woman.

Question 16 (3 marks)

Team α faces team β in a 7-match series. Matches are either won or lost, i.e., there are no draws. It takes 4 wins to win the series. Team α has probability $p \in (0, 1)$ of winning a match. They have lost the first two matches of the series already. What is the probability that they will lose the whole series?

Question 17 (3 marks)

Suppose you are taking an exam consisting of 100 multiple choice questions. Each question has three possible answers exactly one of which is correct. A correct answer scores 1, an incorrect answer scores $-1/3$ and blank scores 0. You did not study at all, and decide to randomly guess all the answers and leave no blanks. What should you expect to score in the exam? Derive the correct answer to this question mathematically.

Question 18 (3 marks)

Calculate a weakest proposition ϕ that validates the Hoare triple

$$\{\phi\}\, y := y + 3; x := f(x, y)\, \{x > 5 \wedge y < x\} \quad .$$

Question 19 (3 marks)

Find expressions e_1 and e_2 such that

$$\left\{ \begin{array}{l} x_0 \in \mathbb{P} \land x = x_0 \land \\ y_0 \in \mathbb{P} \land y = y_0 \land \\ x \neq y \end{array} \right\} \begin{array}{l} \textbf{if } x > y \\ \textbf{then } x := e_1 \\ \textbf{else } y = e_2 \textbf{ fi} \end{array} \left\{ \begin{array}{l} x + y < x_0 + y_0 \land \\ \forall p \in \mathbb{P}\,((p|x_0 \land p|y_0) \Leftrightarrow (p|x \land p|y)) \end{array} \right\}$$

is valid. Prove that your choices work. (Recall that x_0 and y_0 are so-called *ghost* variables that cannot occur in program code.)

Question 20 (4 marks)

Use Hoare logic to prove the validity of

$$\begin{array}{l} \{n \in \mathbb{N}\} \\ x := 0; \\ k := 1; \\ \textbf{while } k \leq n \textbf{ do } x := x + b[k]; k := k + 1 \textbf{ od} \\ \left\{ x = \sum_{i=1}^{n} b[i] \right\} \end{array}$$

that is, fully annotate the code with intermediate assertions such that all resulting Hoare triples are valid.