Problem1

(a) list all possible function: *f*: {a, b, c} → {0, 1}

1. $f_1(x) = 0$ for all $x \in \{a, b, c\}$
2. $f_2(x) = 1$ for all $x \in \{a, b, c\}$
3. $f_3: \{a, b, c\} \rightarrow \{0, 1\}$ where $f(a) = 1, f(b) = 0, f(c) = 0$
4. $f_4: \{a, b, c\} \rightarrow \{0, 1\}$ where $f(a) = 0, f(b) = 1, f(c) = 0$
5. $f_5: \{a, b, c\} \rightarrow \{0, 1\}$ where $f(a) = 0, f(b) = 0, f(c) = 1$
6. $f_6: \{a, b, c\} \rightarrow \{0, 1\}$ where $f(a) = 1, f(b) = 1, f(c) = 0$
7. $f_7: \{a, b, c\} \rightarrow \{0, 1\}$ where $f(a) = 0, f(b) = 1, f(c) = 1$
8. $f_8: \{a, b, c\} \rightarrow \{0, 1\}$ where $f(a) = 1, f(b) = 0, f(c) = 1$

(b) Describe a connection between your answer for (a) and Pow({ a, b, c}).

1. The number of all listed functions equals to |Pow({a, b, c})|
2. Assume 0 represent True, being included in the set. And 1 represent False, not included in the set. Then the functions listed in (a) and elements of Pow({a, b, c}) are of one-to-one correspondence. And the set related to the functions, which are listed below, are all subsets of Pow({a, b, c}).

$f_1. \rightarrow \{\}$	$f_5 \rightarrow \{c\}$
$f_2. \rightarrow \{a, b, c\}$	$f_6 \rightarrow \{a, b\}$
$f_3 \rightarrow \{a\}$	$f_7 \rightarrow \{b, c\}$
$f_4 \rightarrow \{b\}$	$f_8 \rightarrow \{a, c\}$

(c) In general, if card(A) = m and card(B) = n, how many:

(i) functions are there from A to B?

Since a functiton f: S → T require for each element s in S, we can find exactly one element t from T that make (s, t) in the relationship

Then in this case, each of n elements in B maybe mapped by m elements from A

So the number of possibilities are:

$$m \cdot n = mn$$

There are mn functions from A to B

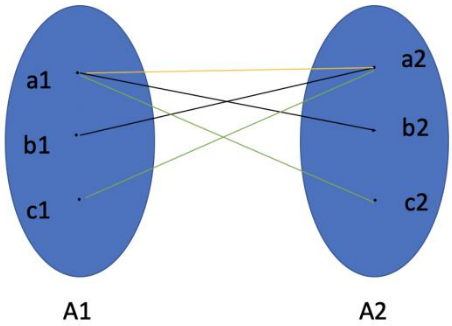
(ii) relations are there between A and B?

From (i), there are mn posible pairs of element from A to B and each of them may be included in a relation or not

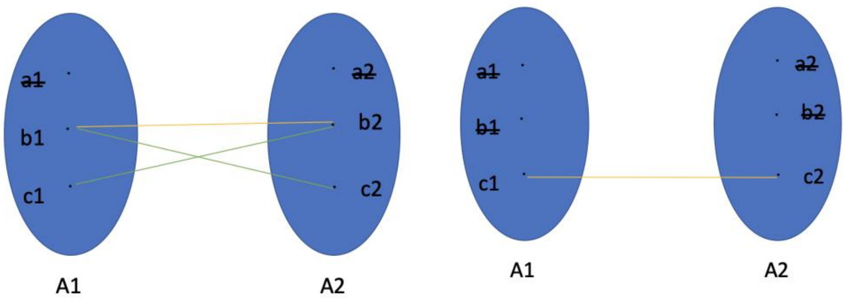
So the possible number of relations: $2^{m \cdot n}$

(iii) symmetric relations are there on A?

For A = {a, b, c} , if i need to achieve a symmertric relation on A_1 x A_2, everytime when I include a element a_1 from A_1 mapping any element b_2 of A_2. Then i need to exclude b_1 from A_1 because it should map a_2 of A_2.



And for each pair of them, a_1 may map |A| elements from A_2 or not. So the posibilty is 2^3. Then i need to consider the left elements in A_1 and A_2 and ignore the situations refering a_1 or a_2 because they have been included in the first mapping mentioned above. similarly, the posibilities are 2^2 and 2^1 respectivly for the second and third maping.



At last, the total posibility is ： $2^3 \times 2^2 \times 2^1 = 2^6 = 64$

For A which contain m elements, the number of symmetric relations are:

$$2^m + 2^{m-1} + 2^{m-2} ... + 2 = 2^{1+2+3+...+m} = 2^{(1+m)m/2}$$

Problem 2

For x, y $\in$ Z we define the set:

$S_{x,y}$ = {mx+ny: m, n $\in$ Z}

(a) Give five elements of $S_{2,-3}$

$S_{2,-3}$ = {2m - 3n:m, n $\in$ Z}

1)When m=n=0 , $S_{2,-3}$={0}

2)When m=n=1 , $S_{2,-3}$={-1}

3)When m=n=2 , $S_{2,-3}$={-2}

4)When m=n=3 , $S_{2,-3}$={-3}

5)When m=n=4 , $S_{2,-3}$={-4}

(b) Give five elements of $S_{12,16}$

$S_{12,16}$= {12m + 16n : m, n $\in$ Z}

1)When m=n=0 , $S_{12,16}$={0}

2)When m=n=1 , $S_{12,16}$={28}

3)When m=0, n=1 , $S_{12,16}$={16}

4)When m=1, n=0 , $S_{12,16}$={12}

5)When m=2, n=0 , $S_{12,16}$={24}

(c) Show that $S_{x,y} \subseteq$ {n : n $\in$ Z and d|n}. Let d = gcd(x, y).

Since d = gcd(x, y)

Assume that there are g_x, g_y $\in$ Z such that:

$x = g_x d \quad y = g_y d$

So $S_{x,y}$ = {mg_xd+ng_yd: m, n $\in$ Z}

= {$(mg_x+ng_y)d$: m, n $\in$ Z}

For all m, n $\in$ Z, we can have d | $(mg_x+ng_y)d$

So $S_{x,y}$ = {n : n $\in$ Z and d|n} when k = mg_x+ng_y

and $S_{x,y} \subseteq$ {n : n $\in$ Z and d|n}

(d) Show that $\{n : n \in Z \text{ and } z|n\} \subseteq S_{x,y}$.Let z be the smallest positive number in $S_{x,y}$.

Asuume there are some $k \in Z$ such that $n = zk$

Then $\{n: n \in Z \text{ and } z \mid n\} = \{zk: k \in Z\}$

Since $S_{x,y} = \{mx + ny: m, n \in Z\}$

Let $x = x_1$, $y = y_1$ such that $(mx + ny)$ is least:

$$mx_1 + ny_1 = z$$

$$k(mx_1+ny_1) = kz$$

$$kmx_1+kny_1 = kz$$

$$kx_1 \cdot m + ky_1 \cdot n = kz$$

For any $k,m,n \in Z$, $\{kx_1 \cdot m + ky_1 \cdot n\} \in S_{x,y}$

So $\{zk: k \in Z\} = S_{x,y}$, when $x = kx_1$, $y = ky_1$

And $\{n: n \in Z \text{ and } z \mid n\} \subseteq S_{x,y}$

(e) Show that $d \leq z$. (Hint: use (c))

From (c) we have $S_{x,y} = \{mx + ny: m, n \in Z\} \subseteq \{n : n \in Z \text{ and } d|n\}$

Thus for all $x, y, m, n \in Z$, $d \mid (mx + ny)$ as $S_{x,y}= \{(mg_x+ng_y)d: m, n \in Z\}$

For x_1, y_1 in (d) such that:

$$mx_1 + ny_1 = z$$

So $d \mid (mx_1 + ny_1)$

Then $d \mid z$

Thus $d \leq z$

(f) Show that $z \leq d$. (Hint: use (d))

Let $q = \lfloor \frac{m}{z} \rfloor$

Then $r = m \bmod z$

$= m - q(mx + ny)$

$= m(1-qx) + n(-qy)$

Thus r is also a linear combination of m, n

Since z is the smallest positive number in $S_{x,y}$, $0 \leq r < z$

So $r = 0$

Thus $z \mid m$

And similarly we can prove:

$$z \mid n$$

Thus $z \mid d$

So $z \leq d$

Problem 3

We define the operation $*$ on subsets of a universal set U as follows. For any two sets A and B:

$$A * B := A^c \cup B^c$$

Answer the following questions using the Laws of Set Operations (and any derived results given in lectures) to justify your answer:

First, we have:

(I) $A * B = A^c \cup B^c = (A \cap B)^c$ (de Morgan's laws)

(II) $A * A = A^c \cup A^c = A^c$ (Idempotence)

(a) What is $(A * B) * (A * B)$?

	$(A * B) * (A * B)$	
=	$(A \cap B)^c * (A \cap B)^c$	(I)
=	$((A \cap B)^c)^c$	(I)
=	$A \cap B$	(Double complementation)

(b) Express A^c using only A, $*$ and parentheses (if necessary).

$A^c = A^c \cup A^c = A * A$ (II)

(c) Express $\varnothing$ using only A, $*$ and parentheses (if necessary).

	$\varnothing$	
=	$A \cap A^c$	(Idempotence)
=	$((A \cap (A * A))^c)^c$	(II)
=	$(A * (A * A))^c$	(I)
=	$(A * (A * A)) * (A * (A * A))$	(I)

(d) Express $A \setminus B$ using only A, B, $*$ and parentheses (if necessary).

	$A \setminus B$	
=	$A \cap B^c$	
=	$(A \cap B^c) * (A \cap B^c)$	(I)
=	$[A * (B * B)] * [A * (B * B)]$	(I)

Problem 4

Let $\Sigma = \{a, b\}$. Define $R \subseteq \Sigma^* \times \Sigma^*$ as follows:

$$(w, v) \in R \text{ if there exists } z \in \Sigma^* \text{ such that } v = wz.$$

(a) Give two words $w, v \in \Sigma^*$ such that $(w, v) \notin R$ and $(v, w) \notin R$.

If w = bab, v = aba, $w \neq v$ is constantly true
If w = aba, v = bab, $w \neq v$ is constantly true

(b) What is $R^{\leftarrow}(\{aba\})$?

Suppose v = aba,

Since $(w, v) \in R^{\leftarrow}$,So: $wz = aba, z \in \Sigma^*$

List all possible answer:
1 .$w = \lambda$, z = aba
2 .w = a, z = ba
3. w = ab, z = a
4. w = aba, $z = \lambda$

(c) Show that R is partial order.

If R satisfies Reflexity, Antisymmertry, Transitivity then it is a partial order

1) for any $x \in \Sigma^*$ such that:

$$x = xz$$

When $z = \lambda$, it is true. For all $x \in \Sigma^*$, $(x, x) \in R$, R satisfies reflexity

2) for any $x, y \in \Sigma^*$ such that: $(x, y) \in R$ and $(y, x) \in R$

Then
$$\begin{cases} y = xz & z \in Z \\ x = yz & z \in Z \end{cases}$$

Since
$$\begin{cases} |y| = |x| + |z| & z \in Z \\ |x| = |y| + |z| & z \in Z \end{cases}$$

So $|z| = 0, z = \lambda$

And
$$\begin{cases} y = x & z \in Z \\ x = y & z \in Z \end{cases}$$

R satisfies Antisymmertry such that if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$

3) assume (x, y) $\in$ R and (y, h) $\in$ R, then we have:

$$\begin{cases} y = xz_1 & z \in Z \\ h = yz_2 & z \in Z \end{cases}$$

Let $z_3 = z_1 z_2$

Then $xz_3 = (xz_1)z_2 = yz_2 = h$

$$h = xz_3$$

So $(x, h) \in R$

R satisfies Transitivity such that for (x, y) $\in$ R and (y, h) $\in$ R, then (x, h) $\in$ R

Thus R satisfies Reflexity(R), Antisymmertry(AS) and Transitivity(T) and it is a partial order

Problem 5

Show that for all x, y, z $\in$ Z:

If x | yz and gcd(x, y) = 1

then x | z. (Hint: Use the connection between gcd(x, y) and $S_{x,y}$ shown in Problem 2.)

From Problem 2, we have proved that z ≤ d and d ≤ z, so z = d

For any x, y $\in$ Z, there are some m, n $\in$ Z such that:

$$mx + ny = 1 \ (a)$$

Since x | yz, it can be represented as:

$$yz = kx, k \in Z$$

When z = 0, x | z is constantly true, then when z ≠ 0:

$$y = \frac{kx}{z}, k \in Z \, and \, z \neq 0 \ (b)$$

Then Assign (b) to (a):

$$mx + \frac{nkx}{z} = 1$$
$$zmx + nkx = z$$
$$(zm + nk)x = z$$

Since z, m, n, k $\in$ Z, then zm + nk $\in$ Z and x | z

Thus For both z = 0 and z ≠ 0, x | z is true.