The University of New South Wales

Final Exam

Session 2, 2015

COMP9020

Foundations of Computer Science

Time allowed: **2 hours**
Total number of questions: **10**
Maximum number of marks: **50** (to be scaled to 100)

Not all questions are worth the same.

Answer all questions.

Textbooks, lecture notes, etc. are not permitted, except for up to 2 double-sided A4 sheets containing handwritten notes.

Calculators may not be used.

Answers must be written in ink. Use a pencil or the back of the booklet for rough work. Your rough work will not be marked.

You can answer the questions in any order.

You may take this question paper out of the exam.

Write your answers into the answer booklet provided.

Question 1 (5 marks)

Prove or disprove $(A \Rightarrow B) \wedge B \Rightarrow A$.

Answer: 1 mark for the right claim, 4 marks for the proof.

In general, this does not hold. Consider $A = \mathbf{F}$ and $B = \mathbf{T}$: then the LHS is $\mathbf{T}$ whereas the RHS is $\mathbf{F}$.

Question 2 (5 marks)

Prove or disprove soundness of the following case analysis scheme as a proof technique.

$$(A \Rightarrow C) \wedge (B \Rightarrow C) \wedge (\neg A \wedge \neg B \Rightarrow C) \Leftrightarrow C \tag{1}$$

Answer: 1 mark for the right claim, 4 marks for the proof.

The scheme is sound. Proof by truth table:

A	B	C	$\neg A$	$\neg B$	$A \Rightarrow C$	$B \Rightarrow C$	$\neg A \wedge \neg B \Rightarrow C$	LHS
F	**F**	**F**	**T**	**T**	**T**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**F**
F	**T**	**T**	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**	**T**	**T**	**F**
T	**F**	**T**	**F**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**F**	**F**	**F**	**F**	**T**	**F**
T	**T**	**T**	**F**	**F**	**T**	**T**	**T**	**T**

The columns labeled LHS and C are identical, which proves the claimed equivalence.

Alternative proofs include:

$$\begin{aligned}
(A \Rightarrow C) \wedge (B \Rightarrow C) \wedge (\neg A \wedge \neg B \Rightarrow C) &\Leftrightarrow (\neg A \vee C) \wedge (\neg B \vee C) \wedge (\neg(\neg A \wedge \neg B) \vee C) \\
&\Leftrightarrow ((\neg A \wedge \neg B) \vee C) \wedge (\neg(\neg A \wedge \neg B) \vee C) \\
&\Leftrightarrow ((\neg A \wedge \neg B) \wedge \neg(\neg A \wedge \neg B)) \vee C \\
&\Leftrightarrow \mathbf{F} \vee C \\
&\Leftrightarrow C
\end{aligned}$$

Question 3 (5 marks)

Prove or disprove that the sequential composition operator ";" distributes over union, that is, for all binary relations $r, s \subseteq X \times Y$ and $t \subseteq Y \times Z$ we have that $(r \cup s); t = (r; t) \cup (s; t)$.

Answer: 1 mark for the right claim, 4 marks for the proof.

Claim: ";" distributes over union.
Proof: Let $x \in X$ and $z \in Z$.

$$\begin{aligned}
(x,z) \in (r \cup s);t &\Leftrightarrow \exists y \in Y\,((x,y) \in (r \cup s) \wedge (y,z) \in t) && \text{Def. of ;}\\
&\Leftrightarrow \exists y \in Y\,(((x,y) \in r \vee (x,y) \in s) \wedge (y,z) \in t) && \text{Def. of } \cup\\
&\Leftrightarrow \exists y \in Y\,(((x,y) \in r \wedge (y,z) \in t) \vee ((x,y) \in s \wedge (y,z) \in t)) && \text{distributivity}\\
&\Leftrightarrow \exists y \in Y\,((x,y) \in r \wedge (y,z) \in t) \vee \exists y \in Y\,((x,y) \in s \wedge (y,z) \in t) && \text{distributivity}\\
&\Leftrightarrow (x,z) \in r;t \vee (x,z) \in s;t && \text{Def. of ;}\\
&\Leftrightarrow (x,z) \in r;t \cup s;t && \text{Def. of } \cup
\end{aligned}$$

Question 4 (5 marks)

Let $P(n)$ be a predicate on natural numbers such that

$$\exists k \in \mathbb{N}\,(P(k)) \tag{2}$$
$$\forall k \in \mathbb{N}\,(P(k) \Rightarrow P(k+2)) \tag{3}$$

For P's that satisfy (2) and (3), some of the assertions below **C**an hold for some, but not all, such P, other assertions **A**lways hold no matter what the P may be, and some **N**ever hold for any such P. Indicate which case applies for each of the assertions and briefly justify why.

(a) $\forall k \in \mathbb{N}\,(P(k+2))$

(b) $\forall k \in \mathbb{N}\,(\neg P(k+1) \vee P(k+3))$

(c) $P(0) \Rightarrow \forall k \in \mathbb{N}\,(P(k+2))$

(d) $P(0) \wedge \neg P(1) \wedge P(3) \Rightarrow \forall k \in \mathbb{N}\,(P(k) \Leftrightarrow \neg P(k+1))$

(e) $\exists k \in \mathbb{N}\,(P(2k) \wedge \forall \ell \in \mathbb{N}\,(P(2(k+\ell))))$

Answer: For each of the five parts, $\frac{1}{2}$ mark for the right claim and $\frac{1}{2}$ mark for the justification.

(a) **C** a positive example is $P(n) = \mathbf{T}$, a negative example is $P(n) = \text{even}(n)$ where for $k = 1$ we have $\neg P(k+2)$

(b) **A** follows from (3) when recognising the definition of $\Rightarrow$

(c) **C** a positive example is $P(n) = \mathbf{T}$, a negative example is $P(n) = \text{even}(n)$: instance $k = 1$ requires $P(3)$.

(d) **C** positive example are obtained by making the LHS false, e.g., $P(n) = \text{even}(n)$; a negative example is $P(n) = (n \neq 1)$ where for $k = 2$ we have $P(k)$ and $P(k+1)$

(e) **C** a positive example is $P(n) = \mathbf{T}$, a negative example is $P(n) = \text{odd}(n)$. We have $\neg P(2k)$ for all $k \in \mathbb{N}$.

Question 5 (4 marks)

Define the sequence of numbers

$$a_n = \begin{cases} 1 & \text{for } n \leq 3 \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{for } n > 3 \end{cases}.$$

Prove that, for all $n \in \mathbb{N}$, dividing a_n by 3 leaves the remainder 1.

Answer: 1 mark for the right proof strategy, 3 marks for executing it.

Proof by strong induction. Base cases $n = 0, 1, 2, 3$ are obvious. Inductive case is $a_n \bmod 3 = (a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}) \bmod 3 = (a_{n-1} \bmod 3 + a_{n-2} \bmod 3 + a_{n-3} \bmod 3 + a_{n-4} \bmod 3) \bmod 3 = (1 + 1 + 1 + 1) \bmod 3 = 4 \bmod 3 = 1$.

Question 6 (5 marks)

Prove that every connected graph $G = (V, E)$ must satisfy $|E| + 1 \geq |V|$.

Answer: (from first principles, using the WOP) The empty graph satisfies the formula. The next-smallest connected graph $G = (\{x\}, \emptyset)$ has one node and no edges: $|e(G)|+1 = 0+1 = 1 = |v(G)|$ and is thus no counterexample. Adding a selfloop wouldn't hurt. We conclude that there's no counterexample with less than two nodes. Let $G = (V, E)$ be a counterexample with the smallest number of nodes. Let $x \in V$. Since G is connected and has $|V| > 1$, we must have $\deg(x) \geq 1$. Just removing x and all its incident edges will not necessarily result in a connected graph. What we can do though is remove x and its $\deg(x)$ incident edges and then add up to $\deg(x) - 1$ edges to connect all neighbours of x to form a string. The resulting graph is connected, has $|V| - 1$ nodes, and has at most $|E| - 1$ edges. It is thus a smaller counterexample than G, contradicting our choice of G.

Question 7 (6 marks)

For each of the following recurrences, what is $T(n)$'s order of growth?

$$T(n) = 27\,T\left(\frac{n}{3}\right) + 3n^3 + 3 \tag{4}$$

$$T(n) = T(n-1) + 3n^3 + 3n + 3 \tag{5}$$

$$T(n) = 3\,T(n-1) + 3n^3 + 3n + 3 \tag{6}$$

Answer: by the master theorem:

(a) $\mathcal{O}(n^3 \log n)$

(b) $\mathcal{O}(n^4)$

(c) $\mathcal{O}(3^n)$

Question 8 (5 marks)

How many 5-letter words over the alphabet $\Sigma = \{\mathtt{b}, \mathtt{d}, \mathtt{u}, \mathtt{x}, \mathtt{z}\}$

(a) include the substring `bud`?

(b) contain all letters of Σ and have `d` occur before `u`?

Answer:

(a) (2.5 marks) 3 possible positions for the substring $\times\ 5^2$ strings over Σ for the remaining 2 positions $= 75$.

(b) (2.5 marks) Before eliminating those violating the order constraint, there are $5! = 120$ strings to consider. By symmetry, half of them violate the order constraint. So we have 60 as answer.

Question 9 (5 marks)

Two dice are rolled.

(a) What is the probability that the sum of the values is even?

(b) What is the probability that the sum of the values is prime?

Justify your answers briefly.

Answer:

(a) No real need to add up outcomes to derive the answer. It suffices to realise that whatever the first die value is, there's a $\frac{1}{2}$ chance for the second die to be of the same parity.

(b) This time we need to add up outcomes in $(1..6)^2$. Possible primes are 2 (1 outcome), 3 (2 outcomes), 5 (4 outcomes), 7 (6 outcomes), 11 (2 outcomes). The probability is thus $\frac{1+2+4+6+2}{36} = \frac{15}{36} = \frac{5}{12}$.

Question 10 (5 marks)

Suppose student X is taking an exam consisting of 100 multiple choice questions. Each question has five possible answers, exactly one of which is correct. A correct answer scores 3, an incorrect answer scores $-\frac{1}{2}$ and blank scores 0. X did not study at all, and decides to randomly guess all the answers and leave no blanks. What should X expect to score in the exam? Derive the correct answer to this question mathematically.

Answer:

$3 \cdot \frac{1}{5} \cdot 100 - \frac{1}{2} \cdot \frac{4}{5} \cdot 100 = 60 - 40 = \mathbf{20}$