Álgebra Linear. 2025-1

Avaliação Final VR.

Turma **B1**.

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Problema 1: (3 Pontos) Determine quais das seguintes matrizes são invertíveis. Caso a matriz seja invertível calcule sua inversa:

a) $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & 0 \\ 3 & -5 & 4 \end{pmatrix}$, **b)** $B = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}$.

Problema 2: (3 Pontos) Considere a aplicação $f : \mathbb{K}^3 \to \mathbb{K}^3$ dada por:

$$f(x,y,z) = (2x - y + z, -2x + y - z, 6x - 3y + 3z).$$

a) Prove que f é uma aplicação linear.

b) Calcule a dimensão do núcleo de f.

c) Calcule a dimensão da imagem de f.

Problema 3: (4 Pontos) Determine si existe uma matriz $D \in M_3(\mathbb{R})$ diagonal e uma matriz invertível $P \in M_3(\mathbb{R})$ tais que $A = P^t D P$. Em caso afirmativo calcule as matrizes D e P.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Ponto Extra (Opcional): Seja $A \in M_n(\mathbb{K})$ uma matriz quadrada de $n \times n$. Mostre que existe um polinômio não nulo $p(x) \in \mathbb{K}[x]$ tal que $p(A) = 0$.

Problema 1:

a) $A=\begin{pmatrix}1&-1&2\\-1&3&0\\3&-5&4\end{pmatrix}$ $\det(A)=\begin{vmatrix}1&-1&2\\-1&3&0\\3&-5&4\end{vmatrix}=\begin{vmatrix}1&-1&2\\-1&3&0\\1&&0\end{vmatrix}=2\begin{vmatrix}-1&3\\1&-3\end{vmatrix}=2\cdot 0=0$

$\det(A)=0 \Rightarrow A$ não é invertível

b) $B=\begin{pmatrix}-1&2&-3\\2&1&0\\4&-2&5\end{pmatrix}$ $\det(B)=\begin{vmatrix}-1&2&-3\\2&1&0\\4&-2&5\end{vmatrix}=\begin{vmatrix}-5&2&-3\\0&1&0\\8&-2&5\end{vmatrix}=1\begin{vmatrix}-5&-3\\8&5\end{vmatrix}=-25+24=-1\neq 0$

$\det(B)=-1\neq 0 \Rightarrow B$ é invertível $B^{-1}=\frac{1}{\det(B)}\,\mathrm{Cof}(B)^t=\frac{1}{-1}\,\mathrm{Cof}(B)^t=-\mathrm{Cof}(B)^t$

$$\mathrm{Cof}(B)=\begin{pmatrix}\begin{vmatrix}1&0\\-2&5\end{vmatrix} & -\begin{vmatrix}2&0\\4&5\end{vmatrix} & \begin{vmatrix}2&1\\4&-2\end{vmatrix}\\ -\begin{vmatrix}2&-3\\-2&5\end{vmatrix} & \begin{vmatrix}-1&-3\\4&5\end{vmatrix} & -\begin{vmatrix}-1&2\\4&-2\end{vmatrix}\\ \begin{vmatrix}2&-3\\1&0\end{vmatrix} & -\begin{vmatrix}-1&-3\\2&0\end{vmatrix} & \begin{vmatrix}-1&2\\2&1\end{vmatrix}\end{pmatrix}=\begin{pmatrix}5&-10&-8\\-4&7&6\\3&-6&-5\end{pmatrix},\ \boxed{B^{-1}=-\begin{pmatrix}5&-4&3\\-10&7&-6\\-8&6&-5\end{pmatrix}=\begin{pmatrix}-5&4&-3\\10&-7&6\\8&-6&5\end{pmatrix}}$$

Problema 2:

$f(x,y,z)=(2x-y+z,\ -2x+y-z,\ 6x-3y+3z)$

a) se $(x_1,y_1,z_1),(x_2,y_2,z_2)\in\mathbb{R}^3$, $\alpha,\beta\in K$ temos

$f(\alpha(x_1,y_1,z_1)+\beta(x_2,y_2,z_2))=f(\alpha x_1+\beta x_2,\ \alpha y_1+\beta y_2,\ \alpha z_1+\beta z_2)=$

$=(2(\alpha x_1+\beta x_2)-(\alpha y_1+\beta y_2)+(\alpha z_1+\beta z_2),\ -2(\alpha x_1+\beta x_2)+(\alpha y_1+\beta y_2)-(\alpha z_1+\beta z_2),\ 6(\alpha x_1+\beta x_2)-3(\alpha y_1+\beta y_2)+3(\alpha z_1+\beta z_2))$

$=(\alpha(2x_1-y_1+z_1)+\beta(2x_2-y_2+z_2),\ \alpha(-2x_1+y_1-z_1)+\beta(-2x_2+y_2-z_2),\ \alpha(6x_1-3y_1+3z_1)+\beta(6x_2-3y_2+3z_2))$

$=\alpha(2x_1-y_1+z_1,\ -2x_1+y_1-z_1,\ 6x_1-3y_1+3z_1)+\beta(2x_2-y_2+z_2,\ -2x_2+y_2-z_2,\ 6x_2-3y_2+3z_2)$

$=\alpha f(x_1,y_1,z_1)+\beta f(x_2,y_2,z_2)$ $\therefore$ f é uma aplicação linear.

b) $A=M(f,\{e_i\})=\begin{pmatrix}2&-1&1\\-2&1&-1\\6&-3&3\end{pmatrix}$ $\mathrm{Ker}\,f=\{(x,y,z)\in K^3:\ A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\}$

$\begin{pmatrix}2&-1&1\\-2&1&-1\\6&-3&3\end{pmatrix}\sim\begin{pmatrix}2&-1&1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$2x-y+z=0 \Rightarrow y=2x+z$

$\mathrm{Ker}\,f=\{(x,2x+z,z):\ x,z\in K\}=\{(x,2x,0)+(0,z,z):\ x,z\in K\}$

$\mathrm{Ker}\,f=\{x(1,2,0)+z(0,1,1):\ x,z\in K\}$

$\mathrm{Ker}\,f=\langle\{(1,2,0),(0,1,1)\}\rangle \Longrightarrow \boxed{\dim(\mathrm{Ker}\,f)=2}$

c) Pelo Teorema do Posto temos $\dim K^3=\dim(\mathrm{Ker}\,f)+\dim(\mathrm{Im}\,f)$

$3 = 2 + \dim(\mathrm{Im}\,f) \Longrightarrow \boxed{\dim(\mathrm{Im}\,f)=1}$

Problema 3:

$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ $A \in M_3(\mathbb{R})$ é simétrica logo representa um endomorfismo autoadjunto de $\mathbb{R}^3$ com o produto escalar usual na base canónica $\{e_1, e_2, e_3\} \Rightarrow$ Pelo Teorema Espectral A é diagonalizável em base ortonormal.

Calculando autovalores

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = -(\lambda-1)[\lambda^2 - 2\lambda + 1 - 1] = -(\lambda-1)(\lambda^2 - 2\lambda) = -\lambda(\lambda-1)(\lambda-2)$$

Autovalores $\lambda = 0$, $\lambda = 1$, $\lambda = 2$

Calculando autoespaços:

$V_0 = \operatorname{Ker}(A - 0I) = \operatorname{Ker} A$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{cases} x - z = 0 \\ y = 0 \end{cases} \Rightarrow z = x$$

$V_0 = \{(x,0,x) : x \in \mathbb{R}\} = \langle (1,0,1) \rangle$

$a_1 = (1,0,1)$ $\quad \|a_1\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

Gram-Schmidt: $v_1 = \dfrac{a_1}{\|a_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$V_1 = \operatorname{Ker}(A - I)$

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{cases} -z = 0 \\ -x = 0 \end{cases}$$

$V_1 = \{(0,y,0) : y \in \mathbb{R}\} = \langle (0,1,0) \rangle$

$a_2 = (0,1,0)$ $\quad \|a_2\| = 1$

$v_2 = a_2 = (0,1,0)$

$V_2 = \operatorname{Ker}(A - 2I)$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{cases} -x - z = 0 \\ -y = 0 \end{cases} \Rightarrow z = -x$$

$V_2 = \{(x,0,-x) : x \in \mathbb{R}\} = \langle (1,0,-1) \rangle$

$a_3 = (1,0,-1)$ $\quad \|a_3\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

Gram-Schmidt $v_3 = \dfrac{a_3}{\|a_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

$\therefore \left\{ v_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),\ v_2 = (0,1,0),\ v_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \right\}$ é base ortonormal de $\mathbb{R}^3$ formada por autovetores de A.

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$P = M(\{e_i\}, \{v_i\})$

$$P^t = P^{-1} = M(\{v_i\}, \{e_i\}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{P^t} \cdot \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D} \cdot \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{P}$$

Ponto Extra:

$A \in M_n(K)$ Provar que existe $p(x) \in K[x]$, $p(x) \neq 0$ tal que $p(A) = 0$

Prova:

Considere as matrizes $\underbrace{I, A, A^2, A^3, \dots, A^n, \dots, A^{n^2}}_{n^2+1 \text{ matrizes}}$.

Como $\dim_K M_n(K) = n^2 \Rightarrow n^2+1$ matrizes necessariamente são L.D.

$\Downarrow$

Existem coeficientes não nulos $a_0, a_1, \dots, a_{n^2} \in K$ tais que

$$a_{n^2} A^{n^2} + \dots + a_n A^n + \dots + a_1 A + a_0 I = 0$$

Basta considerar o polinômio $p(x) = a_{n^2} x^{n^2} + \dots + a_n x^n + \dots + a_1 x + a_0$

$p(A) = 0$

$\square$