Black-Scholes and Further Suggestions

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1 Introduction

The Black-Scholes Option Pricing model, is profound in its nature, for it gives an analyctic (good looking) formula for pricing options and other corporate liabilities (the latter ones we will not adress much) without needing to investigate the tastes (preferences) or the expectations (belief in the movement of market variable) of stock prices.

This article will summarize concisely with necessary theory and equations the famous Black-Scholes partial differential equation for the pricing of options in markets which are European in their options market functioning. I shall then give my comments on the current developments which are much advanced for the scope of this article and provide some extensions of my own and hope to have hopefully contributed something.

2 Concepts and Definitions

Definition 1. *(Call Option) A call option is a financial contract that gives the holder the right, but not the obligation, to buy an underlying asset at a predetermined price (strike price) on or before a specified expiration date.*

Definition 2. *(Put Option) A put option is a financial contract that gives the holder the right, but not the obligation, to sell an underlying asset at a predetermined price (strike price) on or before a specified expiration date.*

A point to note would be that both the definitions talk about the American option, and the European option can only be exercised on the expiry date.

Definition 3. *(Short Position) Shorting a stock is the process of borrowing stocks from another party, and selling them, hoping to repurchase them at a lower rate.*

Definition 4. *(Long Position) Longing a stock is the process of purchasing or holding an already purchased stock in the hope that the stock price goes up in the near future.*

Definition 5. *(Hedging) Hedging is the act of combining two investment types together in such a manner that the position after combining holds lesser risk than the position before.*

Longing a call option is from the buyer's point of view- whereas shorting the call option is from the seller's- the seller is short a call option.The same logic applies to Put options too.
So usually, stock prices that have an observed inverse relation are hedged, or two types of stock holding strategies that tend to have the opposite effects are hedged- the latter, we shall soon see.

3 The Build Up

We start from intuition: how do I write down the value of the option? I think that the option must be a function of two things, the price of the stock, and the time period. I also have a constant time which is the expiration date. so:

$$\textit{option value} \approx w(x,t)$$

I need the value of option at any time period. But I notice that the strike price is fixed and agreed upon. This would also mean that the agreed upon price must be somewhat inflated- ie. the agreed upon price is an agreed prediction of what the price on the day of purchase would have been, rationally speaking. Hence, to find the value of the strike price, we can discount it using the formula for present value. We take it to be the compounded value of the stock had it acted like a zero coupon bond.

we notice that if the price of the stock is greater than the PV of strike price, we are more inclined towards buying it, and the opposing statement follows.

notice also if the time period to maturity (the degree of the denominator) is very large, then the PV becomes quite small and exercising the call option is almost assured. again, the opposing statement follows.

it must hence have a relation of the following form, though not exact for reasons we shall see:

$$w(x,t) \approx x - \frac{K}{(1+r)^{T-t}} \tag{1}$$

where K is the striking price, T is the date of maturity, x is the price of the stock. We make one more claim, **that the option is more volatile than the stock.**

To see this, we need to show that the percentage change in w wrt percentage change in x has to be more than 1. and it does follow:

$$\frac{\partial w}{\partial x} * \frac{x}{w} \approx \frac{x}{w} \tag{2}$$

and as w is always less than x, for $x - c$ (where c is always a nonnegative quantity) is w.

Note that, due to coninuous time compounding, equation (1) can be written as:

$$w(x,t) \approx x - Ke^{-r(T-t)} \tag{3}$$

From equations (1) and (2), we have actually made quite wonderful claims, and our final precise formula for valuation incorporating a bit more practicality, which is on its way soon, will also exhibit the same features.

Forget for now, the structure of equations (1) to (3), except the understanding.

4 The Derivation

4.1 Assumptions Utmost Necessary

At this point the authors make a bunch of assumptions, which are paramount to the derivation, but not to you, the reader, so i shall only state the most important ones and those too, as concise as possible, and the rest, whenever necessary.

We note that we work in continuous time, and it is very common (not much these days), to say that the stock prices usually follow the pattern of a drunk man walking- A random walk process- however that is only defined is discrete time, and the analog of it, what is used in this paper, in continous time domain, is a stochastic differential equation (differential equations in random variables), and is called **Geometric Brownian Motion:**

$$dx(t) = \mu x(t)\,dt + \sigma x(t)\,dW(t) \tag{4}$$

without much confusion, take $dx(t)$ to be dependent on $x(t)$, just like a differential equation and also on $dW(t)$ a **random** "shock" parameter. An assumption is that the interest rate r is constant, and known at all times.

(Some more comments?): the μ in the GBM equation is the "drift rate" or the expected rate of return over time, σ is the standard deviation of the asset returns over time, assumed to be constant. σ here, is a measure of risk. The reader is to note that standard deviation of the distribution of the returns of the asset, though a measure of risk, is quite primitive, and many methods are used in addition to observing the standard deviation.

4.2 Portfolio Structure and Δ

Π will be our portfolio (value) and our portfolio is a hedged one. our portfolio is going to have one longed stock and Δ shorted call options. A shorted call option is a liability, as if the option is exercised by the third party you sold it to, you need to pay your premium back. The second term, including the shorted options, will hence be subtracted from the one longed stock.

$$\Pi = x - \Delta * w(x,t) \tag{5}$$

The most natural thing to do would be to maximise portfolio value- I'm sure we will get some condition:

$$\text{FOC} : \frac{\partial \Pi}{\partial x} = 0 \tag{6}$$

$$\implies \Delta = \frac{1}{\frac{\partial w}{\partial x}} \tag{7}$$

We now see that to maximise the portfolio value, I must continuously hedge (literally) as $\frac{\partial w}{\partial x}$ changes, and match Δ with it. Hence, **Δ is the number of options I need to short for one stock I long.** The value of the portfolio at its maximum is hence:

$$\Pi = x - \frac{1}{\frac{\partial w}{\partial x}} * w(x,t) \tag{8}$$

We then look at the change in the value of Π in a small time period Δt

The authors then proceed to take the equivalent of a stochastic total derivative- mostly due to the fact that changing t would imply that there would be a change in x- after all, x is a variable which randomly changes over time.

Δ from now means the usual "change" and not the previously derived meaning, unless specified.

$$\Delta\Pi = \Delta x - \frac{1}{\frac{\partial w}{\partial x}} * \Delta w(x,t) \tag{9}$$

We do not involve Δ with the denominator for the denominator is negligible enough, and the change of that with respect to time might as well be ignored.

Now the real investigation is in Δw, for $\Delta w = w(x + \Delta x, t + \Delta t) - w(x,t)$.

From calculus, we know that the Taylor series of the function $w(x+\Delta x, t+\Delta t)$ is:

$$w(x+\Delta x, t+\Delta t) = w(x,t) + \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(\Delta x)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial t^2}(\Delta t)^2 + ... \tag{10}$$

We truncate it after the first order terms are done, and it is our familiar total derivative as $\Delta t, \Delta x \to 0$. However, notice that x is stochastic and t is deterministic- so we use "stochastic calculus" and keep it till the third term, for the logic that higher order coefficients of the terms with derivative operators are negligible works only for deterministic variables.

From Itô calculus (Kiyosi Itô contributed so heavily to stochastic calculus, branches of it are named after him.), we know,

$$(\Delta W)^2 = \Delta t$$

Using this, we take our GBM equation (4), and square both sides. d is Δ for all practical reasons, it is a notational difference between the author and the general expression. We then substitute it into the Taylor series up to 3 terms:

$$dx = \mu x \, dt + \sigma x \, dW \tag{11}$$

$$\begin{aligned} (dx)^2 &= (\mu x \, dt + \sigma x \, dW)^2 \\ &= (\mu x \, dt)^2 + 2(\mu x \, dt)(\sigma x \, dW) + (\sigma x \, dW)^2 \\ &= \sigma^2 x^2 \, dt \end{aligned} \tag{12}$$

$$\begin{aligned} w(x+dx, t+dt) &= w(x,t) + \frac{\partial w}{\partial x}\, dx + \frac{\partial w}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(dx)^2 \\ &= w(x,t) + \frac{\partial w}{\partial x}\, dx + \left(\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2}\right) dt \end{aligned} \tag{13}$$

$$\implies \Delta w = \frac{\partial w}{\partial x}\, dx + \left(\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2}\right) dt \tag{14}$$

Our goal is to substitute (14) in (9): we need $\Delta w * \frac{1}{\frac{\partial w}{\partial x}}$.

$$\Delta w * \frac{1}{\frac{\partial w}{\partial x}} = dx + \left(\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2}\right) dt * \frac{1}{\frac{\partial w}{\partial x}} \tag{15}$$

After substituting (15) in (9), we get:

$$\Delta \Pi = -\left(\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2}\right) dt * \frac{1}{\frac{\partial w}{\partial x}} \tag{16}$$

We realise that now, the change in the value of the portfolio is actually risk free (no stochastic term), given that the hedging process is continuous. Its growth can hence be described by the constant interest rate attributed to the change in time, $r\Delta t$. This following relation then holds true:

$$\Delta\Pi = \Pi * r\Delta t$$

4.3 Solutions for Call Option

After some cumbersome algebra, we arrive at the famous Black-Scholes PDE for the value of an option:

Result (Black-Scholes PDE)**.**

$$\boxed{\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2} + rx\frac{\partial w}{\partial x} - rw = 0}$$

We skip the process of solving the actual differential equation as it is not of much use to us.

However, we look into the boundary conditions of this problem, which helps solve in a more "precise" manner the differential equation:

$$w(x, t^*) = \begin{cases} x - c & \mathit{if}\ x \geq c \\ 0 & \mathit{if}\ x < c \end{cases}$$

t^* is the date of expiration, c is the strike price (not discounted on the date of expiration as $t^* - t = 0$).

The above condition simply says that the call option is exercised at the expriation date if the price of the stock is more than the strike price, meaning the value of the option is then the difference between the stocks value and how much you paid for it (the striking price). The boundary condition is from the point of view of the buyer of contract.

The solution is as follows:

$$\begin{aligned} w(x,t) &= xN(d_1) - ce^{r(t-t^*)}N(d_2) \\ d_1 &= \frac{\ln x/c + (r + \frac{1}{2}v^2)(t^* - t)}{v\sqrt{t^* - t}} \\ d_2 &= \frac{\ln x/c + (r - \frac{1}{2}v^2)(t^* - t)}{v\sqrt{t^* - t}} \end{aligned}$$

Note that $N(d_i)$ are the Cumulative Density Functions of the normal distribution.

4.4 Solutions for Put options

The portfolio optimisation problem here is as follows:

$$\Pi = x - \Delta * u(x,t)$$

The PDE is similar, the boundary value conditions for this is as follows:

$$u(x,t^*) = \begin{cases} 0 & \textit{if } x \geq c \\ c - x & \textit{if } x < c \end{cases}$$

That is, the put option is exercised if the striking price is more than the value of the stock. From the point of view of the buyer, buyer profits if the price of stock falls below the strike price, because then the buyer has the right to sell the asset at the striking price.

The solution is as follows:

$$u(x,t) = -xN(-d_1) + ce^{-rt^*}N(-d_2)$$

The interpretations of $N(.)$, c, t^* remain the same.

4.5 Comments and criticisms

Notice the solution for the call option value is actually very similar to our continuously compounded intuitive formula we developed in the beginning of the article. The difference is, we moved more towards practicality by defining the random movement of the stock price, and more assumptions which helped us get an analyctic (a good looking) solution to the PDE.

However, the Black-Scholes model has its fair share of critisisms, starting from the conclusion in their own paper, where they informed us that there are systematic deviations in price predictions by this model- of course, the reasons for this are clear. The restrictive assumptions are the culprit.

So, the finance people in the late 90s took it upon themselves to solve this, and came up with modifications of the same model with lesser errors. Merton(1973), in his Theory of Rational Option Pricing, relaxes the assumption that stocks do not pay dividends (yes, that was an assumption of Black-Scholes).

Merton(1975), again relaxed one more assumption- that the stock price follows GBM. This assummption did not account for random shocks and abnormal events in the stock market, and hence had to be modelled differently. He introduced "Jump-diffusions" as an extension of the GBM model. The difference is simply:

$$dx_t = \mu x_t \, dt + \sigma x_t \, dW_t + x_t \, dJ_t$$

Where, the extra dJ_t can be understood as an extra random shock, making the variance of the movement non constant

5 Ideas

5.1 On the interest rate assumption

One other criticism is that the interest rate is fixed, whereas the need for volatility exists. My proposal is the following:

- The striking price should depend on the stock price and interest rate **expectations.**
- On the second part of the previous sentence, as an economics student, I can see potential in modelling nominal interest rate from the macroeconomics of it.
- The Fisher equation: relating nominal interest rate, real interest rate, and inflation expectations, is as follows:

$$(1 + R_t) = (1 + r_t)(1 + E(\pi_t))$$

to solve models in expectations, from my limited knowledge we would need a rule for the behaviour of expectations-ie, the rule which describes how people will behave in the market. I propose the following rule as i found this interesting and realistic immediately:

A 1998 paper "Heterogeneous beliefs and routes to chaos in a simple asset pricing model", looks at how asset prices change when agents have different beliefs about future prices. These beliefs evolve over time based on past profits, creating price fluctuations driven by an ongoing battle between different forecasting methods.