CS5785 Homework 2

Applied Machine Learning

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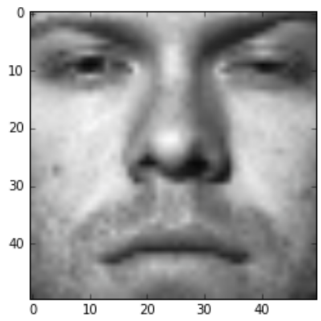
Programming Exercises

1. Eigenface Method

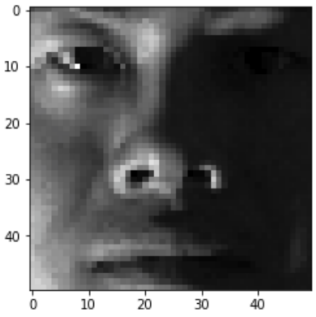
   (a) See `./Faces/HW2_Q1_Faces.ipynb` for solution code.

   (b) Pick a face image from X and display that image in grayscale. Do the same thing for the test set. The size of X_{train} should be 540 2500. The size of matrix X_{test} should be 100 2500.

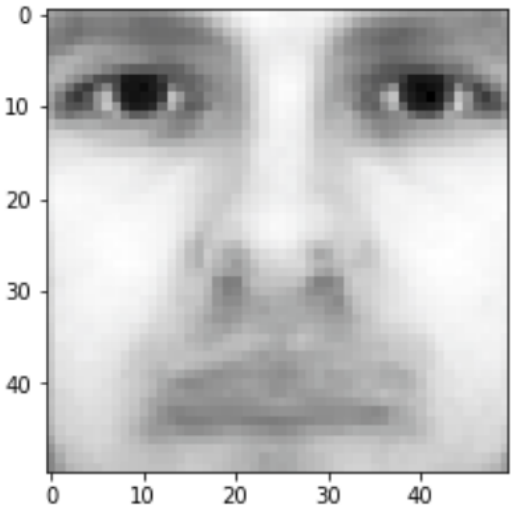
```
training attributes matrix shape:  (540, 2500)
training labels matrix shape:  (540,)
```



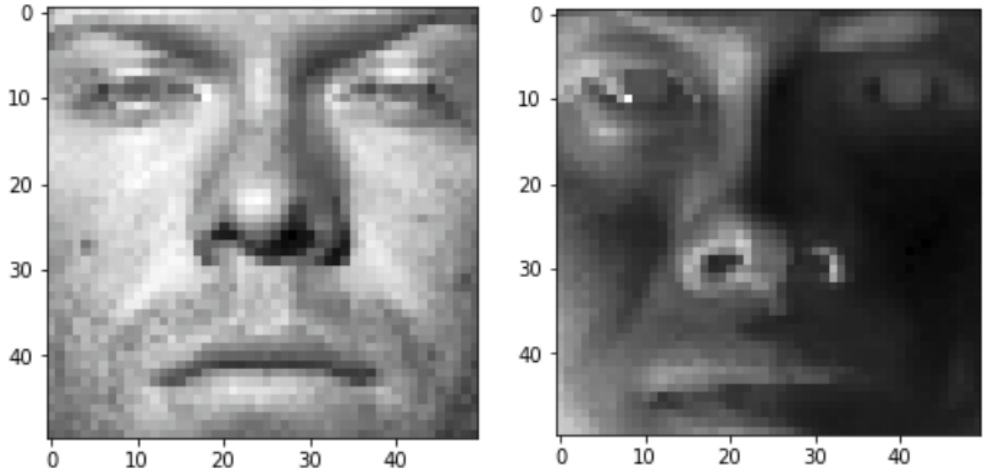

```
test attributes matrix shape:  (100, 2500)
test labels matrix shape:  (100,)
```



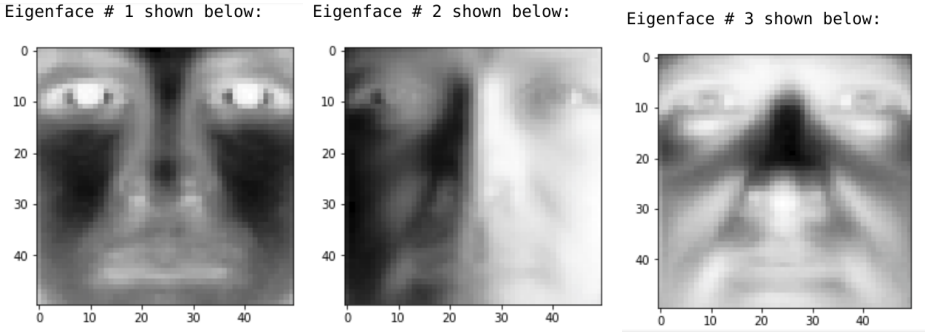
   (c) Compute the average face from the whole training set by summing up every column in X then dividing by the number of faces. Display the average face as a grayscale image.



(d) Subtract average face μ from every column in X. Pick a face image after mean subtraction from the new X and display that image in grayscale. Do the same thing for the test set X_{test} using the pre-computed average face μ in (c).

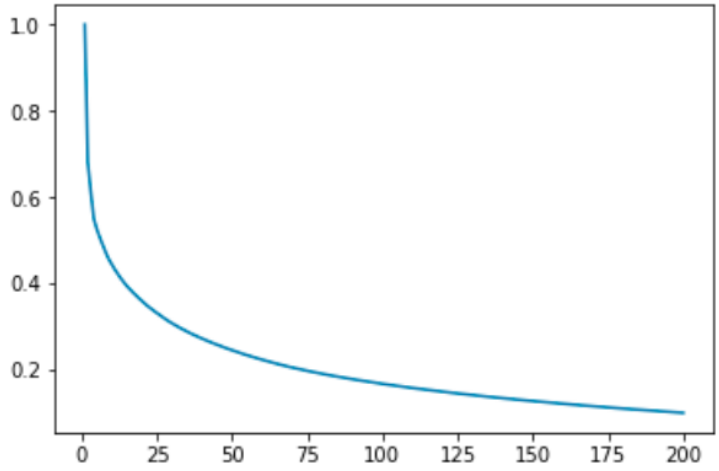


(e) Perform Singular Value Decomposition (SVD) on the training set X ($X = UV^T$) to get matrix V^T, where each row of V^T has the same dimension as the face image... Display the first 10 eigenfaces as 10 images in grayscale.



See `./Faces/HW2_Q1_Faces.ipynb` for the remaining 10 faces.

(f) Plot the rank-r approximation error $||X - X_r||_F$ as a function of r when $r = 1, 2, ..., 200$.
See file: `./Faces/HW2_Q1_Faces.ipynb` for source.
x-axis: rank, y-axis: approximation error



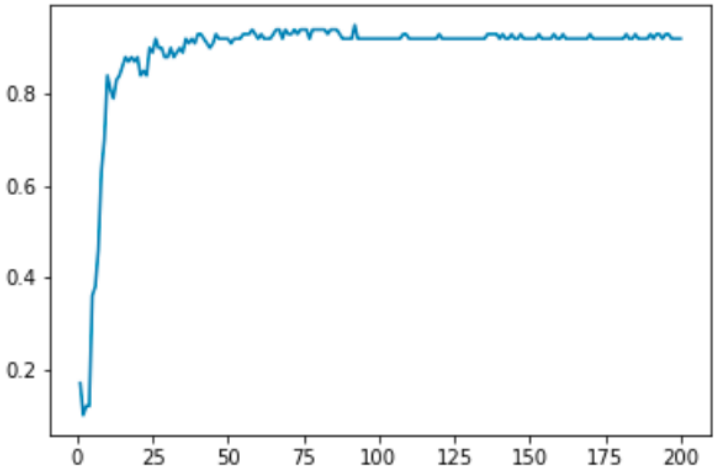
(g) Write a function to generate r-dimensional feature matrix F and F_{test} for training images X and test images X_{test}, respectively.

See `./Faces/HW2_Q1_Faces.ipynb` for full context.

```
def convert_dataset(data, r, V_tr):
    lower_data = np.dot(data, np.transpose(V_tr[:r,:]))
    return lower_data
```

(h) Extract training and test features for $r = 10$. Train a Logistic Regression model using F and test on F_{test}. Report the classification accuracy on the test set. Plot the classification accuracy on the test set as a function of r when $r = 1, 2, ..., 200$. Use one-vs-rest logistic regression, where a classifier is trained for each possible output label.

See file: `./Faces/HW2_Q1_Faces.ipynb` for source.
Accuracy when $r = 10$ *is* 0.84.



2. Cooking

(a) See `./Cooking/HW2_Q2_Cooking.ipynb` for solution code.

(b) Tell us about the data. How many samples (dishes) are there in the training set? How many categories (types of cuisine)? Use a list to keep all the unique ingredients appearing in the training set. How many unique ingredients are there?

There are 39774 dishes in the dataset.
There are 6714 unique types of ingredients in the dataset.
There are 20 types of cuisine in the dataset.

(c) Represent each dish by a binary ingredient feature vector. Use $n * d$ feature matrix to represent all the dishes in training set and test set, where n is the number of dishes.

Generated training feature matrix of shape: (39774, 6714).
Generated test feature matrix of shape: (9944, 6714).

(d) Using Nave Bayes Classifier to perform 3 fold cross-validation on the training set and report your average classification accuracy. Try both Gaussian distribution prior assumption and Bernoulli distribution prior assumption.

Nave Bayes with Gaussian prior average classification accuracy: 0.379493793821
Nave Bayes with Bernoulli prior average classification accuracy: 0.683587657646

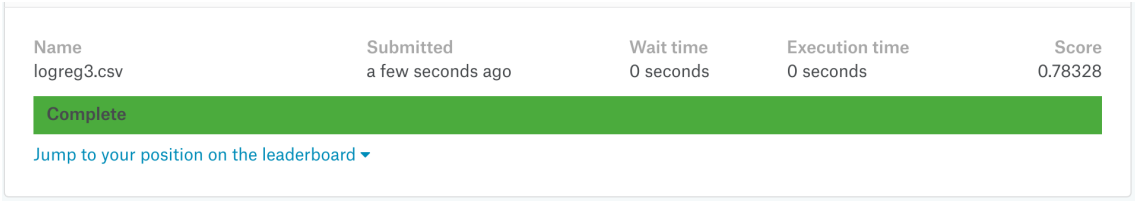
(e) For Gaussian prior and Bernoulli prior, which performs better in terms of cross-validation accuracy? Why? Please give specific arguments.

The Bernoulli prior performs better in terms of cross-validation accuracy. This is because the Bernoulli is a discrete (binomial) distribution, while the Gaussian is a continuous distribution. Our feature matrix is modeled to better suit a discrete distribution, for each ingredient is modeled as a member of a discrete set, rather than a quantitative or continuous measure. Furthermore, food is discrete. An ingredient is not 90% bananna, but either a bananna or not a bananna. Determining what kind of cuisine a food originated from based on the ingredients is thus based entirely on discrete probability.

(f) Using the Logistic Regression Model, perform 3 fold cross-validation on the training set and report your average classification accuracy.

Logistic Regression average accuracy: 0.7756

(g) Train your best-performed classifier with all of the training data, and generate test labels on test set. Submit your results to Kaggle and report the accuracy.



Note: the submission file was renamed to `./Cooking/submission.csv`

Written Exercises

1. HTF Exercise 4.1
   Show how to solve the generalized eigenvalue problem $max\ a^T\mathbf{B}a$ subject to $a^T\mathbf{W}a = 1$ by transforming to a standard eigenvalue problem.

   From HTF p.116:

   Fisher's problem therefore amounts to maximizing the *Rayleigh quotient*,

   $$\max_a \frac{a^T\mathbf{B}a}{a^T\mathbf{W}a}, \tag{4.15}$$

   or equivalently

   $$\max_a a^T\mathbf{B}a \text{ subject to } a^T\mathbf{W}a = 1. \tag{4.16}$$

   Therefore, use Lagrangian Optimization. $L(a, \lambda) = (a^T Ba) - \lambda(aWa - 1)$

   $$\begin{aligned} \frac{\delta L}{\delta a}(a^T Ba) - \lambda(aWa - 1) &= 0 \\ 2Ba + \lambda(2Wa) &= 0 \\ Ba + \lambda Wa &= 0 \\ W^{-1}(Ba + \lambda Wa) &= 0 \\ W^{-1}Ba = \lambda a &=> Av = \lambda v \end{aligned}$$

   This is the generalized eigenvalue problem. To solve the generalized eigenvalue problem, solve the characteristic polynomial for λ.

   $$p(\lambda) = det(A - \lambda I) = 0$$

2. HTF Exercise 4.2

   (a) $\delta_2(x) > \delta_1(x)$
   $x^T\Sigma^{-1}\mu_2 - \frac{1}{2}\mu_2^T\Sigma^{-1}\mu_2 + log(\frac{N_2}{N_{tot}}) > x^T\Sigma^{-1}\mu_1 - \frac{1}{2}\mu_1^T + log(\frac{N_1}{N_{tot}})$
   $x^T\Sigma^{-1}\mu_2 - X^T\Sigma^{-1}\mu_1 > \frac{1}{2}\mu_2^T\Sigma^{-1}\mu_2 - \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 + log(\frac{N_2}{N_{tot}})$
   $x^T\Sigma^{-1}\mu_2 - X^T\Sigma^{-1}\mu_1 > \frac{1}{2}(\mu_2^T\Sigma^{-1}\mu_2 - \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1) + log(\frac{N_1}{N_{tot}} - \frac{N_2}{N_{tot}})$
   ...
   $\frac{1}{2}(\mu_2^T\Sigma^{-1}\mu_2 - \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1) = (\mu_2 + \mu_1)^T\Sigma^{-1}(\mu_2 - \mu_1)$
   ...
   $x^T\Sigma^{-1}(\mu_2 - \mu_1) > (\mu_2 + \mu_1)^T\Sigma^{-1}(\mu_2 - \mu_1) - log(\frac{N_2}{N_1})$

   (b) no answer

   (c) Show that $\Sigma_B\beta$ is in the direction $(\mu_2 - \mu_1)$
   $\Sigma_B = \frac{N_1N_2}{N_2}(\mu_2 - \mu_1)(\mu_2 - \mu_1)^T$
   $\Sigma_B\beta = \frac{N_1N_2}{N_2}(\mu_2 - \mu_1)(\mu_2 - \mu_1)^T\beta$
   $(\frac{N_1N_2}{N_2})(\mu_2 - \mu_1)(\mu_2 - \mu_1)^T\beta$
   ...
   $(\frac{N_1N_2}{N_2})$ is scalar and $(\mu_2 - \mu_1)^T\beta$ is scalar
   ...
   $(\frac{N_1N_2}{N_2})(\mu_2 - \mu_1)(\mu_2 - \mu_1)^T\beta = (\mu_2 - \mu_1) * scalar$
   Thus, $\Sigma_B\beta$ is parallel to $(\mu_2 - \mu_1)$

   (d) no answer

(e) no answer

3. SVD of Rank Deficient Matrix. Consider matrix M . It has rank 2, as you can see by observing that there times the first column minus the other two columns is 0.

$$M = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 7 & 2 \\ 2 & -2 & 8 \\ 0 & -1 & 1 \\ 5 & 8 & 7 \end{bmatrix}$$

(a) Compute the matrices $M^T M$ and MM^T.

$$M^T = \begin{bmatrix} 1 & 3 & 2 & 0 & 5 \\ 0 & 7 & -2 & -1 & 8 \\ 3 & 2 & 8 & 1 & 7 \end{bmatrix}$$

$$MM^T = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 7 & 2 \\ 2 & -2 & 8 \\ 0 & -1 & 1 \\ 5 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 0 & 5 \\ 0 & 7 & -2 & -1 & 8 \\ 3 & 2 & 8 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 9 & 26 & 3 & 26 \\ 9 & 62 & 8 & -5 & 85 \\ 26 & 8 & 72 & 10 & 50 \\ 3 & -5 & 10 & 2 & -1 \\ 26 & 85 & 50 & -1 & 138 \end{bmatrix}$$

$$M^T M = \begin{bmatrix} 1 & 3 & 2 & 0 & 5 \\ 0 & 7 & -2 & -1 & 8 \\ 3 & 2 & 8 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 3 & 7 & 2 \\ 2 & -2 & 8 \\ 0 & -1 & 1 \\ 5 & 8 & 7 \end{bmatrix} = \begin{bmatrix} 39 & 57 & 60 \\ 57 & 118 & 53 \\ 60 & 53 & 127 \end{bmatrix}$$

(b) Find the eigenvalues for your matrices of part (a).

$det(MM^T - \lambda I) = 0$
$\lambda_1 = 214.67$
$\lambda_2 = 69.33$
$\lambda_3 = 0$
$\lambda_4 = 0$
$\lambda_5 = 0$

$det(M^T M - \lambda I) = 0$
$\lambda_1 = 214.67$
$\lambda_2 = 69.33$
$\lambda_3 = 0$

(c) Find the eigenvectors for the matrices of part (a).

For $\lambda_1 ... \lambda_5$:
$MM^T v = \lambda v$
$(MM^T - \lambda)v = 0$

$$v_{1...5} = \begin{bmatrix} 0.16492942 \\ 0.47164732 \\ 0.33647055 \\ 0.00330585 \\ 0.79820031 \end{bmatrix}, \begin{bmatrix} 0.95539856 \\ 0.03481209 \\ -0.27076072 \\ -0.04409532 \\ -0.10366268 \end{bmatrix}, \begin{bmatrix} -0.24497323 \\ 0.45330644 \\ -0.82943965 \\ -0.16974659 \\ 0.13310656 \end{bmatrix}, \begin{bmatrix} 0.06456841 \\ -0.75012833 \\ -0.32834497 \\ -0.04619795 \\ 0.56850131 \end{bmatrix}, \begin{bmatrix} -0.08432693 \\ 0.16709883 \\ 0.27420212 \\ -0.9233105 \\ -0.19307476 \end{bmatrix}$$

For $\lambda_1 ... \lambda_3$:
$M^T M v = \lambda v$
$(M^T M - \lambda)v = 0$

$$v_{1\ldots3} = \begin{bmatrix}0.42615127\\0.61500884\\0.66344497\end{bmatrix}, \begin{bmatrix}0.90453403\\-0.30151134\\-0.30151134\end{bmatrix}, \begin{bmatrix}-0.01460404\\-0.72859799\\0.68478587\end{bmatrix}$$

(d) Find the SVD for the original matrix M from parts (b) and (c). Note that there are only two nonzero eigenvalues, so your matrix Σ should have only two singular values, while U and V have only two columns.

$X = UsV^T$

$$U = \begin{bmatrix}0.16492942 & -0.24497323 & 0.06456841\\0.47164732 & 0.45330644 & -0.75012833\\0.33647055 & -0.82943965 & -0.32834497\\0.00330585 & -0.16974659 & -0.04619795\\0.79820031 & 0.13310656 & 0.56850131\end{bmatrix}$$

$$s = \begin{bmatrix}14.65163776 & 0 & 0\\0 & 8.32643446 & 0\\0 & 0 & 0\end{bmatrix}$$

$$V^T = \begin{bmatrix}0.42615127 & 0.61500884 & 0.66344497\\-0.01460404 & -0.72859799 & 0.68478587\\0.90453403 & -0.30151134 & -0.30151134\end{bmatrix}$$

(e) Set your smaller singular value to 0 and compute the one-dimensional approximation to the matrix M.

Rank 1 approximation:

$X_1 = U_1 s_1 V_1^T$

$$X_1 = \begin{bmatrix}0.16492942\\0.47164732\\0.33647055\\0.00330585\\0.79820031\end{bmatrix} \begin{bmatrix}14.65163776\end{bmatrix} \begin{bmatrix}0.42615127 & 0.61500884 & 0.66344497\end{bmatrix}$$

$$X_1 = \begin{bmatrix}1.02978864 & 1.48616035 & 1.60320558\\2.94487812 & 4.24996055 & 4.58467382\\2.10085952 & 3.031898 & 3.27068057\\0.02064112 & 0.02978864 & 0.0321347\\4.9838143 & 7.19249261 & 7.75895028\end{bmatrix}$$