Assignment-2

(1)

$$X_1 \ldots X_n \mid \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\pi(\mu, \sigma^2) = \pi(\mu \mid \sigma^2)\, \pi(\sigma^2)$$

$$\sigma^2 \sim IG(a, b)$$

$$\mu \mid \sigma^2 \sim N(0, c\sigma^2)$$

$$\pi(\mu, \sigma^2 \mid x_1 \cdots x_n) \propto f(x_1 \ldots x_n \mid \mu, \sigma^2)\, \pi(\mu, \sigma^2)$$

$$\propto \underbrace{f(x_1 \ldots x_n \mid \mu, \sigma^2)\, \pi(\mu \mid \sigma^2)\, \pi(\sigma^2)}$$

$$\cancel{= (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma}}$$

$$\propto (\sigma^2)^{-n/2} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2} \cdot (\sigma^2)^{-1/2}\, e^{-\frac{\mu^2}{2c\sigma^2}} \cdot (\sigma^2)^{-a-1}\, e^{-\frac{b}{\sigma^2}}$$

$$\propto (\sigma^2)^{-\overbrace{(a + \frac{n}{2} + \frac{1}{2})}^{A} - 1} \cdot e^{-\frac{1}{\sigma^2}\underbrace{\left\{\sum_{i=1}^{n}(x_i - \mu)^2 + \frac{\mu^2}{2c} + b\right\}}_{B}}$$

$$\propto (\sigma^2)^{-A-1}\, e^{-\frac{1}{\sigma^2}\{B\}}$$

$$\cancel{\sim IG(A, B)}$$

Now

$$\pi(\mu \mid x_1 \ldots x_n) \propto \int_0^\infty \pi(\mu, \sigma^2 \mid x_1 \ldots x_n)\, d\sigma^2$$

$$\propto \int_0^\infty \sigma^{2^{-A-1}}\, e^{-\frac{1}{\sigma^2}\{B\}}\, d\sigma^2$$

$$\propto \Gamma A\; B^{-A}$$

$$\propto B^{-A}$$

$$\propto \left\{\sum_{i=1}^{n}(x_i - \mu)^2 + \frac{\mu^2}{2c} + b\right\}^{-(a + \frac{n}{2} + \frac{1}{2})}$$

$$\propto \left\{ \sum_{i=1}^{n} (x_i - \mu)^2 + \frac{\mu^2}{2c} + b \right\}^{-\left(\frac{2a+n+1}{2}\right)}$$

this is similar to the kernel to a student t-distribution with df $\underline{2a+n}$

<u>part-2</u>

~~$f(x_{n+1}|x$~~

$\mu \in \mathbb{R}, \sigma^2 > 0$

$$\pi(x_{n+1}|x_1, \ldots x_n) = \iint \pi(x_{n+1}|\mu, \sigma^2)\, \pi(\mu, \sigma^2|x_1 - x_n)\, d\mu\, d\sigma^2$$

$$= \iint \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x_{n+1}-\mu)^2}{\sigma^2}} k^{**} (\sigma^2)^{-A-1} e^{-\frac{1}{\sigma^2}\{B\}} d\mu\, d\sigma^2$$

(3). For Analytical part

$n = m$

$X_1 \cdots X_n | \lambda_1 \overset{iid}{\sim} \text{Poisson}(\lambda_1)$

$Y_1 \cdots Y_n | \lambda_2 \overset{iid}{\sim} \text{Poisson}(\lambda_2)$

$\lambda_1, \lambda_2 \overset{iid}{\sim} \text{Gamma}(a, b)$

$$\pi(\lambda_1 \lambda_2 | x_1 \cdots x_n, y_1 \cdots y_n)$$

$$\propto f(x_1 \cdots x_n, y_1 \cdots y_n | \lambda_1, \lambda_2)\, \pi(\lambda_1, \lambda_2)$$

$$\propto f(x_1 \cdots x_n | \lambda_1)\, f(y_1 \cdots y_n | \lambda_2)\, \pi(\lambda_1)\, \pi(\lambda_2)$$

$$\propto e^{-n\lambda_1} \lambda_1^{\sum_{i=1}^n x_i} e^{-n\lambda_2} \lambda_2^{\sum_{i=1}^n y_i} \lambda_1^{+a-1} e^{-b\lambda_1} \lambda_2^{a-1} e^{-b\lambda_2}$$

$$\propto \lambda_1^{(\sum_{i=1}^n x_i)+a-1} e^{-(n+b)\lambda_1} \quad \lambda_2^{\sum_{i=1}^n y_i + a-1} e^{-(n+b)\lambda_2}$$

$$\downarrow$$

$$\text{Gamma}\left(\sum_{i=1}^n x_i + a,\ n+b\right) . \text{Gamma}\left(\sum y_i + a,\ n+b\right)$$

So, $(\lambda_1 | x_1 \cdots x_n, y_1 \cdots y_n)$ and $(\lambda_2 | x_1 \cdots x_n, y_1 \cdots y_n)$ are independent

$$\boxed{\lambda_1 | x_1 \cdots x_n \sim \text{Gamma}\left(\sum_{i=1}^n x_i + a,\ n+b\right)}$$

Result

independent $\to X \sim$ ~~Gamma(α, θ)~~ $\text{Gamma}(\alpha, \theta)$

$\to Y \sim \text{Gamma}(\beta, \theta)$

then $\dfrac{X}{X+Y} \sim \text{Beta}(\alpha+\beta, \theta)$

and λ_2 for $y_1 \cdots y_n \sim \text{Gamma}\left(\sum_{i=1}^n y_i + a,\ n+b\right)$

$$\theta = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$\pi(\theta \mid x_1 \cdots x_n, y_1 \cdots y_n) \sim \text{Beta}\left\{\left(\sum_{i=1}^{n} x_i + y_i\right) + 2a,\ n+b\right\}$$

(4) $X_1 \sim N(0,1)$,

$X_{t+1} \mid X_t \sim N(\rho X_t, 1-\rho^2)$, $t = 1, 2, \ldots T$

$\rho \sim U(-1, 1)$

~~For Accept - Reject~~

$$\cancel{\pi(\rho \mid x_1 \ldots x_T) = k^* \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \cdot \frac{1}{\quad}}$$

$$\cancel{\pi(\rho \mid x_1 \ldots x_T) \propto \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \frac{1}{\sqrt{2\pi(1-\rho^2)}}}$$

$$\pi(\rho \mid x_1 \ldots x_T) \propto \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(x_t - \rho x_{t-1})^2}{2(1-\rho^2)}}$$

$$= k^* \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \frac{1}{\left(\sqrt{2\pi(1-\rho^2)}\right)^{T-1}} e^{-\frac{\sum_{t=2}^{T} (x_t - \rho x_{t-1})^2}{2(1-\rho^2)}}}_{\to f}$$

Prospod Density Uniform $(-1, 1) \to g$

for accept - Reject

$$\sup_{\rho} \frac{k^* f}{g} \leq c$$

$$\Rightarrow \sup_{\rho} \frac{f}{g} \leq \frac{c}{k^*} = c^*$$

$\sup_{x}(2f) \leq c^*$

$\sup_{x} f \leq \frac{c^*}{2} = c^{**}$

$\sup_{x} f \leq c^{**}$

Accept Reject Algo.

(1) Generate $U \leftarrow U(0,1)$

(2) $Y \leftarrow$ proposal $\cdot$ $U(-1, 1)$

(3) ratio $= \frac{f(Y)}{c^{*} g(Y)}$

(4) if $U \leq$ ratio

then $X = Y$

else

Repeat (1) to (4)

(5) for multivariate

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$$X_1 | X_2 = x_2 \sim N\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2),\ \Sigma_{11.2}\right)$$

for univariate.

$$X^{(b)} | Y = y^{(b-1)} \sim N\left(\mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y^{(b-1)} - \mu_2),\right.$$

$\Sigma_{11.2} \rightarrow$ for univariate

$$\boxed{\sigma_{12} = \rho\sqrt{\sigma_1^2 \sigma_2^2}}$$

$$\sigma_1^2 - \frac{\sigma_{12}}{\sigma_2^2}\sigma_{12}$$

$$= \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}$$

$$= \sigma_1^2 - \frac{\rho^2 \sigma_1^2 \cancel{\sigma_2^2}}{\cancel{\sigma_2^2}}$$

$$= \sigma_1^2(1 - \rho^2)$$

Similarly for other part.

$\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

$$Y^{(b)} | X = x^{(b)} \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x^{(b)} - \mu_1),\ \Sigma_{22.1}\right)$$

$\Sigma_{22.1}$ for univariate

$$\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}$$

$$= \sigma_2^2 - \frac{\rho^2 \cancel{\sigma_1^2} \sigma_2^2}{\cancel{\sigma_1^2}}$$

$$= \sigma_2^2 - \rho^2 \sigma_2^2$$

$$= \sigma_2^2 (1 - \rho^2)$$

So,

$$X^{(b)} \mid Y = y^{(b-1)} \sim N\left(\mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y^{(b-1)} - \mu_2),\ \sigma_1^2(1-\rho^2)\right)$$

$$Y^{(b)} \mid X = x^{(b)} \sim N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x^{(b)} - \mu_1),\ \sigma_2^2(1-\rho^2)\right)$$