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MGF of bernoulli(x) = $(1-\theta) + \theta e^t = E(e^{tx})$

MGF of binomial$(n, \theta)_{(x)}$ = $(1-\theta+\theta e^t)^n = E(e^{tx})$

$$\overline{Y}_N \; \cancel{\overline{X}_N} = \frac{\sum_{n=1}^{N} x_n}{N}$$

$$M_{\bar{Y}_N}(t) = E(e^{t\bar{x}_N})$$

$$= E\left(e^{\frac{t\sum_{n=1}^{N} x_n}{N}}\right)$$

$$= E\left(e^{\frac{t(x_1 + x_2 + \cdots x_n)}{N}}\right)$$

$$= E\left(e^{\frac{t}{N}x_1} e^{\frac{t}{N}x_2} \cdots e^{\frac{t}{N}x_n}\right)$$

$$= \left(E\, e^{\frac{t}{N}x_i}\right)^N$$

because of iid x_i

$$= \left(1-\theta+\theta e^{\frac{t}{N}}\right)^N$$

$\overline{Y}_N \; \cancel{\overline{X}_N} \sim$ ~~Gam~~ Binomial $(\cancel{N}, \theta)$

$$E(\bar{x}_n) = \theta$$

$$V(\bar{x}_n) = \frac{\theta(1-\theta)}{N}$$

② $X_i \overset{iid}{\sim} \text{Gamma}(\alpha, \lambda)$

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha} = E(e^{tX})$$

$$\bar{X}_N = \sum_{n=1}^{N} X_n$$

$$M_{\bar{X}_N}(t) = E\left(e^{t\bar{X}_N}\right)$$

$$= E\left(e^{t\frac{\sum_{n=1}^{N} X_n}{N}}\right)$$

$$= E\left(e^{\frac{t(X_1 + X_2 + \cdots X_n)}{N}}\right)$$

$$= E e^{\frac{tX_1}{N}} E^{t\frac{X_2}{N}} \cdots E^{t\frac{X_n}{N}} \quad (\because \text{iid})$$

$$= \left(E e^{\frac{t}{N} X_i}\right)^N$$

$$= \left(1 - \frac{t}{N\lambda}\right)^{-N\alpha}$$

$$\bar{X}_N \sim \text{Gamma}(N\alpha, N\lambda)$$

③ $f_x(x) = 2K_0(2\sqrt{x})$, $x \in (0, \infty)$

$$K_0(2x) = \frac{1}{2}\int_0^\infty y^{-1} \exp\left(-y - \frac{x^2}{y}\right) dy$$

$K_0(2x)$ f^n of ~~expon~~ exponential

so, $K_0(2x) \geq 0$ for $x > 0$

$f_x(x) = 2K_0(2\sqrt{x}) \geq 0$ for $x > 0$

Now

$$\int_0^\infty f_x(x)\, dx$$

$$= \int_0^\infty 2K_0(2\sqrt{x})\, dx$$

$$= \int_0^\infty 2 \times \int_0^\infty \frac{1}{2} y^{-1} e^{-y} e^{-\frac{x}{y}} dy \cdot dx$$

$$= \int_0^\infty \int_0^\infty y^{-1} e^{-y} e^{-\frac{x}{y}} dy\, dx$$

$$= \int_0^\infty y^{-1} e^{-y} \int_0^\infty e^{-\frac{x}{y}} dx\, dy$$

$$= \int_0^\infty y^{-1} e^{-y} \left[-y e^{-\frac{x}{y}}\right]_0^\infty dy$$

$$= \int_0^\infty y^{-1} y e^{-y} dy = \int_0^\infty e^{-y} dy$$

$$= \left[-e^{-y}\right]_0^\infty = \left[0 + e^{-0}\right] = 1$$

f_x is valid PDf

④ $X_1 \sim N(0,1)$

$X_t | X_{t-1} \sim N(\rho X_{t-1}, 1-\rho^2)$, $t = 2, \dots T$

$X_2 | X_1 \sim N(\rho X_1, 1-\rho^2)$

$X_3 | X_2 \sim N(\rho X_2, 1-\rho^2)$

we can also write $X_3 | X_2, X_1 \sim N(\rho X_2, 1-\rho^2)$

similarly

$X_t | X_{t-1}, \dots, X_1 \equiv X_t | X_{t-1} \sim N(\rho X_{t-1}, 1-\rho^2)$

The joint p.d.f of $\underline{X} = (X_1, \dots, X_T)'$

$$f_{X_1, X_2 \dots X_T} = f_{X_T | X_{T-1}, \dots X_1} \cdot f_{X_{T-1}, \dots X_1}$$

$$= f_{X_T | X_{T-1} \dots X_1} \cdot f_{X_{T-1} | X_{n-2} \dots X_1} \cdot f_{X_{T-2} \dots X_1}$$

Now spliting also

$$f_{X_1, X_2 \dots X_T} = f_{X_T | X_{T-1}} f_{X_{T-1} | X_{T-2}} \dots f_{X_2 | X_1} f_{X_1}$$

$$= f_{X_1} \prod_{t=2}^{T} f_{X_t | X_{t-1}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x_1^2} \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2} \left\{ \frac{(x_t - \rho x_{t-1})^2}{1-\rho^2} \right\}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x_1^2} \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2} \frac{(x_t - \rho x_{t-1})^2}{1-\rho^2}}$$

$E(X_t) = \rho E(X_{t-1}) = 0$

$$\underline{\mu} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{T\times 1}$$

This is AR(1)

$$X_t = \rho X_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim N(0, 1-\rho^2)$$

$$\mathrm{cov}(X_t, X_s) = \mathrm{cov}(\rho X_{t-1} + \varepsilon_t, \rho X_{s-1} + \varepsilon_s)$$
$$= \rho^2 \mathrm{Cov}(X_{t-1} + \varepsilon_t, X_{s-1} + \varepsilon_s)$$

$$\mathrm{cov}(X_t, X_t) = \mathrm{Var}(X_t) = 1$$

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & & \rho^{T-2} \\ \vdots & & \ddots & & \vdots \\ & & & \ddots & \\ \rho^{T-1} & & & & 1 \end{bmatrix}$$

So, $X \sim N_T(\underline{0}, \Sigma)$

and Conditional distribution

X_t depends only on X_{t-1} and X_{t+1}

So, $X_t | X_1 \ldots X_{t-1}\, X_{t+1} \ldots X_T \sim N(\rho X_{t-1}, 1-\rho^2)$

6 $Y_1, \ldots Y_T \mid X \overset{iid}{\sim} \text{Poisson}(X)$

$X \sim \text{Gamm}(a, b)$

$f(x \mid Y_1 \ldots Y_T) = ?$

$f(x \mid Y_1 \ldots Y_T) \propto \text{Likelihood} \cdot \text{prior}$

$\propto \cancel{f_{X|Y}}\, f(Y_1 \ldots Y_T \mid X) \cdot \pi(X)$

$\propto e^{-Tx} \cdot x^{\sum_{i=1}^{T} y_i} \cdot x^{a-1} e^{-bx}$

$\propto e^{-(T+b)x}\, x^{\sum_{i=1}^{T} y_i + a - 1}$

$\propto \text{Gamma}\left(\sum_{i=1}^{T} y_i + a,\ T+b\right)$

⑦

$$(x^2+y^2-1)^3 = x^2y^3$$

$$\Rightarrow x^2+y^2-1 = x^{2/3}y$$

$$\Rightarrow y^2 - x^{2/3}y + x^2 - 1 = 0$$

quadratic in y

$$y = \frac{x^{2/3} \pm \sqrt{x^{4/3} - 4(x^2-1)}}{2}$$

$$y = \frac{1}{2}x^{2/3} \pm \sqrt{\frac{1}{4}x^{4/3} - x^2 + 1}$$

real solution

$$\frac{1}{4}x^{4/3} - x^2 + 1 \geq 0$$

by solving this equation

we get

$$-1.139 \leq x \leq 1.139$$

$$y' = \frac{1}{3}x^{-1/3} \pm \frac{1}{2}\left(\frac{1}{4}x^{4/3} - x^2 + 1\right)^{-1/2}\left(\frac{1}{3}x^{1/3} - 2x\right)$$

$$= 0$$

difficult find

using R

at $x = 0$ $\quad y_{min} = -1$

at $x \approx 0.5144$, $\quad y_{max} \approx 1.236$