Assgn - 3

Q.

1.

let $P_i \to$ success probability
probability Approve of HOV lane
of County i

$n_i \to$ # commuters

$Y_i \to$ no. of Approve.

this will
Data follow binomial

$$Y_i \mid P_i \sim Bin(n_i, p_i)$$

Liklihood →

$$f(Y_i \mid P_i) = {}^{n_i}C_{y_i} P_i^{y_i} (1-P_i)^{n_i - y_i}$$

$$l(f) = \log(f(y_i \mid P_i))$$

$$= \log {}^{n_i}C_{y_i} + y_i \log(P_i) + (n_i - y_i) \log(1-P_i)$$

(a) for Jeffrey's priors.

$$\frac{\partial l}{\partial P_i} = \frac{y_i}{P_i} - \frac{n_i - y_i}{1-P_i}$$

$$\frac{\partial^2 l}{\partial P_i^2} = -\frac{y_i}{P_i^2} - \frac{(n_i - y_i)}{(1-P_i)^2}$$

$$I(P_i) = -E\left(\frac{\partial^2 l}{\partial P_i}\right)$$

$$= \frac{n_i P_i}{P_i^2} + \frac{1}{(1-P_i)^2}(n_i - n_i P_i)$$

$$= \frac{n_i}{p_i(1-p_i)}$$

Jeffrey's prior.

$$\pi(p_i) \propto \sqrt{I(p_i)} = \sqrt{\frac{1}{p_i(1-p_i)}} = p_i^{-1/2}(1-p_i)^{-1/2}$$

$$\propto p_i^{-1/2}(1-p_i)^{-1/2}$$

$$\searrow \text{Beta}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$p_i \mid y_i \propto f(y_i \mid p_i)\, \pi(p_i)$$

$$\propto p_i^{y_i}(1-p_i)^{n_i - y_i}\, p_i^{-1/2}(1-p_i)^{-1/2}$$

$$\propto p_i^{y_i + \frac{1}{2} - 1}(1-p_i)^{n_i - y_i + \frac{1}{2} - 1}$$

$$\searrow \text{Beta}\left(y_i + \tfrac{1}{2}, n_i - y_i + \tfrac{1}{2}\right)$$

95% Credible interval for county i

$$(L, U)$$

$$L = \text{qbeta}\left(0.025, y_i + \tfrac{1}{2}, n_i - y_i + \tfrac{1}{2}\right)$$

(b) $\hat{p} = 0.48$

$var\ p = 0.02$

$$\frac{a}{a+b} = 0.48$$

$$\frac{ab}{(a+b)^2(a+b+1)} = 0.02$$

$$\Rightarrow a = c(a+b) = \left(\frac{c}{1-c}\right) b = c^* b$$

$$\Rightarrow \frac{c^* bb}{(c^* b+b)^2(c^* b+b+1)} = 0.02$$

$$\Rightarrow \frac{1}{\underbrace{(c^*+1)^2}_{d}(c^* b+b+1)} = \underbrace{\frac{0.02}{c^*}}_{e}$$

$$\Rightarrow (c^* b+b+1) = \frac{1}{ed}$$

$$\Rightarrow b = \frac{\left(\frac{1}{ed} - 1\right)}{(c^*+1)} = \frac{f}{(c^*+1)}$$

$$\boxed{\begin{aligned} c^* &= \frac{c}{1-c} \\ e &= \frac{0.02}{c^*} \\ d &= (c^*+1)^2 \\ f &= \left(\frac{1}{ed} - 1\right) \end{aligned}}$$

$a = 5.5104$

$b = 5.9696$

Q.2

(a) $\hat{u}_{MAP} = \arg\max \left[\log(f(Y|u)) + \log(\overset{\nearrow 0}{\pi(u)}) \right]$

log

$f(y_1 \ldots y_n | u)$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \sigma_i^2}} e^{-\frac{(y_i - u)^2}{2\sigma_i^2}}$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \sigma_i^2}} \cdot e^{-\sum \frac{(y_i - \mu)^2}{2\sigma_i^2}}$$

$$= c\, e^{-\sum \frac{(y_i - \mu)^2}{2\sigma_i^2}}$$

$l = \log(f(y_1 \ldots y_n | u))$

$$= \log c + \left(-\frac{1}{2} \sum \frac{(y_i - u)^2}{\sigma_i^2}\right)$$

$$\frac{dl}{du} = \frac{1}{2} \sum_{i=1}^{n} \frac{(y_i - u)}{\sigma_i^2} \Rightarrow 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{y_i}{\sigma_i^2} = \sum_{i=1}^{n} \frac{\mu}{\sigma_i^2}$$

$$\Rightarrow \hat{\mu}_{MAP} = \frac{\sum_{i=1}^{n} \left(\frac{y_i}{\sigma_i^2}\right)}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$$

$$\pi(\mu \mid y_1 \cdots y_n) \propto f(y_1 \cdots y_n \mid \mu)\, \pi(\mu)$$

$$\propto e^{-\frac{1}{2}\sum \frac{(y_i-\mu)^2}{\sigma_i^2}}$$

$$\propto e^{-\frac{1}{2}\left(\mu^2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2} - 2\mu \sum_{i=1}^{n} \frac{y_i}{\sigma_i^2}\right)}$$

$$\propto e^{-\frac{1}{2}\mu^2\left[\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right]} e^{\mu\left[\sum_{i=1}^{n} \frac{y_i}{\sigma_i^2}\right]}$$

$$\rightarrow N\left(\frac{\sum_{i=1}^{n} \frac{y_i}{\sigma_i^2}}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}, \frac{1}{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}\right)$$

(8). $Y_i \mid \theta \overset{iid}{\sim}$ Laplace (μ, σ) , $\theta = (\mu, \sigma)$

$$f(y \mid \mu, \sigma) = \frac{1}{2\sigma} e^{-\frac{|y-\mu|}{\sigma}}$$

$\mu \in \mathbb{R}$

$\pi(\theta \mid y_1 \ldots y_n)$

$$\propto f(y_1 \ldots y_n \mid \theta)\, \pi(\theta)$$

$$\propto \frac{1}{\sigma^n} e^{-\frac{\sum_{i=1}^{n} |y_i - \mu|}{\sigma}} \quad \mathbb{I}(\sigma \in (0, 10^5), 1$$

$\theta_j^c \sim$ Normal (θ_j^*, s_j^2)

$$R = \min\left\{1, \frac{p(\theta_j^c \mid \theta_{(j)}, y)}{p(\theta_j^* \mid \theta_{(j)}, y)}\right)$$

$$\frac{e^{\frac{-\sum_{i=1}^{n} |y_i - \mu^c|}{\sigma}}}{e^{-\frac{\sum |y_i - \mu^*|}{\sigma}}} = e^{\frac{\sum_{i=1}^{n} |y_i - \mu^*| - \sum |y_i - \mu^c|}{\sigma}}$$