3.3 Supercharges as operators in momentum space

We now present a description of the supercharges in terms of operators in momentum space. Consider as before an infinite chain of fields Z. The vacuum state, written before as $|0;J\rangle = \mathrm{Tr}\left(Z^J\right)|0\rangle$, can be rewritten, in the "hamiltonian formalism" introduced in [4] as $|0;J\rangle = \left(b_z^\dagger\right)^J|0\rangle$, where $b_z^\dagger$ creates an extra Z field in the string.[9] Then we can write a state with K impurities as:

$$|\Psi\rangle = \sum_{n_1,\ldots,n_K} e^{ip_jn_j} b^\dagger(n_1)\cdots b^\dagger(n_K)|0;J\rangle = b^\dagger(p_1)\cdots b^\dagger(p_K)|0;J\rangle\,.$$

We are imposing dilute gas approximation, in which we consider $n_1 \ll n_2 \ll \cdots \ll n_K$. We will now assume $p_1 < p_2 < \cdots < p_K$.

In the last expression for $|\Psi\rangle$ we used the creation operators $b^\dagger(n) = \left(b_z^\dagger\right)^n b^\dagger(b_z)^n$, which create a boson b at position n in the string of $Z's$. One can also introduce $c^\dagger(n) = \left(b_z^\dagger\right)^n c^\dagger(b_z)^n$ as a creation operator for a fermion at position n. The action of the hamiltonian in this framework can be found in [4], and a further comparison with lattice strings can be found in [36].

To write the action of the supercharges in terms of these operators, we also need to introduce a partial momentum operator

$$\hat{\mathcal{P}}(p) = \int_0^p dp'\,p'\left[b^\dagger(p')\,b(p') + c^\dagger(p')\,c(p')\right],$$

or the discrete momentum version

$$\hat{\mathcal{P}}(p) = \sum_{k=0}^{p-1} k\left[b^\dagger(k)\,b(k) + c^\dagger(k)\,c(k)\right].$$

The total momentum operator is just $\hat{P} = \hat{\mathcal{P}}(p_{max})$, where p_{max} is either ∞ in the continuum case, or finite (but large) in the lattice. Also, define an operator $\hat{\Theta}$ conjugate to the "R-charge operator" $\hat{\mathcal{J}}$. In the spin-chain formalism, $\hat{\mathcal{J}}$ effectively measures the length of the chain of Z fields, and $\hat{\Theta}$ changes that length:

$$\hat{\mathcal{J}}\,e^{\pm i\hat{\Theta}}|0,J\rangle = (J\pm 1)\,e^{\pm i\hat{\Theta}}|0,J\rangle\,.$$

We can now proceed to the action of the supercharges. In momentum space, they become:

$$\begin{aligned}
Q_b^\beta &= -\frac{\sqrt{2N}}{M^3}\varepsilon_{bb'}\varepsilon^{\beta\beta'}e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p b^{b'\dagger}(p)\left(e^{ip}-1\right)e^{i\hat{\mathcal{P}}(p)}c_{\beta'}(p) + \sum_p c^{\beta\dagger}(p)\,b_b(p)\,;\\
S_\alpha^a &= \frac{\sqrt{2N}}{M^3}\varepsilon_{\alpha\alpha'}\varepsilon^{aa'}e^{-i\hat{\Theta}}e^{i\hat{P}}\sum_p c^{\alpha'\dagger}(p)\left(1-e^{-ip}\right)e^{-i\hat{\mathcal{P}}(p)}b_{a'}(p) - \sum_p b^{a\dagger}(p)\,c_\alpha(p)\,.
\end{aligned} \tag{22}$$

It is not hard to check that these definition give us the results obtained in the previous section. In the above expression the sum over momenta has increments of $\frac{2\pi}{J}$.[10]

Commuting two central charges Q will give us the central charge $\mathcal{P}$:

$$\{Q,Q\} = e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p b^\dagger(p)\left(e^{ip}-1\right)e^{i\hat{\mathcal{P}}(p)}b(p) + e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p c^\dagger(p)\left(e^{ip}-1\right)e^{i\hat{\mathcal{P}}(p)}c(p) = \mathcal{P}. \tag{23}$$

[9] The subscript z is used in this section, to distinguish the creation operator $b_z^\dagger$ for the boson Z from the creation operator $b^{a\dagger}$ for the two bosonic impurities.

[10] The operator $e^{\pm i\hat{\Theta}}$ does not commute with the sum over the momenta, as it changes the increments in the sum. But in the limit J very large, this change will be negligible.