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Lagrangian Formulation of the Zachariasen Model

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A Lagrangian formulation of the Zachariasen model is discussed. It contains a field with a continuous mass distribution which describes pairs of particles. The significance of this model within the framework of axiomatic field theory is pointed out.

I. INTRODUCTION

A MODEL of a relativistic quantum field which can be solved exactly and which illustrates several features of field theory[1] has been proposed recently by Zachariasen.[2] In this model two kinds of particles A and B are considered and the recipe is to calculate Feynman diagrams using only the vertices $A+\bar{A}\leftrightarrow B$ and $A+\bar{A}\rightarrow A+\bar{A}$. Thus, if only the first vertex exists, for instance, the propagator for the B particle is first the usual sum of bubbles (Fig. 1).

It is to be noted that this does not simply correspond to an interaction Hamiltonian where these vertices are generated by the corresponding creation and absorption operators since that would only lead to the nonrelativistic energy denominators and not to Feynman propagators in the internal lines. On the other hand, Zachariasen showed that the results of this theory could be derived by dispersion theory with an additional prescription as to what intermediate states are to be used. Thus, it was concluded that this model provides the interesting example of a relativistic theory without Hamiltonian which is defined in terms of dispersion relations. However, Lee[3] made the remark that the calculations made by Zachariasen correspond to the use of a quadratic Hamiltonian where pairs of A particles are replaced by a field with a continuous mass distribution. This can be seen most easily by noting that the product of the two propagators in the $A-\bar{A}$ bubble can be represented as a superposition of Feynman propagators with different masses according to the formula[4]

$$\Delta_F(x,m_A{}^2)\Delta_F(x,m_A{}^2)=-\int dm^2\, f^2(m^2)\Delta_F(x,m^2),$$
$$f^2(m^2)=\frac{1}{16\pi^2}\theta(m^2-4m_A{}^2)\left(\frac{m^2-4m_A{}^2}{m^2}\right)^{\frac{1}{2}}. \tag{1}$$

Thus, if $A\bar{A}$ is replaced by one field with the propagator given by (1) all Feynman graphs considered in this model correspond to a quadratic L'.

In the present note we will work out a theory with a continuous mass distribution and demonstrate that it is in fact equivalent to the Zachariasen model. This theory is perfectly local and satisfies all axioms of field theory except the asymptotic condition in its usual form.[5] Since the A pairs are represented by one field with a mass distribution from $2m_A$ to infinity, there is no way to project out the single A particle. Thus, the most interesting feature of the model is that it provides an example which illustrates the importance of the asymptotic condition.

II. FORMALISM

In this section we will formulate a theory with a continuous mass distribution which is equivalent to the Zachariasen model. For simplicity we limit ourselves

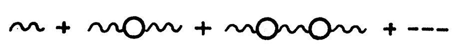

FIG. 1. Diagrams for the π-particle propagator which are taken into account in the model. Wavy lines $=B$ lines; straight lines $=A$ lines.

to the case of the $A\bar{A}B$ vertex only, the more general case can be treated in the same way as will be discussed below. We shall also ignore divergent integrals since the renormalization procedure goes completely in the standard way.

In the model, one single A particle has no interaction and can therefore be disregarded. The B particle will be described by a field $\varphi(x,\mu_0{}^2)$ and the pairs of A particles by fields $\varphi(x,s)4m^2<s<\infty$. This field replaces $A(x)\bar{A}(x)$ and thus creates an A pair in an s state with (total mass)$^2=s$ and center of mass at x. Putting $4m_A{}^2=1$ and suppressing the argument x, we can write L_0 in the condensed form (up to a divergence)

$$L_0=\int ds\;\varphi(s)(\Box-s)p(s)\varphi(s), \tag{2}$$

with

$$p(s)=\delta(s-s_0)+\theta(s-1),\quad s_0=\mu_0{}^2.$$

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[1] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

[2] F. Zachariasen, Phys. Rev. **121**, 1851 (1961).

[3] T. D. Lee, remark at a CERN seminar.

[4] See, for instance, W. Thirring, *Principles of Quantum Electrodynamics* (Academic Press Inc., New York), Appendix 2.

[5] See H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 1425 (1955).

In this notation the interaction corresponding to the Zachariasen model is given [with f as defined in (1)] by

$$L'=g_0\int dsds'\ \varphi(s)[\delta(s-s_0)f(s')+\delta(s'-s_0)f(s)]\varphi(s'). \quad (3)$$

It is apparent that treating L' in perturbation theory yields propagators given by (1). However, since $L=L_0+L'$ is quadratic it can be diagonalized to the form (always understanding that the Φ also depend on x)

$$L=\int ds\,\Phi(s)[\Box-M^2(s)]p(s)\Phi(s), \quad (4)$$

where the $\Phi(s)$ are linear combinations of the $\varphi(s)$:

$$\varphi(s)=\int ds\,K(s,s')\Phi(s),$$
$$\int ds\,K^\dagger(s'',s)p(s)K(s,s')=\delta(s''-s')p(s'). \quad (5)$$

This diagonalization leads to the usual separable integral equation for K.[6] Writing L in the form (4), we have anticipated that the eigenvalue spectrum of this equation consists of one discrete point $M^2(s_0)$ and a continuum. This happens if the equation,

$$\mu^2-\mu_0^2=g_0^2\int\frac{ds\,f^2(s)}{\mu^2-s}, \quad (6)$$

has a solution for $0<\mu^2<1$. Then we have two solutions $K_\pm$ with $M^2(s_0)=\mu^2$, $M^2(s)=s$ for $s>1$, namely,

$$K_\pm(s,s')=a(s)\delta(s'-s_0)+G_\pm(s,s')\theta(s'-1),$$
$$a(s')=\frac{g_0f(s')}{\mu^2-s'}a(s_0),$$
$$G_\pm(s_0,s)=\frac{g_0f(s)}{s-s_0}\frac{1}{D_\pm(s)}, \quad (7)$$
$$D_\pm(s)=1+\frac{g_0^2}{s-s_0}\int\frac{ds'\,f(s')}{s'-s\pm i\epsilon},$$
$$G_\pm(s',s)=\delta(s'-s)+\frac{g_0^2f(s')f(s)}{(s-s_0)(s-s'\pm i\epsilon)D_\pm(s)}\quad\text{for}\quad s'>1,$$

a_0 will be determined by the orthogonality relation (5) later on. Thus, there are two kinds of K's which diagonalize L and satisfy (5).

Correspondingly, there will be two Φ's defined by $\varphi=K^\pm\Phi^\pm$. Since the K's are complex the Φ's will no longer be Hermitian fields but differ from such fields by a phase factor. To study their physical significance we observe that the matrix elements of $\Phi(s)$ will have a time dependence $\exp[i(s-k^2)^{\frac{1}{2}}t]$ and therefore we suspect from (7) that in some sense φ approaches Φ^- for $t\to-\infty$ and Φ^+ for $t\to+\infty$; to make this a little more precise we define states generated by the Φ's as

$$\Phi^+(s)|0\rangle=|A\bar{A}\text{ out}\rangle,$$
$$\Phi^-(s)|0\rangle=|A\bar{A}\text{ in}\rangle, \quad (8)$$

for $s>1$. With (9) and (7) we see that for $K=0$ and $s>1$,

$$\langle\text{in }A\bar{A}|\varphi(s)|0\rangle=\langle\text{in }A\bar{A}|\Phi^-(s)|0\rangle+g_0^2\int_1^\infty\frac{ds'\exp[i(s')^{\frac{1}{2}}t]}{s'-s-i\epsilon}\frac{f(s)f(s')}{s-s_0}\to\langle\text{in }A\bar{A}|\Phi^-(s)|0\rangle, \quad (9)$$

for $t\to-\infty$. An analogous relation holds for the "out" states, Φ^+ and $t\to\infty$. Thus, in this sense φ approaches Φ for $t\to\mp\infty$, which suggests the interpretation that the states $\langle A\bar{A}\text{ out}|$ and $\langle A\bar{A}\text{ in}|$ for $t\to\mp\infty$ correspond to incoming or outgoing pairs of particles. Since only relative s states are involved the two kinds of states differ just by a phase factor. This can be seen by observing that the G's can also be written[6]

$$G_\pm(s',s)=\frac{D_1(s)}{D_\pm(s)}\left\{\delta(s-s')+P\frac{g_0^2f(s)f(s')}{(s-s_0)(s-s')D_1(s)}\right\}, \quad (10)$$

where P means principal value and D_1 is the real part of $D_\pm$. From this and (5) we infer for $s>1$:

$$|A\bar{A}\text{ out}\rangle=\frac{D_+(s)}{D_-(s)}|A\bar{A}\text{ in}\rangle=e^{2i\delta(s)}|A\bar{A}\text{ in}\rangle, \quad (11)$$

or

$$\tan\delta(s)=16\pi\left(\frac{s}{s-1}\right)^{\frac{1}{2}}\frac{s-\mu_0^2}{g_0^2}\times\left\{1+\frac{g_0^2}{s-\mu_0^2}P\int\frac{ds'}{s'-s}\left[\frac{s-1}{s(16\pi^2)^2}\right]^{\frac{1}{2}}\right\}. \quad (12)$$

This is exactly the unrenormalized expression for the phase shift one obtains in the Zachariasen model[7] for the case of no direct interaction. Since the asymptotic condition does not hold in its usual form but only in the way specified by (9) the reader may not be completely convinced by our physical interpretation of $\delta(s)$.

[6] A similar integral equation is met in meson pair theory. See, for instance, A. Klein and B. McCormick, Phys. Rev. **98**, 1428 (1955).

[7] See also S. Okubo, Phys. Rev. **118**, 357 (1960).

An alternative way of deriving δ would be to use finite normalization volume and calculate the shift of the energy levels due to L'. Using the well-known formula[8] energy shift/level spacing$=-\delta/\pi$, one can derive the same expression for δ.

Another quantity of interest is the propagator for the B particles. This can be simply derived in our formulation by using (5):

$$\varphi(s_0)=a(s_0)\Phi(s_0)+\int_1^\infty ds\, G_-(s_0,s)\Phi^-(s). \qquad (13)$$

Hence we obtain

$$\langle 0|\varphi(s_0)\varphi(s_0)|0\rangle=|a(s_0)|^2\Delta(\mu^2)+\int_1^\infty |G_-(s_0,s)|^2\Delta(s). \qquad (14)$$

Thus, we see that the (unrenormalized) Lehmann-Källén spectral function is given by

$$\sigma(s)=|G_\mp(s_0,s)|^2=\frac{g_0^2|f(s)|^2}{(s-s_0)^2|D_\pm(s)|^2}. \qquad (15)$$

To calculate the renormalization constant we have to use (5):

$$Z_3=|a(s_0)|^2=1-\int_1^\infty ds\,|a(s)|^2$$
$$=1-|a(s_0)|^2g_0^2\int_1^\infty\frac{dsf^2(s)}{(\mu^2-s)^2}$$
$$=1\Big/\left[1+g_0^2\int_1^\infty\frac{ds\,|f(s)|^2}{(\mu^2-s)^2}\right]. \qquad (16)$$

Since in our case in the Zachariasen model $Z_3=g^2/g_0^2$ we see that the renormalized spectral function is obtained by replacing g_0^2 by g^2. Our results (14) to (16) agree with the corresponding formulas derived by Zachariasen.

So far, we have illustrated that in the simplest case our formalism agrees with the calculations of Zachariasen. The case of a direct interaction, i.e., vertices $A+\bar{A}\rightarrow A+\bar{A}$, can be treated by an interaction $\varphi(s)\varphi(s')$ with both s and s' greater than 1. Zachariasen also remarks on the $A-B$ scattering where he proposes to consider diagrams of the form shown in Fig. 2. Since their calculation is not so elementary they are not treated in his paper. To include them in our formulation we have to introduce a field for a single A particle and an interaction which yields the processes $A+(A\bar{A})\leftrightarrow A+B$. This will be quadrilinear in the fields and hence cannot be solved explicitly any more. However, we do

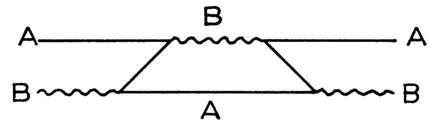

FIG. 2. Diagram for AB scattering. Wavy lines$=B$ lines; straight lines$=A$ lines.

not have to introduce these terms if we want to have an explicitly soluble model.

III. DISCUSSION

Our results have shown that there exists a Lagrangian for the Zachariasen model which has the same shortcomings as the one of other relativistic field theories: It contains renormalization constants when expressed in terms of bare fields but everything is perfectly finite when the incoming or outcoming fields are introduced. The peculiarity of this model lies in the fact that the asymptotic condition is not met in its usual form.[5] The only asymptotic field in the usual sense is the $\Phi(s_0)$ which describes the physical B particle. Obviously this field does not give a complete set of operators. Since the interacting $\bar{A}-A$ pairs are described by a single field with a continuous mass distribution, there is no way to project out the single A particles. Thus the model fills a certain gap in the axiomatic scheme of field theory as we shall discuss now.

The usual axioms of quantum field theory can be listed in the following way[9]:

(1) Lorentz invariance;
(2) positive energies;
(3) positive probabilities;
(4) locality;
(5) asymptotic condition.

In four-dimensional space-time continuum there are no realizations of all five axioms except the free fields. Leaving out (1), there are several well-known examples which satisfy the other axioms.[10]

If one leaves out (2) or (3), explicit realizations have been given by Glaser.[11] They are of the type

$$S=T\exp\left\{ig\int d^4x[A(x)+B(x)]^4\right\},$$

where A and B are fields which commute with each other and their propagators have opposite signs. Such

[8] See, for instance, M. Baker, Ann. Phys. (New York) **4**, 271 (1958).

[9] For a full discussion see K. Symanzik, *Celebration Volume for Heisenberg's 60th Birthday*. (Friedrich Vieweg Sohn, Brauschweig Germany).

[10] Nonrelativistic models will be treated extensively by E. Henley and W. Thirring in *Elementary Quantum Field Theory* [McGraw-Hill Book Company, Inc., New York, 1962 (to be published)].

[11] V. Glaser (private communication).

a problem can be solved exactly and it shows the expected unphysical features. The case where (4) is omitted has been discussed by Lehmann[12] and an example has been given. Our model fits into this scheme inasmuch as it satisfies all axioms except (5). Thus it demonstrates that the five axioms listed above are independent.[13]

Another case where the asymptotic condition does not hold has recently been proposed by Sudarshan.[14] There the Wightman functions are simply sums of the ones for free fields. This case can be excluded by supplementing Wightman's scheme by the requirement of a certain cluster property of the Wightman functions. This, however, does not rule out the present example. Fields with continuous mass distributions where the asymptotic fields are not complete have been studied by Greenberg.[15] The merit of these models is that they indicate what exists mathematically. Furthermore, they illustrate what kind of theories have to be taken into account when one tries to widen the framework of field theory as outlined by the axioms. Our findings make one suspect that field theories without Hamiltonian, which are defined only by dispersion relations, actually have an underlying Hamiltonian structure.

ACKNOWLEDGMENT

I would like to thank Professor T. D. Lee for the remark which stimulated this investigation.

[12] H. Lehmann, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics, 1957* (Interscience Publishers, New York, 1957).

[13] This point has been emphasized by D. Fivel (private communication).

[14] E. C. G. Sudarshan (to be published).

[15] L. O. W. Greenberg (to be published).