It is easy to verify by explicit calculation that the contribution of radiation reaction to $\langle dH_A/d\tau\rangle$ is given indeed by (17), as stated by Theorem 1.

The contribution of vacuum fluctuations is more difficult to obtain. It is calculated from Eq. (4) via the residue theorem so that we need to know the zeroes of the denominator. This leads to an equation of the form $x^2 = v^2 \sin^2 x$, which is not analytically solvable. The problem becomes tractable in the "high velocity limit" [9, 10]. For $v \gtrsim 0.85$, we can expand the sine to find the zero with the smallest imaginary part (besides $x = 0$). Because the exponential in the numerator of the resulting integral, zeroes with larger imaginary part can be neglected. In this way, we obtain

$$\begin{aligned} \left\langle \frac{dH(\tau)}{d\tau} \right\rangle_{vf} &= -\frac{\mu^2}{2\pi} \sum_{\omega_a > \omega_b} \left(\frac{\omega_{ab}^2}{2} + \frac{a\omega_{ab}}{4\sqrt{3}} A e^{-2\sqrt{3}B\frac{\omega_{ab}}{a}} \right) |\langle a|R_2^f(0)|b\rangle|^2 \\ &\quad -\frac{\mu^2}{2\pi} \sum_{\omega_a < \omega_b} \left(\frac{\omega_{ab}^2}{2} + \frac{a|\omega_{ab}|}{4\sqrt{3}} A e^{-2\sqrt{3}B\frac{|\omega_{ab}|}{a}} \right) |\langle a|R_2^f(0)|b\rangle|^2 \end{aligned} \tag{43}$$

where

$$A = 1 + \frac{3}{5}(v\gamma)^{-2}, \qquad B = 1 - \frac{1}{5}(v\gamma)^{-2}. \tag{44}$$

The total rate of change of the atomic excitation energy can be found by adding the contributions of vacuum fluctuations and radiation reaction:

$$\begin{aligned} \left\langle \frac{dH(\tau)}{d\tau} \right\rangle_{tot} &= -\frac{\mu^2}{2\pi} \sum_{\omega_a > \omega_b} \left(\omega_{ab}^2 + \frac{a\omega_{ab}}{4\sqrt{3}} A e^{-2\sqrt{3}B\frac{\omega_{ab}}{a}} \right) |\langle a|R_2^f(0)|b\rangle|^2 \\ &\quad -\frac{\mu^2}{2\pi} \sum_{\omega_a < \omega_b} \frac{a|\omega_{ab}|}{4\sqrt{3}} A e^{2\sqrt{3}B\frac{|\omega_{ab}|}{a}} |\langle a|R_2^f(0)|b\rangle|^2 \end{aligned} \tag{45}$$

We recognize that spontaneous excitation ($\omega_a < \omega_b$) is possible as well as spontaneous de-excitation ($\omega_a > \omega_b$). As stated by the contents of theorems 1 and 2, the balance between vacuum fluctuations and radiation reaction which is present for an inertial atom becomes upset.

From Eq. (45), it is possible to get in second order in μ a differential equation for $\langle H_A\rangle$, (cf. [2])

$$\left\langle \frac{dH(\tau)}{d\tau} \right\rangle = -\frac{\mu^2}{8\pi}\omega_0 \left\{ \frac{\omega_0}{2} + \left(1 + \frac{A}{2\sqrt{3}} \frac{a}{\omega_0} e^{-2\sqrt{3}B\frac{\omega_0}{a}} \right) \langle H(\tau)\rangle \right\}. \tag{46}$$

The solution

$$\begin{aligned} \langle H(\tau)\rangle &= -\frac{\omega_0}{2} + \frac{\omega_0}{2} \left[1 + \frac{2\sqrt{3}}{A} \frac{\omega_0}{a} e^{2\sqrt{3}B\frac{\omega_0}{a}} \right]^{-1} \\ &\quad + \left(\langle H(0)\rangle + \frac{\omega_0}{2} - \frac{\omega_0}{2} \left[1 + \frac{2\sqrt{3}}{A} \frac{\omega_0}{a} e^{2\sqrt{3}B\frac{\omega_0}{a}} \right]^{-1} \right) e^{-\Gamma\tau} \end{aligned} \tag{47}$$

gives the time evolution of the mean atomic excitation energy. The decay rate is

$$\Gamma = \frac{\mu^2}{8\pi}\omega_0 \left(1 + \frac{A}{2\sqrt{3}} \frac{a}{\omega_0} e^{-2\sqrt{3}B\frac{\omega_0}{a}} \right). \tag{48}$$

The second term in the bracket represents the modification of the inertial decay constant

$$\Gamma_{inert} = \frac{\mu^2}{8\pi}\omega_0. \tag{49}$$

From (47) we see that the system evolves towards an equilibrium population

$$\langle H_A\rangle = -\frac{1}{2}\omega_0 + \frac{\omega_0}{2} \left(1 + \frac{2\sqrt{3}}{A} \frac{\omega_0}{a} e^{2\sqrt{3}B\frac{\omega_0}{a}} \right)^{-1}, \tag{50}$$