Homework 4

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Problem 1

$$\sum_{1\le m\le n}\mu(m)\lfloor\frac{n}{m}\rfloor = 1$$

Proof:

Counting the squarefree number in $[n]$ which is not a multiplication of any primes. It turns out there is only one number 1 satisfied this condition.

On the other hand, count them by PIE:

Assume that $P_1, P_2, \cdots, P_r$ are the primes less or equal than n.

$$\sum_{S\in[r],m=\Pi_{i\in S}P_i}(-1)^{|S|}\lfloor\frac{n}{m}\rfloor = 1 = \sum_{S\in[r],m=\Pi_{i\in S}P_i}\mu(m)\lfloor\frac{n}{m}\rfloor$$

And for non-squarefree number $\mu(m) = 0$.

$$\sum_{1\le m\le n}\mu(m)\lfloor\frac{n}{m}\rfloor = 1$$

Problem 2 $20121009 = 3\times 599\times 11197$ Where $3, 599$ and 11197 are all primes.

Therefore

$$\sum_{d|20121009}\mu(d)P(\frac{20121009}{d}) = \sum_{S\in 2^3}(-1)^{|S|}(3-|S|) = 0$$

Problem 3 $x_1 + x_2 + \cdots + x_t = \frac{1}{2}t(x_1+x_t) = (x_1 - 1 + t)t = n$

For every $t|n$ there is an unique x_1 which defines an integer partition. Therefore $f(n)$ equals the number of divisors of n.

Problem 4

(a) For $n = 1$, $1 = 1$.
For $n = 2$, $2 = 1 + 1 = 2$.
For $n = 3$, $3 = 1 + 1 + 1 = 1 + 2 = 3$.
For $n = 4$, $4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 3 = 2 + 2 = 4$.
For $n = 5$, $5 = 1+1+1+1+1 = 1+1+1+2 = 1+1+3 = 1+2+2 = 1+4 = 2+3 = 5$.

(b) **Proof by induction:**
First $p(0)\le F_0, p(1)\le F_1$ holds.
Assume that $\forall i < n, p(i)\le F_i$.
For every partitions of n, if there is a 1 in it, remove it then this partition lies in partitions of $n-1$. If there is no 1 in it, pick the smallest element and decrease it by 2 then it lies in partitions of $n-2$. And the two transformation above will not produce a collision.
Which means $p(n)\le p(n-1)+p(n-2)\le F_{n-1}+F_{n-2} = F_n$.
So $p(n)\le F_n$ forall $n\in\mathbb{N}$.

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