Homework 9

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Problem 1 (a)

$$E(X_v) = \frac{\binom{6}{3} \times 2 \times 3!}{6!} = \frac{1}{3}$$

$$E(X) = \sum_v E(X_v) = 2$$

And no matter what σ is, each triangle has one and only one seed. Therefore the probability is 1.

(b) $E(X)$ is the same as above.

Fix the position of 1. If another seed is 2, considering the position of 2 there are $4! + 2(4! - 3! - 2)$ outcomes. Otherwise 2 is adjacent to 1. If another seed is 3, there are $2(3! \times 2)$ outcomes. Otherwise 3 is adjacent to 2 or 1. If another seed is 4, there are $2 \times 2 \times 2$ outcomes.

So $Pr(X = 2) = \frac{4!+2(4!-3!-2)+2(3!+2)+2\times 2\times 2}{5!} = \frac{11}{15}$.

Problem 2 For odd n, considering two coloring in C_n, events E_i are whether one edge is not monochromatic. Then $Pr(\cap_{i\in S} E_i) = \frac{1}{2^{|S|}}(S \neq [n])$, but $Pr(\cap_{i\in[n]} E_i) = 0$.

Otherwise provide $n-1$ random variables a_i. The first $n-1$ events show that $a_i = 1$. And the n-th event shows $\sum_{i=1}^{n-1} a_i \equiv 1(mod 2)$

Then $Pr(\cap_{i\in S} E_i = \frac{1}{2^{|S|}})(S \subseteq [n-1])$.

$Pr(\cap_{i\in S} E_i \cap E_n) = \frac{1}{2^{|S|+1}}(S \subset [n-1])$,

But $Pr(\cap_{i\in[n]} E_i) = \frac{1}{2^{n-1}}$.

Problem 3 Color the vertices uniformly at random.

Events: for each edge (u, v) where $S(u) \cap S(v) \neq \emptyset$, one event denotes that u and v are colored with c. At most $|V| \times 10d^2$ events.

Dependency: One edge is conditional independent to another set of edges if the two vertices is not adjacent to any edge of that set. Because the probability for one edge (u, v) to be monochromatic is $p = Pr(S_u = c)Pr(S_v = c) = \frac{1}{100d^2}$, no matter what's the color of other vertices.

Degrees: So in the dependency graph, for each events there are at most $20d^2$ events adjacent to it. Since each vertex has at most d vertices for one color so that the edge between them can be monochromatic.

$4 \times \frac{1}{100d^2} \times 20d^2 = \frac{4}{5} < 1$. By L. L. L., $Pr(\text{every edge is not monochromatic}) > 0$. So there exists an outcome which is a proper colouring.

Problem 4 (a) Since every Z_a has fixed length k, the common difference will be no greater than $\frac{n}{k-1}$.

And for every number, it could lie in at most $\frac{n}{k-1}k$ a.p.s with length k since a.p. is uniquely determined by a starter and a common difference.

So for any Z_a, there are at most $\frac{n}{k-1}k^2$ Z_as can have non-empty intersection with it. Which is less than $1.25kn$ when $k \geq 10$.

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(b) Color every elements uniformly at random.

Events: for each Z_a an event shows whether it's monochromatic. The probability for one event to be true is 2^{1-k}.

Dependency: One event Z_a is independent to another set of Z_as if this Z_a do not share any points with that set. In dependency graph one event has degree less than $1.25kn = 2^{k-3}$.

$4pd < 4 \times 2^{1-k} \times 2^{k-3} = 1$. By L. L. L., Pr(every Z_a is not monochromatic) > 0. So there exists an outcome satisfies the condition.