Homework 1

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Problem 1 (a)

$$\binom{10}{8} = \binom{10}{2} = \frac{10 \times 9}{2 \times 1} = 45$$

(b) The inverse of $1, 2, 3, 4, 5, 6$ in $\mathbb{Z}_7$ are $1, 4, 5, 2, 3, 6$.

(c) Easily that $\binom{449}{137} \equiv 0 (\text{mod } 2)$ and $\binom{449}{137} \equiv 3 (\text{mod } 5)$.
By Chinese Remainder Theorem $\binom{449}{137} \equiv 8 (\text{mod } 10)$.

Problem 2 For every ordered pair (A, B) which $A \cap B = \emptyset$, each element $x \in [n]$ has one of these three states: $x \in A$, $x \in B$ and $x \in [n] - A - B$.

And clearly that the map between ordered pairs and state vector of elements is bijective.

So the total number of ordered pairs is 3^n.

Problem 3

$$\begin{aligned}
(x+y)^n &= \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \\
\frac{d(x+y)^n}{dx} &= \sum_{r=0}^{n} r \binom{n}{r} x^{r-1} y^{n-r} && = n(x+y)^{n-1} \\
\frac{d(nx(x+y)^{n-1})}{dx} &= \sum_{r=0}^{n} r^2 \binom{n}{r} x^{r-1} y^{n-r} && = n(x+y)^{n-1} + n(n-1)x(x+y)^{n-2}
\end{aligned}$$

Let $x = y = 1$, $\sum_{r=0}^{n} r^2 \binom{n}{r} = n2^{n-1} + n(n-1)2^{n-2} = n(n+1)2^{n-2}$

Combinatorial proof:

$\sum_{r=0}^{n} r^2 \binom{n}{r}$ means choose r elements from $[n]$ and then pick 2 special elements (not necessary distinct) from these r elements. That is the number of triples $\{(a, b), A\}$ where $A \subseteq [n]$ and $a, b \in A$.

There is another way to count the number above: If $a = b$, the number of ways to pick a and then determine whether the other element is in A is $n2^{n-1}$. Otherwise the number of ways to pick a and b and then determine whether the other element is in A is $n(n-1)2^{n-2}$. Therefore the total number of ways to count it is $n(n+1)2^{n-2}$.

Problem 4 (a) Let i denotes the size of the blue square.

$$\sum_{i=1}^{9} (9 - i + 1)^2 = 285$$

(b) First, count the number of disjoint squares which can be divide via a horizontal(or vertical) line (i enumerates the line, j enumerates the size of square):

$$2 \times \sum_{i=1}^{8} ((\sum_{j=1}^{i} (9 - j + 1)(i - j + 1)) \times (\sum_{j=1}^{9-i} 9 - j + 1)) = 37752$$

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Then count the number of disjoint squares which can be divice via a horizontal line and a vertical line.

$$2 \times \sum_{i=1}^{8} \sum_{j=1}^{8} ((\sum_{k=1}^{min(i,j)} (i - k + 1) \times (j - k + 1)) \times \min(9 - i, 9 - j)) = 11748$$

So the total number is $2 \times (37752 - 11748) = 52008$.

Problem 5 Fill the 81 entries one-by-one using the ascending sequence from 1 to 81. Consider the first time every row or every column are filled with at least one number. WLOG, let's discuss with the first situation, and the last number placed into the entries is x.

Clearly every row has at least one empty entry which is adjcent to the one was filled before. Otherwize there is some rows full-filled, that is the situation that every column are filled with at least one number. So there are at least 9 empty entries which is adjcent to an entry filled with number less or equal than x. And there is no way to put the numbers left without violating the condition given.