Homework 10

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Problem 1 Let $\mathcal{B} = \{(A, x) | A \in \mathcal{A}, x \in A\}$.

There is a way to count $|\mathcal{B}|$: First choose one $\bar{A} \in \partial\mathcal{A}$, then choose x from $[n] \setminus \bar{A}$, $(\bar{A} \cap \{x\}, x)$ is what we want. Clearly every entries from $\mathcal{B}$ would be count at least once.

Therefore $\partial\mathcal{A}(n-r) \geq \mathcal{A}(r+1)$.

$\binom{n}{r+1} = \frac{\binom{n}{r}(n-r)}{r+1}$.
So

$$\frac{\partial\mathcal{A}}{\binom{n}{r}} \geq \frac{\mathcal{A}}{\binom{n}{r+1}}$$

Problem 2 (a) Chain1 : $\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}$
Chain2 : $\{2\}, \{2,3\}, \{2,3,4\}$
Chain3 : $\{3\}, \{1,3\}, \{1,3,4\}$
Chain4 : $\{4\}, \{1,4\}, \{1,2,4\}$
Chain5 : $\{2,4\}$
Chain6 : $\{3,4\}$

(b) For $2^{[n]}$, construct the graph with $V = 2^{[n]}$. There is an edge between A and B $(WLOG, |A| \leq |B|)$ iff $\exists x, A \cup \{x\} = B$.

Consider the subgraph induced by $\binom{[n]}{r}$ and $\binom{[n]}{r+1}$. The vertex in the former has a degree of $n-r$ and in the latter has a degree of $r+1$. The vertices in smaller side has a higher degree. Which means there is always a perfect matching between them (number of matches equal to the size of smaller side).

And $\binom{n}{r} = \binom{n}{n-r}$, each time we can pick one longest chain start at the very beginning (lies in $\binom{[n]}{k}$) and end at the last (lies in $\binom{[n]}{n-k}$). Which satisfies the condition.

Problem 3 (a)

$$\binom{10}{5} = 252 > 200$$

(b)

$$\binom{9}{4} = 126 > 100$$

Problem 4 Create a new matrix M_i for each line. The entry at row j column k represents that line i has a number k set at column j in the original matrix A.

Since there is no line contains the same number twice. M_i is a permutation matrix. Let $M = \sum_{i=1}^{k} M_i$, M has $n-k$ 0s in each row and column.

Let C be a matrix with all entries set to 1. Then $C - M$ can be factored into the sum of $n-k$ permutation matrices which implies the result.

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