Homework 7

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Problem 1 Let $N = (c+1)^{c+1} + 1$.

Consider the first $c+1$ columns. There are only $(c+1)^{c+1}$ ways to color one row. And N rows imply that there are two rows with the same color of their first $c+1$ elements. (By pigeonhole principle) Take a look at those two rows. Since we have only c colors by pigeonhole principle there must be two elements have the same color, which construct a axis-parallel rectangle with the same color.

Problem 2 Proof by v.d.W:

There exists $N = N(c, r(r+1))$ such that no matter how we c-color [N], there is an monochromatic A.P. with length $r(r+1)$. Then assign x with the minimum of A.P. and $y_i = i \times d$ where d is the common difference. Easy to check that the 2^r sums are no bigger than $r(r+1) \le N$ and have the same color.

Problem 3 For any given graph with colors on N nodes. Coloring the edge of K_n in the following way: edge between a and b ($a > b$) are colored with the color of node $b > a$ in previous graph.

Take $N = N_c(3;2)$, there exists a monochromatic triangle with nodes A, B and C ($A > B > C$), which means $A - B$, $B - C$ and $A - C$ has the same color in previous graph $(A - B) + (B - C) = A - C$.

The only problem happens when $A - B = B - C$. And the solution is to add another c colors, for every A.P. in $[N]$, color the edge between them alternatively (with x and $-x$). Therefore the theorem holds and $S^*(c) \le N_{2c}(3;2)$.

Problem 4 (a) Every vertex of Q_n can be represented with a binary number with length n (leading zero is allowed). And there is one edge between two numbers iff they differ from exactly one bit.

Let $c(n)$ be the number of '1's in the binary representation of n.

Color the edge between a and b (WLOG, $a < b$) with Y iff $c(a)$ is odd, with B iff $c(a)$ is even.

For every $Q_2(a, b, c, d)$, the four numbers differ from only two bits. WLOG consider $a = 00, b = 01, c = 10, d = 11$, it's not monochromatic according the color made above.

(b) If T_2 has a chain with more then 3 nodes then it does not hold. Since we can randomly pick one node to be root in T_1, and color the edge alternatively according to its distance from the root (odd to be Y and even to be B). Clearly that there is no monochromatic chain with length 4 (nodes).

So the only possible T_2 is star-shaple: one n-degree node and n 1-degree nodes. And easy to see that the property holds iff there exists a node in T_1 with degree greater than $(n-1)^2$ by pigeonhole principle.

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