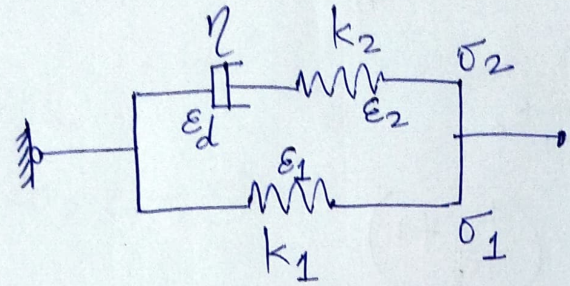
Project- Rheology

Rough Calculations

(A)



$$\sigma_1 = k_1 \varepsilon_1$$
$$\sigma_2 = k_2 \varepsilon_2$$
$$\sigma_d = \eta \dot{\varepsilon}_d$$

$$\varepsilon_1 = \varepsilon_2 + \varepsilon_d = \varepsilon$$
$$\sigma_1 + \sigma_M = \sigma$$
$$\sigma_2 = \sigma_d = \sigma_M$$

$$k_1 \varepsilon + \sigma_M = \sigma$$

$$\frac{\dot{\sigma}_2}{k_2} + \frac{\sigma_d}{\eta} = \dot{\varepsilon}$$

$$\frac{\dot{\sigma}_M}{k_2} + \frac{\sigma_M}{\eta} = \dot{\varepsilon}$$

$$\frac{k_1 \varepsilon + \sigma_M}{\eta} + \frac{k_1 \dot{\varepsilon} + \dot{\sigma}_M}{k_2} = \frac{\sigma}{\eta} + \frac{\dot{\sigma}}{k_2}$$

$$\boxed{\frac{k_1}{\eta} \varepsilon + \left(\frac{k_1}{k_2} + 1\right) \dot{\varepsilon} = \frac{\sigma}{\eta} + \frac{\dot{\sigma}}{k_2}}$$

Comparing with

$$p_0\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon}$$

$$p_0 = \frac{1}{\eta}, \quad p_1 = \frac{1}{K_2}, \quad q_0 = \frac{K_1}{\eta}, \quad q_1 = \frac{K_1 + K_2}{K_2}$$

(B) I.C $\Rightarrow$ $p_1\sigma_0 = q_1\varepsilon_0$

$$\Rightarrow \frac{\sigma_0}{K_2} = \varepsilon_0\left(\frac{K_1}{K_2} + 1\right)$$

$$\boxed{\sigma_0 = \varepsilon_0(K_1 + K_2)}$$

i) Constant strain $= \varepsilon_0 \Rightarrow \dot{\varepsilon} = 0$

& at $t=0$, $\sigma = \sigma_0$

$$p_0\sigma + p_1\dot{\sigma} = q_0\varepsilon_0$$

$$\sigma = -\frac{1}{p_0}\left[c_1 \exp\left(-\frac{p_0 t}{p_1}\right) - \varepsilon_0 q_0\right]$$

At, $t=0$, $\sigma = \sigma_0 \Rightarrow c_1 = \varepsilon_0 q_0 - \sigma_0 p_0$

$$\sigma = \frac{-1}{p_0}\left[(\varepsilon_0 q_0 - \sigma_0 p_0)\exp\left(-\frac{p_0 t}{p_1}\right) - \varepsilon_0 q_0\right]$$

$$\sigma(t) = \varepsilon_0\left\{\frac{q_0}{p_0} + \left(\frac{q_1}{p_1} - \frac{q_0}{p_0}\right)\exp\left(\frac{-p_0 t}{p_1}\right)\right\}$$

$$\boxed{\sigma(t) = \varepsilon_0\left\{K_1 + K_2 \exp\left(\frac{-K_2 t}{\eta}\right)\right\}}$$

ii) Constant stress $= \sigma_0 \Rightarrow \dot{\sigma} = 0$ and at $t=0$, $\varepsilon = \varepsilon_0$

$$p_0\sigma_0 = q_0\varepsilon + q_1\dot{\varepsilon}$$

$$\dot{\varepsilon} = \frac{p_0}{q_1}\sigma_0 - \frac{q_0}{q_1}\varepsilon$$

$$\varepsilon = -\frac{1}{q_0}\left[c_1\exp\left(\frac{-q_0 t}{q_1}\right) - p_0\sigma_0\right]$$

At $t=0$, $\varepsilon = \varepsilon_0 \Rightarrow c_1 = p_0\sigma_0 - q_0\varepsilon_0$

$$\therefore \varepsilon(t) = \sigma_0\left[\frac{p_0}{q_0} + \left(\frac{p_1}{q_1} - \frac{p_0}{q_0}\right)\exp\left(-\frac{q_0 t}{q_1}\right)\right]$$

$$\varepsilon(t) = \sigma_0\left\{\frac{1}{k_1} + \left(\frac{1}{k_1+k_2} - \frac{1}{k_1}\right)\exp\left(\frac{-k_1k_2t}{\eta(k_1+k_2)}\right)\right\}$$

(c) Oscillatory stress input:- $\sigma = \sigma_0\sin(\omega^c t)$

$$p_0\sigma_0\sin(\omega^c t) + p_1\omega^c\sigma_0\cos(\omega^c t) = q_0\varepsilon + q_1\dot{\varepsilon}$$

$$\dot{\varepsilon} = \frac{p_0}{q_1}\sigma_0\sin(\omega^c t) + \frac{p_1}{q_1}\omega^c\sigma_0\cos(\omega^c t) - \frac{q_0}{q_1}\varepsilon$$

Using Matlab,

$$\varepsilon = \left(\exp\left(\frac{-q_0 t}{q_1}\right)\right)\left\{c_1 + \left(\frac{\sigma_0\exp\left(\frac{q_0 t}{q_1}\right)}{q_0^2 + q_1^2\omega^2}\right)\right\}\left[\sin(\omega^c t)\left(p_1q_1\omega^{c2} + p_0q_0\right) - \omega^c\cos(\omega^c t)\left(p_0q_1 - p_1q_0\right)\right]$$

At $t=0$, $\varepsilon = \varepsilon_0$

$$\Rightarrow c_1 = \frac{\varepsilon_0}{\omega^c\sigma_0}\left(\frac{q_0^2 + q_1^2\omega^{c2}}{p_1q_0 - p_0q_1}\right)$$

$$\therefore \boxed{\varepsilon(t) = \varepsilon_\alpha\sin(\omega^c t) + \varepsilon_\beta\cos(\omega^c t) - f(t)}$$

where, $\varepsilon_\alpha = \sigma_0\left(\dfrac{p_1q_1\omega^{c2} + p_0q_0}{q_0^2 + q_1^2\omega^{c2}}\right)$

$$\varepsilon_\beta = \omega\sigma_0 \left(\frac{p_1 q_0 - p_0 q_1}{q_0^2 + q_1^2 \omega^{c2}} \right)$$

$$f(t) = \left(\frac{p_1}{q_1}\right)\left(\frac{q_0^2 + q_1^2 \omega_c^2}{\omega^c (p_0 q_1 - p_1 q_0)}\right) \exp\left(\frac{q_0 t}{q_1}\right)$$

(D) Oscillatory strain input:-

$$\varepsilon = \varepsilon_0 \sin(\omega^R t)$$

$$p_0 \sigma + p_1 \dot{\sigma} = q_0 \varepsilon_0 \sin(\omega^R t) + q_1 \omega^R \varepsilon_0 \cos(\omega^R t)$$

$$\dot{\sigma} = \frac{q_0}{p_1} \varepsilon_0 \sin(\omega^R t) + \frac{q_1}{p_1} \omega^R \varepsilon_0 \cos(\omega^R t) - \left(\frac{p_0}{p_1}\right)\sigma$$

Using Matlab,

$$\sigma = \left(\exp\left(-\frac{p_0 t}{p_1}\right)\right)\left[c_1 + \frac{\varepsilon_0 \exp\left(\frac{p_0 t}{p_1}\right)}{p_0^2 + p_1^2 \omega^{R2}} \left[\sin(\omega^R t)(p_1 q_1 \omega^{R2} + p_0 q_0) + \omega^R \cos(\omega^R t)(p_0 q_1 - p_1 q_0)\right]\right]$$

At $t=0$, $\sigma = \sigma_0 \Rightarrow$ $c_1 = \left(\frac{\sigma_0}{\varepsilon_0}\right)\left(\frac{p_0^2 + p_1^2 \omega^{R2}}{\omega^R (p_0 q_1 - p_1 q_0)}\right)$

$$\therefore \boxed{\sigma(t) = \sigma_\alpha \sin(\omega^R t) + \sigma_\beta \cos(\omega^R t) - f(t)}$$

where $\sigma_\alpha = \varepsilon_0 \left(\frac{p_1 q_1 \omega^{R2} + p_0 q_0}{p_0^2 + p_1^2 \omega^{R2}}\right)$

$$\sigma_\beta = \varepsilon_0 \omega^R \left(\frac{p_0 q_1 - p_1 q_0}{p_0^2 + p_1^2 \omega^{R2}}\right)$$

$$f(t) = \left(\frac{q_1}{p_1}\right)\left(\frac{p_0^2 + p_1^2 \omega^{R2}}{\omega (p_1 q_0 - p_0 q_1)}\right) \exp\left(\frac{-p_0 t}{p_1}\right)$$

(F) $G(t) = \frac{\sigma(t)}{\varepsilon_0} = \frac{q_0}{p_0} + \left(\frac{q_1}{p_1} - \frac{q_0}{p_0}\right) \exp\left(\frac{-p_0 t}{p_1}\right)$

$$\boxed{G(t) = k_1 + k_2 \exp\left(-\frac{k_2 t}{\eta}\right)}$$

$J(t) = \frac{\varepsilon(t)}{\sigma_0} = \frac{p_0}{q_0} + \left(\frac{p_1}{q_1} - \frac{p_0}{q_0}\right) \exp\left(\frac{-q_0 t}{q_1}\right)$

$$\boxed{J(t) = \frac{1}{k_1} + \left(\frac{1}{k_1+k_2} - \frac{1}{k_1}\right) \exp\left(\frac{-k_1 k_2 t}{\eta(k_1+k_2)}\right)}$$

i) L.H.S. $= G(0) \times J(0)$

$= (k_1+k_2) \times \left(\frac{1}{k_1} + \frac{1}{k_1+k_2} - \frac{1}{k_1}\right)$

$= 1 =$ R.H.S.

$\Rightarrow \boxed{G(0) \times J(0) = 1}$

ii) $G(\infty) \times J(\infty) = k_1 \times \frac{1}{k_1} = 1$

$\Rightarrow \boxed{G(\infty) \times J(\infty) = 1}$

iii) $G(t) = k_1 + k_2 \exp\left(-\frac{k_2 t}{\eta}\right)$

$\mathcal{L}(G) = \frac{k_1}{s} + \frac{k_2}{\left(s + \frac{k_2}{\eta}\right)} = \frac{k_1}{s} + \frac{k_2 \eta}{s\eta + k_2}$

$$\mathcal{L}(J) = \frac{1}{sk_1} + \left(\frac{1}{k_1+k_2} - \frac{1}{k_1}\right)\left[\frac{1}{s+\left(\frac{k_1k_2}{\eta(k_1+k_2)}\right)}\right]$$

$$\mathcal{L}(J) = \frac{1}{sk_1} - \frac{k_2\eta}{s\eta(k_1+k_2)k_1 + k_1^2k_2}$$

$$\frac{dJ}{dt} = \left(\frac{-k_2}{k_1(k_1+k_2)}\right)\left(\frac{-k_1k_2}{\eta(k_1+k_2)}\right)\exp\left(\frac{-k_1k_2t}{\eta(k_1+k_2)}\right)$$

$$\left.\frac{dJ}{dt}\right|_{t=s} = \left(\frac{k_1k_2^2}{k_1\eta(k_1+k_2)^2}\right)\exp\left(\frac{-k_1k_2s}{\eta(k_1+k_2)}\right)$$

Applying Laplace transform to R.H.S:-

$$\mathcal{L}(R.H.S) = J(0)\mathcal{L}(G) + \mathcal{L}(G)\,\mathcal{L}\left(\frac{dJ}{dt}\right)_{t=s}$$

$$= \mathcal{L}(G)\left[\frac{1}{k_1+k_2} + \frac{k_1k_2^2}{k_1\eta(k_1+k_2)^2}\left(\frac{1}{s+\frac{k_1k_2}{\eta(k_1+k_2)}}\right)\right]$$

$$= \mathcal{L}(G)\left(\frac{1}{k_1+k_2}\right)\left\{1 + \frac{k_2^2}{s\eta(k_1+k_2)+k_1k_2}\right\}$$

$$= \left(\frac{k_1}{s} + \frac{k_2\eta}{s\eta+k_2}\right)\left(\frac{1}{k_1+k_2}\right)\left(1 + \frac{k_2^2}{s\eta(k_1+k_2)+k_1k_2}\right)$$

$$\mathcal{L}(R.H.S) = \frac{(s\eta+k_2)(k_1+k_2)}{s(k_1+k_2)(s\eta+k_2)} = \frac{1}{s}$$

Taking Laplace inverse $\Rightarrow$ $R.H.S = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1 = L.H.S$

$$\Rightarrow \boxed{1 = J(0)\,G(\tau) + \int_0^\tau G(\tau - s)\left.\frac{dJ}{dt}\right|_{t=s} ds}$$

(b) (i) $\eta = \int_0^\infty G(t)\,dt = \int_0^\infty \left[k_1 + k_2 \exp\left(\frac{-k_2 t}{\eta}\right)\right] dt$

$= [k_1 t]_0^\infty + \left[\left(\frac{-k_2 \eta}{k_2}\right) \exp\left(\frac{k_2 t}{\eta}\right)\right]_0^\infty$

$\boxed{\eta = \infty}$

(ii) $\tau_1^c = \dfrac{J(\infty) - J(0)}{\left.\frac{dJ}{dt}\right|_{t=0}} = \dfrac{\left(\frac{1}{k_1}\right) - \left(\frac{1}{k_1+k_2}\right)}{\frac{k_2^2}{\eta(k_1+k_2)^2}} = \dfrac{\left(\frac{k_2}{k_1\cancel{(k_1+k_2)}}\right)}{\left(\frac{k_2^{\cancel{2}}}{\eta(k_1+k_2)^{\cancel{2}}}\right)}$

$$\boxed{\tau_1^c = \frac{\eta(k_1+k_2)}{k_1 k_2}}$$

$$\tau_2^c = \frac{\int_0^\infty s\,[J(\infty) - J(s)]\,ds}{\int_0^\infty [J(\infty) - J(s)]\,ds}$$

$$= \frac{\int_0^\infty s\left\{\frac{1}{k_1} - \left[\frac{1}{k_1} + \left(\frac{1}{k_1+k_2} - \frac{1}{k_1}\right)\exp\left(\frac{-k_1 k_2 s}{\eta(k_1+k_2)}\right)\right]\right\} ds}{\int_0^\infty \left\{\frac{1}{k_1} - \left[\frac{1}{k_1} + \left(\frac{1}{k_1+k_2} - \frac{1}{k_1}\right)\exp\left(\frac{-k_1 k_2 s}{\eta(k_1+k_2)}\right)\right]\right\} ds}$$

$$= \frac{\int_0^\infty s\,\cancel{\left(\frac{-k_2}{k_1(k_1+k_2)}\right)} \exp\left(\frac{-k_1 k_2 s}{\eta(k_1+k_2)}\right) ds}{\int_0^\infty \cancel{\left(\frac{-k_2}{k_1(k_1+k_2)}\right)} \exp\left(\frac{k_1 k_2 s}{\eta(k_1+k_2)}\right) ds}$$

$$= \left(\frac{\eta(k_1+k_2)}{k_1k_2}\right)^2 + \left(\frac{k_1k_2}{\eta(k_1+k_2)}\right)$$

$$\boxed{\tau_2^c = \frac{\eta(k_1+k_2)}{k_1k_2}}$$

(iii) $$\tau_1^R = \frac{[G(0)-G(\infty)]}{\left(\frac{dG}{dt}\right)_{t=0}} = \frac{(k_1+k_2)-k_1}{\left(\frac{-k_2}{\eta}\right)}$$

$$\boxed{\tau_1^R = \frac{-\eta}{k_2}}$$

$$\tau_2^R = \frac{\int_0^\infty s\,[G(s)-G(\infty)]\,ds}{\int_0^\infty [G(s)-G(\infty)]\,ds}$$

$$= \frac{\int_0^\infty s\left(\cancel{k_1}+k_2\exp\left(\frac{-k_2s}{\eta}\right)-\cancel{k_1}\right)ds}{\int_0^\infty \left(\cancel{k_1}+k_2\exp\left(\frac{-k_2s}{\eta}\right)-\cancel{k_1}\right)ds}$$

$$= \frac{\int_0^\infty s.\exp\left(-\frac{k_2s}{\eta}\right)ds}{\int_0^\infty \exp\left(-\frac{k_2s}{\eta}\right)ds}$$

$$\boxed{\tau_2^R = \frac{\eta}{k_2}}$$

(I) i) Phase lag:-

$$\varepsilon(t) = \varepsilon_\alpha \sin(\omega^c t) + \varepsilon_\beta \cos(\omega^c t) - f(t)$$

wher $\varepsilon_\alpha = \sigma_0 \left(\dfrac{\eta^2 (k_1+k_2) \omega^{c^2} + k_1 k_2^2}{k_1 k_2^2 + (k_1+k_2)^2 \eta^2 \omega^{c^2}} \right)$

$$\varepsilon_\beta = -\omega \sigma_0 \left[\frac{\eta k_2^2}{k_1^2 k_2^2 + (k_1+k_2)^2 \eta^2 \omega^{c^2}} \right]$$

substituting $\varepsilon_\alpha = \bar{\varepsilon}_\alpha \cos\phi$; $\varepsilon_\beta = \bar{\varepsilon}_\alpha \sin\phi$

$$\therefore \varepsilon(t) = \bar{\varepsilon}_\alpha \cos\phi \sin(\omega^c t) + \bar{\varepsilon}_\alpha \sin\phi \cos(\omega^c t) - f(t)$$

$$= \bar{\varepsilon}_\alpha \sin(\omega^c t + \phi) - f(t)$$

$$\tan\phi = \frac{\bar{\varepsilon}_\alpha \sin\phi}{\bar{\varepsilon}_\beta \cos\phi} = \frac{\varepsilon_\beta}{\varepsilon_\alpha}$$

$$= -\omega \left(\frac{\eta k_2^2}{\eta^2 (k_1+k_2) \omega^{c^2} + k_1 k_2^2} \right)$$

$$\boxed{\text{phase lag} = \phi = \tan^{-1} \left[-\omega \left(\frac{\eta k_2^2}{\eta^2 (k_1+k_2) \omega^{c^2} + k_1 k_2^2} \right) \right]}$$