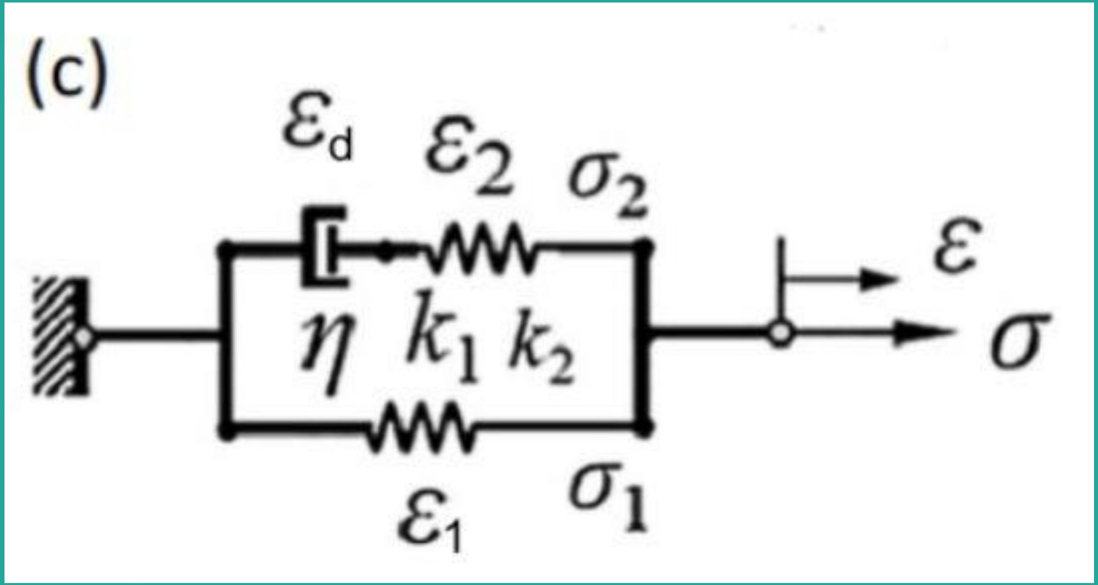
PROJECT-1

CE7620-Rheology of Civil Engineering Materials

ME21S030 - VIJAY HAJARE

Spring and dash-pot arrangement



Question A

Find the relation between the stress, stress rate, strain, and strain rate

$$\sigma_1 = k_1 \varepsilon_1$$

$$\sigma_2 = k_2 \varepsilon_2$$

$$\sigma_d = \eta \dot{\varepsilon}_d$$

$$\varepsilon = \varepsilon_1 = \varepsilon_2 + \varepsilon_d$$

$$\sigma_1 + \sigma_M = \sigma$$

$$\sigma_2 = \sigma_d = \sigma_M$$

where,

σ	=	stress in spring 1
σ	=	stress in spring 2
σ	=	stress in dash-pot
σ	=	stress in maxwell model
ε	=	strain in spring 1
ε	=	strain in spring 2
ε	=	strain in dash-pot
η	=	dash-pot constant
k_1	=	spring constant in spring 1
k_2	=	spring constant in spring 2

Question A

$$\frac{k_1\varepsilon}{\eta} + \left(\frac{k_1}{k_2} + 1\right)\dot{\varepsilon} = \frac{\sigma}{\eta} + \frac{\dot{\sigma}}{k_2}$$

Comparing above equation with

$$q_0\varepsilon + q_1\dot{\varepsilon} = p_0\sigma + p_1\dot{\sigma}$$

$$q_0 = \frac{k_1}{\eta}, \quad q_1 = \left(\frac{k_1}{k_2} + 1\right), \quad p_0 = \frac{1}{\eta}, \quad p_1 = \frac{1}{k_2}$$

Question B-i)

Jump condition:-

$$p_1\sigma_0 = q_1\varepsilon_0$$

$$\sigma_0 = \varepsilon_0(k_1 + k_2)$$

i) Find the stress response for an applied constant strain:-

$$\varepsilon = \varepsilon_0$$

$$q_0\varepsilon = p_0\sigma + p_1\dot{\sigma}$$

Question B

$$\sigma(t) = \varepsilon_0 \left\{ \frac{q_0}{p_0} + \left(\frac{q_1}{p_1} - \frac{q_0}{p_0} \right) exp\left(\frac{-p_0 t}{p_1} \right) \right\}$$

$$\sigma(t) = \varepsilon_0 \left\{ k_1 + \left(k_2 \right) exp\left(\frac{-k_2 t}{\eta} \right) \right\}$$

Question B-ii)

ii) Find the strain response for an applied constant stress

$$\varepsilon(t) = \sigma_0 \left\{ \frac{p_0}{q_0} + \left(\frac{p_1}{q_1} - \frac{p_0}{q_0} \right) exp\left(\frac{-q_0 t}{q_1} \right) \right\}$$

$$\varepsilon(t) = \sigma_0 \left\{ \frac{1}{k_1} + \left(\frac{1}{k_1 + k_2} - \frac{1}{k_1} \right) exp\left(\frac{-k_1 k_2 t}{\eta(k_1 + k_2)} \right) \right\}$$

Question C

Find the response for oscillatory loading, $\sigma = \sigma_o \sin(\omega^c t)$

$$p_o \sigma_o \sin(\omega^c t) + p_1 \omega^c \sigma_o \cos(\omega^c t) = q_o \varepsilon + q_1 \acute{\varepsilon}$$

$$\varepsilon(t) = \varepsilon_\propto \sin(\omega^c t) + \varepsilon_\beta \cos(\omega^c t) - f(t)$$

where,

$$\varepsilon_\propto = \sigma_0 \left(\frac{p_1 q_1 w^{c^2} + p_0 q_0}{q_0^2 + q_1^2 w^{c^2}} \right) = \sigma_0 \left(\frac{\eta^2 (k_1 + k_2) w^{c^2} + k_1 k_2^2}{k_1^2 k_2^2 + (k_1 + k_2)^2 \eta^2 w^{c^2}} \right)$$

$$\varepsilon_\beta = \sigma_0 w^c \left(\frac{p_1 q_0 - p_0 q_1}{q_0^2 + q_1^2 w^{c^2}} \right) = -\, \sigma_0 w^c \left(\frac{\eta k_2^2}{k_1^2 k_2^2 + (k_1 + k_2)^2 \eta^2 w^{c^2}} \right)$$

$$f(t) = \left(\frac{p_1}{q_1} \right) \left(\frac{q_0^2 + q_1^2 w^{c^2}}{w^c (p_0 q_1 - p_1 q_0)} \right) exp\left(\frac{q_0 t}{q_1} \right) = \left(\frac{1}{k_1 + k_2} \right) \left(\frac{\eta^2 (k_1 + k_2)^2 w^{c^2} + k_1 k_2^2}{k_2^2 + \eta^2 w^{c^2}} \right) exp\left(\frac{k_1 k_2 t}{(k_1 + k_2)\eta} \right)$$

Question D

Find the response for oscillatory displacement, $\varepsilon = \varepsilon_o \sin(\omega^R t)$

$$p_o \sigma + p_1 \dot{\sigma} = q_o \varepsilon_o \sin(\omega^R t) + q_1 \omega^R \varepsilon_o \cos(\omega^R t)$$

$$\sigma(t) = \sigma_\propto \sin(\omega^R t) + \sigma_\beta \cos(\omega^R t) - f(t)$$

where,

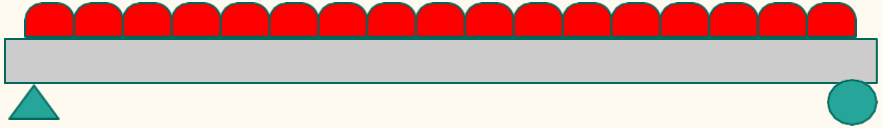
$$\sigma_\propto = \varepsilon_0 \left(\frac{p_1 q_1 w^{R^2} + p_0 q_0}{p_0^2 + p_1^2 w^{R^2}} \right) = \varepsilon_0 \left(\frac{\eta^2 (k_1 + k_2) w^{R^2} + k_1 k_2^2}{k_2^2 + \eta^2 w^{R^2}} \right)$$

$$\sigma_\beta = \varepsilon_0 w^R \left(\frac{p_0 q_1 - p_1 q_0}{p_0^2 + p_1^2 w^{R^2}} \right) = \varepsilon_0 w^R \left(\frac{k_1 k_2^2 - k_1 k_2 \eta}{k_2^2 + \eta^2 w^{R^2}} \right)$$

$$f(t) = \left(\frac{q_1}{p_1} \right) \left(\frac{p_0^2 + p_1^2 w^{R^2}}{w^R (p_1 q_0 - p_0 q_1)} \right) exp\left(\frac{-p_0 t}{p_1} \right) = (k_1 + k_2) \left(\frac{k_2^2 + \eta^2 w^{R^2}}{w^R (k_1 k_2^2 - k_1 k_2 \eta\ )} \right) exp\left(\frac{-k_2 t}{\eta} \right)$$

Question E

Find the response for a simply supported beam subjected to constant uniformly distributed load.



$$G(t) = k_1 + k_2 \exp\left(\frac{-k_2 t}{\eta}\right)$$

$$\varepsilon = -\frac{d^2\Delta}{dx^2}(y - y_0)$$

$$\mathcal{L}(\sigma_{xx}) = s\,\mathcal{L}(G)\,\mathcal{L}(\varepsilon)$$ valid for linear model

Question E

$$\mathcal{L}(\sigma_{xx}) = s\left(\frac{k_1}{s} + \frac{k_2\eta}{(s\eta + k_2)}\right)\left(-y\mathcal{L}\left(\frac{d^2\Delta}{dx^2}\right) + \mathcal{L}\left(y_0\frac{d^2\Delta}{dx^2}\right)\right)$$

$$= \left(k_1 + \frac{k_2\eta s}{(s\eta + k_2)}\right)\left(-y\mathcal{L}\left(\frac{d^2\Delta}{dx^2}\right) + \mathcal{L}\left(y_0\frac{d^2\Delta}{dx^2}\right)\right)$$

$\sum F_x = 0$

$$P = \int \sigma_{xx}\, da = 0$$

$$\mathcal{L}(P) = \int \mathcal{L}(\sigma_{xx})\, da = \int \left(k_1 + \frac{k_2\eta s}{(s\eta + k_2)}\right)\left(-y\mathcal{L}\left(\frac{d^2\Delta}{dx^2}\right) + \mathcal{L}\left(y_0\frac{d^2\Delta}{dx^2}\right)\right) da$$

$$A y_c \mathcal{L}\left(\frac{d^2\Delta}{dx^2}\right) - A\mathcal{L}\left(y_0 \frac{d^2\Delta}{dx^2}\right) = \frac{k_1(s\eta + k_2)}{k_2 \eta s} \quad \text{—} \; (1)$$

$$\Sigma M_z = 0, \qquad M = \int \sigma_{xx}\, y \, da$$

$$\mathcal{L}(M) = \int \mathcal{L}(\sigma_{xx})\, y \, da$$

$$= \int \left(k_1 + \frac{k_2 \eta s}{(s\eta + k_2)}\right)\left(-y^2 \mathcal{L}\left(\frac{d^2\Delta}{dx^2}\right) + y\mathcal{L}\left(y_0 \frac{d^2\Delta}{dx^2}\right)\right) da$$

$$= k_1 + \frac{k_2 \eta s}{(s\eta + k_2)}\left[\bar{I}_{zz}\, \mathcal{L}\left(\frac{d^2\Delta}{dx^2}\right) - A y_c\, \mathcal{L}\left(y_0 \frac{d^2\Delta}{dx^2}\right)\right]$$

$$= \frac{wlx}{2s} - \frac{wx^2}{2s}$$

Question E

$$I_{zz}\,\alpha\left(\frac{d^2\Delta}{dx^2}\right) - A y_c\left(y_0\frac{d^2\Delta}{dx^2}\right) = \frac{\left(\frac{\omega l x}{2s} - \frac{\omega x^2}{2s} - K_1\right)(s\eta + k_1)}{k_2\,\eta\, s} \quad \text{--- (2)}$$

By solving eq^{n}s (1) and (2) solutions for $\frac{d^2\Delta}{dx^2}$ and $y_0\frac{d^2\Delta}{dx^2}$ can be obtained.

Question F-i)

Using the above results find the stress relaxation function, G and creep compliance function, J. Then verify the following:

i. G(0)×J(0) = 1

$$G(t) = k_1 + k_2 exp\left(\frac{-k_2 t}{\eta}\right)$$

$$J(t) = \frac{1}{k_1} + \left(\frac{1}{k_1+k_2} - \frac{1}{k_1}\right)exp\left(\frac{-k_1 k_2 t}{\eta(k_1+k_2)}\right)$$

$$LHS = G(0)\, X\, J(0)$$

$$LHS = \left(k_1 + k_2\right) X \frac{1}{k_1} + \left(\frac{1}{k_1+k_2} - \frac{1}{k_1}\right) = 1$$

$$RHS = G(0)\, X\, J(0) = 1$$

Question F-ii)

ii. G(∞)×J(∞) = 1

$$G(\infty)\ X\ J(\infty) = k_1\ X\ \frac{1}{k_1} = 1$$

$$G(\infty)\ X\ J(\infty) = 1$$

Question F-iii)

iii. $1 = J(0) \times G(\tau) + \int G(\tau - s) \left.\frac{dJ}{dt}\right|_{t=s} ds$

$$\mathcal{L}(G) = \frac{k_1}{s} + \frac{k_2}{\left(s + \frac{k_2}{\eta}\right)} = \frac{k_1}{s} + \frac{k_2\eta}{\left(s\eta + k_2\right)}$$

$$\mathcal{L}(J) = \frac{1}{sk_1} - \frac{k_2\eta}{s\eta\left(k_1 + k_2\right)k_2 + k_1^{\,2}k_2}$$

$$\left.\frac{dJ}{dt}\right|_{t=s} = \frac{k_1 k_2^{\,2}}{k_1\eta\left(k_1 + k_2\right)^2} exp\left(\frac{-k_1k_2s}{\eta(k_1 + k_2)}\right)$$

Question F-iii)

iii. $1 = J(0) \times G(\tau) + \int G(\tau - s) \frac{dJ}{ds} ds$

Applying Laplace transformation to RHS

$$\mathcal{L}\,(RHS) = \frac{(s\eta + k_2)(k_1 + k_2)}{s(k_1 + k_2)(s\eta + k_2)} = \frac{1}{s}$$

Applying inverse Laplace transformation,

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

Question G-i)

Also compute the following:

i. Apparent viscosity, $\eta = \int G(t)dt$

$$\eta_a = \int_0^{\infty} \left\{ k_1 + k_2 exp\left(\frac{-k_2 t}{\eta} \right) \right\} \mathrm{dt}$$

Question G-ii)

ii. Characteristic creep times,

$$\tau_1^c = \left(\frac{J(\infty) - J(0)}{\frac{dJ}{dt}_{(t=0)}} \right) = \frac{\eta(k_1 + k_2)}{k_1 k_2}$$

$$\tau_2^c = \frac{\int_0^\infty s[J(\infty) - J(s)]ds}{\int_0^\infty [J(\infty) - J(s)]ds} = \frac{\eta(k_1 + k_2)}{k_1 k_2}$$

Question G-iii)

iii. Characteristic relaxation times,

$$\tau_1^R = \left(\frac{G(0) - G(\infty)}{\frac{dG}{dt}_{(t=0)}} \right) = \frac{-\eta}{k_2}$$

$$\tau_2^R = \frac{\int_0^\infty s[G(s) - G(\infty)]ds}{\int_0^\infty [G(s) - G(\infty)]ds} = \frac{\eta}{k_2}$$

Question H

Comment on the obtained characteristic times and the apparent viscosity.

$$\tau_1^c = \frac{\eta(k_1+k_2)}{k_1k_2} \text{ and } \tau_1^R = \frac{\eta}{k_2}$$

From above expressions it is clear that the characteristic creep time is more than characteristic relaxation time and as

$$G(0) = \left(k_1 + k_2\right) \text{ and } G(\infty) = k_1$$

We can write relation as

$$\tau_1^c = \frac{G(0)}{G(\infty)}\,\tau_1^R$$

Question I-i)

Compute the following for the oscillatory loading and oscillatory displacement:

i. Phase lag between the stress and strain

$$\varepsilon_{\alpha} = \sigma_0\left(\frac{p_1 q_1 w^{c^2} + p_0 q_0}{q_0^2 + q_1^2 w^{c^2}}\right) = \sigma_0\left(\frac{\eta^2 (k_1 + k_2) w^{c^2} + k_1 k_2^2}{k_1^2 k_2^2 + (k_1 + k_2)^2 \eta^2 w^{c^2}}\right)$$

$$\varepsilon_{\beta} = \sigma_0 w^c\left(\frac{p_1 q_0 - p_0 q_1}{q_0^2 + q_1^2 w^{c^2}}\right) = -\sigma_0 w^c\left(\frac{\eta k_2^2}{k_1^2 k_2^2 + (k_1 + k_2)^2 \eta^2 w^{c^2}}\right)$$

Question I-i)

$$\varepsilon(t) = \overline{\varepsilon}_{\propto} cos\Phi\, sin(w^{c}t) + \overline{\varepsilon}_{\beta} sin\Phi\, cos(w^{c}t) - f(t)$$

substituting $\varepsilon_{\propto} = \overline{\varepsilon}_{\propto} \cos\Phi$ and $\varepsilon_{\beta} = \overline{\varepsilon}_{\beta} \sin\Phi$

$$\varepsilon(t) = \overline{\varepsilon}_{\propto} sin(w^{c}t + \Phi) - f(t),\ \text{where } \Phi \text{ is phase lag}$$

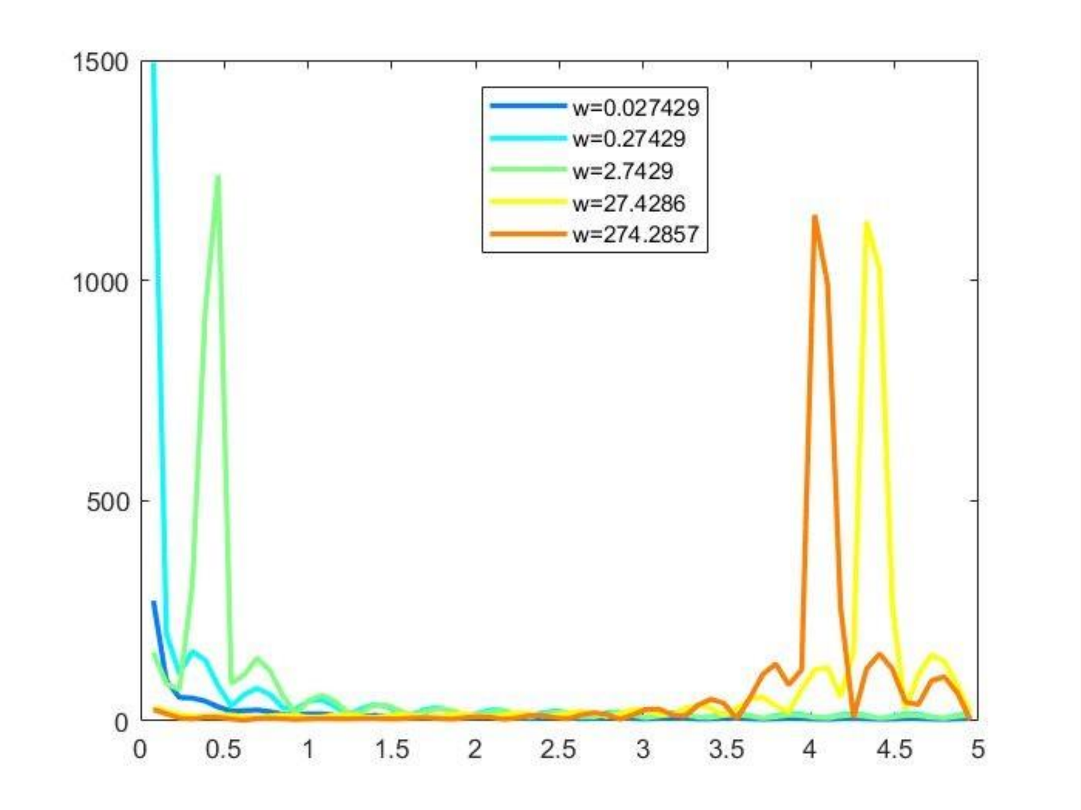
$$tan\Phi = \frac{sin\Phi}{cos\Phi} = -w^{c}\left(\frac{\eta k_{2}^{2}}{k_{1}k_{2}^{2}+(k_{1}+k_{2})\ \eta^{2}w^{c^{2}}}\right)$$

$$\Phi = tan^{-1}\left[-w^{c}\left(\frac{\eta k_{2}^{2}}{k_{1}k_{2}^{2}+(k_{1}+k_{2})\ \eta^{2}w^{c^{2}}}\right)\right]$$

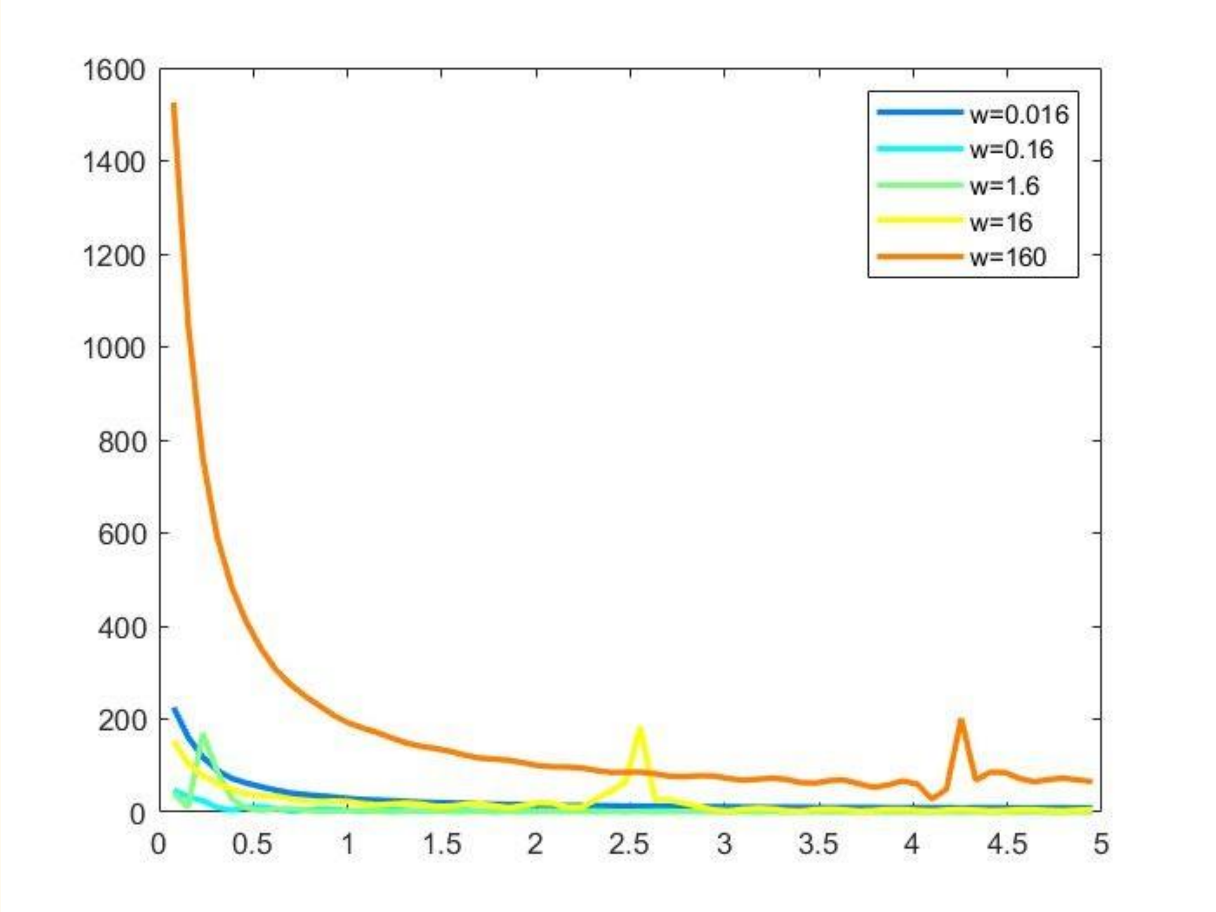
Question I-ii)

ii. Do a Fourier transform of the response and plot the amplitude versus frequency for the following: $\omega_j = \{1/100, 1/10, 1, 10, 100\}\tau_{1j}$, where j = C or R

Oscillatory stress input:-



Oscillatory strain input:-



THANK YOU!