Project Elasticity

Vijay Gajanan Hajare - ME21S030

PART A – Sant Venant Kirchhoffs CR Model and Treloar Model

$$x = a_1 X + kY \; ; \; y = a_2 Y \; ; \; z = a_3 Z$$

$$\underline{\underline{F}} \begin{pmatrix} a_1 & k & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \; ; \; \underline{\underline{F}}^t \begin{pmatrix} a_1 & 0 & 0 \\ k & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

$$\underline{\underline{B}} = \underline{\underline{F}}\,\underline{\underline{F}}^T = \begin{pmatrix} a_1 + k^2 & ka_2 & 0 \\ a_2 k & a_2^2 & 0 \\ 0 & 0 & a_3^2 \end{pmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{pmatrix} a_1^2 & ka_1 & 0 \\ ka_1 & k^2 + a_2^2 & 0 \\ 0 & 0 & a_3^2 \end{pmatrix}$$

$$\underline{\underline{B}}^{-1} = \begin{pmatrix} 1/a_1^2 & -k/a_1^2 a_2 & 0 \\ -k/a_1^2 a_2 & (a_1^2 + k^2)/a_1^2 a_2^2 & 0 \\ 0 & 0 & 1/a_3^2 \end{pmatrix}$$

$$\underline{J_1} = tr(\underline{B}) = a_1^2 + a_2^2 + a_3^2 + k^2$$

$$\underline{J_2} = tr(\underline{B})^{-1} = \frac{1}{a_1^2} + \frac{a_1^2 + k^2}{a_1^2 a_2^2} + \frac{1}{a_3^2}$$

$$\underline{J_3} = det(\underline{F}) = a_1 \cdot a_2 \cdot a_3$$

Saint venant Kirchhoff's constitutive relation

$$\mu = 0.5 \; ; \; \nu = 2$$

$$\underline{\underline{\sigma}} = -\mu J_2 J_3 \underline{\underline{I}} + \left[\mu\left(\frac{J_1 - 1}{J_3}\right) + \nu\left(\frac{J_1 - 3}{2J_3}\right)\right] \cdot \underline{\underline{B}} + \mu J_3 \underline{\underline{B}}^{-1}$$

UNIAXIAL LOADING

For uniaxial loading:- $k=0$

$$\underline{\underline{\sigma}} = \begin{pmatrix} 3a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Forming the equations

$$3a = \left(\frac{-a_1a_2a_3}{2}\right)\left(\frac{1}{a_1^2} + \left(\frac{a_1^2+k^2}{a_2^2a_3^2}\right) + \frac{1}{a_3^2}\right) + \left(\frac{3a_1(a_1^2+a_2^2+a_3^2+k^2) - 7a_1}{2a_2a_3a_1}\right) + \frac{a_2a_3}{2a_1} \quad \text{— ③}$$

$$0 = \left(\frac{-a_1a_2a_3}{2}\right)\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2}\right) + \left(\frac{3a_2(a_1^2+a_2^2+a_3^2) - 7a_2}{2a_1a_3}\right) + \frac{a_1a_3}{2a_2} \quad \text{— ①}$$

$$\left(\frac{-a_1a_2a_3}{2}\right)\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2}\right) + \left(\frac{3a_3(a_1^2+a_2^2+a_3^2) - 7a_3}{2a_1a_2}\right) + \frac{a_1a_2}{2a_3} \quad \text{— ②}$$

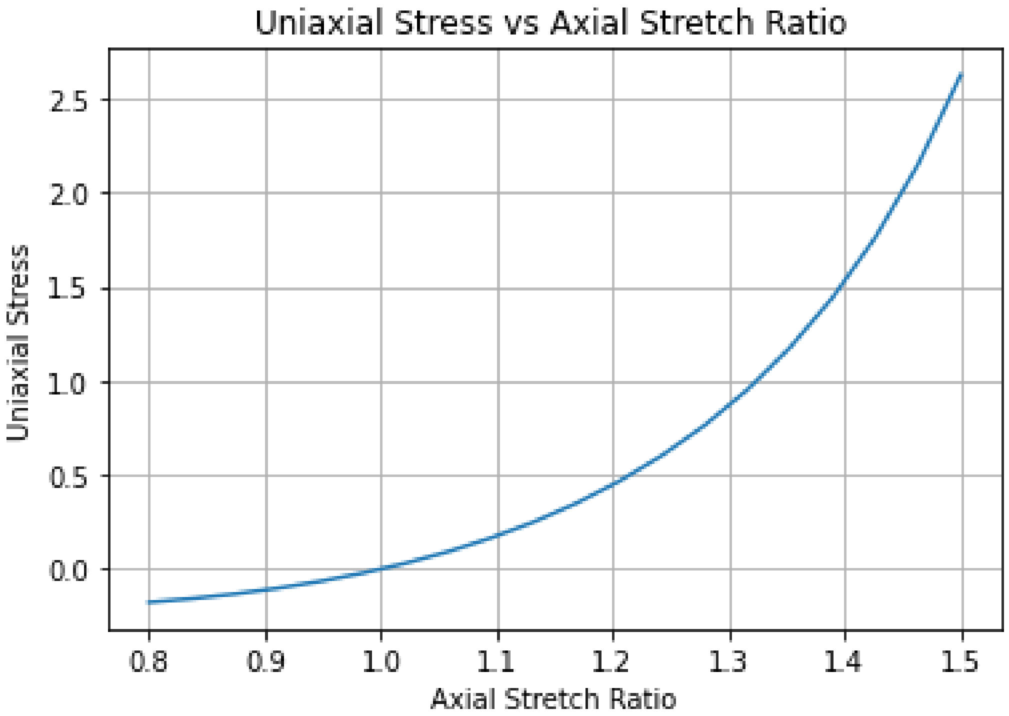
got $a_2 = a_3$; varry a_1 to calculate a_2 & a_3 then $\underline{\underline{\sigma}}$

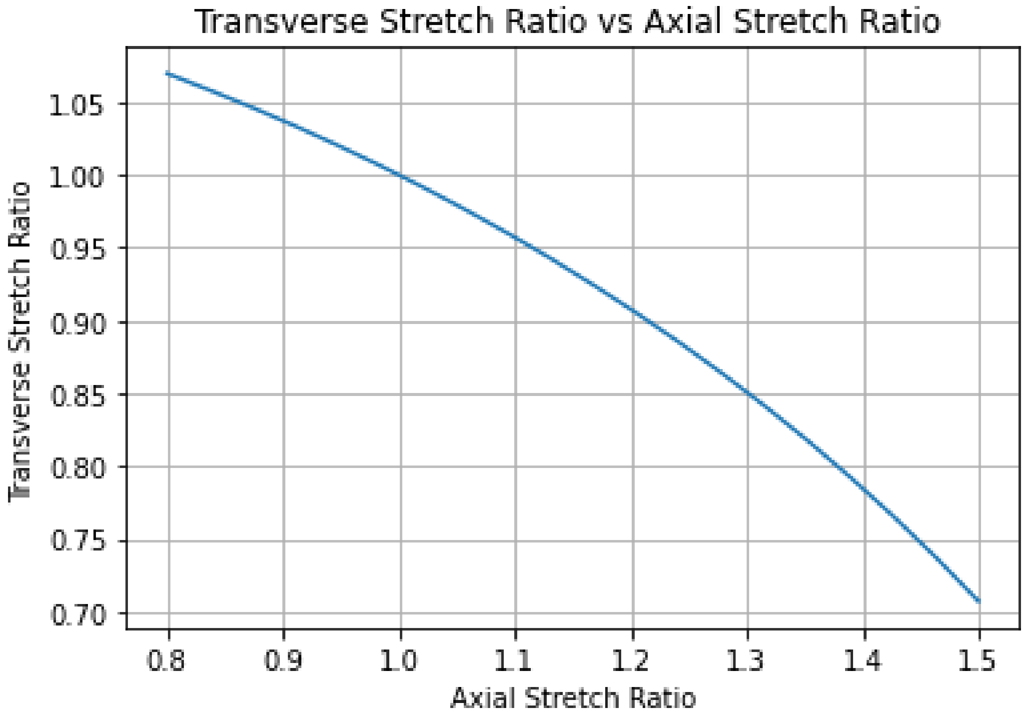
$$\sqrt{\underline{\underline{C}}\,\underline{e}x\cdot\underline{e}x} = a_1$$

$$\sqrt{\underline{\underline{C}}\,\underline{e}y\cdot\underline{e}y} = a_2$$

$$\sqrt{\underline{\underline{C}}\cdot\underline{e}z\cdot\underline{e}z} = a_3$$

PLOT FOR UNIAXIAL LOADING





PURE SHEAR

$$0 = -\left(\frac{a_2^2(a_1^2+a_3^2) + a_3^2(a_1^2+k^2)}{2a_1a_2a_3}\right) + \left[\frac{3(a_1^2+a_2^2+a_3^2+k^2) - 7}{2a_1a_2a_3}\right](a_1^2+k^2)$$

$$+\left[\frac{a_2a_3}{2a_1}\right]$$

$$0 = -\left(\frac{a_2^2(a_1^2+a_3^2) + a_3^2(a_1^2+k^2)}{2a_1a_2a_3}\right) + \left[\frac{3(a_1^2+a_2^2+a_3^2+k^2) - 7}{2a_1a_2a_3}\right](a_2^2)$$

$$+\frac{a_3(a_1^2+k^2)}{2a_1a_2}$$

$$0 = -\left(\frac{a_2^2(a_1^2+a_3^2) + a_3^2(a_1^2+k^2)}{2a_1a_2a_3}\right) + \left[\frac{3(a_1^2+a_2^2+a_3^2+k^2) - 7}{2a_1a_2a_3}\right]a_3^2$$

$$+\left[\frac{a_1a_2}{2a_3}\right]$$

b) Pure shear -

$$\sigma = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

get equation from CR

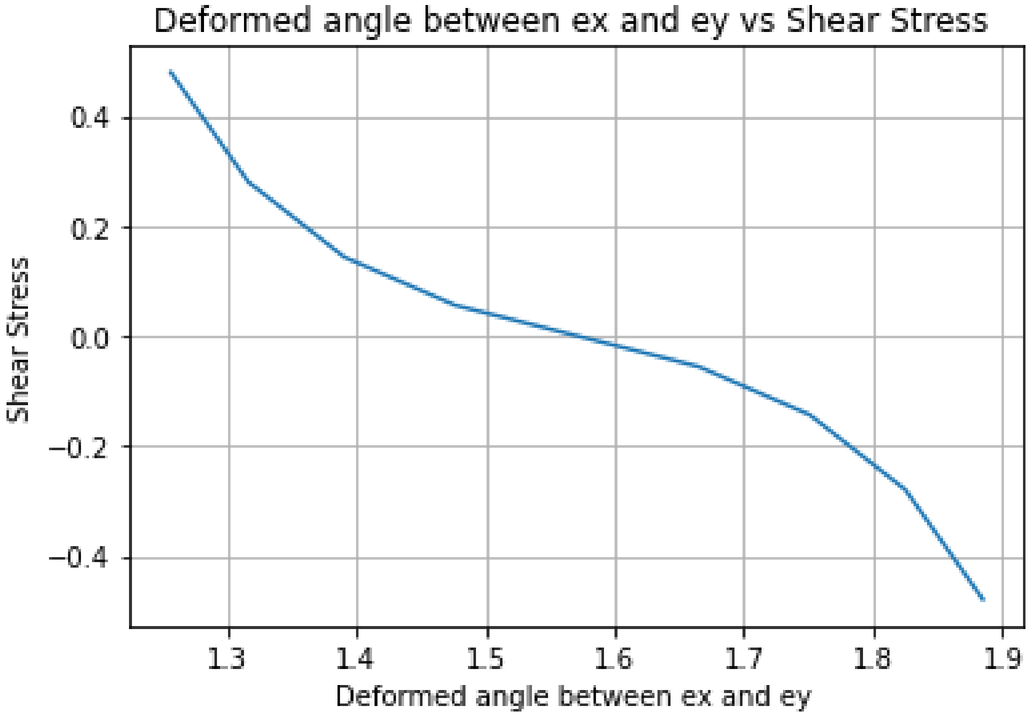
$$\tau = \frac{3ka_2(a_1^2+a_2^2+a_3^2+k^2) - 7ka_2}{2a_1a_2a_3} - \frac{ka_3}{2a_1}$$

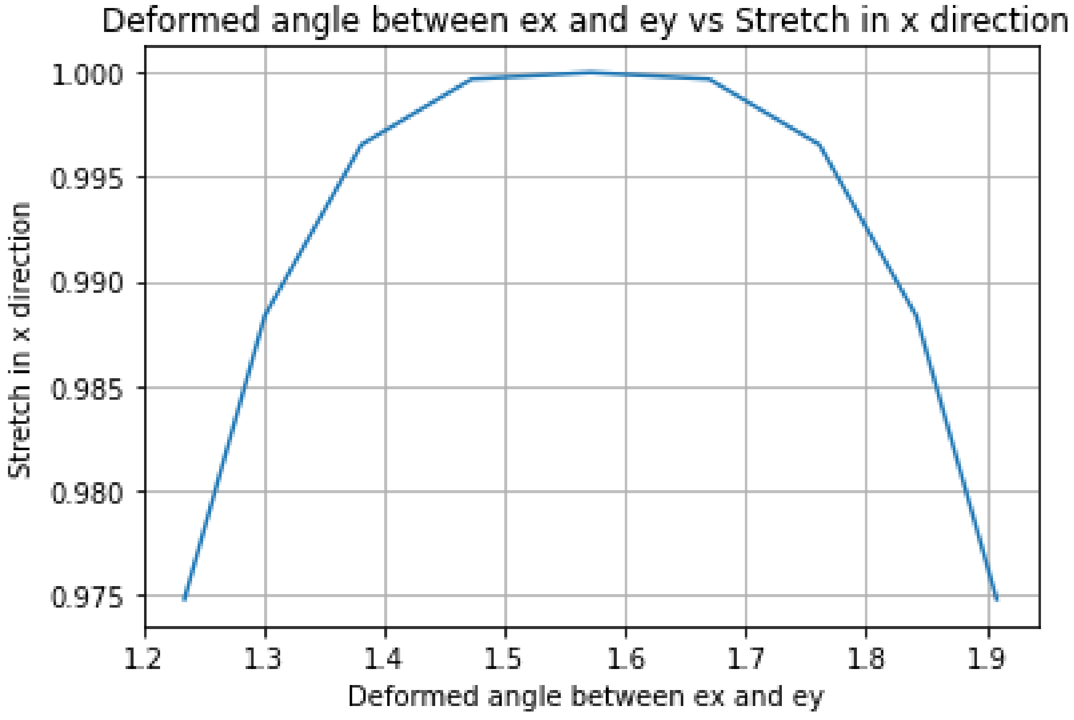
$$\lambda x = \sqrt{\underline{\underline{C}}\,\underline{e}x\cdot\underline{e}x} = a_1 \qquad \theta c = \cos^{-1}\left(\frac{\underline{\underline{C}}\,ex\cdot ex}{\sqrt{\underline{\underline{C}}\,\underline{e}x\cdot\underline{e}x}\cdot\sqrt{\underline{\underline{C}}\,ex\cdot ex}}\right)$$

$$\lambda y = \sqrt{\underline{\underline{C}}\,\underline{e}y\cdot\underline{e}y} = \sqrt{a_2^2+k^2} \qquad \theta c = \cos^{-1}\left(\frac{k\cancel{a_2}}{\sqrt{k^2+a_2^2}\,a_1}\right)$$

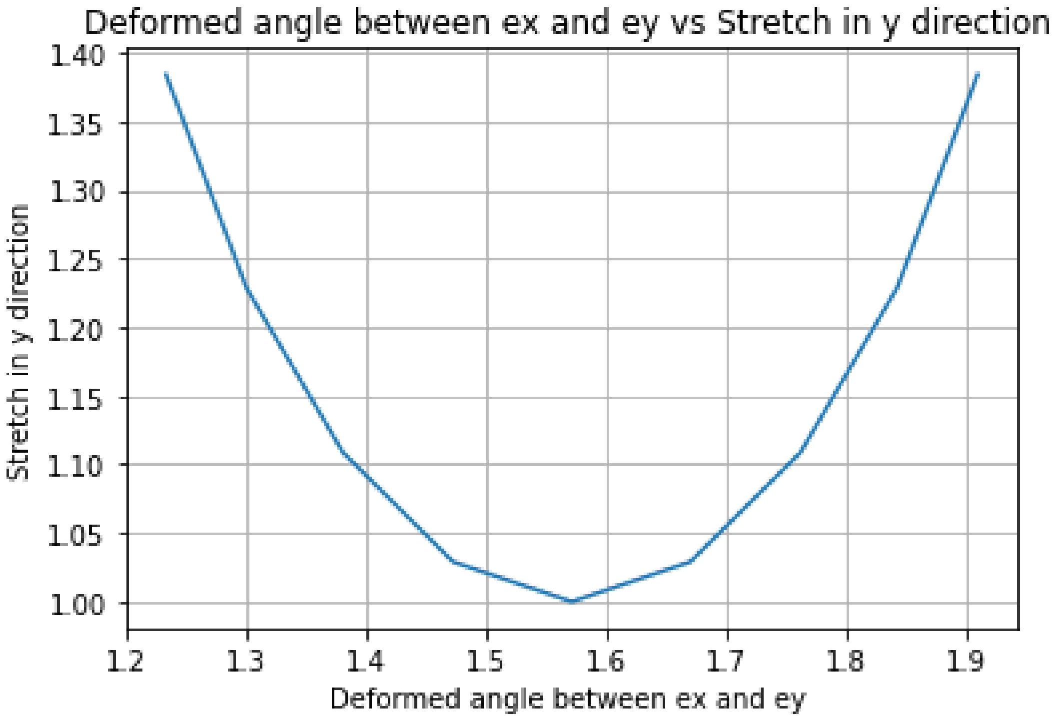
$$\lambda z = \sqrt{\underline{\underline{C}}\cdot\underline{e}z\cdot\underline{e}z} = a_3 \qquad \theta c = \cos^{-1}\left(\frac{ka_1}{\sqrt{k^2+a_2^2}}\right)$$

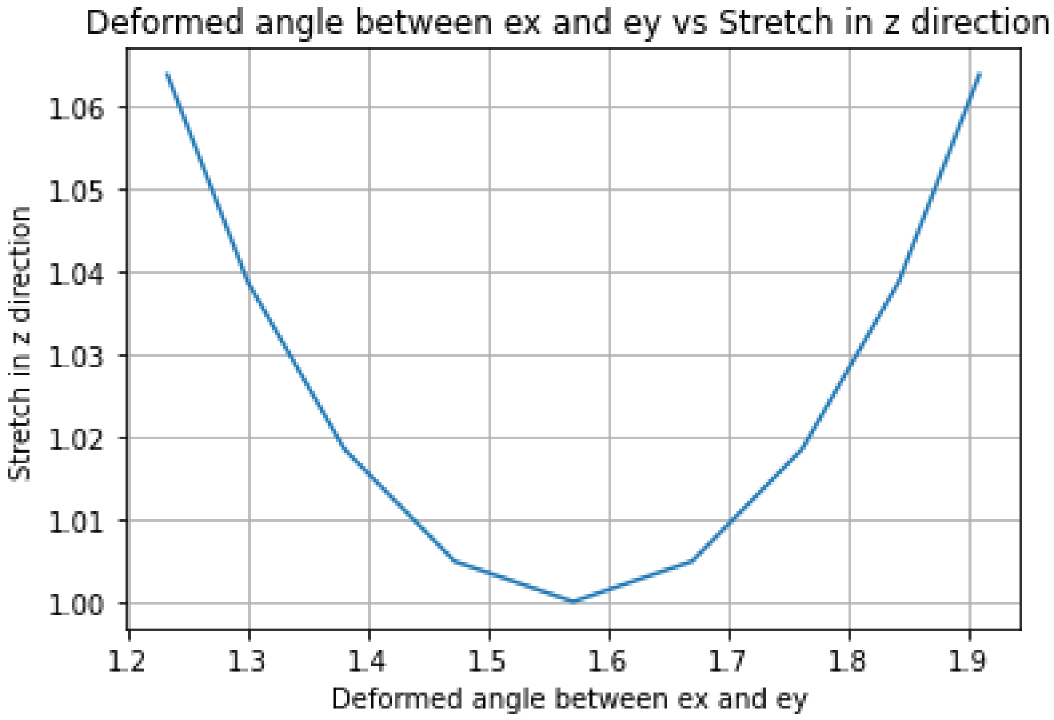
PLOT FOR PURE SHEAR





PLOT FOR PURE SHEAR





FOR INCOMPRESSIBLE MATERIAL

Trelor Model

② For incompressible Material - Treloar model.

CR :- $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + C_1 \underline{\underline{B}} = -p\underline{\underline{I}} + 0.838\,\underline{\underline{B}}$

$J_3 = 1 \Rightarrow a_1 = \frac{1}{a_2 a_3}$

$$\underline{\underline{\sigma}} = -p\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0.838\begin{pmatrix} a_1^2+k^2 & ka_2 & 0 \\ a_2 k & a_2^2 & 0 \\ 0 & 0 & a_3^2 \end{pmatrix}$$

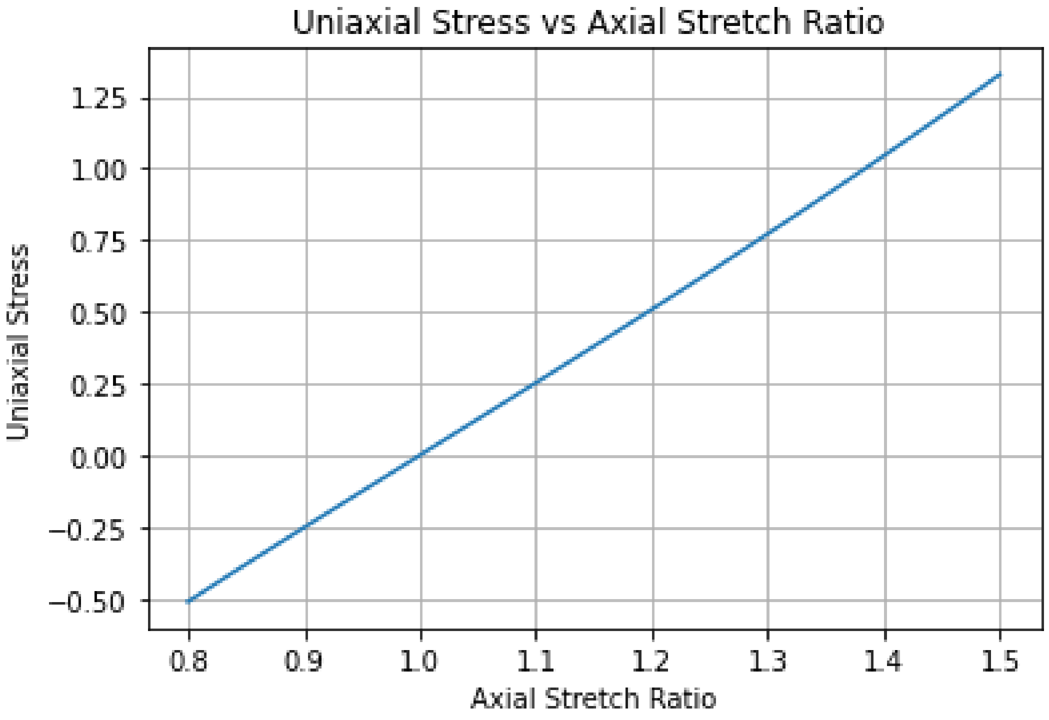
a) Uniaxial loading :- $k=0$

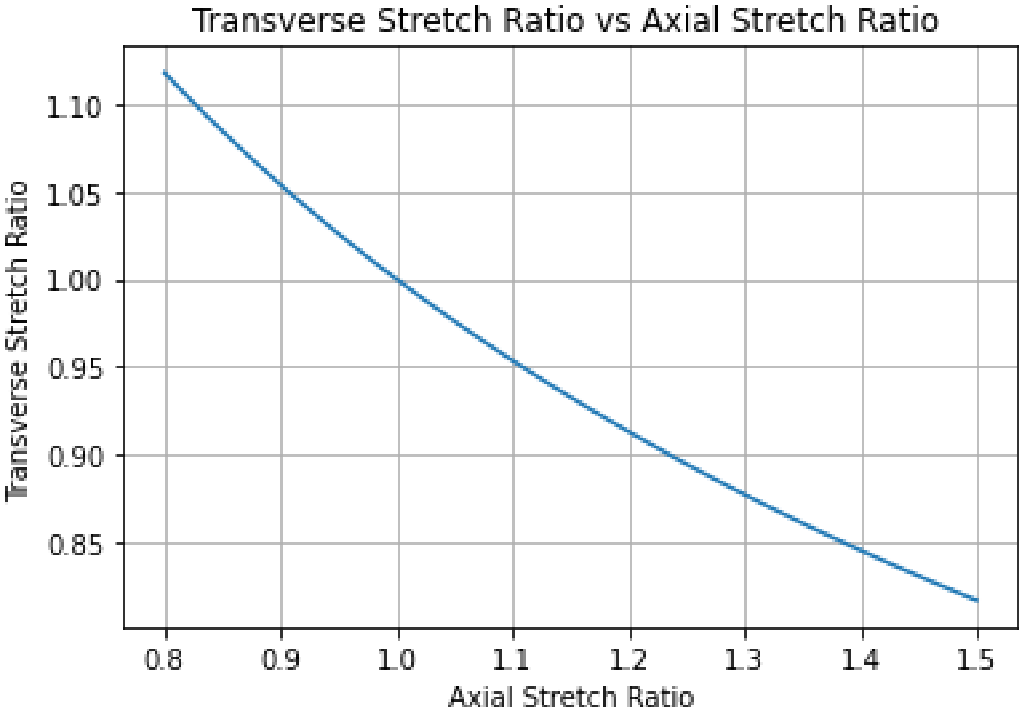
$$\begin{aligned} \sigma_a &= -p + 0.838\, a_1^2 \\ 0 &= -p + 0.838\, a_2^2 \\ 0 &= -p + 0.838\, a_3^2 \end{aligned} \Bigg\} \quad a_2 = a_3$$

b) Pure shear stress :-

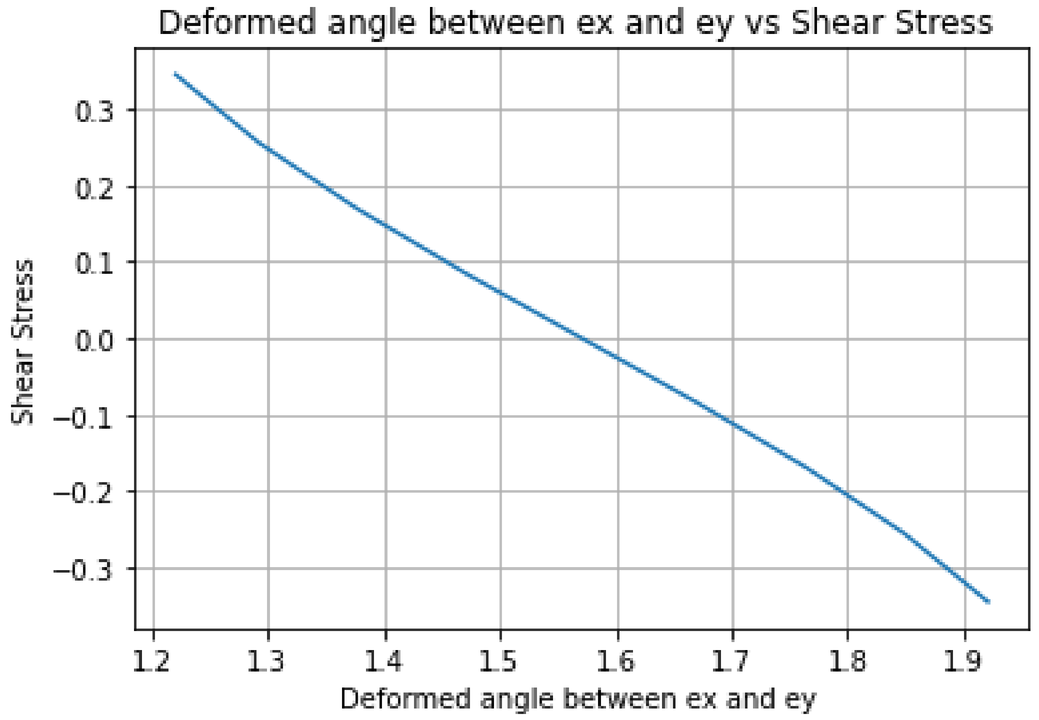
$$\begin{aligned} 0 &= -p + 0.838(a_1^2 + k^2) \\ 0 &= -p + 0.838\, a_2^2 \\ 0 &= -p + 0.838\, a_3^2 \\ \tau &= 0.838\, k a_2 \end{aligned}$$

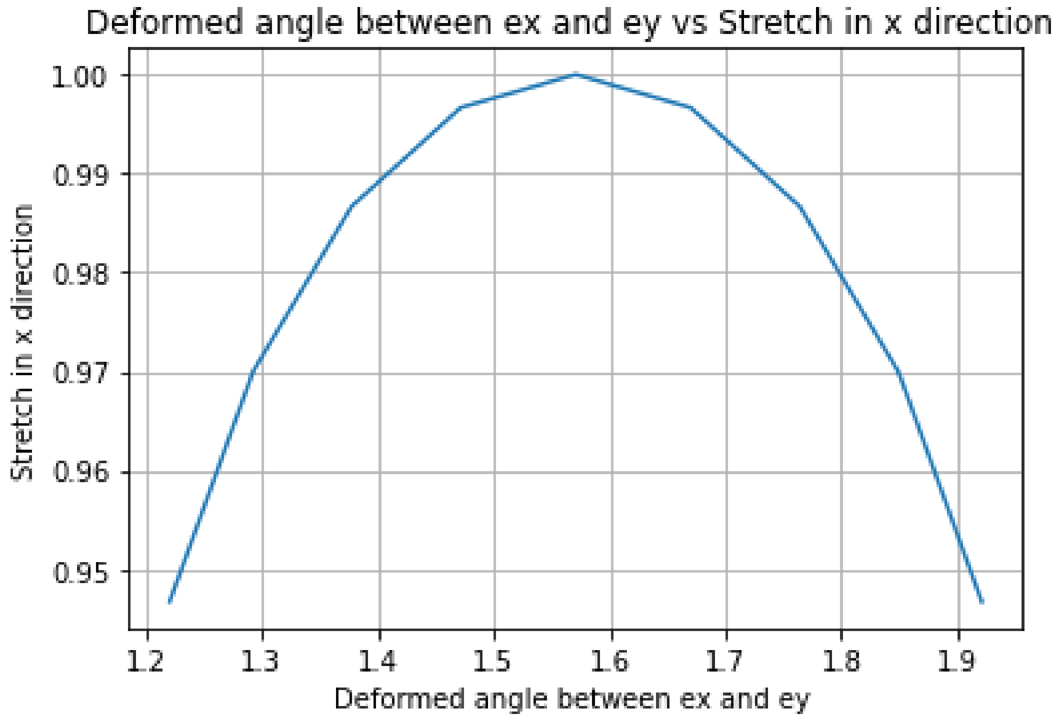
PLOT FOR UNIAXIAL LOADING



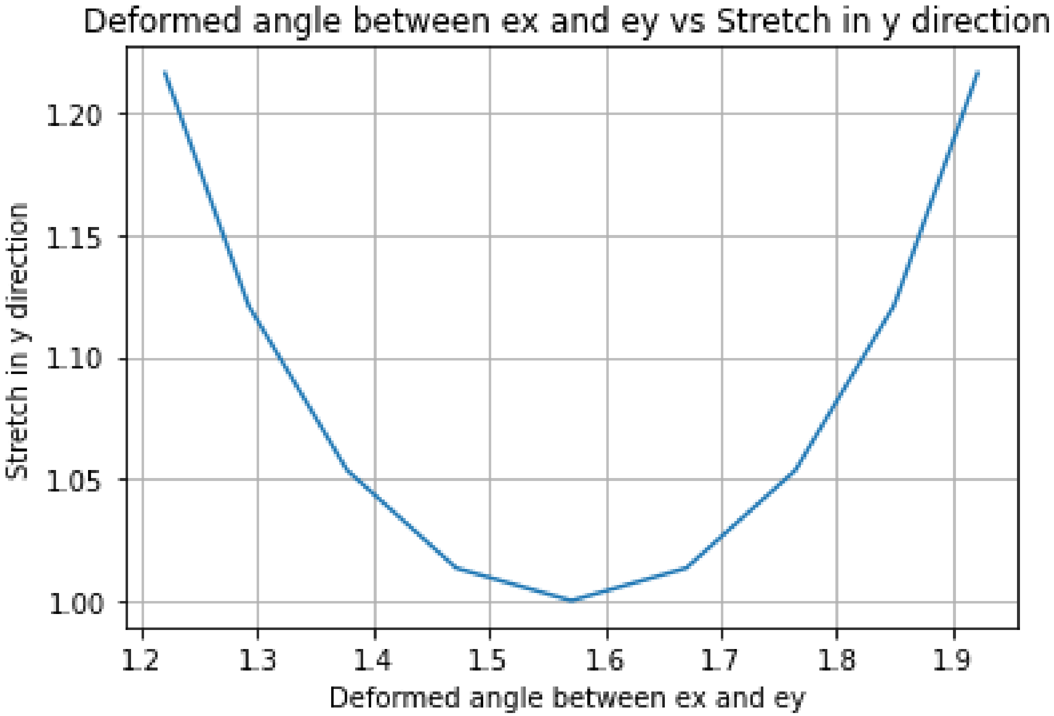


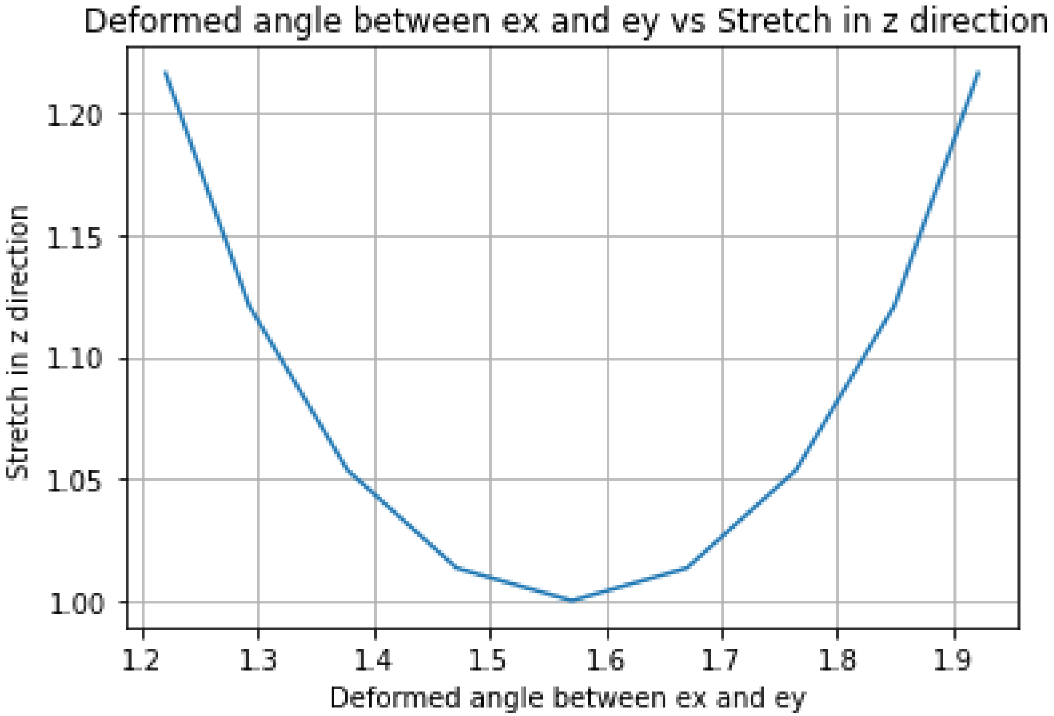
PLOT FOR PURE SHEAR





PLOT FOR PURE SHEAR





PART B
Inflation of Annular Sphere

$(R, \theta, \phi) \longrightarrow (r, \theta, \phi)$

$r = f(R) \qquad \theta = \theta \qquad \phi = \phi$

$$\underline{\underline{F}} = \begin{pmatrix} f_{,R} & 0 & 0 \\ 0 & r/R & 0 \\ 0 & 0 & r/R \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} f_{,R}^2 & 0 & 0 \\ 0 & (r/R)^2 & 0 \\ 0 & 0 & (r/R)^2 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1/f_{,R}^2 & 0 & 0 \\ 0 & R^2/r^2 & 0 \\ 0 & 0 & R^2/r^2 \end{pmatrix}$$

$$J_1 = tr(\underline{\underline{B}}) = f_{,R}^2 + 2\left(\frac{r}{R}\right)^2$$

$$J_2 = tr(\underline{\underline{B}}^{-1}) = \frac{1}{f_{,R}^2} + 2\left(\frac{R}{r}\right)^2$$

$$J_3 = f_{,R} \cdot \left(\frac{r}{R}\right)^2$$

Saint venant Kirchhoff's constitutive relation

$\mu = 0.5 \; ; \; \nu = 2$

$$\underline{\underline{\sigma}} = -\mu J_2 J_3 \, \underline{\underline{I}} + \left[\mu\left(\frac{J_1 - 1}{J_3}\right) + \nu\left(\frac{J_1 - 3}{2J_3}\right)\right] \cdot \underline{\underline{B}} + \mu J_3 \, \underline{\underline{B}}^{-1}$$

$$\operatorname{div}(\underline{\underline{\sigma}}) = 0$$

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) = 0$$

$$\frac{d\sigma_{rr}}{\partial R} \cdot \frac{\partial R}{\partial r} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) = 0$$

$$\frac{d\sigma_{rr}}{\partial R}\,\frac{1}{f_{,R}} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) = 0$$

$$\underline{\sigma_{rr}} = \alpha_0 + \alpha_1 f_{,R}^2 + \alpha_2 / f_{,R}^2$$

$$\sigma_{\theta\theta} = -\mu J_2 J_3 + \left[\mu\frac{(J_1-1)}{J_3} + \gamma\,\frac{(J_1-3)}{2J_3}\right]\left(\frac{r}{R}\right)^2 + \mu J_3 \left(\frac{R}{r}\right)^2$$

$$\alpha_0 = -\mu J_2 J_3$$

$$\sigma_{\phi\phi} = -\mu J_2 J_3 + \left[\mu\frac{(J_1-1)}{J_3} + \gamma\,\frac{(J_1-3)}{2J_3}\right]\left(\frac{r}{R}\right)^2 + \mu J_3 \left(\frac{R}{r}\right)^2$$

$$\alpha_1 = \mu\,\frac{(J_1-1)}{J_3} + \upsilon\,\frac{(J_1-3)}{2J_3}$$

$$\alpha_2 = \mu J_3$$

$$\frac{\partial J_1}{\partial R} = \frac{\partial}{\partial R}\left(f_{,R}^2 + 2\frac{r^2}{R^2}\right) = 2 f_{,R} f_{,RR} + \frac{2r^2(-2)}{R^3} = 2 f_{,R} \cdot f_{,RR} - 4\frac{r^2}{R^3}$$

$$\frac{\partial J_2}{\partial R} = \frac{\partial}{\partial R}\left(\frac{1}{f_{,R}^2} + \left(\frac{R}{r}\right)^2 + \left(\frac{R}{r}\right)^2\right) = -2 f_{,R}^{-3} f_{,RR} + 2 \times 2\frac{R}{r^2} = -2\frac{f_{,RR}}{f_{,R}^3} + 4\frac{R}{r^2}$$

$$\frac{\partial J_3}{\partial R} = \frac{\partial}{\partial R}\left(f_{,R} \cdot \frac{r^2}{R^2}\right) = \frac{r^2}{R^2} f_{,RR} + f_{,R} - 2\frac{r^2}{R^3}$$

$$\frac{\partial \alpha_0}{\partial J_1} = 0 \qquad \frac{\partial \alpha_0}{\partial J_2} = -\mu J_3 \qquad \frac{\partial \alpha_0}{\partial J_3} = -\mu J_2$$

$$\frac{\partial \alpha_1}{\partial J_1} = 0 \qquad \frac{\partial \alpha_1}{\partial J_2} = 0 \qquad \frac{\partial \alpha_1}{\partial J_3} = -\mu\frac{(J_1-1)}{J_3^2} - \upsilon\frac{(J_1-3)}{2J_3^2}$$

$$\frac{\partial \alpha_2}{\partial J_1} = 0 \qquad \frac{\partial \alpha_2}{\partial J_2} = 0 \qquad \frac{\partial \alpha_3}{\partial J_3} = \mu$$

$$\frac{\partial \sigma_{rr}}{\partial R} = \left(\frac{\partial \alpha_0}{\partial J_1}\frac{\partial J_1}{\partial R} + \frac{\partial \alpha_0}{\partial J_2}\cdot\frac{\partial J_2}{\partial R} + \frac{\partial \alpha_0}{\partial J_3}\cdot\frac{\partial J_3}{\partial R}\right) + \left(\frac{\partial \alpha_1}{\partial J_1}\frac{\partial J_1}{\partial R} + \frac{\partial \alpha_1}{\partial J_2}\cdot\frac{\partial J_2}{\partial R} + \frac{\partial \alpha_1}{\partial J_3}\cdot\frac{\partial J_3}{\partial R}\right) f_{,R}^2$$

$$+ \alpha_1 \cdot 2 f_{,R} \cdot f_{,RR} + \frac{\left(\frac{\partial \alpha_2}{\partial J_1}\frac{\partial J_1}{\partial R} + \frac{\partial \alpha_2}{\partial J_2}\cdot\frac{\partial J_2}{\partial R} + \frac{\partial \alpha_2}{\partial J_3}\cdot\frac{\partial J_3}{\partial R}\right) f_{,R}^2 - \alpha_2\, 2 \cdot f_{,R} f_{,RR}}{(f_{,R})^4}$$

SUBSITUTING ALL EQUATION IN EQUILIBRIUM EQUATION

$$-\mu J_3\left(-2\frac{f_{,RR}}{f_{,R}^3}+4\frac{R}{r^2}\right)+\left(\frac{r^2}{R^2}f_{,RR}+f_{,R}-2\frac{r^2}{R^3}\right)\left\{-\mu J_2+f_{,R}^2\left(-\mu\frac{(J_1-1)}{J_3^2}-\upsilon\frac{(J_1-3)}{2J_3^2}\right)+\frac{\mu}{f_{,R}^2}\right\}$$

$$+2f_{,R}f_{,RR}\frac{(\lambda_1-\lambda_2)}{(f_{,R})^4}+\frac{2f_{,R}}{r}\left[-\mu J_2J_3+\left[\mu\frac{(J_1-1)}{J_3}+\upsilon\frac{(J_1-3)}{2J_3}\right].f_{,R}^2+\frac{\mu J_3}{f_{,R}^2}-\right.$$

$$\left.\left(-\mu J_2J_3+\left[\mu\frac{(J_1-1)}{J_3}+\upsilon\frac{(J_1-3)}{2J_3}\right]\left(\frac{r}{R}\right)^2+\mu J_3\left(\frac{R}{r}\right)^2\right)\right]=0$$

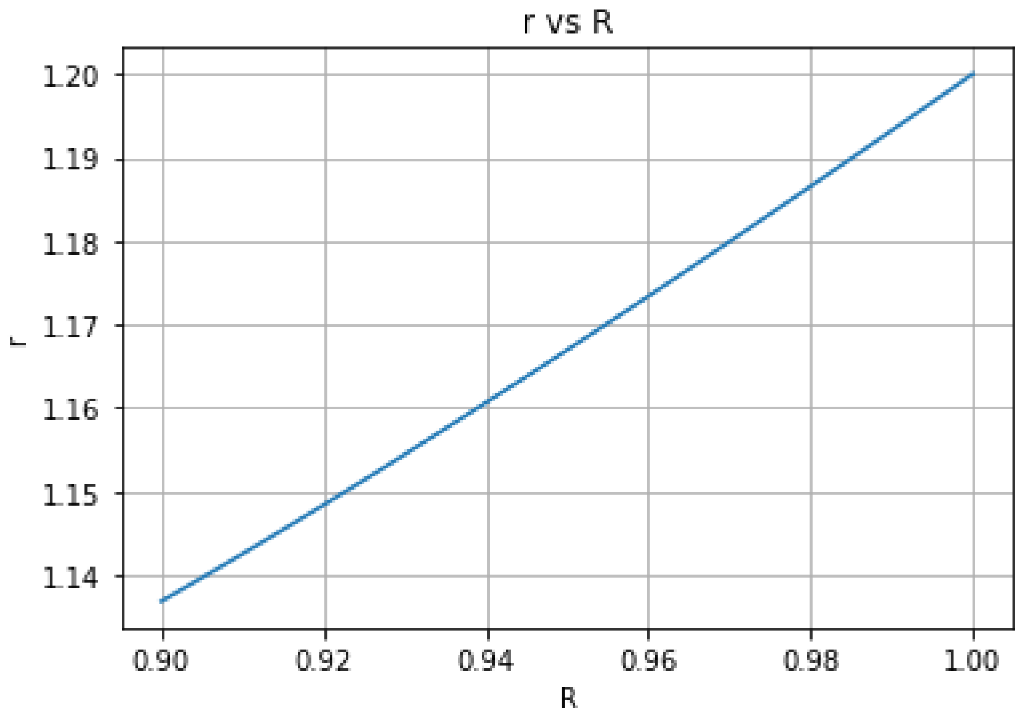
f1*$f_{,RR}$ + f2 = 0

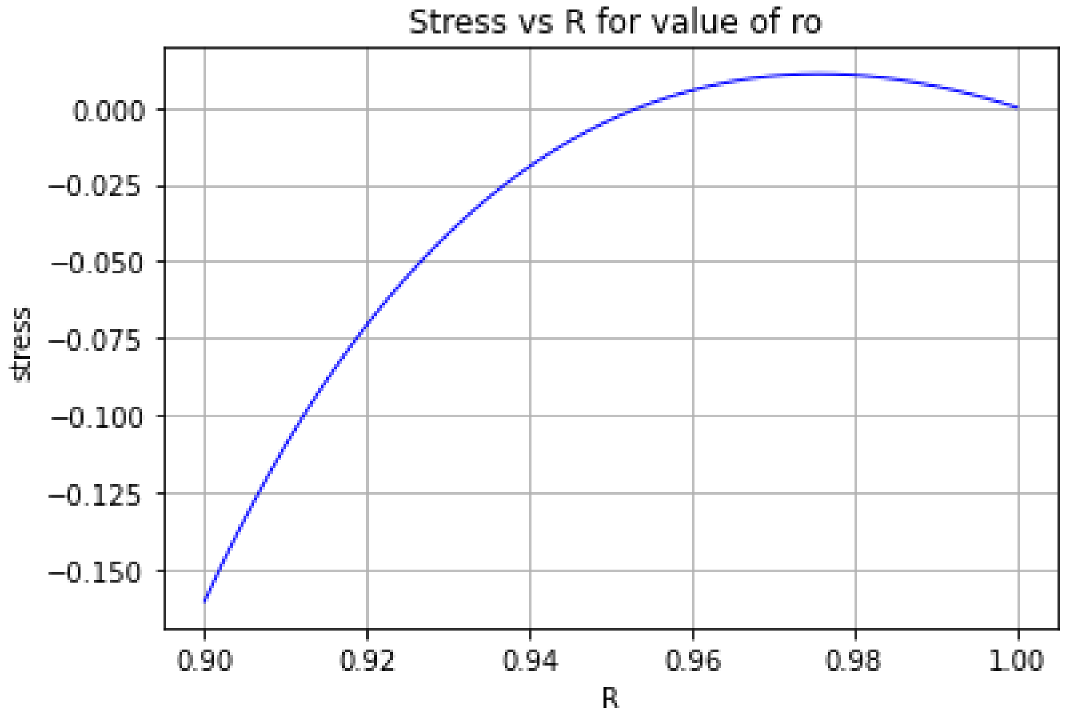
$$f_1=\frac{2\mu y_1^2}{y_2^2.R^2}+\frac{y_1^2}{R^2}\left\{-\mu\left[\left(\frac{1}{y_2^2}\right)+2\left(\frac{R}{y_1}\right)^2\right]+y_2^2\left(-\mu\frac{\left[y_2^2+2\left(\frac{y_1}{R}\right)^2-1\right]}{y_2^2\times\frac{y_1^4}{R^4}}-\upsilon\frac{\left[y_2^2+2\left(\frac{y_1}{R}\right)^2-3\right]}{2\left(y_2^2\times\frac{y_1^4}{R^4}\right)}\right)+\frac{\mu}{y_2^2}\right\}$$

$$+2y_2\left(\mu\frac{\left(y_2^2+2\left(\frac{y_1}{R}\right)^2-1\right)}{\left(y_2\times\frac{y_1^2}{R^2}\right)}+\upsilon\frac{\left[\left(y_2^2+2\left(\frac{y_1}{R}\right)^2-3\right]}{y_2^4\;2\left(y_2\times\frac{y_1^2}{R^2}\right)}\right)$$

$$
\begin{aligned}
f_2 = {} & -4\mu\left(y_2 \times \frac{y_1^2}{R^2}\right) \times \frac{R}{y_1^2} \\
& + \left(y_2 - \frac{2y_1^2}{R^3}\right)\Bigg\{ -\mu\left[\left(\frac{1}{y_2^2}\right) + 2\left(\frac{R}{y_1}\right)^2\right] + y_2^2 \left(-\mu \frac{\left[y_2^2 + 2\left(\frac{y_1}{R}\right)^2 - 1\right]}{y_2^2 \times \frac{y_1^4}{R^4}} - \nu \frac{\left[y_2^2+2\left(\frac{y_1}{R}\right)^2 - 3\right]}{2\left(y_2^2 \times \frac{y_1^4}{R^4}\right)}\right) + \frac{\mu}{y_2^2}\Bigg\} \\
& + \frac{2y_2}{r}\Bigg[-\mu\left[\left(\frac{1}{y_2^2}\right)+2\left(\frac{R}{y_1}\right)^2\right]\left[y_2\times\frac{y_1^2}{R^2}\right] + \left(-\mu\frac{\left[y_2^2+2\left(\frac{y_1}{R}\right)^2-1\right]}{y_2\times \frac{y_1^2}{R^2}} - \nu \frac{\left[y_2^2+2\left(\frac{y_1}{R}\right)^2-3\right]}{2\left(y_2\times\frac{y_1^2}{R^2}\right)}\right)\cdot y_2^2 \\
& + \frac{\mu\left(y_2\times \frac{y_1^2}{R^2}\right)}{y_2^2} - \Bigg\{ \mu\left[\left(\frac{1}{y_2^2}\right)+2\left(\frac{R}{y_1}\right)^2\right]\left(y_2\times\frac{y_1^2}{R^2}\right) + \left(\mu\frac{\left[y_2^2+2\left(\frac{y_1}{R}\right)^2-1\right]}{y_2\times\frac{y_1^2}{R^2}} + \nu\frac{\left[y_2^2+2\left(\frac{y_1}{R}\right)^2-3\right]}{2\left(y_2\times\frac{y_1^2}{R^2}\right)}\right) \\
& \times\left(\frac{y_1}{R}\right)^2 + \mu\left(y_2\times\frac{y_1^2}{R^2}\right)\left(\frac{R^2}{y_1^2}\right)\Bigg\}\Bigg]
\end{aligned}
$$

PLOT FOR COMPRESSIBLE MATERIAL





Trelor Model

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 0.838\,\underline{\underline{B}}$$

$$\underline{\underline{\sigma}}_{rr} = -p + 0.838\, f_{,R}^{2}$$

$$\underline{\underline{\sigma}}_{\theta\theta} = -p + 0.838\left(\frac{r}{R}\right)^2 = \sigma_{\phi\phi}$$

$J_3 = 1$ — incompressible

$$f_{,R}\cdot\left(\frac{r}{R}\right)^2 = 1$$

$$\frac{\partial r}{\partial R} = \frac{R^2}{r^2}$$

$$\int_r^{r_0} r^2\,\partial r = \int_R^{R_0} R^2\,\partial R$$

$$\left.\frac{r^3}{3}\right|_r^{r_0} = \left.\frac{R}{3}\right|_R^{R_0}$$

$$r_0^3 - r^3 = R_0^3 - R^3$$

$$\boxed{r^3 = r_0^3 + R^3 - R_0^3}$$

$$\frac{dr}{dR} = \frac{\partial}{\partial R}\left(r_0^3 - R^3 - R_0\right)^{1/3}$$

$$\boxed{\frac{dr}{dR} = \frac{R^2}{\left(R^3 - R_0^3 + r_0^3\right)^{2/3}}}$$

We have equilibrium equation as;

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) = 0$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0 \qquad \because \sigma_{\theta\theta} = \sigma_{\phi\phi}$$

Substituting σ in form of $-p+\sigma^e$

$$\frac{\partial}{\partial r}\left(-p+\sigma^e_{rr}\right) + \frac{2}{r}\left(-\cancel{p} + \sigma^e_{rr} + \cancel{p} - \sigma^e_{\theta\theta}\right) = 0$$

$$-\frac{\partial p}{\partial r} + \frac{\partial \sigma^e_{rr}}{\partial r} + \frac{2}{r}\left(\sigma^e_{rr} - \sigma^e_{\theta\theta}\right) = 0$$

$$-\frac{\partial p}{\partial r} + \frac{\partial \sigma^e_{rr}}{\partial r} + \frac{2}{r}\left(\sigma^e_{rr} - \sigma^e_{\theta\theta}\right) = 0 \rightarrow \text{integrating equation.}$$

$$-\left[p(f_0) - p(f)\right] + \sigma^e_{rr}(f_0) - \sigma_{rr}(f) + \int_f^{f_0} \frac{2}{r}\left(\sigma^e_{rr} - \sigma^e_{\theta\theta}\right) dr = 0$$

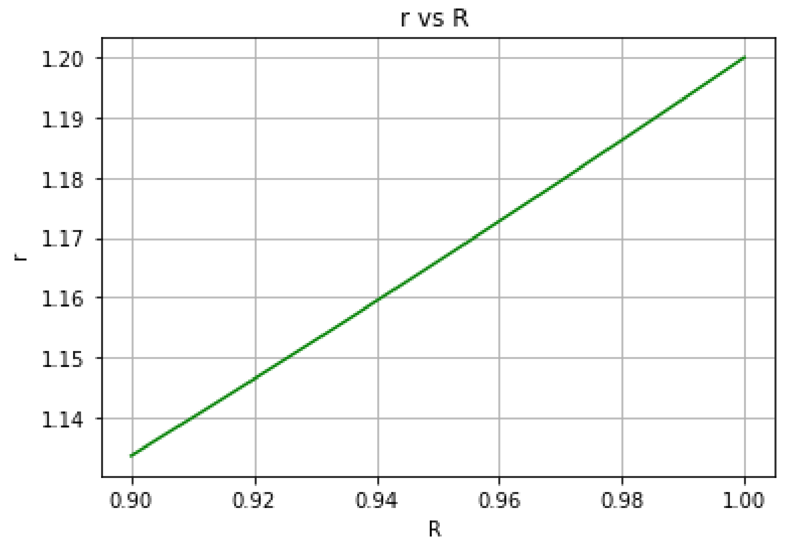
$$p(f) + \sigma_{rr}(f_0) - \sigma^e_{rr}(f) + \int_f^{f_0} \frac{2}{r}\left(\sigma^e_{rr} - \sigma^e_{\theta\theta}\right) dr = 0$$

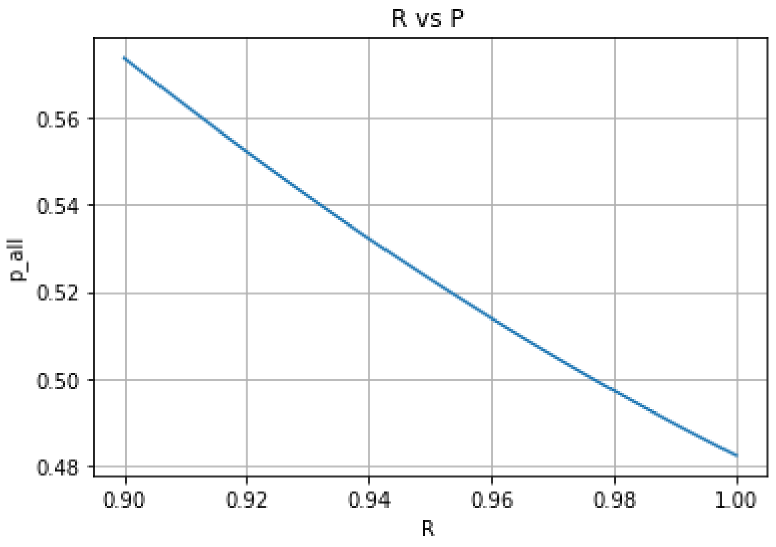
$$p(f) = \sigma^e_{rr}(f) - \int_f^{f_0} \frac{2}{r}\left(\sigma^e_{rr} - \sigma^e_{\theta\theta}\right) dr - \cancelto{0}{\sigma_{rr}(f_0)}$$

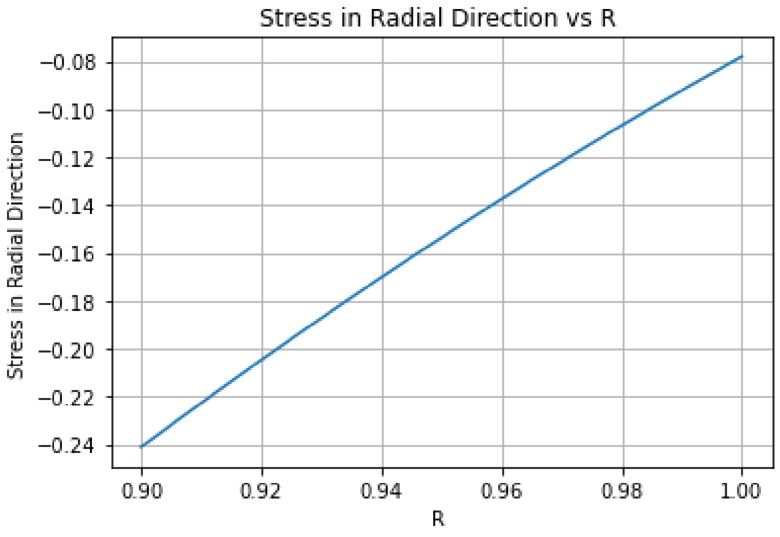
$$\boxed{p(f) = \sigma^e_{rr}(f) - \int_f^{f_0} \frac{2}{r}\left(\sigma^e_{rr} - \sigma^e_{\theta\theta}\right) dr}$$

$p(f) = f(R)$ because $r = f(R) \rightarrow$ we calculate.

Plot with Numerical Integration by Trapezoidal Method







Plot with Numerical Integration by Simpson Method