CE7640: Elastic and Plastic Stress Analysis
Project – 1: Elastic deformations

Tasks:

A. Formulate and solve the pure shear stress boundary value problem (deformation field: $x = a_1X + k Y$, $y = a_2Y$, $z = a_3Z$) for the constitutive relation that you are going to use. By making appropriate assumptions and using the general solution plot (i) the axial stress versus axial stretch ratio for uniaxial loading (ii) Transverse stretch ratio versus axial stretch ratio for uniaxial loading (iii) shear stress versus deformed angle between $\mathbf{e}_x$ and $\mathbf{e}_y$ for a pure shear stress loading (iv) stretch along coordinate directions versus the deformed angle between $\mathbf{e}_x$ and $\mathbf{e}_y$ for a pure shear stress loading.

B. Formulate and solve the boundary value problem assigned below and plot (i) the relevant force / moment versus displacement / rotation (ii) the spatial variation of the non-zero components of the stresses (only as 2D plots no 3D plots) (iii) Visualize the deformation field. Compare and contrast the response of compressible and incompressible materials.

1. **CE20D406 & CE21D016 & CE21D402**: Circumferential shearing of an annular cylinder made up of a material that obeys (a) Hart-Smith and Crisp model (b) Blatz-Ko model for foam rubber (Refer to Saravanan and Rajagopal (2005))
2. **CE21D042 & CE21D060**: Torsion of an annular cylinder made up of a material that obeys (a) Mooney model (b) Blatz-Ko model for foam rubber (Refer to Saravanan and Rajagopal (2005))
3. **CE21S007 & CE21S009**: Longitudinal shearing of an annular cylinder made up of a material that obeys (a) Khajehsaeid et al (2013) model (b) Blatz-Ko model for polyurethane (Refer to Saravanan and Rajagopal (2005))
4. **ME21S030 & ME21S032**: Inflation of an annular sphere made of a material that obeys (a) Treloar model (b) Saint Venant Krichhoff model (Refer to Saravanan and Rajagopal (2007))

For details about the deformation field that needs to be assumed refer to the suggested journal article or any other journal articles. (Suggested articles are also uploaded in courses.iitm.ac.in.)

1. Saravanan, U and Rajagopal, K.R.: Inflation, extension, torsion and shearing of an inhomogeneous compressible elastic right circular annular cylinder, Mathematics and mechanics of solids 10, (2005) 603-650.
2. Saravanan, U and Rajagopal, K.R.: On some finite deformations of inhomogeneous compressible elastic solids, Mathematical proceedings of the royal Irish academy 107A(1), (2007) 43-72.

The constitutive models for incompressible materials are listed in table 1 and the value for the material parameters is listed in table 2.

Blatz- Ko constitutive relation:

$$\boldsymbol{\sigma} = \frac{\mu_1}{J_3}\left[\mu_m \mathbf{1} + \mu_2 \mathbf{B} - (1-\mu_2)\mathbf{B}^{-1}\right],$$

For polyurethane: $\mu_2 = 1$, $\mu_3 = 1$, $\mu_1 = 0.5$ MPa

For foam rubber: $\mu_2 = 0$, $\mu_3 = 1$, $\mu_1 = 0.25$ MPa

$$\mu_m = J_3^{2\mu_3} - \mu_2\left[J_3^{2\mu_3} + J_3^{-2\mu_3}\right].$$

Saint Venant Kirchhoff constitutive relation:

$\mu = 0.5$ MPa, $\gamma = 2$ MPa

$$\boldsymbol{\sigma} = -\mu J_2 J_3 \mathbf{1} + \left[\mu \frac{J_1 - 1}{J_3} + \gamma \frac{J_1 - 3}{2J_3}\right] \mathbf{B} + \mu J_3 \mathbf{B}^{-1}.$$

Table 1: Constitutive relations for incompressible materials (C_i material parameters)

Constitutive model	Cauchy Stress, $\boldsymbol{\sigma}$
Mooney (1940)	$-p\mathbf{1} + C_1\mathbf{B} - C_2\mathbf{B}^{-\mathbf{1}}$
Treloar (1943)	$-p\mathbf{1} + C_1\mathbf{B}$
Hart-Smith and Crisp (1967)	$-p\mathbf{1} + C_1 e^{C_2(I_1-3)^2}\mathbf{B} - C_1\frac{C_3}{I_2}\mathbf{B}^{-\mathbf{1}}$
Anand (1986)	$-p\mathbf{1} + \sum_{i=1}^{3} \frac{C_1 ln(\lambda_i)}{\lambda_i}\mathbf{n}_i \otimes \mathbf{n}_i$
Arruda and Boyce (1993)	$-p\mathbf{1} + R\mathbf{B}$ where $R = C_1\left[\frac{1}{2} + \frac{I_1}{10C_2} + \frac{11I_1^2}{350(C_2)^2} + \frac{19I_1^3}{1750(C_2)^3} + \frac{519I_1^4}{134750(C_2)^4}\right]$
Puso (1994)	$-p\mathbf{1} + R\left(R_0\mathbf{B} - \frac{2\sqrt{3}}{5}I_1^{-1.5}\mathbf{B}^{\mathbf{2}}\right)$ where $R = \frac{0.3C_1\sqrt{3}(3I_1^{0.5} + \frac{I_2}{I_1^{1.5}})}{1-(\frac{0.1}{\sqrt{C_2}}\sqrt{3}(3I_1^{0.5}+\frac{I_2}{I_1^{1.5}}))^3}$ $R_0 = \frac{\sqrt{3}}{2}(I_1^{-0.5} - \frac{3}{5}\frac{I_2}{I_1^{2.5}})$
Gent (1996)	$-p\mathbf{1} + [\frac{C_1}{6}][\frac{1}{1-(\frac{I_1-3}{C_2-3})}]\mathbf{B}$
Yeoh and Fleming (1997)	$-p\mathbf{1} + \left[\frac{C_1}{1-R} - C_3(1 - e^{C_4 R})\right]\mathbf{B}$ where $R = \frac{I_1-3}{(C_2-3)}$
Subhani and Kumar (2009)	$-p\mathbf{1} + R(C_2 I_1^2\mathbf{B} - (1 - C_2)I_2^2\mathbf{B}^{-\mathbf{1}}$ where $R = \frac{6C_1}{C_2 I_1^3 + (1-C_2)I_2^3}$
Khajehsaeid et al. (2013)	$-p\mathbf{1} + R\mathbf{B}$ where $R = C_1(\exp(C_2(I_1 - 3)) + C_3\ln(I_1 - 2))$

Table 2: Value of material parameters (C_i)

Constitutive model	Material parameters
Invariants of $\boldsymbol{\eta}$	$D_2^0 = 0.13$ MPa, $D_2^1 = 0.106$ MPa, $E_1 = 1.696$ MPa, $E_2 = 0.039$ MPa
Invariants of $\mathbf{C}$	$D_0^0 = 0.343$ MPa, $D_1^0 = 0.02$ MPa, $E_0 = 0.007$ MPa
Hart-Smith and Crisp	$C_1 = 0.828$ MPa, $C_2 = 0.174$, $C_3 = 0.285$
Khajehsaeid et al.	$C_1 = 0.808$ MPa, $C_2 = -0.429$, $C_3 = -0.65$
Ogden	$C_1 = 0.075$ MPa, $\alpha_1 = 5.15$, $C_2 = -8.794$ MPa, $\alpha_2 = -0.136$,
Yeoh and Fleming	$C_1 = 0.815$ MPa, $C_2 = 35$, $C_3 = 0.126$ MPa, $C_4 = 20$
Mooney	$C_1 = 0.66$ MPa, $C_2 = 0.15$ MPa
Gent	$C_1 = 5.00$ MPa, $C_2 = 50$
Arruda and Boyce	$C_1 = 1.639$ MPa, $C_2 = 30$
Treloar	$C_1 = 0.838$ MPa
Subhani and Kumar	$C_1 = 0.438$ MPa, $C_2 = 0.872$
Puso	$C_1 = 1.82$ MPa, $C_2 = 35$
Anand	$C_1 = 1.43$ MPa