ENS1161 Computer Fundamentals

Lecture 11

Matrices and Applications

Part 1

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Outline

01. Some definitions and terminology
02. Order, square matrices, column and row matrices
03. Equality of matrices
04. Addition of matrices
05. Multiplication by a number
06. Multiplying two matrices
07. Which matrices can be multiplied?
08. Some rules of matrix algebra
09. An important exception
10. Non-commutativity in robotics
11. An application of matrices to directed graphs
12. "Reachability" of vertices

Lecture’s Major Objectives

After completing this section, students should be able to:

- use subscript notation
- add matrices
- multiply a matrix by a number
- multiply two matrices
- apply the "domino rule"
- find the adjacency matrix of a digraph
- find the indegree and outdegree of a vertex of a digraph
- use logical multiplication and addition to calculate the reachability matrix of a digraph

Some definitions and terminology

A matrix is a rectangular array of numbers.
We have seen that such an array of numbers could represent:

- relations between sets;
- connections in a graph (an adjacency matrix).

But they could also represent:

- 2D and 3D geometric transformations (rotations, reflections, etc.)
- stresses in the framework that holds a building together
- voltages in an electrical network;
- densities of body tissue in a CAT scan;
- intensities of infra-red radiation in a satellite image;
- monthly sales of various types of goods.

Some definitions and terminology

The numbers that form a matrix are usually called elements, and sometimes it is convenient to refer to particular elements by specifying the row and column that they occupy.
For example, suppose we have a matrix

$$\mathbf{M} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$\mathbf{m_{14}}$

$\mathbf{m_{23}}$

In general the element in the i^{th} row and j^{th} column is called m_{ij}.
This method of naming elements is called subscript notation.

Some definitions and terminology

Using subscript notation we would write a 2×3 matrix A like this:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

and a 3×2 matrix B like this:

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

Order, square matrices, column and row matrices

The order of a matrix tells us how many rows and columns it has.
For example a 10×3 matrix has 10 rows and 3 columns.
So the orders of these three matrices

$$\begin{pmatrix} 2 & 5 & 1 \\ 0 & 2 & 6 \end{pmatrix} \qquad \begin{pmatrix} 4 & 5 \\ 1 & 3 \\ 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 5 & 3 & 0 & 4 \end{pmatrix}$$

are 2×3, 3×2 and 3×4 respectively.

Order, square matrices, column and row matrices

For obvious reasons, a matrix that has the same number of rows and columns, for example 3×3 is called square, a matrix with just one column is called a column matrix and a matrix with just one row is called a row matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 9 \\ 2 & 2 & 7 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 & 8 \end{pmatrix}$$

Equality of matrices

Two matrices are said to be equal if they have the same order, and the corresponding elements are equal.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \quad \textbf{\textit{means that}} \quad \begin{matrix} x = 3 \\ y = 5 \\ z = 0 \end{matrix}$$

Addition of matrices

Matrices may be added, provided that they are the same size and shape.

To add two matrices, simply add the corresponding elements.

Examples

$$\begin{pmatrix} 0 & 2 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 7 & 10 \end{pmatrix}$$

***If* A = $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \end{pmatrix}$ *and* B = $\begin{pmatrix} 3 & 5 & -4 \\ 0 & 2 & 3 \end{pmatrix}$ *then* A + B = $\begin{pmatrix} 4 & 4 & -2 \\ 0 & 5 & 5 \end{pmatrix}$**

Multiplication by a number

A matrix may be multiplied by a number.
To do this we simply multiply every element by the number.

Examples

1.

$$10 \begin{pmatrix} 2 & 3 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} 20 & 30 \\ 70 & -40 \end{pmatrix}$$

2.

$$\textit{If} \quad A = \begin{pmatrix} 4 & 0 \\ 3 & 1 \\ -2 & 2 \end{pmatrix} \quad \textit{then} \quad 6A = \begin{pmatrix} 24 & 0 \\ 18 & 6 \\ -12 & 12 \end{pmatrix}$$

Multiplication by a number

Examples

This is convenient, for example, when fractions are involved, e.g.

$$\begin{pmatrix} \frac{7}{15} & \frac{-4}{15} \\ \frac{2}{15} & \frac{1}{15} \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 7 & -4 \\ 2 & 1 \end{pmatrix}$$

Multiplying two matrices

The appendix for Week 5 gave a detailed explanation of the method for multiplying two matrices.

For example

$$\begin{pmatrix} 1 & 2 & 0 \\ -2 & 0 & 3 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 7 & -2 \\ 5 & -7 & 6 \\ 14 & 10 & 7 \end{pmatrix}$$

Multiplying two matrices

To multiply matrices we combine the rows of the first matrix with the columns of the second in a process that involves multiplication and addition.

Suppose we multiply a matrix A by a matrix B to obtain a matrix C, as shown:

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ a_{21} & a_{22} & a_{23} \\ \cdots & \cdots & \cdots \end{pmatrix} \times \begin{pmatrix} b_{11} & \cdots \\ b_{21} & \cdots \\ b_{31} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots \\ c_{21} & \cdots \\ \cdots & \cdots \end{pmatrix}$$

Multiplying two matrices

The element in the 2nd row and 1st column of C comes from the 2nd row of A and the 1st column of B, and is

$$c_{21} = a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23}\, b_{31}$$

We can abbreviate this to:

$$c_{21} = \sum_{k=1}^{3} a_{2k} b_{k1}$$

Similarly the element in the 1st row and 2nd column of C, for example, would be

$$c_{12} = \sum_{k=1}^{3} a_{1k} b_{k2}$$

Multiplying two matrices

More generally, if we were multiplying an $m \times n$ matrix A by an $n \times q$ matrix B to obtain an $m \times q$ matrix C, the formula for the typical element c_{ij} of C is:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Which matrices can be multiplied?

If we try to multiply the following matrices, we find that it is not possible:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$$

The number of columns of the first matrix is not the same as the number of rows of the second.

Two matrices can be multiplied only if the number of columns of the first is equal to the number of rows of the second.

The order of the result also depends on the orders of the two matrices being multiplied.

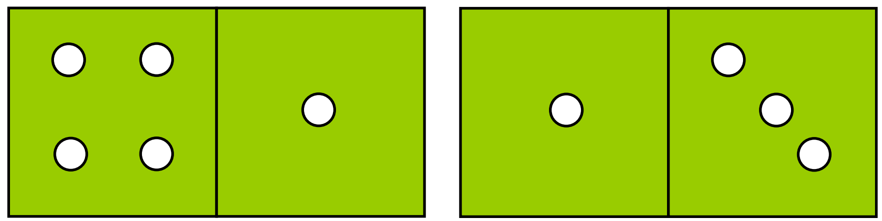
Which matrices can be multiplied?

The simplest way to remember the rules is as follows:

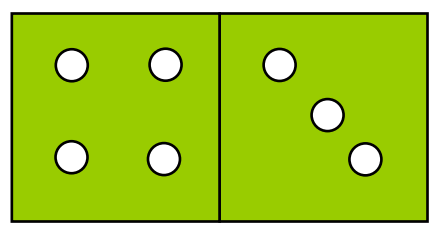
- Suppose we have an $m \times n$ matrix and a $p \times q$ matrix
- Multiplication is possible only if $n = p$
- and if this is the case, then the order of the result is $m \times q$

Which matrices can be multiplied?

This is sometimes called the domino rule, because it is similar to the rule for the game of dominoes.



Then the matching numbers "cancel" and the resulting arrangement is equivalent to:



Which matrices can be multiplied?

Similarly we can multiply a 4×1 matrix by a 1×3 matrix, and the result is 4×3.

Or, we can multiply a 3×5 matrix by a 5×4 matrix, and the result is 3×4.

Or, we can multiply a 1×2 matrix by a 2×6 matrix, and the result is 1×6.

Which matrices can be multiplied?

Examples 1

$$\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 3 & 1 \end{pmatrix} = ?$$

Which matrices can be multiplied?

Examples 1

$$\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 3 & 1 \end{pmatrix} = ?$$

The matrices have orders 2×3 and 3×2, so the product exists and is 2×2:

$$\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 0 & 1 \end{pmatrix}$$

Which matrices can be multiplied?

Examples 2

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 4 \end{pmatrix} = ?$$

Which matrices can be multiplied?

Examples 2

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 4 \end{pmatrix} = ?$$

The matrices have orders 2×2 and $2x2$, so the product exists and is 2×2:

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -2 & 13 \end{pmatrix}$$

Which matrices can be multiplied?

Examples 3

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -2 & 1 & 3 & 0 \\ 3 & 5 & -4 & 0 & 2 \end{pmatrix} = ?$$

Which matrices can be multiplied?

Examples 3

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -2 & 1 & 3 & 0 \\ 3 & 5 & -4 & 0 & 2 \end{pmatrix} = ?$$

The matrices have orders 2×2 and 2×5, so the product exists and is 2×5.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -2 & 1 & 3 & 0 \\ 3 & 5 & -4 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 & -4 & 0 & 2 \\ 4 & -2 & 1 & 3 & 0 \end{pmatrix}$$

Which matrices can be multiplied?

Examples 4

$$\begin{pmatrix} 1 & 5 & 3 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = ?$$

Which matrices can be multiplied?

Examples 4

$$\begin{pmatrix} 1 & 5 & 3 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = ?$$

Cannot multiply 2×3 by 2×1.

Which matrices can be multiplied?

Examples 5

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 & 2 \\ -3 & 0 & 1 & 1 \end{pmatrix} = ?$$

Which matrices can be multiplied?

Examples 5

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 & 2 \\ -3 & 0 & 1 & 1 \end{pmatrix} = ?$$

The matrices have orders 3×2 and 2×4, so the product exists and is 3×4:

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 & 2 \\ -3 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 4 & 3 & 4 \\ 4 & 8 & 2 & 4 \\ -7 & -8 & -1 & -3 \end{pmatrix}$$

Some rules of matrix algebra

Provided that the orders of the matrices allow the indicated operations to be performed, the following rules of matrix algebra are valid.

(i) $A + B = B + A$

(ii) $A + (B + C) = (A + B) + C$

(iii) $A\,(B\,C) = (A\,B)\,C$

(iv) $A\,(B + C) = A\,B + A\,C$

(v) $(B + C)\,A = B\,A + C\,A$

Some rules of matrix algebra

Example

Suppose that:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix}$$

Some rules of matrix algebra

(i) Show that A (B + C) = A B + A C

$$B + C = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\text{LHS} = A\,(B + C) = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 7 & 13 \end{pmatrix}$$

$$\text{RHS} = A\,B + A\,C = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 4 & 10 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 7 & 13 \end{pmatrix}$$

So LHS = RHS

Some rules of matrix algebra

(ii) Show that $(B + C)A = BA + CA$

$$B + C = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\text{LHS} = (B + C)A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 7 & 9 \end{pmatrix}$$

$$\text{RHS} = BA + CA = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 \\ 6 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 7 & 9 \end{pmatrix}$$

So LHS = RHS

An important exception

Many of the rules for matrices look like the familiar rules of the ordinary algebra of numbers.

For numbers a and b, it is always true that $ab = ba$.

However this rule does not hold for matrices.
For matrices A and B, it may be that $AB = BA$, but very often $AB \neq BA$.

$$\textit{If}\ \ A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\ \textit{and}\ \ B = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

$$\textit{then}\ \ AB = \begin{pmatrix} 11 & -2 \\ 2 & 11 \end{pmatrix}\ \textit{and}\ \ BA = \begin{pmatrix} 11 & -2 \\ 2 & 11 \end{pmatrix}$$

So in this case $AB = BA$, but this is the exception rather than the usual result.

An important exception

More typical is the next example:

$$\textit{If}\ \ C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\ \textit{and}\ \ D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\textit{then}\ \ CD = \begin{pmatrix} 2 & 6 \\ 6 & 12 \end{pmatrix}\ \textit{and}\ \ DC = \begin{pmatrix} 2 & 4 \\ 9 & 12 \end{pmatrix}$$

So $CD \neq DC$

An important exception

Even more obvious examples of non-commutativity are the following:

Suppose that L and M are matrices such that L is 2×3 and M is 3×2.

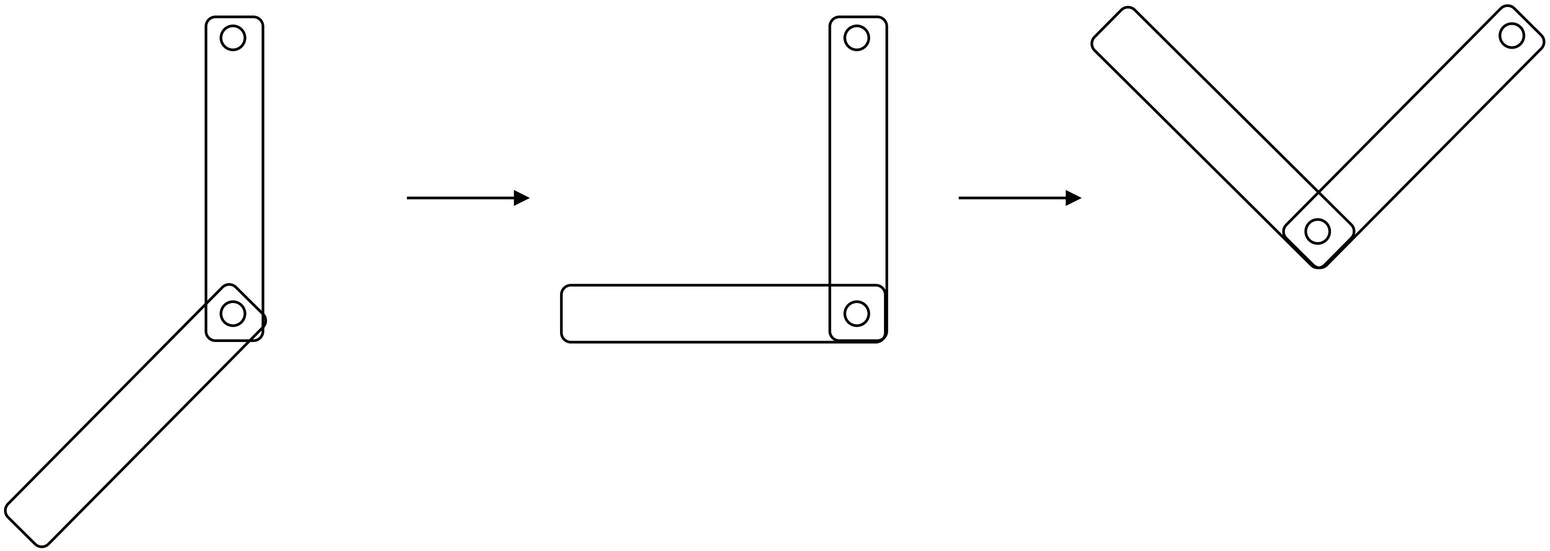
Then LM is 2×2, while ML is 3×3, so clearly $LM \neq ML$.

Suppose that P and Q are matrices such that P is 2×3 and Q is 3×1.

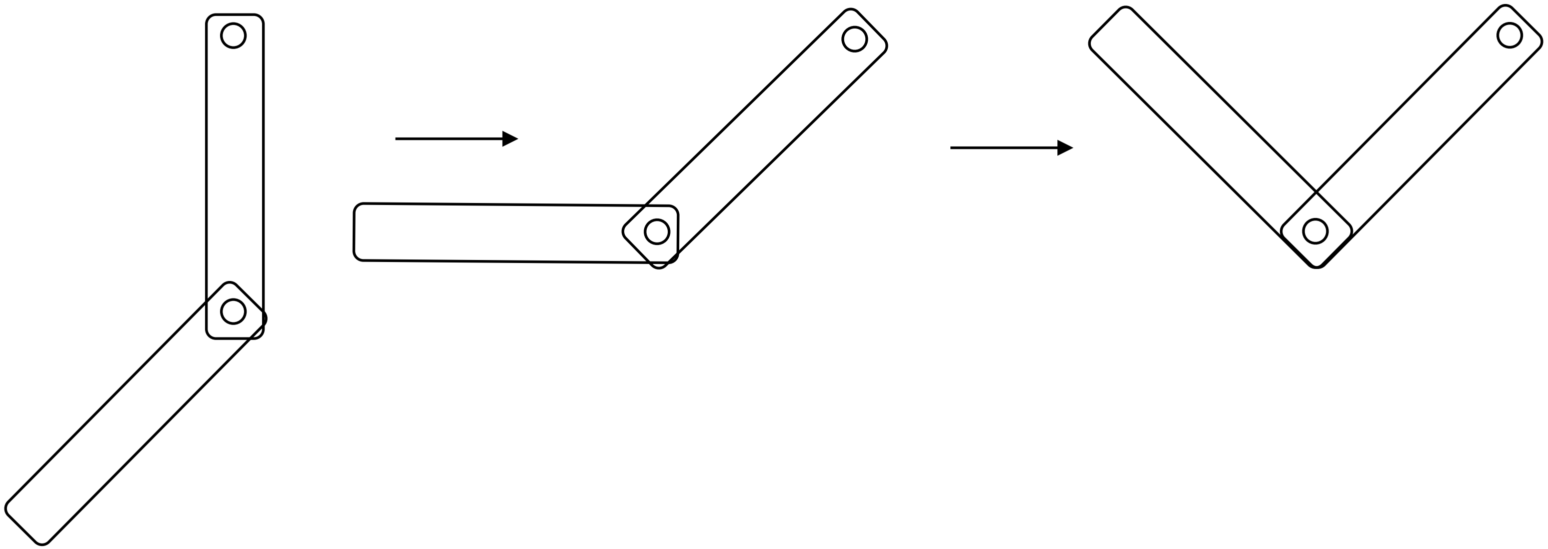
Then PQ is 2×1, but QP does not even exist. So obviously $PQ \neq QP$.

Non-commutativity in robotics

Non-commutativity of operations has important consequences in many applications, for example in robotics where they refer to holonomic or non-holonomic systems.
For example a robot arm with two pivot points, as shown below. In the first sequence the lower arm rotates through 45° and then the upper arm rotates through 45°:



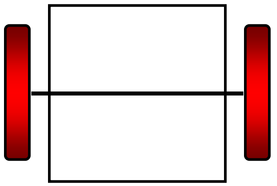
Non-commutativity in robotics



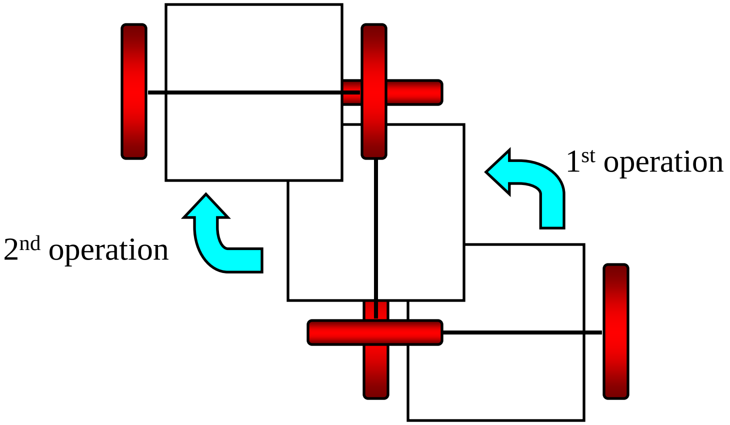
Notice that the final position is unchanged.
This is an example of a holonomic system, in which the order of operations does not affect the final result.

Non-commutativity in robotics

Contrast the robot arm with a wheel chair that has independent motors on each of its two main wheels.

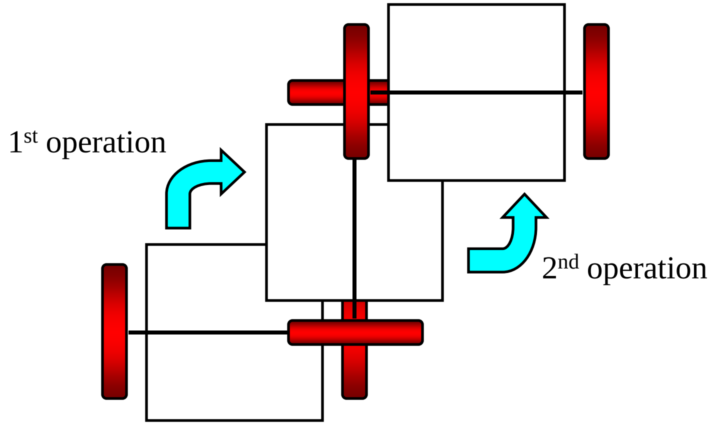


Suppose the left wheel is kept fixed while the motor on the right wheel turns the chair through 90°, and then the right wheel is kept fixed while the motor on the left wheel turns the chair through 90°.



Non-commutativity in robotics

Compare the final position with that obtained when the operations are reversed. In other words, suppose the right wheel is kept fixed while the motor on the left wheel turns the chair through 90°, and then the left wheel is kept fixed while the motor on the right wheel turns the chair through 90°.

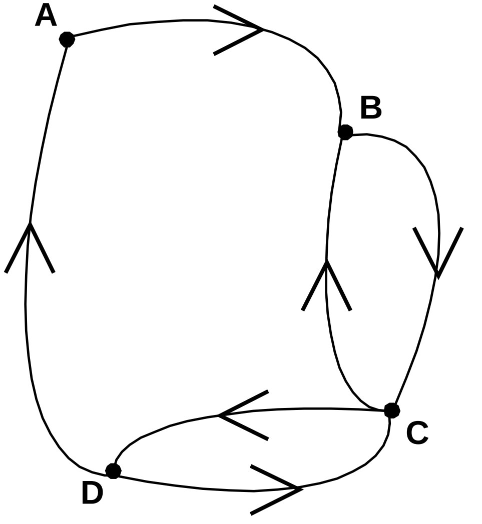


The final positions are quite different. This is an example of a non-holonomic system where the order of operations affects the final outcome.

An application of matrices to directed graphs

Directed graphs

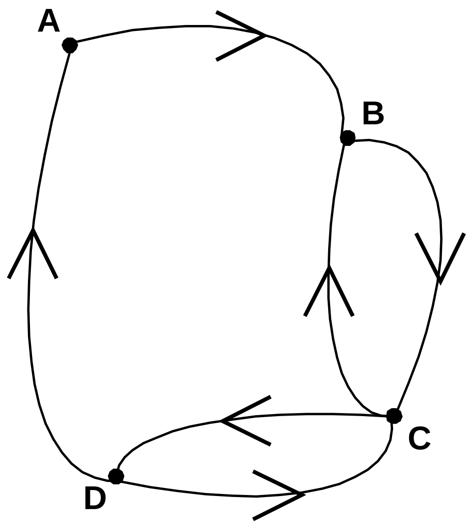
In a directed graph (or digraph), each edge has a certain direction.



In this digraph, it is possible to move directly from D to A. However it is not possible to move directly from A to D. There is a path from A through B and C to D. So A is "reachable" from D in just one step, whereas D is "reachable" from A in three steps.

An application of matrices to directed graphs

The adjacency matrix for this digraph is



$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

This 1 represents the edge from D to C

The vertices where the edges start are shown on the left side of the matrix, and the vertices where the edges end are shown across the top.

An application of matrices to directed graphs

In this digraph there is an edge from D to A, but there is no edge from A to D.

As a check on the accuracy of the matrix, notice that:

- the row totals give the number of edges leaving each vertex (the so called "outdegree" of the vertex), and
- the column totals give the number of edges arriving at the vertex (the so called "indegree" of the vertex)

An application of matrices to directed graphs

It follows that the sum of the indegrees must equal the sum of the outdegrees.

In this example:

Vertex	Indegree	Outdegree
A	1	1
B	2	1
C	2	2
D	1	2
Total	6	6

"Reachability" of vertices

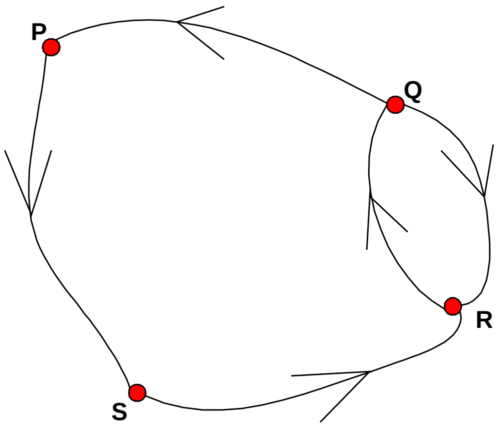
Suppose there are several buildings on a university campus, each with a computer laboratory, and we want to design a communication system (a local area network, or LAN) so that it is possible for a computer in any lab to communicate with a computer in any other lab.

It is not necessary that every pair of buildings is directly linked, since messages could be sent from one building to another via other buildings.

An important practical problem that needs to be considered is, given a proposed set of connections (a digraph), how do we determine which sites are "reachable" from other sites?

"Reachability" of vertices

Example: Consider a simple digraph with four vertices



The adjacency matrix is

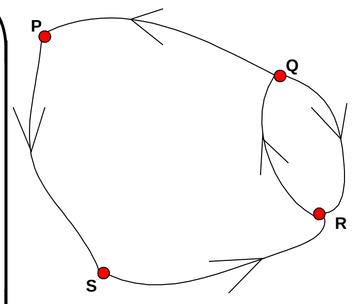
$$
M = \begin{array}{c c}
 & \begin{array}{cccc} P & Q & R & S \end{array} \\
\begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

"Reachability" of vertices

From the matrix we can see that it is possible to move directly from P to S, from Q to P or R, from R to Q, and from S to R.
It is not possible to move directly from Q to S.
However S is reachable from Q in two steps, by going via P.

To see which vertices are reachable in two steps, we calculate M^2, using logical multiplication

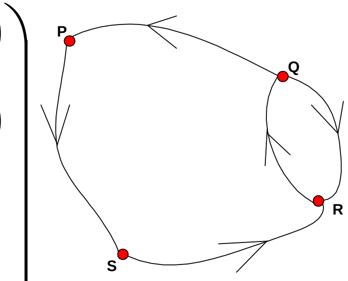
$$M^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$



"Reachability" of vertices

The matrix M^2 shows that P can reach R in two steps, Q can reach S in two steps, and so on.
To determine 3-step reachability, we calculate M3, again using logical multiplication.

$$M^3 = M^2 M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

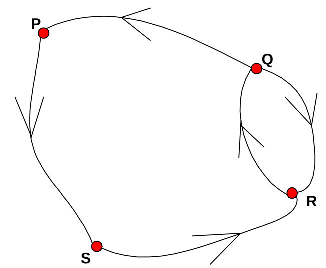


From M^3 we can see that, for example, P can reach Q in three steps, S can reach P in three steps, R can reach S in three steps, and so on.

"Reachability" of vertices

For a digraph with 4 vertices, we calculate as far as M^4, because for such a digraph, if a vertex is not reachable in 1, 2, 3 or 4 steps, then it cannot be reached at all.

For this digraph,



$$M^4 = M^2 M^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Combining all these results, we calculate the reachability matrix M* that shows which vertices are reachable in 1, 2, 3 or 4 steps.

"Reachability" of vertices

To find M* we add M and M^2 and M^3 and M^4, that is

$$M^* = M + M^2 + M^3 + M^4$$

using logical addition:

$$M^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

"Reachability" of vertices

which gives

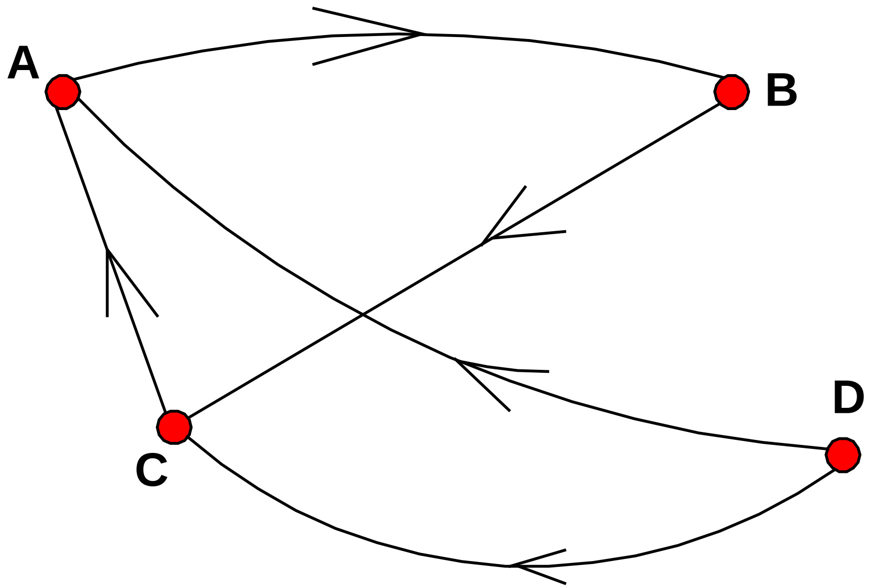
$$M^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

So, for this digraph, every vertex is reachable from every other vertex in no more than 4 steps.

"Reachability" of vertices

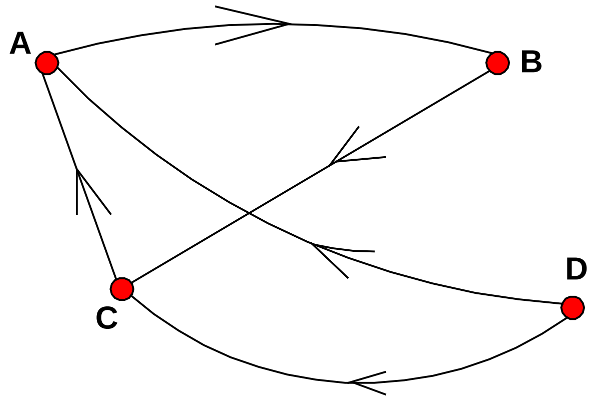
Example:

Calculate the reachability matrix M* for the following digraph:



"Reachability" of vertices

The adjacency matrix is



$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Then, using logical multiplication

$$M^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

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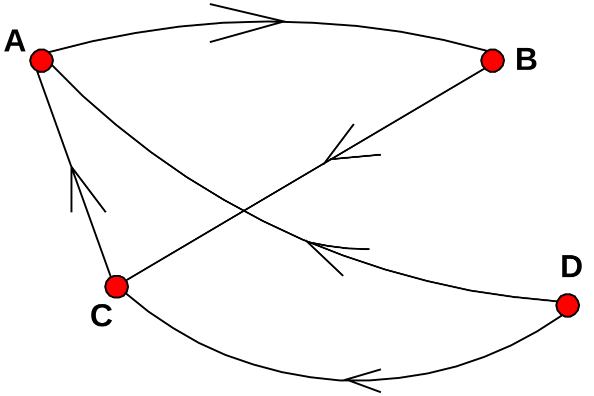


Then, using logical multiplication

$$M^3 = M^2 M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$M^4 = M^2 M^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

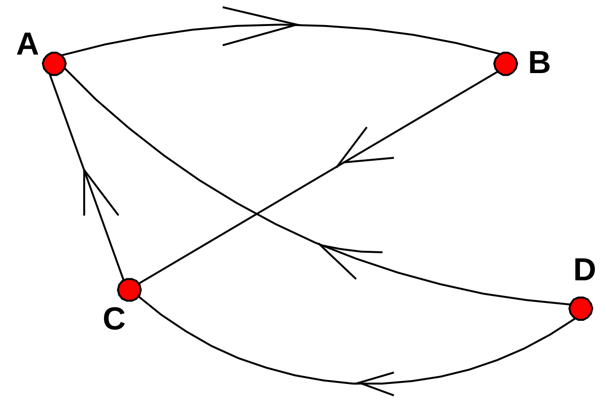
"Reachability" of vertices

Logically adding M , M^2 , M^3 and M^4 , we find the reachability matrix

$$M^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

which gives

$$M^* = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



"Reachability" of vertices

NOTE:
The order of the reachability matrix M* of a digraph, and the number of matrices added to find M*, depends on the number of vertices.

If the digraph has five vertices, then M* will be 5×5, and is obtained from

$$M^* = M + M^2 + M^3 + M^4 + M^5$$

If the digraph has n vertices, then M* will be $n \times n$, and is obtained from

$$M^* = M + M^2 + ... + M^n$$

The End