Lect 7: 29^{th} Jan '21

Proof by existence

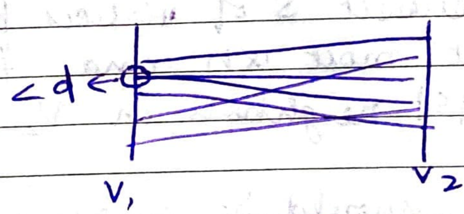
Statement 1: If a RV has a finite expected value $E[X] = a$, then certainly there exists a realisation of X with value $\geq a$ and a realisation of X with value $\leq a$.

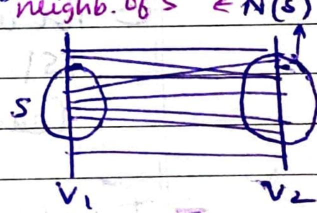
Statement 2: If a random object drawn from some universe of objects has a certain property $\bar{c}$ non-zero prob. then there must exist an object $\bar{c}$ that property in that universe.

Example to study st. 2:

α, β, n, d - Expander: A bipartite graph $G = (V_1 \cup V_2, E)$ on 'n' nodes is an (α, β, n, d) expander if
① every vertex in V_1 has degree at most d
② For any subset S of vertices ~~in~~ from V_1 such that $|S|$ is at most αn, there are at least $\beta|S|$ neighbours in V_2.

neighb. of S ← N(S)





$$|V_1| = |V_2| = n$$

Ideally, d should be small & β as large as possible.

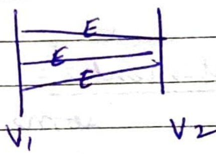
Study how Statement 2 helps –

<u>EXPANDER</u> – (α, β, n, d)

└ a bipartite graph $G = (V_1 \cup V_2, E)$.

on n nodes is an (α, β, n, d) expander if –

$E \subseteq V_1 \times V_2$
(subset)
so no edges within V_1 & V_2



① Every vertex in V_1 has degree at most d

$(|V_1| = |V_2| = n)$

<u>What are α and β?</u>

② For any subset S of vertices from V_1 s.t. $|S|$ is at most αn and there are at least $\beta|S|$ neighbors in V_2. ☆

- Ideally, d should be small & β as large as possible

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Advantage of expanders = no matter where the sources are, you can reach out to large portion of destinations very quickly also $\bar{c}$ very low condit^n if the underlying graph is an expander.

- How do we construct these expander graphs?

- we use randomisation to build expanders but verification that it is a expander is difficult.

- difficult in deterministic manner.

- So simple randomisat^n construct^n & its verificat^n.

No. of possible S sets $= \underbrace{\sum_{i=1}^{\alpha n} \binom{n}{i}}$

so $i = 1$ to $n/3$ (very large)
→ not possible to verify for each set

in our eg. $d = 18$
$\alpha = 1/3$
$\beta = 2$
& any value of 'n'

Proof: (PTO)

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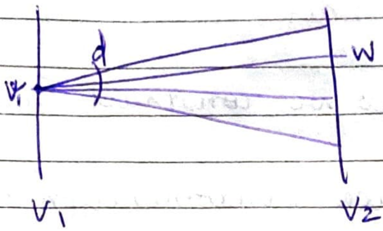
Let each vertex v_i in V_1 choose d neighbours in V_2 by sampling independently and uniformly at random.

Pr(of picking 'w' in a round) $= \frac{1}{n}$



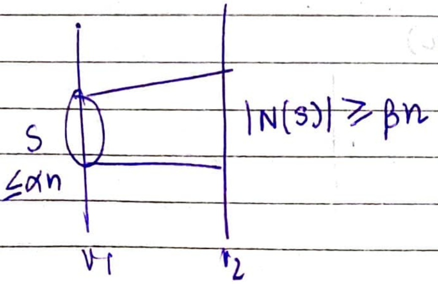
→ sample $\bar{c}$ replacement

↳ so 'w' can be picked more than once but we will keep only one copy of edge b/w v_i & w

So, $deg(v_i) \le d$
where d is the no. of choices

⇒ each vertex has degree at most 'd'

2. $N(S) \geq \beta|S|$



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→ Let us fix a parameter $s \le \alpha n$

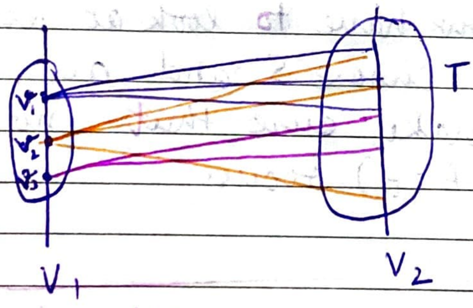
Consider any subset S of V_1 with $|S| = s$

Let T be any subset of V_2 of size $|T| = \beta s$

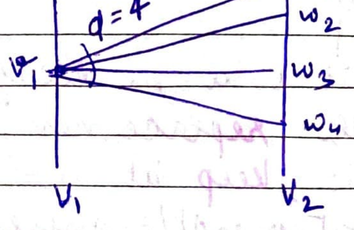
Now, consider the event that all the neighbours of S are in T — $\{E_{ST}\}$

E_{ST} is a bad event, because it means the neighbours have size at most βs, but we want $|T| \ge \beta s$ (at least).

So,



So, let us look at how these v_i made these choices s.t. all in set T. Let us look at V_1:



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So, P(at a trial v_1 picks only within set T)
(out of d trials)

$$= \frac{|T|}{n} = \frac{\beta s}{n}$$

Now, this choice was made 'd' times

so, $\left(\frac{\beta s}{n}\right)^d$

Now, not only v_1 all vertices in set S did this —

$$\text{So, } P_r(E_{ST}) = \left(\left(\frac{\beta s}{n}\right)^d\right)^s = \left(\frac{\beta s}{n}\right)^{ds} \quad \text{— (i)}$$

→ We now have to look at all possible ways to choose S and all possible T and make sure that no such bad event (E_{ST}) occurs.

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why replacement allowed?

↳ otherwise to find prob. we have to subtract one from ~~num~~ ~~nu~~ num & denom

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it will be messy.

and with replacement we will be able to keep "at most d" expander statement/requirement

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So, now we want to find Pr that no E_{ST} occurs.

Many ways of choosing S ($|S| = s$)

$$= {}^{n}C_{s}$$

ways of choosing T $= {}^{n}C_{\beta s}$

So many possible E_{ST} and we are looking at the union of these events:
(to see if at least one E_{ST} happens) —
and then we dont want this to happen.

Let us use Boole's inequality to upper bound the prob. of the event that for some S all its neighbour in T.

$$E_s = \bigcup_{S,T} E_{ST} \quad \text{—— (ii)}$$

$$Pr(E_s) \le \sum_{S,T} Pr(E_{ST})$$

$$Pr(E_s) \le {}^{n}C_{s}\, {}^{n}C_{\beta s}\, Pr(E_{ST})$$

$$Pr(E_s) \le {}^{n}C_{s}\, {}^{n}C_{\beta s}\, (\beta s/n)^{ds}$$

simplifying this using the stirling inequality / approximat^n

$$\left(E = \bigcup_{s} E_s\right) \text{ — (iii)}$$

because in eq. we fixed s but our condition said s could be 1 to αn.

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Sterling approximation

for any n & k $(1 \le k \le n)$

$${}^{n}C_k \le (en/k)^k$$

So,

$$P(E_s) \le {}^{n}C_s \cdot {}^{n}C_{\beta s} \cdot (\beta s/n)^{ds}$$

$$\le \left(\frac{en}{s}\right)^s \times \left(\frac{en}{\beta s}\right)^{\beta s} \times \left(\frac{\beta s}{n}\right)^{ds}$$

$$\le e^{s+\beta s} \cdot n^{s+\beta s-ds} \cdot \beta^{ds-\beta s} \cdot s^{ds-s-\beta s}$$

$$\le \left(e^{1+\beta} \cdot n^{1+\beta-d} \cdot \beta^{d-\beta} \cdot s^{d-1-\beta}\right)^s$$

$$\le \left[\left(\frac{s}{n}\right)^{d-\beta-1} \cdot e^{1+\beta} \cdot \beta^{d-\beta}\right]^s \quad \text{--- (iv)}$$

a little insight on how we prove existence of expander - α, β, n, d =

$$\because E_s = \bigcup_{S,T} E_{S,T}$$

and $E = \bigcup_s E_s$

we want $Pr(E) < 1$. Why? Because $\bar{E}$ (read E-complement) is a good event. and $\bar{E} = 1 - P(E)$

So, if $Pr(E) < 1$ we can say $Pr(\bar{E}) > 0$ and by st. 2 we can say it exists.

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Now, using '$s \le \alpha n$' for $\alpha = 1/3$

$$s \le \alpha n$$

$$\frac{s}{n} \le \alpha = \frac{1}{3} \quad \text{—— (v)}$$

(v) in (iv)

$$P(E_s) \le \left[\left(\frac{1}{3}\right)^{d-\beta-1} \cdot e^{1+\beta} \cdot \beta^{d-\beta} \right]^s$$

$$\cancel{\le \left[\left(\frac{1}{3}\right)^{-1} \cdot e^{1+\beta} \cdot \left(\frac{\beta}{3}\right)^{d-\beta} \right]^s}$$

$$\le \left[\left(\frac{1}{3}\right)^{-(1+\beta)} \cdot e^{1+\beta} \cdot \left(\frac{\beta}{3}\right)^{d} \cdot \overset{1}{\cancel{\beta^{-\beta}}} \right]^s$$

when $\beta > 1$, ~~this term~~ is a fraction

$$\le \left[(3e)^{1+\beta} \cdot (\beta/3)^d \right]^s$$

$(d = 18, \ \beta = 2)$

$$\le \left[\left(\frac{2}{3}\right)^{18} (3e)^3 \right]^s$$

$(e < 3, \ (3e)^3 \le 3^6)$

$$\le \left(\left(\frac{2}{3}\right)^{18} \cdot 3^6 \right)^s$$

$$\le \left(2^{18} \cdot 3^{-12}\right)^s$$

$$\le (1/2)^s$$

(if $\beta\uparrow$, $\alpha\uparrow$)

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We used specific s. But, we need to show the result for all s b/w 1 to αn.

Apply Boole's inequality

$$\Pr(E) \le \sum_{s=1}^{\alpha n} \Pr(E_s)$$

$$\le \sum_{s=1}^{\alpha n} \left(\frac{1}{2}\right)^s$$

$$\le \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{\alpha n+1}}{1 - (1/2)}$$

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lec 7 + Lec 8 : 2nd Feb '21

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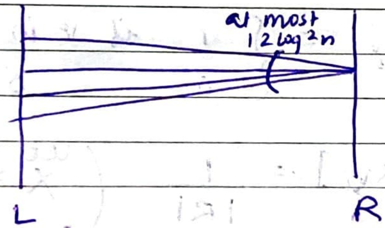
Another Example

Bipartite graph $G = (L \cup R, E)$ such that
$|L| = n$
$|R| = 2^{\log^2 n}$ } so, $|L| \neq |R|$

Condition 1: Every subset of $n/2$ vertices of L has at least $2^{\log^2 n} - n$ neighbours in R. $\Rightarrow$ $\underbrace{N(S) \geq 2^{\log^2 n} - n}_{*}$

Condition 2: No vertex of R has more than $12 \log^2 n$ neighbours.



* this means except n vertices in R all other vertices in R are neighbours of set S. ($|S| = n/2$)

Use Proof by existence to prove =
Here, is the experiment we will do-
→ let every vertex of L choose d neigh-bours in R independently and uniformly at random.
→ choices are made with replacement
→ Multiple edges are dropped in favour of one edge.

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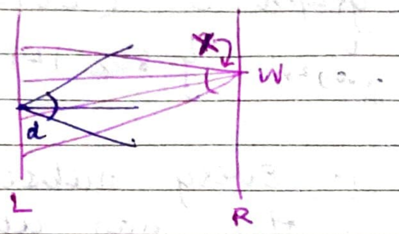
Cond 2: Consider any vertex w in R. Show that

$E(\deg(w)) = 4 \log^2 n$

Chernoff →

$\deg(w) \leq 12 \log^2 n$.

find a 'd' for the above to happen.



Every $v \in L$ makes 'd' choices.

$$\underset{w \in R}{E}[\deg(w)] = ?$$

$$X_{vi} = \begin{cases} 1 & \text{if } v \text{ in } i^{th} \text{ choice picks } w \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_{v,i}] = \frac{1}{|R|}$$

$\left(\text{let } X = \sum_{v \in L} \sum_{i=1}^{d} X_{v,i}\right)$

$\deg(w) \leq X$

considering replacement

$\boxed{n = |L|}$

$$E[X_w] = \sum_{v \in L} \sum_{i=1}^{d} E[X_{v,i}]$$

$$E[X_w] = n \times d \times \frac{1}{2^{\log^2 n}}$$

so X is all vertices that chose w ⇒ deg(w)

(We need to find a d s.t. E(degree (w)) = $4 \log^2 n$)

So, $n \times d \times \frac{1}{2^{\log^2 n}} = 4 \log^2 n$

$$\Rightarrow d = \frac{4 \log^2 n \times 2^{\log^2 n}}{n}$$

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$u \in [0,1]$
$k \geq 1$

Lecture 8: 2nd Feb '21

Applying chernoff bound –

$$\Pr(X_w \geq 12 \log^2 n)$$
$$= \Pr(X_w \geq 3 \cdot E[X]) \leftarrow \begin{pmatrix}(\delta+1)\mu \\ (2+1)\mu\end{pmatrix}$$
$$\leq e^{-\frac{\mu}{4}\delta \ln \delta}$$
$$\leq e^{-4\log^2 n \times 2\ln 2}$$

(But here X is defined only for w)
So, we use BOOLE'S INEQUALITY

$$\Pr(\text{some } u \text{ has degree} \geq 12\log^2 n) = \Pr\left(\bigcup_{w \in R} X_w\right)$$
$$\leq |R| \; e^{-4\log^2 \cdot 2\ln 2} \qquad \left[\; |R| = 2^{\log^2 n} \leq e^{\log^2 n} \;\right]$$
$$\leq e^{-3\log^2 n}$$
$$< \frac{1}{n^c}$$

Cond 1: Every subset of L of size $\leq n/2$ has $\geq$ (at least) $2^{\log^2 n} - n$ neighbours in R.

Pick a subset S of size $n/2$ in L and pick a subset T of size $2^{\log^2 n} - n$ in R

Now, we are interested in the event
$E_{ST} = \{N(S) \subseteq T\}$

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$$Pr(E_{S,T}) = \left(\frac{|T|}{|R|}\right)^{d\times\frac{n}{2}}$$

$$Pr(E_{S,T}) = \left(1-\frac{n}{|R|}\right)^{dn/2} \qquad (\because |T| = |R|-n)$$

\# ways of choosing S : ${}^{n}C_{n/2}$

\# ways of choosing T : ${}^{2\log^2 n}C_{2\log^2 n-n}$

$$E_S = \bigcup_{S,T} E_{S,T} \quad \text{(Boole's)}$$

$$Pr(E_S) \le \sum_{S,T} Pr(E_{S,T})$$

$$Pr(E_S) \le {}^{n}C_{n/2}\ {}^{|R|}C_{|R|-n}\ Pr(E_{S,T})$$

$$Pr(E_S) \le {}^{n}C_{n/2}\ {}^{|R|}C_{n}\ Pr(E_{S,T})$$

$$Pr(E_S) \le {}^{n}C_{n/2}\ {}^{|R|}C_{n} \left(1-\frac{n}{|R|}\right)^{dn/2}$$

$$Pr(E_S) < (2e)^{n/2}\left(\frac{e|R|}{n}\right)^{n} e^{-\frac{n^2 d}{2|R|}}$$

substitue d

$$e^{-\frac{n^2 d}{2|R|}} \qquad d = \frac{|R|}{|L|}\,4\log^2 n$$

$$\le e^{-\frac{n^2 \cancel{|R|}}{2\cancel{|R|}\,|L|}\,4\log^2 n}$$

$$\rightarrow \boxed{2^{\log^2 n} = n^{\log n} = e^{\log^2 n}}$$

$$\rightarrow \boxed{\log|R| = \log^2 n}$$

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$$\leq 2^{n/2} \cdot e^{\frac{n}{2}+n-\frac{n}{2}\cdot 2\log^2 n} \cdot |R|^n \cdot n^{-n}$$

$$\cdot e^{n\log^2 n} \leftarrow (n^{\log n})^n \leftarrow (2^{\log^2 n})^n \quad e^{-n\log n}$$

$$\leq 2^{n/2} \cdot e^{\frac{n}{2}+n-\frac{n}{2}\cdot 2\log^2 n + n\log^2 n - n\log n}$$

(replace 2 ~ e)

$$\leq e^{n+n-2n\log^2 n + n\log^2 n - n\log n}$$

$$\leq e^{2n - 2n\log^2 n - n\log n}$$

$$\leq e^{-\frac{n}{2}\log^2 n} \quad \text{(HOW?)}$$

So, $-\frac{n}{2}\log^2 n$ will cancel $2n - n\log n$

and $-\frac{n}{2}\log^2 n$ will remain

$$\leq \frac{1}{n^{O(1)}} \quad \Rightarrow P(E) < 1$$

$$\Rightarrow Pr(E) > 0$$