- Complexity and Advanced Algorithms
- Spring 2020

Fractional Independent Set (FIS)

- Let G = (V, E) be an undirected graph.
- We say that a set $S \subseteq V$ is an independent set if no two vertices of S are mutual neighbors in G.
- The notion of an independent set is very popular in graph theory.
- Several variants are also studied:
  - Maximal Independent Set (MIS)
  - Maximum Independent Set
  - Fractional Independent Set

Fractional Independent Set

- We now define an FIS.
- Let $G = (V, E)$ be an undirected graph. A set $S \subseteq V$ is called a (c,d)-fractional independent set of G if it satisfies:
  - S is an independent set
  - For every vertex v in S, degree(v) is at most d.
  - $|S|$ is at least $|V|/c$.

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- Not every graph may have an FIS.
- Planar graphs have an FIS.
  - We will see that today, along with a way to construct such an FIS.

Planar Graphs

- Recall the theorem of Euler concerning planar graphs.
- Theorem (Euler): If G = (V, E) is planar with |V| at least 3, then |E| is at most 3|V| - 6.
- Using the above theorem, can show that in a planar graph G, there are lots of vertices of a degree at most d.
- Theorem: Let G = (V, E) be a planar graph and d be an integer at least 6. Let Vd be the set of vertices of degree at most d. Then, |Vd| is at least |V|/c for some constant c.

Planar Graphs

- Theorem: Let G = (V, E) be a planar graph and d be an integer at least 6. Let V_d be the set of vertices of degree at most d. Then, $|V_d|$ is at least $|V|/c$ for some constant c.
- Proof: Let V_h be the complement of V_d.
- We will estimate an upper bound on the size of Vh as follows.
- Consider Σ_v degree(v) $\geq \Sigma_{v\in V_h}$ degree(v) $\geq (d+1)|V_h|$.

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- Using Euler's theorem, we get that $(d+1)\ |V_h| \leq 2(3|V| - 6)$.
- So, $|V_d| \geq |V| - |V_h| \geq |V|. (d - 5)/(d+1)$.

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- With $d = 6$, we get that $|V_6|$ is at least $|V|/7$.

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- With d = 6, we get that $|V_6|$ is at least $|V|/7$.
- Can be used to show that in a sequential setting, an FIS for a planar graph can be found.
  - Start with any vertex v.
  - If v has a degree at most 6, add v to the set S.
  - Remove all the neighbors of v.
  - Continue until there are more vertices.
- Can show that $|S|$ is at least $|V_d|/7$.

FIS

- The sequential algorithm is not efficient in parallel.
- Need a better approach where multiple nodes decide to join the FIS or not on their own.

Parallel FIS on Planar Graphs

- Consider each vertex of degree at most 6.
- For each such vertex, set label(v) = 1 with probability ½ and set label(v) = 0 with probability 0.
- Note that several vertices and their neighbors may choose their label as 1.
- So, the set of vertices with label set to 1 is not independent.

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- To make this set independent, we proceed as follows.

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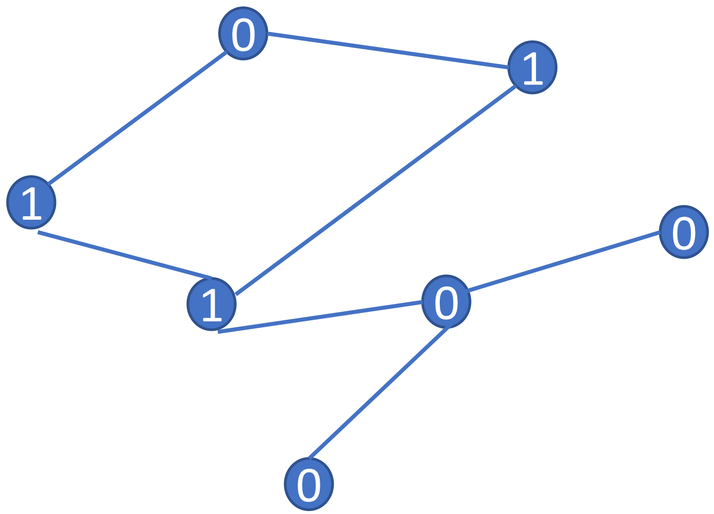

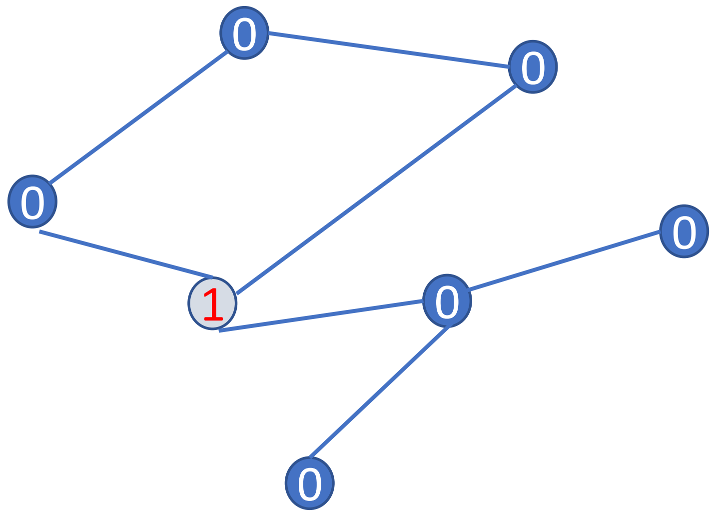

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- So, the set of vertices with label set to 1 is not independent.
- To make this set independent, we proceed as follows.
- If a node v of degree at most 6 has label(v) = 1 and all its neighbors have label 0, then v enters a set S.
- Otherwise, v drops out.

Parallel FIS on Planar Graphs

- We want to claim that S is a (c, 6)—FIS for some constant c.
  - Only for planar graphs of course.
- S is indeed an independent set.
- Moreover, the degree of any vertex in S is at most 6.
- So, we only have to find a suitable value for c.

Parallel FIS on Planar Graphs

- For a vertex v with degree at most 6, note that Pr(v has label 1 and v is in S) is at least $1/2^7$.
  - For the event to occur, v has to pick 1 as its label, and the neighbors of v (of degree at most 6) have to pick 0 as their label.
- Let us use 1/128 as the actual probability of the above event going forward.
- Now, we can note that since $|V_6|$ is at least $|V|/7$, on expectation, the number of vertices in S is at least $|V|/(7 \times 128)$.

Parallel FIS on Planar Graphs

- We wish that S has a large size not just in expectation, but also with high probability.
- We have E|S| is at least |V|/(7x128).
- It appears that we can use Chernoff bounds to show that the size of S is close to its expectation.
- But, the random variables that we use are not independent.
- In particular, for two neighbors v and w of small degree, if v is in S then w cannot be in S.
- The events v in S, and w in S are therefore not always independent.

Parallel FIS in Planar Graphs

- There are several ways to deal with this lack of independence.
- One such way is to consider only a subset of random variables that are then independent of each other.
- In the present case, we will consider only vertices of degree at most 6 and are at least a distance of 3 apart from each other.

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- One such way is to consider only a subset of random variables that are then independent of each other.
- In the present case, we will consider only vertices of degree at most 6 and are at least a distance of 3 apart from each other.
- The random variables corresponding to such vertices are independent.

Parallel FIS in Planar Graphs

- Let V’ be the set of vertices of degree at most 6 .
- We observe that |V’| is at least |V|/36.
- Now, define an indicator random variable for each v in V’ so that this RV takes value 1 if v is in S.
- Define X as the sum of these random variables.
- Note that EX is at least |V’|/(7x128), that is now at least |V|/(36x7x128).
- Use Chernoff bounds to show that Pr(X $\leq$ EX/2) is at most exp{-EX.1/12} which is polynomially small.