Advanced Algorithms Assignment 1

Vijayraj Shanmugaraj, 20171026

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Exercise 1

1.1 The event that every 10th throw lands on Heads. Assume that n is divisible by 10, and also that we are not

For any 10 throws, we would want the following sequence:

H/T H/T ... H/T (9 times) H

The probability of such a sequence = $1^9 \times 0.5 = 0.5$ There will be $\frac{n}{10}$ such sequences. Probability of getting $\frac{n}{10}$ such sequences in a row would be $0.5^{0.1n}$ (we multiply probabilities as the individual events are independent).

1.2 The event that there is no head in any consecutive 10 log n throws.

Taking the question literally, the answer is as follows:
Let us take k instead of $10logn$ first. The only sequence that would not have any head in any consecutive k throws would be

T T ... T (n times) ,

since even a single head at position i would include it the window $[max(i, n-k+1), min(i+k-1, n)]$, among other possible windows.

The probability of this sequence would be 0.5^n. Assuming that $10\log(n) = \lfloor 10\log(n) \rfloor$, we need to make sure $10\log(n) \leq n$, assuming $\log_{10}(n)$, $n \leq 1$ or $n \geq 10$.

$$\begin{aligned} P(\text{No head in any consecutive 10log(n) throws}) &= 1 \text{ , n = 1} \\ &= 0 \text{ , n > 1 and n < 10} \\ &= 0.5^n \text{ , n} \geq 10 \end{aligned}$$

(Since n = 1, 10log(n) would be 0, and hence the probability of getting 0 throws with no heads is 1).
Assuming that the question meant the existence of at least 1 consecutive sequence of 10log(n) throws without any heads in it, then this would be the answer:

Let us take k instead of $10log(n)$ first. We will first see the probability that a set of throws starting from throw i would be all tails and at least k consecutive throws are tails. In this case, throw $i-1$ should be a head, and the sequence $1...i-2$ should not have a sequence of $length \geq k$. For the case where the sequence starts from the first coin, we don't need the extra 0.5 for a head in the $i-1$th throw. Also, when $i = 2$, there is no sequence $1...i-2$. So the probability of that sequence would be P(head in throw 1)*P(k tails after) = $0.5 \cdot 0.5^k$ So, the formula would be as follows:

$$\begin{aligned} P(X, n, k) &= 0.5^k + 0.5^{k+1} + \sum_{i=3}^{n-k+1} (1 - P(X, i-2, k)).0.5 \times (0.5^k) \\ &= 0.5^k + 0.5^{k+1} + 0.5^{k+1} \sum_{i=3}^{n-k+1} (1 - P(X, i-2, k)) \end{aligned}$$

Where P(X, z, i) is the probability of z coins having at least one sequence of tails with length i. And, P(X, z, i) = 0 if i > z and P(X, i, i) = $\frac{1}{2^i}$.

$$\begin{aligned} P(\text{Required sequence}) &= 1 \text{ , n = 1} \\ &= 0 \text{ , n > 1 and n < 10} \\ &= 0.5^{10} \text{ , n = 10} \\ &= 0.5^{10log(n)} + 0.5^{10log(n)+1} + 0.5^{10log(n)+1} \sum_{i=2}^{n-10log(n)+1} (1 - P(X, i-2, 10log(n))) \text{ , n > 10} \end{aligned}$$

Again, assuming that $10\log(n) = \lfloor 10\log(n) \rfloor$, and the base of log is 10.

Exercise 2

The function for conditional probability is as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad [2.1]$$

Axiom 1:The probability of an event is a non-negative real number
Let B be such that $P(B) \neq 0$, since $P(B) = 0$ would imply that $P(A \cap B) = 0$ which would render the function meaningless.

Now by the first axiom of probability we know that $P(B) \geq 0$, so $P(B) > 0$ and $P(A \cap B) \geq 0$, which implies that $P(A|B) \geq 0$

Axiom 2:The probability of the whole sample space is equal to one
Let Ω be the sample space

$$\begin{aligned} P(\omega|B) &= \frac{P(\Omega \cap B)}{P(B)} \\ P(\omega|B) &= \frac{P(B)}{P(B)} = 1 \text{ (Since, } \Omega \cap B = \text{B)} \end{aligned}$$

Hence, it satisfies the 2nd axiom as well.

Axiom 3: If a set of events are disjoint (i.e., mutually exclusive), then the probability of their union must be the summations of their probabilities.

Take n events A_1, A_2, ... A_n such that they are all mutually exclusive. Since they are mutually exclusive, the events $(A_1 \cap B)$, $(A_2 \cap B)$, ... $(A_n \cap B)$ are mutually exclusive as well. Now,

$$\begin{aligned}
P(A_1 \cup A_2 \cup ...A_n|B) &= \frac{P((A_1 \cup A_2 \cup A_n) \cap B)}{P(B)} \\
&= \frac{P((A_1 \cap B) \cup (A_2 \cap B) \cup (A_n \cap B))}{P(B)} \\
&= \frac{P(A_1 \cap B) + P(A_2 \cap B) + ...P(A_n \cap B)}{P(B)} \text{Since they are mutually exclusive} \\
&= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} + ...\frac{P(A_n \cap B)}{P(B)} \\
&= P(A_1|B) + P(A_2|B) + ...P(A_n|B)
\end{aligned}$$

Hence the conditional probability function satisfies the third axiom as well.

Exercise 3

Note: I have assumed that the T-shirts are distinct in this following question

3.1 The expected number of participants who do not get any T-shirt.

Let us define a random variable X_i such that $X_i = 1$ if person i ends with no shirts, and 0 otherwise. Let us define random variable X such that

$$X = \sum_{i=1}^{n} X_i \qquad [3.1]$$

The expectation of X is what we're looking for, since X is the total number of people who end up with zero shirts. So,

$$E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) \qquad [3.2]$$

Now, to calculate $E(X_i)$. The probability that person i doesn't get chosen for a T-Shirt = $\frac{n-1}{n}$. The probability that person i doesn't get chosen for all m T-shirts is $(\frac{n-1}{n})^m$. $E(X_i) = 1 \times P(X_i = 1) + 0 \times P(X_i = 0) = P(X_i = 1)$ Hence,

$$E(X_i) = \left(\frac{n-1}{n}\right)^m \text{, and hence}$$
$$E(X) = n \times E(X_i) = n. \left(\frac{n-1}{n}\right)^m$$

3.2 The expected number of participants who get exactly one T-shirt.

Here, let's define Y_i such that $Y_i = 1$ if person i ends with exactly 1 shirt, and 0 otherwise, and

$$Y = \sum_{i=1}^{n} Y_i \qquad [3.3]$$

Similar to X_i, $E(Y_i) = P(Y_i = 1)$ Now, there are $\binom{m}{1}$ ways to chose a T-shirt for that person, and the probability that the person gets exactly a particular T-shirt is $\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^{m-1}$. Hence,

$$P(Y_i = 1) = \binom{m}{1}\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^{m-1}$$

Hence,

$$\begin{aligned} E(Y) &= n \times E(y_i) \\ &= \binom{m}{1}\left(\frac{n-1}{n}\right)^{m-1} \end{aligned}$$

3.3 The probability that some participant gets more than $10m\log(n)/n$ shirts

When a particular shirt is taken, the probability that it ends up with a particular person i is $\frac{1}{n}$. Let X_{ij} be a random variable such that $X_{ij} = 1$ if person i gets the j'th t-shirt, and 0 otherwise. Now,

$$E(X_{ij}) = 1 \times P(\text{Person i gets the T-shirt}) + 0 \times P(\text{Person i doesn't the T-shirt}) = \frac{1}{n}$$

Let $X_i = \sum i = 1mX_{ij}$. X_i will give us the expected number of T-shirts person i will get.

$$X_i = \sum_{i=1}^{m} X_{ij} = m \times \frac{1}{n} = \frac{m}{n} \tag{3.4}$$

Now applying Chernoff bounds on X_i,

$$\begin{aligned} \mu &= \frac{m}{n} \\ \mu(1+\delta) &= 10m\log(n)/n + 1 \\ 1 + \delta &= 10\log(n) + \frac{n}{m} \end{aligned}$$

By chernoff bounds,

$$\begin{aligned} P(X_i \geq \mu(1+\delta)) &\leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu} \\ &= \left(\frac{e^{10\log(n)+\frac{n}{m}-1}}{(10\log(n)+\frac{n}{m})^{10\log(n)+\frac{n}{m}}}\right)^{\frac{m}{n}} \end{aligned}$$

This is applicable only when $n > 1$. If n = 1, the probability that someone gets more than 0 shirts is 1.

Exercise 4

Say X and Y are defined as follows:

- X = 1 with probability 0.5, and 4 with probability 0.5.

- Y = 2 with probability 0.5, and 3 with probability 0.5.

$$E(X) = 1 \times 0.5 + 4 \times 0.5 = 2.5$$
$$E(Y) = 2 \times 0.5 + 3 \times 0.5 = 2.5$$

E(X) = E(Y), but clearly, both X and Y have different distributions. So, having the same expectation does not mean both variables have the same distribution.

Now take X and Y as follows

- X = -1 with probability 0.5, and 1 with probability 0.5.
- Y = $\sqrt{3}$ with probability $\frac{1}{3}$, 0 with probability $\frac{1}{3}$, and $-\sqrt{3}$ with probability $\frac{1}{3}$.

$$\mu_X = 0.5 \times 1 + 0.5 \times -1 = 0$$
$$\mu_Y = \sqrt{3} \times \frac{1}{3} + 0 \times \frac{1}{3} + (-\sqrt{3}) \times \frac{1}{3} = 0$$

$$\text{Since } \mu_X = 0, \sigma_X^2 = E(X^2) = 0.5 \times 1^2 + 0.5 \times (-1)^2 = 2$$
$$\text{Since } \mu_Y = 0, \sigma_Y^2 = E(Y^2) = 3 \times \frac{1}{3} + 0 \times \frac{1}{3} + 3 \times \frac{1}{3} = 2$$

E(X) = E(Y) and var(X) = var(Y), but clearly, both X and Y have different distributions. So, having the same expectation and variation does not mean both variables have the same distribution.