Advanced Algorithms Homework 3

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Exercise 1

Say X_{vu} is an indicator random variable that takes the value of 1 if the edge from vertex v (in L) to vertex u (in R) exists, else takes the value of 0.

$$\begin{aligned} E(X_{vu}) &= 1 \cdot P(X_{vu} == 1) + 0 \cdot P(X_{vu} == 0) \\ &= \frac{n^{3/4}}{|R|} \text{(Since vertex v has } n^{3/4} \text{ choices to make out of |R| vertices)} \\ &= \frac{n^{3/4}}{n} \\ &= n^{-1/4} \end{aligned}$$

Now, let's define X_u for some u in R as follows:

$$X_u = \sum_{v \in L} X_{uv}$$

So,

$$\begin{aligned} E(X_u) &= \sum_{v \in L} E(X_{uv}) \\ &= |L| \cdot E(X_{uv}) \\ &= n \cdot n^{-1/4} \\ &= n^{3/4} \end{aligned}$$

So,

$$\begin{aligned} P(X_u \geq 3n^{3/4}) &= P(X_u \geq 3 \cdot E(X_u)) \\ &\leq e^{\mu \delta ln(\delta)} \end{aligned}$$

Now

$$1 + \delta = 3 \implies \delta = 2$$

Hence,

$$\begin{aligned} P(X_u \geq 3n^{3/4}) &\leq e^{-n^{3/4} \cdot 2ln(2)} \\ &= e^{-n^{3/4} \cdot ln(4)} \\ &= (e^{ln(4)})^{-n^{3/4}} \\ &= \left(\frac{1}{4}\right)^{n^{3/4}} \\ &\leq \left(\frac{1}{2}\right)^{n^{3/4}} \end{aligned}$$

Now by boole's inequality,

$$\begin{aligned} P(\text{Any } u \in R \text{ has degree } \geq 3n^{3/4}) &\leq \sum_{u \in R} P(X_u \geq 3n^{3/4}) \\ &= \frac{n}{2^{n^{3/4}}} \\ &= \left(\frac{n^{4/3}}{2^n}\right)^{3/4} \end{aligned}$$

Now we know that the above function is positive for all n. So we can say that the function lies between zero and it's maxima. To find the maximum value, differentiate the function within the power (since $x^{0.75}$ is an increasing function, the maxima of the function coincides with the maxima of the function inside the power)

$$\frac{d(f(n))}{dn} = 0, \text{Where } f(n) = \left(\frac{n^{4/3}}{2^n}\right)$$

$$f'(n) = \frac{(4/3)n^{1/3} \cdot 2^n - ln(2) \cdot 2^n \cdot n^{4/3}}{2^{2n}} = 0 \implies n = \frac{4}{3ln(2)} \approx 1.92$$

Now the slope for the function is negative when

$$\frac{(4/3)n^{1/3} \cdot 2^n - ln(2) \cdot 2^n \cdot n^{4/3}}{2^{2n}} < 0 \implies n > \frac{4}{3ln(2)}$$

So, the function is decreasing from n = 1.92. Now, $f(1.92) \approx 0.6$. And f(1) = 0.5. Since the function is decreasing, and the above function tends to 0 as $t \longrightarrow \infty$, we can say that Any $u \in R$ has degree $\geq 3n^{3/4}) = P(E) < 1$ for all n. Hence, $P(\overline{E}) > 0 \; \forall n$

Now for the second part, let us choose $S \subseteq L$ such that $|S| = n^{3/4}$ and $T \subseteq R$ such that $|T| = n - n^{3/4}$. Say E_{ST} is the event where Neighbours(S) $\subseteq T$, then (say d is the number of neighbour choices a vertex in L can make):

$$P(E_{ST}) = \left(\frac{|T|}{|R|}\right)^{|S| \cdot d}$$

$$\text{No. of ways of choosing S} = \binom{n}{n^{3/4}}$$

$$\text{No. of ways of choosing T} = \binom{n}{n - n^{3/4}} = \binom{n}{n^{3/4}}$$

Let E be the event where $Neighbours(S) \subseteq T$ for any S and any T. Then,

$$P(E) = \binom{n}{n^{3/4}} \cdot \binom{n}{n^{3/4}} \cdot P(E_{ST})$$

(Substituting $|S| = n^{3/4}$, $|T| = n - n^{3/4}$, $d = n^{3/4}$, $|R| = n$),

$$P(E) = \binom{n}{n^{3/4}}^2 \cdot \left(1 - n^{-1/4}\right)^{n^{3/2}}$$

Using Stirling's inequality ($\binom{n}{k}$ is at most $\left(\frac{en}{k}\right)^k$ for k in [1, n]).

$$P(E) \leq (en^{1/4})^{2n^{3/4}} \cdot \left(1 - n^{-1/4}\right)^{n^{3/2}}$$

Now using $1 + x \leq e^x$,

$$\begin{aligned} P(E) &\leq (en^{1/4})^{2n^{3/4}} \cdot e^{-n^{5/4}} \\ (\text{Note that } n^{1/4} &= e^{0.25 \ln(n)}) \\ &= \exp((2 + 0.5 \ln(n) - n^{1/2}) \cdot n^{3/4}) \end{aligned}$$

Say

$$f(n) = 2 + 0.5 \ln(n) - n^{1/2}$$

Let us analyse this function. Now,

$$f'(n) = \frac{1}{2n} - \frac{1}{2\sqrt{n}}$$

$f'(n) < 0$ for n > 1. Also, by graphing methods, we can determine that

$$f(n) = 0 \text{ for } n \approx 9.89$$

Since $f'(n) < 0$ for n > 1 (it is a decreasing function), We can say that f(n) < 0 for $n \geq 10$. Hence, for $n \geq 10$, $\exp((2 + 0.5 \ln(n) - n^{1/2}) \cdot n^{3/4}) < 1$. And hence $P(\bar{E}) > 0$ (i.e probability of every subset S such that $|S| = n^{3/4}$ $S \subseteq L$ has at least $n - n^{3/4}$ neighbors in R) $\forall n \in \mathbb{N}, n \geq 10$.

Exercise 2

We know that a function is concave on an interval if and only if its derivative f' is a monotonically decreasing function on that range. Now,

$$\begin{aligned} f(x) &= 1 - \left(1 - \frac{x}{k}\right)^k \\ f'(x) &= -k \cdot \left(1 - \frac{x}{k}\right)^{k-1} \cdot \left\{\frac{-1}{k}\right\} \\ &= \left(1 - \frac{x}{k}\right)^{k-1} \\ f''(x) &= (k-1) \cdot \left(1 - \frac{x}{k}\right)^{k-2} \cdot \left\{\frac{-1}{k}\right\} \\ &= -\left(\frac{k-1}{k}\right) \cdot \left(1 - \frac{x}{k}\right)^{k-2} \end{aligned}$$

Now,

$$\frac{k-1}{k} \geq 0 \; \forall k > 1$$

and

$$1 - \frac{x}{k} \geq 0 \; \forall x \in [0, 1]$$

(For k = 1, f''(x) = 0), making the slope monotonically decreasing (since it remains the same), so this applies for k = 1)

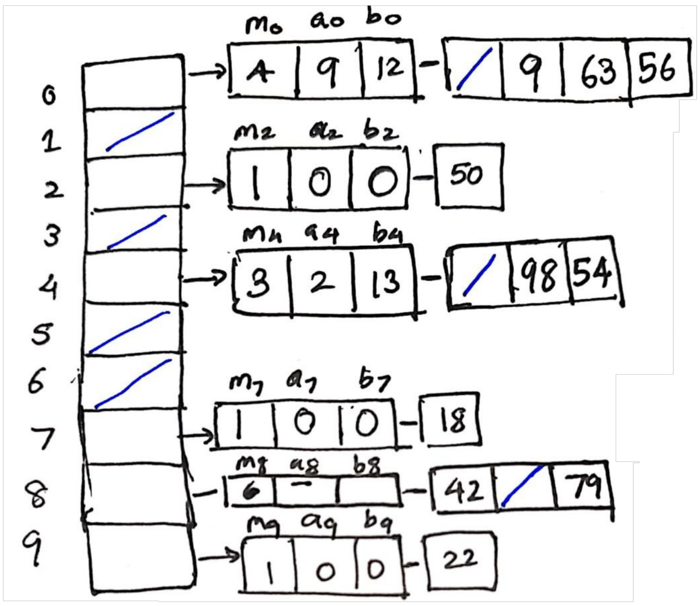
For $k > 1$, $\frac{x}{k} < 1$ and hence $1 - \frac{x}{k} > 0$. So, $(1 - \frac{x}{k})^k \geq 0$ for all x in [0, 1], k in $\mathbb{Z}^+ - \{1\}$. Hence $-\left(\frac{k-1}{k}\right) \cdot \left(1 - \frac{x}{k}\right)^{k-2} < 0$ for all k > 1. Hence f' is monotonically decreasing. And hence we have proved that f is concave for $k \geq 1$ and $x \in [0, 1]$.

Exercise 3

Keys to be hashed:

$$
\begin{aligned}
n(\text{number of keys}) &= m = 10\\
p &= 101\\
K &= \{42, 50, 9, 18, 22, 54, 98, 79, 63, 56\}\\
a &= 3, b = 23\\
\text{Base hash function } h(x) &= ((ax + b) mod p) mod m\\
\text{Second level hash functions:}&\\
h_i(x) &= ((a_i x + b_i) mod p) mod m_i
\end{aligned}
$$

Note that $m_8 = 3$, $a_8 = 6$, $b_8 = 7$, and blue dashes denote empty blocks.



Exercise 4

Say the variables are x_1, x_2 ... x_n. We take the boolean formula

$$C = C_1 \wedge C_2 \wedge \ldots \wedge C_2 n$$

Where

$$C_i = x_i, \text{ for } 1 \leq x \leq n$$
$$C_i = x'_{i-n}, \text{ for } n+1 \leq x \leq 2n$$

The variable x_k can only satisfy either C_k or C_{k+n} at a time, and therefore, only half of the clauses can be satisfied for a given truth value combination on all variables.

Now for exactly $\frac{3}{4}$ of the clauses to be satisfiable, we will first take two variables. We will take 4 clauses as shown below and we will construct the truth table for the same.

x_1	x_2	$x_1 \vee x_2$	$x_1 \vee x'_2$	$x'_1 \vee x_2$	$x'_1 \vee x'_2$
T	T	T	T	T	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	T	T	T

We can see that in any unique combination of the Boolean values taken by x_1 and x_2, only any three out of the four given clauses are satisfiable. Now, for n variables, we take $4(n-1)$ clauses as follows:

$$C = C_{11} \wedge C_{12} \wedge C_{13} \wedge C_{14} \wedge C_{21} \wedge \ldots \wedge C_{(n-1)1} \wedge C_{(n-1)2} \wedge C_{(n-1)3} \wedge C_{(n-1)4}$$

Where

$$C_{i1} = x_i \vee x_{i+1}$$
$$C_{i2} = x_i \vee x'_{i+1}$$
$$C_{i3} = x'_i \vee x_{i+1}$$
$$C_{i4} = x'_i \vee x'_{i+1}$$

Similar to the previous argument, for any boolean value picked up by x_i and x_{i+1}, only any three out of the four clauses C_{i1}, C_{i2}, C_{i3} or C_{i4} can be satisfied, and hence by that logic, only exactly $\frac{3}{4}$ of the $4(n-1)$ clauses can be satisfied for a given truth value combination on all variables.