Advanced Algorithms Homework 2

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Exercise 1

$X = \sum_{i=1}^{n} X_i$ where each X_i is a random variable that takes value 1 or -1 with probability 0.5. Take $Y_i = e^{t.X_i}$.

$$E(Y_i) = E(e^{t.X_i}) = 0.5(e^t + e^{-t}) \tag{1.1}$$

We take $Y = \prod_{i=1}^{n} Y_i$. Now,

$$\begin{aligned} E(Y) &= E(Y_1.Y_2...Y_n) \\ &= \prod_{i=1}^{n} E(Y_i) \\ &= (0.5(e^t + e^{-t}))^n \end{aligned}$$

Now, $Y = e^{t(X_1+X_2+...X_n)} = e^{TX}$ and

$$\begin{aligned} X \geq a &\implies e^{tX} \geq e^{ta} \\ &\implies Y \geq e^{ta} \end{aligned}$$

So,

$$\begin{aligned} Pr(Y \geq e^{ta}) &\leq \frac{E(Y)}{e^{ta}} \text{ (By Markov inequality)} \\ &= \frac{(0.5(e^t + e^{-t}))^n}{e^{ta}} \\ &= \frac{\left(0.5 \times 2 \times \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + ...\right)\right)^n}{e^{ta}} \text{ (Expanding using Taylor series for } e^x \text{ and } e^{e-x}\text{)} \\ &= \frac{\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + ...\right)^n}{e^{ta}} \end{aligned}$$

Now,

$$e^{\frac{t^2}{2}} = 1 + \frac{t^2}{2 \times 1!} + \frac{t^4}{2 \times 2!} + ...$$

Clearly,

$$\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + ...\right) \leq e^{\frac{t^2}{2}}$$

(Comparing coefficients of corresponding powers). So, we can write

$$\begin{aligned} Pr(Y \geq e^{ta}) &\leq \frac{e^{0.5.nt^2}}{e^{ta}} \\ &= e^{0.5.nt^2 - ta} \end{aligned}$$

Now for the minima, we take log of the above expression and find the minima of that. (The minima will lie at the same point for both equations since the log function is a strictly increasing function.

$$\begin{aligned} \frac{d(0.5.nt^2 - ta)}{dt} &= 0 \\ \implies t &= \frac{a}{n} \end{aligned}$$

Checking for sign of second derivative, second derivative = n > 0. So it is a minima.

Substituting it into the equation for $Pr(Y \geq e^{ta})$, we get

$$Pr(Y \geq e^{ta}) \leq e^{0.5\frac{a^2}{n} - \frac{a^2}{n}}$$
$$Pr(Y \geq e^{ta}) \leq e^{-\frac{a^2}{2n}}$$

Hence,

$$Pr(X \geq a) = e^{-\frac{a^2}{2n}} \qquad [1.2]$$

Exercise 2

Say $E(X_i) = \mu_i = (c + d)/2$

Take any random variable $M_i = X_i - \mu_i$. Clearly, $E(M_i) = 0$ Let us declare a variable Y_i such that $Y_i = e^{tM_i}$, t > 0. We will first try getting a simple upper bound on the expected value of Y_i, since the actual expression of Y_i would consist of the geometric progression of the terms $e^{t(c-\mu_i)}$, $e^{t(c-\mu_i+1)}$...$e^{t(d-\mu_i)}$, making it difficult to deal with in the later stages.

The equation of a line between two points (x_1, y_1) and (x_2, y_2) is:

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \qquad [2.1]$$

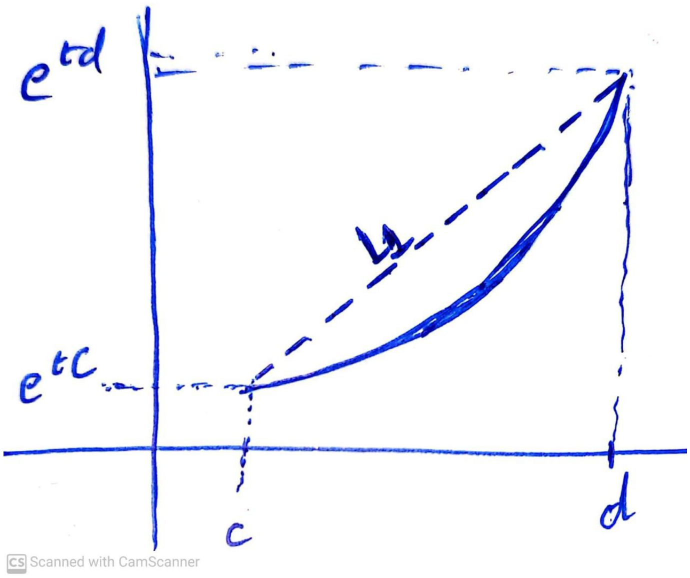
This can be refactored as follows:

$$\begin{aligned} y - y_1 &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \\ y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x - \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x_1 + y_1\frac{x_2 - x_1}{x_2 - x_1} \\ y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + \frac{y_1.x_2 - y_2.x_1}{x_2 - x_1} \end{aligned}$$

Refactoring terms, we get the final form as

$$y = \left(\frac{x_2 - x}{x_2 - x_1}\right) y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right) y_2 \qquad [2.2]$$

Now, take the two points $e^{t(c-\mu_i)}$ and $e^{t(d-\mu_i)}$. Say $C = c - \mu_i$ and $D = d - \mu_i$. Now, since e^{tx} is a convex function, the line between the points (c, e^{tc}) (Line L1 in the picture) and (d, e^{td}) will always be above $f(x) = e^{tx}$ for x ∈ (c, d).



So,

$$\begin{aligned} e^{tM_i} &\le \left(\frac{D - x}{D - C}\right) e^{tC} + \left(\frac{x - C}{D - C}\right) e^{tD} \\ E(e^{tM_i}) &\le \left(\frac{D}{D - C}\right) e^{tC} - \left(\frac{C}{D - C}\right) e^{tD} = f(t) \qquad \text{(Note that } E(M_i) = 0\text{)} \end{aligned}$$

Now, let $\gamma = \frac{-C}{D-C}$. $\gamma > 0$ since $C \le 0 \le D$. So,

$$\begin{aligned} f(t) &= (1 - \gamma)e^{tC} + \gamma e^{tD} \\ &= (1 - \gamma + \gamma e^{t(D-C)})e^{-t\gamma(D-C)} \end{aligned}$$

Let u = t(D-C). Define Q such that Q = log(f(t))

$$Q(u) = \ln(1 - \gamma + \gamma e^{u}) - u\gamma$$

Now, by Taylor's theorem, we know that for every real u, there exists v such that

$$Q(u) = Q(0) + u.Q'(0) + \frac{u^2}{2}.Q''(v) \qquad [2.3]$$

Now, Q(0) = 0. $Q'(0) = -\gamma + \frac{\gamma e^0}{1-\gamma+\gamma e^0} = 0$

$$Q''(v) = \frac{\gamma e^v(1-\gamma+\gamma e^v) - (\gamma e^v)^2}{1-\gamma+\gamma e^v)^2}$$
$$= \left(\frac{\gamma e^v}{1-\gamma+\gamma e^v}\right)\left(1 - \frac{\gamma e^v}{1-\gamma+\gamma e^v}\right)$$

Taking $t = \frac{\gamma e^v}{1-\gamma+\gamma e^v}$

$$Q''(v) = t(1-t) \leq \frac{1}{4} \qquad [2.4]$$

Using the above result in 2.3,

$$Q(u) \leq 0 + u.0 + \frac{u^2}{2}.\frac{1}{4} = \frac{u^2}{8} = \frac{t^2(D-C)^2}{8} = \frac{t^2(d-c)^2}{8} \qquad [2.5]$$

$(D - C = (d - \mu_i) - (c - \mu_i) = d - c)$ So,

$$E(e^{tM_i}) \leq exp\left(\frac{t^2(d-c)^2}{8}\right)$$
$$\implies E(e^{t(X_i - E(X_i))}) \leq exp\left(\frac{t^2(d-c)^2}{8}\right)$$
$$\begin{aligned} Pr(X \geq E(X)(1+\delta)) &= Pr(X - E(X) \geq E(X)\delta) \\ &= Pr(n(X_i - E(X_i)) \geq E(X)\delta) \\ &= Pr(e^{tn\times(X_i - E(X_i))} \geq e^{tE(X)\delta}) \\ &= Pr(e^{tnM_i} \geq e^{tE(X)\delta}) \\ &\leq \frac{E(e^{tnM_i})}{e^{tE(X)\delta}} \end{aligned}$$

Numerator = $E(e^{tM_i})^n$. So,

$$E(e^{tnM_i} \geq e^{tE(X)\delta}) \leq exp\left(\frac{nt^2(d-c)^2}{8} - tE(X)\delta\right)$$

Now to find the minima, we differentiate the log(RHS).

$$nt\frac{(d-c)^2}{4} - E(X)\delta = 0$$
$$\implies t = \frac{4E(X)\delta}{n(d-c)^2}$$

Double differentiating the log(RHS), we get $\frac{n(d-c)^2}{4} > 0$ for all t. So it is a minima. Substituting the value of t obtained into the final equation, we get

$$Pr(X \geq E(X)(1+\delta)) \leq exp\left(-\frac{2\delta^2 E(X)^2}{n(d-c)^2}\right) \qquad [2.6]$$

Where E(X) = $n\frac{(c+d)}{2}$

Exercise 3

Now, similar to the original Chernoff bounds proof, we take $Y_i = e^{-tX_i}$, and $Y = \prod_{i=1}^{n} Y_i$ ($Y = e^{-tX}$).

Now, assuming positive t just like the original Chernoff proof,

$X \leq \mu(1-\delta) \implies -tX \geq -t\mu(1-\delta) \implies (e^{-tX} = Y) \geq e^{-t\mu(1-\delta)}$

Note:$E[Y_i] = E(e^{-tX_i}) = p_i e^{-t} + (1-p_i).e^0 = p_i e^{-t} + (1-p_i)$ ($p_i = 0.5$ here) and $E[X_i] = (1\times0.5 + 0\times0.5) = 0.5 = p_i$, hence $E(X) = 0.5 \times n = n \times p_i$

$$\begin{aligned} Pr(Y \geq e^{-t\mu(1-\delta)}) &\leq \frac{E(Y)}{e^{-t\mu(1-\delta)}} \\ &= \frac{\prod_{i=1}^{n}(1 - p_i + p_i e^{-t})}{e^{-t\mu(1-\delta)}} \\ &\leq \frac{\prod_{i=1}^{n} exp(-p_i + p_i e^{-t})}{e^{-t\mu(1-\delta)}} \text{ ,Since } e^x \geq (1+x)\forall x \\ &= \frac{e^{-\mu(1-e^{-t})}}{e^{-t\mu(1-\delta)}} = Q(say) \end{aligned}$$

Let f(t) = ln(Q). Now do $\frac{df}{dt} = 0$ to find the minima.

$$\begin{aligned} Q &= exp(-\mu(1-e^{-t}) + t\mu(1-\delta)) \\ \ln Q &= -\mu(1-e^{-t}) + t\mu(1-\delta) = f \\ \frac{df}{dt} &= 0 \\ \implies -\mu(e^{-t}) + \mu(1-\delta) &= 0 \\ e^{-t} &= 1-\delta \end{aligned}$$

Checking sign of $\frac{d^2f}{dt^2}$, $\frac{d^2f}{dt^2} = \mu e^{-t} = 1 - \delta > 0, (\mu = 0.5, 0 < \delta < 1)$. Hence we know that it is a minima.

Substituting the value for minima,

$$\begin{aligned} Pr(X \leq \mu(1-\delta)) &\leq \frac{e^{-\mu(1-e^{-t})}}{e^{-t\mu(1-\delta)}} \\ \text{(Substituting } e^{-t} &= 1-\delta) \\ Pr(X \leq \mu(1-\delta)) &\leq \frac{e^{-\mu(\delta)}}{(1-\delta)^{\mu(1-\delta)}} \end{aligned}$$

So,

$$Pr(X \leq \mu(1-\delta)) \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu} \qquad [3.1]$$

Where $\mu = E(X) = \frac{n}{2}$