Introduction to Coding Theory
EC5.205 Spring 2025
Mid Sem Exam

3rd February, 2025

Please note:

- There are 2 pages in this question paper.
- No source material is to be used.
- Go through every question carefully before answering. Please write down all the steps in your solutions.

Maximum Marks: 30

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1. A codeword of the code $C = \{01010, 10101\}$ is transmitted through a binary symmetric channel with cross-over probability $p = 0.1$.

   (a) Specify completely the maximum a-posteriori (MAP) decoding rule given that $P(01010) = \frac{1}{4}$ and $P(10101) = \frac{3}{4}$.
   (4 marks)

   (b) Suppose $P(01010) = P(10101) = \frac{1}{2}$ and a minimum distance decoder is applied, then compute the probability of decoding error.
   (3 marks)

2. Let $G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ be the generator matrix of a code.

   (a) What is the minimum distance of this code?
   (2 marks)

   (b) What is the maximum number of errors which can be detected by the code t_d and what is the maximum number of errors which can be corrected by this code t_c?
   (2 marks)

   (c) Find the parity-check matrix of the code.
   (2 marks)

3. Let C_1 be a binary (n_1, k) linear code with generator matrix $G_1 = [P_1 \ I_k]$. Similarly C_2 be a binary (n_2, k) code with generator matrix $G_2 = [P_2 \ I_k]$. Consider an $(n_1 + n_2, k)$ linear code C_3 with the following parity check matrix:

$$H = \left[\begin{array}{c|c} I_{n_1+n_2-k} & \begin{array}{c} P_1^T \\ I_k \\ P_2^T \end{array} \end{array} \right].$$

   Show that C_3 has minimum distance at least $d_1 + d_2$.
   (5 marks)

4. Consider an (n, k) linear block code. For any set of k independent columns of the generator matrix G, the corresponding set of coordinates $S \subseteq \{1, 2, \ldots, n\}$ is defined as an information set. Determine five information sets of the $(7, 4)$ Hamming code discussed in the class.
   (4 marks)

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5. Show that for every (n, k, d) code C over $\mathbb{F}_2$ and for every decoder $\mathcal{D} : \mathbb{F}_2^n \to C$, there is a codeword $\underline{c} \in C$ and $\underline{y} \in \mathbb{F}_2^n$ such that $d_H((\underline{y}), (\underline{c})) \leq \lfloor \frac{d+1}{2} \rfloor$ and $\mathcal{D}(\underline{y}) \neq \underline{c}$. (3 marks)

6. For an integer $m > 1$, let C ne the $[n, n - m, 3]$ Hamming code over $\mathbb{F}_2$, where $n = 2^m - 1$ and whose parity-check matrix is the matrix with all possible non-zero m tuples as columns. With this context, answer the following:

   (a) Show that C contains a codeword of Hamming weight n (namely C contains the all-one codeword) (1 mark)

   (b) How many codewords are there in C of Hamming weight $n - 1$ and $n - 2$? (3 marks)