

1. (Sets) [4]

- 1 For all sets  $A$  and  $B$ , prove that

$$(A \cap B) \cup (A \cap B') = A.$$

Here  $A'$  denotes complement of set  $A$ .

- 2 Suppose  $A = \{1, 2\}$  and  $B = \{2, 3\}$ . Find power set  $P(A \times B)$ . Here  $A \times B$  denotes the Cartesian product of  $A$  and  $B$ .

- 3 Let  $\mathbb{R}$  be the set of real numbers. Is  $\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$  a partition of  $\mathbb{R}$ ? Here  $\mathbb{R}^+$  denote set of positive reals,  $\mathbb{R}^-$  denote set of negative reals. Explain your answer.

- 4 Let  $S_i = \{x \in \mathbb{R} \mid 1 < x < 1 + \frac{1}{i}\} = (1, 1 + \frac{1}{i})$  for all positive integers  $i$ . Then find the following:

- (a)  $\bigcup_{i=1}^{\infty} S_i = ?$   
 (b)  $\bigcap_{i=1}^{\infty} S_i = ?$

2. (Induction Proofs) [4]

Prove the following using induction.

1. Suppose  $e_0, e_1, \dots$  is a sequence defined as follows

$$e_0 = 12, e_1 = 29, e_k = 5e_{k-1} - 6e_{k-2}, \forall k \geq 2$$

Prove that  $e_n = 5 \cdot 3^n + 7 \cdot 2^n$  for all integers  $n \geq 0$ .

- 2 Show that

$$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \cdots + \frac{(m+n)!}{n!} = \frac{(m+n+1)!}{n!(m+1)},$$

where  $m, n = 0, 1, 2$ .

3 (Pigeon hole principle) [3]

The pigeon-hole principle states that:

If we put  $N+1$  pigeons in  $N$  pigeon-holes, then there will be atleast one pigeon hole with at least two pigeons. Prove this statement using contrapositive proof.

A general pigeon-hole principle is stated as follows:

If we must put  $Nk+1$  or more pigeons into  $N$  pigeon holes, then some pigeon-hole must contain at least  $k+1$  pigeons. Prove this using contrapositive proof.

Prove the following using pigeon-hole principle.

1. Show that among 5 people at a dinner table, there are two that have an identical number of friends among those at the table.