Science-1, Monsoon 2024: Endsem

180 min. Total 100 points.

ROLL NUMBER: ______________________ SEAT: ______________________

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Instructions:

- A simple calculator IS ALLOWED in this exam.
- Write your roll number and seat number on top-page.
- Question paper consists of 5 questions of equal weightage.
- Answer in the space given for the question.
- Following values for some constants:
  Mass of electron = $9.1 \times 10^{-31} kg$. Planck's constant $= 6.626 \times 10^{-34} J\ s$.
  Boltzmann constant = $1.381 \times 10^{-23} J/K$. Stefan-Boltzmann constant= $5.670 \times 10^{-8} W/(m^2K^4)$.
  Avogadro Constant = 6.022×10^{23}. Speed of light in vacuum = $3 \times 10^8\ m/s$

  $$\texttt{Spherical Coordinates:}\quad \nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

  For constant $a > 0$, given that: $\int_{-\infty}^{\infty} dx\ e^{-ax^2} = \sqrt{\pi/a}$

PART-I (50 points) (Answer ONLY in space provided.

(Answer all question in the space given; without exceptions. Answers outside this space will NOT be considered; answer in one/two sentences clearly. Each question is worth 2 points)

1. A particle has an external force given by $\vec{F}(x,y,z) = (ze^{-x},\ \ln z,\ y/z)$, and moves from $\vec{r}_0$ at $t = t_0$ to $\vec{r}_1$ at $t = t_1$. Is the change in kinetic energy independent of path taken? If yes, find the potential. If no, why?

$F_x = ze^{-x}$; $\frac{\partial F_x}{\partial y} = 0$, $\frac{\partial F_x}{\partial z} = e^{-z}$

$F_y = \ln z$; $\frac{\partial F_y}{\partial x} = 0$, $\frac{\partial F_y}{\partial z} = \frac{1}{z}$.

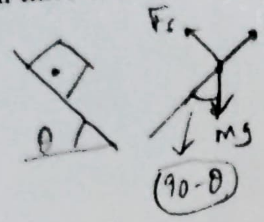
$F_z = y/z$; $\frac{\partial F_z}{\partial x} = 0$, $\frac{\partial F_z}{\partial y} = \frac{1}{z}$

Since $\left(\frac{\partial F_x}{\partial z} = e^{-x}\right) \neq \left(\frac{\partial F_z}{\partial x} = 0\right)$,

F is not from a potential.

Hence $\boxed{\text{KE IS DEPENDENT ON PATH}}$

2. A block is on an incline plane. If the coefficient of friction is 0.3, find the angle of the incline that will have the block start sliding if it was initially at rest.

① 

② $N = mg\sin\theta$.

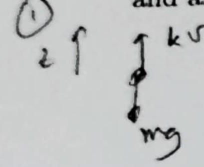
③ $F_{\text{static friction}} = \mu N = \mu mg \sin\theta$

④ Require $\{$ for (not) sliding $\}$: $F_s < mg\cos\theta \Rightarrow \mu\, mg\sin\theta < mg\cos\theta$

$\tan\theta < \frac{1}{0.3} \Rightarrow \theta = \tan^{-1}\frac{1}{0.3} = 150^\circ$

⑤ Block will start sliding at $\boxed{\theta = 150^\circ}$.

3. Find the terminal velocity of a vertically falling rain drop (mass m) falling from rest and experiencing air resistance; the retarding force is proportional to velocity. Assume that evaporation is negligible and air is still (no turbulence).

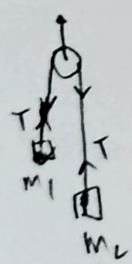
① 

② $m\ddot{z} = -mg + k\dot{z}$

③ $\cancel{\ddot{z} + g}$ - Terminal vel.: $\ddot{z} = 0$

④ $\boxed{\dot{z}_{\text{term}} = \frac{mg}{k}}$ where k measures air resistance

4. A pulley with two masses suspended from light string at each end (m_1 and m_2) is hung in an elevator. If the elevator is moving vertically up with acceleration α, find the tension in the string (on which the masses are suspended).



① ~~acc~~ $(g+\alpha)$ → Apparent acceleration. Hence $g_a = g + \alpha$.

② $m_1\ddot{z}_1 = -m_1 g_a + T$
$m_2\ddot{z}_2 = -m_2 g_a + T$ $\}$ and $z_1 + z_2 = \text{const}$: $\ddot{z}_1 = -\ddot{z}_2$.

→ $(m_1 - m_2)\ddot{z}_1 = -(m_1 + m_2) g_a + 2T$

③ $T = \frac{2 m_1 m_2 (g+\alpha)}{m_1 + m_2}$

5. A charged particle (charge q) has velocity $\vec{v}_0$ at $t = t_0$. It is known (question in Quiz-2) that when magnetic field $\vec{B}(\vec{r}) = 1\hat{z}$ is present, the particle undergoes helical motion. What is the pitch of the helical path of this particle?

① $\vec{v}$; $v_{\parallel} = v_z$, $v_{\perp} = \sqrt{v_x^2 + v_y^2}$

$\vec{v} = v_{\perp}\hat{\theta} + v_{\parallel}\hat{z}$

② $F = q\vec{v}\times\vec{B} = q v_{\perp} B_0\, \hat{r}$

③ $\frac{m v_{\perp}^2}{r} = q v_{\perp} B_0 \Rightarrow \frac{m v_{\perp}}{r} = qB_0 \Rightarrow m\dot{\theta} = qB_0 \Rightarrow$

$\Rightarrow T = \frac{2\pi}{qB_0/m}$.

④ Pitch $= v_z \cdot \frac{2\pi}{qB_0/m} = \boxed{\frac{2\pi v_z m}{qB_0}}$

6. Write the inverse Lorentz transforms. ; write $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ in term of $\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}$

$$\boxed{\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{x'v}{c^2}\right) \end{aligned}}$$

where $\boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$

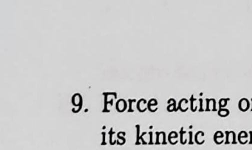
7. A muon detecter when located on top of a 2km (above sea level) mountain counts the number of muons traveling at speed of 0.98c to be 1000 in one hour. This detector is moved to sea level and the number of muons at that speed measured over a one hour interval is found to be 542! What is the half-life of muon (in its rest frame)?

Time elapsed $= \frac{2\,km}{0.98c} = \frac{2000}{v}$ sec. in earth frame.

In muon's frame: $\Delta t = \frac{2000}{v} \times \sqrt{1-\frac{v^2}{c^2}} = \frac{2000}{3\times10^8 \frac{m}{s}} \sqrt{1-0.98^2} = 132.6\times10^{-8}$ sec

$542 \sim \frac{1000}{2} \rightarrow$ This is half life: $\boxed{T_0 = 1.31\times10^{-6}\,s}$

8. A rod makes an angle θ with x-axis in the rest frame of rod. What will be angle the rod makes in a frame that moves with velocity v along x-axis?



; In moving frame: $L_x = L_0 \cos\theta \cdot \sqrt{1-v^2/c^2}$

$L_y = L_0 \sin\theta$

so angle: $\tan\theta' = \frac{L_0 \sin\theta}{L_0 \cos\theta \sqrt{1-v^2/c^2}} = \frac{\tan\theta}{\sqrt{1-v^2/c^2}}$

$$\boxed{\theta' = \tan^{-1}\left[\frac{\tan\theta}{\sqrt{1-v^2/c^2}}\right]}$$

9. Force acting on a gold atom causes its velocity to change from 0.6c to 0.8c. Consider the change in its kinetic energy, in classical mechanics and special theory of relativity. Calculate ratio of changes in KE (relativistic divided by classical):

$\Delta K_{classical} = \frac{1}{2}mc^2(0.8^2 - 0.6^2)$.

$\Delta K_{STR} = (\gamma_2 - 1)mc^2 - (\gamma_1 - 1)mc^2 = (\gamma_2 - \gamma_1)mc^2$

Ratio $= \frac{\gamma_2 - \gamma_1}{\frac{1}{2}(v_2^2 - v_1^2)} = \frac{\frac{0.8-0.6}{0.8\times0.6}}{\frac{1}{2}(0.8^2-0.6^2)}$

$\gamma_1 = \frac{1}{\sqrt{1-0.6^2}} = \frac{1}{0.8}$, $\gamma_2 = \frac{1}{\sqrt{1-0.8^2}} = \frac{1}{0.6}$

$= \frac{2}{0.8\times0.6} \times \frac{1}{(0.8+0.6)} = \boxed{2.9762 = \text{Ratio}}$

10. Find the de Broglie wavelength of an electron in the ground state of Hydrogen atom.

① $F_c = \frac{mv^2}{r}$, $F_e = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$ ⇒ $F_c = F_e$ ⇒ $v = \frac{e}{\sqrt{4\pi\varepsilon_0 m r}}$

② $mvr = n\hbar = \hbar$, $m \cdot \frac{e}{\sqrt{4\pi\varepsilon_0 m r}} \cdot r = \hbar$ → $\sqrt{r} = \frac{\hbar}{e}\sqrt{\frac{4\pi\varepsilon_0}{m}}$ ⇒ $r = \frac{\hbar^2}{e^2}\frac{4\pi\varepsilon_0}{m}$.

③ $\lambda = 2\pi r$ ⇒ $\boxed{\lambda = \frac{8\pi\hbar^2\varepsilon_0}{e^2 m}.}$

11. A scientific instrument company claims that its newly released microscope can position the proton with an accuracy of $\pm 10^{-11}$ m. Find the uncertainty of the proton's position 1 second later.

① $\Delta x \cdot (m \cdot \Delta v) \geq \frac{1}{2}\hbar$ → $\Delta v \geq \frac{\hbar}{2\Delta x\, m}$

② Δx 1 sec later ≈ $\Delta v \cdot \Delta t \geq \frac{\hbar}{2\Delta x\, m} \cdot 1\,sec. = \frac{6\times10^{-34}\,J s.s}{(2\times3.142)\times10^{-11}\,m \times (2000\times 9.1\times10^{-31}\,kg)}$

$= 2.6\times10^{-5}\times10^{8}\ \frac{J.s^2}{m\,kg.} = \boxed{2.6\times10^{3}\ m = \Delta x}$

12. Moun is an unstable elementary particle with mass of $207m_e$; when this muon with a negative charge of $-e$ is 'captured' by proton it forms a 'muonic' atom. Find the ionization energy of this 'muonic' atom in eV.

① $E_H = -13.6eV$, ② Note: $E_H \propto -\frac{m_e}{n^2}$ where m_e is reduced mass of e^-, p system.

③ $E_{\mu m} \propto -\frac{m_\mu}{n^2}$ ⇒ Ionisation $= +13.6\,eV \times 187.6$

$m_e = \frac{mM}{m+M} = \frac{1\times2000}{2001}$.

$m_\mu = \frac{207\times2000}{207+2000} \approx 187.6$.

$\boxed{= +2{,}551\ eV}$

2.5 keV

13. When an electron with energy of 1.0 eV is incident on a barrier of 10.0 eV energy and 0.50 nanometer, find the transmission probability of this electron.

① $T \sim e^{-2k_2 L}$. where $\hbar k_2 = \sqrt{2m(U-E)}$

② $k_2 = \frac{1}{\hbar}\sqrt{2\times9.1\times10^{-31}\,kg \times (10.0\ eV)}$; $1\,eV = 1.6\times10^{-19}C \times 1\frac{J}{C} = 1.6\times10^{-19}J$.

$= \frac{\sqrt{2\times9.1\times10^{-31}\times10\times1.6\times10^{-19}}}{6.626\times10^{-34}/2\pi} = \frac{16.184\times10^{-25}}{10^{-34}}\,/m. = 16.184\times10^{9}$.

③ $T = \exp(-2\times16.184\times10^{9}\times0.5\times10^{-9}) = \boxed{9.2\times10^{-9}}$

14. For a harmonic oscillator, find the expectation value of x^2 in $n = 0$ state.

① $\psi_0 = C e^{-y^2/2}$ where $y = \alpha x$. $\alpha^2 = \frac{\sqrt{km}}{\hbar}$.

② $\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} dx\, e^{-y^2/2} \cdot x^2}{\int_{-\infty}^{\infty} dx\, e^{-y^2}} = \frac{1}{\alpha^2} \frac{\int_{-\infty}^{\infty} dy\, y^2 e^{-y^2/2}}{\int_{-\infty}^{\infty} dy\, e^{-y^2/2}}$

③

15. In Radio astronomy, clouds of hydrogen atoms in galaxy are detected by the 21 cm spectral line; in hydrogen atom, this line corresponds to the flipping of electron spin, from it being parallel to becoming anti-parallel to spin of the proton (nucleus). Find the magnetic field experienced by this electron.

① $u = m_l \left(\frac{e\hbar}{2m} B\right)$ ② $m_l : -\frac{1}{2} \to +\frac{1}{2}$. $\Rightarrow \Delta m_l = 1$.

$\Delta u = (\Delta m_l)\left(\frac{e\hbar}{2m} B\right)$ ③ $\Delta u = \frac{hc}{\lambda} = $

$B = \frac{\Delta u}{\Delta m_l} \times \left(\frac{2m}{e\hbar}\right)$ ④ $B = \frac{hc}{\lambda} \times \frac{2m}{eh/2\pi} = \frac{4\pi m c}{\lambda e} = \frac{4 \times 3.14 \times 9.1 \times 10^{-31} \times 3 \times 10^{8}}{0.21 \times 1.6 \times 10^{-19}}$ T

$\boxed{B = 1009.3 \times 10^{-4} = 0.1\,T.}$

16. What is the physical significance of each quantum number associated with electron in Hydrogen atom?

① n – principal quantum # → energy of e^-

② l → orbital quantum # → Total Angular Momentum of e^-.

③ m → magnetic quantum # → z-component of angular momentum of e^-.

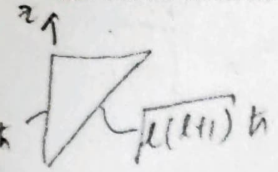
④ m_s → spin quantum # → spin of the e^-.

17. Phosphorous has atomic number of 15. What will be electronic configuration of this element?

$1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p_x^1 3p_y^1 3p_z^1$

2	2
2+6	8
2	10

18. For an electron in hydrogen atom, find the value/s of angle between L and z-axis for $l = 2$.



so . $\cos\theta = \frac{m_l}{\sqrt{l(l+1)}} = \frac{m_l}{\sqrt{6}}$, $\theta = \cos^{-1}\left(\frac{m_l}{\sqrt{6}}\right)$.

θ values are $\left(\pm\cos^{-1}\left(\frac{\pm 2}{\sqrt{6}}\right), \pm\cos^{-1}\left(\frac{1}{\sqrt{6}}\right), \cos^{-1}\left(\frac{0}{\sqrt{6}}\right)\right)$

ie $\boxed{(\pm 65.9°, \pm 35.2, 0) \text{ 5 angles}}$

19. For an electron in the hydrogen atom, what are the allowed transitions?

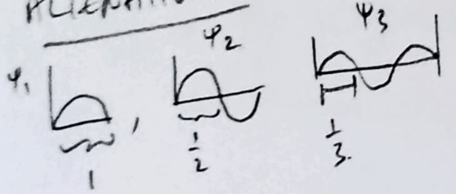
$\boxed{\begin{array}{l}\Delta L = \pm 1 \\ \Delta m_l = 0, \pm 1.\end{array}}$

20. What is the probability that a particle in a box L wide can be found between $x = 0$ and $x = L/n$ when it is in n–th state?

① $\psi_n = \sqrt{\frac{2}{L}} \sin\left(n\frac{\pi}{L}x\right)$ ② $\text{Prob}\left(x \in [0, \frac{L}{n}]\right) = \int_0^{L/n} dx \left(\frac{2}{L}\right) \sin^2\left(\frac{n\pi x}{L}\right) = \frac{1}{L}\int_0^{L/n} dx \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)$

$= \frac{1}{L}\left[\frac{L}{n} - 0\right] = \boxed{\frac{1}{n} = \text{Prob}}$

ALTERNATIVELY



$\rightarrow \boxed{\text{Ans } \frac{1}{n} = \text{Prob}}$

21. For a particle in a square of side L, at what temperature is the probability of finding the particle in its *second* excited state equals one half of its probability in ground state?

① $E_n = (n_1^2 + n_2^2) \cdot \frac{h^2}{8mL^2}$ ③ Second exited state: 8C, Two states

(1,3) (3,1) 10C

(2,2) 8C

② (1,2) (2,1) → 5C

(1,1) → 2C

④ $\frac{P_2}{P_0} = \frac{2 \cdot e^{-8C/kT}}{1 \cdot e^{-2C/kT}} = \frac{1}{2} \Longrightarrow 4 = e^{+6C/kT}$

or $\frac{6C}{kT} = \ln 4 \Rightarrow T = \frac{6}{\ln 4} \cdot \frac{C}{k} = \boxed{\frac{6}{\ln 4} \cdot \frac{h^2}{8mL^2 k} = T}$

22. What is the most probable speed of a oxygen molecule in air at 25 C (assume ideal gas)? Is it

mx -

faster/slower than an aeroplane (500km/hr)?

$$v = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\times 1.381\times10^{-23}\,J/k\;\;298K}{32\times1.67\times10^{-27}\,kg}} = 3.92\times10^{2}\,m/s = \boxed{392\,m/s = v}$$

$$500\frac{km}{hr} = \frac{500\times10^{3}}{60\times60}\frac{m}{s} = 138\,m/s$$ → so O_2 speed is about 3 times more than aeroplane!

23. Microprocessors in a computer produces heat at rate of 10 W per square centimeter of surface area. At what temperature would a blackbody be at this radiance?

$\frac{Power}{area} = \sigma T^4$. (Stephan Boltzman law) → $10\frac{W}{m^2} = 5.67\times10^{-8}\frac{W}{m^2T^4}\cdot T^4$.

$$\Rightarrow T = \sqrt[4]{\frac{10}{5.67}}\times10^{2}\,K = 132.8\,K.$$

24. From the Plank's radiation law for $u(\lambda)$, find the equation satisfied by peak wavelength λ_p

① $u(\nu)d\nu = K\cdot\frac{\nu^3 d\nu}{e^{h\nu/kT}-1}$; ② $\nu = \frac{c}{\lambda} \rightarrow d\nu = -\frac{c}{\lambda^2}d\lambda$. $\Rightarrow u(\lambda)d\lambda = \frac{K}{\lambda^5}\frac{1}{(e^{hc/\lambda kT}-1)}d\lambda$.

③ Peak: $\boxed{\frac{d}{d\lambda}(\lambda^5(e^{hc/\lambda kT}-1)) = 0}$

25. The smallest angle of Bragg scattering in potassium chloride (KCl) is 28.4 degrees for 0.30 nm X-Rays. Find the distance between the atomic planes in KCl.

$\lambda = 0.30\,nm$

① $2d\sin\theta = 1\cdot\lambda \rightarrow d = \frac{\lambda}{2\sin\theta} = \frac{0.30\,nm}{2\times} = \boxed{0.3157\,nm. = d}$

PART-II: 50 points

Answer in the answer book given. Each question is worth 5 points.

1. A damped oscillator is defined as modified simple harmonic oscillator; modification is the additional damping force, that is proportional to velocity but opposite to direction of motion (the proportionality constant is $b > 0$). Find the relation between b, m and k which still has oscillatory motion about the equilibrium position and find the frequency of particle hitting equilibrium position $x = 0$.

2. Consider a single particle Lagrangian given by $\mathcal{L} = -m\sqrt{c^2 - \vec{u}\cdot\vec{u}} - V(\vec{r})$ where $\vec{r}$ is the position vector of the particle and has velocity $\vec{u} = \frac{d}{dt}\vec{r}$. And c is a constant. Find (a) conjugate momentum $\vec{p}$ of the particle with Lagrangian $\mathcal{L}$, and hence calculate $\vec{p}\cdot\vec{p}$ in terms of $\vec{u}\cdot\vec{u}$ (b) For this Lagrangian, find the Hamiltonian. (c) Simplify to get a formula for H that looks like the famous Einstein energy-mass relation.

3. Light of frequency (ν_0 in rest frame of the source) is received on a space ship located a very large distance away which notes that source is moving perpendicular to the line of sight with speed of $0.6c$. What is the apparent frequency of light on spaceship?

4. A uniform rope of total length $2a$ hangs in equilibrium over a smooth nail. A very small impulse causes the rope to slowly roll off the nail; additionally, assume that the rope does *not* lift off the nail and is in free fall. Find the velocity of rope as it just clears the nail using (a) principle of conservation of energy and (b) using forces and Newton's second law.

5. A hollow cylinder of radius R is lying on its curved edge on a horizontal plane; assume that it is fixed. Another small cylinder of radius r rolls without slipping inside it; find the frequency of oscillation.

6. Light of wavelength λ scatters off an electron; when scattering angle is θ, the scattered light has wavelength $\lambda + \Delta\lambda$, derive the relation between $\Delta\lambda$, λ, scattering angle θ (and any other universal constants) in this experiment.

7. Gas of hydrogen atoms is at temperature T. The $(n = 2) \to (n = 1)$ line photon will be seen to have different frequencies due to Doppler shift by a fixed observer (detector). Find the distribution of frequencies (this broadening of spectral line is called thermal broadening). [You may simplify by assuming that only radial Doppler shift exits by neglecting transverse Doppler shift]

8. If $\psi(r, \theta, \phi)$ given below is a solution to hydrogen atom, find the quantum numbers for this wave function using the hydrogen atom Hamiltonian.

$$\psi(r, \theta, \phi) = C\, r e^{-r/(2a)} \sin\theta e^{-i\phi}$$

where C and a are constants, and $i = \sqrt{-1}$.

9. For the the particle in 1-box, write the wavefunction that is 50/50 superposition of $n = 1$ and $n = 2$ wave function. For the wave function you have written down, calculate the expectation values for (a) position, (b) momentum, (c) energy.

10. Electrons in a metal can be modeled as particles in 3-d box. Using the fact that electrons are Fermions, and if the system is in lowest energy, find the energy of the highest energy electron.

assume non-interacting electron.

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Roll No. .... Additional Sheet No. .... Invigilator's Signature ....

① $m\ddot{x} = -b\dot{x} - kx \Rightarrow \ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0 \Rightarrow \ddot{x} + \beta\dot{x} + \omega^2 x = 0$

Ansatz: $x = e^{\alpha t} \to \alpha^2 e^{\alpha t} + \beta\alpha e^{\alpha t} + \omega^2 e^{\alpha t} = 0 \to \alpha^2 + \beta\alpha + \omega^2 = 0$

$\Rightarrow \alpha = \frac{-\beta \pm \sqrt{\beta^2 - 4\omega^2}}{2} \Rightarrow x = e^{\alpha t}$

Oscillatory when $(\beta^2 - 4\omega^2) < 0 \Rightarrow x = c_1 e^{-\frac{\beta}{2}t}\left[c_1 e^{ic_3 t} + c_2 e^{-ic_3 t}\right] =$

where $c_3 = \frac{1}{2}\sqrt{4\omega^2 - \beta^2}$

$\Rightarrow x(t) = e^{-\frac{\beta}{2}t}\left[c_1 \cos(c_3 t)\right] + i\,0].$

clearly freq $= c_3 = \frac{1}{2}\sqrt{4\omega^2 - \beta^2} = \sqrt{\omega^2 - b^2/m^2} = \frac{\sqrt{k^2 - b^2}}{m}$

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② $\mathcal{L} = -m\sqrt{c^2 - u^2} - V(\bar{r})$; $\bar{p} = \frac{\partial \mathcal{L}}{\partial \bar{u}} = \frac{-m}{2\sqrt{c^2 - u^2}} \cdot (-2\bar{u}) - 0.$

ⓐ so $\bar{p} = \frac{m\bar{u}}{\sqrt{c^2 - u^2}} \to p^2 = \frac{m^2 u^2}{c^2 - u^2} \to \frac{p^2}{m^2} = \frac{u^2}{c^2 - u^2} \Rightarrow \frac{p^2}{m^2} + 1 = \frac{c^2}{c^2 - u^2}$

ⓑ Hamiltonian $H = \bar{u}\cdot\bar{p} - \mathcal{L} = \frac{mu^2}{\sqrt{c^2 - u^2}} - \left(-m\sqrt{c^2 - u^2} - V(r)\right)$

$H = \frac{mu^2 + m(c^2 - u^2)}{\sqrt{c^2 - u^2}} + V(\bar{r}) = \boxed{\frac{mc^2}{\sqrt{c^2 - u^2}} + V(\bar{r}) = H}$

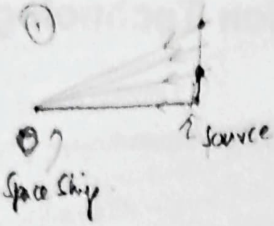
ⓒ $H = \gamma m c^2 + V(r)$.

ⓒ $\to$ mistake in given Lagrangian. It should have been

$\mathcal{L} = -mc\sqrt{c^2 - u^2} - V(\bar{r})$

3

(i)



(ii)

Lets say τ is time diff between flashes.

$\tau = \tau_0 \gamma = \frac{1}{\nu_0} \frac{1}{\sqrt{1-0.6^2}} = \frac{10}{8\nu_0}$

(iii) Let t_n time for n^{th} flash reaches O.

$$t_n = n\tau + \frac{\sqrt{x_0^2 + (n v \tau)^2}}{c}$$

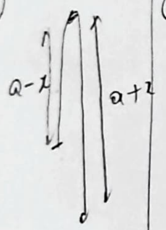
$\uparrow$ flash happen — flash travels to reach O.

$$t_n = n\tau + \frac{x_0}{c}\sqrt{1 + \left(\frac{n v \tau}{x_0}\right)^2} = n\tau + \frac{x_0}{c}\left(1 + \frac{1}{2}\frac{n v \tau}{x_0}\right)$$

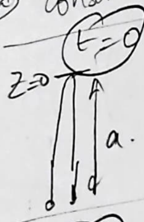
So $t_{n+1} - t_n = \tau + \frac{1}{2}\frac{v\tau}{x_0} = \tau\left[1 + \frac{v}{2x_0}\right] = \tau$.

So frequency of light is $\frac{1}{\tau} = \frac{1}{\tau_0\gamma} = \boxed{\frac{8}{10}\nu_0 = \nu}$

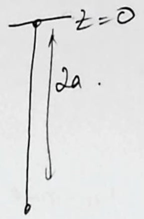
4



(a) Conservation of energy.



$E = \frac{1}{2}m v^2 - m g \cdot \frac{a}{2}$, $v = 0$



$E = \frac{1}{2}m v^2 - m g \cdot a$

$-mg\frac{a}{2} = \frac{1}{2}mv^2 - mga$

$\Downarrow$

$v^2 = ag$

$\boxed{v = \sqrt{ag}}$

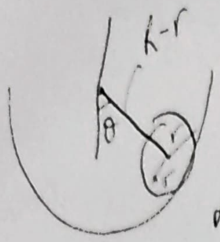
(b) Net force on rope: $\left(\frac{M}{2a}\right)\left[(a+x) - (a-x)\right] \cdot g$. downwards

$= \frac{M}{2a} 2x g = \frac{Mg}{a} \cdot x$

$m \cdot \ddot{x} = \frac{mg}{a} \cdot x \rightarrow \frac{m\,dv}{dx} \cdot \frac{dx}{dt} = \frac{mg}{a}x \Rightarrow m v \frac{dv}{dx} = mgx$.

$\frac{1}{2}mv^2 = \frac{1}{2}mgx^2 + C \Rightarrow \boxed{v = \sqrt{ag}}$

(5)



① $PE = -mg(R-r)\cos\theta$

② KE: Rolling w/o slipping

① velocity $|v| = (R-r)\dot\theta \to$ Translational KE

② Rolling $\to$ So rotational KE $= \frac{1}{2}I\left|\frac{v}{r}\right|^2$

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2 = \frac{1}{2}v^2\left[m + \frac{I}{r^2}\right]$$

$$\mathcal{L} = \frac{1}{2}\left(m+\frac{I}{r^2}\right)(R-r)^2\dot\theta^2 + mg(R-r)\cos\theta$$

Euler Lagrange: $\frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{1}{2}\left(m+\frac{I}{r^2}\right)(R-r)^2 \cdot 2\dot\theta$ | $\frac{\partial \mathcal{L}}{\partial \theta} = -mg(R-r)\sin\theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \Rightarrow \frac{1}{2}\left(m+\frac{I}{r^2}\right)(R-r)^2\, 2\ddot\theta = -mg(R-r)\sin\theta$$

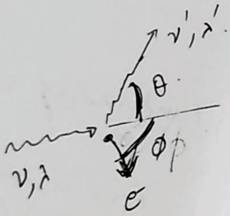
$$\Downarrow$$

$$\ddot\theta = -C\sin\theta$$

for small oscillation, $\ddot\theta = -C\theta \Rightarrow \boxed{freq = \sqrt{C}}$

where $\boxed{C = \frac{mg}{\left(m+\frac{I}{R^2}\right)(R-r)}}$

(6)



① Energy conservation.

(a) $h\nu + m_e c^2 = h\nu' + \gamma m_e c^2$

② Momentum Conservation.

(b) x-dir: $\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + p\cos\phi$

(c) y-dir: $p\sin\phi = \frac{h\nu'}{c}\sin\theta$

3 Eqns. eliminate γ, p, ϕ.

$B^2 + C^2$: $p^2 = p^2\cos^2\phi + p^2\sin^2\phi = \frac{h^2}{c^2}\left[(\nu-\nu'\cos\theta)^2 + (\nu'\sin\theta)^2\right]$

$$\Rightarrow p^2c^2 = h^2\left[\nu^2+\nu'^2 - 2\nu\nu'\cos\theta\right]$$

$$(\gamma^2 m_e^2 c^4 - m_e^2c^4) = h^2(\nu^2+\nu'^2-2\nu\nu'\cos\theta)$$

A: $[h(\nu-\nu') + m_e c^2]^2 - m_e^2c^4 = h^2((\nu-\nu')^2 + 2\nu\nu'(1-\cos\theta))$

$$h^2(\nu-\nu')^2 + 2h(\nu-\nu')m_e c^2$$

$$\Rightarrow 2h(\nu-\nu')m_e c^2 = 2h\nu \cdot h\nu'(1-\cos\theta)$$

Rearrange $\left(\frac{1}{\nu'} - \frac{1}{\nu}\right)\frac{mc^2}{h} = 1-\cos\theta \Rightarrow \lambda'-\lambda = \frac{h}{mc}(1-\cos\theta)$

$$\boxed{\Delta\lambda = \frac{h}{mc}(1-\cos\theta)}$$

⑦ If ν_0 is frequency, source moving with velocity V, then Doppler shift is $\nu = \nu_0 \sqrt{\frac{1-V/c}{1+V/c}}$. Since $\frac{V}{c} \ll 1$, $\frac{V}{c} \sim 0$.

$$\nu \approx \nu_0 \left(\frac{1-\frac{1}{2}\frac{V}{c}}{1+\frac{1}{2}\frac{V}{c}} \right) \approx \nu_0\left(1-\frac{v}{c}\right).$$

Neglect transverse Doppler effect. So if V_x velocity gives. freq, $\nu = \nu_0\left(1-\frac{V}{c}\right)$

then $\text{prob}(\text{frequency} = \nu) = \text{prob}(\text{velocity} = V_x)$

$$= \exp\left(-\frac{\frac{1}{2} m V_x^2}{kT}\right) = \exp\left(-\frac{m}{2kT} V_x^2\right)$$

$$\boxed{\nu = \nu_0\left(1-\frac{V}{c}\right) \Rightarrow 1-\frac{\nu}{\nu_0} = \frac{V}{c}}$$

$$\text{prob}(\nu) \propto \exp\left(-\frac{m\nu_0^2 c^2}{2kT}\left(1-\frac{\nu}{\nu_0}\right)^2\right)$$

$$\boxed{\text{prob}(\nu) \propto \exp\left[-\frac{mc^2}{2kT}(\nu_0-\nu)^2\right]}$$

⑧ $\psi(r,\theta,\phi) = c\, r e^{-r/2a} \cdot \sin\theta \cdot e^{-i\phi}$.

ⓐ $e^{-i\phi} \Rightarrow m_l = -1$.

ⓑ θ component is non-zero, so clearly $l \neq 0$. [confirmed by $m_l = -1$].
θ component $\propto \sin\theta \rightarrow l=1$ component;
θ component $\propto \sin\theta \rightarrow l=1$ component

ⓒ r component is $re^{-r/2a}$. $\rightarrow f(r)e^{-r}$ where $f(r)$ is linear.
Clearly $n=1$

we have $(n, l, m_l) = (1, 1, -1)$ wave function

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Roll No. .... Additional Sheet No. .... Invigilator's Signature ....

9. 50/50 Superposition of n=1 and n=2 PiB wave functions

$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$, $E_n = n^2 C$, $C = \frac{h^2}{8mL^2}$

$\cancel{\psi = \psi_1}$

$\cancel{\psi = N(\psi_1 + \psi_2)}$ $\quad \psi = N\left(\psi_1 e^{-\frac{i}{\hbar}E_1 t} + \psi_2 e^{-\frac{i}{\hbar}E_2 t}\right)$

$\int_0^L dx\, \psi^* \psi = 1 \Rightarrow \int_0^L dx (N^* N)\left(\psi_1 e^{+\frac{i}{\hbar}E_1 t} + \psi_2 e^{+\frac{i}{\hbar}E_2 t}\right)\left(\psi_1 e^{-i\frac{E_1 t}{\hbar}} + \psi_2 e^{-i\frac{E_2 t}{\hbar}}\right) = 1$

$\Rightarrow N^* N \int_0^L dx \left(\psi_1^2 + \psi_2^2 + \psi_1\psi_2\left[e^{\frac{it}{\hbar}(E_1 - E_2)} + e^{\frac{it}{\hbar}(E_2 - E_1)}\right]\right) = 1$

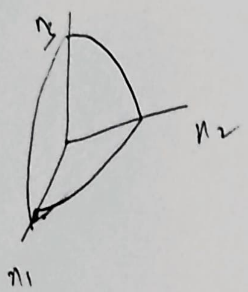
$|N|^2 \left[1 + 1 + 0\right] = 1 \Rightarrow |N| = \frac{1}{\sqrt{2}}$

$\psi(x,t) = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{L}}\right]\left[\sin\left(\frac{\pi x}{L}\right) e^{-\frac{it}{\hbar}E_1} + \sin\left(\frac{2\pi x}{L}\right) e^{-\frac{it}{\hbar}E_2}\right]$

$\langle x \rangle = \int_0^L dx\, \psi^* x \psi = \frac{1}{2}\int_0^L dx \left(\psi_1^2 x + \psi_2^2 x + x\psi_1\psi_2 \cdot 2\cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\right)$

$= \frac{1}{2}\left[\frac{L}{2} + \frac{L}{2} + 2\cos\left(\frac{E_1 - E_2}{\hbar} t\right)\left(\frac{2}{L}\right)\int_0^L dx\, x \sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi x}{L}\right)\right.$

(10) $E = (n_1^2+n_2^2+n_3^2)\dfrac{h^2}{8mL^2} = c\cdot(n_1^2+n_2^2+n_3^2)$.



# of states below energy E = # of integer triples with length less than $\frac{E}{c}$.

(i.e. $n_1^2+n_2^2+n_3^2 < \frac{E}{c}$.)

$$= \frac{1}{8}\cdot\frac{4\pi}{3}\left(\sqrt{\frac{E}{c}}\right)^3.$$

# of electrons below energy $E = 2\cdot\frac{1}{8}\cdot\frac{4}{3}\pi\left(\frac{E}{c}\right)^{3/2} = Ne$.

or $E = \left(\frac{3}{\pi}Ne\right)^{2/3} c = \left(\frac{3}{\pi}Ne\right)^{2/3}\frac{h^2}{8mL^2}$.

This is the energy of highest energy electron.