CPSC 4430/5440: ML

Lesson A02: Linear Algebre (Eigenvalu / SVD)

Outline:

① Review of last class.

② Eigendecomposition. ($A \in R^{n \times n}$ square Matrix)

③ Hands-on of Eigendecomposition

③ SVD. (Singular Value Decomposition) ($A^{m \times n}$).

④ Hands on of SVD.

Review of last class: Metrix

$$A \in R^{m \times n}.$$

Operations of Metrix.

- $+$, $-$, $*$ (Metrix * scala, Metrix * vector, Metrix * Metrix).
- Norm of Metrix: (Magnitude)

$$\left\| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \right\|_2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2.$$

Solution of Linear Eq.

(Eigen value, Eigenvector)

$\underbrace{\qquad\qquad\qquad\qquad}$

Eigen Pair. (λ_i, v_i)
$\in R \quad \in R$

$A \in R^{n\times n}$ (square)

— $(A \cdot \vec{v_i} = \lambda_i \cdot \vec{v_i})$ & $\|v_i\| = 1$
normalized

$\Longrightarrow (\lambda_i, v_i)$ is Eigen-pair.

Concepts: 0- Eigenval $\rightarrow$ Singular

① if A is non singular

$\rightleftharpoons$ No 0- Eigenvalue
$(\lambda_i \neq 0)$

② if A $\in R^{n\times n}$ is Symmetrix.

$\longrightarrow \lambda_i \in R$ (Real)

③ if A is Non-symmetrix $\begin{matrix} 2+1i \\ 2+3i \end{matrix}$

$\longrightarrow \lambda_i \in \mathbb{C}$ (Complex)

symmetric

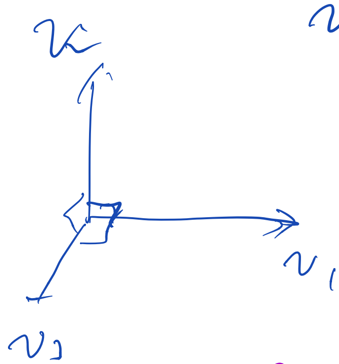
A Non-singular Metrix $A \in \mathbb{R}^{n \times n}$.

→ it should have n Eigenpair

$(\lambda_1, \vec{v}_1)$,
$(\lambda_2, \vec{v}_2)$,
⋮
$(\lambda_n, \vec{v}_n)$.

orthogonal to each other

$$\vec{v}_i \circ \vec{v}_j = \begin{cases} 0 & (i \neq j). \\ 1 & (i = j). \end{cases}$$



unit Length.
(Normalize)

orthonormal

a sequence of unit vector are orthogonal against each other
↓
orthonormal.

$$\left\{ \begin{array}{l} ① \; A \vec{v}_1 = \lambda_1 \vec{v}_1 \\ ② \; A \vec{v}_2 = \lambda_2 \cdot \vec{v}_2 \\ \vdots \\ ⓝ \; A \vec{v}_n = \lambda_n \cdot \vec{v}_n \end{array} \right\} \Longrightarrow$$

Metrix Version

‖

Metrix

$$\Downarrow$$

$$A\,[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n] = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]\underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix}}_{\Lambda}$$

Multiply $[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]^T$ on the both side of Eq.

$$\underbrace{A\,[\vec{v}_1\ \vec{v}_2 \cdots v_n] \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix}}_{I} = [\vec{v}_1 \cdots \vec{v}_n]\cdot\Lambda \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_2 \end{bmatrix}.$$

$$\Rightarrow\quad A = [\vec{v}_1 \cdots \vec{v}_n]\,\Lambda \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$(\because\ v_i \circ v_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}.$$

$$\Rightarrow\quad \boxed{A = V\,\Lambda\,V^T}$$

diagonalization

Eigen-decomposit

Application of Eigenvalue in Signal Compression

$$A = V \cdot \lambda \cdot V^T$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix}$$

$$= \underbrace{\lambda_1}_{1000} \cdot \vec{v}_1 * \vec{v}_1^T + \underbrace{\lambda_2}_{10} \cdot \vec{v}_2 * \vec{v}_2^T + \cdots$$

primary component

primary term

$$+ \underbrace{\lambda_n}_{0.01} \cdot \vec{v}_n * \vec{v}_n^T$$

trivial term

trivial

Hands on of Eigen solution

$$A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}.$$

square? 2x2.
symmetric? Ye.

step 1:

Solution: ① find the characteristic polynomial.

$$\det(A - \lambda \underline{I}) = \det\left(\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix}\right)$$

$$= (\lambda - 25)(\lambda - 9).$$

② Solve the characteristic Eq.

$$(\lambda - 25)(\lambda - 9) = 0.$$

$$\Rightarrow \lambda_1 = 25, \quad \lambda_2 = 9.$$

(Eigen value).

step 3:

③ Find the Eigenvector.

for $\lambda_1 = 25$.

$$\boxed{A \cdot \vec{v}_1 = \overset{25}{\lambda_1} \cdot \vec{v}_1}$$

$$\Rightarrow (A - \overset{25}{\lambda_1} I) \cdot \vec{v}_1 = 0$$

$$\Rightarrow \left(\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} - 25 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \vec{v}_1 = 0$$

$$\Rightarrow \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

$\Rightarrow$ $x_1 = x_2$ is the solution

infinite # of solution

$\Rightarrow$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution

$$\sqrt{1^2 + 1^2}$$

$\sqrt{1^2+1^2} = \sqrt{2}$.

$$\Rightarrow \quad \vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix} \Big/ \underline{\underline{\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|_2}}.$$

$$= \begin{bmatrix}1\\1\end{bmatrix} \Big/ \underline{\underline{\sqrt{2}}} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}.$$

$$\therefore \quad (\underset{\lambda_1}{25}, \underset{\vec{v}_1}{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}}) \text{ is } \underline{\underline{1^{st}}} \ \underline{\underline{\text{eigenpair}}}.$$

for $\underline{\underline{\lambda_2 = 9}}$.

$$\left(\begin{bmatrix}17 & 8\\8 & 17\end{bmatrix} - 9\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0.$$

$$\Rightarrow \quad \begin{bmatrix}8 & 8\\8 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0.$$

$$\Rightarrow \quad x_1 = -x_2 \quad \text{the solution}.$$

$$\Rightarrow \quad \begin{bmatrix}1\\-1\end{bmatrix} \text{ is a soln}$$

$$\Rightarrow \vec{v}_2 = \underline{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \Big/ \underline{\left\|\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\|_2} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow \left(\underset{\lambda_2}{8}, \ \underset{\vec{v}_1}{\begin{bmatrix} +1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}} \right) \text{ is the}$$

second pair

$$\Rightarrow \underbrace{\begin{bmatrix} 17 & 17 \\ 8 & 8 \end{bmatrix}}_{A} = \underline{V \cdot \begin{bmatrix} 21 & 0 \\ 0 & 8 \end{bmatrix} V^T}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$\vec{v}_1$ $\quad$ $\vec{v}_2$

$$\vec{v}_1 \circ \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \circ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$= 1/\sqrt{2} * 1/\sqrt{2} - 1/\sqrt{2} \cdot 1/\sqrt{2} = 0.$$

in addition, $\|\vec{v}_1\| = 1$. $\|\vec{v}_2\| = 1$.

$\Rightarrow$ $\vec{v}_1$ & $\vec{v}_2$ orthonormal against each other.

Singular Value Decomposition

$$A = U \Sigma V^T$$

$A \in \mathbb{R}^{m \times n}$

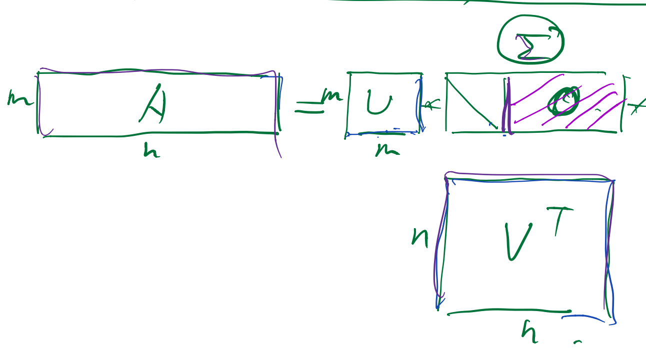
Non-square

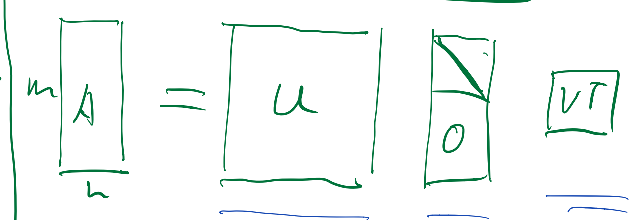
U → Left singular vectors

Σ → singular values

V^T → Right singular vectors

Fat Matrix A $(m < n)$	Thin Matrix A.





Large singular value → primary component

0.1

1000

$$A = \sigma_1 \cdot \vec{u}_1 \vec{v}_1^T + \sigma_2 \cdot \vec{u}_2 \vec{v}_2^T + \cdots$$

pattern mode

10

$$+ \sigma_k \cdot \vec{u}_k \cdot \vec{v}_k.$$

pattern mode

$(k = \min(m, n))$.

ML: primery component Analysis

↓

Large Singular Value

Hands on: Find the SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$. (Ref)

$U : \mathbb{R}^{2\times 2}$, $\Sigma : \mathbb{R}^{2\times 3}$.

$V^T : \mathbb{R}^{3\times 3}$.

Step 1

Solution: Compute the Σ. (singular val)

$\equiv$ compute the Eigenval of $A * A^T$. $(\mathbb{R}^{2\times 2})$

$$A \cdot A^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -1 \end{bmatrix} * \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

The characteristic polynomial is

$$\det(A \cdot A^T - \lambda I) = \lambda^2 - 34\lambda + 225$$

$$= (\lambda - 25)(\lambda - 9) = 0.$$

$\therefore$ Singular values

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{25} = 5.$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{9} = 3.$$

Step 2: Compute the Eigen-vector of $A \cdot A^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$.

$$\Rightarrow \quad \vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}. \quad \vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

Step 3: Compute the Right singular Vector $[\vec{v}_1, \vec{v}_2, \vec{v}_3]$.

$\equiv$ compute the Eigen-vector of $A^T \cdot A$

$$A^T \cdot A = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} \lambda_1 = 25. & \vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}. \\ \lambda_2 = 9. & \vec{v}_2 = \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}. \\ \vec{v}_3 = \vec{v}_1 \underbrace{\times}_{\text{cross product}} \vec{v}_2 \Big/ \|\vec{v}_1 \times \vec{v}_2\| & \end{cases}$$

$$= \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}.$$

Left singular vector

Right singular vector.

$$\therefore \quad A = U \Sigma V^T.$$

$$= \begin{bmatrix} \vec{u}_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T & v_2^T & v_3^T \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{bmatrix}$$

expand it:

primary

$$A = 5 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} +$$

$$3 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{18} & -1/\sqrt{18} & + 4/\sqrt{18} \end{bmatrix}$$