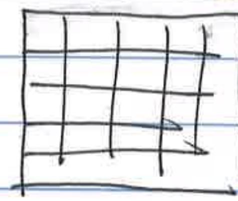
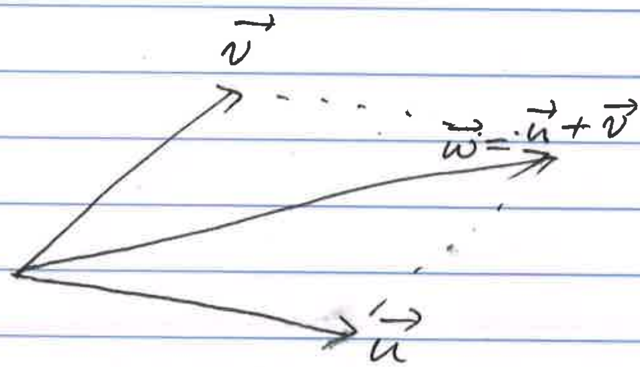
Lesson A 02 : Linear Algebra.

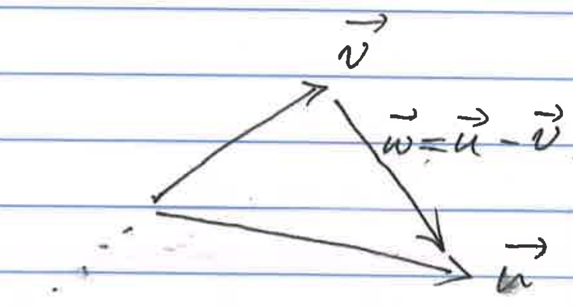
Tensor of various dimensionality.

0-D tensor Scalar	1 D tensor Vector	2 D tensor. Matrix
R.	R^n	$R^{m \times n}$.
α, β, γ.	$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. $\vec{w}^T = [2\ 3\ 1]$.	m  n.

The Operations about vector.

① Addition and Substraction.
$(R^n \times R^n \to R^n)$.

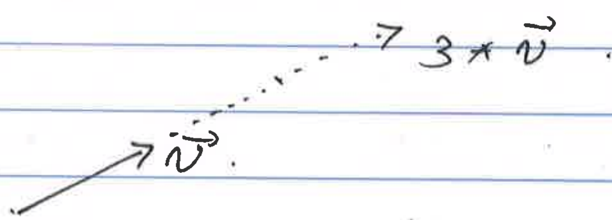




(2) Vector Multiplication

column vector times row vector	row vector times column
$\begin{bmatrix}1\\0\\3\end{bmatrix} * \begin{bmatrix}2 & 1 & 3\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 3\end{bmatrix} * \begin{bmatrix}2\\1\\3\end{bmatrix}$.
$= \begin{bmatrix}1*2 & 1*1 & 1*3\\0*2 & 0*1 & 0*3\\3*2 & 3*1 & 3*3\end{bmatrix}$.	$= 1*2 + 0*1 + 3*3$.
	$=$ inner product of $\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\\3\end{bmatrix}$.

(3) Scalar * Vector.



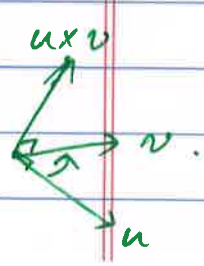
(4). Inner product and Cross product.

($R^n \times R^n \to R$) ($R^n \times R^n \to R^n$)

inner product :

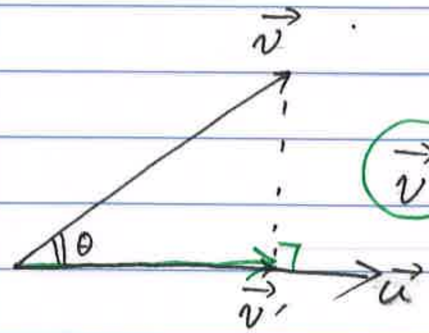
$$\vec{u} \circ \vec{v} = \vec{u}^T * \vec{v} \quad \left(\begin{bmatrix}1\\0\\3\end{bmatrix} \circ \begin{bmatrix}2\\1\\3\end{bmatrix} = 2+0+9 \right)$$

cross product.



$$\vec{u} \times \vec{v}. \left(\begin{bmatrix}1\\0\\3\end{bmatrix} \times \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}i & j & k\\1 & 0 & 3\\2 & 1 & 3\end{bmatrix}\right.$$

Operations related to inner product.



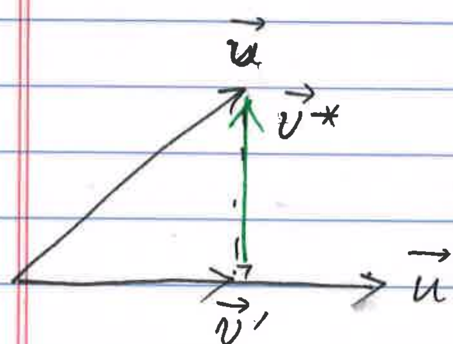
① the projection of $\vec{v}$ over $\vec{u}$

$$\vec{v}' = \frac{\vec{u}}{\|\vec{u}\|} \cdot \|\vec{v}\| \cdot \cos\theta$$

$$= \frac{\vec{u}}{\|\vec{u}\|} \cdot \cancel{\|\vec{v}\|} \cdot \frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \cdot \cancel{\|\vec{v}\|}}$$

$$= \left(\frac{\vec{u} \circ \vec{v}}{\|\vec{u}\|^2} \right) \cdot \vec{u}$$

② Orthogonalization of $\vec{v}$ against $\vec{u}$.



$$\vec{v}^* = \vec{v} - \vec{v}'$$

(projection vector)

③

Magnitude (Length) of Vector

—— Norm of Vector.

$$\|\vec{v}\| = \begin{cases} L_2\text{-norm}: \left\| \begin{bmatrix}1\\2\\3\end{bmatrix} \right\|_2 = \sqrt{1^2+2^2+3^2} \\ \qquad = \sqrt{\begin{bmatrix}1\\2\\3\end{bmatrix} \circ \begin{bmatrix}1\\2\\3\end{bmatrix}} \\ L_1\text{-norm}: \left\| \begin{bmatrix}1\\2\\3\end{bmatrix} \right\| = |1|+|2|+|3| \\ L_\infty\text{-norm}. \left\| \begin{bmatrix}1\\2\\3\end{bmatrix} \right\| = 3 \end{cases}$$

L_p-norm.

$$\|\vec{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}.$$

Applications about Norm.

① The Error of Neural Network.

② Normalize a vector

$$\frac{\vec{v}}{\|v\|}$$. a vector with unit length.

Matrix. (pattern + Magnitude)

① Matrix Norm.

Frobenius norm

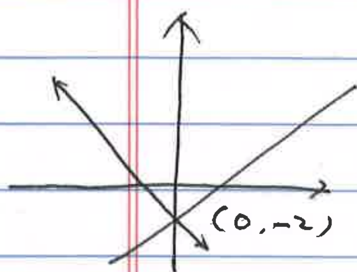
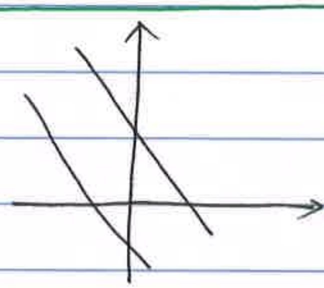
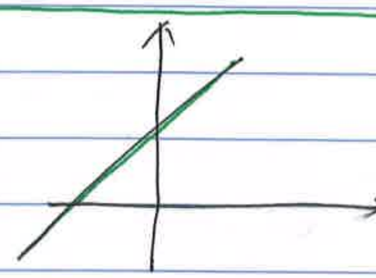
$$\| A \|_F = \sqrt{\Sigma \Sigma |a_{ij}|^2}$$
$$= \sqrt{\mathrm{trace}\ (A^T A)}.$$

(Trace: the sum of the diagonal Elements).

② Operation of Matrix.

- Matrix addition/ substruction
- Matrix vector product
- Matrix scalar product
- Matrix – Matrix product
- Solution of Linear Equation
- Eigen-decomposition.
- Singular Value Decomposition

Solution of Linear Equation.

$\begin{cases} 2x - y = 2. \\ x + y = -2. \end{cases}$	$\begin{cases} 3x + 2y = 3 \\ 3x + 2y = -4 \end{cases}$	$\begin{cases} x - y = 3 \\ 2x - 2y = 6 \end{cases}$
		

One Solution   No solution ∞- many solution.

Example :

$$\begin{cases} x - y = 3 & ① \\ 2x - 2y = 6 & ②. \end{cases}$$

② $-$ ① $* 2$. $\Longrightarrow$ $\begin{cases} x - y = 3. \\ 0 + 0 = 0. \end{cases}$

Assume y is free. $(y+3, \ y)$ is the solution.

Eigen Value / Vector Problem

$Av_i = \lambda_i v_i$. ((λ_i, v_i) is eigen-pair).

$$\begin{cases} A\ v_1 = \lambda_1 \cdot v_1 \\ A\ v_2 = \lambda_2 \cdot v_2 \\ \quad \vdots \\ A\ v_n = \lambda_n\ v_n. \end{cases} \Longrightarrow A V = V \cdot \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}.$$

or.

$$A \cdot \begin{bmatrix} v_1\ v_2 \cdots v_n \end{bmatrix} = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda}.$$

$$\Longrightarrow A = V \cdot \Lambda \cdot V^T.$$

Example : compute the eigen value / vector of $A : \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$.

Step 1: Find the characteristic polynomial.

$\det(A - \lambda I) = \det \begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = (\lambda - 25)(\lambda - 9)$

2 Eigen value are 25 and 9.

Step 2. Find Eigenvalue by solving $(\lambda-25)(\lambda-9)=0$.

Step 3: Find the Eigenvector Associated with each eigenvalue.

for $\lambda_1 = 25$.

$$(A - \lambda_1 I)\ \vec{v}_1 = 0$$

$$\Rightarrow \left(\begin{bmatrix} 17 & +8 \\ 8 & 17 \end{bmatrix} - 25 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$$\Rightarrow \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \Rightarrow (x_2, x_2) \text{ is solution.}$$

After normalization. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is $\vec{v}_1$.

for $\lambda_2 = 9$.

$$\left(\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} - 9 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$$\Rightarrow \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow (-x_2, x_2) \text{ is solut}$$

After Normalizat. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is $\vec{v}_2$.

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Find the Eigenvalue and Eigenvector.

$$A = \begin{bmatrix} 2 & +1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}.$$

Step 1: Find the characteristic Eq.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda \end{vmatrix} = -\lambda^3 + 24\lambda^2 - 65\lambda + 42$$

Step 2. Solve the characteristic eq to find Eigen value.

$$-\lambda^3 + 24\lambda^2 - 65\lambda + 42 = 0.$$

$$\Rightarrow (\lambda - 1)(\lambda - 2)(-\lambda + 21) = 0.$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 21$$

Step 3: Find the Eigenvector associated with each eigenvalue.

for $\lambda_1 = 1$. $$\left(\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot v_1 = 0.$$

$$\Rightarrow \quad v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \ldots$$

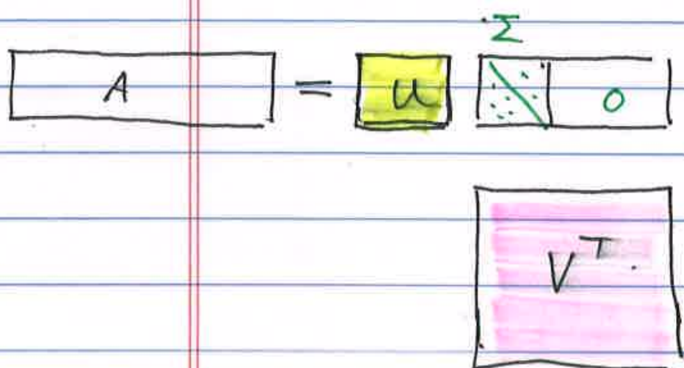
Singular Value Decomposition

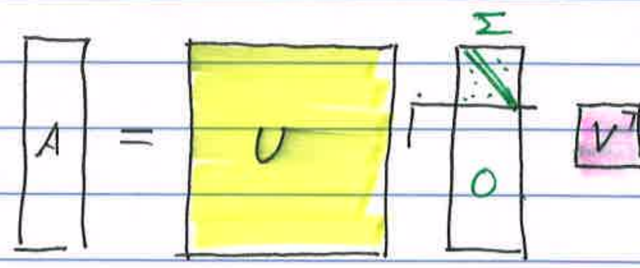
$$\underset{(R^{m\times n})}{A} = U \Sigma V^T$$

Left singular vector

Right singular vector.

Fat Matrix A	Thin Matrix A.





$$A = \sigma_1 u_1 \cdot v_1^T + \sigma_2 \cdot u_2 \cdot v_2^T + \sigma_3 \cdot u_3 v_3^T + \cdots$$

$$+ \sigma_n \cdot u_n \cdot v_n^T .$$

Find the SVD of A, $U\Sigma V^T$, $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$
(Fat Matrix).

Step 1: compute the singular value σ_i by computing the Eigenvalue of $A \cdot A^T$.

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} * \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}.$$

the characteristic polymormial is

$$\det(AA^T - \lambda I) = \lambda^2 - 34\lambda + 225$$
$$= (\lambda - 25)(\lambda - 9)$$

$\therefore$ the singulcr values are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{25} = 5. \quad \sigma_2 = \sqrt{9} = 3.$$

Step 2: Compute the Eigenvector of A^TA (Right Vector)

$$AA^T = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}.$$

for $\lambda_1 = 25$. $\left(\begin{bmatrix} -12 & 12 & 2 \\ 12 & -2 & -2 \\ 2 & -2 & -17 \end{bmatrix} - 25 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot v_1 = 0.$

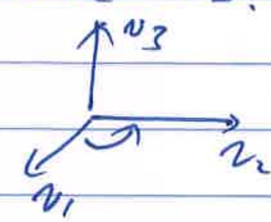
$$\Rightarrow \quad v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

for $\lambda_2 = 9$. $\left(\begin{bmatrix} -12 & 12 & 2 \\ 12 & -2 & -2 \\ 2 & -2 & -17 \end{bmatrix} - 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) v_2 = 0.$

$$\Rightarrow \quad v_2 = \begin{pmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{pmatrix}.$$

for the last right singular vector v_3.

$$v_3 = \frac{v_1 \times v_2}{\| v_1 \times v_2 \|} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}.$$



Step 3 : Compute the left singular vecter

(3.1) $u_1 = \dfrac{A v_1}{\| A v_1 \|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. $\quad u_2 = \dfrac{A v_2}{\| A v_2 \|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

(3.2) or in other way. compute the eigenvector of $AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$.

for $\lambda_1 = 25$. $\left(\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} - 25 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) u_1 = 0$

$$\Rightarrow \quad u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

for $\lambda_2 = 9$. $\Rightarrow \quad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$$\therefore \; A = \underbrace{U}_{\text{left singular vectors}} \Sigma \underbrace{V^T}_{\text{Right singular vectors}} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{bmatrix}$$

Singular Value Decomposition.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = U \Sigma V^T.$$

(Thin Matrix)

Solution:

Right Singular value Vector

$$V^T = [\text{eigenvector}(A^TA)]^T = [v_1\ v_2]^T.$$

with λ_1, λ_2 as eigenvalue.

Left singular Vector

$$U = \left[\frac{1}{\sigma_1} A v_1 \quad \frac{1}{\sigma_2} A v_2 \quad \frac{NS(A^T)}{|NS(A^T)|}\right] \quad (\sigma_i = \sqrt{\lambda_i}).$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}.$$

$$\therefore A = \left[\frac{1}{\sigma_1} A v_1 \quad \frac{1}{\sigma_2} A v_2 \quad \frac{NS(A^T)}{|NS(A^T)|}\right] \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} [v_1\ v_2]^T.$$

In detail:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = 3. & v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \\ \lambda_2 = 2. & \lambda_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

$$\therefore V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T = [v_1\ v_2]^T.$$

$$U = \left[\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \frac{NS(A^T)}{|NS(A^T)|}\right].$$

$NS(A^T) \Rightarrow A^T x = 0.$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Assume that x_3 is free. $x_3 = 1 \Rightarrow x_1 = 1,$
$x_2 = -2.$

$$\therefore \; u_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} / \sqrt{6} \; (\|u_3\|).$$

$$\Rightarrow U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

As a result,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$