Kuttaka
Aryabhata's algorithm

R Gopu

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Varahamihira Science Forum
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Determinate Equations

- Consider this
- $5x - 35 = 0$
- What is x?
- $x = 7$
- How?
- $5 * 7 - 35 = 0$

- Consider this
- $5y^2 + 1 = 21$
- What is y?
- $y = (2, -2)$
- How?
- $5 * 2^2 + 1 = 21$
- $5 * (-2)^2 + 1 = 21$

- You can determine x
- This x only has ONE value

- You can determine y
- This y only has TWO values

LINEAR determinate equation

QUADRATIC determinate equation

Indeterminate Equations

- Consider this
- $5x + 4 = N$
- What is x?
- x can have many values
- N can also have many values
- If $x = 1, N = 9$
- If $x = 2, N = 14$
- If $x = 7, N = 39$

LINEAR indeterminate equation

- Consider this
- $5y^2 + 1 = N$
- What is y?
- Both y and N can have many values

QUADRATIC indeterminate equation

Integer solutions

- $2x + 7 = N$
- $5y^2 + 1 = N$
- N, x, y <u>need not</u> be integers
- They can also be fractions or whole numbers

But in Indian astronomy, and for this course,
We will confine to <u>positive integer solutions</u> ONLY

Chinese Remainder Problem

A grandmother has some bananas, which she distributed equally to 7 grandchildren. 1 banana was left over.

A neighbour's child joined them, so grandma took back all the bananas, and redistributed bananas to all 8 children. Now, 3 bananas were left over

How many bananas did the grandma have originally?

This can be stated as

$$N = 7x + 1 = 8y + 3$$

Simultaneous Linear indeterminate equations

Chinese Remainder problem

This is easy for small values

$$N = 7x + 1$$
$$N = 8y + 3$$

Very hard for large values

$$N = 1841\,x + 245$$
$$N = 8205\,y + 9732$$

Aryabhata's kuTTaka algorithm is a simple method to solve such problems. It was the FIRST such algorithm

Infinite solutions

$$N = ax + r = by + s$$

If you find x, y

Then $(x+b, y+a)$, $(x+2b, y+2a)$, $(x+3b,y+3a)$

… are solutions for (x,y)

For k=1,2,3,4….. any $(x+bk, y+ak)$ are solutions

$N = akx + r = bky + s$

So there are infinite solutions

The trick is to find the SMALLEST solution

Guessing

Let us first solve this problem with guessing.

Remember, we want integer solutions for x, y

$$N = 8x + 3$$

$$N = 7y + 1$$

$$7y + 1 = 8x + 3$$

$$7y = 8x + 3 - 1$$

$$7y = 8x + 2$$

$$y = \frac{8x + 2}{7}$$

$x=1, \quad y = 10/7$

$x=2, \quad y = 18/7$

$x=3, \quad y = 26/7$

$x=4, \quad y = 34/7$

$x=5, \quad y = 42/7 = 6$

$N = 8x + 3 = 43$

$N = 7y + 1 = 43$

Grandma had 43 bananas

kuTTaka - Pulverizer

Can you use guessing method for these two?

$$N = 1841\,x + 245$$

$$N = 8205\,y + 9732$$

No. It will take far too long for such large numbers.

kuTTaka algorithm solves without guessing

kuTTaka means pounding or pulverizing

Breaking a large problem into smaller problems

In Computer science, this type of method is called **Divide and Conquer** algorithm

kuTTaka stanzas 32, 33

अधिकाग्रभागहारं छिन्द्यादूनाग्रभागहारेण ।
adhikaagra-bhaaga-haaram chindyat-UnAgra-bhaaga-haareNa
शेषपरस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तम् ॥ 32 ॥
sheSha-paraspara-bhaktam mati-guNa-agra-antare-kShiptam
अधउपरिगुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेषम् ।
Adha-upari-guNitam-antya-yug-UnAgra-cheda-bhAjite sheSham
अधिकाग्रच्छेदगुणं द्विच्छेदाग्रमदिकाग्रयुतम् ॥ 33 ॥
adhika-agra-ccheda-guNam dvicchedAgram-adhikAgra-yutam

kuTTaka 32

अधिकाग्रभागहारं छिन्द्यादूनाग्रभागहारेण ।
adhikaagra-bhaaga-haaram chindyat-UnAgra-bhaaga-hareNa
शेषपरस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तम् ॥ 32 ॥
sheSha-paraspara-bhaktam mati-guNa-agra-antare kShiptam

adhika	Greater remainder
una	Lesser remainder
agra-bhaaga-haaram	Divisor corresponding to
sheSha	Obtained remainder
paraspara	By one another
bhaktam	divide
mati	Optional (intelligent) number
agraantare	Difference of remainder
kshiptam	add

This is too complicated to understand merely by semantic parsing or grammatic interpretation. Bhaskara 1st's commentary Arybhatiya Bhashya helped gaNaka s understand this algorithm kuTTaka algorithm

Translation

by

KV Sarma, KS Sukla

32-33. Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. (Discard the quotient). Divide the remainder obtained (and the divisor) by one another (until the number of quotients of the mutual division is even and the final remainder is small enough). Multiply the final remainder by an optional number and to the product obtained add the difference of the remainders (corresponding to the greater and smaller divisors; then divide this sum by the last divisor of the mutual division. The optional number is to be so chosen that this division is exact. Now place the quotients of the mutual division one below the other in a column; below them write the optional number and underneath it the quotient just obtained. Then reduce the chain of numbers which have been written down one below the other, as follows): Multiply by the last but one number (in the bottom) the number just above it and then add the number just below it (and then discard the lower number). (Repeat this process until there are only two numbers in the chain). Divide (the upper number) by the divisor corresponding to the smaller remainder, then multiply the remainder obtained by the divisor corresponding to the greater remainder, and then add the greater remainder: the result is the *dvicchedāgra* (i.e., the number answering to the two divisors). (This is also the remainder corresponding to the divisor equal to the product of the two divisors).[1]

kuTTaka

Just like Aryabhata's square root and cube root algorithms, his kuTTaka algorithm is far too complicated and cryptic

Bhaskara 1 (c 630), who wrote Aryabhatiya-bhashya, gave us an explanation, with additional algorithmic notes, which was translated by KV Sarma and KS Sukla

It is much easier to understand by example

kuTTaka Problem 1 – Reduction

23) 63 (2
46
17)23 (1
17
6)17 (2
12
5)6 (1
5
1)5 (5
5
0

$$y = \frac{63x + 7}{23}$$

Take the quotient list

2,1,2,1,5

DROP the first quotient

1,2,1,5

Form valli upa-samhaara

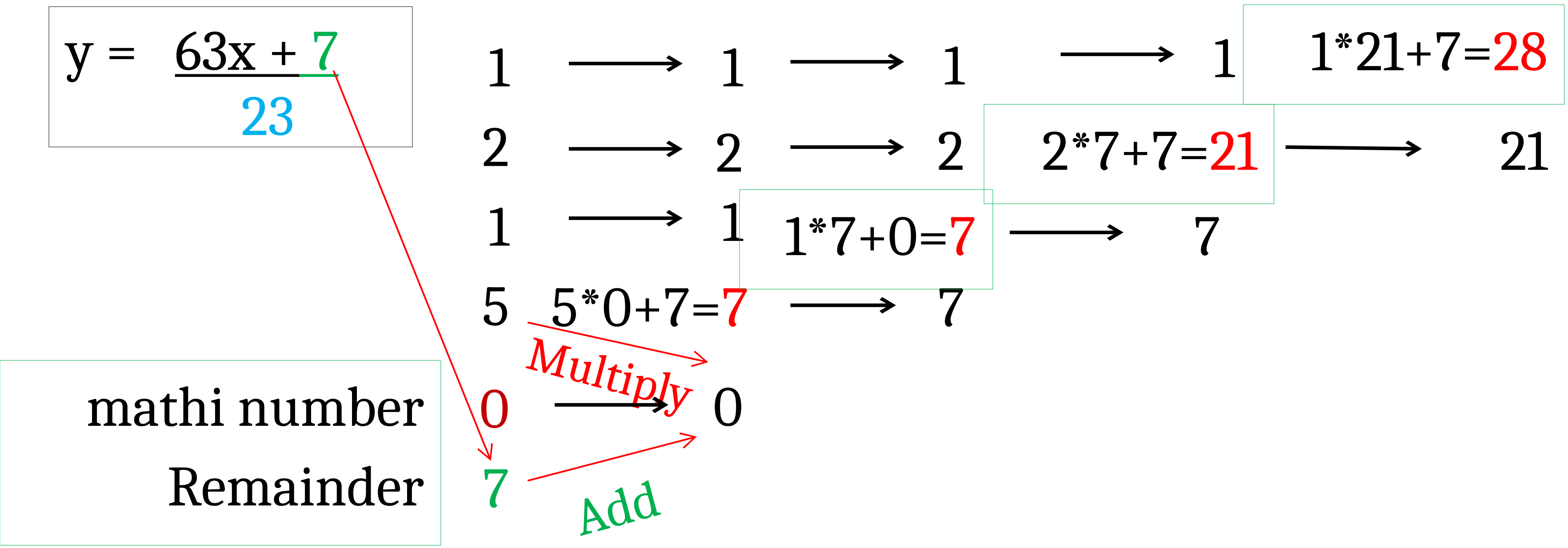
1

2

1

5

Problem 1 – valli upasamhaara



1. Copy second last number of valli
2. Multiply by number above by it
3. Add number below it
4. Copy all numbers above
5. Repeat until top of valli (x value) is reached

kuTTaka – Problem 1 : Find x,y

$y = \dfrac{63x + 7}{23}$

$$\begin{array}{ccccccccc} 1 & \to & 1 & \to & 1 & \to & 1 & & \\ 2 & \to & 2 & \to & 2 & & 21 & \to & 7 \\ 1 & \to & 1 & & 7 & \to & 7 & & \\ 5 & & 7 & \to & 7 & & & & \end{array}$$

x= 28

mathi number $0 \to 0$

Remainder 7

Divide 28 / 23

23)28 (1
23
5

Remainder is 5

So x=5

y = (63x+7)/23 = 328/23 = 14

Solution x=5, y=14

What if we use different mathi?

Problem 1N - Negative remainder

$$y = \frac{63x - 7}{23}$$

	1	1	1	1	-28
	2	2	2	-21	-21
	1	1	-7	-7	
	5	-7	-7		
mathi number	0	0			
Remainder	-7				

5*0-7=-7

1*(-7)+0=-7

2*(-7)-7=-21

1*(-21)-7=-28

kuTTaka finds the smallest POSITIVE INTEGER solutions
Negative numbers are NOT acceptable
If you get FRACTIONS, you have made an arithmetic mistake

We resolve by using better MATHI number
That avoids negative results in valli upasamhara

Prob 1N – Mathi – intelligent number

$$y = \frac{63x - 7}{23}$$

	1	1	1	1	18
	2	2	2	13	13
	1	1	5	5	
	5	3	3		
mathi	2	2			
Remainder	-7				

5*2-7=3

1*3+2=5

2*5+3=13

1*13+5=18

Divide 18/ 23

```
23)18 (0
    0
   --
   18
```

Remainder is 18

So x=18

y = (63x-7)/23 = 1127/23 = 49

Solution x=18, y=49

Simultaneous Equations

Solve simultaneous equations by re-arranging

$N = 12x + 8$

$N = 7y + 3$

Keep <u>greater remainder</u> in numerator (of 8 and 3)

In other words,

$$7y + 3 = 12x + 8$$

$$7y = 12x + 5$$

$$y = \frac{12x + 5}{7}$$

Mahavira's invention – Odd/Even factor

$$y = \frac{12x + 5}{7}$$

Now let us solve this problem using kuTTaka

$$
\begin{array}{l}
7\ )\ 12\ (1 \\
\quad\ \ 7 \\
\quad\ \overline{\ \ 5\ }\ )7\ (1 \\
\qquad\quad\ 5 \\
\qquad\ \overline{\ \ 2\ }\ )5\ (2 \\
\qquad\qquad\ 4 \\
\qquad\quad\ \overline{\ \ 1\ }\ )2\ (2 \\
\qquad\qquad\quad\ 2 \\
\qquad\qquad\ \overline{\ \ 0\ }
\end{array}
$$

Take the quotient list: 1,1,2,2

Always DROP first quotient: 1,2,2

Form valli upa-samhaara

1

2

2

We have an **odd** (3) number of quotients

Aryabhata's kuTTaka method is for **even** number of quotients

Mahavira invented Odd/Even Factor for ODD number of quotients

Problem 2 - Odd/even factor

$$y = \frac{12x + 5}{7}$$

1	$\longrightarrow$ 1	$\longrightarrow$ 1		1*5+1=6	
2	$\longrightarrow$ 2	2*1+3 =5		$\longrightarrow$ 5	
2	2*3-5=1	$\longrightarrow$ 1			
3	$\longrightarrow$ 3				
-5					

mathi

O/E * Remainder
-1 * 5

Choose mathi such that valli numbers are positive

Divide 6/ 7
Remainder is 6
So x=6
y = (12x+5)/7 = 77/7 = 11

Solution x=6, y=11

Odd/even factor is
-1 if odd number of quotients
1 if even number of quotients

Solution for Problem 2

From kuTTaka and valli-upasamhaara

We get $x=6$, $y= 11$

So

$N = 12x + 8 = 80$

$N = 7y + 3 = 80$

kuTTaka – Larger Denominator

$$y = \frac{4x + 1}{13}$$

What if denominator is greater than coefficent(x) ?

$$
\begin{array}{l}
13\ )\ 4\ (0 \\
\quad \underline{\ 0\ } \\
\quad 4\ )\ 13\ (3 \\
\qquad \underline{\ 12\ } \\
\qquad\ 1\ )\ 4\ (4 \\
\qquad\quad \underline{\ 4\ } \\
\qquad\quad\ 0
\end{array}
$$

Take the quotient list

0,3,4

DROP the first quotient

3,4

Form valli upa-samhaara

3

4

In this case, I will stop here

And let you complete Valli Upasamhaara

Problem 3
Using mathi and shortcut

Aryabhata did not go all way to zero in his algorithm

Stop after an even number of quotients (excluding first quotient), to solve using a good *mathi* number

Let us solve, with Aryabhata's method

$$y = \frac{144x + 5}{91}$$

Prob 3 – mathi & shortcut

91) 144 (1
91
53) 91 (1
53
38) 53 (1
38
15) 38 (2
30
8) 15 (1 ← Even
8
7) 8 (1
7
1) 7 (7 ← Even
7
0

$$y = \frac{144x + 5}{91}$$

Valli Upasamhaara

$7m + 5 = k$

8

m=5
k=5

1		1
1		1
2		2
1		1
1	m	5
7	k	5
0		
5		

Quotient list
1,1,2,1,1,7

Instead of dividing upto remainder 0
Stop after even number of quotients

Prob 3 – mathi & shortcut

$$y = \frac{144x + 5}{91}$$

91) 144 (1
91
53) 91 (1
53
38) 53 (1 ← Even
38
15) 38 (2
30
8) 15 (1 ← Even
8
7) 8 (1
7
1) 7 (7 ← Even
7
0

$\underline{15m+5} = k$
38

m=25
k=10

Valli Upasamhaara

1	1		1
1	1		1
2	2	m	25
1	1	k	10
1	5		
7	5		
0			
5			

Quotient list
1,1,2,1,1,7

Instead of dividing upto remainder 0
Stop after even number of quotients

Problem 4
Using mathi and shortcut

Let us solve, another problem, with Aryabhata's method

$$y = \frac{447x + 9}{11}$$

kuTTaka – Prob 4

$$y = \frac{447x + 9}{11}$$

```
11 ) 447 (40
     440
     ---
       7 )11 (1
           7
          --
           4) 7 (1
              4
              -
              3
```

Valli Upasamhaara

	1	1	5
	1	4	4
mathi	1	1	
k	3		

Stop after EVEN number of quotients excluding first quotient

Solve $\frac{3m+9}{4} = k$ [k must be integer]

So m = 1, k=3

Quotient list is 40,1, 1

Solution

x=5

y= (447x+9)/11=204

Large Remainder – Later Aryabhata's method

Aryabhata 2nd (who lived in 10th century, and is far less famous Aryabhata 1st) had a great insight

He realized that if you could solve

$y = (ax+1)/b$ → Equation Alpha

Then you could solve :

$y = (ax + R)/b$ → Equation Beta

If (m,n) are solutions for Remainder 1

Then (R*m, R*n) are solutions for Remainder R

Problem 5
Use 1 instead of large remainder

Take this problem

$$N = 234x + 41 = 173y + 13$$

We should solve: $$y = \frac{234x + 28}{173}$$

Instead we solve: $$y = \frac{234x + 1}{173}$$

Prob 5 (Solve for Remainder 1)

$$
\begin{array}{l}
173\,)\,234\,(1 \\
\quad\;\; \underline{173} \\
\quad\quad 61\,)\,173\,(2 \\
\quad\quad\quad\quad \underline{122} \\
\quad\quad\quad\quad 51\,)\,61\,(1 \\
\quad\quad\quad\quad\quad\quad \underline{51} \\
\quad\quad\quad\quad\quad\quad 10
\end{array}
$$

$$y = \frac{234x + 1}{173}$$

Valli Upasamhaara

	2	2	17
	1	6	6
mathi	5	5	
k	1		

Stop after EVEN number of quotients excluding first quotient

Solve $\frac{10m+1}{51} = k$ [k must be integer]

So m = 5, k=1

Quotient list is 1,2, 1

Solution

x=17

y= (234x+1)/173=23

Problem 5 – Solve for R from 1

If (m,n) are solutions for Remainder 1

Then (R*m, R*n) are solutions for Remainder R

Original problem: $N = 234x + 41 = 173y + 13$

For $y = \dfrac{234x + 1}{173}$

We got x = 17, y = 23

For remainder R=28, solution set is (R*x, R*y)

That is (17*28, 23*28) → (x=476,y=644)

But it may not be the SMALLEST solution!

Problem 5 – Smallest Solution

Original problem: $N = 234x + 41 = 173y + 13$

<u>Aryabhata 2nd's Solution</u> is $(x=476, y=644)$

For SMALLEST solution,

Divide x by coefficient (y) → 476/173

Divide y by coefficient (x) → 644/234

173)476(2
<u>346</u>
130

234) 644 (2
<u>468</u>
176

<u>Smallest Solution</u>

x=130

y=176

Substituting in $N = 234x + 41 = 173y + 13$

$N = 234*130 + 41 = 173*176 + 13 = 30461$

kuTTaka after Aryabha

Brahmagupta

Mahavira (O/E factor)

Aryabhata the 2nd

Sripati

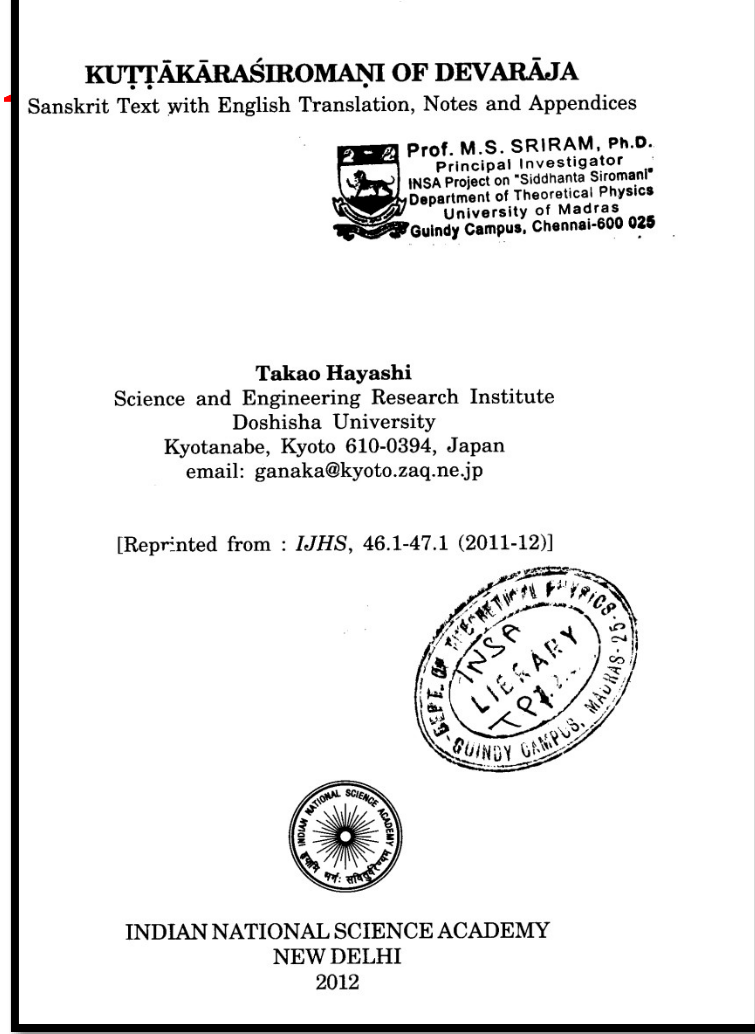
Bhaskara the 2nd

Narayana Pandita (13th century)

Devaraja (15th century)

have all commented on or

improved Aryabhata's method



KUṬṬĀKĀRAŚIROMAṆI OF DEVARĀJA

Sanskrit Text with English Translation, Notes and Appendices

Takao Hayashi
Science and Engineering Research Institute
Doshisha University
Kyotanabe, Kyoto 610-0394, Japan
email: ganaka@kyoto.zaq.ne.jp

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- In their books, kuTTaka is one or more <u>chapters,</u> not stanzas
- Devaraja of Kanchipuram, wrote a book called **KuttakaaShiromani**, dedicated to Kuttaka problems

Devaraja's Saturn problem

At Lanka, when 701 *vighatika-s* (*vinaadi-s*) have elapsed since midnight, the residue of the minute of Saturn was kha-kha-rasa-guna-raama-veda-shara-ashta-aakaasha-achala-eka-sarpa (006334580718, which is 817085433600 in modern notation). O kuTTaka-expert, tell me the aharga Na accompanied by fractional part

Actual Equation to be solved is

Find smallest x, y where

$$y = \frac{12*30*60*146568*x - 817085433600}{5680504180800}$$

146,568 and 5,680,504,180,800 are astronomical constants from Surya Siddhaanta

Devaraja's Saturn solution

Actual Equation to be solved is

Find smallest x, y where

$$y = \frac{12*30*60*146568*x - 817085433600}{5680504180800}$$

- Kuttaka solution using Aryabhata algorithm and Devaraja's modification is:
- x = 22507901
- y = 12544

This is why such huge numbers are called Astronomical numbers

Thank You

<u>References</u>

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