# Algorithms: COMP3121/9101

School of Computer Science and Engineering
University of New South Wales

10. LINEAR PROGRAMMING

# Linear Programming problems - Example 1

**Problem:**

- You are given a list of food sources $f_1, f_2, \dots, f_n$;
- for each source $f_i$ you are given:
  - its price per gram $p_i$;
  - the number of calories $c_i$ per gram, and
  - for each of 13 vitamins $V_1, V_2, \dots, V_{13}$ you are given the content $v(i, j)$ of milligrams of vitamin $V_j$ in one gram of food source $f_i$.
- Your task: to find a combination of quantities of food sources such that:
  - the total number of calories in all of the chosen food is equal to a recommended daily value of 2000 calories;
  - the total intake of each vitamin $V_j$ is at least the recommended daily intake of $w_j$ milligrams for all $1 \leq j \leq 13$;
  - the price of all food per day is as low as possible.

# Linear Programming problems - Example 1 cont.

- To obtain the corresponding constraints let us assume that we take $x_i$ grams of each food source $f_i$ for $1 \leq i \leq n$. Then:
  - the total number of calories must satisfy
  $$\sum_{i=1}^{n} x_i c_i = 2000;$$
  - for each vitamin $V_j$ the total amount in all food must satisfy
  $$\sum_{i=1}^{n} x_i v(i,j) \geq w_j \quad (1 \leq j \leq 13);$$
  - an implicit assumption is that all the quantities must be non-negative numbers,
  $$x_i \geq 0, \quad 1 \leq i \leq n.$$
- Our goal is to minimise the objective function which is the total cost
$$y = \sum_{i=1}^{n} x_i p_i.$$
- Note that all constraints and the objective function, are **linear**.

# Linear Programming problems - Example 2

**Problem:**

- Assume now that you are politician and you want to make certain promises to the electorate which will ensure that your party will win in the forthcoming elections.
- You can promise that you will build
  - a certain number of bridges, each 3 billion a piece;
  - a certain number of rural airports, each 2 billion a piece, and
  - a certain number of olympic swimming pools each a billion a piece.
- You were told by your wise advisers that
  - each bridge you promise brings you 5% of city votes, 7% of suburban votes and 9% of rural votes;
  - each rural airport you promise brings you no city votes, 2% of suburban votes and 15% of rural votes;
  - each olympic swimming pool promised brings you 12% of city votes, 3% of suburban votes and no rural votes.
- In order to win, you have to get at least 51% of each of the city, suburban and rural votes.
- You wish to win the election by cleverly making a promise that **appears** that it will blow as small hole in the budget as possible, i.e., that the total cost of your promises is as low as possible.

# Linear Programming problems - Example 2

- We can let the number of bridges to be built be $x_b$, number of airports $x_a$ and the number of swimming pools $x_p$.
- We now see that the problem amounts to minimising the objective $y = 3x_b + 2x_a + x_p$, while making sure that the following constraints are satisfied:

$$\begin{aligned}
0.05x_b \phantom{{}+0.02x_a} + 0.12x_p &\geq 0.51 \quad \text{(securing majority of city votes)}\\
0.07x_b + 0.02x_a + 0.03x_p &\geq 0.51 \quad \text{(securing majority of suburban votes)}\\
0.09x_b + 0.15x_a \phantom{{}+0.03x_p} &\geq 0.51 \quad \text{(securing majority of rural votes)}\\
x_b, x_a, x_p &\geq 0.
\end{aligned}$$

- However, there is a very significant difference with the first example:
  - you can eat 1.56 grams of chocolate, but
  - you cannot promise to build 1.56 bridges, 2.83 airports and 0.57 swimming pools!
- The second example is an example of an **Integer Linear Programming problem**, which requires all the solutions to be integers.
- Such problems are MUCH harder to solve than the "plain" Linear Programming problems whose solutions can be real numbers.

# Linear Programming problems

- In the **standard form** the *objective* to be maximised is given by

$$\sum_{j=1}^{n} c_j \, x_j$$

- the *constraints* are of the form

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad 1 \leq i \leq m; \tag{1}$$

$$x_j \geq 0, \qquad 1 \leq j \leq n, \tag{2}$$

- Let the boldface $\mathbf{x}$ represent a (column) vector, $\mathbf{x} = \langle x_1 \ldots x_n \rangle^{\intercal}$.
- To get a more compact representation of linear programs we introduce a partial ordering on vectors $\mathbf{x} \in \mathbf{R}^n$ by $\mathbf{x} \leq \mathbf{y}$ if and only if the corresponding inequalities hold coordinate-wise, i.e., if and only if $x_j \leq y_j$ for all $1 \leq j \leq n$.

# Linear Programming

- Letting $\mathbf{c} = \langle c_1 \dots c_n \rangle^\intercal \in \mathbf{R}^n$ and $\mathbf{b} = \langle b_1 \dots b_m \rangle^\intercal \in \mathbf{R}^m$, and letting $A$ be the matrix $A = (a_{ij})$ of size $m \times n$, we get that the above problem can be formulated simply as:
  - maximize $\mathbf{c}^\intercal \mathbf{x}$
  - subject to the following two (matrix-vector) constraints:

$$A\mathbf{x} \le \mathbf{b}$$

  and

$$\mathbf{x} \ge \mathbf{0}.$$

- Thus, to specify a Linear Programming optimisation problem we just have to provide a triplet $(A, \mathbf{b}, \mathbf{c})$;
- This is the usual form which is accepted by most standard LP solvers.

- The value of the objective for any value of the variables which makes the constraints satisfied is called a *feasible solution* of the LP problem.
- Equality constraints of the form $\sum_{i=1}^{n} a_{ij}x_i = b_j$ can be replaced by two inequalities: $\sum_{i=1}^{n} a_{ij}x_i \geq b_j$ and $\sum_{i=1}^{n} a_{ij}x_i \leq b_j$; thus, we can assume that all constraints are inequalities.
- In general, a "natural formulation" of a problem as a Linear Program does not necessarily produce the non-negativity constrains for all of the variables.
- However, in the standard form such constraints are required for all of the variables.
- This poses no problem, because each occurrence of an unconstrained variable $x_j$ can be replaced by the expression $x'_j - x^*_j$ where $x'_j, x^*_j$ are new variables satisfying the constraints $x'_j \geq 0,\ x^*_j \geq 0$.
- If $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector, we let $|\mathbf{x}| = (|x_1|, \ldots, |x_n|)$. Some problems are naturally translated into constraints of the form $|A\mathbf{x}| \leq \mathbf{b}$. This also poses no problem because we can replace such constraints with two linear constraints: $A\mathbf{x} \leq \mathbf{b}$ and $-A\mathbf{x} \leq \mathbf{b}$ because $|x| \leq y$ if and only if $x \leq y$ and $-x \leq y$.

# Linear Programming - Standard Form

- Standard Form: maximize $\mathbf{c}^{\intercal}\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
- Any vector $\mathbf{x}$ which satisfies the two constraints is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^{\intercal}\mathbf{x}$ might be.
- As an example, let us consider the following optimisation problem:

$$
\begin{aligned}
&\text{maximize} && z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 && (3)\\
&\text{subject to the constraints} && && \\
& && x_1 + x_2 + 3x_3 \leq 30 && (4)\\
& && 2x_1 + 2x_2 + 5x_3 \leq 24 && (5)\\
& && 4x_1 + x_2 + 2x_3 \leq 36 && (6)\\
& && x_1, x_2, x_3 \geq 0 && (7)
\end{aligned}
$$

- How large can the value of the objective $z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3$ be, without violating the constraints?
- If we add inequalities (4) and (5), we get

$$3x_1 + 3x_2 + 8x_3 \leq 54 \qquad (8)$$

- Since all variables are constrained to be non-negative, we are assured that

$$3x_1 + x_2 + 2x_3 \leq 3x_1 + 3x_2 + 8x_3 \leq 54$$

# Linear Programming - Standard Form

$$\begin{aligned}
\text{maximize:} \quad & z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 & (3)\\
\text{with constraints:} \quad & x_1 + x_2 + 3x_3 \leq 30 & (4)\\
& 2x_1 + 2x_2 + 5x_3 \leq 24 & (5)\\
& 4x_1 + x_2 + 2x_3 \leq 36 & (6)\\
& x_1, x_2, x_3 \geq 0 & (7)
\end{aligned}$$

- Thus the objective $z(x_1, x_2, x_3)$ is bounded above by 54, i.e., $z(x_1, x_2, x_3) \leq 54$.
- Can we obtain a tighter bound? We could try to look for coefficients $y_1, y_2, y_3 \geq 0$ to be used to for a linear combination of the constraints:

$$\begin{aligned}
y_1(x_1 + x_2 + 3x_3) &\leq 30y_1\\
y_2(2x_1 + 2x_2 + 5x_3) &\leq 24y_2\\
y_3(4x_1 + x_2 + 2x_3) &\leq 36y_3
\end{aligned}$$

- Then, summing up all these inequalities and factoring, we get

$$x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3) \leq 30y_1 + 24y_2 + 36y_3$$

# Linear Programming - Standard Form

$$\begin{aligned}
\text{maximize:} \quad & z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 && (3)\\
\text{with constraints:} \quad & x_1 + x_2 + 3x_3 \leq 30 && (4)\\
& 2x_1 + 2x_2 + 5x_3 \leq 24 && (5)\\
& 4x_1 + x_2 + 2x_3 \leq 36 && (6)\\
& x_1, x_2, x_3 \geq 0 && (7)
\end{aligned}$$

- So we got

$$x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3) \leq 30y_1 + 24y_2 + 36y_3 \qquad (9)$$

- If we compare this with our objective (3) we see that if we choose $y_1, y_2$ and $y_3$ so that:

$$\begin{aligned}
y_1 + 2y_2 + 4y_3 &\geq 3\\
y_1 + 2y_2 + y_3 &\geq 1\\
3y_1 + 5y_2 + 2y_3 &\geq 2
\end{aligned}$$

then

$$3x_3 + x_2 + 2x_3 \leq x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3)$$

Combining this with (9) we get:

$$30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$$

# Linear Programming - Standard Form

- Consequently, in order to find as tight upper bound for our objective $z(x_1, x_2, x_3)$ of the problem $P$:

$$
\begin{aligned}
\text{maximize:} \quad & z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 && (3)\\
\text{with constraints:} \quad & x_1 + x_2 + 3x_3 \leq 30 && (4)\\
& 2x_1 + 2x_2 + 5x_3 \leq 24 && (5)\\
& 4x_1 + x_2 + 2x_3 \leq 36 && (6)\\
& x_1, x_2, x_3 \geq 0 && (7)
\end{aligned}
$$

  we have to look for $y_1, y_2, y_3$ which solve problem $P^*$:

$$
\begin{aligned}
\text{minimise:} \quad & z^*(y_1, y_2, y_3) = 30y_1 + 24y_2 + 36y_3 && (10)\\
\text{with constraints:} \quad & y_1 + 2y_2 + 4y_3 \geq 3 && (11)\\
& y_1 + 2y_2 + y_3 \geq 1 && (12)\\
& 3y_1 + 5y_2 + 2y_3 \geq 2 && (13)\\
& y_1, y_2, y_3 \geq 0 && (14)
\end{aligned}
$$

  then $z^*(y_1, y_2, y_3) = 30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$ will be a tight upper bound for $z(x_1, x_2, x_3)$
- The new problem $P^*$ is called the *dual problem* for the problem $P$.

# Linear Programming - Standard Form

- Let us now repeat the whole procedure with $P^*$ in place of $P$, i.e., let us find the dual program $(P^*)^*$ of $P^*$.
- We are now looking for $z_1, z_2, z_3 \geq 0$ to multiply inequalities (11)-(13) and obtain

$$z_1(y_1 + 2y_2 + 4y_3) \geq 3z_1$$
$$z_2(y_1 + 2y_2 + y_3) \geq z_2$$
$$z_3(3y_1 + 5y_2 + 2y_3) \geq 2z_3$$

- Summing these up and factoring produces

$$y_1(z_1 + z_2 + 3z_3) + y_2(2z_1 + 2z_2 + 5z_3) + y_3(4z_1 + z_2 + 2z_3) \geq 3z_1 + z_2 + 2z_3 \tag{15}$$

- If we choose multipliers $z_1, z_2, z_3$ so that

$$z_1 + z_2 + 3z_3 \leq 30 \tag{16}$$
$$2z_1 + 2z_2 + 5z_3 \leq 24 \tag{17}$$
$$4z_1 + z_2 + 2z_3 \leq 36 \tag{18}$$

we will have:

$$y_1(z_1 + z_2 + 3z_3) + y_2(2z_1 + 2z_2 + 5z_3) + y_3(4z_1 + z_1 + 2z_3) \leq 30y_1 + 24y_2 + 36y_3$$

- Combining this with (15) we get

$$3z_1 + z_2 + 2z_3 \leq 30y_1 + 24y_2 + 36y_3$$

# Linear Programming - Standard Form

- Consequently, finding the dual program $(P^*)^*$ of $P^*$ amounts to maximising the objective $3z_1 + z_2 + 2z_3$ subject to the constraints

$$\begin{aligned} z_1 + z_2 + 3z_3 &\leq 30 \\ 2z_1 + 2z_2 + 5z_3 &\leq 24 \\ 4z_1 + z_2 + 2z_3 &\leq 36 \end{aligned}$$

- But note that, except for having different variables, $(P^*)^*$ is exactly our starting program $P$. Thus, the dual program $(P^*)^*$ for program $P^*$ is just $P$ itself, i.e., $(P^*)^* = P$.

- So, at the first sight, looking for the multipliers $y_1, y_2, y_3$ did not help much, because it only reduced a maximisation problem to an equally hard minimisation problem.
- It is now useful to remember how we proved that the Ford - Fulkerson Max Flow algorithm in fact produces a **maximal flow**, by showing that it terminates only when we reach the capacity of a **minimal cut**.

# Linear Programming - primal/dual problem forms

- The original, *primal* Linear Program $P$ and its *dual* Linear Program can be easily described in the most general case:

$$
\begin{aligned}
P: \quad & \text{maximize} & z(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j, & \\
& \text{subject to the constraints} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i; & \quad 1 \leq i \leq m \\
& & x_1, \ldots, x_n \geq 0; & \\
P^*: \quad & \text{minimize} & z^*(\mathbf{y}) = \sum_{i=1}^{m} b_i y_i, & \\
& \text{subject to the constraints} & \sum_{i=1}^{m} a_{ij} y_i \geq c_j; & \quad 1 \leq j \leq n \\
& & y_1, \ldots, y_m \geq 0, &
\end{aligned}
$$

or, in matrix form,

$$
\begin{aligned}
P: \quad & \text{maximize} \quad z(\mathbf{x}) = \mathbf{c}^{\mathsf{T}}\mathbf{x}, \ \text{subject to the constraints} \quad A\mathbf{x} \leq \mathbf{b} \ \text{and} \ \mathbf{x} \geq 0; \\
P^*: \quad & \text{minimize} \quad z^*(\mathbf{y}) = \mathbf{b}^{\mathsf{T}}\mathbf{y}, \ \text{subject to the constraints} \quad A^{\mathsf{T}}\mathbf{y} \geq \mathbf{c} \ \text{and} \ \mathbf{y} \geq 0.
\end{aligned}
$$

# Weak Duality Theorem

- Recall that any vector $\mathbf{x}$ which satisfies the two constraints, $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^{\mathsf{T}}\mathbf{x}$ might be.
- **Theorem** If $x = \langle x_1 \dots x_n \rangle$ is any basic feasible solution for $P$ and $y = \langle y_1 \dots y_m \rangle$ is any basic feasible solution for $P^*$, then:

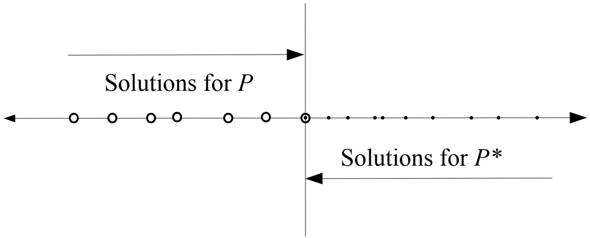
$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

**Proof:** Since $x$ and $y$ are basic feasible solutions for $P$ and $P^*$ respectively, we can use the constraint inequalities, first from $P^*$ and then from $P$ to obtain

$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{n} b_i y_i = z^*(y)$$

- Thus, the value of (the objective of $P^*$ for) any feasible solution of $P^*$ is an upper bound for the set of all values of (the objective of $P$ for) all feasible solutions of $P$, and
- every feasible solution of $P$ is a lower bound for the set of feasible solutions for $P^*$.

# Weak Duality Theorem




- Thus, if we find a feasible solution for $P$ which is equal to a feasible solution to $P^*$, such solution must be the maximal feasible value of the objective of $P$ and the minimal feasible value of the objective of $P^*$.
- If we use a search procedure to find an optimal solution for $P$ we know when to stop: when such a value is also a feasible solution for $P^*$.
- This is why the most commonly used LP solving method, the SIMPLEX method, produces optimal solution for $P$, because it stops at a value of the primal objective which is also a value of the dual objective.
- See the Lecture Notes for the details and an example of how the SIMPLEX algorithm runs.

# PUZZLE!!

Five sisters are alone in their house. Sharon is reading a book, Jennifer is playing chess, Cathrine is cooking and Ana is doing laundry. What is Helen, the fifth sister, doing?