ADVANCED SOLID MECHANICS

Project Statement 2021 – 2022

Laruelle Cedric
Assistant
E-mail: cedric.laruelle@uliege.be

University of Liege
Aerospace and Mechanical engineering department/LTAS
B52/3 (office +2/541)
B-4000 Liege - Belgium

Master in Aerospace engineering, Master in Mechanical engineering and Isolated Students
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Consider a cube subjected to a surface traction (positive or negative) t and whose geometry is defined by (cf. Figure 1) :

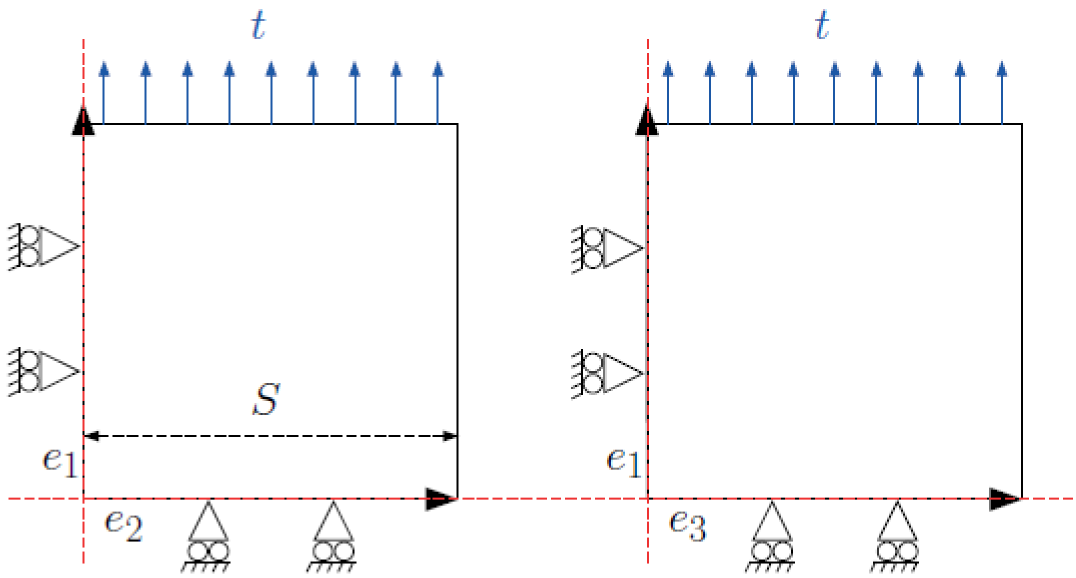

Figure 1: Schematic view of a cube subjected to a pressure t in simple tension or compression

The inertial forces and the temperature effects are neglected (isothermal problem and quasi-static loading).

You are asked to study the behavior of the cube according to several hardening laws included in Table (1).

Hardening law			
Hardening	Isotropic ($\theta = 1$)	linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0 + h_i \bar{\varepsilon}^{vp}$ et $\alpha_{ij} = 0$
		non-linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^\infty - \left(\sigma_{\mathrm{y}}^\infty - \sigma_{\mathrm{y}}^0\right) e^{\left(-\frac{h_i \bar{\varepsilon}^{vp}}{\sigma_{\mathrm{y}}^\infty - \sigma_{\mathrm{y}}^0}\right)}$ et $\alpha_{ij} = 0$
	Kinematic ($\theta = 0$)	linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0$ et $\dot{\alpha}_{ij} = \frac{2}{3} h_k D_{ij}^{vp}$
		non-linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0$ et $\dot{\alpha}_{ij} = \frac{2}{3} h_k D_{ij}^{vp} - \eta_k \dot{\bar{\varepsilon}}^{vp} \alpha_{ij}$
	Mixed ($\theta = \theta^m$)	linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0 + h_i \bar{\varepsilon}^{vp}$ et $\dot{\alpha}_{ij} = \frac{2}{3} h_k D_{ij}^{vp}$
Viscoplastic law (Perzyna)			$\lambda = \sqrt{\frac{3}{2}} \left\langle \frac{\bar{\sigma}^{VM} - \sigma_{\mathrm{y}}}{\eta} \right\rangle$
Hardening parameters			$h_i = \theta h$ $h_k = (1-\theta) h$

Table 1: Hardening law.

The current yield stress σ_{y} represents, at about $\sqrt{\dfrac{2}{3}}$, the radius of von Mises' yield surface. The components of the backstress tensor α_{ij} represents the current position of the yield surface center. (The equivalent backstress is defined as $\bar{\alpha} = \sqrt{\dfrac{3}{2}\, \alpha_{ij} \alpha_{ij}}$). The equation of von Mises' yield criterion is $\bar{\sigma}^{\mathrm{VM}} = \sqrt{\dfrac{3}{2} (s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})} = \sigma_{\mathrm{y}}$. The equivalent viscoplastic strain is defined as $\bar{\varepsilon}^{vp}(t) = \bar{\varepsilon}^{vp}(t_0) + \displaystyle\int_{t_0}^{t} \sqrt{\dfrac{2}{3}\, D_{ij}^{vp} D_{ij}^{vp}}\, dt$.

The geometry and material data, as well as the loading for each group, are included in Tables (2), (3) and (4).

	S $[mm]$	$\bar{\varepsilon}^{\text{vp}}{}_{\text{max}}$ [%]
Geometry 1	100	0.25
Geometry 2	80	0.3
Geometry 3	120	0.35
Geometry 4	50	0.4
Geometry 5	75	0.45

Table 2: Geometry and loading data.

	ρ $[kg/m^3]$	E $[MPa]$	ν $[-]$	σ_{y}^{0} $[MPa]$	h $[MPa]$	θ $[-]$	$\sigma_{\text{y}}^{\infty}$ $[MPa]$
Material 1	7850	205000	0.3	200	30000	0.2	300
Material 2	2700	70000	0.33	100	16000	0.35	250
Material 3	4500	110000	0.25	300	40000	0.75	425

Table 3: Material parameter data.

Group	Geometry and Loading	Material
1	1	1
2	1	2
3	1	3
4	2	1
5	2	2
6	2	3
7	3	1
8	3	2
9	3	3
10	4	1
11	4	2
12	4	3
13	5	1
14	5	2
15	5	3

Table 4: Geometry, loading and material data associated to the group number.

Metafor (80%).

The following tasks will be performed with the Metafor software

- **Part 1 : Study of elasto-plastic behavior with linear hardening.**
  By studying several loading/unloading cycles (A cycle corresponds to a linear evolution of the imposed load from 0 to $t_{\max}$, from $t_{\max}$ to $-t_{\max}$ and from $-t_{\max}$ to 0.), you are asked to:

  - For a plastic model with isotropic hardening, determine the maximum surface traction $t_{\max}$ needed in a loading cycle to reach a permanent equivalent viscoplastic deformation $\bar{\varepsilon}^{\mathrm{vp}}{}_{\max}$;
  - The cube represented in Figure 1 is in plane stress. Determine the boundary conditions needed to have a state of plane strain instead of a state of of plane stress;
  - Compare the behavior of the cube if you consider a perfectly plastic model, a linear isotropic hardening, a linear kinematic hardening and a linear mixed hardening in both a state of plane stress and a state of plane strain ;
  - Describe and explain the evolution of the relevant variables included hereafter for a loading/unloading cycle ;

  For further guidance, follow these guidelines :

  - The relevant variables are the equivalent backstress $\bar{\alpha}(t)$, the equivalent stress $\sqrt{3J_2(s_{ij})}(t)$, von Mises' equivalent stress $\bar{\sigma}^{\mathrm{VM}}(t) = \sqrt{\dfrac{3}{2}\left(s_{ij}-\alpha_{ij}\right)\left(s_{ij}-\alpha_{ij}\right)}$, the current yield stress $\sigma_{\mathrm{y}}(t)$, the equivalent plastic strain $\bar{\varepsilon}^{\mathrm{p}}(t)$, all evaluated at a material point of the cube;
  - What happens in the case of the perfectly plastic model ?
  - Do the equivalent stress $\sqrt{3J_2}$ and the equivalent backstress $\bar{\alpha}$ take a zero value when the imposed loads $p(t)$ evolves from $t_{\max}$ to $-t_{\max}$? (What happens according to the elasto-plastic model, when the number of loading/unloading cycles approaches the infinity ?) Does the equivalent stress $\sqrt{3J_2}$ take a zero value when the imposed load $t(t)$ is equal to 0 at the end of the cycles ?
  - Explain the reason why the equivalent plastic strain $\bar{\varepsilon}^{\mathrm{vp}}$ in the cube takes the same value when the imposed loads reaches for the first time its maximal value $t_{\max}$, whatever the linear hardening model (Tip : use the expression of the plastic multiplier λ); Is there a difference between the state of plane strain and plane stress ? Why ?
  - Study the problem in Haigh Westergaard's space in order to compare the different models and to comment the evolution of the relevant variables ;
  - What happens if you invert the loading direction, i.e. $t(t) := -t(t)$?
  - How does the plastic dissipation ($\mathbb{D} = \sigma_{ij} D_{ij}^{vp}$) evolve according to the elasto-plastic model ?

- **Part 2 : Study of elasto-plastic behavior with non-linear hardening.**
  While studying several loading/unloading cycles $t_{\max}$, you are asked to study the non-linear effects of the hardening law by considering an elasto-plastic behavior with a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor in plane stress state. You are asked to:

- Analyze the influence of the dynamic recovery parameter η_k and determine the limit case(s) ;
- For a given value of the dynamic recovery parameter η_k (different from the limit case(s)), study the influence of the maximum prescribed load $t_{\max}$ on the behavior of the cube in the case of :
  * a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor,
  * a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor combined with a linear isotropic hardening ($\theta = \theta^m$),
  * a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor combined with a non-linear isotropic hardening described by Voce's saturated law ($\theta = \theta^m$);
- If one combines a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor with a linear isotropic hardening ($\theta = \theta^m$), does the yield stress evolve linearly ? Justify (Tip : determine the analytical expression of the plastic multiplier λ) ;
- If one combines a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor with a non-linear isotropic hardening described by Voce's saturated law ($\theta = \theta^m$), does the equivalent backstress always tend toward zero when the number of loading/unloading cycles approaches the infinity ? Justify.

For further guidance, follow these guidelines :

- Determine the unit of the parameter η_k ;
- Write down the evolution laws of the hardening parameters under the general form $\overset{\bullet}{\boldsymbol{q}}^{(k)} = \lambda \times r^{(k)}(\boldsymbol{\sigma}, \boldsymbol{q})$;
- Write down the expression of the generalized plastic modulus H^p (cf. Appendix) in the case of a non-linear mixed hardening (isotropic and kinematic) (Tip : use the consistency condition $\overset{\bullet}{f}\,((\boldsymbol{\sigma}, \boldsymbol{q}) = 0$ and insert the expression derived for $\dfrac{\partial f}{\partial \sigma_{ij}}\,\overset{\bullet}{\sigma}_{ij}$ into the expression of H^p) ;
- From the previous results, deduce graphically in Haigh Westergaard's space, the influence of the dynamic recovery term $-\eta_k\,\overset{\bullet}{\bar{\varepsilon}}{}^{\mathrm{p}}\,\alpha_{ij}$ on the generalized plastic modulus H^p for a loading/unloading cycle (Tip : study the influence of the term $(\boldsymbol{s} - \boldsymbol{\alpha}) : \boldsymbol{\alpha}$ on H^p at crucial steps of one loading/unloading cycle) ?
- How does the plastic dissipation evolve through the loading/unloading cycles ? Compare your findings with a linear kinematic hardening.
- Study the non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor (The components α_{ij} of the backstress tensor are chosen such that $\bar{\alpha} = |\alpha|$ and the components ε^p_{ij} of the plastic strain tensor are chosen such that $\bar{\varepsilon}^{\mathrm{p}}(t) = \bar{\varepsilon}^{\mathrm{p}}(t_0) + \displaystyle\int_{t_0}^{t} \left|\overset{\bullet}{\varepsilon}{}^p\right| dt$). Write down the first order differential equation in α with respect to ε^p and deduce the analytical expression of $\alpha = f(\varepsilon^p)$ for the tensile and compression parts (Inelastic loading tensile/compression cycle) and plot the curve $\bar{\sigma} = f(\bar{\varepsilon}^{\mathrm{p}})$ for the tensile and compression parts knowing that

$\frac{d\,\bar{\sigma}}{d\,\bar{\varepsilon}^{\mathrm{p}}} = h = \sqrt{\frac{3}{2}}H^p$. In the case of a loading/unloading cycle, does the curve $\bar{\sigma} = f(\bar{\varepsilon}^{\mathrm{p}})$ represent a closed cycle in the space $\bar{\sigma} - \bar{\varepsilon}^{\mathrm{p}}$?

- Determine the asymptotic value of the backstress tensor α_{ij}^u if the plastic multiplier $\lambda > 0$, i.e. when $\dot{\alpha}_{ij} = 0$ for $\alpha_{ij} = \alpha_{ij}^u$. Deduce the asymptotic value of the equivalent backstress $\bar{\alpha}^u$? What is the influence of η_k ?
- Determine the asymptotic value of the yield stress σ_{y}^u if the plastic multiplier $\lambda > 0$, i.e. when $\dot{\sigma}_{\mathrm{y}} = 0$ for $\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^u$,
- From these results, deduce the upper bound of the equivalent stress $\sqrt{3J_2}$ in Haigh-Westergaard's space (Tip : use the expression of von Mises' yield stress $\bar{\sigma}^{\mathrm{VM}}$ and the triangular inequality $\left\|\vec{a} + \vec{b}\right\| \leq \|\vec{a}\| + \left\|\vec{b}\right\|$). Write down the expression of the asymptotic surface described by the upper bound of the equivalent stress $\sqrt{3J_2}$. Draw the yield surface and the asymptotic surface in Haigh-Westergaard's space !
- What happens if/when the equivalent backstress $\bar{\alpha}$ and/or the yield stress σ_{y} reaches their respective asymptotic values ?

- **Part 3 : Study of elasto-viscoplastic behavior.**
  While studying several loading/unloading cycles, you are asked to study the viscous effects by considering elasto-viscoplastic behaviors with isotropic and mixed linear hardening:

  - Analyze the influence of the viscosity parameter η and determine the limit case(s) ,
  - Show the effects on the cube behavior, if one modifies the preceding sawtooth loading by adding steps at the imposed displacement extrema, i.e. by keeping the imposed load constant at $t_{\max}$, 0 and -$t_{\max}$ for a certain duration of time.

  For further guidance, follow these guidelines :

  - Determine the unit of parameter η ;
  - Does the presence of viscoplasticity in the material delay the first entrance into plasticity of the cube ?
  - After a loading/unloading inelastic cycle, the imposed load is kept at a non-zero value for a long time. How do the components of the deformation tensor evolve in the cube ?
  - Demonstrate in the case of an elasto-viscoplastic behavior without hardening, how will the plastic deformation evolve if the imposed load is kept at its maximum value $t_{\max}$ for an infinitely long time ? Deduce the influence of the viscosity parameter on the cube behavior;
  - Demonstrate in the case of an elasto-viscoplastic behavior with an isotropic hardening , how will the plastic deformation evolve if the imposed load is kept at its maximum value $t_{\max}$ for an infinitely long time ?
  - Extrapolate your findings in the case of a linear mixed hardening : how do the equivalent backstress $\bar{\alpha}$ and the yield stress σ_{y} evolve if the imposed load is kept at its maximum value $t_{\max}$ for an infinitely long time ?
  - Study the evolution of the signed distance with respect to the yield surface center $d = \bar{\sigma}^{\mathrm{VM}} - \sigma_{\mathrm{y}}$ during the loading/unloading cycles ? Study the problem in Haigh Westergaard's space ;

  - Does the maximum signed distance $d_{\max}$, i.e. when the imposed load reaches its maximum value $t_{\max}$, depend on the loading speed ? (Tip : refer to the observations performed about the experimental data for elasto-viscoplastic material.)

- **Part 4 : Sensitivity study of numerical parameters**
  You are asked to analyze the influence of the following numerical parameters on the numerical simulation :

  - Influence of loading speed (Keep $t_{\max}$ at its prescribed value !) ,
  - Influence of spatial discretization : mesh refinement and mesh distribution ,
  - Influence of temporal discretization : maximum time step .

  For further guidance, follow these guidelines :

  - Are the best suited numerical parameters identical for the three parts of study (elasto-platic with a linear hardening, elasto-plastic with a non-linear hardening, elasto-viscoplastic with hardening) ?
  - Is it possible to validate the numerical solution based on the total potential energy ? Is the work done by the internal forces $W^{Int} = \int_0^{t_f} \left(\vec{F}^{Int}(t) . \overset{\bullet}{\vec{q}}(t) \right) dt$ equal to the work done by the external forces $W^{Ext} = \int_0^{t_f} \left(\vec{F}^{Ext}(t) . \overset{\bullet}{\vec{q}}(t) \right) dt$ at the end of the numerical simulation, if $\overset{\bullet}{\vec{q}}(t)$ denotes the structural nodal velocity vector ? Which final value should they take in elasticity ?
  - Is an elasto-plastic model independent of the loading speed from a numerical point of view ? (Tip : influence of the temporal discretization on the loading history.)

Report quality (20%).

A peculiar attention will be paid to the quality of report, the ability to summarize and the scientific rigor. For further information, please refer to the document "Report instructions" on eCampus.

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Appendix : generalized plastic modulus.

The general expression of a plastic flow rule (Melan's hypothesis) is given by

$$D_{ij}^p = \lambda \frac{\frac{\partial g}{\partial \boldsymbol{\sigma}}}{\sqrt{\frac{\partial g}{\partial \boldsymbol{\sigma}} : \frac{\partial g}{\partial \boldsymbol{\sigma}}}}. \tag{1}$$

This flow rule states that the direction of D_{ij}^p is given by the unit normal $\frac{\frac{\partial g}{\partial \boldsymbol{\sigma}}}{\sqrt{\frac{\partial g}{\partial \boldsymbol{\sigma}} : \frac{\partial g}{\partial \boldsymbol{\sigma}}}}$ to plastic potential $g(\boldsymbol{\sigma})$, whereas the plastic multiplier λ determines the magnitude of D_{ij}^p.

The consistency condition states that during development of plastic strains, the yield criterion is fulfilled, i.e.

$$\overset{\bullet}{f} = \frac{\partial f}{\partial \sigma_{ij}} \overset{\bullet}{\sigma}_{ij} + \frac{\partial f}{\partial \boldsymbol{q}^{(k)}} * \overset{\bullet}{\boldsymbol{q}}^{(k)} = 0. \tag{2}$$

In Eq. (2), the increments of the internal/hidden variables $\boldsymbol{q}^{(k)}$, which enter in the consistency condition, evolvse with the plastic strain according to the evolution laws :

$$\overset{\bullet}{\boldsymbol{q}}^{(k)} = \lambda r^{(k)}(\boldsymbol{\sigma}, \boldsymbol{q}), \tag{3}$$

where the evolution functions $r^{(k)}(\boldsymbol{\sigma}, \boldsymbol{q})$, in general are allowed to depend on the same variables as the yield function f and the potential function g.

Insertion of Eq. (3) into Eq. (2) leads to

$$\frac{\partial f}{\partial \sigma_{ij}} \overset{\bullet}{\sigma}_{ij} - H^p \lambda = 0, \tag{4}$$

where the quantity H^p is defined by

$$H^p = -\frac{\partial f}{\partial \boldsymbol{q}^{(k)}} * r^{(k)}(\boldsymbol{\sigma}, \boldsymbol{q}). \tag{5}$$

This quantity is termed as the generalized plastic modulus. If $H^p \neq 0$, i.e. if it is assumed some hardening, then the intensity of plastic flow is given by

$$\lambda = \frac{1}{H^p} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\bullet}{\boldsymbol{\sigma}} \right). \tag{6}$$

If we assume associated plasticity $f = g$, then the associated flow rule becomes

$$\boldsymbol{D}^p = \frac{1}{H^p} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\bullet}{\boldsymbol{\sigma}} \right) \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}}}{\sqrt{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}}}. \tag{7}$$

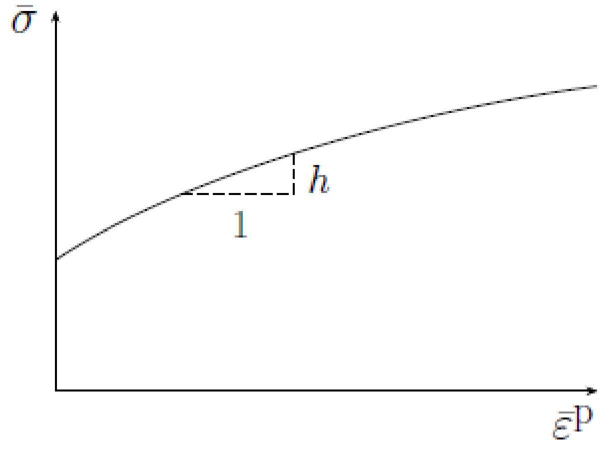

Figure 2: Interpretation of plastic modulus h.

This equation implies that the plastic flow is caused by the normal component of $\overset{\bullet}{\boldsymbol{\sigma}}$ and is in the direction to the normal to the subsequent or initial yield surface $\dfrac{\frac{\partial f}{\partial \boldsymbol{\sigma}}}{\sqrt{\frac{\partial f}{\partial \boldsymbol{\sigma}}:\frac{\partial f}{\partial \boldsymbol{\sigma}}}}$, which denotes the unit normal to f at each loading point $\boldsymbol{\sigma}$. Once the yield function f and the generalized plastic modulus H^p are known then the plasticity formulation is complete.

By definition, the equivalent plastic strain rate is written as

$$\overset{\bullet}{\bar{\varepsilon}}{}^{\mathrm{p}} = \sqrt{\frac{2}{3}D^p_{ij}D^p_{ij}} = \sqrt{\frac{2}{3}}\frac{1}{H^p}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}:\overset{\bullet}{\boldsymbol{\sigma}}\right) = \sqrt{\frac{2}{3}}\lambda. \tag{8}$$

To obtain an interpretation of the generalized plastic modulus H^p, we multiply the consistency condition Eq. (4) by λ and divide by $\sqrt{\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}}$, and use the associated flow rule to obtain in small deformation

$$\Rightarrow \quad \overset{\bullet}{\sigma}_{ij}\overset{\bullet}{\varepsilon}{}^{p}_{ij} = H^p\frac{\frac{3}{2}\left(\overset{\bullet}{\bar{\varepsilon}}{}^{\mathrm{p}}\right)^2}{\sqrt{\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}}}. \tag{9}$$

Von Mises' yield criterion $f = \sqrt{\frac{3}{2}\left(s_{ij}-\alpha_{ij}\right)\left(s_{ij}-\alpha_{ij}\right)} - \sigma_{\mathrm{y}}$ leads us to

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial s_{ij}} = \sqrt{\frac{3}{2}}\frac{s_{ij}-\alpha_{ij}}{\sqrt{\left(s_{kl}-\alpha_{kl}\right)\left(s_{kl}-\alpha_{kl}\right)}}. \tag{10}$$

Thus, one gets

$$\Rightarrow \quad \overset{\bullet}{\sigma}_{ij}\overset{\bullet}{\varepsilon}{}^{p}_{ij} = \sqrt{\frac{3}{2}}\left(\overset{\bullet}{\bar{\varepsilon}}{}^{\mathrm{p}}\right)^2 H^p. \tag{11}$$

Considering uniaxial stress conditions, the term $\overset{\bullet}{\sigma}_{ij}\overset{\bullet}{\varepsilon}{}^{p}_{ij}$ reduces to $\overset{\bullet}{\sigma}\overset{\bullet}{\varepsilon}{}^{p}$ and $\overset{\bullet}{\bar{\varepsilon}}{}^{\mathrm{p}} = \left|\overset{\bullet}{\varepsilon}{}^{p}\right|$. If the loading is tensile, Eq. (9) reduces to

$$\Rightarrow \quad \frac{d\,\bar{\sigma}}{d\,\bar{\varepsilon}^{\mathrm{p}}} = \sqrt{\frac{3}{2}}H^p = h(\bar{\varepsilon}^{\mathrm{p}}). \tag{12}$$

The interpretation of h is illustrated in Figure 2.

Therefore, the key to describe an uniaxial stress-strain curve $\sigma - \varepsilon$ correctly is to find out the evolution of h with the plastic strain.