Advanced Solid Mechanics

Project Statement 2021 − 2022

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Consider a cube subjected to a surface traction (positive or negative) t and whose geometry is defined by (cf. Figure 1) :
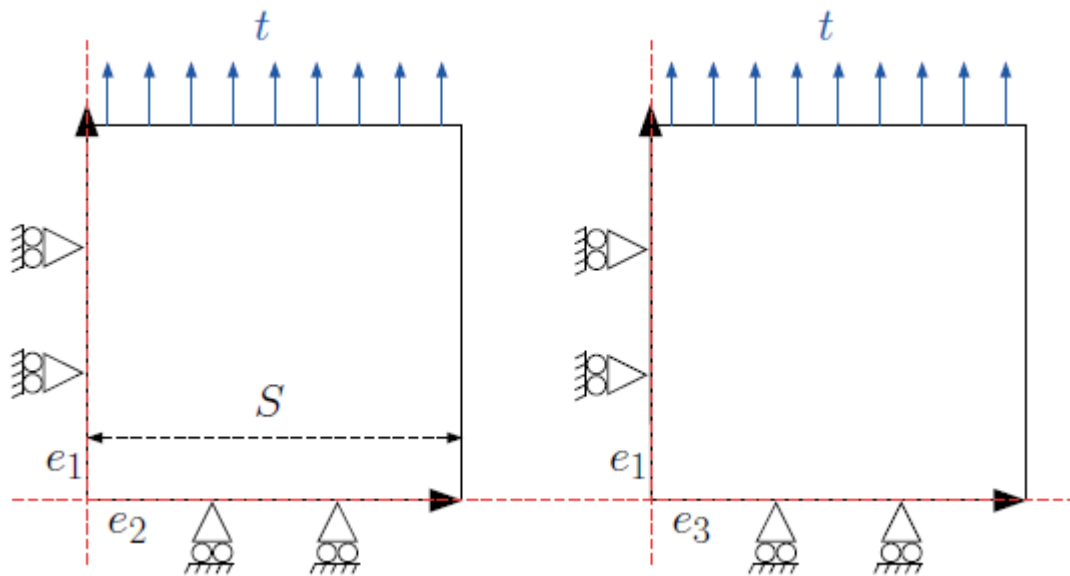

Figure 1: Schematic view of a cube subjected to a pressure t in simple tension or compression

The inertial forces and the temperature effects are neglected (isothermal problem and quasi-static loading).

You are asked to study the behavior of the cube according to several hardening laws included in Table (1).

Hardening law			
Hardening	Isotropic ($\theta = 1$)	linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0 + h_i \bar{\varepsilon}^{vp}$ et $\alpha_{ij} = 0$
		non-linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^\infty - \left(\sigma_{\mathrm{y}}^\infty - \sigma_{\mathrm{y}}^0\right) e^{\left(-\frac{h_i \bar{\varepsilon}^{vp}}{\sigma_{\mathrm{y}}^\infty - \sigma_{\mathrm{y}}^0}\right)}$ et $\alpha_{ij} = 0$
	Kinematic ($\theta = 0$)	linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0$ et $\dot{\alpha}_{ij} = \frac{2}{3} h_k D_{ij}^{vp}$
		non-linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0$ et $\dot{\alpha}_{ij} = \frac{2}{3} h_k D_{ij}^{vp} - \eta_k \dot{\bar{\varepsilon}}^{vp} \alpha_{ij}$
	Mixed ($\theta = \theta^m$)	linear	$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^0 + h_i \bar{\varepsilon}^{vp}$ et $\dot{\alpha}_{ij} = \frac{2}{3} h_k D_{ij}^{vp}$
Viscoplastic law (Perzyna)			$\lambda = \sqrt{\frac{3}{2}} \left\langle \frac{\bar{\sigma}^{VM} - \sigma_{\mathrm{y}}}{\eta} \right\rangle$
Hardening parameters			$h_i = \theta h$ $h_k = (1 - \theta)h$

Table 1: Hardening law.

The current yield stress σ_{y} represents, at about $\sqrt{\dfrac{2}{3}}$, the radius of von Mises' yield surface. The components of the backstress tensor α_{ij} represents the current position of the yield surface center. (The equivalent backstress is defined as $\bar{\alpha} = \sqrt{\dfrac{3}{2} \alpha_{ij}\alpha_{ij}}$). The equation of von Mises' yield criterion is $\bar{\sigma}^{\mathrm{VM}} = \sqrt{\dfrac{3}{2}(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})} = \sigma_{\mathrm{y}}$. The equivalent viscoplastic strain is defined as $\bar{\varepsilon}^{vp}(t) = \bar{\varepsilon}^{vp}(t_0) + \displaystyle\int_{t_0}^{t} \sqrt{\dfrac{2}{3} D_{ij}^{vp} D_{ij}^{vp}} \, dt$.

The geometry and material data, as well as the loading for each group, are included in Tables (2), (3) and (4).

	$S\ [mm]$	$\bar{\varepsilon}^{\mathrm{vp}}_{\mathrm{max}}\ [\%]$
Geometry 1	100	0.25
Geometry 2	80	0.3
Geometry 3	120	0.35
Geometry 4	50	0.4
Geometry 5	75	0.45

Table 2: Geometry and loading data.

	$\rho\ [kg/m^3]$	$E\ [MPa]$	$\nu\ [-]$	$\sigma_{\mathrm{y}}^{0}\ [MPa]$	$h\ [MPa]$	$\theta\ [-]$	$\sigma_{\mathrm{y}}^{\infty}\ [MPa]$
Material 1	7850	205000	0.3	200	30000	0.2	300
Material 2	2700	70000	0.33	100	16000	0.35	250
Material 3	4500	110000	0.25	300	40000	0.75	425

Table 3: Material parameter data.

Group	Geometry and Loading	Material
1	1	1
2	1	2
3	1	3
4	2	1
5	2	2
6	2	3
7	3	1
8	3	2
9	3	3
10	4	1
11	4	2
12	4	3
13	5	1
14	5	2
15	5	3

Table 4: Geometry, loading and material data associated to the group number.

Metafor (80%).

The following tasks will be performed with the Metafor software

- **Part 1 : Study of elasto-plastic behavior with linear hardening.**
  By studying several loading/unloading cycles (A cycle corresponds to a linear evolution of the imposed load from 0 to $t_{\max}$, from $t_{\max}$ to $-t_{\max}$ and from $-t_{\max}$ to 0.), you are asked to:

  - For a plastic model with isotropic hardening, determine the maximum surface traction $t_{\max}$ needed in a loading cycle to reach a permanent equivalent viscoplastic deformation $\bar{\varepsilon}^{\mathrm{vp}}{}_{\max}$;
  - The cube represented in Figure 1 is in plane stress. Determine the boundary conditions needed to have a state of plane strain instead of a state of of plane stress;
  - Compare the behavior of the cube if you consider a perfectly plastic model, a linear isotropic hardening, a linear kinematic hardening and a linear mixed hardening in both a state of plane stress and a state of plane strain ;
  - Describe and explain the evolution of the relevant variables included hereafter for a loading/unloading cycle ;

  For further guidance, follow these guidelines :

  - The relevant variables are the equivalent backstress $\bar{\alpha}(t)$, the equivalent stress $\sqrt{3J_2(s_{ij})}(t)$, von Mises' equivalent stress $\bar{\sigma}^{\mathrm{VM}}(t) = \sqrt{\dfrac{3}{2}\left(s_{ij} - \alpha_{ij}\right)\left(s_{ij} - \alpha_{ij}\right)}$, the current yield stress $\sigma_{\mathrm{y}}(t)$, the equivalent plastic strain $\bar{\varepsilon}^{\mathrm{p}}(t)$, all evaluated at a material point of the cube;
  - What happens in the case of the perfectly plastic model ?
  - Do the equivalent stress $\sqrt{3J_2}$ and the equivalent backstress $\bar{\alpha}$ take a zero value when the imposed loads $p(t)$ evolves from $t_{\max}$ to $-t_{\max}$? (What happens according to the elasto-plastic model, when the number of loading/unloading cycles approaches the infinity ?) Does the equivalent stress $\sqrt{3J_2}$ take a zero value when the imposed load $t(t)$ is equal to 0 at the end of the cycles ?
  - Explain the reason why the equivalent plastic strain $\bar{\varepsilon}^{\mathrm{vp}}$ in the cube takes the same value when the imposed loads reaches for the first time its maximal value $t_{\max}$, whatever the linear hardening model (Tip : use the expression of the plastic multiplier λ); Is there a difference between the state of plane strain and plane stress ? Why ?
  - Study the problem in Haigh Westergaard's space in order to compare the different models and to comment the evolution of the relevant variables ;
  - What happens if you invert the loading direction, i.e. $t(t) := -t(t)$?
  - How does the plastic dissipation ($\mathbb{D} = \sigma_{ij} D_{ij}^{vp}$) evolve according to the elasto-plastic model ?

- **Part 2 : Study of elasto-plastic behavior with non-linear hardening.**
  While studying several loading/unloading cycles $t_{\max}$, you are asked to study the non-linear effects of the hardening law by considering an elasto-plastic behavior with a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor in plane stress state. You are asked to:

- Analyze the influence of the dynamic recovery parameter η_k and determine the limit case(s) ;

- For a given value of the dynamic recovery parameter η_k (different from the limit case(s)), study the influence of the maximum prescribed load $t_{\max}$ on the behavior of the cube in the case of :

  * a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor,

  * a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor combined with a linear isotropic hardening ($\theta = \theta^m$),

  * a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor combined with a non-linear isotropic hardening described by Voce's saturated law ($\theta = \theta^m$);

- If one combines a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor with a linear isotropic hardening ($\theta = \theta^m$), does the yield stress evolve linearly ? Justify (Tip : determine the analytical expression of the plastic multiplier λ) ;

- If one combines a non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor with a non-linear isotropic hardening described by Voce's saturated law ($\theta = \theta^m$), does the equivalent backstress always tend toward zero when the number of loading/unloading cycles approaches the infinity ? Justify.

For further guidance, follow these guidelines :

- Determine the unit of the parameter η_k ;

- Write down the evolution laws of the hardening parameters under the general form $\overset{\bullet}{\boldsymbol{q}}^{(k)} = \lambda \times r^{(k)}(\boldsymbol{\sigma}, \boldsymbol{q})$;

- Write down the expression of the generalized plastic modulus H^p (cf. Appendix) in the case of a non-linear mixed hardening (isotropic and kinematic) (Tip : use the consistency condition $\overset{\bullet}{f}((\boldsymbol{\sigma}, \boldsymbol{q}) = 0$ and insert the expression derived for $\dfrac{\partial f}{\partial \sigma_{ij}} \overset{\bullet}{\sigma}_{ij}$ into the expression of H^p) ;

- From the previous results, deduce graphically in Haigh Westergaard's space, the influence of the dynamic recovery term $-\eta_k \overset{\bullet}{\bar{\varepsilon}}^{\mathrm{p}} \alpha_{ij}$ on the generalized plastic modulus H^p for a loading/unloading cycle (Tip : study the influence of the term $(\boldsymbol{s} - \boldsymbol{\alpha}) : \boldsymbol{\alpha}$ on H^p at crucial steps of one loading/unloading cycle) ?

- How does the plastic dissipation evolve through the loading/unloading cycles ? Compare your findings with a linear kinematic hardening.

- Study the non-linear kinematic hardening described by Armstrong Frederick's evolution law of the backstress tensor (The components α_{ij} of the backstress tensor are chosen such that $\bar{\alpha} = |\alpha|$ and the components ε_{ij}^p of the plastic strain tensor are chosen such that $\bar{\varepsilon}^{\mathrm{p}}(t) = \bar{\varepsilon}^{\mathrm{p}}(t_0) + \int_{t_0}^{t} \left| \overset{\bullet}{\varepsilon}^p \right| \, dt$). Write down the first order differential equation in α with respect to ε^p and deduce the analytical expression of $\alpha = f(\varepsilon^p)$ for the tensile and compression parts (Inelastic loading tensile/compression cycle) and plot the curve $\bar{\sigma} = f(\bar{\varepsilon}^{\mathrm{p}})$ for the tensile and compression parts knowing that

$\dfrac{d\,\bar{\sigma}}{d\,\bar{\varepsilon}^{\mathrm{p}}} = h = \sqrt{\dfrac{3}{2}}H^p$. In the case of a loading/unloading cycle, does the curve $\bar{\sigma} = f(\bar{\varepsilon}^{\mathrm{p}})$ represent a closed cycle in the space $\bar{\sigma} - \bar{\varepsilon}^{\mathrm{p}}$?

– Determine the asymptotic value of the backstress tensor α_{ij}^{u} if the plastic multiplier $\lambda > 0$, i.e. when $\overset{\bullet}{\alpha}_{ij} = 0$ for $\alpha_{ij} = \alpha_{ij}^{u}$. Deduce the asymptotic value of the equivalent backstress $\bar{\alpha}^{u}$? What is the influence of η_k ?

– Determine the asymptotic value of the yield stress σ_{y}^{u} if the plastic multiplier $\lambda > 0$, i.e. when $\overset{\bullet}{\sigma}_{\mathrm{y}} = 0$ for $\sigma_{\mathrm{y}} = \sigma_{\mathrm{y}}^{u}$,

– From these results, deduce the upper bound of the equivalent stress $\sqrt{3J_2}$ in Haigh-Westergaard's space (Tip : use the expression of von Mises' yield stress $\bar{\sigma}^{\mathrm{VM}}$ and the triangular inequality $\left\| \vec{a} + \vec{b} \right\| \leq \|\vec{a}\| + \left\| \vec{b} \right\|$). Write down the expression of the asymptotic surface described by the upper bound of the equivalent stress $\sqrt{3J_2}$. Draw the yield surface and the asymptotic surface in Haigh-Westergaard's space !

– What happens if/when the equivalent backstress $\bar{\alpha}$ and/or the yield stress σ_{y} reaches their respective asymptotic values ?

- **Part 3 : Study of elasto-viscoplastic behavior.**
  While studying several loading/unloading cycles, you are asked to study the viscous effects by considering elasto-viscoplastic behaviors with isotropic and mixed linear hardening:

  – Analyze the influence of the viscosity parameter η and determine the limit case(s) ,

  – Show the effects on the cube behavior, if one modifies the preceding sawtooth loading by adding steps at the imposed displacement extrema, i.e. by keeping the imposed load constant at $t_{\max}$, 0 and -$t_{\max}$ for a certain duration of time.

For further guidance, follow these guidelines :

– Determine the unit of parameter η ;

– Does the presence of viscoplasticity in the material delay the first entrance into plasticity of the cube ?

– After a loading/unloading inelastic cycle, the imposed load is kept at a non-zero value for a long time. How do the components of the deformation tensor evolve in the cube ?

– Demonstrate in the case of an elasto-viscoplastic behavior without hardening, how will the plastic deformation evolve if the imposed load is kept at its maximum value $t_{\max}$ for an infinitely long time ? Deduce the influence of the viscosity parameter on the cube behavior;

– Demonstrate in the case of an elasto-viscoplastic behavior with an isotropic hardening , how will the plastic deformation evolve if the imposed load is kept at its maximum value $t_{\max}$ for an infinitely long time ?

– Extrapolate your findings in the case of a linear mixed hardening : how do the equivalent backstress $\bar{\alpha}$ and the yield stress σ_{y} evolve if the imposed load is kept at its maximum value $t_{\max}$ for an infinitely long time ?

– Study the evolution of the signed distance with respect to the yield surface center $d = \bar{\sigma}^{\mathrm{VM}} - \sigma_{\mathrm{y}}$ during the loading/unloading cycles ? Study the problem in Haigh Westergaard's space ;

– Does the maximum signed distance $d_{\max}$, i.e. when the imposed load reaches its maximum value $t_{\max}$, depend on the loading speed ? (Tip : refer to the observations performed about the experimental data for elasto-viscoplastic material.)

- **Part 4 : Sensitivity study of numerical parameters**
  You are asked to analyze the influence of the following numerical parameters on the numerical simulation :

  – Influence of loading speed (Keep $t_{\max}$ at its prescribed value !) ,

  – Influence of spatial discretization : mesh refinement and mesh distribution ,

  – Influence of temporal discretization : maximum time step .

For further guidance, follow these guidelines :

– Are the best suited numerical parameters identical for the three parts of study (elasto-platic with a linear hardening, elasto-plastic with a non-linear hardening, elasto-viscoplastic with hardening) ?

– Is it possible to validate the numerical solution based on the total potential energy ? Is the work done by the internal forces $W^{Int} = \int_0^{t_f} \left(\vec{F}^{Int}(t) . \dot{\vec{q}}(t) \right) dt$ equal to the work done by the external forces $W^{Ext} = \int_0^{t_f} \left(\vec{F}^{Ext}(t) . \dot{\vec{q}}(t) \right) dt$ at the end of the numerical simulation, if $\dot{\vec{q}}(t)$ denotes the structural nodal velocity vector ? Which final value should they take in elasticity ?

– Is an elasto-plastic model independent of the loading speed from a numerical point of view ? (Tip : influence of the temporal discretization on the loading history.)

Report quality (20%).

A peculiar attention will be paid to the quality of report, the ability to summarize and the scientific rigor. For further information, please refer to the document "Report instructions" on eCampus.

Appendix : generalized plastic modulus.

The general expression of a plastic flow rule (Melan's hypothesis) is given by

$$D_{ij}^p = \lambda \frac{\dfrac{\partial g}{\partial \boldsymbol{\sigma}}}{\sqrt{\dfrac{\partial g}{\partial \boldsymbol{\sigma}} : \dfrac{\partial g}{\partial \boldsymbol{\sigma}}}}. \tag{1}$$

This flow rule states that the direction of D_{ij}^p is given by the unit normal $\dfrac{\dfrac{\partial g}{\partial \boldsymbol{\sigma}}}{\sqrt{\dfrac{\partial g}{\partial \boldsymbol{\sigma}} : \dfrac{\partial g}{\partial \boldsymbol{\sigma}}}}$ to plastic

potential $g(\boldsymbol{\sigma})$, whereas the plastic multiplier λ determines the magnitude of D_{ij}^p.

The consistency condition states that during development of plastic strains, the yield criterion is fulfilled, i.e.

$$\overset{\bullet}{f} = \frac{\partial f}{\partial \sigma_{ij}} \overset{\bullet}{\sigma}_{ij} + \frac{\partial f}{\partial \boldsymbol{q}^{(k)}} * \overset{\bullet}{\boldsymbol{q}}^{(k)} = 0. \tag{2}$$

In Eq. (2), the increments of the internal/hidden variables $\boldsymbol{q}^{(k)}$, which enter in the consistency condition, evolvse with the plastic strain according to the evolution laws :

$$\overset{\bullet}{\boldsymbol{q}}^{(k)} = \lambda r^{(k)}(\boldsymbol{\sigma}, \boldsymbol{q}), \tag{3}$$

where the evolution functions $r^{(k)}(\boldsymbol{\sigma}, \boldsymbol{q})$, in general are allowed to depend on the same variables as the yield function f and the potential function g.

Insertion of Eq. (3) into Eq. (2) leads to

$$\frac{\partial f}{\partial \sigma_{ij}} \overset{\bullet}{\sigma}_{ij} - H^p \lambda = 0, \tag{4}$$

where the quantity H^p is defined by

$$H^p = -\frac{\partial f}{\partial \boldsymbol{q}^{(k)}} * r^{(k)}(\boldsymbol{\sigma}, \boldsymbol{q}). \tag{5}$$

This quantity is termed as the generalized plastic modulus. If $H^p \neq 0$, i.e. if it is assumed some hardening, then the intensity of plastic flow is given by

$$\lambda = \frac{1}{H^p} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\bullet}{\boldsymbol{\sigma}} \right). \tag{6}$$

If we assume associated plasticity $f = g$, then the associated flow rule becomes

$$\boldsymbol{D}^p = \frac{1}{H^p} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\bullet}{\boldsymbol{\sigma}} \right) \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}}{\sqrt{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}}}. \tag{7}$$
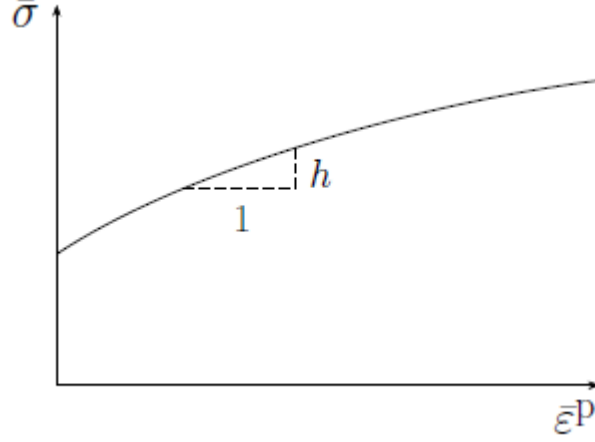
Figure 2: Interpretation of plastic modulus h.

This equation implies that the plastic flow is caused by the normal component of $\dot{\boldsymbol{\sigma}}$ and is in

the direction to the normal to the subsequent or initial yield surface $\dfrac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}}{\sqrt{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}:\dfrac{\partial f}{\partial \boldsymbol{\sigma}}}}$, which denotes

the unit normal to f at each loading point $\boldsymbol{\sigma}$. Once the yield function f and the generalized plastic modulus H^p are known then the plasticity formulation is complete.

By definition, the equivalent plastic strain rate is written as

$$\dot{\bar{\varepsilon}}^{\mathrm{P}} = \sqrt{\frac{2}{3} D_{ij}^p D_{ij}^p} = \sqrt{\frac{2}{3}}\frac{1}{H^p}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}}\right) = \sqrt{\frac{2}{3}}\lambda. \tag{8}$$

To obtain an interpretation of the generalized plastic modulus H^p, we multiply the consistency condition Eq. (4) by λ and divide by $\sqrt{\dfrac{\partial f}{\partial \sigma_{kl}}\dfrac{\partial f}{\partial \sigma_{kl}}}$, and use the associated flow rule to obtain in small deformation

$$\Rightarrow \quad \dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^p = H^p\frac{\frac{3}{2}\left(\dot{\bar{\varepsilon}}^{\mathrm{P}}\right)^2}{\sqrt{\dfrac{\partial f}{\partial \sigma_{kl}}\dfrac{\partial f}{\partial \sigma_{kl}}}}. \tag{9}$$

Von Mises' yield criterion $f = \sqrt{\dfrac{3}{2}\left(s_{ij}-\alpha_{ij}\right)\left(s_{ij}-\alpha_{ij}\right)} - \sigma_{\mathrm{y}}$ leads us to

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial s_{ij}} = \sqrt{\frac{3}{2}}\frac{s_{ij}-\alpha_{ij}}{\sqrt{\left(s_{kl}-\alpha_{kl}\right)\left(s_{kl}-\alpha_{kl}\right)}}. \tag{10}$$

Thus, one gets

$$\Rightarrow \quad \dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^p = \sqrt{\frac{3}{2}}\left(\dot{\bar{\varepsilon}}^{\mathrm{P}}\right)^2 H^p. \tag{11}$$

Considering uniaxial stress conditions, the term $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^p$ reduces to $\dot{\sigma}\dot{\varepsilon}^p$ and $\dot{\bar{\varepsilon}}^{\mathrm{P}} = \left|\dot{\varepsilon}^p\right|$. If the loading is tensile, Eq. (9) reduces to

$$\Rightarrow \quad \frac{d\bar{\sigma}}{d\bar{\varepsilon}^{\mathrm{P}}} = \sqrt{\frac{3}{2}}H^p = h(\bar{\varepsilon}^{\mathrm{P}}). \tag{12}$$

The interpretation of h is illustrated in Figure 2.

Therefore, the key to describe an uniaxial stress-strain curve $\sigma - \varepsilon$ correctly is to find out the evolution of h with the plastic strain.