Dimensional analysis - Problem Set solutions

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1 Thermal transfer in a building

(a)

$$Q \sim \frac{k\Delta\theta}{e} \tag{1}$$

(b)

$$P \sim \frac{k\Delta\theta S}{e} \tag{2}$$

(c) The total surface of the walls (+roof) in contact with the outside air is $S_1 = 10 \cdot 10 + 4 \cdot 10 \cdot 6 = 340\text{m}^2$, while the surface of the ground is $S_2 = 100\text{m}^2$. The total heating power is

$$P \sim \frac{k}{e}\Big[S_1\Delta\theta_1 + S_2\Delta\theta_2\Big] \tag{3}$$

By using the numerical values [$e = 0.1$m, $k = 0.04$W/(m.s), $P = 4256$W, $\Delta\theta_1 = 26°$ F and $\Delta\theta_2 = 18°$ F], we infer that the proportionality constant is 1. Since the power is invertly proportional to the thickness of the walls, the heating power should be three times less, so 1419W.

(d) This scaling is the same as the cooking time presented in lecture. It results from a balance between thermal inertia and thermal conductivity, although it can also be found by using Vaschy-Buckingham.

$$\tau \sim \frac{\rho C e^2}{k} \simeq 72000\text{s} = 20\text{h} \tag{4}$$

These walls are therefore able to correctly smooth the daily variations of the outdoor temperature. Note that this characteristic time does not depend on $\Delta\theta$. A wall thickness about 1.5m would be required to increase this time shift to 6 months, which is not very realistic. So practically, bioclimatic buildings combine different materials in the walls, some of them having a large inertia ρC (like stone or concrete) while the others have good insulating properties (like wood or straw). Even better is the Trombe wall, consisting of an inertial wall (e.g. in stone) in front of which a window is placed in order to create a greenhouse effect.

2 Stride frequency

According to the graphic, $f \sim M^{-1/8}$. The kinetic energy of the legs scales as MV^2, where the velocity $V \sim Lf$ and the mass $M \sim \rho L r^2$. This kinetic energy is balanced by the elastic energy in the legs Er^4/L, so we infer

$$f^2 \sim \frac{Er^2}{\rho L^4} \tag{5}$$

Using Kleiber's law of elastic similarity ($r \sim M^{3/8}$ and $L \sim M^{1/4}$), we deduce

$$f^2 \sim \frac{\rho^{1/2} g^{5/4}}{E^{1/4}} M^{-1/4} \tag{6}$$

as observed experimentally.

3 Flying animals

(a) The lift must balance the weight, so

$$\rho_a V^2 S \sim Mg \Rightarrow V \sim \sqrt{\frac{Mg}{\rho_a S}} \sim \rho^{1/3} \left(\frac{g}{\rho_a}\right)^{1/2} M^{1/6} \tag{7}$$

(b) The power to be supplied to balance drag scales as

$$P \sim (\rho_a V^2 S) \cdot V \sim \frac{g^{3/2} \rho^{1/3}}{\rho_a^{1/2}} M^{7/6} \tag{8}$$

(c) As the metabolic rate for isometric animals scales as $M^{2/3}$, the ratio of flight power to metabolic power scales as $M^{1/2}$. Consequently, it is about 70 times more exhausting for a 15kg andean condor to flap and sustain steady flight than for a 3g hummingbird. That is why hummingbirds can hover (the most demanding way of flying) while condors can only soar.

4 Can you detect the Coriolis effect in your sink ?

The Coriolis force scales as $M\Omega U$, where $\Omega = 2\pi/86400 \simeq 7 \times 10^{-5}$rad/s is the angular frequency of the earth rotation. So the Rossby number is

$$Ro = \frac{\rho L^2 U^2}{\rho L^3 \Omega L U} = \frac{U}{L\Omega} \tag{9}$$

The ratio L/U is the characteristic time of the emptying (i.e. about 1 minute), while $1/\Omega$ is the characteristic time of the earth rotation (i.e. one day). Since the first is more than 1000 times smaller than the second, the Rossby number is more than 1000, and the Coriolis force is about three orders of magnitude lower than any other force that drives the emptying motion. It is very likely that any other tiny perturbation (e.g a deposition of limestone that slightly obstructs the hole) would have much more impact than the Coriolis force in determining the direction of rotation of the vortex. So the legend is false: in your sink the direction is random. Nevertheless, when considerable experimental care is taken and when the flow is slowed down by using a smaller hole, it is possible to observe the tiny Coriolis effect. Such an experiment was made for the first time at MIT in 1962 by prof. Shapiro.

5 Brownian motion

There are 4 variables $k_B T$ [kg m^2 s^{-2}], μ [kg m^{-1} s^{-1}], R [m] and D [m^2 s^{-1}], but only three are independent. Since these variables involve three independent units [kg, m, s], Vaschy-Buckingham theorem says that the number of independent dimensionless groups should be 0. So every dimensionless group formed with these 4 variables should be constant. One of these groups is

$$\frac{k_B T}{\mu R D} \tag{10}$$

from which we infer the Einstein's relation

$$D \sim k_B T/(\mu R).$$