Dimensional analysis
and
Design of Experiments

Experimental Project

Master Engineering Physics

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Table of contents

Main goal of this workshop

Learn how to work less…

i.e. reduce the number of experiments needed to characterize a system.

Two complementary tools

1. Dimensional analysis

   ➔ Reduce the number of parameters to vary

2. Design of Experiments

   ➔ Choose how to vary these parameters

Introduction: cooking time

10kg turkey, to be roasted

Heat equation

$$\rho C_P \frac{\partial \theta}{\partial t} = k \nabla^2 \theta$$

Boundary conditions

Complex geometry

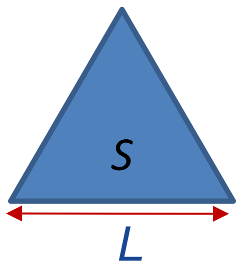
Numerical simulations

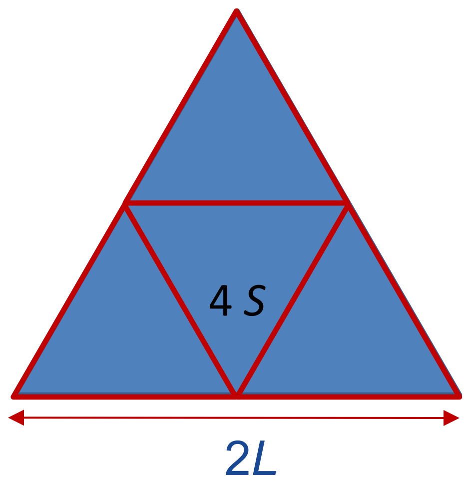
Finite element method

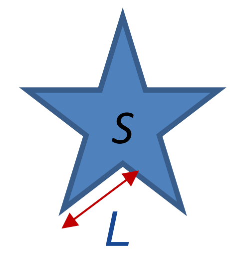
HELP !!!

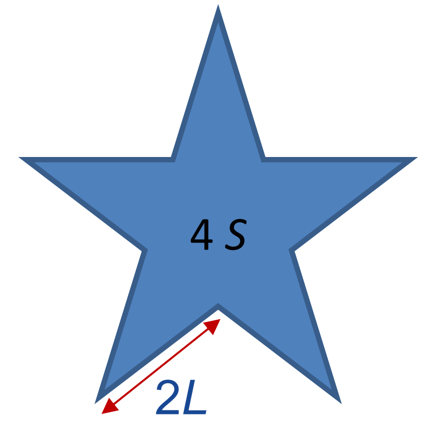
Dimensional analysis

- Surfaces : $S \sim L^2$









$$S = \frac{\sqrt{3}}{4} L^2$$

$[S] = \mathrm{m}^2$ and $[L] = \mathrm{m}$

- Free fall : $x \sim g\, t^2$ (numerical factor ½) ➔ $[x] = \mathrm{m}$, $[g] = \mathrm{m/s^2}$ and $[t] = \mathrm{s}$

➔ Scaling laws, in agreement with physical units

Physical laws DO NOT DEPEND on the units we use to formulate them !

Dimensional analysis

Here (and only here…),

There Is No Alternative (TINA)

Laws of physics must satisfy dimensional analysis

Back to cooking…

You must always know a little bit of physics about the phenomenon…

- Conduction of heat ➔ thermal conductivity $k \approx 2$ W / (m.K)
- Storage of heat ➔ specific heat capacity $C \approx 3500$ J / (kg.K)

Other significant variables

- Temperature (difference) $\Delta\theta \approx 160$ K
- Mass (or size) of the turkey $M = 10$ kg
- Density $\rho \approx 1000$ kg/m^3
- Cooking time T_{cook} = ???

Build a formula in agreement with units (how?)

$$T_{cook} \sim \frac{\rho^{1/3} C M^{2/3}}{k} \propto M^{2/3}$$

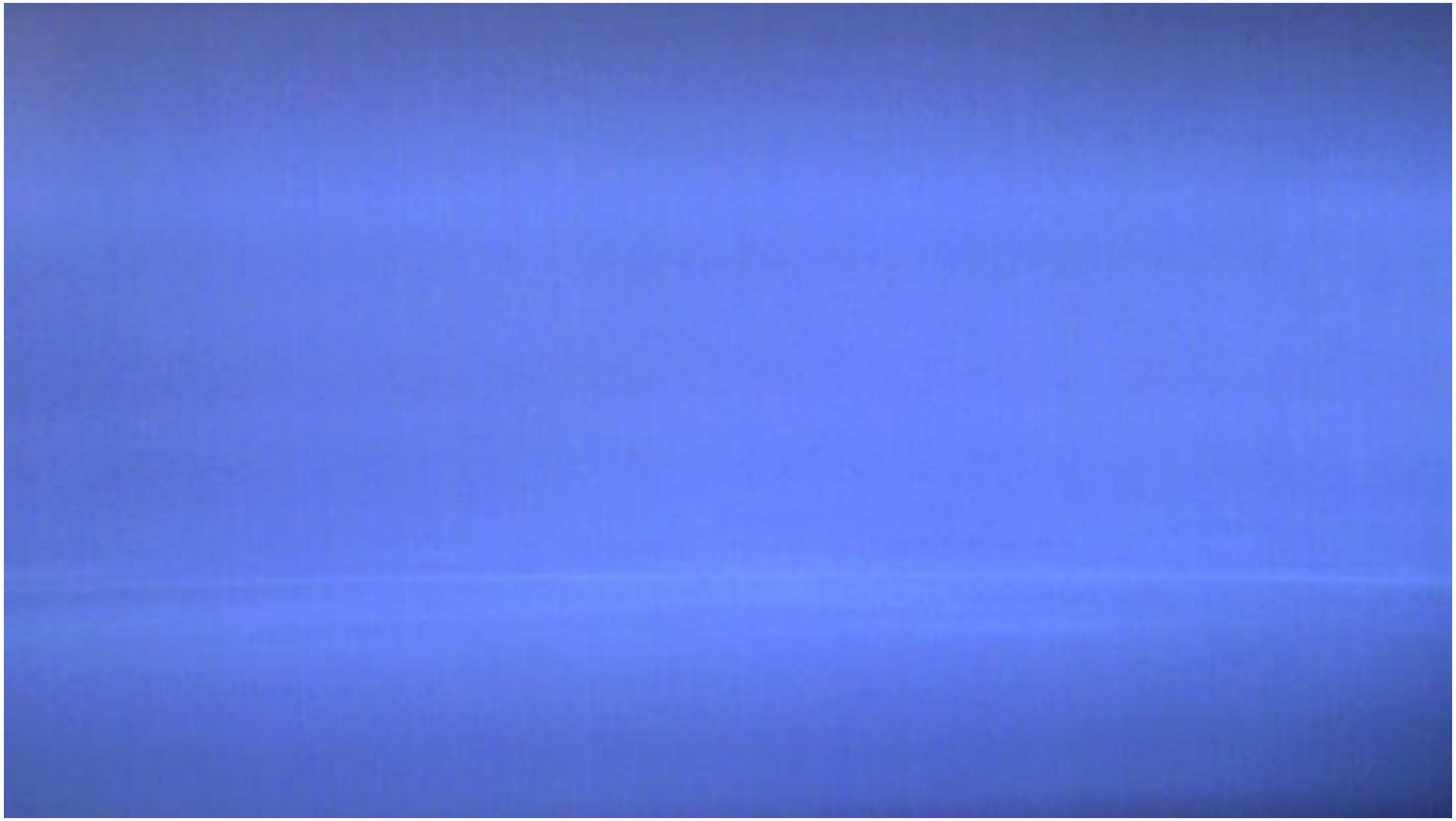
Unknown proportionality factor ➔ what's T_{cook} ???

Experience / cooking book / internet ➔ 3 hours for a 5kg turkey at 180°C

What about a 10kg turkey ? Not 6 hours, but 3 x $2^{2/3} \approx$ 4 hours and 45min

Bon appetit !

An academic example of dimensional analysis



Partial coalescence

3 relevant forces : gravity, surface tension, viscous shear

6 relevant parameters: initial radius R_i, final radius R_f, viscosity μ, surface tension σ, density ρ and gravity g.

Question: How does R_f depend on the other parameters ?

Vaschy – Buckingham theorem

When there are more than 2 effects, or no evident balance…

\# indep. dimensionless groups = # indep. variables - # indep. units involved

Other groups are dependent

Partial coalescence : R_i [m], μ [kg/m.s], σ [kg/s^2], ρ [kg/m^3] and g [m/s^2]

- 5 independent variables
- 3 independent units involved (kg, m, s)

➔ 2 independent dimensionless groups

1st : Bond number = $$\frac{\text{Grav. en.}}{\text{Surface en.}} \sim \frac{\rho g R_i^4}{\sigma R_i^2} \sim \frac{\rho g R_i^2}{\sigma} = Bo$$

2nd: Involve μ ??? Balance between viscosity and surface tension ?

How to form dimensionless groups

Partial coalescence ➔ we need a number out of R_i, μ, σ and ρ

$[R_i]$ = m $[\mu]$ = kg/m.s $[\sigma]$ = kg/s^2 $[\rho]$ = kg/m^3

$N = R_i^a . \mu^b . \sigma^c . \rho^d$ ➔ $[N] = \mathrm{m}^a\, \mathrm{kg}^b\, \mathrm{m}^{-b}\, \mathrm{s}^{-b}\, \mathrm{kg}^c\, \mathrm{s}^{-2c}\, \mathrm{kg}^d\, \mathrm{m}^{-3d} = 1$

➔ $a - b - 3d = 0$ $b + c + d = 0$ $-b - 2c = 0$

We choose $b = 1$ ➔ $a = c = d = -1/2$

Ohnesorge number = $$\frac{\text{Viscosity}}{\text{Surface tension}} \sim \frac{\mu}{\sqrt{\rho\sigma R_i}} = Oh$$

Vaschy-Buckingham : Other dimensionless groups are dependent

➔ $$\frac{R_f}{R_i} = F\Big(Bo, Oh\Big)$$

Understanding from dimensionless groups

Formation of a new droplet driven by surface tension (complex shape)

Both gravity and viscosity favor the collapse and oppose the daughter droplet formation

➔ Partial coalescence only when

$$Bo \ll 1 \text{ and } Oh \ll 1 \Rightarrow \frac{\mu^2}{\rho\sigma} \ll R_i \ll \sqrt{\frac{\sigma}{\rho g}}$$

➔ Range of droplet size for partial coalescence, in which both *μ* and *g* are negligible

If they are, # indep. variables = 3 ➔ # indep. dimensionless groups = 0 !

$$\frac{R_f}{R_i} = \text{cst.}$$

➔ Self-similar cascade, until $R_i \sim \frac{\mu^2}{\rho\sigma}$

About saving time

Experiments on partial coalescence:

5 parameters, 10 levels for each ➔ 10^5 experiments ➔ take students! ☹

OR use dimensional analysis ☺

➔ 2 dimensionless parameters ➔ 10^2 experiments

Same for numerical simulations ! E.g. diffusion equation: $\boxed{\rho C_P \frac{\partial \theta}{\partial t} = k \nabla^2 \theta}$

➔ 3 dimensional parameters ρ, C_P and k (+ Boundary Conditions)

➔ 10^3 simulations to run per BC

Dimensionless equation, involving dimensionless variables

$$\tilde{\theta} = \frac{\theta - \theta_0}{\theta_1 - \theta_0}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{t} = t \frac{k}{\rho C_P L^2} \Rightarrow \boxed{\frac{\partial \tilde{\theta}}{\partial \tilde{t}} = \tilde{\nabla}^2 \tilde{\theta}}$$

➔ 0 dimensionless parameters ➔ only 1 simulation per BC !!!

Dimensional analysis is a powerful weapon

GI Taylor's blast (1949)

Energy E of an atomic explosion ?

➔ Classified information…

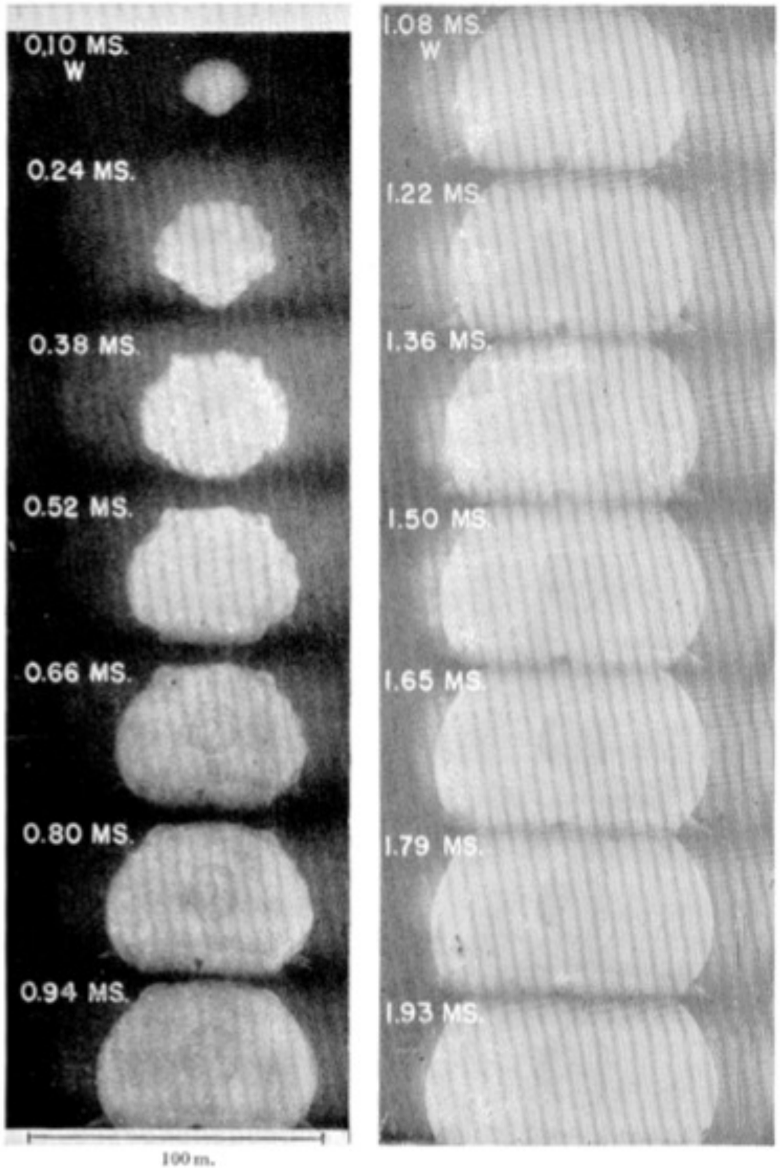
… accessible from a single snapshot !

Official (classified) value: $E = 84.10^{12}$ J

Taylor's prediction: $E \approx 10^{14}$ J …

(from pictures and dimensional analysis)

Oops!



Dimensional analysis is a powerful weapon

Energy E of an atomic explosion ?

Energy released almost instantaneously

➔ Spherical shock wave (radius R)

expanding with time

Variables : E, ρ_{air}, t, R (depends on the others)

➔3 indep. variables

Units: 3 (kg, m, s)

➔ 0 indep. dimensionless groups

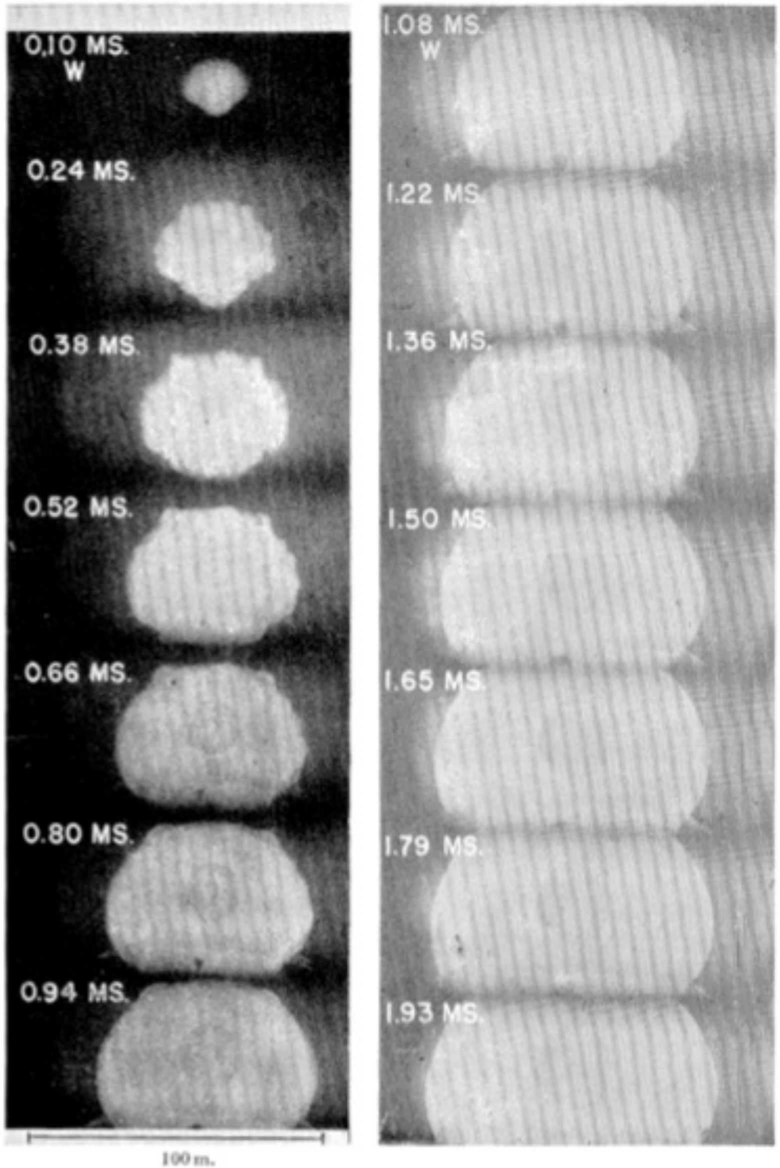
$N = E^a.\rho^b.t^c.R^d$

➔ $[N]=\text{kg}^a\text{m}^{2a}\text{s}^{-2a}\text{kg}^b\text{m}^{-3b}\text{s}^c\text{m}^d=1$

➔ a=1 (chosen), b=-1, c=2, d=-5

$$N = \frac{Et^2}{\rho R^5} = const. \sim 1$$

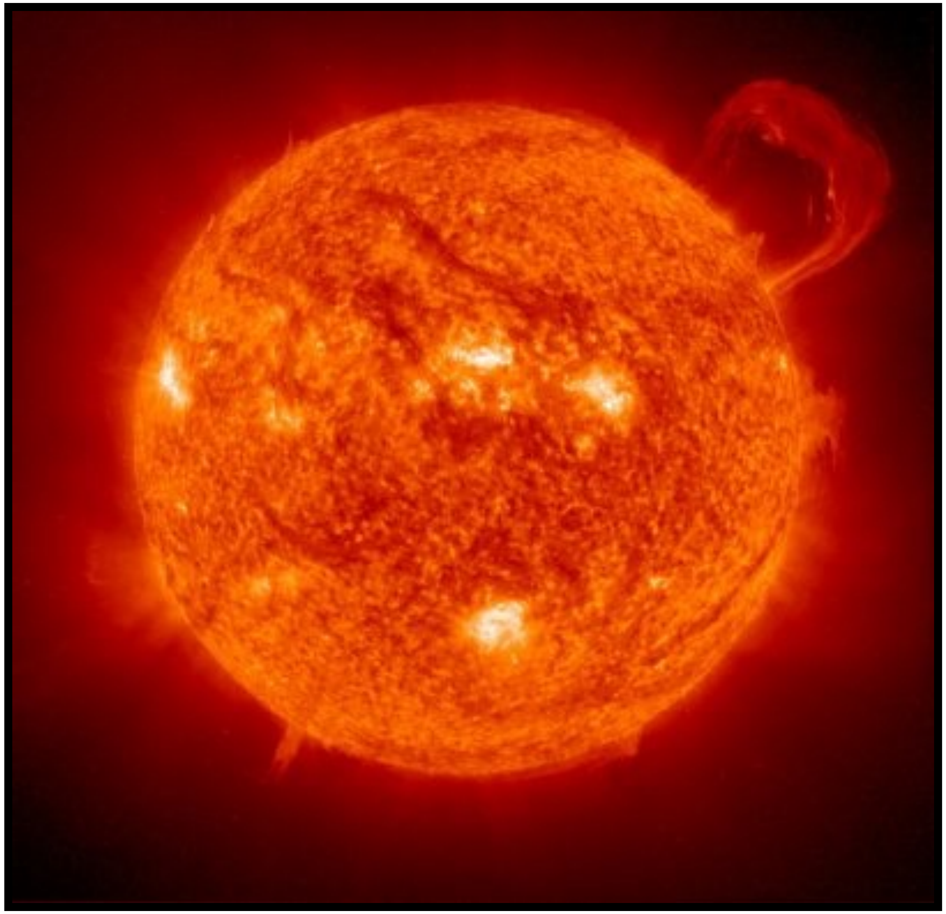
Pictures ➔ $R(t)$ ➔ $E \approx 10^{14}$J… oops!



Your turn: Vibration of a star

Question: what is the characteristic vibration frequency of a star ?

Rayleigh (1915) – before thermonuclear physics ➔ Only gravity is considered (g-modes)

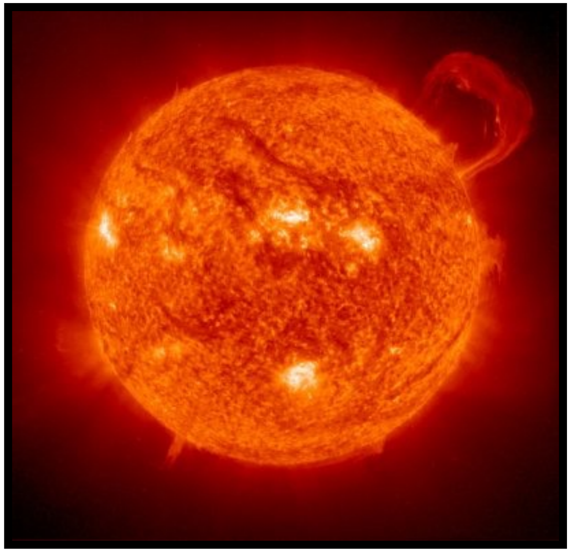


Your turn: Vibration of a star

Variables : radius R, density ρ, gravitational constant G,
and frequency f (that depends on the others)

Vaschy-Buckingham : # indep. dim. groups = 3 (R, G, ρ) – 3 (kg, m, s) = 0

$[R]$ = m $[G] = \mathrm{m}^3/\mathrm{kg.s}^2$ $[\rho] = \mathrm{kg/m}^3$ $[f] = \mathrm{s}^{-1}$



$N = R^a . G^b . \rho^c . f^d$ ➔ $[N] = \mathrm{m}^a\, \mathrm{m}^{3b}\, \mathrm{kg}^{-b}\, \mathrm{s}^{-2b}\, \mathrm{kg}^c\, \mathrm{m}^{-3c}\, \mathrm{s}^{-d} = 1$

➔ $a + 3b - 3c = 0$; $-b + c = 0$; $-2b - d = 0$
➔ $a = 0$, and we choose $d = 1$ ➔ $b = c = -1/2$

$$\frac{f}{\sqrt{\rho G}} = \mathrm{cst.} \Rightarrow f \sim \sqrt{\rho G}$$ indep. of the size R !

E.g. : Sun ➔ $\rho \sim 1400$ kg/m^3 in average ➔ $f \sim$ 3kHz (observed ~ 2.5kHz)

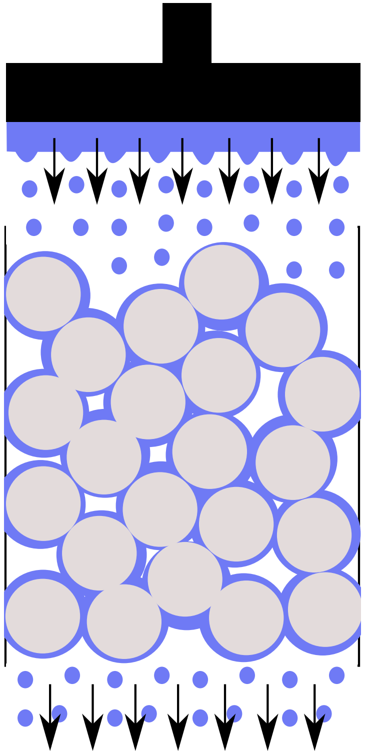
The importance of educated guessing

Dimensional analysis does not only work for academic cases. It can also reveal crucial in industrial experiments.

BUT educated guessing is required to turn the messy problem into a framework suitable for dimensional analysis.

e.g. Trickle bed reactor (suggested by D. Toye to C. Gommes)

(very common in chem. Eng.)



Unknown: volume of liquid V held in the packing (steady state)

Parameters:

- Diameter d and height H of the reactor
- Surface area per unit volume a_V
- Flow rate Q of liquid poured
- Dynamic viscosity μ and density ρ
- Gravity g

➔ Without dimensional analysis: 10^7 experiments

The importance of educated guessing

7 dimensional parameters & 3 independent units ➔ 4 dimensionless numbers

$$\frac{V}{d^3} = F\left(\frac{H}{d}, \frac{\mu d}{\rho Q}, \frac{g d^5}{Q^2}, a_V d\right)$$

➔ With such poor dimensional analysis, 10^4 experiments left

1st educated guess: the liquid distribution is homogeneous

➔ Hold-up per unit volume $h_L = \dfrac{V}{\pi d^2 H}$ and superficial velocity $U = \dfrac{Q}{\pi d^2}$

both constant in space

2nd educated guess: the flow is laminar

➔ Liquid inertia of negligible influence ➔ *ρ* only involved in *ρg*

➔ 4 dimensional parameters left: *ρg*, *μ*, a_V, *U*

$$h_L = F(\beta)$$

➔ 1 dimensionless group $\beta = \dfrac{\mu U a_V^2}{\rho g}$

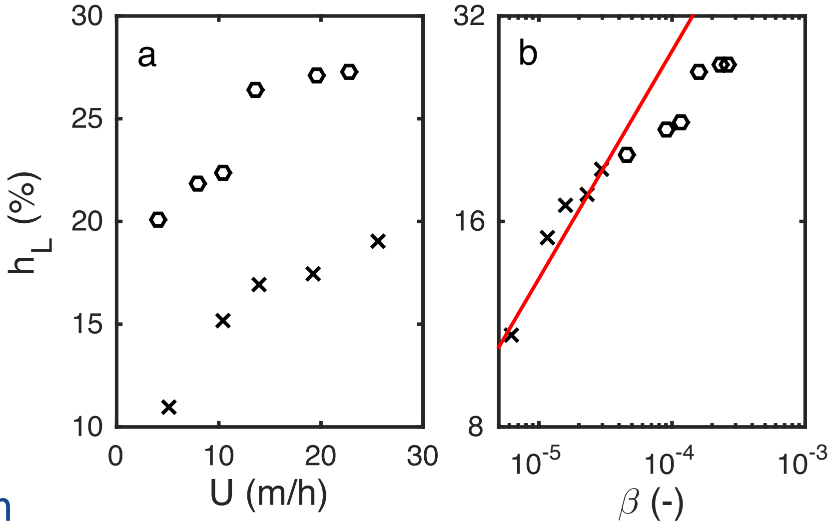
➔ 10 experiments left !

The importance of educated guessing

Does it work ? YES !

Can more guessing yield to the function F ?

YES !



3rd educated guess:

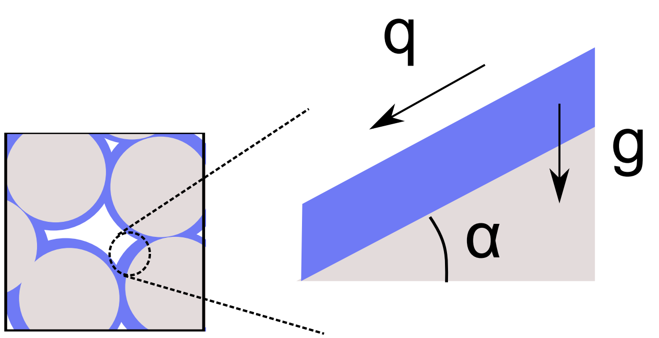
At the local scale, the liquid flows as a thin film

➔ Relation between film thickness *t* and flow rate per unit width *q*

$$t = F\left(\rho, g, \alpha, \mu, q\right)$$

Laminar flow ➔ $t = F\left(\rho g \sin\alpha, \mu, q\right)$

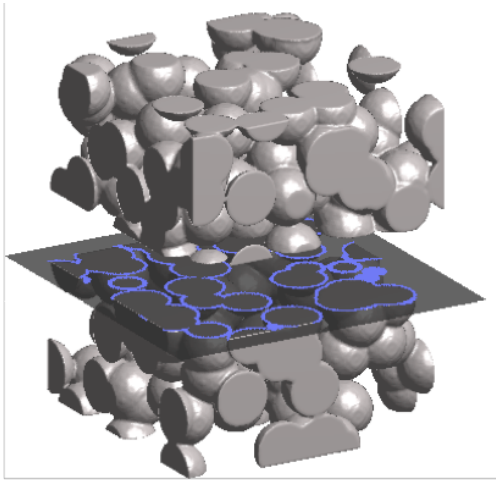
➔ $$\frac{t^3 \rho g \sin\alpha}{\mu q} = cst. \sim 1$$



The importance of educated guessing

Geometrical consequences of the thin film assumption:

1) Any cross section of the packing ➔ convoluted perimeter of length *L* (in blue)



$$Q \sim qL \Rightarrow U \sim q\frac{L}{\pi d^2} \sim q a_V$$

2) $$t = \frac{h_L}{a_V}$$

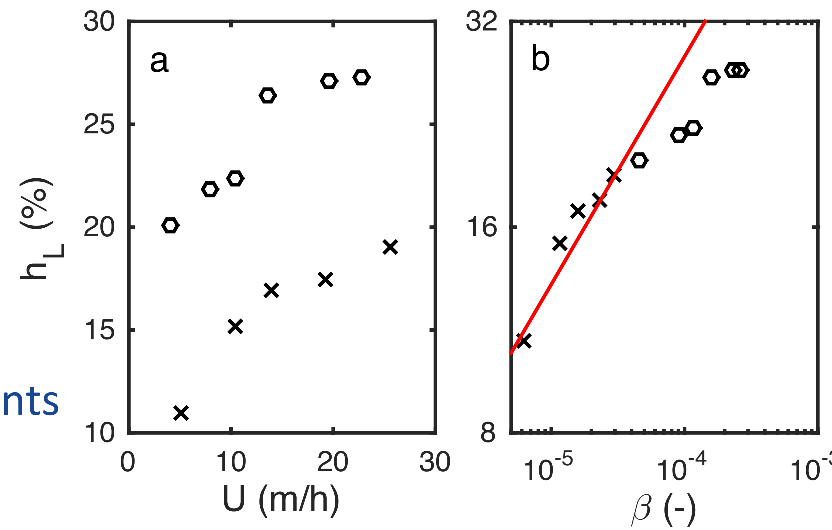
➔ … ➔ $$h_L \sim \left(\frac{\mu U a_V^2}{\rho g}\right)^{1/3} = \beta^{1/3}$$

Only one experiment left !

(proportionality constant betweeen $h_L{}^3$ and β)

Conclusion:

Every educated guess reduces the # of experiments



Dimensional analysis to check assumptions

Thin film assumption: valid only if $t \ll 1/a_V \Rightarrow h_L \ll 1$

(for h_L ~ 1, thick film ➔ preferential flow paths)

Laminar flow assumption:

Inertial force per unit volume ~ $\rho a_V (q/t)^2$

Viscous force per unit volume ~ $\mu \frac{q}{t^3}$

Laminar if inertial << viscous ➔ … ➔ $\beta \ll \left(\frac{\mu^2 a_V^3}{\rho^2 g}\right)^{3/4}$

➔ This defines the range of validity of the obtained scaling laws. Other scaling laws can be obtained outside this range.

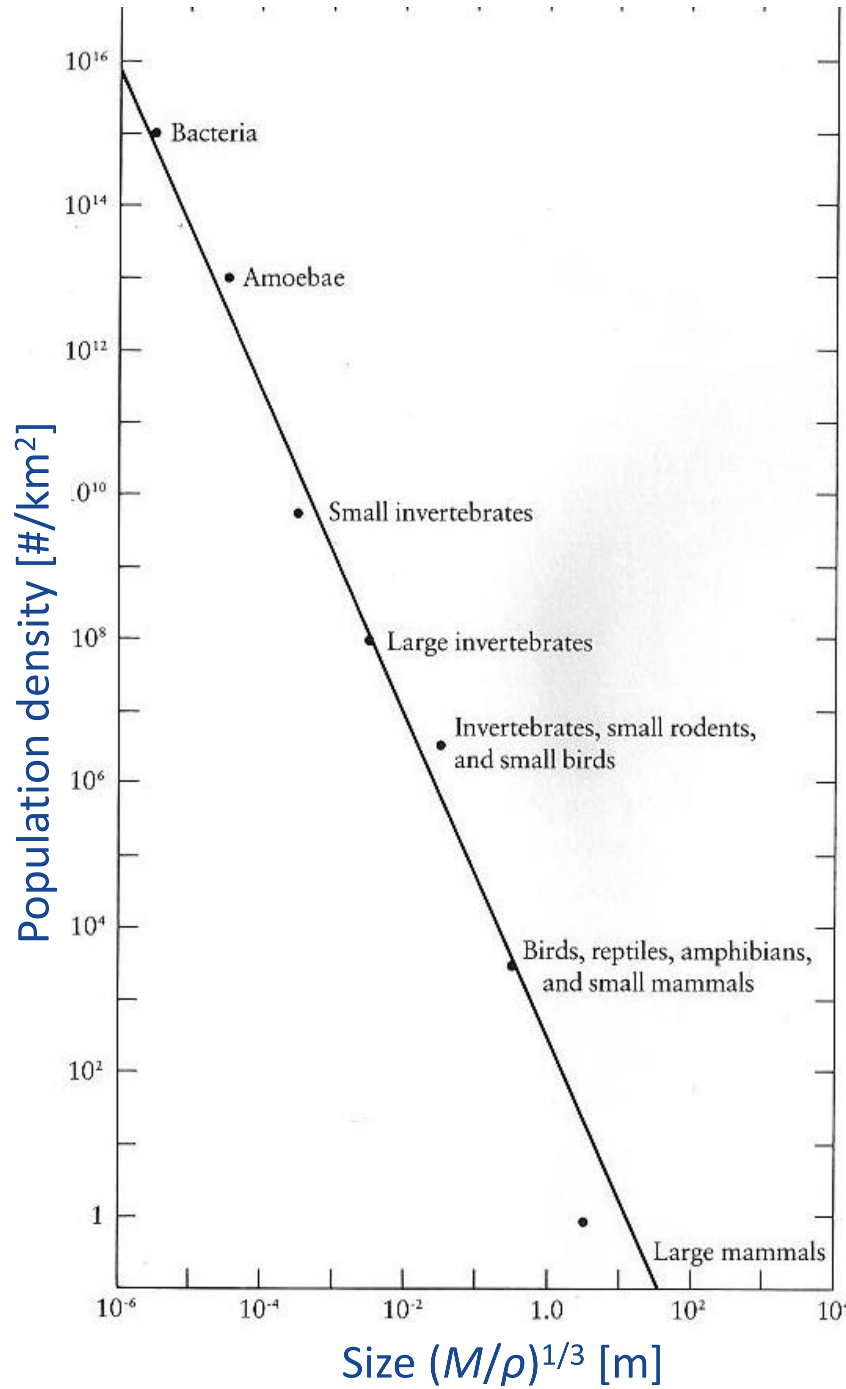
The significance of scaling

One might say that dimensional analysis is only applicable in physics…

Well, not really !

e.g. Damuth's rule in ecology (empirical)
Population density ~ (Body mass)$^{-3/4}$
(for a given species)

What does it have to do with physics ?
Can we show this with dimensional analysis ?



From elastic similarity…

The maximum size of trees (and terrestrial animals)

2 different lengths *r* (radius) and *L* (height)

No buckling if elasticity > weight

Criterion → balancing both (energy)

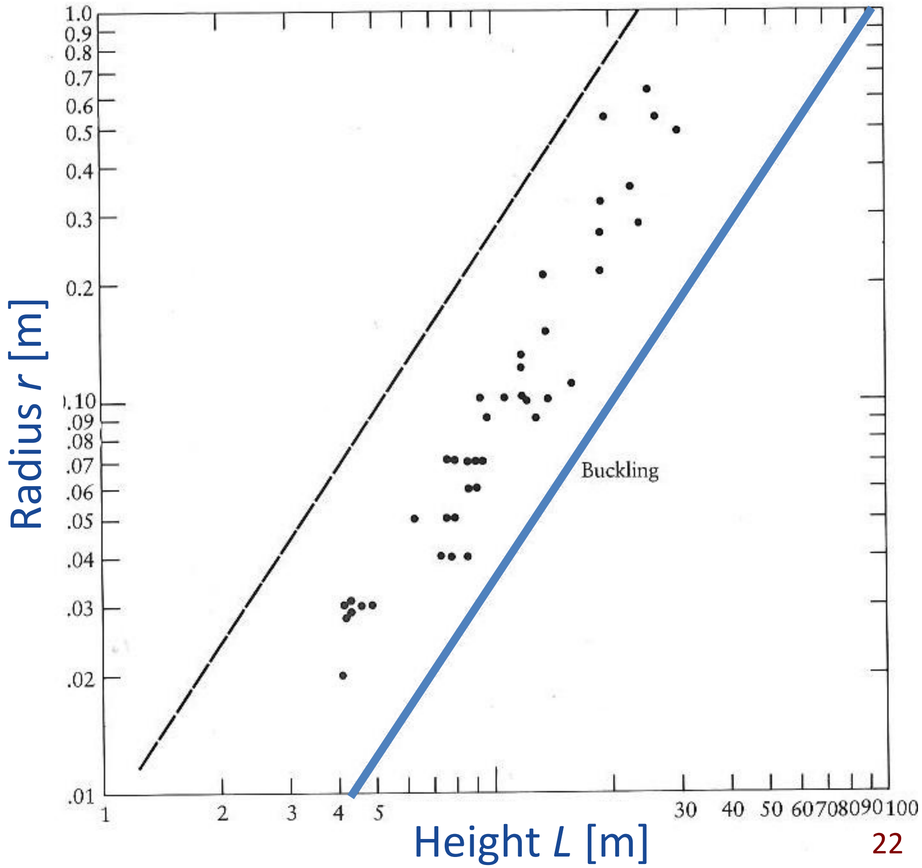
Gravity $$E_{grav} \sim MgL \sim \rho g r^2 L^2$$

Elasticity $$E_{elast} \sim \frac{Er^4}{L}$$

$$E_{grav} \lesssim E_{elast} \Rightarrow \boxed{L^3 \lesssim \frac{E}{\rho g} r^2}$$

Kleiber's law: $\boxed{r \sim M^{3/8}}$

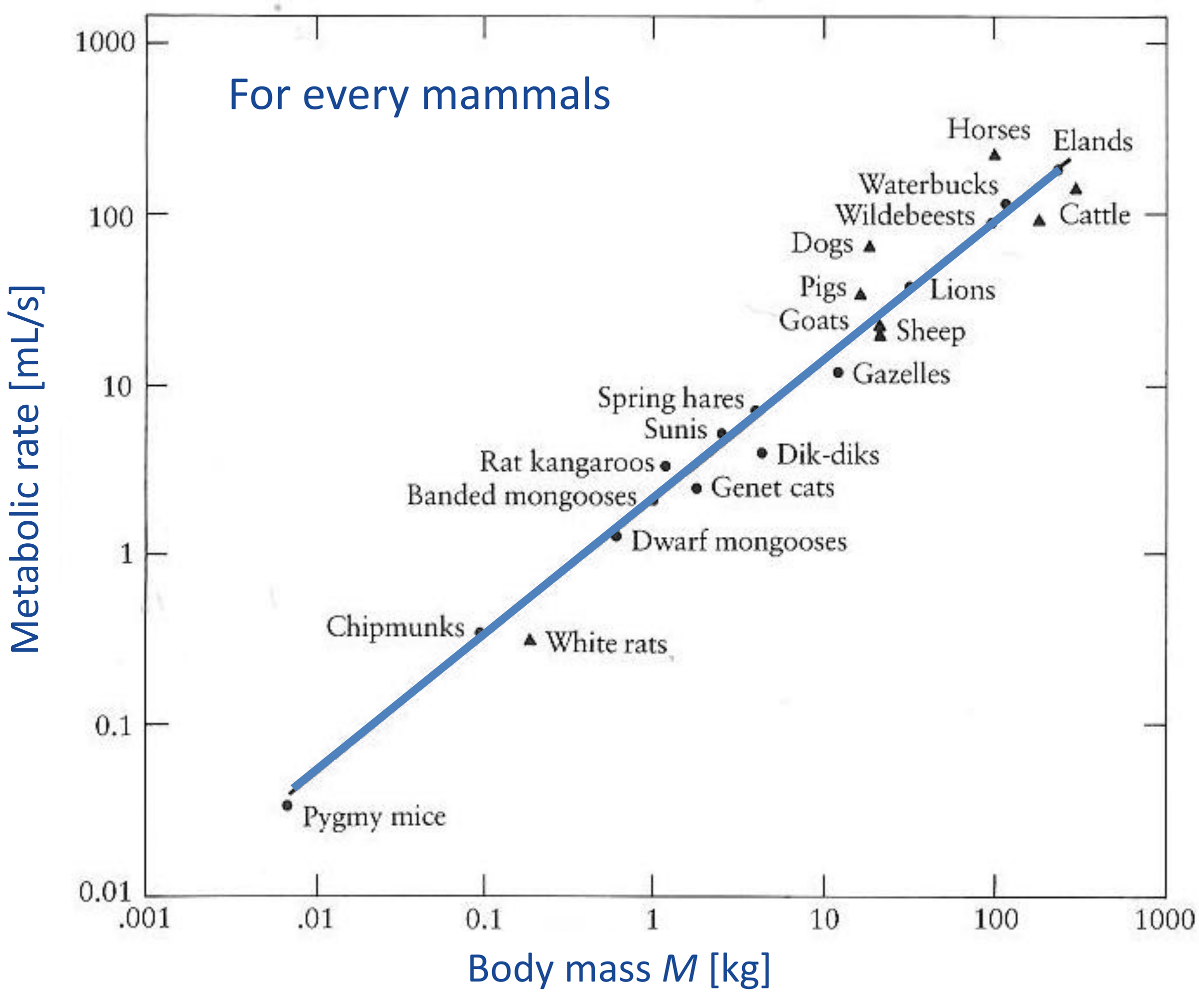
Fagus Grandifolia (increasing age)



… to the metabolic rate…

Metabolic rate (e.g. flow rate of O_2)

Scales as the surface of the lungs ~ $r^2 \sim \boxed{M^{3/4}}$



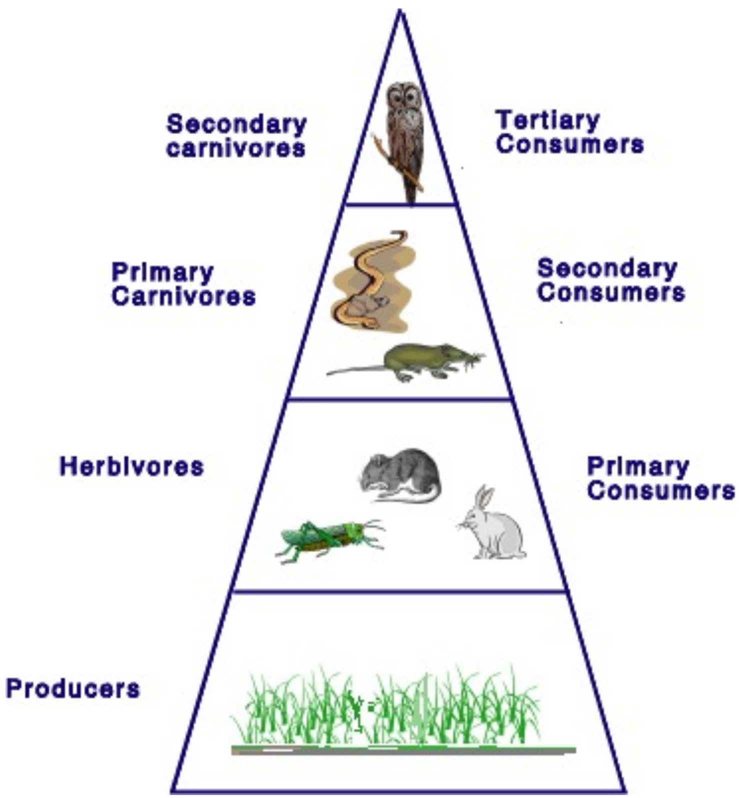
… back to Damuth's rule

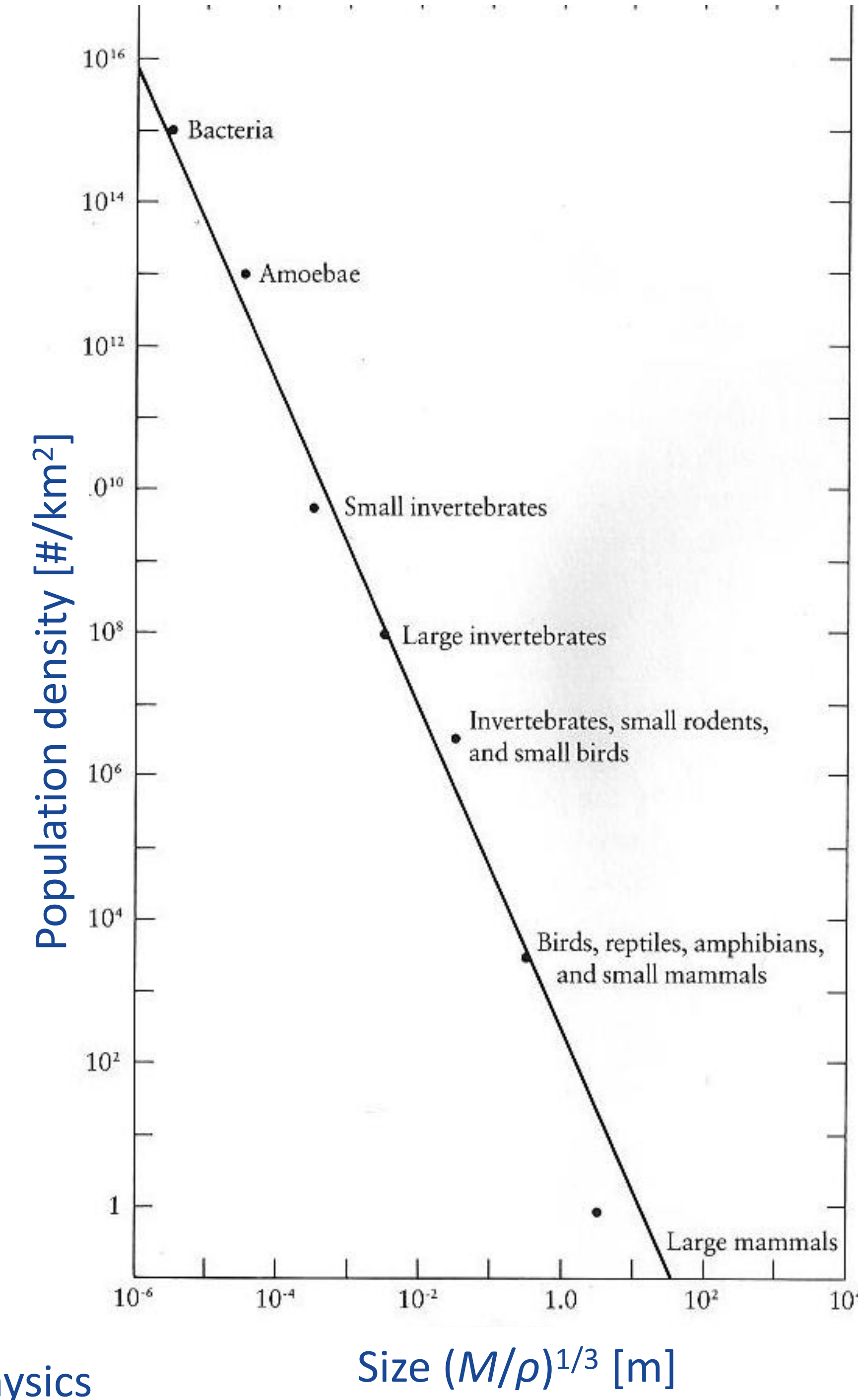
Population density (~ $M^{-3/4}$)

x Metabolic rate (~ $M^{3/4}$)

~ Total amount of food

➔ indep. of M in a balanced food chain





➔ Scaling laws are applicable well beyond physics

Similitude and scaled models

Naval engineering ➔ scaled models of ships

Main indep. variables : V, L, μ, ρ and g

➔ 2 dimensionless groups

- Reynolds $$Re = \frac{\text{Inertia}}{\text{Viscosity}} = \frac{\rho L V}{\mu}$$
- Froude $$Fr = \frac{\text{Inertia (kin. en.)}}{\text{Gravity}} = \frac{V^2}{gL}$$

Realistic scaled model ➔ EVERY dimensionless groups are reproduced

Scale 1:100 ➔ L divided by 100

➔ V multiplied by 10 to conserve Fr

➔ Re divided by 10 (decreasing μ/ρ by 10 is impossible)

➔ Viscous drag overestimated in the scaled version ☹

Trick : Roughness ➔ triggers transition to turbulence prematurely

➔ reduces the importance of viscous drag

Dimensional analysis – summary

Costs:

- A pen and sheet of paper
- One hour

Benefits:

- Applies to the whole realm of science and engineering
- Reduces the number of parameters to vary (especially with educated guessing)
- Provides orders of magnitude for about everything
- Helps comparing different ingredients and identifying which ones are relevant to explain a phenomenon + checking such assumptions
- Helps identifying and rationalizing scaling laws
- Helps building relevant scaled models

Dimensional analysis: why not ?

THE PRINCIPLE OF SIMILITUDE.

I HAVE often been impressed by the scanty attention paid even by original workers in physics to the great principle of similitude. It happens not infrequently that results in the form of "laws" are put forward as novelties on the basis of elaborate experiments, which might have been predicted *a priori* after a few minutes' consideration. However useful verification may be, whether to solve doubts or to exercise students, this seems to be an inversion of the natural order. One reason for the neglect of the principle may be that, at any rate in its applications to particular cases, it does not much interest mathematicians. On the other hand, engineers, who might make much more use of it than they have done, employ a notation which tends to obscure it.

Lord Rayleigh, Nature 1915

Table of contents

Main goal of this workshop

Learn how to work less…

i.e. reduce the number of experiments needed to characterize a system.

Two complementary tools

1. Dimensional analysis

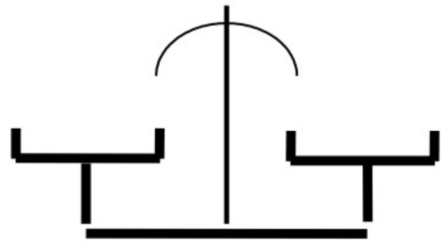
   ➔ Reduce the number of parameters to vary

2. Design of Experiments

   ➔ Choose how to vary these parameters

Design of Experiments

Introductory example: Weigh 3 objects

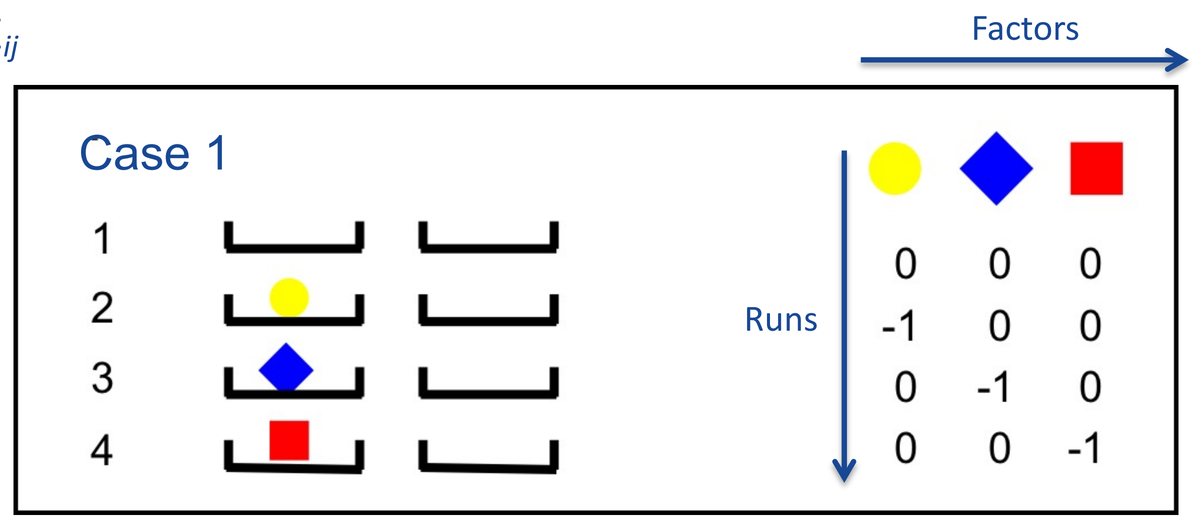


The Balance

The Three Objects

- Result R_i of measurement *i*
  = weight to add to bring the balance back to equilibrium
- 3 factors = weight P_j of each object *j*
- 3 levels for each factor *j*:
  - (-1) if placed on the left
  - (0) if not placed
  - (+1) if placed on the right
- For each experimental run *i* ➔ specify the level chosen for each factor *j*
  ➔ Experimental matrix E_{ij}

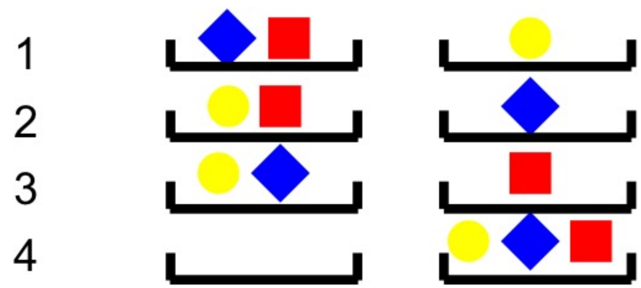
Here, design =
One factor at a time (OFAT)
(most naïve)



Design of Experiments

Case 2

A more elaborated design (called Plackett & Burman) with only 2 levels (+1 / -1)
➔ Is it better ?



1	-1	-1
-1	1	-1
-1	-1	1
1	1	1

Measurement results obtained as

$$R_i = \sum_j E_{ij} P_j$$

In both chosen cases, columns of *E* have been chosen orthogonal:

$$E^T \cdot E = nI$$

where *n*=1 in case 1, and *n*=4 in case 2.

Consequently,

$$P_j = \frac{1}{n} \sum_i E_{ij} R_i$$

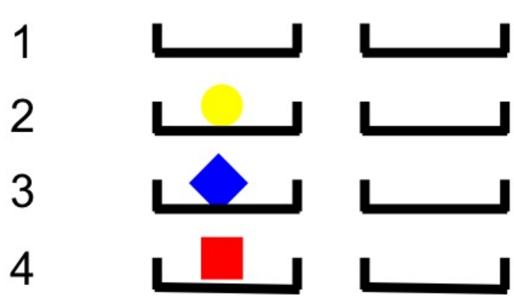
Design of Experiments

If σ^2 is the variance on each measurement (due to errors),

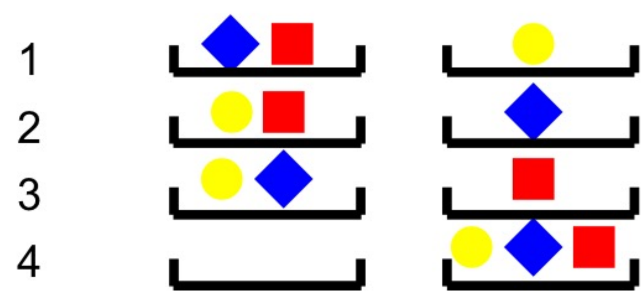
$$\mathrm{var}(P_j) = \frac{1}{n^2}\sum_i E_{ij}^2 \,\mathrm{var}(R_i) = \frac{\sigma^2}{n^2}\sum_i E_{ij}^2$$

Here, the design is such that the sum goes to *n,* so $\mathrm{var}(P_j) = \dfrac{\sigma^2}{n}$

With the same experimental setup and number of experiments, the second design will produce four times less variance on the unknowns P_j than the first design !



Case 1: var(P) = σ^2



Case 2: var(P) = $\sigma^2/4$

Full factorial design

➔ Take every possible combination of levels.

e.g. with two levels and three factors $E = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

Interest: capture coupling between factors, best accuracy

For *N* factors (*j*), each having L_j levels, the number of experiments is $M = \prod_{j=1}^{N} L_j$

which can quickly become prohibitively large.

Generation in Matlab: *L*=[3 2 4]; % # of levels of each factor

E=fullfact(*L*); % Full factorial design
Er=*E*(randperms(prod(*L*)),:); % Randomization
E2=*E*-ones(prod(*L*),1)*mean(*E*); % Centering
E2'**E2* % Orthogonality check

ATTENTION: Need to randomize, i.e. to perform runs in a random order

D-optimal design

If we cannot run as many experiments as in the full factorial design,
what is the best combination ? (which would minimize variance on measurements)

Optimal design ➔ iterative algorithm that looks for this best selection

e.g. in Matlab: 3 factors with 3, 2 and 4 levels respectively. Maximum *M=10* runs

L=[3 2 4]; *M*=10;
[*E*,*X*] = cordexch(numel(*L*),*M*,'linear','categorical',1:numel(*L*),'levels',*L*);
E2=*E*-ones(size(*E*,1),1)*mean(*E*);
E2'**E2*

$$E = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 2 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

Remark: columns of *E2* are not orthogonal anymore !
➔ data analysis more complex

In addition to D-optimal designs, many other possibilities
(Latin square, fractional factorial, Box-Behnken, Taguchi, etc.)
➔ Check online + advanced courses

Now, your turn

Apply dimensional analysis to your project

- List the variables that you think could be relevant in your system, and try to provide a range of variation for each. Separate variables that you measure (Cat. 1) from those that you impose (Cat. 2). In these latter, separate variables that you can easily vary (Cat. 2a) from those that are more or less fixed (Cat. 2b).

- Apply Vaschy-Buckingham's theorem brutally. How many dimensionless numbers do you get ?

- Rethink your dimensionless groups in terms of ratios of timescale, force, energy, etc. If possible, try to have one dimensionless group for each variable of category 1 and 2a. Considering the range of each variable, are some of these dimensionless numbers much smaller / larger than unity ?

- Make educated guesses, i.e. hypotheses dictated by your knowledge of the physics. How many key dimensionless numbers are you left with and what is their range of variation ? How many experiments should you plan ? Is a full-factorial design achievable ? Are your hypotheses supposedly valid in the whole range of these dimensionless numbers ?