Eigenfunctions of the Edge-Based Laplacian on a Graph

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Abstract

In this paper, we analyze the eigenfunctions of the edge-based Laplacian on a graph and the relationship of these functions to random walks on the graph. We commence by discussing the set of eigenfunctions supported at the vertices, and demonstrate the relationship of these eigenfunctions to the classical random walk on the graph. Then, from an analysis of functions supported only on the interior of edges, we develop a method for explicitly calculating the edge-interior eigenfunctions of the edge-based Laplacian. This reveals a connection between the edge-based Laplacian and the adjacency matrix of backtrackless random walk on the graph. The edge-based eigenfunctions therefore correspond to some eigenfunctions of the normalised Hashimoto matrix.

Keywords: graph eigenfunctions, edge-based Laplacian, backtrackless random walk, graph calculus

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1 Introduction

The traditional discrete graph Laplacian operator[2] has proved to be a useful tool in the analysis of graphs and has found application in a number of areas. For example, the heat kernel, which is derived from the graph Laplacian, has been used to define graph kernels[10, 12] in the machine learning literature. Sun et al[15] used the heat kernel on the mesh representing a 3D shape to create a heat kernel signature for describing shape. In [1], Aubry et al used wave-like solutions of Schrödinger's equation to construct an alternative shape descriptor, referred to as the wave kernel signature. The graph Laplacian was used by Coifman and Lafon[3] for dimensionality reduction of data. There are many other applications graph Laplacian in the literature.

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In [5, 6], Friedman and Tillich developed a calculus on graphs which provides strong connections between graph theory and analysis. This approach has a number of advantages. It allows the application of many results from analysis directly to the graph domain, and opens up the use of many new partial differential equations on graphs. As an example, they define a wave equation[7] which has a finite speed of propagation, in contrast to the usual wave equation on a graph.

In the graph calculus of Friedman and Tillich, the graph is given a geometric realization by associating an interval with each edge of the graph. Functions may therefore exist both at the vertices and on the interior of edges. From this starting point they develop a divergence and, most importantly, a graph Laplacian. This type of Laplacian has found application in the physics literature where the interpretation is as the limiting case of a "quantum wire" [9, 14, 11].

The graph Laplacian consists of two parts; namely a vertex-based Laplacian and an edge-based Laplacian. Friedman and Tillich also demonstrate that for edgewise-linear functions the edge-based Laplacian is zero and the graph Laplacian reduces to the traditional discrete graph Laplacian. On the other hand, for functions where the vertex-based Laplacian is zero, they obtain the edge-based Laplacian only. This results in a setting which is substantially different from the traditional approach. In the remainder of this paper, we will concern ourselves with the edge-based Laplacian.

While Friedman and Tillich find the eigenvalues of the edge-based Laplacian, and give some of its eigenfunctions explicitly, they do not give a method for computing the entire eigensystem of the graph. In this paper, we demonstrate the relationship between the simple eigenfunctions and the random walk on the graph. We then give a method for explicit calculation of the remaining eigenfunctions. This analysis reveals a link between the remaining eigenfunctions and the backtrackless random walk on the graph, which in turn is linked to some properties of the quantum walk on a graph[4] and its Ihara zeta function[13]. Because of this link, the edge-based eigenfunctions correspond naturally to eigenfunctions of the normalised adjacency matrix of the oriented line graph (the normalised Hashimoto matrix[8]).

The remainder of this paper is organized as follows. In Section 2, we briefly introduce the formalism of Friedman and Tillich. In Section 3, we detail the calculation of the vertex-supported eigenfunctions of the edge-based Laplacian and describe their relationship to the random walk. Section 4 contains our main result; we provide a method for calculating the remaining eigenfunctions, an approach which is linked to the backtrackless random walk. Together with the eigenfunctions detailed in Section 3, this completely describes the eigensystem in terms of finite matrices on the graph. Finally in Section 5 we give some conclusions and suggest future directions of research.

2 The Edge-based Laplacian

In this section, we give brief details of the formalism of Friedman and Tillich[6]. Let $G = (V, E)$ be a graph with a geometric realization $\mathcal{G}$. The geometric realization is the metric space consisting of the vertices V and a closed interval of length l_e associated with each edge $e \in E$. The graph has a (possibly empty) boundary set $\partial G \subset G$. We assume that the boundary is separated, i.e. that each boundary vertex is incident on only one edge. The graph $\mathring{G}=\{\mathring{V},\mathring{E}\}$ excludes boundary vertices and any incident edges $\mathring{G}= G\backslash\partial G$. We associate an edge variable with each edge. Let $e = (u, v)$ be an edge with interval variable x_e. The edge variable x_e equals zero where the edge meets vertex u and equals one at vertex v. The start and end vertices are determined by assigning an arbitrary orientation to each edge.

Definition 2.1 (Friedman and Tillich[6])**.** A *vertex measure*, $\mathcal{V}$ is a measure supported on the vertices with $\mathcal{V}(v) > 0$ for all $v \in V$. For our purposes it suffices to take $\mathcal{V}(v) = 1$ for all $v \in V$. An *edge measure*, $\mathcal{E}$ is a measure supported on the interior of edges. $\mathcal{E}(v) = 0$ for all $v \in V$ and the restriction to the interior of edges is the Lebesgue measure.

Let f be a function defined on the graph (on both edges and vertices). We take $f(u)$ to mean the value of f at vertex u and $f(e, x_e)$ to mean the value of f at position x_e along edge e. Since we have different volume measures on the edges and vertices, we must take care in dealing with integrating factors since they are different on edges and vertices. We use $d\mathcal{V}$ for the vertex integration factor and $d\mathcal{E}$ for the edge integration factor. As a result, we have a two-part Laplacian:

$$\Delta f = \Delta_V f d\mathcal{V} + \Delta_E f d\mathcal{E}, \tag{1}$$

where Δ_V is the *vertex-based Laplacian* and Δ_E is the *edge-based Laplacian*. Since graph Laplacians are usually given as positive definite operators, the edge-based Laplacian is minus the usual calculus Laplacian

$$\Delta_E f = -\nabla_{\text{calc}} \cdot \nabla f. \tag{2}$$

The vertex-based Laplacian turns out to be

$$\Delta_V f = \frac{1}{\mathcal{V}(v)} \sum_{e \ni v} \mathbf{n}_{e,v} \cdot \nabla f|_e(v). \tag{3}$$

Here $\mathbf{n}_{e,v}$ is the *outward-pointing* unit normal. In other words, for an edge (a, b) it points from a to b at the vertex b, and from b to a at the vertex a.

Definition 2.2. A function is said to be *edge-based* if $\Delta_V f = 0$. For edge-based functions, the Laplacian consists of only the edge-based part, $\Delta f = \Delta_E f d\mathcal{E}$

For a function f to be edge-based, the following condition applies.

$$\sum_{e \ni v} (-1)^{1-x_{e,v}} \nabla f(e, x_{e,v}) = 0 \, \forall v, \tag{4}$$

or in other words, the sum of the outward-pointing gradients must be zero.

3 Vertex-supported Edge-based Eigenfunctions

For the remainder of this paper, we assume that the edge lengths on the graph are equal. Friedman and Tillich[7] demonstrate the connection between the eigenfunctions of the edge-based Laplacian and the eigenvectors of the row-normalised adjacency matrix. This follows directly from the observation that Δ_E is essentially the familiar Laplacian of calculus and therefore admits eigenvectors of the form $f(e, x_e) = C(e)\cos(\omega x_e + B(e))$ where ω is the frequency of the eigenfunction, corresponding to an eigenvalue of ω^2. The eigenfunction is edge-based and so applying condition (4) to the eigenfunction gives

$$\sum_{e \ni v} \frac{f(u) - f(v)\cos\omega}{\sin\omega} = 0. \tag{5}$$

for any $\{\omega, f\}$ for Δ_E, when ω is not a multiple of π. We call ω the *frequency* of the eigenfunction and $\{\omega, f\}$ an *eigenpair* while noting that the corresponding eigenvalue is ω^2.

Definition 3.1. A *principal eigenpair* is an eigenpair of Δ_E with $0 \leq \omega \leq 2\pi$ where ω is the smallest magnitude frequency of a sequence of eigenpairs of the form $\{\omega + 2n\pi, C(e)\cos[(\omega + 2n\pi)x_e + B(e)]\}$, $n \in \mathbb{N}$ with the same coefficients $B(e)$ and $C(e)$.

Let the vector $\mathbf{g} \neq 0$ be the restriction of an eigenfunction f to the vertices, taken in some particular order, i.e. $g_u = f(u)$. All eigenfunctions in the same sequence as f have a vertex restriction equal to $\mathbf{g}$.

The row-normalised adjacency matrix $\tilde{\mathbf{A}}$ for interior vertices $v \in \mathring{V}$ is given by

$$\tilde{A}_{ij} = \frac{A_{ij}}{\sum_j A_{ij}}, \tag{6}$$

where $\mathbf{A}$ is the usual graph adjacency matrix.

Theorem 3.2 (Friedman and Tillich[7])**.** *Let $\mathcal{G}$ be the geometric realization of a graph and let $\tilde{\mathbf{A}}$ be its row-normalized adjacency matrix. Each eigenvalue λ of $\tilde{\mathbf{A}}$, with $\lambda \notin \{-1, 1\}$ corresponds to two principal frequencies of $\mathcal{G}$*

$$\cos^{-1}\lambda \quad \text{and} \quad 2\pi - \cos^{-1}\lambda, \tag{7}$$

each with the same multiplicity as λ. The corresponding eigenvector $\mathbf{g}$ is the vertex restriction of the sequence of eigenfunctions based on the principal eigenvalue ω.

Equation (5) allows us to determine the eigenfunctions. For a principal frequency ω the principal eigenfunction is $f(e, x_e) = C(e)\cos(B(e) + \omega x_e)$ with

$$C(e)^2 = \frac{g_v^2 + g_u^2 - 2g_u g_v \cos\omega}{\sin^2\omega}, \tag{8}$$

$$\tan B(e) = \frac{g_v \cos\omega - g_u}{g_v \sin\omega}. \tag{9}$$

There are two solutions to Equations (8) and (9) which are $\{C(e), B_0(e)\}$ or $\{-C(e), B_0(e)+\pi\}$ but both give the same eigenfunction. The sign of $C(e)$ must be chosen correctly to match the phase, i.e. so that $C(e)\cos(B(e)) = g_u$. Since B and C are uniquely determined, this comprises all the eigenvectors of this form.

Friedman and Tillich give a pair of frequencies as above for each eigenvalue of the row-normalized adjacency matrix, but we note that the eigenpair with $\omega = 2\pi - \cos^{-1}\lambda$ actually corresponds to the same sequence of eigenpairs as those with $\omega = \cos^{-1}\lambda$, but with negative values of n. There is therefore a single sequence for each eigenvalue with

$$\omega = \cos^{-1}\lambda, \tag{10}$$

$$f(e, x_e) = C(e)\cos\left[B(e) + (\omega + 2\pi n)x_e\right], n \in \mathbb{Z}. \tag{11}$$

The value $\lambda = 1$ is always an eigenvalue of $\tilde{\mathbf{A}}$. This corresponds to a principal frequency of $\omega = 0$ in Δ_E and therefore the corresponding eigenfunction is $f(e, x_e) = C(e)\cos(B(e))$ which is constant on the vertices. If $\partial G \neq \emptyset$ then this eigenfunction must be zero everywhere. If $\partial G = \emptyset$ then we obtain a single principal eigenpair with $\omega = 0$ and $f(e, x_e) = C$.

The value $\lambda = -1$ will be an eigenvalue of $\tilde{\mathbf{A}}$ if $\mathring{G}$ is bipartite. If $\partial G \neq \emptyset$ then the eigenfunction must be zero on the vertices. We defer the evaluation of eigenfunctions not supported on V for the next section. If $\partial G = \emptyset$ then a single principal eigenpair exists with $\omega = \pi$ and $f(e, x_e) = C\cos(\pi x_e)$. The eigenfunction alternates in sign between the two partitions of the graph.

This comprises all principal eigenpairs which are supported on the vertices ($\mathbf{g} \neq \mathbf{0}$)[7]. The eigenfunctions supported on the vertices are therefore directly determined by the eigensystem of $\tilde{\mathbf{A}}$.

3.1 Random Walks and Line Graphs

The line graph $\mathrm{LG}(G) = (V_l, E_l)$ of a graph G is a graph construction in which we replace the edges of G with the vertices of $\mathrm{LG}(G)$ as follows. Firstly we create the symmetric digraph $\mathrm{SDG}(G)$ of G by replacing each undirected edge with a pair of oriented edges. Each oriented edge of the SDG then becomes a vertex of $\mathrm{LG}(G)$. These vertices are connected if the head of one oriented edge meets the tail of another. The reverse pair of oriented edges are connected, i.e. $((u,v),(v,u))$ is an edge in the LG.

$$V_l = \{(u,v) \in E(SDG)\},$$

$$E_l = \{((u,v),(v,w)), (u,v) \in E(SDG), (v,w) \in E(SDG)\}.$$

The *oriented line graph* $\mathrm{OLG}(G) = (V_o, E_o)$ is constructed in the same way as the LG except that reverse pairs of oriented edges are not connected, i.e. $((u,v),(v,u))$ is *not* an edge. The vertex and edge sets of $\mathrm{OLG}(G)$ are therefore

$$V_o = \{(u,v) \in E(SDG)\},$$

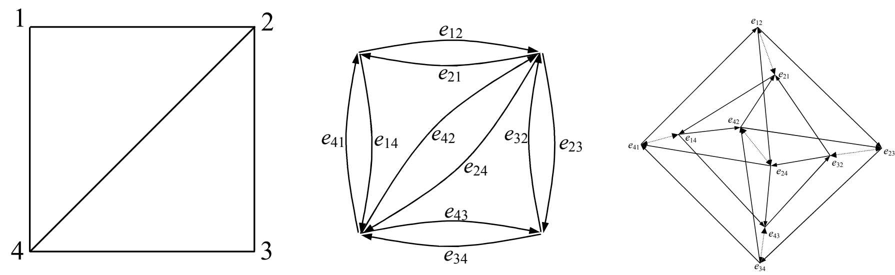

Figure 1: Left: A graph. Middle: Its symmetric digraph. Right: The line graph and the oriented line graph; the dotted edges exist in the line graph but not in the oriented line graph.

$$E_o = \{((u,v),(v,w)), (u,v) \in E(SDG), (v,w) \in E(SDG) : u \neq w\}.$$

Figure 1 illustrates these concepts. A random walk on the vertices of LG represents the sequence of edges traversed in a random walk on the original graph G. Similarly, a random walk on the OLG represents the sequence of edges in a random walk on G where backtracking steps are not allowed (a *backtrackless walk*).

Proposition 3.3. *Let $\tilde{\mathbf{A}}$ be the row-normalised adjacency matrix of G and $\tilde{\mathbf{U}}$ be the row-normalised adjacency matrix of the line graph of G. Each eigenpair $\{\lambda, \mathbf{g}\}$ of $\tilde{\mathbf{A}}$ corresponds to an eigenpair $\{\mu, \mathbf{h}\}$ of $\tilde{\mathbf{U}}$ with*

$$\begin{aligned} \mu &= \lambda, & (12) \\ h_{uv} &= A_{uv} g_v. & (13) \end{aligned}$$

Proof. We may write the row-normalized adjacency matrix of the LG as

$$\tilde{\mathbf{U}}_{uv,wx} = \frac{A_{uv} A_{wx} \delta_{vw}}{d_x} = A_{uv} \tilde{A}_{wx} \delta_{vw} \tag{14}$$

We have

$$\begin{aligned} \sum_{w,x} \tilde{U}_{uv,wx} h_{wx} &= \sum_{w,x} A_{uv} \tilde{A}_{wx} \delta_{vw} A_{wx} g_x, & (15) \\ &= A_{uv} \sum_{x} \tilde{A}_{vx} g_x, & (16) \\ &= \lambda A_{uv} g_v = \mu h_{uv}. & (17) \end{aligned}$$

□

The vertex-supported eigenfunctions of the edge-based Laplacian are therefore determined by the structure of the random walk on the graph. As we shall show later, the remaining eigenfunctions are determined by the structure of the backtrackless random walk.

4 Edge-interior Eigenfunctions

We now proceed to our main result. All remaining eigenfunctions of the edge-based Laplacian are zero on the vertices of $\mathcal{G}$ and therefore must have a principal frequency of $\omega \in \{\pi, 2\pi\}$. We start with the case $\omega = \pi$.

Proposition 4.1. *Principal eigenfunctions of the edge-based Laplacian with principal frequency $\omega = \pi$ and which are zero on the vertices of $\mathcal{G}$ are of the form $f(e, x_e) = C(e)\cos(\frac{\pi}{2} + \pi x_e)$, $e \in \mathring{E}$ with*

$$\sum_{e \ni v} C(e) = 0 \; \forall v \in \mathring{V}. \tag{18}$$

Proof. Since the boundary is separated, the eigenfunctions must be zero on any boundary edge and any edge incident on a boundary vertex. As a result, we may concern ourselves only with $\mathring{G}$. The eigenfunction $f(e, x_e)$ is zero at both vertices incident on edge e, giving values for $B(e)$ of $B(e) \in \{\pi/2, 3\pi/2\}$, which both give the same eigenfunction (with a different sign for $C(e)$). We may therefore take $B(e) = \pi/2$. The gradients at either end of the edge are $\nabla f(e, x_e = 0) = -\pi C(e)\sin(\frac{\pi}{2})$ and $\nabla f(e, x_e = 1) = -\pi C(e)\sin(\frac{3\pi}{2})$. Applying condition (4) to this eigenfunction, we obtain

$$\sum_{e \ni v} C(e) = 0.$$

□

Hence in order to find eigenfunctions of this type, we must find a set of coefficients attached to the edges which sum to zero at every vertex.

Proposition 4.2. *The principal eigenfunctions of the edge-based Laplacian with principal frequency $\omega = 2\pi$ and which are zero on the vertices of $\mathcal{G}$ are of the form $f(e, x_e) = C(e)\cos(\frac{\pi}{2} + 2\pi x_e)$, $e \in \mathring{E}$ with*

$$\sum_{e \ni v} (-1)^{1 - x_{e,v}} C(e) = 0 \; \forall v \in \mathring{V}. \tag{19}$$

Proof. The proof is essentially the same as for the previous proposition. However the gradients at either end of the edge are $\nabla f(e, x_e = 0) = -2\pi C(e)\sin(\frac{\pi}{2})$ and $\nabla f(e, x_e = 1) = -2\pi C(e)\sin(\frac{5\pi}{2})$. Applying Condition (4), we obtain

$$\sum_{e \ni v} (-1)^{1 - x_{e,v}} C(e) = 0.$$

□

We may interpret this condition as follows: Consider each undirected edge of $\mathring{G}$ as a pair of directed edges. Associate a value $C(e)$ with each directed edge in the direction of increasing x_e (i.e. from $x_e = 0$ to $x_e = 1$) and a value $-C(e)$

with the reverse edge. Then the condition above is that the sum of incoming directed edges at a vertex must be zero.

We may write these conditions for both principal frequencies in the following form. Let $w_{uv} = \pm C(e)$ be the values associated with each edge, as described above. The sign is always positive for $\omega = \pi$ and alternates in sign depending on the direction of edge traversal for $\omega = 2\pi$. Then we may form a matrix $\mathbf{W}$ with elements $W_{uv} = w_{uv}A_{uv}$. The conditions for $\omega = \pi$ are then

$$\mathbf{W1} = 0, \tag{20}$$
$$\mathbf{W} = \mathbf{W}^T, \tag{21}$$

and for $\omega = 2\pi$ they are

$$\mathbf{W1} = 0, \tag{22}$$
$$\mathbf{W} = -\mathbf{W}^T, \tag{23}$$

where $\mathbf{1}$ is the vector of all-ones.

The oriented line graph of G (OLG(G)) represents the structure of a backtrackless random walk on G in the sense that a random walk on the vertices of OLG(G) generate a sequence of edges which can be traversed in a backtrackless random walk on G. In [13] we demonstrated that the adjacency matrix of the oriented line graph ($\mathbf{T}$) is equal to the positive support of a quantum walk on the graph G. This matrix is also related to the Ihara zeta function of the graph since it is equal to the Perron-Frobenius operator on OLG(G) and therefore can be used to calculate the Ihara zeta function $Z_G(u)$ using $Z_G^{-1}(u) = \det(\mathbf{I} - u\mathbf{T})$. We now show that the eigenvectors of $\mathbf{T}$ corresponding to eigenvalues of $\lambda = \pm 1$ determine the structure of the edge-interior eigenfunctions.

Theorem 4.3. *Let $\mathbf{T}$ be the adjacency matrix of OLG(G) and $\mathbf{s}$ be an eigenvector of $\mathbf{T}$ with eigenvalue $\lambda = 1$. Then $s_{uv} = -s_{vu}$ and $\sum_u s_{uv} = 0$, and $\mathbf{s}$ provides a solution for $\mathbf{W}$ in the case of $\omega = 2\pi$. Similarly, if $\lambda = -1$ then $s_{uv} = s_{vu}$ and $\sum_u s_{uv} = 0$, and $\mathbf{s}$ provides a solution for $\mathbf{W}$ in the case of $\omega = \pi$.*

Proof. In [13] we demonstrate that $s_{uv} = A_{uv}w_{uv}$ is an eigenvector of $\mathbf{T}$ if $\sum_v A_{uv}w_{uv} = 0$ and either $w_{uv} = -w_{vu}$ or $w_{uv} = w_{vu}$. If $w_{uv} = -w_{vu}$ the eigenvalue is $\lambda = 1$, and if $w_{uv} = w_{vu}$ then the eigenvalue is $\lambda = -1$. These are precisely the solutions for $\mathbf{W}$ (and for $C(e)$). Since the eigenvectors with $\lambda = \pm 1$ span the space of possible solutions, we obtain $|E| - |V| + 1$ linearly independent solutions for $\lambda = 1$ and $|E| - |V|$ linearly independent solutions for $\lambda = -1$ which are all the available solutions according to [7] □

The structure of the eigenfunctions which are not supported on the vertices is therefore determined by the eigenvectors of the backtrackless random walk on the graph G.

5 Conclusions

We have analyzed and completely determined the eigenfunctions of the edge-based Laplacian and given explicit forms for all of these eigenfunctions. Our analysis provides a method of computing the eigenfunctions which are zero on the vertices from the eigenvectors of the oriented line graph. We demonstrate the connection between the eigenfunctions and both the classical random walk and the backtrackless random walk. The eigensystem of edge-based Laplacian contains eigenfunctions which are related to both the adjacency matrix of the line graph of G and the adjacency matrix of the oriented line graph of G.

As noted by Friedman and Tillich[7], this approach is closer to traditional analysis than the usual discrete graph Laplacian. In particular it allows us to formulate wave equations and relativistic heat equations which have the more usual properties associated with these equations on a manifold (for example they will have a finite speed of propagation). The eigensystem of the edge-based Laplacian may be of great use in the study of networks where distance and propagation speed are important. The current analysis is currently limited to the case of uniform edge lengths. Future work will focus on the case where the edge lengths may vary.

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