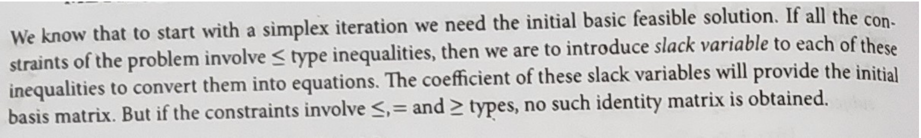
Linear Programming Problem (II)

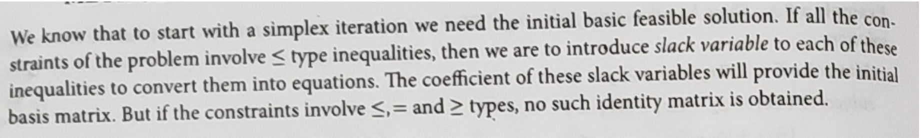
February 10, 2022

Artificial Variables and Charnes' Big-M Method

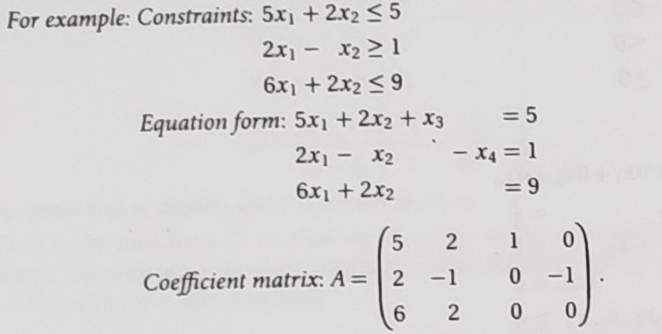


We know that to start with a simplex iteration we need the initial basic feasible solution. If all the constraints of the problem involve ≤ type inequalities, then we are to introduce *slack variable* to each of these inequalities to convert them into equations. The coefficient of these slack variables will provide the initial basis matrix. But if the constraints involve ≤, = and ≥ types, no such identity matrix is obtained.

Artificial Variables and Charnes' Big-M Method



We know that to start with a simplex iteration we need the initial basic feasible solution. If all the constraints of the problem involve $\leq$ type inequalities, then we are to introduce *slack variable* to each of these inequalities to convert them into equations. The coefficient of these slack variables will provide the initial basis matrix. But if the constraints involve $\leq, =$ and $\geq$ types, no such identity matrix is obtained.



For example: Constraints:

$$\begin{aligned} 5x_1 + 2x_2 &\leq 5 \\ 2x_1 - x_2 &\geq 1 \\ 6x_1 + 2x_2 &\leq 9 \end{aligned}$$

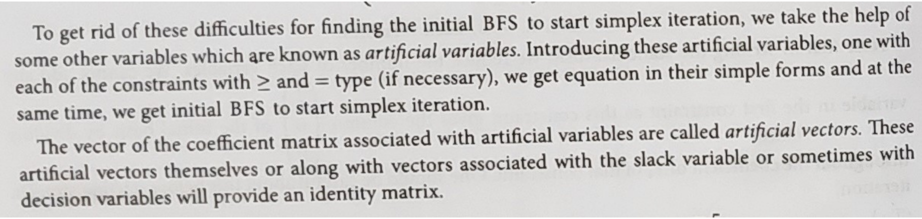
Equation form:

$$\begin{aligned} 5x_1 + 2x_2 + x_3 \phantom{{}- x_4} &= 5 \\ 2x_1 - x_2 \phantom{{}+ x_3} - x_4 &= 1 \\ 6x_1 + 2x_2 \phantom{{}+ x_3 - x_4} &= 9 \end{aligned}$$

Coefficient matrix:

$$A = \begin{pmatrix} 5 & 2 & 1 & 0 \\ 2 & -1 & 0 & -1 \\ 6 & 2 & 0 & 0 \end{pmatrix}.$$

Cont...



To get rid of these difficulties for finding the initial BFS to start simplex iteration, we take the help of some other variables which are known as *artificial variables*. Introducing these artificial variables, one with each of the constraints with $\geq$ and $=$ type (if necessary), we get equation in their simple forms and at the same time, we get initial BFS to start simplex iteration.

The vector of the coefficient matrix associated with artificial variables are called *artificial vectors*. These artificial vectors themselves or along with vectors associated with the slack variable or sometimes with decision variables will provide an identity matrix.

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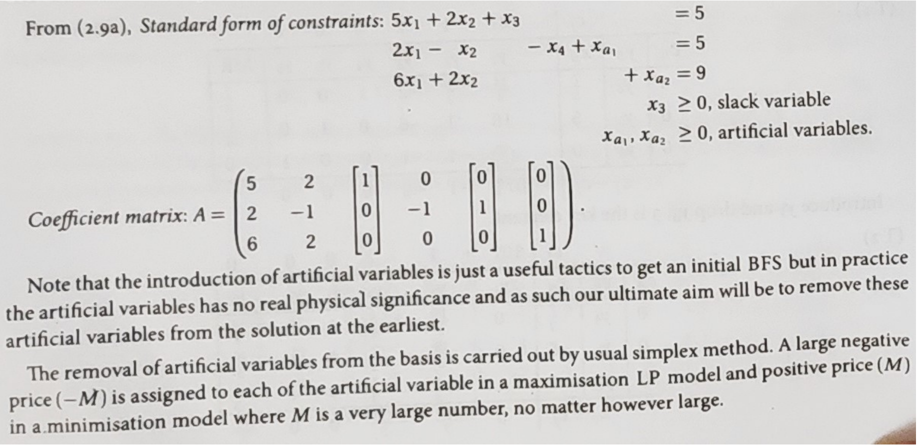

From (2.9a), *Standard form of constraints:*

$$
\begin{aligned}
5x_1 + 2x_2 + x_3 & & &= 5 \\
2x_1 - x_2 & - x_4 + x_{a_1} & &= 5 \\
6x_1 + 2x_2 & & + x_{a_2} &= 9 \\
& & x_3 &\geq 0, \text{ slack variable} \\
& & x_{a_1}, x_{a_2} &\geq 0, \text{ artificial variables.}
\end{aligned}
$$

Coefficient matrix: $A = \left(\begin{matrix} 5 \\ 2 \\ 6 \end{matrix} \quad \begin{matrix} 2 \\ -1 \\ 2 \end{matrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} 0 \\ -1 \\ 0 \end{matrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$

Note that the introduction of artificial variables is just a useful tactics to get an initial BFS but in practice the artificial variables has no real physical significance and as such our ultimate aim will be to remove these artificial variables from the solution at the earliest.

The removal of artificial variables from the basis is carried out by usual simplex method. A large negative price $(-M)$ is assigned to each of the artificial variable in a maximisation LP model and positive price (M) in a minimisation model where M is a very large number, no matter however large.

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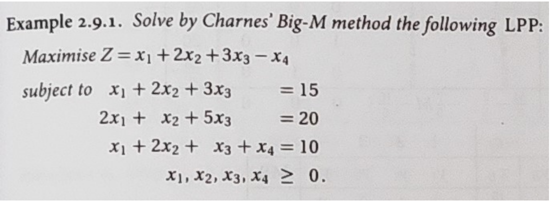
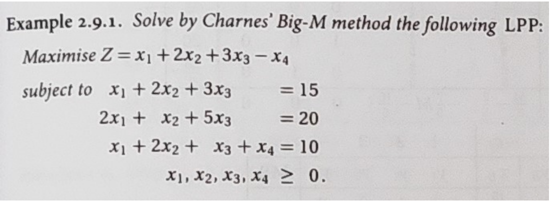


Example 2.9.1. *Solve by Charnes' Big-M method the following* LPP:

$$\text{Maximise } Z = x_1 + 2x_2 + 3x_3 - x_4$$

$$\begin{aligned} \text{subject to} \quad x_1 + 2x_2 + 3x_3 \qquad &= 15 \\ 2x_1 + x_2 + 5x_3 \qquad &= 20 \\ x_1 + 2x_2 + x_3 + x_4 &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Cont...



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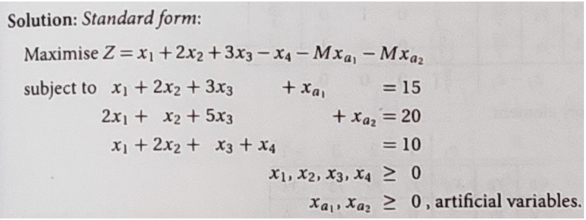
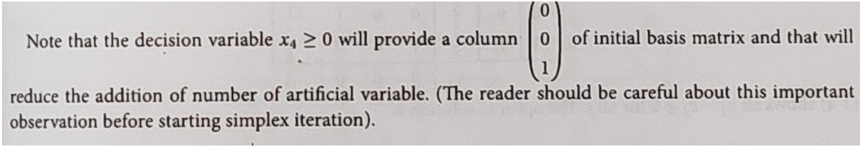


Solution: *Standard form:*

$$\text{Maximise } Z = x_1 + 2x_2 + 3x_3 - x_4 - Mx_{a_1} - Mx_{a_2}$$

$$\begin{aligned} \text{subject to} \quad x_1 + 2x_2 + 3x_3 + x_{a_1} \phantom{+ x_{a_2}} &= 15 \\ 2x_1 + x_2 + 5x_3 \phantom{+ x_4 + x_{a_1}} + x_{a_2} &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \phantom{+ x_{a_1} + x_{a_2}} &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \\ x_{a_1}, x_{a_2} &\geq 0\text{, artificial variables.} \end{aligned}$$

Cont...



Note that the decision variable $x_4 \geq 0$ will provide a column $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ of initial basis matrix and that will reduce the addition of number of artificial variable. (The reader should be careful about this important observation before starting simplex iteration).

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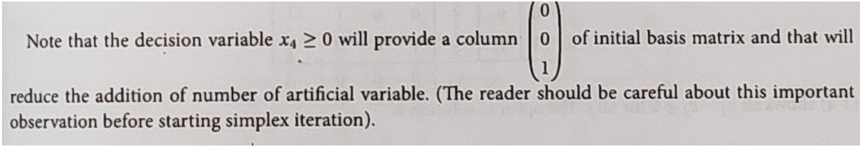

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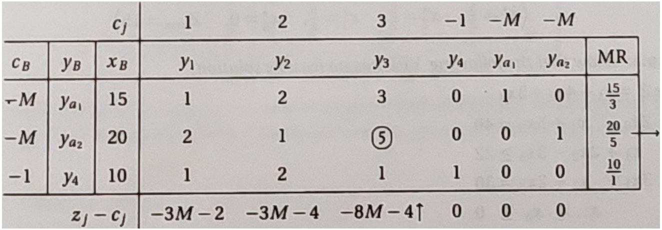
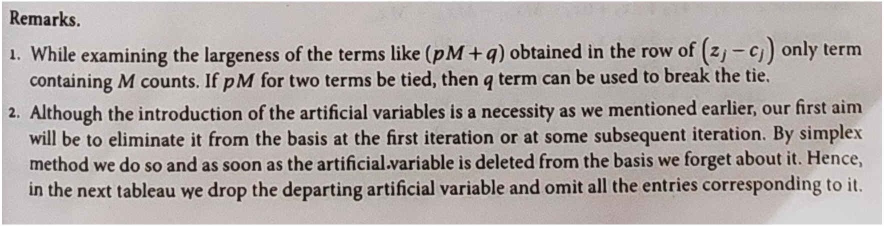


		c_j	1	2	3	-1	$-M$	$-M$	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_{a_1}	y_{a_2}	MR
$-M$	y_{a_1}	15	1	2	3	0	1	0	$\frac{15}{3}$
$-M$	y_{a_2}	20	2	1	⑤	0	0	1	$\frac{20}{5}$ →
-1	y_4	10	1	2	1	1	0	0	$\frac{10}{1}$
		$z_j - c_j$	$-3M-2$	$-3M-4$	$-8M-4\uparrow$	0	0	0	

Cont...



Remarks.

1. While examining the largeness of the terms like $(pM+q)$ obtained in the row of $\left(z_j - c_j\right)$ only term containing M counts. If pM for two terms be tied, then q term can be used to break the tie.
2. Although the introduction of the artificial variables is a necessity as we mentioned earlier, our first aim will be to eliminate it from the basis at the first iteration or at some subsequent iteration. By simplex method we do so and as soon as the artificial.variable is deleted from the basis we forget about it. Hence, in the next tableau we drop the departing artificial variable and omit all the entries corresponding to it.

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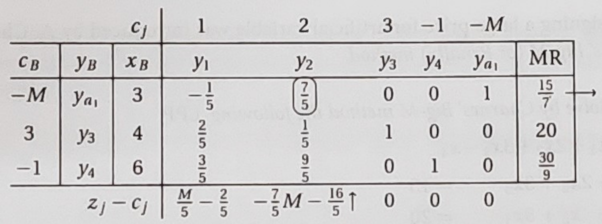

		c_j	1	2	3	-1	$-M$	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_{a_1}	MR
$-M$	y_{a_1}	3	$-\frac{1}{5}$	$\boxed{\frac{7}{5}}$	0	0	1	$\frac{15}{7} \longrightarrow$
3	y_3	4	$\frac{2}{5}$	$\frac{1}{5}$	1	0	0	20
-1	y_4	6	$\frac{3}{5}$	$\frac{9}{5}$	0	1	0	$\frac{30}{9}$
		$z_j - c_j$	$\frac{M}{5} - \frac{2}{5}$	$-\frac{7}{5}M - \frac{16}{5} \uparrow$	0	0	0	

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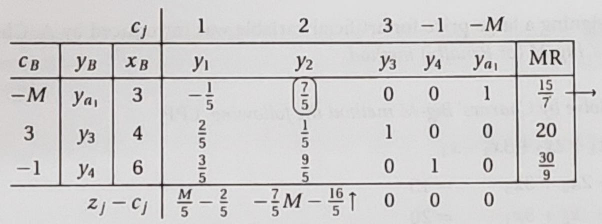
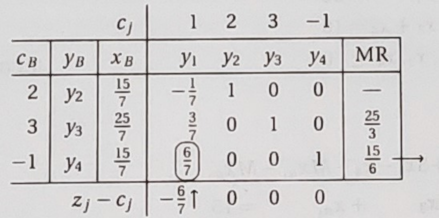


		c_j	1	2	3	-1	$-M$	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_{a_1}	MR
$-M$	y_{a_1}	3	$-\frac{1}{5}$	$\boxed{\frac{7}{5}}$	0	0	1	$\frac{15}{7} \longrightarrow$
3	y_3	4	$\frac{2}{5}$	$\frac{1}{5}$	1	0	0	20
-1	y_4	6	$\frac{3}{5}$	$\frac{9}{5}$	0	1	0	$\frac{30}{9}$
		$z_j - c_j$	$\frac{M}{5} - \frac{2}{5}$	$-\frac{7}{5}M - \frac{16}{5} \uparrow$	0	0	0	



		c_j	1	2	3	-1	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	MR
2	y_2	$\frac{15}{7}$	$-\frac{1}{7}$	1	0	0	—
3	y_3	$\frac{25}{7}$	$\frac{3}{7}$	0	1	0	$\frac{25}{3}$
-1	y_4	$\frac{15}{7}$	$\boxed{\frac{6}{7}}$	0	0	1	$\frac{15}{6} \longrightarrow$
		$z_j - c_j$	$-\frac{6}{7} \uparrow$	0	0	0	

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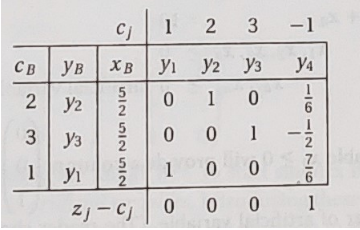
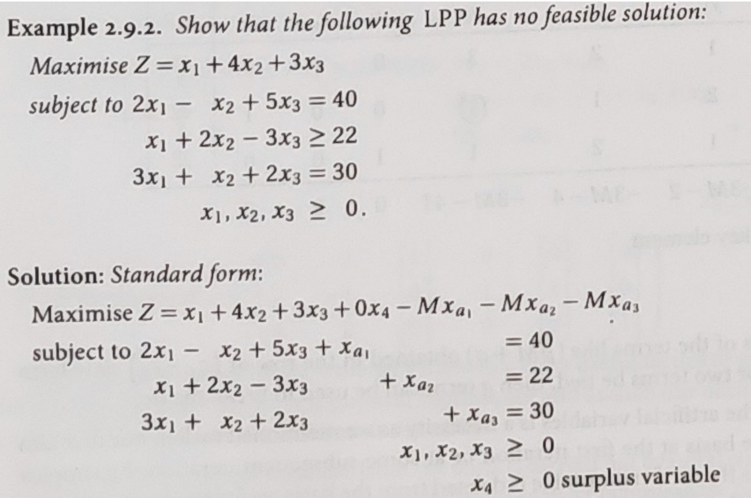


		c_j	1	2	3	-1
c_B	y_B	x_B	y_1	y_2	y_3	y_4
2	y_2	$\frac{5}{2}$	0	1	0	$\frac{1}{6}$
3	y_3	$\frac{5}{2}$	0	0	1	$-\frac{1}{2}$
1	y_1	$\frac{5}{2}$	1	0	0	$\frac{7}{6}$
		$z_j - c_j$	0	0	0	1

Example



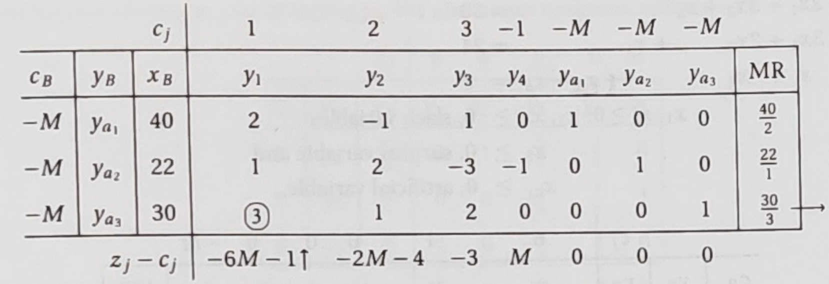
Example 2.9.2. *Show that the following* LPP *has no feasible solution:*

$$
\begin{aligned}
\textit{Maximise } Z &= x_1 + 4x_2 + 3x_3 \\
\textit{subject to } 2x_1 - x_2 + 5x_3 &= 40 \\
x_1 + 2x_2 - 3x_3 &\geq 22 \\
3x_1 + x_2 + 2x_3 &= 30 \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

Solution: *Standard form*:

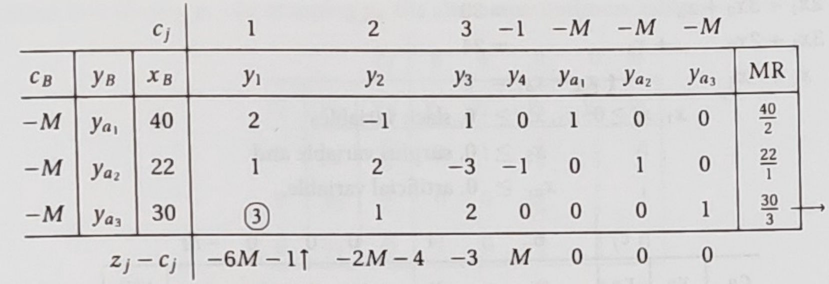
$$
\begin{aligned}
\text{Maximise } Z &= x_1 + 4x_2 + 3x_3 + 0x_4 - Mx_{a_1} - Mx_{a_2} - Mx_{a_3} \\
\text{subject to } 2x_1 - x_2 + 5x_3 + x_{a_1} &= 40 \\
x_1 + 2x_2 - 3x_3 \qquad + x_{a_2} &= 22 \\
3x_1 + x_2 + 2x_3 \qquad + x_{a_3} &= 30 \\
x_1, x_2, x_3 &\geq 0 \\
x_4 &\geq 0 \text{ surplus variable}
\end{aligned}
$$

Example

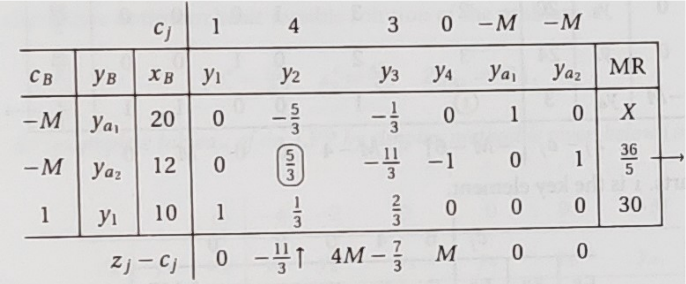


		c_j	1	2	3	-1	$-M$	$-M$	$-M$	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_{a_1}	y_{a_2}	y_{a_3}	MR
$-M$	y_{a_1}	40	2	-1	1	0	1	0	0	$\frac{40}{2}$
$-M$	y_{a_2}	22	1	2	-3	-1	0	1	0	$\frac{22}{1}$
$-M$	y_{a_3}	30	③	1	2	0	0	0	1	$\frac{30}{3}$ →
		$z_j - c_j$	$-6M-1$↑	$-2M-4$	-3	M	0	0	0	

Example

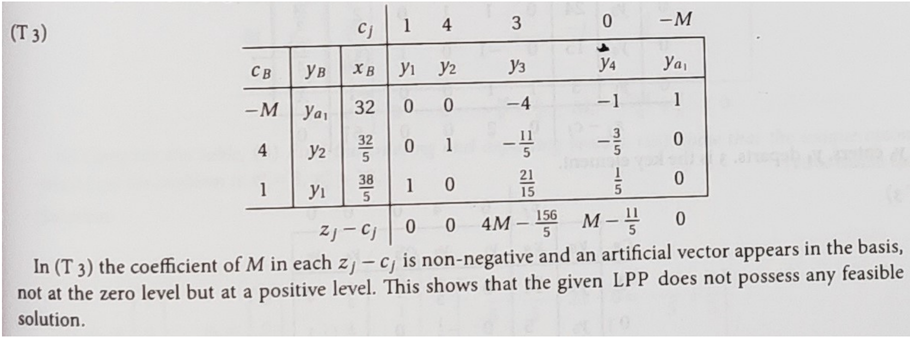


		c_j	1	2	3	-1	$-M$	$-M$	$-M$	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_{a_1}	y_{a_2}	y_{a_3}	MR
$-M$	y_{a_1}	40	2	-1	1	0	1	0	0	$\frac{40}{2}$
$-M$	y_{a_2}	22	1	2	-3	-1	0	1	0	$\frac{22}{1}$
$-M$	y_{a_3}	30	$\boxed{3}$	1	2	0	0	0	1	$\frac{30}{3}$ →
		$z_j - c_j$	$-6M-1\uparrow$	$-2M-4$	-3	M	0	0	0	



		c_j	1	4	3	0	$-M$	$-M$	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_{a_1}	y_{a_2}	MR
$-M$	y_{a_1}	20	0	$-\frac{5}{3}$	$-\frac{1}{3}$	0	1	0	X
$-M$	y_{a_2}	12	0	$\boxed{\frac{5}{3}}$	$-\frac{11}{3}$	-1	0	1	$\frac{36}{5}$ →
1	y_1	10	1	$\frac{1}{3}$	$\frac{2}{3}$	0	0	0	30
		$z_j - c_j$	0	$-\frac{11}{3}\uparrow$	$4M-\frac{7}{3}$	M	0	0	

Example



(T 3)

		c_j	1	4	3	0	$-M$
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_{a_1}
$-M$	y_{a_1}	32	0	0	-4	-1	1
4	y_2	$\frac{32}{5}$	0	1	$-\frac{11}{5}$	$-\frac{3}{5}$	0
1	y_1	$\frac{38}{5}$	1	0	$\frac{21}{15}$	$\frac{1}{5}$	0
		$z_j - c_j$	0	0	$4M - \frac{156}{5}$	$M - \frac{11}{5}$	0

In (T 3) the coefficient of M in each $z_j - c_j$ is non-negative and an artificial vector appears in the basis, not at the zero level but at a positive level. This shows that the given LPP does not possess any feasible solution.