Lec-12

Fibonacci Method

This is a very useful elimination technique. To eliminate some region of the interval of uncertainty we need to assume the function as unimodal. This method has use of Fibonacci numbers. Therefore, we try to define Fibonacci sequence first and then discuss about the Fibonacci Method.

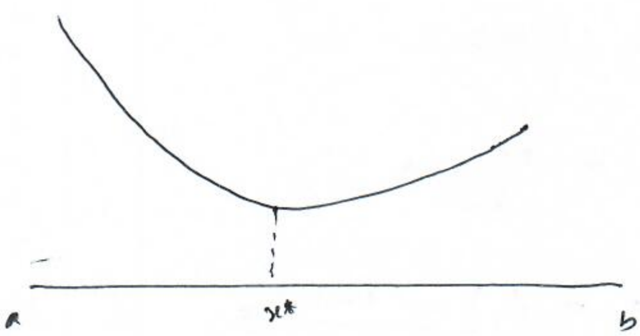
Def^n:- $\boxed{\text{Fibonacci sequence}}$:-

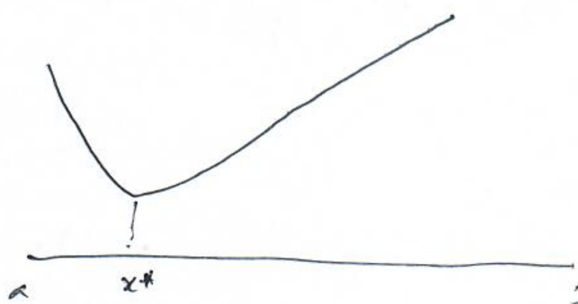
Let $F_0 = 1$, $F_1 = 1$ and $F_i = F_{i-1} + F_{i-2}$ $(i \geq 2)$.

Then $\{F_n\}$ is called the sequence of Fibonacci numbers or in short, the Fibonacci sequence.

Thus the Fibonacci sequence is $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$.

Min $f(x)$
s.t. $x \in [a, b]$. (Region Elemination Technique)





ii) Step-1 :- Given the initial interval of uncertainty $L_0 = [a, b]$. We consider the number of experiments be n.

Step-2 :- To determine initial points x_1 & x_2 we define

$$L_2^* = \frac{F_{n-2}}{F_n} L_0 .$$

and select x_1, x_2 such a way that ~~p~~ both the points are L_2^* apart from the ~~en~~ both ends.

$$\boxed{\begin{aligned} x_1 &= a + L_2^* = a + \frac{F_{n-2}}{F_n} L_0 \\ x_2 &= b - L_2^* = b - \frac{F_{n-2}}{F_n} L_0 \end{aligned}}$$

Now, see x_2^* can be rewritten as

$$\begin{aligned} x_2 = b - \frac{F_{n-2}}{F_n} L_0 &= b - \frac{F_{n-2}}{F_n}(b-a) \\ &= \left(1 - \frac{F_{n-2}}{F_n}\right) b + \frac{F_{n-2}}{F_n} a \\ &= \left(\frac{F_n - F_{n-2}}{F_n}\right) b + \left(\frac{F_n - F_{n-1}}{F_n}\right) a \qquad \left[\because F_n = F_{n-1} + F_{n-2}\right] \\ &= \frac{F_{n-1}}{F_n} b + \left(\frac{F_n - F_{n-1}}{F_n}\right) a \end{aligned}$$

$$= \frac{F_{n-1}}{F_n}(b-a) + a$$

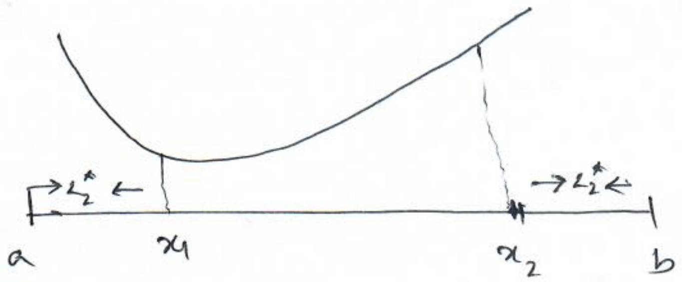
$$= a + \frac{F_{n-1}}{F_n} L_0 .$$

$$\therefore \boxed{x_1 = a + \frac{F_{n-2}}{F_n} L_0}$$

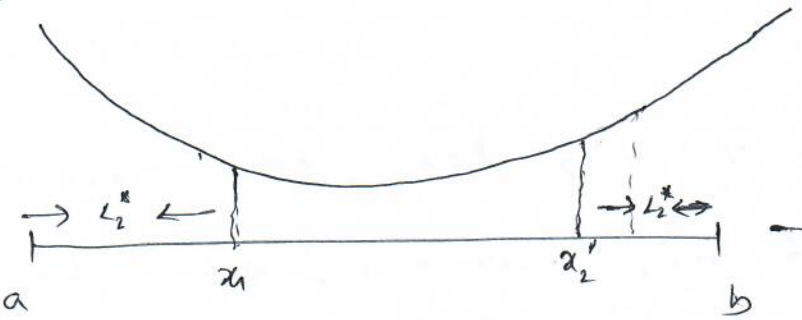
$$\boxed{x_2 = a + \frac{F_{n-1}}{F_n} L_0}$$

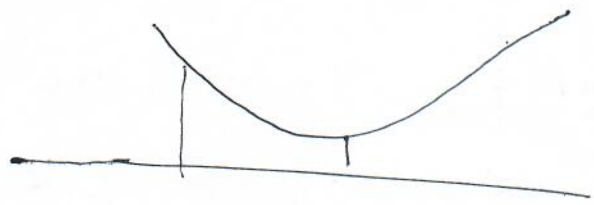
Case-I

Step-3 :-



Case-II





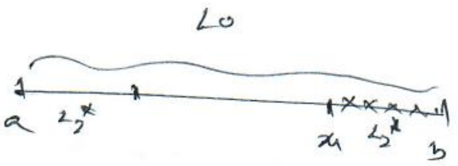
• Find values of $f(x_1)$ & $f(x_2)$

and do the region elemination by unimodality of the function according to $f(x_1) > f(x_2)$ or $f(x_1) < f(x_2)$.

Therefore, next interval of uncertainity would be

$L_2' = [a, x_2]$ or $[x_1, b]$

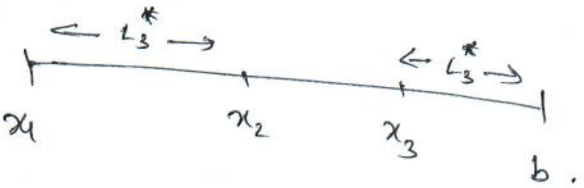
Length of $L_2 = L_0 - L_2^*$



$$= L_0 - \frac{F_{n-2}}{F_n} L_0$$

$$= \frac{F_n - F_{n-2}}{F_n} L_0 = \frac{F_{n-1}}{F_n} L_0$$

Step-4 : To evaluate x_3



$$L_3^* = \frac{F_{n-3}}{F_n} L_0$$

Locate x_3 in such a way that current two experiments are L_3^* distance apart from both the ends of $L_2 = [x_1, b]$

Step-5 : Evaluate the value of $f(x_2)$, $f(x_3)$, and using unimodal property of $f(x)$, eliminate the portion of interval of uncertainty, obtain the new interval L_3.

Where, $L_3 = L_2 - L_3^* = L_2 - \frac{F_{n-3}}{F_n} L_0 = L_2 - \frac{F_{n-3}}{F_{n-1}} L_2$

$$= \frac{F_{n-3} - F_{n-1}}{F_{n-1}} L_2$$

$$\boxed{L_3 = \frac{F_{n-2}}{F_{n-1}} L_2}$$

In general, to obtain the K^{th} experiment

$$L_K^* = \frac{F_{n-K}}{F_{n-(K-2)}} L_{K-1}$$

Length of interval of uncertainity after K^{th} experiment

$$L_K = \frac{F_{n-(K-1)}}{F_n} L_0 .$$

Step-5 :- We repeat the process till the desire number of experiments. Now, the final shortest interval would be the optimal interval and the middle point of that interval would be the optimal point and function value at that point would be the optimal value of the problem.

Example 1: Find the minimum value of $f(x) = x^2 + 2x$ within the interval $[-3, 4]$ using Fibonacci Method obtain the optimal value within 5% of exact value.

$\Rightarrow$

$$\frac{\text{Length of final interval of uncertainty}}{2 \times \text{Length of initial interval of uncertainty}} \le \frac{5}{100}$$

$$\Rightarrow \quad \frac{L_n}{2} \le \frac{1}{20} L_0 .$$

$$\Rightarrow \quad L_n \le \frac{L_0}{10}.$$

$$\Rightarrow \quad \frac{L_n}{L_0} = \frac{1}{F_n} \le \frac{1}{10}$$

$$\Rightarrow \quad F_n \geq 10.$$

$$\Rightarrow \quad F_6 \geq 10.$$

Measure of efficiency $= \frac{L_n}{L_0} = \frac{1}{F_n}$

$F_0 = 1$
$F_1 = 1$
$F_2 = 2$
$F_3 = 3$
$F_4 = 5$
$F_5 = 8$
$F_6 = 13$

$\therefore \; n \equiv 6,$ (minimum number of experiment)

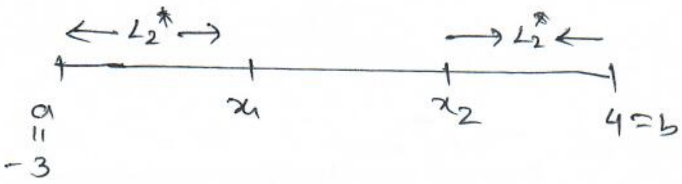
Step-I: $L_0 = [-3 ; 4]$, $n = 6$

Step-2: To obtain x_1, x_2

Let $L_2^* = \frac{F_{n-2}}{F_n} L_0 = \frac{F_4}{F_6} L_0 = \frac{5}{13} \cdot 7 = 2.6923.$

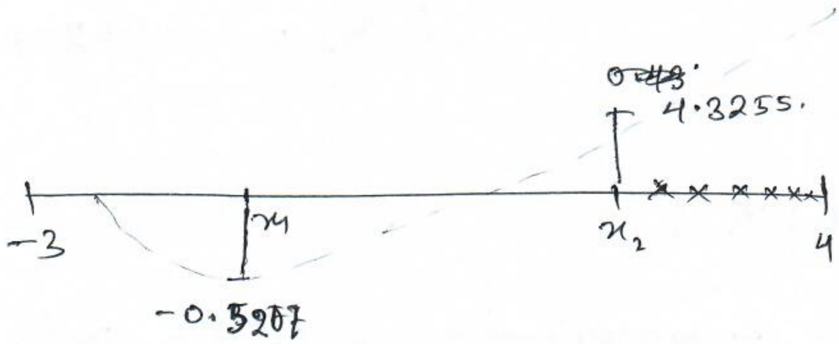
$$x_1 = -3 + 2.6923$$
$$= -0.3077$$

$$x_2 = 4 - 2.6923$$
$$= 1.3077.$$



step-3: Calculate $f(x_1)$ & $f(x_2)$ and check $f(x_1) < f(x_2)$

or $f(x_1) > f(x_2)$



$$f(x_1) = -0.5207, \quad f(x_2) = 4.3255.$$

$\therefore f(x_1) < f(x_2)$

Discard $[x_2, 4]$. Obtain $L_2 = [-3, x_2]$

$= [-3, 1.3077]$.

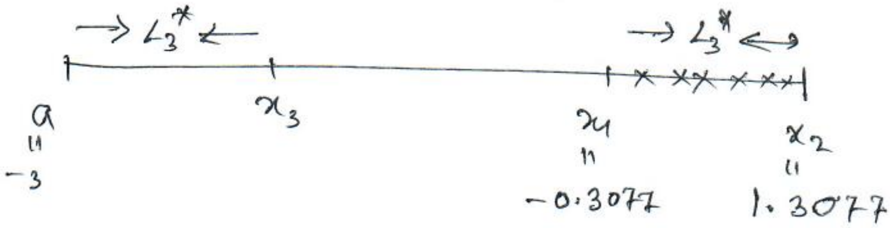
Length of $L_2 = 1.3077 + 3 = 4.3077.$

$L_2 = L_0 - L_2^* = 7 - 2.6923 = 4.3077.$

$$L_2 = \frac{F_{n-1}}{F_n} L_0 = \frac{F_5}{F_6} \times 7 = \frac{8}{13} \times 7 = 4.3077.$$

step-4 : To obtain x_3

$$L_3^* = \frac{F_{n-3}}{F_n} L_0 = \frac{F_3}{F_6} \times 7 = \frac{3}{13} \times 7 = 1.6154.$$



$x_3 = -3 + 1.6154 = -1.3846$, $\quad f(x_3) = -0.8521$

$f(x_4) = -0.5207.$

$f(x_3) < f(x_4)$

$\Rightarrow$ discard $[x_4, x_2]$

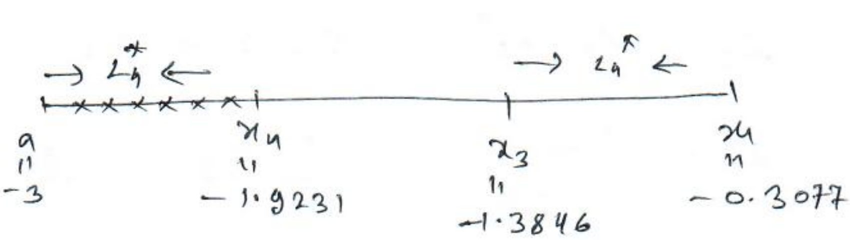
$\therefore$ New interval of uncertainty $L_3 = [a, x_4]$

$= [-3, -0.3077]$

step-5 : $L_3 = L_2 - L_3^*$

$= 4.3077 - 1.6154 = 2.6923.$

step-6 : To obtain x_4.



$$L_4^* = \frac{F_{n-4}}{F_n} L_0 = \frac{F_2}{F_6} \times 7 = 1.0769.$$

$x_4 = -3 + L_4^* = -1.9231$

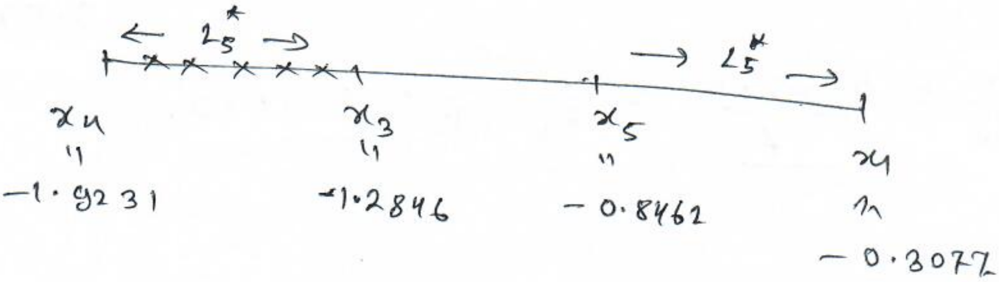
$f(x_4) = -0.1479.$, $f(x_3) = -0.8521$

$f(x_3) < f(x_4) \Rightarrow$ discard $[-3, x_4)$.

$\therefore$ New interval of uncertainty is $[x_4, x_1]$

$= [-1.9231, -0.3077]$

step-7: $L_5^* = \frac{F_{n-5}}{F_n} L_0 = \frac{1}{13} \times 7 = 0.5385$.



$x_5 = -0.3077 - L_5^* = -0.8462$

$f(x_5) = -0.97634556$.

$f(x_3) = -0.8521$

$f(x_5) < f(x_3) \Rightarrow$ Discard $[x_4, x_3)$

$\therefore$ New interval of uncertainty $= [x_3, x_1]$

$= [-1.2846, -0.3077]$.

step-8: $L_5 = [-1.3846, -0.3077]$

$L_6^* = \frac{F_{n-6}}{F_n} L_0 = 0.5385$.

$x_6 = x_3 + L_6^* = -0.8461$

$f(x_6) = -0.97631479$

$L_6 = [-1.3846, -0.8461] = [x_3, x_6]$

$\therefore$ Optimal point ~~solution~~ $= \dfrac{x_3 + x_6}{2} = x^*$

$\therefore$ optimal solution (min $f(x)$) $= f(x^*)$

Note :- $\dfrac{L_n}{L_0}$ = Reduction Ratio $= \dfrac{1}{F_6} = \dfrac{1}{13} = \dfrac{0.5385}{7}$

$= 0.0769.$

$$\frac{L_n}{L_0} = \frac{0.5385}{7} = \frac{1}{F_6} = \frac{1}{13} = \underline{0.0769}.$$

Exercise 1 ! Maximize $f(x) = \begin{cases} x/2 , & \text{for } x \leq 2 \\ -x+3 , & \text{for } x > 2. \end{cases}$

in the interval $(0, 3)$. Given $N = 6$ (number of experiment)