Lecture- 8

Numerical Optimization :- Unconstrained optimization problem :

Ref :- Numerical Optimization with Applications by S. Chandra, Jayadeva, & A. Mehra.

In the previous lectures we have seen the techniques to solve constrained optimization problems (like equality constrained or inequality or mixed).

Now, we study some standard algorithm for unconstrained minimization of a functions of one variables as well as functions of several variables.

The problem formulation :-

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{———} \quad (*)$$

Def^n :- Solution set :- Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuously differentiable function. Then the set

$$\Omega = \{ x \in \mathbb{R}^n : \nabla f(x) = 0 \}$$ is called the solution set of $(*)$.

Remark :- Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable convex function and $\bar{x}$ be a point of the solution set. Then $\bar{x}$ is called the global minima of problem $(*)$.

☒ Before we present any specific algorithm for solving the problem ⊛, we list certain desirable properties which we ideally expect the given algorithm to process.

<u>Def</u>

Defn: (<u>Descent Property</u>)

An algorithm for solving the unconstrained minimization problem ⊛ is said to have the decent property if the objective function value decreases as we go through the sequence $\{x^{(k)}\}$, i.e.

$$f(x^{(k+1)}) < f(x^{(k)}) \ \forall k.$$

☒ <u>Globally Convergent</u> :- An algorithm for the unconstrained minimization problem ⊛ is said to be globally convergent if ~~so~~ starting from any point $x^{(0)} \in \mathbb{R}^n$, the sequence $\{x^{(k)}\}$ always converges to a point of the solution set Ω.

Defn :- (Order of Convergence) :-

Let the sequence $\{x^{(K)}\}$ converge to a point x^* and let $x^{(K)} \neq x^*$ for sufficiently large K. The quantity $\| x^{(K)} - x^* \|$ is called the error of the K^{th} iterate $x^{(K)}$. Suppose that there exists p and $0 < \alpha < \infty$ such that

$$\lim_{K \to \infty} \frac{\| x^{(K+1)} - x^* \|}{\| x^{(K)} - x^* \|^p} = \alpha \ , \ (0 < \alpha < \infty)$$

then p is called the order of convergence of the sequence $\{x^{(K)}\}$. Thus

$$\| x^{(K+1)} - x^* \| = \alpha \| x^{(K)} - x^* \|^p \text{ asymptotically.}$$

Remark :- (i) If $p = 1$ the seqn $\{x^{(K)}\}$ is said to have linear convergence rate

(ii) If $p = 2$, it is called quadratic convergence rate.

Numerical Optimization Technique.

(NLP, with unconstrained)

Single variable optimization

- Numerical Methods
  - Region Elimination Technique
    - Unrestricted Search
    - Exhaustive Search
    - Dichotomous Search
    - Interval halving technique (Bisection Method)
    - Fibonacci Method
    - Golden Section Method
  - Interpolation Method
    - Quadratic Interpolation Method.
    - Derivative based Method.
      - Newton Method
      - Quasi-Newton Method
      - Secant Method.
- Analytical Methods

To use Region Elemination Technique the function should be unimodal in the specified domain of definition.

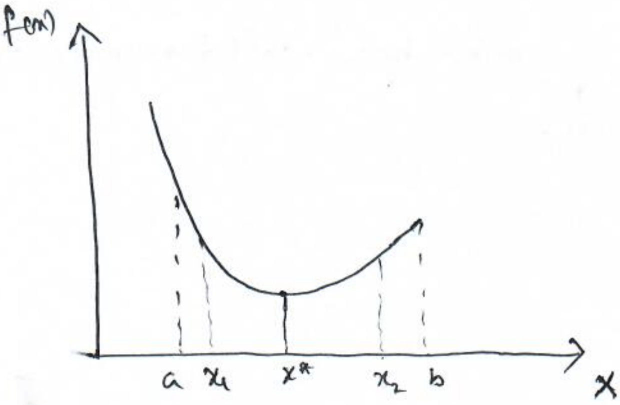
Def^n : $\boxed{\text{Unimodal function}}$:-

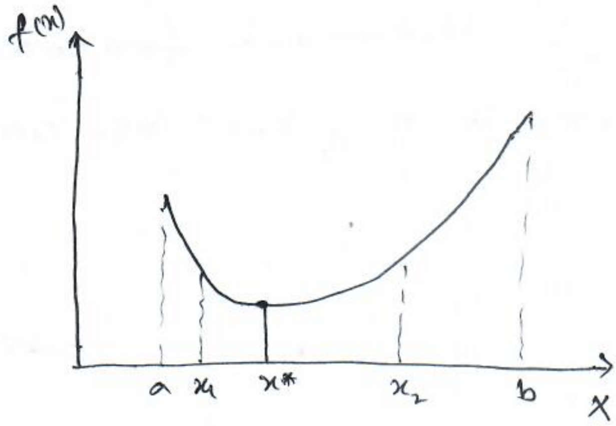
A function $f(x)$ is said to be unimodal in $a \le x \le b$ if it is monoton on either side, x^* is optimal point within the interval

(i) if $f(x_1) > f(x_2) \Rightarrow x^* \in [x_1, b)$ but $x^* \notin (a, x_1)$

(ii) if $f(x_1) < f(x_2) \Rightarrow x^* \in [a, x_2)$ but $x^* \in (x_2, b]$.

(iii) if $f(x_1) = f(x_2) \Rightarrow x^* \in [x_1, x_2]$.





Remark :- f is called strictly unimodal if the function is unimodal and there does not exist any sub interval within the range $[a, b]$, where the function is having constant value.

Remark :- Note that we do not need explicit knowledge of the function f as (continuity or differentiability) as long as it is possible to get its value at a specified point.

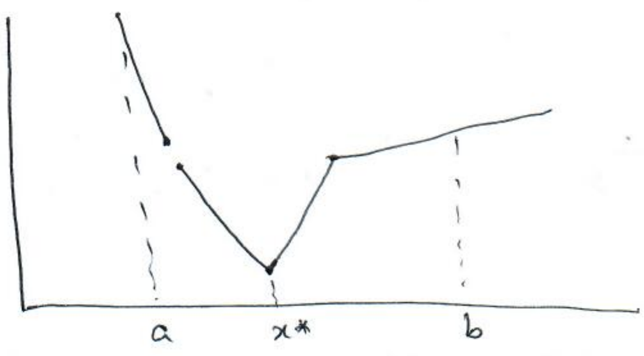
Alternate deffination of Unimodal function :-

The function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be a unimodal (to be specific unimodal min) function if it has only one mode i.e. it has a single relative min, i.e. $\exists\ a \le x^* \le b$ such that

(i) f is strictly decreasing ($\downarrow$) in $[a, x^*)$

(ii) f is strictly increasing ($\uparrow$) in $[x^*, b]$

<u>Note</u> :- Unimodal function may not be differentiable, even it may not be continuous.



Region Elimination Technique.

$$\text{Min } f(x), \quad a \leq x \leq b.$$

Assumption :- f is unimodal function and x^* is min point $a \leq x^* \leq b$.

step-I :- Starting with the initial guess point,

$$a < x_1 < x_2 < b.$$

(i) if $f(x_1) > f(x_2)$ then minimum must lie within (x_1, b)

(ii) if $f(x_1) < f(x_2)$ then minimum must lie within (a, x_2).

(iii) if $f(x_1) = f(x_2)$ then minimum must lie (x_1, x_2). i.e. eleminate the intervals (a, x_1) and (x_2, b).