Interval Halving Method (Bisection Method)

In the previous lecture we have studied about the Dichotomous searching technique. Now, in this lecture we see some example for that technique. After that we will discuss what is called Bisection method or interval halving technique for unconstrained optimization problem,

Example 1: Find the minimum value of $f(x) = 4x^3 + x^2 - 7x + 14$ in the interval $[0, 1]$ within 10% of exact value.

$$\Rightarrow \quad \frac{L_n}{2} \le \frac{L_0}{10} \qquad\qquad L_0 = b - a = 1$$

$$L_n = \frac{L_0}{2^{n/2}} + \delta\left(1 - \frac{1}{2^{n/2}}\right)$$

$$\frac{1}{2^{n/2}} + \delta\left(1 - 1/2^{n/2}\right) \le 1/5 .$$

Take $\delta = 0.001$

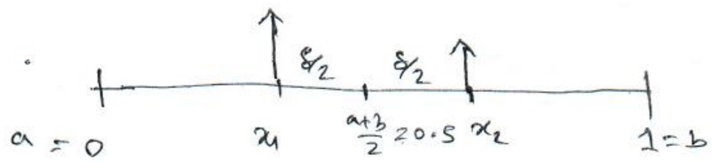
$$\Rightarrow \frac{1}{2^{n/2}}\left(1 - 0.001\right) \le \frac{1}{5} - 0.001$$

$$\Rightarrow \quad \frac{0.999}{2^{n/2}} \le \frac{0.995}{5}$$

$$\Rightarrow \quad 2^{n/2} \geq \frac{0.999}{0.995} \times 5 = 5.02,$$

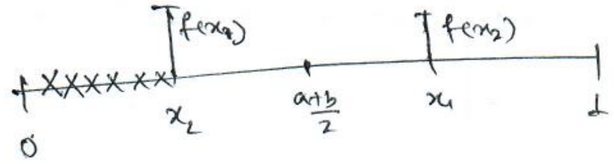
$\because$ n is even, $n \simeq 6$. (minimum value of n)

Step-I :

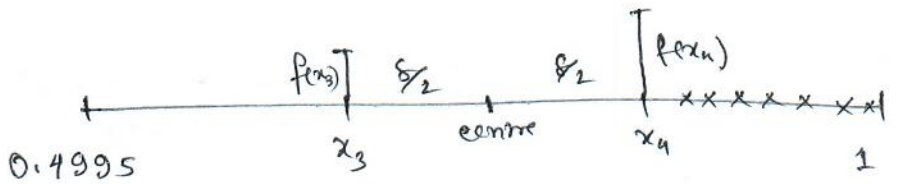


$x_1 = 0.4995$ $\quad f(x_1) = 11.2515$

$x_2 = 0.5005$ $\quad f(x_2) = 11.2485.$



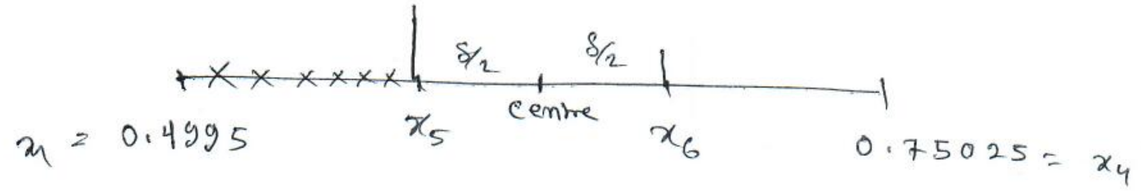
Step-2 :- New interval of uncertainity $[0.4995, 1]$ $\equiv [x_1, b]$



$x_3 = 0.74925$ $\quad f(x_3) = 10.999$

$x_4 = 0.75025$ $\quad f(x_4) = 11.0003.$

Step-3 :- New interval of uncertainity $[0.4995, 0.75025]$



$x_5 = 0.624375$ $\quad f(x_5) = 10.9928$

$x_6 = 0.625375$ $\quad f(x_6) = 10.9918$

Step-4 :- New interval of uncertainity (final one)

$= [x_5, x_4] = [0.624375, 0.75025]$

Middle value of $[x_5, x_4]$ i.e. $x^* = \frac{x_5 + x_4}{2}$

$= 0.6873.$

$\therefore$ The minimum (optimal) value is $x^* = 0.6873$.

and $f(x^*) = 10.99$.

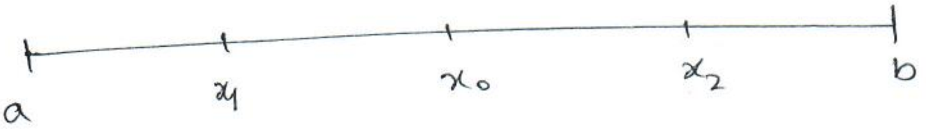
Exercise 1: Find the Maximum ~~minimum~~ value of

$f(x) = -x^2 - 2x$ over the interval $[-3, 6]$

within 20% of exact value and $\delta = 0.01$.

Interval Halving Technique
Bisection Method

Step-I : Divide the initial given interval of uncertainty into four parts (equal length)

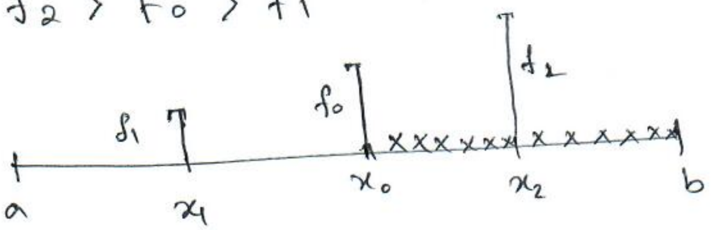


$a < x_1 < x_0 < x_2 < b.$

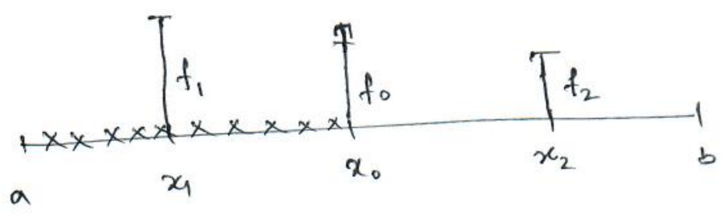
Step-2 : Compute, $f_0 \equiv f(x_0)$, $f_1 \equiv f(x_1)$, $f_2 = (x_2)$.

case-I :

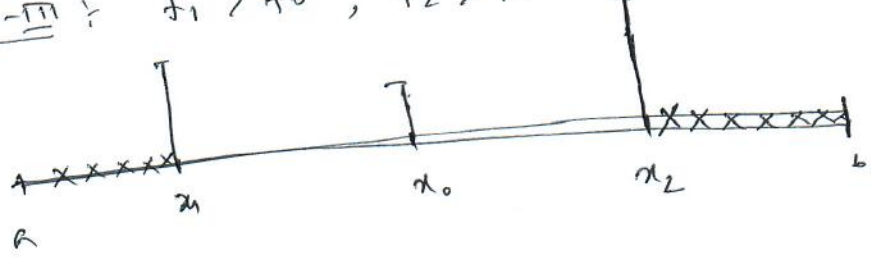
$f_2 > f_0 > f_1$



case-II : $f_2 < f_0 < f_1$



case-III : $f_1 > f_0$, $f_2 > f_0$



Note :- Notice that in each iteration we will discard half of the interval of uncertainity.

Step-3 :- Repeat the process as long as the new interval of uncertainty is very small.

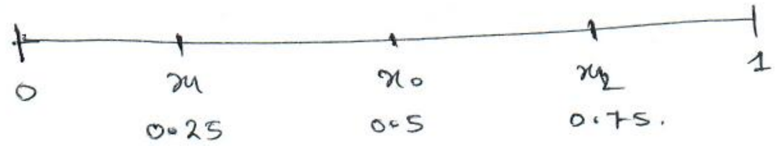
Note :- At each iteration we consider three point. At a particular iteration we discard two points and introduce two points (new) and another point (old) was there. Therefore, $n \geq 3$ and odd numbers.

$$L_n = \left(\frac{1}{2}\right)^{\frac{n-1}{2}} L_0 .$$

Example: Minimize $f(x) = x(x-1.5)$, $x \in [0,1]$. within 10% within exact value

$$\Rightarrow \frac{L_n}{2} \le \frac{L_0}{10} \Rightarrow \frac{1}{2^{\frac{n-1}{2}}} \le \frac{1}{5} \Rightarrow n \geq 7.$$

Step-I:

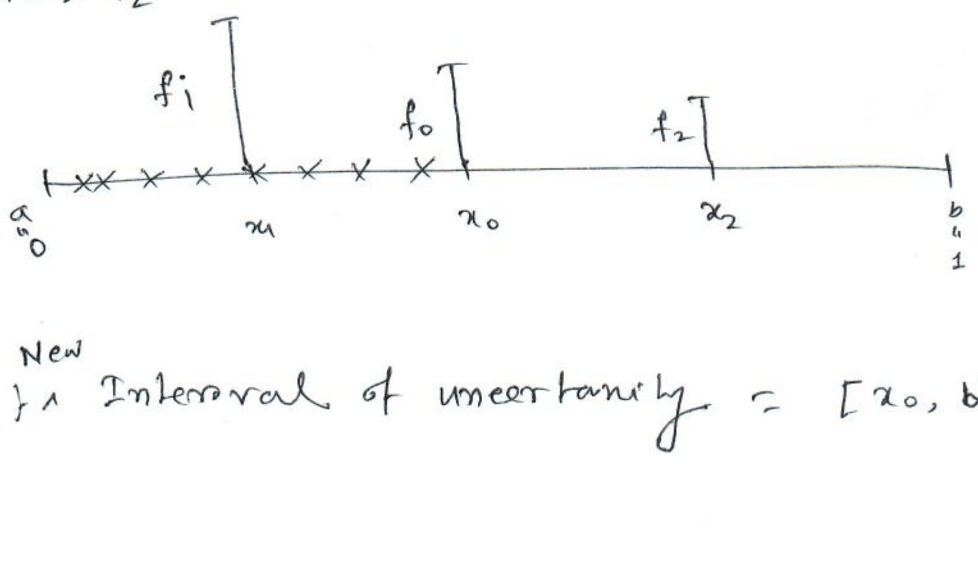


$x_1 = 0.25$ $\quad f(x_1) = -0.3125$

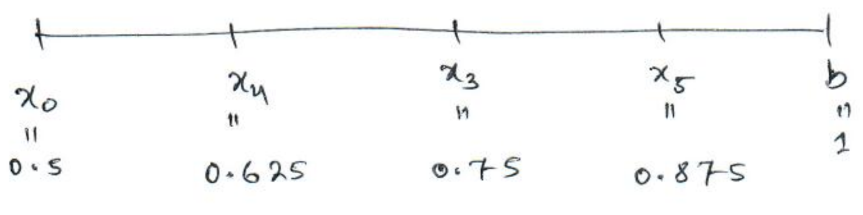
$x_0 = 0.5$ $\quad f(x_0) = -0.5$

$x_2 = 0.75$ $\quad f(x_2) = -0.5625$

$f_1 > f_0 > f_2$



Step-2: New Interval of uncertainty $= [x_0, b] \equiv [0.5, 1]$



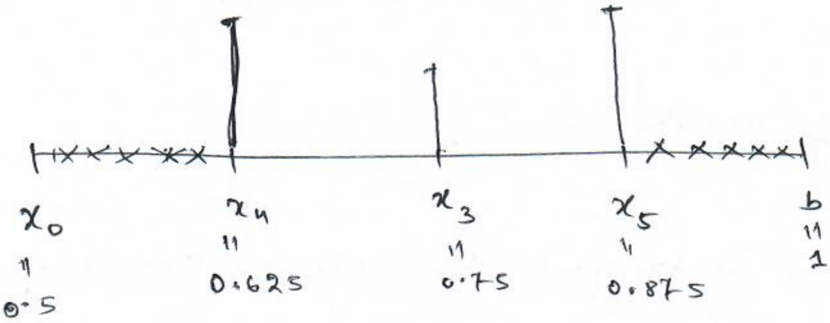
$x_4 = 0.625$ $\quad f(x_4) = -0.5468$

$x_3 = 0.75$ $\quad f(x_3) = -0.5625$

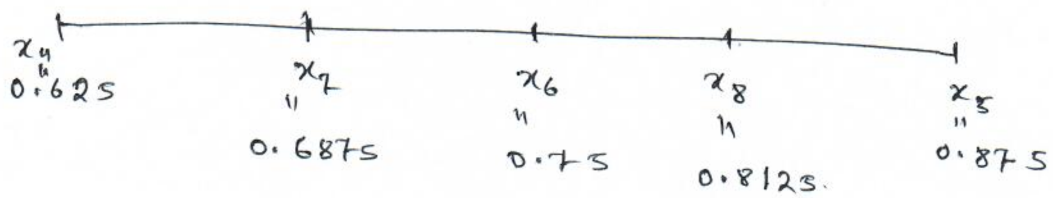
$x_5 = 0.875$ $\quad f(x_5) = -0.5468$

Observe that

$$f_4 > f_3 \ \& \ f_5 > f_3$$



<u>step-3</u>: New level of uncertainty $= [x_4, x_5] \equiv [0.625, 0.875]$

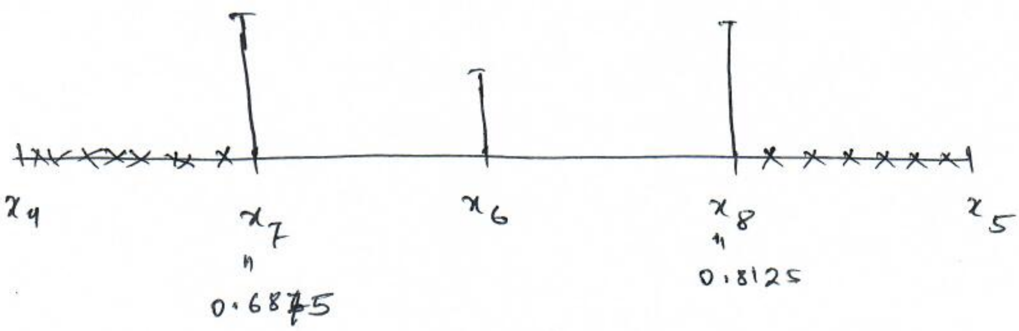


$x_7 = 0.6875 \qquad f(x_7) = -0.5585$

$x_6 = 0.75 \qquad f(x_6) = -0.5625$

$x_8 = 0.8125 \qquad f(x_8) = -0.5585$

$f_7 > f_6$ &
$x_8 > f_6$.



$\therefore$ Final interval of uncertainty $= [0.6875, 0.8125]$

$= [x_7, x_8]$

Optimal value $x^* = \frac{x_7 + x_8}{2} = 0.75$. $f(x^*) = -0.5625$.

Exercise 2:- Find the minimum of the following functions

(i) $f(x) = x^3 - 6x^2 + 4x + 12 \qquad x \in [-2, 6]$.

Using the following methods

(i) Exhaustive search technique to achive accuracy within 5% of the exact value.

(ii) Dichotomous Search

(iii) Interval halving.