Lec- 6

Quadratic Programming Problems.

Ref: Numerical optimization with Applications by S. Chandra, Jayadeva, and A. Mehra.

In the earlier ~~class~~ lectures we studied convex optimization problems, the Lagrange optimization technique when we solve the optimization problems with equality type constraints. In this lecture we will study "Quadratic Optimization Problems".

Quadratic Programming problems :-

A quadratic programming problem (QPP) is the special class of non-linear optimization problems in which the objective function is quadratic and all the constraints are linear.

The general mathematical formulation of a QPP is as follows:

$$\text{Min } f(x) = x^T Q x + c^T x$$
$$\text{s.t.} \quad Ax \le b,$$
$$x \geq 0.$$

where • $Q = [q_{ij}]_{n\times n}$ symmetric positive definite matrix,

- $c \in \mathbb{R}^n$
- $x \in \mathbb{R}^n$
- $b \in \mathbb{R}^m$
- $A := [a_{ij}]_{m\times n}$.

Example 1 :- Min $f(x) = 3x_1^2 + 4x_2^2 + 2x_1x_2 - 2x_1 - 3x_2$,

s.t. $3x_1 + 2x_2 \le 6$

$x_1 + x_2 \le 2$

$x_1, x_2 \geq 0$.

OR $\min f(x) = \overbrace{(x_1\ x_2)}^{x^T} \overbrace{\begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}}^{Q} \overbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{x} + \overbrace{(-2, -3)}^{e^T} \overbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{x}$

s.t. $\overbrace{\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{x} \le \overbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}^{b}$.

$\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x \geq 0.} \geq 0$

$Q = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$

$D_1 = 3 > 0$, $D_2 = 12 - 1 = 11 > 0$

$\therefore$ The matrix Q is positive definite.

Hence, it is positive semi-definite also.

Therefore, the above example is a Quadratic programming problem,

Solution of QPP by graphical Method :-

Example 2 :- Min $(x_1 - 2)^2 + (x_2 - 1)^2$

s.t $x_1 + x_2 \le 2$

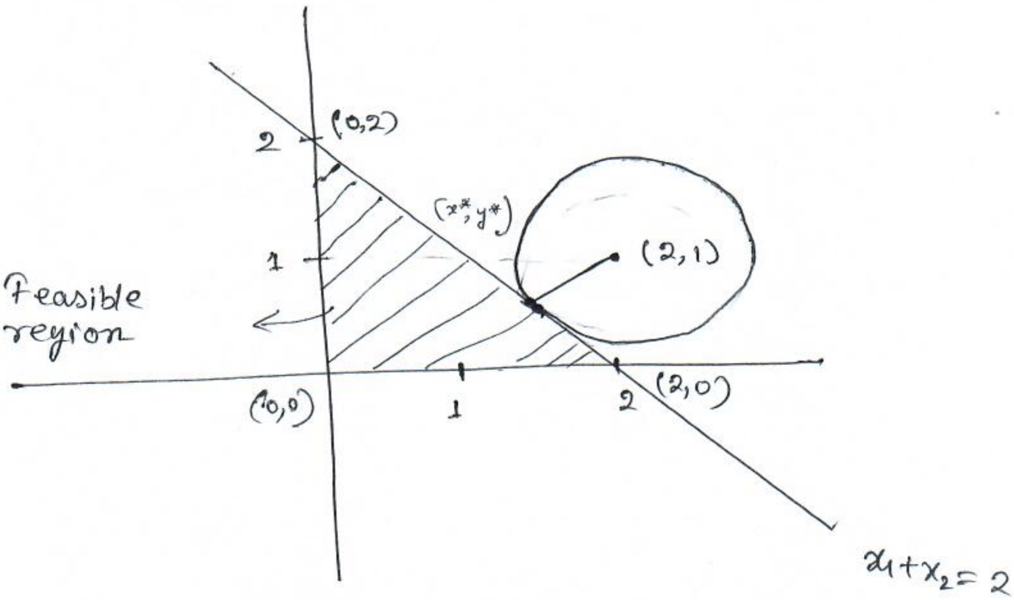
$x_1, x_2 \geq 0$

Some observations are :-

(i) Feasible region of (QPP) is always a convex set.
(since all the constraints are linear)

(ii) Unlike (LPP), the optimal solution of QPP / NLPP may ~~be~~ not be attain at the vertex of the feasible region.

Solution of Example 2 :-



$$x^* + y^* = 2 \quad \text{—— (i)}$$

Slope $\dfrac{y^*-1}{x^*-2} = 1$

$$\Rightarrow x^* - y^* = 1 \quad \text{—— (ii)}$$

Solve equations (i) & (ii) , $x^* = 3/2$, $y^* = 1/2$

Exercise :- check if the following problem is a QPP or NOT?

$$\text{Min } f = 3x_1^2 + 2x_2^2 + x_1x_2 - 4x_1 - 2x_2$$

s.t.

$$x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0.$$

(3)

KKT Conditions for QPP :-

The problem (QPP) can be re-written as

$$\left.\begin{aligned} &\text{Min } f(x) = x^T Q x + c^T x \\ &\text{s.t. } \quad Ax \le b, \quad \text{—— (i)} \\ &\qquad\quad -x \le 0 . \quad \text{—— (ii)} \end{aligned}\right\} \text{—— (1)}$$

Let the KKT multiplier associated with constraints (i) and (ii) be $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, respectively. Then the KKT conditions for the problem (1) are ~~as~~ as follows :

$$c^T + 2x^T Q + u^T A - v^T = 0$$

$$u^T (Ax - b) - v^T x = 0,$$

$$Ax - b \le 0,$$

$$x \geq 0, \; u \geq 0, \; v \geq 0 .$$

From (i)
$Ax - b \le 0$. we need $u \in \mathbb{R}^m$, i.e. m number of variables.
From (ii) $-x \le 0$ we need $v \in \mathbb{R}^n$, i.e., n number of variables.

Therefore, the total number of KKT multipliers are $(m+n)$.

(i) $\boxed{\text{Optimality Condition}}$:-

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$$

For ~~eq~~ Prob. (1)

$$2x^T Q + c^T + u^T A + v^T(-I) = 0 \quad \text{—— (a)}$$

(ii) $\boxed{\text{Feasibility cond}^{n}}$:-

$g_i(x) \leq 0$

For problem (1) $\quad Ax - b \leq 0.$ ——— (b)

(iii) $\boxed{\text{Complementary slackness cond}^{n}}$:-

$\lambda_i^* g_i(x^*) = 0$

For prob (1) $\quad u^T(Ax-b) - v^T x = 0$ ——— (c)

(iv) $\boxed{\text{Non-Negative cond}^{n}}$:-

$\lambda_i^* \geq 0$

for problem (1) $\quad x \geq 0, \ u \geq 0, \ v \geq 0.$ ——— (d)

Rewrite the above conditions :-

For (a) $\quad 2x^T Q = 2Q^T x = 2Qx \quad (\because Q \text{ is symmetric})$

$\therefore$ taking transpose of eq^n. (a), (b), (c), (d)

$2Qx + c + A^T u - vI = 0$

$u^T(Ax-b) - v^T x = 0$

$Ax - b + s = 0 \quad$ (slack variable added)

$x \geq 0, u \geq 0, v \geq 0 \quad$ (all variables are non-negative)

(i) $2Qx + c + A^T u - vI = 0$

(ii) $u^T(-s) - v^T x = 0$ $\quad [\because \; Ax - b = -s]$

$\equiv u^T s + v^T x = 0.$

Since $u \geq 0$, $s \geq 0$ and $v \geq 0$ $x \geq 0$.

therefore, $\left.\begin{array}{l} u^T s = 0 \\ v^T x = 0 \end{array}\right\} \equiv \begin{array}{l} u_1 s_1 + u_2 s_2 + \cdots + u_m s_m = 0 \\ v_1 x_1 + v_2 x_2 + \cdots + v_n x_n = 0 \end{array}$

$\therefore \quad u_i s_i = 0 \quad \forall\, i = 1, 2, \ldots, m$

$v_j x_j = 0 \quad \forall\, j = 1, 2, \ldots, n.$

Therefore, the reduced KKT conditions becomes

$$\left.\begin{array}{ll} \text{(i)} & 2Qx + c + A^T u - vI = 0 \\ \text{(ii)} & Ax - b + s = 0 \\ \text{(iii)} & u_i s_i = 0 \quad \forall\, i = 1, 2, \ldots, m \\ \text{(iv)} & v_j x_j = 0 \quad \forall\, i = 1, 2, \ldots, n. \\ & \forall\; x \geq 0,\; u \geq 0,\; v \geq 0,\; s \geq 0 \end{array}\right\} \text{———} \; (2)$$

The matrix form of the KKT conditions are

$$\begin{bmatrix} 2Q & A^T & -I_n & 0 \\ A & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \\ v \\ s \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

$u_i s_i = 0 = v_j x_j \quad \forall\, i, j, \quad u \geq 0,\; x \geq 0,\; v \geq 0.$

Thm: Let Q be a symmetric positive semi-definite matrix of order n. Then for any $x, y \in \mathbb{R}^n$

$$2x^TQy \leq (x^TQx + y^TQy).$$

Proof: $z^TQz \geq 0 \quad \forall z \in \mathbb{R}^n$ ($\because Q$ is +ve semi-definite) By defination.

$\Rightarrow (x-y)^T Q (x-y) \geq 0, \quad \forall x, y \in \mathbb{R}^n$ ($\because$ it's true for all $z \in \mathbb{R}^n$)

$\Rightarrow x^TQx - x^TQy - y^TQx + y^TQy \geq 0.$

$\Rightarrow x^TQy + y^TQx \leq x^TQx + y^TQy$

$\Rightarrow \boxed{2x^TQy \leq (x^TQx + y^TQy)}$

$x^T_{1\times n} Q_{n\times n} y_{n\times 1}$
$= [d]_{1\times 1} \Rightarrow$ scalar.

$x^TQy = (x^TQy)^T$
[$\because$ (scalar)^T = scalar]

$x^TQy = y^TQ^Tx$
$= y^TQx$ [$\because Q^T = Q$]