Lec- ~~3~~ 4

KKT Optimality conditions.

Ref :- Numerical optimization with applications by S. Chandra, Jayadeva, & A. Mehra.

In the previous lectures we have seen some convex programming problems. We also discussed ~~on~~ the Constraints optimization ~~with~~ ~~using~~ with equality Constraints using Lagrange's optimization technique. Now, in this lecture we will see KKT conditions. Remember when we have equality type constraints then we use Lagrange's Method to find out the optimal solution of that typeof problems, But, suppose we have a constraint optimization problem with inequality type constraints, then Karaush-Kuhn-Tucker proposed some condition to find optimal solutions of the problems. We called that condition as KKT-condition in short. Let's see the following ~~co~~ form of the KKT-Problem:

☒ Karaush-Kuhn-Tucker (KKT) Conditions :-

~~Cond~~ Consider the optimization problem as

$$\left.\begin{array}{l} \text{Min } f(x) \\ \text{s.t. } \quad g_i(x) \le 0, \quad i = 1, 2, \dots, m, \end{array}\right\} \text{———} (*)$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Also, assume that f and g are continuously differentiable functions.

Necessary Condition :-

Let x^* be a local min point of the problem $(*)$ at which basic constraint qualification holds.

Then there exist multipliers (called KKT-multi-pliers) λ_i^*, $i = 1, 2, \dots, m$ such that the following conditions hold :

(i) $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$ (optimality Condⁿ)

(ii) $g_i(x^*) \le 0$; $i = 1, 2, \dots, m$, (Feasibility Condⁿ)

(iii) $\lambda_i^* g_i(x^*) = 0$, $i = 1, 2, \dots, m$, (Complementary slackness)

(iv) $\lambda_i^* \geq 0 \quad \forall i$ (Non-negativity)

These (above four) conditions are called KKT-conditions.

Note :- Here, λ_i^*'s are called KKT-multipliers, x^* called KKT-point.

Sufficient Conditions :-

Let $(x^*, \lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ satisfy the KKT-conditions (i) to (iv). Let f and g_i for $i = 1, 2, \ldots, m$ be differentiable convex functions. Then x^* is a global min point of the problem ⊛.

Note that f and g_i $(\forall i)$ are diff. convex f^ns that means the optimization problem ⊛ becomes convex optimization problem.

Proof :-

⟹ Let us suppose $S = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, 2, \ldots, m\}$

Now, $f(x)$ and $g(x)$ is diff. convex f^n.

then $f(x) - f(x^*) \geq (x - x^*)^T \nabla f(x) \quad \forall x \in S$ — (1)

& $g_i(x) - g_i(x^*) \geq (x - x^*)^T \nabla g_i(x) \quad \forall x \in S.$

Now, ~~from~~ necessary condition (ii) $\lambda_i^* \geq 0$ (non-negative)

Therefor we can multiply λ_i^* in the above inequality and adding them we get

$$\sum_{i=1}^m \lambda_i^* g_i(x) - \sum_{i=1}^m \lambda_i^* g_i(x^*) \geq (x - x^*)^T \sum_{i=1}^m \lambda_i^* \nabla g_i(x) \quad \text{— (2)}$$

Adding equations (1) and (2), we get

$$f(x) - f(x^*) + \sum_{i=1}^m \lambda_i^* g_i(x) - \underbrace{\sum_{i=1}^m \lambda_i^* g_i(x^*)}_{\substack{0 \ \forall i \\ (\text{from (iii)})}} \geq (x - x^*)^T \underbrace{\left(\nabla f(x) + \sum_{i=1}^m \lambda_i^* g_i(x)\right)}_{\substack{= 0 \\ (\text{from (i)})}}$$

Now, $g_i(x) \leq 0 \quad \forall i \quad [\text{Cond}^n \text{ (iii)}]$

$\lambda_i^*(x) \geq 0 \quad \forall i \quad [\text{Cond}^n \text{ (iv)}]$

$\Rightarrow \lambda_i^* g_i(x) \leq 0$

$\Rightarrow \sum_{i=1}^{m} \lambda_i^* g_i(x) \leq 0 \quad \forall i, \ \forall x \in S.$

Therefore, $f(x) - f(x^*) \geq -\sum_{i=1}^{m} \lambda_i^* g_i(x) \geq 0$

$\Rightarrow f(x^*) \leq f(x) \quad \forall x \in S.$

$\Rightarrow x^*$ is the global minimum point.

Example 1 :- Min $f(x_1, x_2) = 2x_1 + x_2$

s.t. $x_1^2 + x_2^2 \leq 4$

$x_1 - x_2 \leq 0$

$\Rightarrow$ Min $f(\underline{x})$

s.t. $g_i(\underline{x}) \leq 0$

where $\underline{x} = (x_1, x_2)$

$f(x_1, x_2) = 2x_1 + x_2$

$g_1(x_1, x_2) = x_1^2 + x_2^2 - 4 \leq 0$

$g_2(x_1, x_2) = x_1 - x_2 \leq 0$

Step-I

Lets first check this problem is a CPP or NOT ?

If convex, then KKT - conditions will be sufficient.

Step-2: Calculate Hessian Matrix for the constraints (which are not linear)

for g_1, $\nabla^2 g_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\lambda_1^* = 2 > 0$
$\lambda_2^* = 2 > 0$.

$\therefore \nabla^2 g_1$ is positive-definite, hence convex.

Step-3: Try to write all the KKT-conditions

(i) $\nabla f(x^*) + \sum_{i=1}^{2} \lambda_i^* g_i(x^*) = 0$

$\Rightarrow (2 \quad 1) + \lambda_1^* (2x_1 \quad 2x_2) + \lambda_2^* (1 \; -1) = (0,0)$

$\Rightarrow 2 + 2x_1 \lambda_1^* + \lambda_2 = 0$

$1 + 2x_2 \lambda_1^* - \lambda_2^* = 0$

(ii) $g_i(x^*) \leq 0 \quad \forall i$, $g_1(x^*) = x_1^2 + x_2^2 - 4 \leq 0$
$g_2(x^*) = x_1 - x_2 \leq 0$

(iii) $\lambda_1^* g_1(x^*) = 0 \Rightarrow \lambda_1^* (x_1^2 + x_2^2 - 4) = 0$

$\lambda_2^* g_2(x^*) = 0 \Rightarrow \lambda_2^* (x_1 - x_2) = 0$

(iv) $\lambda_1^*, \lambda_2^* \geq 0$

Step-4: Try to find x^*

Case-I: $\lambda_1^* = \lambda_2^* = 0$

Case II: $\lambda_1^* = 0$, $x_1 = x_2$, $\lambda_2^* > 0$.

Case-III: $x_1^2 + x_2^2 - 4 = 0$, $\lambda_1^* > 0$, $\lambda_2^* = 0$.

Case IV: $x_1 - x_2 = 0$, $x_1^2 + x_2^2 - 4 = 0$, $\lambda_1^* > 0$ $\lambda_2^* > 0$.

Case - I :-

Case - II :-

Case - III :- $x_1 \lambda_1^* = -1$

$x_2 \lambda_2^* = -1/2$

$x_1^2 + x_2^2 = 4 \Rightarrow 16\lambda_1^2 = 5$

$\Rightarrow \lambda_1^2 = \pm \frac{\sqrt{5}}{4}$

$\therefore (x_1, x_2) = \left(-4/\sqrt{5}, -2/\sqrt{5}\right)$.

Check it satisfy all the KKT conditions or NOT?

If Yes, then (x_1, x_2) is the global min for the CPP.

~~4~~. Similar ~~to~~ to the previous optimization (min) problem, we can exted the logic the sufficient condition for the maximization problem as

if x^* is the local max for the corresponding maximization problem

$$\max f(x)$$
$$\text{s.t. } g_i(x) \leq 0.$$

Then that local maxima would be the global maxima if ~~we~~ the objective function ~~is~~ $f(x)$ is concave and the associated constraints, i.e. $g_i(x)$ (feasible space) is convex. In the other way we can say that KKT conditions are the sufficient condⁿ, if $f(x)$ is concave and feasible space is convex for the maximization problem.

Exercise :- Find the solution of the following optimization problem, $\underline{x} = (x_1, x_2)$
~~Example~~ :-

$$\min f(\underline{x}) = x_1^2 + x_2^2 - 2x_1$$
$$\text{s.t. } x_1^2 + x_2 - 1 = 0$$