Necessary and Sufficient Condition for general non-linear programming Problem. (NLP)

In the previous Lecture we see the KKT condition for constraints (inequality type) optimization Problem and we define necessary and sufficient condition for that problems. Now, when we ~~define~~ stated sufficient condition we assume the objective function as well as the constraints g_i are differentiable and convex.

However, without the convexity assumptions on f and g_i, the KKT conditions are not sufficient for a point x^* to be a local min/global min.

Example :-

$$\text{Min } -x_2$$
$$\text{s.t. } x_1^2+x_2^2 \leq 4$$
$$-x_1^2 + x_2 \leq 0$$

The point $(0,0)$ satisfy KKT conditions but it is NOT a local/global min point.

Step-I

$\Rightarrow$ First observe that $\text{Min } -x_2 \equiv \text{Max } x_2$

and $-x_2$ is concave function.

now, $g_1(x_1, x_2) = x_1^2 + x_2^2 - 4 \le 0 \Rightarrow g_1$ is convex function.

$g_2(x_1, x_2) = -x_1^2 + x_2 \le 0$

$\therefore \nabla^2 g_2(x_1, x_2) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$

Therefore eigen value $\lambda_1 = -2 < 0$

$\lambda_2 = 0$

$\therefore$ Hessian Matrix is negative-semi definite

$\therefore g_2$ is a concave function.

Therefore, this problem is **NOT** a convex programming problem.

Step-2: Write KKT-Conditions

$$\nabla f + \sum_{i=1}^{2} \lambda_i \nabla g_i = 0$$

$$\Rightarrow \frac{\partial f}{\partial x_j} + \sum_{i=1}^{2} \lambda_i \frac{\partial g_i}{\partial x_j} = 0 \qquad j = 1, 2$$

$$\Rightarrow (0, -1) + \lambda_1 (2x_1 \quad 2x_2) + \lambda_2 (-2x_1 \cdot 1) = (0, 0)$$

(i) $2x_1 \lambda_1 - 2x_1 \lambda_2 = 0$

$-1 + 2x_2 \lambda_1 + \lambda_2 = 0$

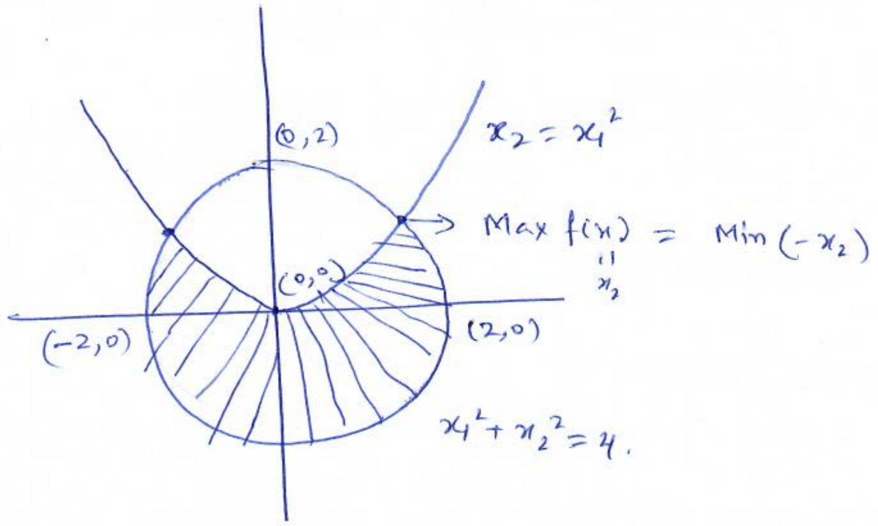
(ii) $\lambda_1 (x_1^2 + x_2^2 - 4) = 0$

$\lambda_2 (-x_1^2 + x_2) = 0$

(iii) $x_1^2 + x_2^2 - 4 \le 0$

$-x_1^2 + x_2 \le 0$

(iv) $\lambda_1, \lambda_2 \geq 0$



<u>Remark</u> :- If KKT conditions holds at a point x^*, it does not mean that the point is the point of global minima because for that point to be a global minima beside KKT conditions, convexity of $f(x)$ and $g_i(x)$ $\forall i$ is required.

Therefore, we will define the necessary condition for general programming problems.

<u>Necessary Conditions for general NLP</u> :-

Suppose,

$$\text{Min } f(x)$$
$$\text{s.t.} \quad g_j(x) \leq 0 \quad (j = 1, 2, \cdots, m)$$
$$h_k(x) = 0 \quad (k = 1, 2, \cdots, l)$$

$X = (x_1, x_2, \cdots, x_n)^T$

If x^* is a regular point, then x^* is also a local minima of f then

(i) Optimality condⁿ :-

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x)}{\partial x_i} + \sum_{K=1}^{l} \lambda'_K \frac{\partial h_K(x)}{\partial x_i} = 0 \quad i=1,2,\ldots,n.$$

Lagrange function

$$L = f + \sum_{j=1}^{m} \lambda_j g_j + \sum_{K=1}^{l} \lambda'_K h_K .$$

(ii) (Feasibility condⁿ) :-

$g_j(x) \leq 0$; $h_K = 0$, $j = 1, 2, \ldots, m$, $K = 1, 2, \ldots, l$.

(iii) complementary slackness condⁿ :-

$\lambda_j g_j(x) = 0$ $j=1,2,\ldots,m$. (only for the inequality constraints)

(iv) Non-Negativity Condⁿ :-

$\lambda_j \geq 0$, $j = 1, 2, \ldots, m$.

(v) λ'_K for $K = 1, 2, \ldots, l$ are unrestricted in sign.
($>$ or $<$).

Example :- Min $f(x) = x_1^2 + x_2^2 - 4x_1 - 6x_2$

s.t. $x_1 + x_2 \leq 3$

$-2x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

(i) Optimality condn :

$$\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$

$$2x_1 - 4 + \lambda_1 - 2\lambda_2 = 0.$$

For g_2, $2x_2 - 6 + \lambda_1 + \lambda_2 = 0$

(ii) Feasibility condn :

$x_1 + x_2 \leq 3$

$-2x_1 + x_2 \leq 2$

(iii) (Comp. slagk. prop.) :-

$\lambda_1 (x_1 + x_2 - 3) = 0$

$\lambda_2 (-2x_1 + x_2 - 2) = 0$

(iv) (Non-negativity) :-

$\lambda_1 \geq 0$, $\lambda_2 \geq 0$.

Case-I :- $\lambda_1 = 0, \lambda_2 = 0$ (both are inactive constraints)

Case-II : $\lambda_1 \neq 0, \lambda_2 = 0$

Case III : $\lambda_1 = 0, \lambda_2 \neq 0$

Case IV : $\lambda_1 \neq 0, \lambda_2 \neq 0$.

Case-I : $\lambda_1 = 0$, $\lambda_2 = 0$, $x_1 = 2$, $x_2 = 3$

This can not be the optimal point.

Case-II : $\lambda_1 \neq 0$, $\lambda_2 = 0$.

$$\left.\begin{array}{l} 2x_1 + \lambda_1 = 4 \\ 2x_2 + \lambda_1 = 6 \\ x_1 + x_2 = 6. \end{array}\right\} \text{ solving } x_1 = 1,\ x_2 = 2,\ \lambda_1 = 2.$$

Then $(1, 2)$ could be a KKT point.

Case-III : $\lambda_1 = 0$, $\lambda_2 \neq 0$.

$$\left.\begin{array}{l} 2x_1 - 2\lambda_2 = 4 \\ 2x_2 + \lambda_2 = 6 \\ -2x_1 + x_2 = 2 \end{array}\right\} \text{ solving } x_1 = 4/6,\ x_2 = 18/5,\ \lambda_2 = -6/5 < 0$$

$[\because \lambda_2 \geq 0]$

$\therefore (x_1, x_2) = (4/6, 18/5)$ will not be a KKT point.

Case-IV : $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.

$$\left.\begin{array}{l} x_1 + x_2 = 3 \\ -2x_1 + x_2 = 2 \end{array}\right\} \quad x_1 = 1/3,\ x_2 = 8/3,\ \lambda_2 = -8/9 < 0.$$

$\therefore (1/3, 8/3)$ is not KKT point.

Exercise: Solve the following NLP

$$\text{Max } 7x_1^2 + 6x_1 + 5x_2^2$$

$$\text{s.t. } x_1 + 2x_2 \leq 10$$

$$x_1 - 3x_2 \leq 9$$

$$x_1, x_2 \geq 0.$$

Advantage of KKT-condition :-

1. If in the ~~NLP~~ non-linear programming problem, the objective function is convex (or concave) for min (or max) problem and the constraints g_i, $i = 1, 2, \cdots, m$ (feasible spaces) are convex for both the cases then if we can find some solution x^* then the local optimal solution would become the global optimal solutions.

2. Using KKT-condition we can also handle the non-linear problems with inequality constraints.

Disadvantage :-

(i) If the functions are continuous, twice differentiable Then only we can go for the optimal solution by KKT-Conditions.

(ii) Ensuring the global optimality's very difficult for some complicated situations because checking the convexity (or concavity) property of the objective function is very difficult for complicated situations.