Lecture- 3

Unconstrained Optimization Problems (UOP)

- Unconstrained optimization problems we will study in this Lecture. If we want to optimize (maximize or minimize) a function without any constrained, then we see such types of problems are called unconstrained optimization problems. The formulation of such problem as.

Suppose, $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$\left.\begin{array}{l} \text{Max / Min } f(x) \\ \quad \text{s.t. } x \in \mathbb{R}. \end{array}\right\} \longrightarrow (*)$$

(Assume that f is twice differentiable over $\mathbb{R}$)

- **Necessary Condition** :-

If $x^* \in \mathbb{R}$ is a local min or local max point of f over $\mathbb{R}$, then $f'(x^*) = 0$.

- **Sufficient Condition** :-

* The point $x^* \in \mathbb{R}$ $(f'(x^*) = 0)$ is a local min point of the problem $(*)$ if $f''(x^*) > 0$.

* The point $x^* \in \mathbb{R}$ $(f'(x^*) = 0)$ is a local max point of the problem $(*)$ if $f''(x^*) < 0$.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ (function of two variable) and the problem is same as ⊛.

Necessary condition :- If $(x^*, y^*) \in \mathbb{R}^2$ is a point of local min or local max of f over $\mathbb{R}^2$, then

$$\left(\frac{\partial f}{\partial x}\right)_{(x^*, y^*)} = \left(\frac{\partial f}{\partial y}\right)_{(x^*, y^*)} = 0$$

Sufficient condition :-

- If $(x^*, y^*) \in \mathbb{R}^2$ satisfy ⊛ and $(\nabla^2 f)_{(x^*, y^*)}$ is positive definite then (x^*, y^*) is local min point of f over $\mathbb{R}^2$.

- If $(x^*, y^*) \in \mathbb{R}$ satisfy ⊛ and $(\nabla^2 f)_{(x^*, y^*)}$ is negative definite then (x^*, y^*) is local max point of f over $\mathbb{R}^2$.

Note :- We can generalized these same condition when our function $f : \mathbb{R}^n \to \mathbb{R}$.

Necessary condition :- Let $x^* \in \mathbb{R}^n$ be a point of local min or local max of f over $\mathbb{R}^n$, then

$$\nabla f(x^*) = 0.$$

Sufficient condition :- Let $x^* \in \mathbb{R}^n$ with $\nabla f(x^*) = 0$,

If f is a strictly convex (strictly concave) function in a nbd of x^* then x^* is a local min (local max) point of $f(x)$ over $\mathbb{R}^n$.

Constrained Optimization Problem with equality constraints

Consider the problem:

$$\left.\begin{array}{l} \text{Min/Max } f(x) \\ \text{s.t. } \quad g_i(x) = 0, \quad i=1,2,\dots,m \\ \text{where } f: \mathbb{R}^n \to \mathbb{R}, \quad g_i: \mathbb{R}^m \to \mathbb{R}. \end{array}\right\} \text{—— } (**)$$

We construct the Lagrange's function or the Lagrangian as;

$$L(x; \lambda_1, \lambda_2, \dots, \lambda_m) = f(x) + \lambda_1 g_1 + \lambda_2 g_2 + \dots + \lambda_n g_m$$

$$= f(x) + \sum_{i=1}^{m} \lambda_i g_i(x).$$

Necessary Condition:

Let $x^* \in \mathbb{R}^n$ be a point of local min or local max of the problem (**), where $m < n$. Let it be possible to choose sets of m variables x_i for which the Jacobian matrix

$$J = \left[\left(\frac{\partial g_i}{\partial x_j}\right)\right]_{m \times m}$$ has an inverse. Then there exists

a unique set of Lagrange multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$ such that

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0. \text{ —— } (***)$$

i.e. $\frac{\partial L}{\partial x_j} = 0$, $j = 1, 2, \dots, n$

& $\frac{\partial L}{\partial \lambda_i} = 0$, $i = 1, 2, \dots, m$.

Example 1: Find the points of local max or local min (if exists) for the function

$$f(x_1, x_2) = 2 + 2x_1 + 3x_2 - x_1^2 - x_2^2$$

$\Rightarrow$

Exercise 1: Find the local maxima and local minima of

$$f(x) = x^3 - 6x^2 + 9x + 40.$$

Sufficient Condition :-

Let $(x^*, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ exists such that condition (***) above holds. Suppose $Z(y^*) = \{ z \in \mathbb{R}^n : z^T \nabla g(x^*) = 0 \}$ and $z^T \nabla^2 L((x^*, \lambda))z > 0$, for all $z \in Z(y^*)$ with $z \neq 0$, then x^* is a local min point of (**).

Similarly, if $z^T \nabla^2 L(x^*, \lambda) z < 0$, $\forall z \in Z(y^*)$ with $z \neq 0$, then x^* is a local maxima point of (**).

Example :- Use the Method of Lagrange multipliers to solve

$$\text{Min} \quad \frac{x^3}{3} - \frac{3y^2}{2} + 2x$$

$$\text{s.t.} \quad x - y = 0$$

$\Rightarrow$ **Method I :-** Here note that $x = y$.

Therefore, putting this value and the objective function becomes single variable optimization (unconstraints) problem. Now, use local min (or local max) technique to solve this problem.

On the other hand, if we want to solve same problem by Lagrange Method then it will be the following :

$$L(x, y, \lambda) = \left(\frac{x^3}{3} - \frac{3y^2}{2} + 2x\right) + \lambda(x - y)$$

$$\frac{\partial L}{\partial x} =$$

$$\frac{\partial L}{\partial y} =$$

$$\frac{\partial L}{\partial \lambda} =$$

The solutions of above system are

$x = 2,\ y = 2,\ \lambda = -6$

& $x = 1,\ y = 1,\ \lambda = -3.$

~~Hessian Matrix~~

First at $(1,1) = y^*$

$$Z(y^*) = \left\{ z \in \mathbb{R}^2 \;\middle|\; (z_1, z_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \right\}$$

$$= \left\{ z \in \mathbb{R}^2 \mid z_1 = z_2 \right\}$$

$$= \left\{ (z_1, z_1) : z_1 \in \mathbb{R} \right\}.$$

$g = x - y$

$\nabla g = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$L = \frac{x^3}{3} - \frac{2y^2}{2} + 2x + \lambda(x-y)$

$$\nabla^2 L = \text{Hessian Matrix} = \begin{pmatrix} 2x & 0 \\ 0 & -3 \end{pmatrix}$$

$$\nabla^2 L \big|_{(1,1)} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

Now we will check $Z^T \nabla^2 L\, Z > 0$ or not?

$$Z^T (\nabla^2 L)\, z = (z_1\ z_1)\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_1 \end{pmatrix}$$

$$= 2z_1^2 - 3z_1^2 = -z_1^2 < 0 \quad \forall z_1$$

$\therefore$ $(1,1)$ is the point of local maxima.

Similarly, check $(2,2) = y^*$

$$Z(y^*) = \left\{ z \in \mathbb{R} \;\middle|\; (z_1\ z_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \right\}$$

$$= \left\{ z \in \mathbb{R} \mid z_1 = z_2 \right\} = \left\{ (z_1, z_1) : z_1 \in \mathbb{R} \right\}$$

$$z^T(\nabla^2 L)z = (z_1 \quad z_1)\begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_1 \end{pmatrix}$$

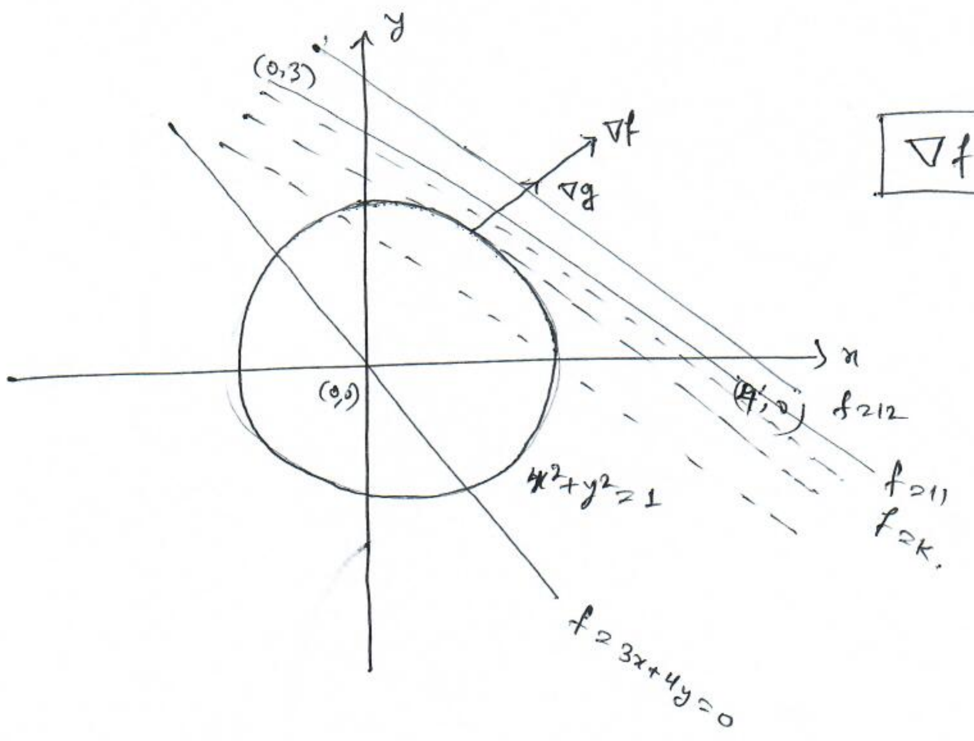
$$= 4z_1^2 - 3z_1^2 = z_1^2 > 0 \quad \forall \; z_1, \; z_1 \neq 0.$$

$\therefore$ (2, 2) is the local minima.

<u>Exercise 1</u>: Find Max/min $f(x,y) = 3x + 4y$

s.t. $x^2 + y^2 = 1$.

<u>Interpretation of Lagrange Multipliers.</u>



$$\boxed{\nabla f = \lambda \nabla g.}$$

Exercise-2:

The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks arround the circle of radius 5. centred at the origin. What are the heighest and lowest temperatures encountered by the ant?

Problem formulation:- $T(x, y) = 4x^2 - 4xy + y^2$

$$\text{s.t.} \quad x^2 + y^2 = 25.$$