Lecture – 2

Convex Programming Problems

Ref :- Numerical optimization with Applications by S. Chandra, Jayadeva and A. Mehra.

In the previous lecture we have seen that what is convex function? what are their properties. In this lecture, we will study some convex programming problems (CPP).

Convex Programming Problem :-

The optimization problem of the form:

$$\left.\begin{array}{ll} & \text{Min } f(x) \\ \text{s.t.} & g_i(x) \le 0, \quad i = 1, 2, \dots, m, \end{array}\right\} \longrightarrow (*)$$

is called a convex programming problem (CPP) if f and g_i $(i = 1, 2, \dots, m)$ are convex functions.

Remark 1:- If the optimization problem of the form:

$$\begin{array}{ll} & \text{Max } f(x) \\ \text{s.t.} & g_i(x) \le 0, \quad i = 1, 2, \dots, m, \end{array}$$

is called a convex programming problem if f is a concave function and (for $i = 1, \dots, m$) g_i's are convex.

Different formats of the ~~prob~~ CPP problem :-

Optimization problem	Conditions for (CPP)
(i) Min $f(x)$ s.t. $g_i(x) \le 0$, $i=1,2,\dots,n$	f and g_i $(i=1,2,\dots,n)$ both are convex.
(ii) Max $f(x)$ s.t. $g_i(x) \le 0$, $i=1,2,\dots,n$	f is concave and g_i's $(i=1,2,\dots,n)$ are convex.
(iii) Min $f(x)$ s.t. $g_i(x) \ge 0$, $(i=1,2,\dots,n)$	f is convex and g_i's $(i=1,2,\dots,n)$ are concave.
(iv) Max $f(x)$ s.t. $g_i(x) \ge 0$, $(i=1,2,\dots,n)$	f and g_i $(i=1,2,\dots,n)$ both are concave.

Theorem 1 :- Let $\bar{x}$ be a local minimum point of the convex optimization problem ⊛. Then $\bar{x}$ is also its global min point.

Proof :- We are given that $\bar{x}$ is a local min point of ⊛ (CPP).

So by definition, there exists $\delta > 0$, s.t. $f(\bar{x}) \le f(y)$,

$\forall\, y \in N_\delta(\bar{x}) \cap S$

$S \to$ (domain of the convex function f which is convex set).

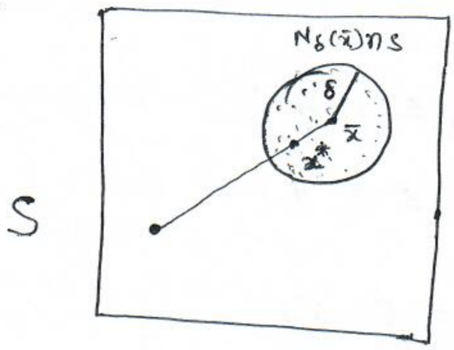
$N_\delta(\bar{x})$ is a ball of radius δ centered at $\bar{x}$.

Let x be an arbitrary point outside the set $N_\delta(\bar{x}) \cap S$.

Then if we show that $f(\bar{x}) \le f(x)$, we are done !.

Now, there certainly exists λ ; $0 < \lambda < 1$ $\ni$

$$x^* = \lambda x + (1-\lambda)\bar{x} \quad \text{and} \quad x^* \in N_\delta(\bar{x}) \cap S.$$



As $x^* \in N_\delta(\bar{x}) \cap S$, and $\bar{x}$ is a local min point of CPP(*)

we have $f(\bar{x}) \le f(x^*)$.

$$\Rightarrow f(\bar{x}) \le f(\lambda x + (1-\lambda)\bar{x})$$

But f is convex on S, and hence the above gives

$$f(\bar{x}) \le \lambda f(x) + (1-\lambda) f(\bar{x}) = \lambda f(x) + f(\bar{x}) - \lambda f(\bar{x}).$$

$$\therefore \lambda f(\bar{x}) \le \lambda f(x).$$

$$\Rightarrow f(\bar{x}) \le f(x) \quad \text{as } 0 < \lambda < 1.$$

Which implies that $\bar{x}$ is a global min point.

Theorem 2 :- Let g_i for each $i=1, 2, \ldots, n$ be a convex function. Then $W = \{x \in \mathbb{R}^n : g_i(x) \le 0, i=1, 2, \ldots, n\}$

is a convex set.

Proof :- $W = \{x \in \mathbb{R}^n : g_i(x) \le 0, i=1, 2, \ldots, n\}$

where all g_i's are convex functions.

Let $x_1, x_2 \in W \Rightarrow g_i(x_1) \le 0 \ \forall i$

$\quad \& \ g_i(x_2) \le 0 \ \forall i$

Suppose ~~Let~~ $x = \lambda x_1 + (1-\lambda)x_2, \ \lambda \in [0, 1]$

$$
\begin{aligned}
g_i(x) &= g_i(\lambda x_1 + (1-\lambda)x_2) \\
&\le \lambda g_i(x_1) + (1-\lambda) g_i(x_2) \quad (\because g_i\text{'s are convex } f^{n}\text{s}) \\
&\le \lambda \times 0 + (1-\lambda) \times 0 = 0
\end{aligned}
$$

$\Rightarrow g_i(x) \le 0 \ \forall i$

$\Rightarrow x \in W$

$\therefore$ W is a convex set.

Theorem 3 :- The set of all optimal solutions of the convex optimization problem (*) is a convex set.

Proof :- $V = \{x^* : x^* \in S,\ f(x^*) \le f(x)\ \forall x \in S\}$

Exercise!!

Theorem 4 :- Let a non-constant convex function f be maximized over a convex set S. Then no interior point of S can be a maximizing point.

i.e. if a maximizing point exists, then it must be a boundary point of S.

Theorem 5 :- Let $S \subseteq \mathbb{R}^n$ be convex and $f : S \to \mathbb{R}$ be a strictly convex function. Then there is a unique minimizing point of f over S.

Proof :- Let $\bar{x}$ and x^* be two distinct minimizing points of f over S, i.e. $\bar{x}, x^* \in S$.

and $f(\bar{x}) \le f(x) \quad \forall x \in S$

also $f(x^*) \le f(x) \quad \forall x \in S$.

$\Rightarrow$ $f(\bar{x}) = f(x^*)$

Let $\hat{x} = \lambda \bar{x} + (1-\lambda) x^*$ for $\lambda \in [0,1]$.

In particular,
$\hat{x} = \frac{\bar{x}+x^*}{2}$
$f(\hat{x}) = f\left(\frac{\bar{x}+x^*}{2}\right)$
$\le \frac{1}{2} f(\bar{x}) + \frac{1}{2} f(x^*)$
$\le f(\bar{x})$

Then by strict convexity of f we have

$$f(\hat{x}) = f(\lambda \bar{x} + (1-\lambda) x^*) < \lambda f(\bar{x}) + (1-\lambda) f(x^*)$$

$\Rightarrow f(\hat{x}) < f(\bar{x}) \quad \Rightarrow\Leftarrow$ that $\bar{x}$ is minimizing point.

$[\because f(\bar{x}) = f(x^*)]$

Examples of convex Programming Problems :-

Example 1 :- Check if the following problem is a convex prog. prob. (CPP).

(i) Min $x_1 + x_2$

s.t. $x_1^2 + x_2^2 \le 1$

$x_1^2 \le x_2$

$\equiv$

Min $f = x_1 + x_2$

s.t. $g_1(x_1, x_2) = x_1^2 + x_2^2 - 1 \le 0$

$g_2(x_1, x_2) = x_1^2 - x_2 \le 0$

(ii) Max $4x_1 + 3x_2$

s.t. $x_1 + x_2 \le 4$

$x_1 x_2 \le 1$

$x_1, x_2 \ge 0$.

$\Rightarrow$

~~Exercise~~ : ~~Max~~ .

Exercise 2:- check if the following is cpp or NOT.

(i) $\quad \text{Max} \quad x_2^2$

$\text{s.t} \quad x_1^2 + x_2^2 \leq 4$

$\quad 4x_1^2 \geq x_2.$

(ii) $\quad \text{Max} \quad 4x_1 + 3x_2$

$\text{s.t.} \quad x_1^2 + x_2^2 \leq 1$

$\quad (x_1-1)^2 + x_2^2 \leq 1.$

(iii) $\quad \text{Min} \quad x_1^2 + x_2$

$\text{s.t.} \quad |x_1| + |x_2| \leq 2$

$\quad x_1^2 - x_2 \geq 0.$