Lecture- 7

▣ In the previous lecture we have seen some QPP and example of some QPP. We see for QPP if if the objective function and all the constraints (which are linear in nature) are convex f^{n}s. Therefore, QPP problem can be solve via KKT condition. We have seen the modification of KKT-condition for a given QPP. In this lecture we study the method to solve some QPP.

▣ Show that $f(x) = x^T Q x + c^T x$, $x \in \mathbb{R}^n$ is a convex function if Q is a positive semi definite symmetric matrix.

$\Rightarrow$ Given $Q \rightarrow$ symmetric
$\quad\quad\quad\quad\rightarrow$ +ve semi definite.

To show, $f(x) = x^T Q x + c^T x$ is a convex f^{n}.

Now, we know, a function $f(x)$ is convex if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \quad \forall x_1, x_2 \in \mathbb{R}^n, \ \lambda \in [0,1].$$

$$\Rightarrow f(\lambda x_1 + (1-\lambda)x_2) - \lambda f(x_1) - (1-\lambda) f(x_2) \leq 0.$$

$$= (\lambda x_1 + (1-\lambda)x_2)^T Q (\lambda x_1 + (1-\lambda)x_2) + c^T[\lambda x_1 + (1-\lambda)x_2]$$
$$- \lambda[x_1^T Q x_1 + c^T x_1] - (1-\lambda)[x_2^T Q x_2 + c^T x_2]$$

$$= \lambda^2 x_1^T Q x_1 + \lambda(1-\lambda) x_1^T Q x_2 + \lambda(1-\lambda) x_2^T Q x_1 + (1-\lambda)^2 x_2^T Q x_2$$
$$- \lambda x_1^T Q x_1 - (1-\lambda) x_2^T Q x_2$$

$$= \lambda(\lambda-1) x_1^T Q x_1 + (1-\lambda)(1-\lambda-1) x_2^T Q x_2$$
$$+ 2\lambda(1-\lambda) x_1^T Q x_2$$

$$\boxed{\begin{array}{c} x_1^T Q x_2 \\ \downarrow \\ \text{scalar} \\ (x_1^T Q x_2)^T \\ = x_2^T Q x_1 \end{array}}$$

$$\le \lambda(\lambda-1) x_1^T Q x_1 - \lambda(1-\lambda) x_2^T Q x_2 + \lambda(1-\lambda)(x_1^T Q x_1 + x_2^T Q x_2)$$

$$\le 0$$

$$\therefore \quad f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

Hence, f is convex.

Remark :- If in the QPP the matrix is positive semi-definite and symmetric matrix then the objective function is convex. Hence, the QPP becomes CPP. Therefore, the KKT-condition becomes sufficient and the local minima will become the global minima point.

Thm 1:- Let Q be a symmetric, positive semi-definite matrix of order n. If there exist $\bar{u}, \bar{v}, \bar{x}$ satisfy KKT Conditions

(i) $2Qx + c + A^T u - vI = 0$

(ii) $Ax - b + s = 0$

(iii) $u_i s_i = 0 \quad \forall\ i = 1, 2, \ldots, m$

(iv) $v_j x_j = 0 \quad \forall\ i = 1, 2, \ldots, n$

$\forall \quad x \geq 0,\ u \geq 0,\ v \geq 0,\ s \geq 0$

Then $\bar{x}$ will be the global minima point of the QPP.

Thm 2:- If Q is negative definite then the quadratic programming problem can not have an unbounded solution.

QPP :-
$$\left.\begin{array}{ll} \text{Max} & x^T Q x + c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0. \end{array}\right\} \quad \text{——} \ (*)$$

Here, Q is negative Semi-definite matrix.

Proof :- See S. Chandra Book for more details.

Note: ~~that~~ In Theorem 2, may not hold if Q is negative semi-definite matrix,

Example: solve the following QPP

$$\text{Max } z = x_1 + x_2 - (2x_1 - x_2)^2$$

$$\text{s.t.} \quad x_1 - x_2 \le 1$$

$$x_1, x_2 \ge 0$$

$\Rightarrow$ We have $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $b = (1)$

$$A = [1 \quad -1]$$

$$Q = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

Step-I: check Q is ~~positive~~ ~~to~~ negative definite or NOT?

$$Q = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

Eigen values are $\lambda_1 = 0$
$\lambda_2 = -5$.

(Principal Minors)

- A matrix A is negative definite iff its odd principal ~~min~~ minors are negative and its even principal minors are positive,

- For positive definite all its leading principle minors are positive.

$\therefore$ The matrix Q is negative - semi-definite,

Now, observe that any $x^* \geq 0$,

$x_1 = x^*$, $x_2 = 2x^*$ is a feasible solution of the problem and the value of the objective function is

$$z = x^* + 2x^* + (2x^* - 2x^*) = 3x^* > 0.$$

Thus, as $x^* \to \infty$, $z \to \infty$. So the given problem has an unbounded solution.

<u>Remark</u> :- If Q is negative definite, the objective function of the QPP ⊛ is strictly concave and so if it has a feasible solution then it has unique optimal solution. However, when Q is negative - semi-definite then ~~the~~ to guarantee that it has bounded optimal solution we not only need that given problem ⊛ is feasible but also require that $c = 0$.

Wolfe's Method

- Wolfe's method is directly based on the KKT condition of the given QPP and is solved by the restricted entry simplex method. This Method was developed by P. Wolfe in 1959.

- Another popular method for solving QPP is the Beal's Method, developed By E. M. L. Beale in 1959. (This method we will not study in this course)

- **Example** :- Solve

$$\text{Min } f(x) = x_1^2 + 2x_2^2 - 2x_1x_2 - 6x_1 - 8x_2$$

$$\text{s.t. } 2x_1 - x_2 \leq 13.$$

$$x_1, x_2 \geq 0.$$

$\Rightarrow$

Step-I :- check that it is a QPP or NOT ?

Therefore, check $Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is positive definite or Not ?

For Q, $|D_1| = 1 > 0$, $|D_2| = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 > 0$

$\therefore$ Q is positive-definite,

Step-II :- Check QPP is a CPP or NOT ?

Here, $f(x)$ is convex, as Q is convex

and linear contraints $\Rightarrow$ convex.

$\therefore$ The QPP is a CPP and the KKT conditions

will be the sufficient.

Step-III :- KKT - Conditions :-

$$L = (x_1^2 + 2x_2^2 - 2x_1x_2 - 6x_1 - 8x_2) + u(2x_1 - x_2 - 13) - v_1(x_1) - v_2 x_2 .$$

Because the problem (CPP) becomes

$$\text{Min } f(x) = x_1^2 + 2x_2^2 - 2x_1x_2 - 6x_1 - 8x_2$$

s.t. $g_1(x_1, x_2) \equiv 2x_1 - x_2 - 13 \le 0$

$g_2(x_1, x_2) \equiv -x_1 \le 0$

$g_3(x_1, x_2) \equiv -x_2 \le 0.$

Lagrangian / KKT ~~slack~~ variables

u

v_1

v_2

$\therefore$ (i) $\frac{\partial L}{\partial x_1} = 2x_1 - 2x_2 - 6 + 2u - v_1 = 0$

$\frac{\partial L}{\partial x_2} = 4x_2 - 2x_1 - 8 - u - v_2 = 0$

(ii) $2x_1 - x_2 + s = 13$

(iii) $x_1, x_2 \geq 0$, $u, v_1, v_2 \geq 0$

(iv) $us = 0$, $v_1 x_1 = v_2 x_2 = 0$

Step-IV : Ignore the complementary slackness conditions in the KKT system (i.e. $us = 0$, $v_i x_i = 0$, $i = 1, 2$) and consider the remaining system of linear equations in the non-negative variables $x_1, x_2, u, v_1, v_2 \geq 0$

$$
\begin{aligned}
2x_1 - 2x_2 + 2u - v_1 &= 6 \\
-2x_1 + 4x_2 - u - v_2 &= 8 \\
2x_1 - x_2 \qquad\qquad + s_1 &= 13
\end{aligned}
$$

$x_1, x_2, u, v_1, v_2, s_1 \geq 0,$

Step-V : Construct the problem in the form of LPP.

$$\text{Max } Z = -a_1 - a_2$$

s.t.

$$
\begin{aligned}
2x_1 - 2x_2 + 2u - v_1 + a_1 &= 6 \\
-2x_1 + 4x_2 - u - v_2 + a_2 &= 8 \\
2x_1 - x_2 + s_1 &= 13
\end{aligned}
$$

$x_1, x_2, u, v_1, v_2, s_1, a_1, a_2 \geq 0,$

Step – VI : Construct simplex table.

B.V	x_1	x_2	u_1	v_1	v_2	a_1	a_2	s_1	Sol^n	Ratio
Z	0	$-2\downarrow$	-1	1	1	0	0	0		
a_1	2	-2	2	-1	0	1	0	0	6	—
$\leftarrow a_2$	-2	4	-1	0	-1	0	1	0	8	2
s_1	2	-1	0	0	0	0	0	1	13	—
Z	$-1\downarrow$	0	$-3/2$	1	$1/2$	0	$1/2$	0		
$\leftarrow a_1$	1	0	$3/2$	-1	$-1/2$	1	$1/2$	0	10	10
x_2	$-1/2$	1	$-1/4$	0	$-1/4$	0	$1/4$	0	2	—
s_1	$3/2$	0	$-1/4$	0	$-1/4$	0	$1/4$	1	15	10
Z	0	0	0	0	0	1	1	0		
x_1	1	0	$3/2$	-1	$-1/2$	1	$1/2$	0	10	
x_2	0	1	$1/2$	$-1/2$	$-1/2$	$1/2$	$1/2$	0	7	
s_1	0	0	$-5/2$	$3/2$	$1/2$	$-3/2$	$-1/2$	1	0	

$\therefore$ Solution is $x_1 = 10$, $x_2 = 7$, $s_1 = 0$, $u = v_1 = v_2 = 0$.