Golden Section Method

In this method the function should be unimodal. If the function is multimodal then there are many local optima (minima / maxima). For that case we have to break the interval of uncertainty into small intervals and apply the region elemination technique to the smaller intervals.

The ~~main~~ Golden section Method is very similar to Fibonacci Method. The main advantage is that ~~is~~ in this method the number of experiment are not pre-fixed before hand. We can fixed this while running the experiments. Let's see the method step. ~~to~~ by ~~se~~ step.

$\boxed{\text{Golden Ratio } \gamma \ (\approx 1.618)}$

$$F_n = F_{n-1} + F_{n-2}$$

$$\Rightarrow \frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

$$\gamma = \lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \lim_{n \to \infty} \frac{F_{n-1}}{F_{n-2}} = \lim_{n \to \infty} \frac{F_{n-j}}{F_{n-(j-1)}}$$

$$\Rightarrow \quad \gamma = 1 + \frac{1}{\gamma}$$

$$\Rightarrow \quad \gamma^2 - \gamma + 1 = 0 \qquad \Rightarrow \quad \gamma = \frac{1 \pm \sqrt{5}}{2}$$

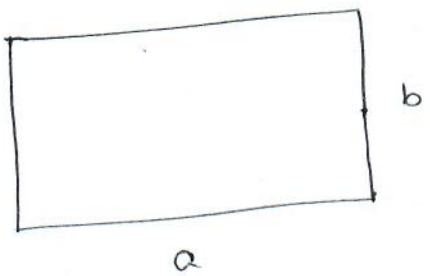
$$\therefore \quad \gamma = \frac{1+\sqrt{5}}{2} = 1.618.$$

$\phi = \frac{1-\sqrt{5}}{2} = 0.618$ (conjugate golden ratio)

$\boxed{1/\phi = 0.618.}$

Golden ratio

(Room construction, envelope and all)
TV screen.



$$\frac{\text{whole length}}{\text{length of biggers}} = \frac{\text{length of bigger}}{\text{length of smaller}}$$

$$\equiv \frac{a+b}{a} = \frac{a}{b} = \phi.$$

In Geometry, if we devide a line segment with unequal parts. The we will see that

$$\frac{\text{length of the whole line segment}}{\text{length of the larger part}} = \frac{\text{length of the larger part}}{\text{length of the smaller part.}}$$

$$= \phi$$

Algorithm for Golden Section Method

Step-I: Given initial interval of uncertainty $L_0 = [a, b]$

Step-2: $L_2^* = \frac{1}{\gamma^2} L_0$

Fibonacci Method $\Rightarrow$ $L_2^* = \frac{F_{n-2}}{F_n} L_0$

$\therefore$ For Golden Section Method we compute large number of experiment.

$$L_2^* = \lim_{n \to \infty} \frac{F_{n-2}}{F_n} L_0$$

$$= \lim_{n \to \infty} \frac{F_{n-2}}{F_{n-1}} \cdot \frac{F_{n-1}}{F_n} \cdot L_0 \qquad \because \lim_{n \to \infty} \frac{F_{n-1}}{F_n} = \frac{1}{\gamma}$$

$$= \frac{1}{\gamma^2} L_0.$$

To generate $x_1 = a + L_2^*$

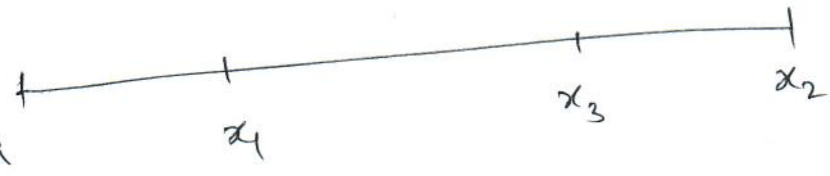
$x_2 = b - L_2^*$

Then,

- $f(x_1) < f(x_2)$
  - Minimization problem discard $(x_2, b]$.
  - Maximization problem discard $[a, x_1)$.

Step-3 :- $L_2 =$ either $[a, x_2)$ or $[x_1, b]$

New interval of uncertainity. $\qquad L_2 = \frac{1}{\gamma} L_0$



$L_3^* = \frac{1}{\gamma^3} L_0.$

Similar like previous

$$\lim_{n\to\infty} \frac{F_{n-3}}{F_n} L_0 = \lim_{n\to\infty} \frac{F_{n-3}}{F_{n-2}} \cdot \frac{F_{n-2}}{F_{n-1}} \cdot \frac{F_{n-1}}{F_n} \cdot L_0$$

$$= \frac{1}{\gamma^3} \cdot L_0$$

Step-4 :- $L_3 = L_2 - L_3^* = \lim_{n\to\infty} \frac{F_{n-2}}{F_n} L_0 = \frac{1}{\gamma^2} L_0.$

To generate the k^{th} experiment

$$\boxed{L_k^* = \frac{1}{\gamma^k} L_0}$$

$$\boxed{L_k = \frac{1}{\gamma^{k-1}} L_0}, \quad k = 1, 2, \ldots$$

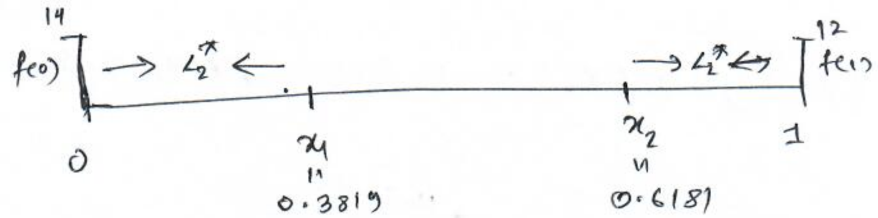
Step-5 :- Length of $L_k < \varepsilon$ (ε is very small)

Then declare that L_k is the final interval of uncertainity. And, certainly, the middle point of L_k would be the optimal solution of the problem.

Example :- Minimize $f(x) = 4x^3 + x^2 - 7x + 14$. within the interval $[0,1]$ using Golden Section Method. Assume that the f^n $f(x)$ is unimodal, $\varepsilon = 0.15$. (Stopping tolerence.)

$\Rightarrow$ Step-I : $L_0 = 1$ (Initial interval of uncertainity)

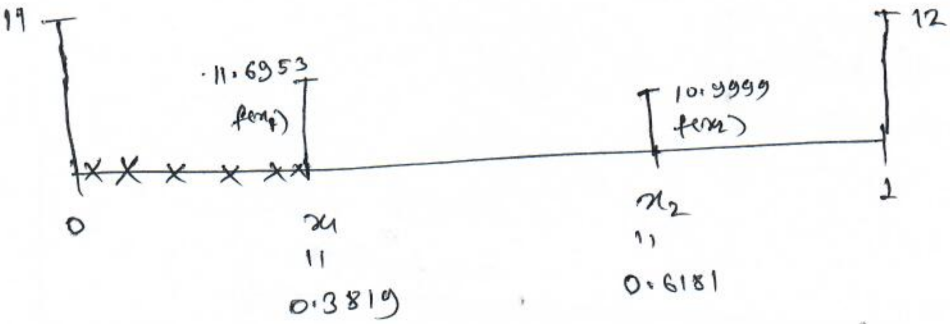
Step-2 :



$$L_2^* = \frac{1}{\nu^2} L_0 = 0.3819.$$

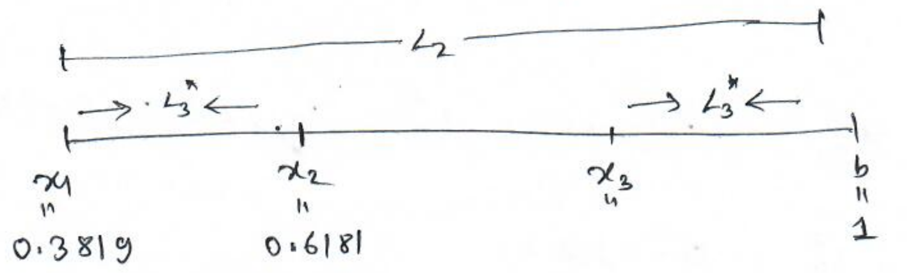
$x_1 = a + L_2^* = 0.3819$, $f(x_1) = 11.6953$

$x_2 = b - L_2^* = 0.6181$, $f(x_2) = 10.9999.$



$\therefore$ Next interval of uncertainity is $(x_1, b] = [0.3819, 1]$

Step-3 :



Generate x_3.

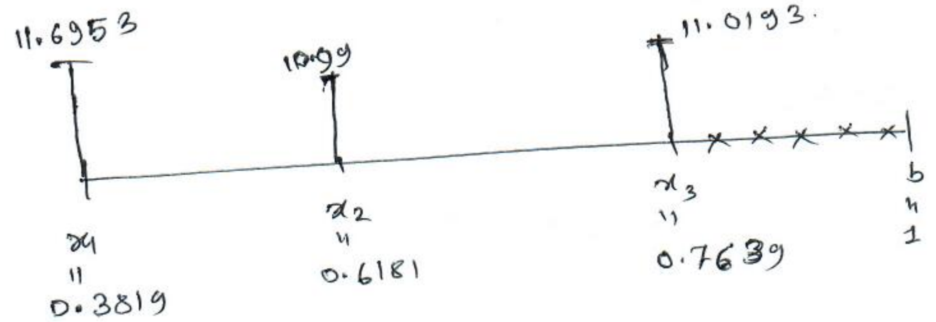
$L_3^* = \frac{1}{\nu^3} L_0 = 0.2361$

$\therefore x_3 = b - L_3^* = 0.7639.$

$\boxed{L_2 = \frac{1}{\nu} L_0}$ → step-2

$\boxed{L_3 = \frac{1}{\nu^2} L_0}$ step-3

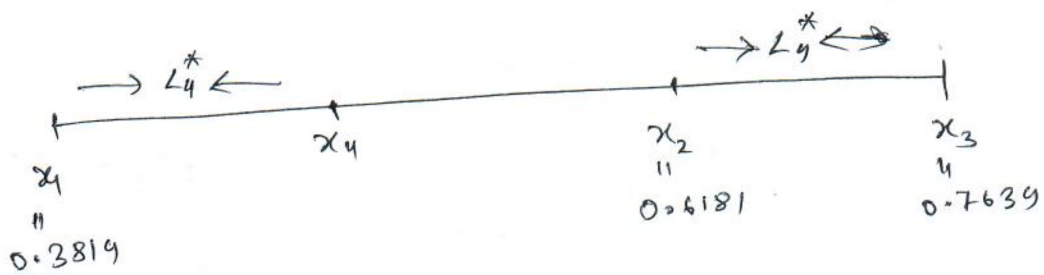
$f(x_1) = 11.6953$, $f(x_3) = 11.0193$, $f(x_2) = 10.9999$



$\therefore$ New interval of uncertainty $(x_1, x_3) \equiv (0.3819, 0.7639)$

$\boxed{L_3 = x_3 - x_1 = \frac{1}{\nu^2} L_0.}$

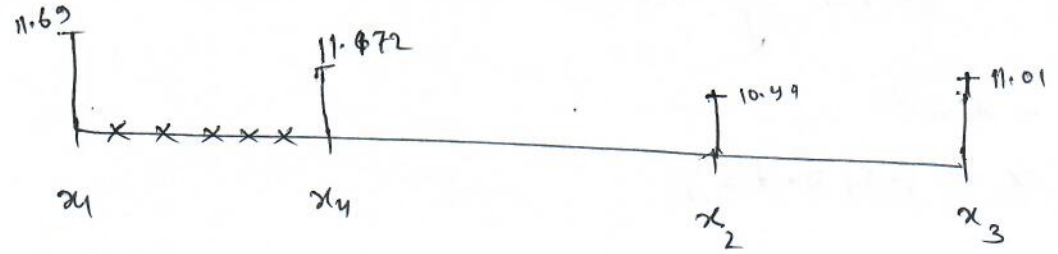
step-4 :



To generate x_4, Calculate $L_4^* = \frac{1}{\nu^4} L_0 = 0.1459$

$x_4 = x_1 + L_4^* = 0.5278.$

$f(x_4) = 11.1721$



$\therefore$ New interval of uncertainty $= (x_4, x_3)$

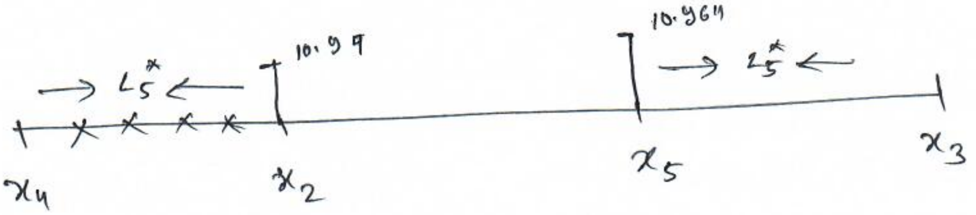
$$\boxed{L_4 = x_3 - x_4 = \frac{1}{\gamma^3} L_0}$$

$L_4 = 0.2360 \not< \varepsilon$

Step-5: To generate x_5, $L_5^* = \frac{1}{\gamma^5} L_0$

$= 0.0902$,

$x_5 = 0.7639$, $f(x_5) = 10.9611$



$L_5 = \frac{1}{\gamma^4} L_0 = 0.1458 < \varepsilon$. (STOP the iteration process)

$\therefore$ The final interval of uncertainty $= (x_2, x_3)$.

$\therefore L_5 = x_2 - x_3 = 0.1458$.

$\therefore$ Optimal point $x^* = \dfrac{x_2 + x_3}{2}$.

☒ Exercise :- Solve by Golden Section Method

$$\text{Max } -x^2-1$$
$$\text{s.t. } x \in [-1, 0.75]$$

with the final interval of uncertainty having a length less than $1/4$.