Integer Linear Programming

February 10, 2022

Introduction

All the problem in the prior chapters that we have treated wherein all basic variables including slack or surplus variables were permitted to take any non-negative real values (continuous or fractional). It was done since in many situations it was quite possible and made sense to have fractional solutions. For example, it is quite sensible to use 10.25 kg of raw material, 2.30 hours of man power, etc.

Introduction

All the problem in the prior chapters that we have treated wherein all basic variables including slack or surplus variables were permitted to take any non-negative real values (continuous or fractional). It was done since in many situations it was quite possible and made sense to have fractional solutions. For example, it is quite sensible to use 10.25 kg of raw material, 2.30 hours of man power, etc.

But there are many problem, particularly in business and industry, where quite often the fractional solutions are unrealistic, because the units are not divisible. For example, it is meaning less to produce 5.76 tables or to open 2.35 shopping mall or to run 1.23 machines to produce a product, etc. In such cases, integer programming plays a key role where some or all the required variables are integer valued. Transportation problems, assignment problems, etc. are IPP, since the decision variables are either zero or one.

Definitions

An LPP in which some or all of the variables are required to be an integer valued is called IPP. If all the variables are to be integer valued then the problem is called a pure integer programming problem. Otherwise, the problem is called mixed integer programming problem. Further, if all the variables in the optimum solution are allowed to take values either 0 or 1 in do or not do type decisions, then the problem is called zero-one integer programming problem.

Why Integer Programming has been Developed

It seems in mind that to obtain an integer solution to a given LPP it is sufficient to find non-integer optimal solution using simplex method (or graphical method) and then rounding off the value. But, in some cases, it is observed that the deviation from the exact optimal integer solution (obtained as a result of rounding off) may become sufficiently large or to give an infeasible solution.

Why Integer Programming has been Developed

It seems in mind that to obtain an integer solution to a given LPP it is sufficient to find non-integer optimal solution using simplex method (or graphical method) and then rounding off the value. But, in some cases, it is observed that the deviation from the exact optimal integer solution (obtained as a result of rounding off) may become sufficiently large or to give an infeasible solution.

Let us consider the following simple problem: **Maximize** $\mathbf{Z = 9x_1 + 5x_2}$, subject to $\mathbf{3x_1 + 4x_2 \leq 8}$, $x_1 \geq 0, x_2 \geq 0$ and x_1, x_2 are integers.

Why Integer Programming has been Developed

It seems in mind that to obtain an integer solution to a given LPP it is sufficient to find non-integer optimal solution using simplex method (or graphical method) and then rounding off the value. But, in some cases, it is observed that the deviation from the exact optimal integer solution (obtained as a result of rounding off) may become sufficiently large or to give an infeasible solution.

Let us consider the following simple problem: **Maximize** $\mathbf{Z} = \mathbf{9x_1} + \mathbf{5x_2}$, subject to $\mathbf{3x_1} + \mathbf{4x_2} \leq \mathbf{8}$, $x_1 \geq 0, x_2 \geq 0$ and x_1, x_2 are integers.

Ignoring the integer valued restriction, we have the optimal solution: **Max** $\mathbf{Z} = \mathbf{24}$, for $x_1 = 8/3, x_2 = 0$ by using graphical method.

Why Integer Programming has been Developed

It seems in mind that to obtain an integer solution to a given LPP it is sufficient to find non-integer optimal solution using simplex method (or graphical method) and then rounding off the value. But, in some cases, it is observed that the deviation from the exact optimal integer solution (obtained as a result of rounding off) may become sufficiently large or to give an infeasible solution.

Let us consider the following simple problem: **Maximize** $\mathbf{Z = 9x_1 + 5x_2}$, subject to $\mathbf{3x_1 + 4x_2 \leq 8}$, $x_1 \geq 0, x_2 \geq 0$ and x_1, x_2 are integers.

Ignoring the integer valued restriction, we have the optimal solution: **Max** $\mathbf{Z = 24}$, for $x_1 = 8/3, x_2 = 0$ by using graphical method.

Then, by rounding off the fractional value of $x_1 = 2\frac{2}{3}$, the optimal solution becomes:

$$Max\ Z = 27,\ for\ x_1 = 3, x_2 = 0.$$

But this solution does not satisfy the constraint $3x_1 + 4x_2 \leq 8$ and so this solution is infeasible.

Cont...

Again if we round off the solution to $x_1 = 2$, $x_2 = 0$, obviously this is feasible as well as integer valued. But, this solution gives $Z = 18$ which is far away from the optimal value of $Z = 24$. Thus this is another disadvantage of getting an integer valued solution by rounding down the exact optimal solution. Still there is no guarantee that the rounding down solution will be an optimal one since it may be far away from the optimal solution.

Again if we round off the solution to $x_1 = 2$, $x_2 = 0$, obviously this is feasible as well as integer valued. But, this solution gives $Z = 18$ which is far away from the optimal value of $Z = 24$. Thus this is another disadvantage of getting an integer valued solution by rounding down the exact optimal solution. Still there is no guarantee that the rounding down solution will be an optimal one since it may be far away from the optimal solution.

Several algorithms have been developed so far for solving IPP. But, in this chapter, we will focus only on the two well known methods: **Gomory's cutting plane method** and **Branch and bound method**

Gomory's cutting plane method

5.3 CUTTING PLANE CONSTRAINTS

Consider a pure integer programme.

$$\text{Maximise } Z = \boldsymbol{c}\boldsymbol{x} \tag{5.3a}$$

$$\text{subject to } A\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \text{ and are integers,}$$

where $A = \left(a_{ij}\right)_{m\times n}$, $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$, $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^T$.

The problem (5.3a) with integer restriction deleted shall be called *related* LPP. The basic idea in cutting plane algorithm are as follows:

Given a pure IPP, solve the related LPP. If the optimum table contains only integer valued variables, then that must be the optimal solution of IPP. But if some variable in the optimum solution is not an integer, then a new constraint is added to the original problem and the new related LPP is solved. Again, we check whether the new optimum solution satisfies the integer requirement solution to the original IPP. If, on the other hand, some of the variables are still not an integer, then again a new constraint is added and the process is continued until one such related LPP is found to have an all integer optimum solution. As these are hyperplanes which cut off a portion of the feasible region of the related LPP, the constraints are called cutting planes.

We now derive the mathematical form of cutting planes.

Mathematical Formulation

Ignoring the integer restriction of the problem (5.3a) let us first solve the related LPP by usual simplex method and for simplicity we shall assume that the optimal solution contains first m columns of A in the basis B. Hence, A can be partitioned into two sub-matrices $A = [B, R]$, where B contains first m columns of A and R contains remaining $(n-m)$ columns of A. The corresponding optimal solution of the problem $\boldsymbol{x}^* = \left[\boldsymbol{x}_B^*, \boldsymbol{x}_R^*\right]^T$, where the non-basic variables $\boldsymbol{x}_R^* = \boldsymbol{0}$.

Now, the constraint equation $A\boldsymbol{x} = \boldsymbol{b}$ gives

$$[B, R]\begin{bmatrix}\boldsymbol{x}_B^*\\ \boldsymbol{x}_R^*\end{bmatrix} = \boldsymbol{b} \Rightarrow B\boldsymbol{x}_B^* + R\boldsymbol{x}_R^* = \boldsymbol{b} \Rightarrow \left(B^{-1}B\right)\boldsymbol{x}_B^* + \left(B^{-1}R\right)\boldsymbol{x}_R^* = B^{-1}\boldsymbol{b} \Rightarrow \boldsymbol{x}_B^* = B^{-1}\boldsymbol{b} - \left(B^{1}R\right)\boldsymbol{x}_R^*. \tag{5.3b}$$

Since R consists of the columns of A we can express (5.3b) in terms of the columns of the simplex table namely

$$\boldsymbol{y}_0 = B^{-1}\boldsymbol{b} \text{ and } \boldsymbol{y}_j = B^{-1}\boldsymbol{a}_j. \tag{5.3c}$$

From (5.3b) using (5.3c),

$$\boldsymbol{x}_B^* = \boldsymbol{y}_0 - \sum_{j\in N} y_j x_j, \tag{5.3d}$$

where $N = \{j \mid x_j$ is a non-basic variable$\}$.

$\boldsymbol{x}_B^*$ represents the optimal basis of the related LPP of problem (5.3a). Suppose that the rth basic variable of $\boldsymbol{x}_B^*$ is not an integer but we want it to be an integer. To do so let us write (5.3d)

$$x_{Br} = y_{r0} - \sum_{j\in N} y_{rj} x_j. \tag{5.3e}$$

In the current optimal solution all x_j for $j \in N$ are zero so that y_{r0} is not an integer.

Gomory's cutting plane method

Let

$$y_{r0} = W_{r0} + f_{r0}, \quad y_{rj} = W_{rj} + f_{rj}, \tag{5.3f}$$

where W_{r0}, W_{rj} are integral parts and $0 < f_{r0}, f_{rj} < 1$ are fractional parts of y_{r0} and y_{rj} respectively. Then (5.3e) becomes

$$x_{Br} = \left(W_{r0} - \sum_{j\in N} W_{rj}x_j\right) + \left(f_{r0} - \sum_{j\in N} f_{rj}x_j\right) = [I] + \{f\}, \tag{5.3g}$$

where $[I]$ and $\{f\}$ are respectively integral part and fractional part of x_{Br}.

If we wish to modify the current solution so that the new solution is integer, then we make $\{f\}$ to be an integer.

Since each $f_{rj} > 0$ and $x_j \geq 0$ imply $\sum_{j\in N} f_{rj}x_j \geq 0$.

Hence, the quantity $\{f\} = f_{r0} - \sum_{j\in N} f_{rj}x_j$ consists of a positive quantity $\sum_{j\in N} f_{rj}x_j$ subtracted from another positive quantity f_{r0} which is less than unity. Therefore, the entire quantity cannot be a positive integer. If it is to be an integer, then it must be a negative integer or zero.

Therefore,

$$\boxed{\{f\} = \left(f_{r0} - \sum_{j\in N} f_{rj}x_j\right) \leq 0.} \tag{5.3h}$$

(5.3h) is the required cutting plane constraint derived by Gomory in the case of pure integer programme. This constraints is added to the optimal table of the related LPP and then solved by *dual-simplex* method.

Remarks

Important Notes

1. Every integer feasible solution of original IPP will satisfy this constraint.
2. If some y_{ij} be negative, even then a non-negative fractional part of it as required may be obtained. For example, $\left(-2\frac{2}{3}\right)$ can be written as $\left(-3+\frac{1}{3}\right)$, whereby the non-negative fractional part $\frac{1}{3}$ has been separated.
3. In our above discussion, we have not considered how to determine the cutting plane constraint if more than one basic variable in the optimum table of the related LPP are not an integer. In most frequently employed technique, the x_{Br} whose fractional part f_{r0} is largest is used to find the cutting plane constraint (5.3h).

Example

Example 5.3.1. *Use Gomory's cutting plane to solve the problem.*

Maximise $Z = x_1 - x_2$

subject to $x_1 + 2x_2 \leq 4$

$6x_1 + 2x_2 \leq 9$

$x_1,\ x_2 \geq 0$ *and are integers.*

Solution: We first find the optimal solution of the related LPP. The standard form of the related LPP is

Maximise $Z = x_1 - x_2 + 0x_3 + 0x_4$

subject to $x_1 + 2x_2 + x_3 = 4$

$6x_1 + 2x_2 + x_4 = 9$

$x_1,\ x_2 \geq 0$, decision variables

$x_3,\ x_4 \geq 0$, slack variables.

Example

Now, solve the problem by simplex method as follows:

		c_j	1	−1	0	0	
c_B	B	x_B	y_1	y_2	y_3	y_4	MR
0	y_3	4	1	2	1	0	$\frac{4}{1}$
0	y_4	9	6	2	0	1	$\frac{9}{6}\rightarrow$
		$z_j - c_j$	$-1\uparrow$	1	0	0	

		c_j	1	−1	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4
0	y_3	$\frac{5}{2}$	0	$\frac{5}{3}$	1	$-\frac{1}{6}$
1	y_1	$\frac{3}{2}$	1	$\frac{1}{3}$	0	$\frac{1}{6}$
		$z_j - c_j$	0	$\frac{4}{3}$	0	$\frac{1}{6}$

All $z_j - c_j \geq 0$ and $x_B > 0$ so optimal basic feasible solution of the related LPP has been reached and is given by $x_1^* = \frac{3}{2}$, $x_2^* = 0$, $Z_{\max} = \frac{3}{2}$.

Observations. The decision variable $x_1^* = \frac{3}{2}$ is not an integer but we require it to be an integer. To obtain the integer solution we use Gomory's cutting plane constraint.

$$\begin{aligned} \text{Here}\quad y_{20} &= \tfrac{3}{2} = 1 + \tfrac{1}{2}, & W_{20} &= 1 & f_{20} &= \tfrac{1}{2} \\ y_{22} &= \tfrac{1}{3} = 0 + \tfrac{1}{3}, & W_{22} &= 0 & f_{22} &= \tfrac{1}{3} \\ y_{24} &= \tfrac{1}{6} = 0 + \tfrac{1}{6}, & W_{24} &= 0 & f_{24} &= \tfrac{1}{6} \end{aligned}$$

$$\therefore\ f_{r0} - \sum_{j\in N} y_{rj}x_j \leq 0 \Rightarrow \frac{1}{2} - \left(\frac{1}{3}x_2 + \frac{1}{6}x_4\right) \leq 0 \Rightarrow -\frac{1}{3}x_2 - \frac{1}{6}x_4 \leq -\frac{1}{2} \Rightarrow -\frac{1}{3}x_2 - \frac{1}{6}x_4 + x_5 = -\frac{1}{2}.$$

We add a new slack variable $x_5 \geq 0$ in the new constraint and compute the following table with additional constraint.

Example

		c_j	1	-1	0	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	$\frac{5}{2}$	0	$\frac{5}{3}$	1	$-\frac{1}{6}$	0
+1	y_1	$\frac{3}{2}$	1	$\frac{1}{3}$	0	$\frac{1}{6}$	0
0	y_5	$-\frac{1}{2}$	0	$-\frac{1}{3}$	0	$-\frac{1}{6}$	1 $\rightarrow$
		$z_j - c_j$	0	$\frac{4}{3}$	0	$\frac{1}{6}\uparrow$	0

We see that the solution is optimum as all $z_j - c_j \geq 0$ but it is not feasible, y_5 being negative. Hence, we apply *dual-simplex* method. y_5 leaves the basis and since

$$\max\left\{\frac{4/3}{-1/3}, \frac{1/6}{-1/6}\right\} = \max\{-4, -1\} = -1,$$

y_4 enters the basis. The new table is thus

		c_j	1	-1	0	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	3	0	2	1	0	-1
1	y_1	1	1	0	0	0	1
0	y_4	3	0	2	0	1	-6
		$z_j - c_j$	0	1	0	0	1

Example

Example 5.3.2. *Using Gomory's cutting plane method solve the following pure* IPP.

Maximum $Z = 2x_1 + 2x_2$

subject to $5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$ *and are integers.* [CU(MSc Appl) 1989, 1992; BESU(MSc) 2001, 2007]

Solution: We first find the optimal solution of the related LPP by introducing slack variables $x_3, x_4 \geq 0$ to the two given constraints respectively. The following is the *optimal table* of the related LPP obtained by usual simplex method.

Optimum table:

		c_j	2	2	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4
2	y_1	$\frac{4}{7}$	1	0	$\frac{2}{7}$	$-\frac{3}{7}$
2	y_2	$\frac{12}{7}$	0	1	$-\frac{1}{7}$	$\frac{5}{7}$
		$z_j - c_j$	0	0	$\frac{2}{7}$	$\frac{4}{7}$

Optimal basic feasible solution of the related LPP: $x_1^* = \frac{4}{7}$, $x_2^* = \frac{12}{7}$, $Z_{\max} = \frac{32}{7}$.

Example

Observation. Neither of the decision variables are non-integer but we require both of them as integral solution.

Here $x_{B1} = x_1^* = \frac{4}{7} = 0 + \frac{4}{7}$, $W_{10} = 0$, $f_{10} = \frac{4}{7}$

$x_{B2} = x_2^* = \frac{12}{7} = 1 + \frac{4}{7}$, $W_{20} = 1$, $f_{20} = \frac{5}{7}$.

Since $f_{20} > f_{10}$ we take $x_{B2} = x_2^*$ to calculate cutting plane: $y_{23} = -\frac{1}{7} = -1 + \frac{6}{7}$, $W_{23} = -1$, $f_{23} = \frac{6}{7}$

$y_{24} = \frac{5}{7} = 0 + \frac{5}{7}$, $W_{24} = 0$, $f_{24} = \frac{5}{7}$.

$$\left(f_{r0} - \sum_{j \in N} f_{rj} x_j\right) \leq 0 \Rightarrow \frac{5}{7} - \left(\frac{6}{7}x_3 + \frac{5}{7}x_4\right) \leq 0 \Rightarrow -\frac{6}{7}x_3 + \frac{5}{7}x_4 \leq -\frac{5}{7} \Rightarrow -\frac{6}{7}x_3 - \frac{5}{7}x_4 + x_5 = -\frac{5}{7}.$$

This constraint is added to the previous optimum table and we apply dual-simplex method.

		c_j	2	2	0	0	0	
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	
2	y_1	$\frac{4}{7}$	1	0	$\frac{2}{7}$	$-\frac{3}{7}$	0	
2	y_2	$\frac{12}{7}$	0	1	$-\frac{1}{7}$	$\frac{5}{7}$	0	
0	y_5	$-\frac{5}{7}$	0	0	$-\frac{6}{7}$	$-\frac{5}{7}$	1	$\rightarrow$
		$z_j - c_j$	0	0	$\frac{2}{7}\uparrow$	$\frac{4}{7}$	0	

$$\max\left\{\frac{2/7}{-6/7}, \frac{4/7}{-5/7}\right\} = \max\left\{-\frac{1}{3}, -\frac{4}{5}\right\} = -\frac{1}{3}.$$

Example

y_5 leaves the basis and y_3 enters the basis.

		c_j	2	2	0	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5
2	y_1	$\frac{1}{3}$	1	0	0	$-\frac{2}{3}$	$\frac{1}{3}$
2	y_2	$\frac{11}{6}$	0	1	0	$\frac{5}{6}$	$-\frac{1}{6}$
0	y_3	$\frac{5}{6}$	0	0	1	$\frac{5}{6}$	$-\frac{7}{6}$
		$z_j - c_j$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$

From this table, we get a feasible and optimal solution but that too is non-integral. We again construct a new cutting plane following the same rule with $x_{B2} = x_2^*$ as the variable and the new constraint is

$$-\frac{5}{6}x_4 - \frac{5}{6}x_5 < -\frac{5}{6} \Rightarrow -\frac{5}{6}x_4 - \frac{5}{6}x_5 + x_6 = -\frac{5}{6}.$$

		c_j	2	2	0	0	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
2	y_1	$\frac{1}{3}$	1	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0
2	y_2	$\frac{11}{6}$	0	1	0	$\frac{5}{6}$	$-\frac{1}{6}$	0
0	y_3	$\frac{5}{6}$	0	0	1	$\frac{5}{6}$	$-\frac{7}{6}$	0
0	y_6	$-\frac{5}{6}$	0	0	0	$-\frac{5}{6}$	$-\frac{5}{6}$	$1 \longrightarrow$
		$z_j - c_j$	0	0	0	$\frac{1}{3}\uparrow$	$\frac{1}{3}$	0

y_6 leaves and y_4 enters the basis.

Example

		c_j	2	2	0	0	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
2	y_1	1	1	0	0	0	1	$-\frac{4}{5}$
2	y_2	1	0	1	0	0	-1	1
0	y_3	0	0	0	1	0	-2	1
0	y_4	1	0	0	0	1	1	$-\frac{6}{5}$
		$z_j - c_j$	0	0	0	0	0	$\frac{2}{5}$

Finally, we arrive at feasible and optimal integral solution as $\hat{x}_1 = 1$, $\hat{x}_2 = 1$, $Z_{\max} = 4$.

Branch and Bound Technique

5.5 BRANCH AND BOUNDS TECHNIQUE

This technique has the potential to become one of the most powerful procedures for solving a pure IPP. Consider a pure integer program defined in (5.3a). The followings are the computational steps to be followed in branch and bounds technique.

Step 1. By ignoring the integer restriction first solve the related LPP. If all x_j are integers, then we obtain the desired optimal solution, but if at least one x_j is not an integer, then go to next step.

Step 2. Select one of the non-integers x_j. Suppose, for some $j = k$, x_k is not an integer. Write $x_k = \omega_k + f_k$, where ω_k is an integral part and $0 < f_k < 1$ is the fractional part of x_k. According to this technique we add each of the constraints $x_k \leq \omega_k$ and $x_k \geq \omega_k + 1$ individually to the constraint set of the original problem and then solve the resulting two related sub-LPPs.

Step 3. For each of the two solutions thus obtained in *Step* 2 we face the following situations:

(i) If the solutions obtained in *Step* 2 for both the cases are feasible integer solutions, compare its objective functional value with the best feasible solution thus found. Save the better of the two. Select another non-integer x_j, if any, and return to *Step* 2.

(ii) If the resulting related sub-LPP has no feasible solution, then select another non-integer x_j, if any, and return to *Step* 2.

(iii) If the solution is not a feasible integer solution, then repeat the process of *Step* 2 on the constraint set of the related LPP.

Step 4. The process terminates when all possibilities in *Step* 2 have been exhausted.

Branch and Bound Technique

Example 5.5.1. *Use branch and bounds technique to solve the following* IPP.

Maximise $Z = x_1 + x_2$

subject to $8x_1 + 63x_2 \leq 273$

$$4x_1 - 3x_2 \leq 10$$

$x_1, x_2 \geq 0$ *and are integers.*

Solution: We begin by solving the related LPP, which will give (Fig. 1) $x_1 = 5\frac{1}{4}$, $x_2 = 3\frac{2}{3}$, $Z_{\max} = 8\frac{11}{12}$.

Since neither x_1 nor x_2 is an integer we choose either variable to begin the next step. We arbitrarily select x_1 to form two branches $x_1 \leq 5$ or $x_1 \geq 6$. These will create two nodes (2A) and (2B) corresponding to two new related sub LPP. The node (2B) has no feasible solution. Hence, no further branches coming out from this node. At node (2A), we have the following optimal feasible solution (Fig. 2): $x_1 = 5$, $x_2 = 3\frac{44}{63}$, $Z_{\max} = 8\frac{44}{63}$.

As x_2 is the only non-integral value, as before we form two branches $x_2 \leq 3$ and $x_2 \geq 4$ coming out from the node (2A) to form two new nodes (3A) and (3B) respectively. Solving the related sub-LPP corresponding to (3A) (Fig. 3) and (3B) (Fig. 4), we get the following optimal basic feasible solutions:

$$x_1 = 4\frac{3}{4}, \quad x_2 = 3, \quad Z_{\max} = 7\frac{3}{4} \quad \text{[from (3A)]}$$
$$x_1 = 2\frac{5}{8}, \quad x_2 = 4, \quad Z_{\max} = 6\frac{5}{8} \quad \text{[from (3B)]}.$$

Branch and Bound Technique

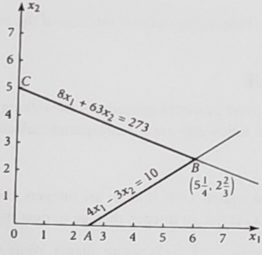


Fig. 1

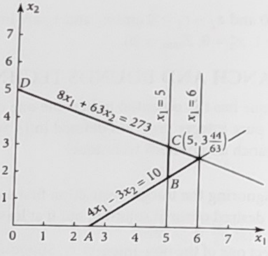


Fig. 2

Observe that node (3A) yields a larger value of Z than node (3B). Therefore, by this technique we use node (3A) since it has a better potential for approaching the ultimate optimal integral solution. At (3A) the only non-integer variable is x_1 so that again another two branches $x_1 \leq 4$ and $x_1 \geq 5$ are coming out from this node to form two new nodes (4A) and (4B) respectively. Upon solving the related sub-LPP, node (4B) indicates no feasible solution and node (4A) gives the ultimate all integral solutions (Fig. 5): $\hat{x}_1 = 4$, $\hat{x}_2 = 3$, $Z_{\max} = 7$.

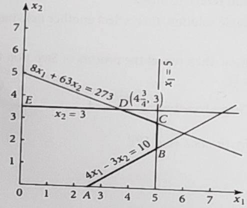


Fig. 3

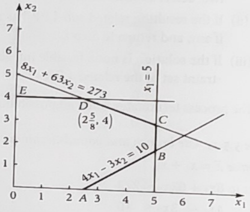


Fig. 4

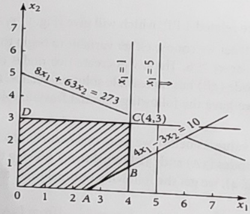
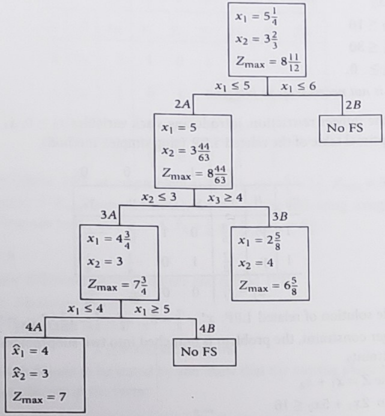


Fig. 5

Branch and Bound Technique

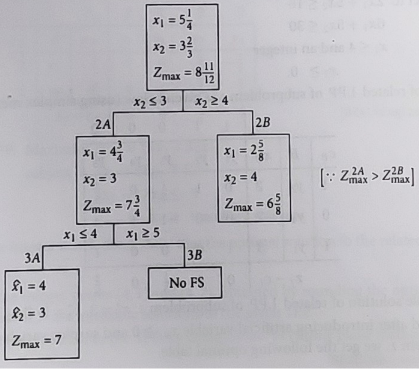
The above illustration can be represented in a diagram known as *tree solution*.



Branch and Bound Technique

Important Note

In the second step of the above example we arbitrarily select x_1 to form two branches $x_1 \leq 5$ and $x_2 \geq 6$. Instead of x_1 if one can take x_2 to form the branches, $x_2 \leq 3$ and $x_2 \geq 4$, then the corresponding *tree solution* is given below (In this technique there exists no particular rule to the above choice).



Branch and Bound Technique

Example 5.5.2. *Solve the following mixed IPP by branch and bounds technique.*

Maximise $Z = x_1 + x_2$

subject to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1, x_2 \geq 0.$

x_1 *is an integer but* x_2 *is not necessarily an integer.*

Solution: Neglecting the integer restriction, introducing slack variables $x_3 \geq 0$, $x_4 \geq 0$ to the constraints, we get the following optimal table of the related LPP (sing simplex method).

		c_j	1	1	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4
1	y_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$-\frac{1}{10}$
1	y_1	$\frac{7}{2}$	1	0	$-\frac{1}{4}$	$\frac{1}{4}$
		$z_j - c_j$	0	0	$\frac{1}{20}$	$\frac{3}{20}$

Optimal basic feasible solution of related LPP: $x_1^* = \frac{7}{2} = 3\frac{1}{2}$, $x_2^* = \frac{9}{5} = 1\frac{4}{5}$, $Z_{\max} = 5\frac{3}{10}$.

Since only x_1 is integer constraint, the problem is branched into two subproblems using one of the following additional constraints.

Subproblem 1. Maximise $Z = x_1 + x_2$

subject to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$0 \leq x_1 \leq 3$ and an integer

$x_2 \geq 0.$

Subproblem 2. Maximise $Z = x_1 + x_2$

subject to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \leq 4$ and an integer

$x_2 \geq 0.$

The optimal table of related LPP of subproblem 1 is given below (using simplex method).

		c_j	1	1	0	0	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5
1	y_2	2	0	1	$\frac{1}{5}$	0	$-\frac{2}{5}$
0	y_4	2	0	0	-1	1	-4
1	y_1	3	1	0	0	0	1
		$z_j - c_j$	0	0	$\frac{1}{5}$	0	$\frac{3}{5}$

Optimal basic feasible solution of related LPP of subproblem 1: $x_1^* = 3, x_2^* = 2, Z_{\max} = 5.$

Using Big-M method after introducing artificial variable $x_{a_1} \geq 0$ and surplus variable $x_5 \geq 0$ to the new constraint of subproblem 2, we get the following optimal table.

		c_j	1	1	0	0	0	$-M$
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5	y_{a_1}
0	y_3	2	1	0	1	1	-4	$\vdots$
1	y_2	$\frac{6}{5}$	0	1	0	$\frac{1}{5}$	$\frac{6}{5}$	$\vdots$
1	y_1	4	0	0	0	0	-1	$\vdots$
		$z_j - c_j$	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	

Optimal solution of the related LPP of subproblem 2: $x_1^* = 4$, $x_2^* = \frac{6}{5} = 1\frac{1}{5}$, $Z_{\max} = 5\frac{1}{5}$.

Since $Z_{\max}$ (subproblem 2) $> Z_{\max}$ (subproblem 1), we accept the following integral optimal solution according to branch and bounds technique: $\hat{x}_1 = 4$, $\hat{x}_2 = \frac{6}{5} = 1\frac{1}{5}$, $Z_{\max} = 5\frac{1}{5}$.