Transportation Problem

A transportation problem (abbreviated as TP) is a special type of LPP, where a **single commodity** is stored at different warehouses (origins) is to be transported to different distribution centers (destinations) in such a way that the transportation cost is minimum.

Consider a particular example: Let there are $\boldsymbol{m}$ **origins (sources)** $\boldsymbol{O_1}, \boldsymbol{O_2}, \cdots, \boldsymbol{O_m}$ and the quantity **available at origin** $\boldsymbol{O_i}$ **be** $\boldsymbol{a_i}(i = 1,2, \dots, m)$ and let there be $\boldsymbol{n}$ **destinations** $\boldsymbol{D_1}, \boldsymbol{D_2}, \cdots, \boldsymbol{D_n}$ and the quantity required i.e., the **demand at** $\boldsymbol{D_j}$ **be** $\boldsymbol{b_j}$ $(j = 1,2, \dots, n)$.

Let us make an assumption that

$$\sum_{i=1}^{m} a_i = M = \sum_{j=1}^{n} b_j.$$

…………………..(1)

This assumption is not restrictive. In a particular problem when

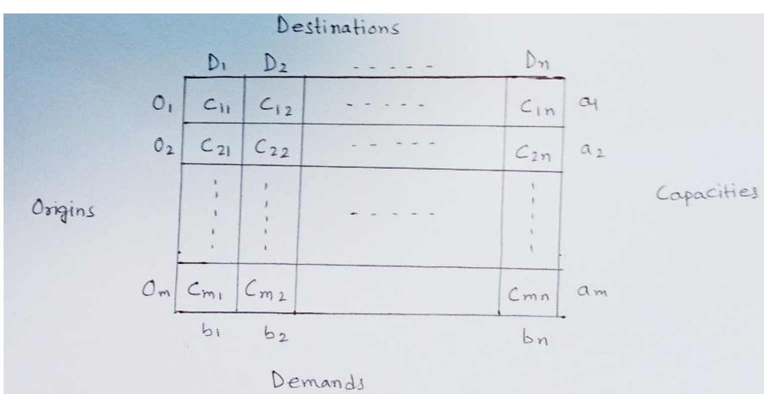
$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

i.e., the total availability is equal to the demand, it is called as balanced transportation problem and when

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

It is called as unbalanced transportation problem.

A transportation problem is completely defined by a tableau of the following type:



c_{ij} := cost of transporting one unit of commodity from origin O_i to the destination D_j is a known quantity. It is assumed in general that $c_{ij} \geq 0$. **The problem before us is to determine the quantity $\boldsymbol{x_{ij}}$ [$\boldsymbol{i = 1, 2, \dots, m}$; $\boldsymbol{j = 1, 2, \dots, n}$] which is to be transported from $\boldsymbol{i-th}$ source to $\boldsymbol{j-th}$ destination such that the transportation cost is minimum** provided the condition (1) is satisfied.

Mathematically, the problem can be written as,

$$minimize, \qquad z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad \dots\dots\dots\dots (2)$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2,\dots,m \dots\dots\dots\dots (3)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2,\dots,n \dots\dots\dots (4)$$

and

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

From the above diagram, the constraints (2) and (3) can be written easily. The sum of the variables of the $i-th$ row is equal to a_i and the sum of the variables of the $j-th$ column is equal to b_j. It is obvious that $x_{ij} \geq 0$ for all i and j.

Therefore, the problem is a minimization problem and $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$ is the objective function which is to be minimized and there are $(m+n)$ constraints of which all the equations of mn variables x_{ij}. Since, in general in a LPP the number of variables are greater than the number of constraints, therefore both m and n must be ≥ 2.

Standard L.P. Form:

A general transportation problem involving m origins and n destinations can be stated as below.

$$minimize \;\; z = \boldsymbol{cX}$$

$$subject\ to\ \boldsymbol{AX} = \boldsymbol{b}, \boldsymbol{X} \geq 0$$

where $\boldsymbol{c} = (c_{11}, c_{12}, \dots, c_{ij}, \dots, c_{mn})$ is an mn-component row vector,

$\boldsymbol{X} = [x_{11}, x_{12}, \dots, x_{ij}, \dots, x_{mn}]$ is an mn-component column vector,

$\boldsymbol{b} = [a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n]$ is an $(m+n)$-component column vector,

$\boldsymbol{A} = (\boldsymbol{a_{11}}, \boldsymbol{a_{12}}, \dots, \boldsymbol{a_{ij}}, \dots, \boldsymbol{a_{mn}})$ is the co-efficient matrix in which $\boldsymbol{a_{ij}}$ is the column vector associated with the variable x_{ij}.

Theorem 1. The transportation problem always has a feasible solution which is given by,

$$x_{ij} = \frac{a_i b_j}{M} \ \ (i = 1,2, \dots, m;\ j = 1,2, \dots, n)$$

where $M = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Proof: Since all a_i and b_j are non-negative quantities therefore $x_{ij} \geq 0$ for all i and j. Observe that,

$$\sum_{j=1}^{n} x_{ij} = \frac{a_i \sum_{j=1}^{n} b_j}{M} = \frac{a_i M}{M} = a_i$$

and

$$\sum_{i=1}^{m} x_{ij} = \frac{b_j \sum_{i=1}^{m} a_i}{M} = \frac{b_j M}{M} = b_j$$

which satisfy the constraints (3) and (4). Hence in each T.P. there exists a feasible solution and $x_{ij} = \frac{a_i b_j}{M}$ $(i = 1,2, \dots, m;\ j = 1,2, \dots, n)$.

Theorem 2. In a balanced T.P. having m origins and n destinations $(m, n \geq 2)$ the exact number of basic variables is $m + n - 1$.

Proof: The balanced transportation problem is

$$minimize, \qquad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2, \dots, m \dots\dots\dots (5)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2, \dots, n \dots \dots \dots (6)$$

and

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

There are $m + n$ linear constraints with mn variables and $mn > m + n - 1$ $(as\, m, n \geq 2)$.

From (5) observe that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \dots \dots \dots (7)$$

Now summing the first $(n - 1)$ constraints of (6), we get

$$\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \dots \dots \dots (8)$$

Now subtracting (8) from (7) we get,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j = b_n.$$

Thus, we get,

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}\right) = b_n, \quad i.e., \sum_{i=1}^{m} x_{in} = b_n$$

which is the last or $n - th$ constraints of (5). Therefore, there are only $(m + n - 1)$ linearly independent equations with mn variables. Thus, from the definitions of the basic solution, we can say that the number of basic variables is exactly $(m + n - 1)$.

Remark: All basic variables may not be positive, some of them may be zero. When all basic variables are positive, the solution is called a non-degenerate B.F.S. When at least one basis variable is zero, the solution is called a degenerate B.F.S.

The number of basic cells will be exactly $(m + n - 1)$ all of which contain $(m + n - 1)$ variables which are either all positive basic variables or some variables may be 0 (we will consider this case later).

Methods for Finding an Initial B.F.S.

So long we have discussed some fundamental properties of a transportation problem next, we will determine the initial B.F.S. of the problem and from this we proceed to find another B.F.S. which will improve the value of the objective function. There are various methods of finding an initial B.F.S. The methods are

1. North-west corner rule
2. Row minima method
3. Column minima method
4. Matrix minima method
5. Vogel's approximation method (VAM)

North-west corner rule:

Various steps of this methods are as follows:

Step 1.: Select the north-west corner cell of the transportation table and allocate as much as possible so that either the capacity (supply) of the first row is exhausted or the demand of the first column is fulfilled, i.e., $x_{11} = \min\{a_1, b_1\}$.

Step 2.

(i) If $b_1 > a_1$, move down vertically to the second row and make the second allocation $x_{21} = \min\{a_2, b_1 - x_{11}\}$ in the cell (2, 1).

(ii) If $b_1 < a_1$, move right horizontally to the second column and make the second allocation $x_{12} = \min\{a_1 - x_{11}, b_1\}$ in the cell (1, 2).

(iii) If $b_1 = a_1$, there is a tie for the second allocation and so one can make the second allocation $x_{12} = \min\{a_1 - a_1, b_1\} = 0$ in the cell (1, 2) or $x_{21} = \min\{a_2, b_1 - b_1\}$ in the cell (2, 1). In this case the solution will be degenerate.

Step 3. Repeat steps 1 and 2 and moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

Notes: 1. The cells which get allocation will be called basic cells.

2. The feasible solution obtained by this method is always a BFS and may be far from optimum solution. Since in this case costs were completely ignored.

Example. Find the initial basic feasible solution of the following transportation problem by North-West corner method:

1.

	D_1	D_2	D_3	D_4	a_i
O_1	4	6	9	5	16
O_2	2	6	4	1	12
O_3	5	7	2	9	15
b_j	12	14	9	8	

2.

	D_1	D_2	D_3	D_4	
O_1	2	5	4	7	4
O_2	6	1	2	5	6
O_3	4	5	2	4	8
	3	7	6	2	

Solution:

1. Since $\sum_i a_i = \sum_j b_j = 43$, so this is a balanced transportation problem. Therefore, a basic feasible solution to the given transportation problem will exists. Following the North-West corner method, the initial BFS is displayed in the following table:

	D_1	D_2	D_3	D_4	
O_1	$\boxed{12}$ 4	$\boxed{4}$ 6	9	5	~~16~~ ~~4~~
O_2	2	$\boxed{10}$ 6	$\boxed{2}$ 4	1	~~12~~ ~~2~~
O_3	5	7	$\boxed{7}$ 2	$\boxed{8}$ 9	~~15~~ 8
	~~12~~	~~14~~ ~~10~~	~~9~~ ~~7~~	8	

Explanation:

1. BFS is displayed in the above table. $Min(16, 12) = 12$. Therefore $x_{11} = 12$ and allocate it in the cell (1, 1). The demand of D_1 is satisfied and hence all other cells in first column remain vacant. Now, as $b_1 = 12 < 16 = a_1$, therefore next allocation will be in cell (1, 2) and $x_{12} = \min(16 - 12,\ 14) = 4$. Now the capacity of O_1 is exhausted. Next allocation will be in cell (2, 2) and $x_{22} = \min(12, 14 - 4) = 10$. Proceeding similarly we get $x_{23} = 2, x_{33} = 7$ and $x_{34} = 8$ and all the rim requirement are satisfied. The solution obtained is a BFS because the set of cells (which contain components of feasible solution i.e. basic cells) do not form a loop and the number of basic variables is $6 = m + n - 1$. The cost due to this assignment is $(4 \times 12) + (6 \times 4) + (6 \times 10) + (4 \times 2) + (2 \times 7) + (9 \times 8) = 226$ units.
2. Since $\sum_i a_i = \sum_j b_j = 18$ so this is a balanced transportation problem. Therefore, a basic feasible solution to the given transportation problem will exists. Following the North-West corner method, the initial BFS is displayed in the following table:

$\boxed{3}$ 2	$\boxed{1}$ 5	4	7	$\cancel{4}$ 1
6	$\boxed{6}$ 1	$\boxed{0}$ 2	5	6
4	5	$\boxed{6}$ 2	$\boxed{2}$ 4	8
3	$\cancel{7}$ 6	6	2	

BFS is displayed in the above table. $\min(4,3) = 3$. Therefore $x_{11} = 3$ and allocate it in the cell $(1, 1)$. The demand of D_1 is satisfied and hence all other cells of first column remain vacant. As $b_1 = 3 < 4 = a_1$, therefore next allocation will be in the cell $(1, 2)$ and $x_{12} = \min(4 - 3, 7) = 1$. Now the capacity of $O_1$is exhausted. Next allocation will be in cell (2,2) $x_{22} = \min(6,7 - 1) = 6$. Now the capacity of $O_3$is exhausted and demand of D_2 is satisfied simultaneously. Therefore either x_{23} or x_{32} will be zero. Let us take $x_{23} = 0$ and proceed similarly until all rim requirement are satisfied.

Row Minima Method:

Step 1. Select the smallest cost in the first row. Let it be c_{1j}; compute $\min\left(a_1, b_j\right)$. Set $x_{1j} = \min(a_1, b_j)$ and allocate it in the cell $(1, j)$. This is the maximum feasible amount which can be allocated in the cell $(1, j)$. If the smallest cost not unique select any one of the minimum costs arbitrarily.

Step 2. It $a_1 < b_j$, the capacity of the origin O_1 will be exhausted. But the demand of the destination D_j remains unsatisfied. Cross off the first row and diminish b_j by a_1. Proceeding similarly, allocate the maximum feasible amount in the cells of the remaining rows starting from the second until all rim requirements are satisfied.

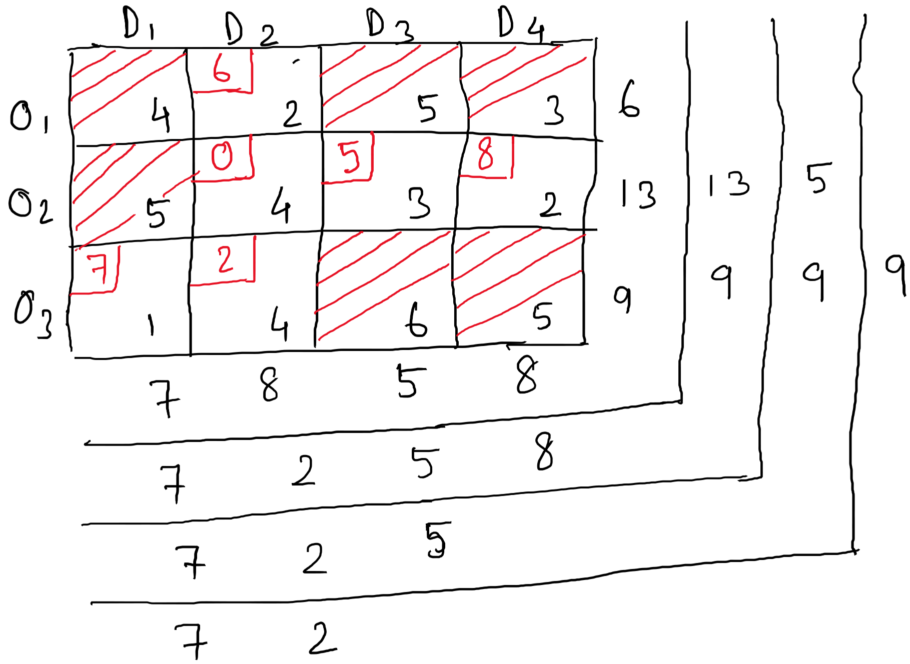
If $b_j < a_1$, the total demand of the destination D_j is satisfied but the capacity of the origin O_1 is not exhausted completely. Cross off the $j-th$ column and diminish a_1 by b_j. Reconsider the first row and select the next smallest cost of this row. Let it be c_{1k}. Compute, $\min(a_1 - b_j, b_k)$ and allocate it in the cell $(1, k)$. Repeat the above procedure for the second row and so on as in the above method until all requirements are satisfied.

Step 3. If $a_1 = b_j$. Set $x_{1j} = a_1 = b_j$ and allocate it in the cell $(1,\ j)$. Due to this allocation, the capacity of the origin O_1 will be exhausted as well as the demand of D_j. In that case solution will be degenerate. Set $x_{1k} = 0$ and display it in thr cell $(1, k)$ with the assumption, that the cost of $(1, k)$ cell is the next minimum cost. Now cross off both the first row and $j-th$ column and procedure similarly.

Example 2. Find out an initial BFS of the following balanced transportation problem using row minima method.

	D_1	D_2	D_3	D_4	
O_1	4	2	5	3	6
O_2	5	4	3	2	13
O_3	1	4	6	5	9
	7	8	5	8	

Solution: It is displayed in the transportation table given below:



Explanation. Lowest cost in first row is $c_{24} = 2$. $\min(a_1, b_2) = \min(6{,}8) = 6$. Set $x_{12} = 6$ and allocate it in cell (1,2), $a_1 < b_2$; therefore the capacity of O_1 is exhausted completely and hence cross off the first row and diminish b_2 by $a_1$which is shown in the above table. Give a shade on the first row and ignore it for future computation.

Lowest cost in the second row is $c_{24} = 2$. $\min(a_2, b_4) = \min(13{,}8) = 8$. Set $x_{24} = 8$ and allocate it in the cell (2,4); $b_4 < a_2$. Therefore the capacity of O_2 will not be exhausted; give a shade on the forth column and ignore the column for future computation since the demand of D_4 is satisfied completely. Cross off the fourth column and diminish a_2 by b_4. Reconsider the second row. The next lowest cost in the row is $c_{23} = 3$. $\min(a_2 - b_4, b_2) = \min(13 - 8, 5) = 5$. Set $x_{23} = 5$ and allocate it in the cell (2,3). As $a_2 - b_4 = 5 = b_3$, the capacity of O_2 is exhausted as well as the demand of D_3 is satisfied completely. Therefore, the solution will be degenerate. The next lowest cost of the second row is $c_{22} = 4$. Set $x_{22} = 0$ and allocate it in the cell (2,2). Cross off the second row and third column simultaneously. Now complete the table and all the rim requirements are satisfied now.

Column Minima Method: The technique used in the column minima method is same as in that in the row minima method.

Matrix Minima Method or Cost Minima Method:

Step 1. Select the smallest cost in the cost matrix. Let it be c_{ij}. Set $x_{ij} = \min(a_i, b_j)$ and allocate it in the cell (i, j). This is the maximum feasible amount that can be allocated in the cell (i, j).

Step 2. If $\boldsymbol{a_i < b_j}$, the capacity of the origin O_i will be exhausted completely. **Cross off the $\boldsymbol{i-th}$ row and diminish $\boldsymbol{b_j}$ by $\boldsymbol{a_i}$.**

If $\boldsymbol{b_j < a_i}$, the demand of the destination D_j will be satisfied completely. **Cross off the $\boldsymbol{j-th}$ column and diminish $\boldsymbol{a_i}$ by $\boldsymbol{b_j}$.**

Step 3. If $\boldsymbol{a_i = b_j}$, the capacity of the origin O_i will be exhausted and the demand of D_j will be satisfied simultaneously. **Set $\boldsymbol{x_{ij} = a_i = b_j}$; allocate it in the cell $\boldsymbol{(i, j)}$. Cross off either the $\boldsymbol{i-th}$ row or the $\boldsymbol{j-th}$ column but not the both**. Of course, we may drop both $i - th$ row or the $j - th$ column by inserting a basic variable 0 at a cell corresponding to the lowest cost of those row and column.

Step 4. Apply the same technique in the reduced transportation table until all rim requirements are satisfied. **At any stage, if the minimum cost is not unique, make an arbitrary choice among the minima**.

Example 3. Determine an initial BFS to the following balanced transportation problem using matrix minima method:

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
b_j	16	18	31	25	

Explanation: BFS is given in the following table.

	D_1	D_2	D_3	D_4				
O_1	5	$\boxed{18}$ 3	$\boxed{1}$ 6	2	19	19	1	1
O_2	4	7	$\boxed{12}$ 9	$\boxed{25}$ 1	37	12	12	12
O_3	$\boxed{16}$ 3	4	$\boxed{18}$ 7	5	34	34	34	18
	16	18	31	25				
	16	18	31					
	16		31					
			31					

The smallest cost is $c_{24} = 1$. Set $x_{24} = \min(a_2, b_4) = \min(37,25) = 25$ and allocate it in the cell (2,4). $b_4 < a_2$, therefore, cross off fourth column and ignore it for future computation.

The smallest cost in the reduced table is c_{12} or c_{31}. Let us select $c_{12} = 3$ as the smallest cost and allocate $x_{12} = \min(a_1, b_2) = \min(19, 18) = 18$ in the cell (1,2), $b_2 < a_1$, therefore cross off the second column; give shade on the 2nd column and diminish a_1 by b_2. Proceed similarly until all rim requirement are satisfied.

Vogel's Approximation Method:

Step 1. Select the lowest and next to lowest cost for each row and determine the difference between them for each row and display them the first bracket against the respective rows. If there are two or more with same lowest costs, difference may be taken to be zero. Compute, similarly, the difference for each column and display them within the bracket against the respective columns.

Step 2. Find the largest value of the differences and find out the row or column for which the difference is maximum. Let the maximum difference corresponding to $i-th$ row. Select the lowest cost in the $i-th$ row. Let it be c_{ij}. Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i,j) which is the maximum feasible amount that can be allocated in the cell (i,j). If the maximum difference is not unique, select any one of them.

Step 3. If $a_i < b_j$, cross off the $i-th$ row and diminish b_j by a_i.

If $b_j < a_i$, cross off the $j-th$ column and diminish a_i by b_j.

If $a_i = b_j$ allocate $x_{ij} = a_i = b_j$ in cell (i, j) and cross off either $i - th$ row or $j - th$ column but not the both. Of course, we can omit both the $i - th$ row and $j - th$ column simultaneously by inserting a basic variable 0 to one of the cell of the corresponding row or column possessing the next minimum cost and the solution will be degenerate then.

Step 4. Re-compute the row and column differences for the reduced transportation table. Repeat the procedure discussed above until all rim requirements are satisfied.

Example 4. Obtain an initial BFS to the balanced TP given below using Vogel's approximation method.

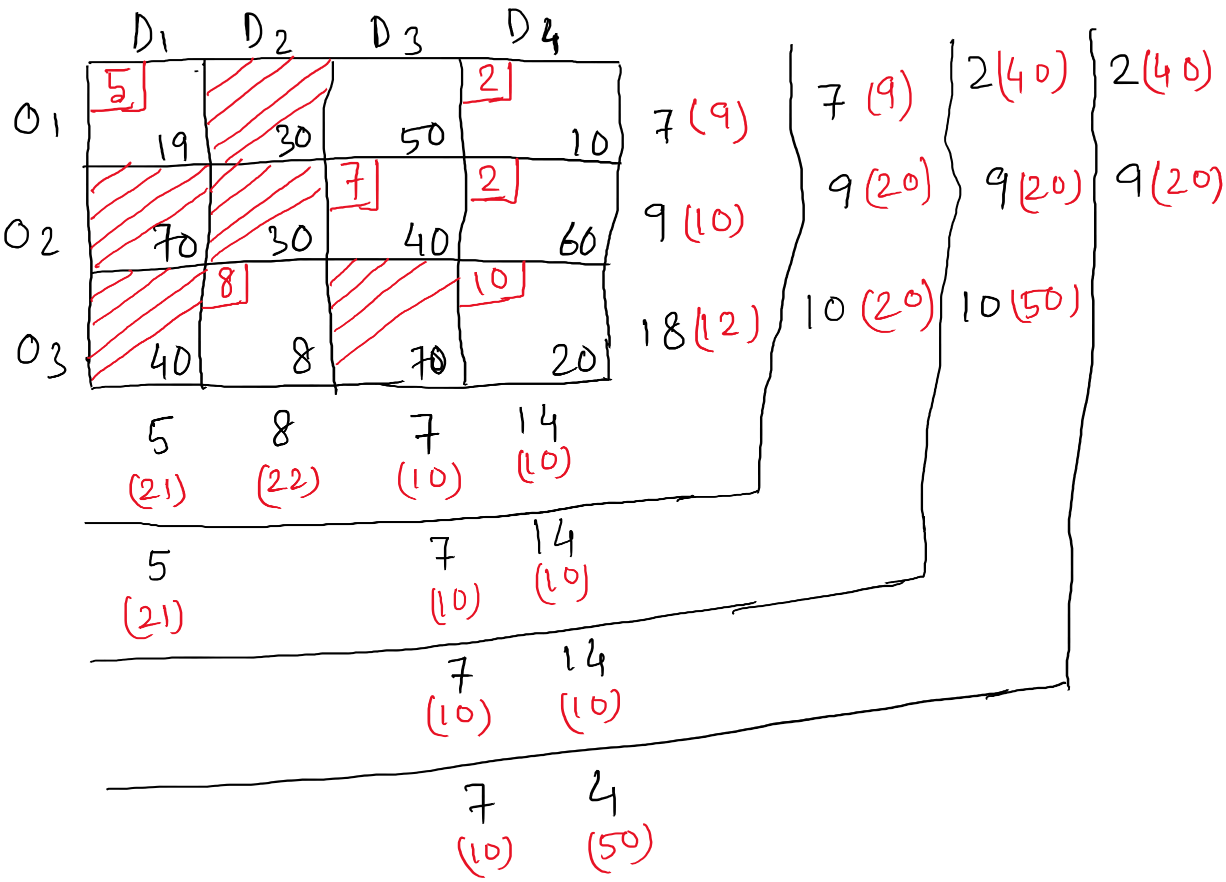
	D1	D2	D3	D4	
O1	19	30	50	10	7
O2	70	30	40	60	9
O3	40	8	70	20	18
	5	8	7	14	

Solution: Here the initial basic feasible solution using by VAM is being shown in a single table in a very compact manner which will save time and labour.

Explanation: Step1. Select the lowest and next to lowest cost for each row and each column and determine the difference between them for each row and column and display them within the first bracket against the respective rows and column. Maximum difference is 22 which occurs at the second column and the minimum cost of that column is $c_{32} = 8$. Allocate $\min(18,8) = 8$ in the cell (3,2). The demand of D_2 has been satisfied and shade the second column as shown in the table below. The resulting cost matrix will be obtain after deleting the cost components of the second column.

Step 2. Appling the same technique in the resulting matrix, the capacities, demands and the differences of the cost components have been shown in the second compartment. Maximum differences is 21 which occurs in the first column and the lowest cost of that column is $c_{11} = 19$. Allocate $\min(7,5) = 5$ in the cell (1,1). The demand of D_1 has been satisfied and shade the first column as shown in the table. The resulting cost matrix will be obtained deleting the cost components of the first column.

Step 3. Proceeding in the same way, we get the maximum difference 50 which occurs in the third row and minimum cost is $c_{34} = 20$. Allocate $\min(10,14) = 10$ in the cell (3,4) and the capacity of O_3 will be exhausted and the resulting matrix will be obtained deleting the cost components of the third row; shade the third row as shown in the table. Using the same technique, ultimately we obtain the initial BFS where all capacities have been exhausted and all the demands will be met.



Optimality Conditions

So long we have discussed the methods for determining the initial BFS. The next problem to us is to find whether the solution obtained, is optimal or not. In the usual simplex method, we stop when all $z_j - c_j \geq 0$ for a maximization problem. So here we need to find the analogue of $z_j - c_j$ for transportation case. For this we shall male use of the duality theory.

The original transportation problem is

$$minimize, \qquad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2,\dots,m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2,\dots,n.$$

where there are $(\boldsymbol{m+n})$ constraints all of which are equations, out of them only $(\boldsymbol{m+n-1})$ equations are independent. Hence there are $(\boldsymbol{m+n})$ dual variables to the primal problem of which one can be selected arbitrarily and all variables are unrestricted in sign.

So, the dual is

$$maximize \quad w = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

Subject to

$$u_i + v_j \le c_{ij}, \qquad i = 1,2,\dots,m, j = 1,2,\dots,n$$

where u_i and v_j are unrestricted in sign.

Let $\{x_{ij}\}$ be a feasible solution of the primal problem and $\{(u_i, v_j), i = 1,2,\dots,m, j = 1,2,\dots,n\}$ be a feasible solution of the dual problem, then for optimality

$$\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$i.e., \quad \sum_{i=1}^{m} u_i \left(\sum_{j=1}^{n} x_{ij}\right) + \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{m} x_{ij}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$i.e., \quad \sum_{i=1}^{m}\sum_{j=1}^{n} (u_i + v_j - c_{ij}) x_{ij} = 0.$$

Observe that $x_{ij} \ge 0$ and $u_i + v_j \le c_{ij}$, so, $(u_i + v_j - c_{ij})x_{ij} = 0$ for all i and j.

Therefore for $x_{ij} > 0$ we have $(u_i + v_j - c_{ij}) = 0$, otherwise $(u_i + v_j - c_{ij}) \le 0$ as u_i, v_j are certainly dual feasible. But $x_{ij} > 0$ means that the (i,j)th cell is a basic cell. Hence to check the optimality of the current basic feasible solution, **we first determine $\boldsymbol{u_i}$ and $\boldsymbol{v_j}$ in such a way that for each basic cell $\boldsymbol{u_i + v_j = c_{ij}}$**. Then we will calculate $(u_i + v_j - c_{ij})$ for the non-basic cell. If

$\boldsymbol{(u_i + v_j - c_{ij}) \leq 0}$ **for all each non-basic cell, then the current basic feasible solution is optimal**.

Determination of $\boldsymbol{u_i}$ and $\boldsymbol{v_j}$:

For the basic cell $u_i + v_j = c_{ij}$. As mentioned earlier one dual variable can be select arbitrarily as there are only $(m + n - 1)$ independent equations. Conventionally, we have select the variable equals to 0 in which row or column the number of allocation is maximum.

Loop:

In a transportation table, an order set of four or more cells are said to form a loop (i) if and only if two consecutive cells in the ordered set lie either in the same row or in the same column and if (ii) the first and last cell of the set also lie either in the same row or in the same column.

Computational Procedure

Assume that we have a transportation table with the origin availability a_i and the destination requirements b_j and the costs c_{ij} in the cell (i, j).

Step 1. First find out a basic feasible solution of the given transportation problem by any one of the methods discussed previously. Without loss of generality let us consider that the solution is a non-degenerate solution.

Step 2. Calculate u_i and v_j's for $i = 1, \dots, m, j = 1, \dots, n$ such that $u_i + v_j = c_{ij}$ for all the basic cell.

In practice this is achieved by choosing arbitrarily any one of the u_i or v_j equal to zero. This choice is made for that u_i or v_j for which the corresponding row or column contains maximum number of occupied cells.

Step 3. Calculate the cell evaluations for the unoccupied cells using the formula $\Delta_{ij} = (u_i + v_j - c_{ij})$.

Three cases can occur

(i) If $\Delta_{ij} < 0$ then the solution is optimal and unique

(ii) If $\Delta_{ij} < 0$ with at least one $\Delta_{ij} = 0$ then the solution is optimal but not unique

(iii) If at least one $\Delta_{ij} > 0$, then the solution is not optimal and we are to seek a new basic feasible solution. Select the non-basic cell having the smallest net evaluation to enter the basis.

Step 4. Let the non-basic cell (r, s) enter the basis. Construct a loop connecting the non-basic cell (r, s) and some of the basic cells. Now allocate a value $\theta > 0$ in the cell (r, s) and readjust the basic variables in ordered set of cells forming the loop by adding and subtracting θ alternately such that all rim requirements are satisfied. Now select the maximum value of θ in such a way that at least one of the values of the readjusted variables vanishes and the variables remain non-negative in the other cells. The basic cell whose allocation has been reduced to zero, leaves the basis.

Step 5. Return to step 2 and repeat the process until an optimal basis feasible solution has been obtained.

Example: Obtain the optimum basic feasible solution to the transportation problem given in Example 3 and find out the corresponding cost of transportation.

Solution. The initial BFS is given in the solution of Example 3 using matrix minima method.

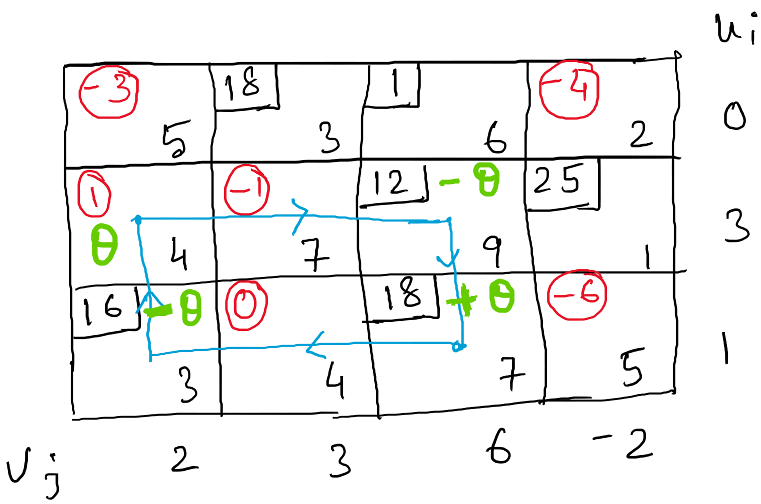


Table-1

We have to test first whether the solution is optimal or not. For this we shall have to calculate the net evaluations for all non-basic cells. The calculated values are displayed within a circle of respective non-basic in Table 1.

During calculation of the net evaluations we have taken arbitrarily $u_1 = 0$ and have calculated all net evaluations. All net evaluations corresponding to the non-basic cells are not less than or equal to zero. Net evaluation corresponding to non-basic cell (2,1) is 1. Therefore, the solution not optimal.

Since there is only one cell (2,1) for which net evaluation is positive therefore the vector a_{21}corresponding to the cell (2,1) is the entering vector and cell (2,1) will be the entering cell.

Construct a loop connecting the cell (2,1) and the set of basic cells or any subset of the basic cells of the original solution. In this problem the ordered set of cells (2,1), (2,4), (3,3), (3,1) are said to form a loop. Now allocate a value $\theta > 0$ in the cell (2,1) and readjusted the basic variables in the ordered set of cells forming the loop, by adding and subtracting θ alternately as given in the Table 1, such that all rim requirements are satisfied properly. Now select the maximum value of θ in such a way that the values of the readjusted variables vanish at least in one cell containing the loop and variables remain non-negative in other cells.

Table - 2

					u_i
	-3, 5	$\boxed{18}$, 3	$\boxed{1}$, 6	-3, 2	0
	$\boxed{12}$, 4	-2, 7	-1, 9	$\boxed{25}$, 1	2
	$\boxed{4}$, 3	0, 4	$\boxed{30}$, 7	-5, 5	1
v_j	2	3	6	-1	

From the table, it is clear that value of $\theta = 12$ and for that the value of variable in the cell (2,3) is zero. Therefore the cell (2,3) will leave the set of basic cells and the vector a_{23}will leave the basis. With this known value $\theta = 12$ construct the new transportation table 2. All basic variables in the cells not in the loop remain unchanged and again calculate the net evaluations corresponding to the non-basic cells. Calculated values are displayed in table 2. All calculations are performed with the assumption that $u_1 = 0$. Here all net evaluations are non-positive quantities. Therefore, the solution obtained is optimal. The optimal solution is given by $x_{12} = 18, x_{13} = 1, x_{21} = 12, x_{24} = 25, x_{31} = 4$, and $x_{33} = 30$ and the corresponding cost of transportation is given by

$$(3 \times 18) + (6 \times 1) + (4 \times 12) + (1 \times 25) + (3 \times 4) + (7 \times 30) = 355$$

Note. As at the optimal stage, net evaluation, corresponding to the non-basic cell (3,2) is zero, then alternative optimal solution exists.

Example: Determine the optimal solution to the problem given in example 4 and find the minimum cost of transportation.

Solution. First we have to test whether the initial solution is optimal or not. For this we shall have to calculate the net evaluations corresponding to all non-basic cells. The calculated values are displayed in the following table. During the computation of net evaluation we have taken arbitrarily $u_1 = 0$. All net evaluations corresponding to non-basic cells are not less than or equal to zero. Net evaluation corresponding to the non-basic cell (2, 2) is 18. Therefore, the solution is not optimal. We shall have to construct the next table to get an optimal solution.

Since there is only one cell (2, 2) for which net evaluation is positive, therefore the vector a_{22}, corresponding to the basic cell (2, 2) is he entering cell. This cell (2, 2) will be a basic cell in the next iteration.

Construct a loop connecting the cell (2, 2) and the set of basic cells or any subset of basic cells. In this problem the ordered set of cells (2, 2), (2, 4), (3, 4) and (3, 2) are said to form a simple loop (ignore the intermediate basic cell (2, 3)). Now allocate a value $\theta > 0$, a variable in the cell (2, 2) and readjust the basic variables in the ordered set of cells forming a simple loop by adding and subtracting θ alternatively as given in the table, such that all rim requirement are satisfied properly. Now select the maximum value of θ in such a way that the values of the readjusted variables vanish at least in one cell containing the loop and variables remain non-negative in other cells. From the table it is clear that $\theta = 2$ and for that the value of the variable in the cell (2, 4) is zero. Therefore the cell (2, 4) will leave the set of basic cells and the vector a_{24} will leave the basis in the next iteration.

					u_i
	$\boxed{5}$ 19	(−32) 30	(−60) 50	$\boxed{2}$ 10	0
	(−1) 70	(18) θ 30	$\boxed{7}$ 40	$\boxed{2}$ $-\theta$ 60	50
	(−11) 40	$\boxed{8}$ $-\theta$ 8	(−70) 70	$\boxed{10}$ $+\theta$ 20	10
v_j	19	−2	−10	10	

With the known value $\theta = 2$ construct the new transportation table and again calculate the net evaluations corresponding to non-basic cells. Calculated values are displayed in the table below and all calculations are non-positive quantities. Hence the solution obtained is optimal. The optimal solution is given by $x_{11} = 5, x_{14} = 2, x_{22} = 2, x_{23} = 7, x_{32} = 6$ and $x_{34} = 12$ and the corresponding cost of transportation is given by (19x5)+(10x2)+(30x2)+(40x7)+(8x6)+(20x2)= 743.

$\boxed{5}$ 19	(−32) 30	(−42) 50	$\boxed{2}$ 10	0
(−19) 70	$\boxed{2}$ 30	$\boxed{7}$ 40	(−18) 60	32
(−11) 40	$\boxed{6}$ 8	(−52) 70	$\boxed{12}$ 20	10
19	−2	8	10	

Example. Obtain the initial BFS to the following transportation problem by matrix minima method and then find out an optimal solution and the corresponding cost of transportation.

	D_1	D_2	D_3	D_4	
O_1	5	4	6	14	15
O_2	2	9	8	6	4
O_3	6	11	7	13	8
	9	7	5	6	

Solution: Initial BFS calculated with the help of matrix minima method is given in table 1. Now calculate all net evaluations corresponding to the non-basic cells with the assumption $u_1 = 0$. All net evaluations are not non-positive. Hence the solution is not optimal.

Cell (2, 4) has the positive net evaluation 3. Thus in the next iteration, cell (2, 4) will be the new basic cell. Construct the loop as shown in the table 1. Loop is simple and unique. Insert the value $\theta > 0$ in the cell (2, 4) and readjust the basic variables in the cells containing the loop accordingly as given in the table 1.

					u_i
	$\boxed{5}+\theta$ 5	$\boxed{7}$ 4	$\boxed{3}-\theta$ 6	-2 14	0
	$\boxed{4}-\theta$ 2	-8 9	-5 8	3 θ 6	-3
	0 6	-6 11	$\boxed{2}+\theta$ 7	$\boxed{6}-\theta$ 13	1
v_j	5	4	6	12	

Now the maximum value of θ will be 3 and the cell (1, 3) will leave basic cell and all other variables remain non-negative. Construct the table 2 with the value $\theta = 3$. Calculate all net evaluations corresponding to the non-basic cells with the assumption $u_1 = 0$. Cell (3, 1) will be the new basic cell. Construct a simple loop with cell (3, 1) as shown in the table 2. Insert the value $\theta > 0$ in the cell (3, 1) and readjust the basic variables as shown in the table 2. Maximum value of

$\theta = 1$, cell (2, 1) will leave the set of basic cells and all other variables remain non-negative. With $\theta = 1$ construct the table 3.

$\boxed{8}$ 5	$\boxed{7}$ 4	-3 6	-5 14	0
$\boxed{1}$ $-\theta$ 2	-8 9	-8 8	$\boxed{3}$ $+\theta$ 6	-3
3 θ 6	-3 11	$\boxed{5}$ 7	$\boxed{3}$ $-\theta$ 13	4
5	4	3	9	

u_i

$\boxed{8}$ 5	$\boxed{7}$ 4	0 6	-2 14	-1
-3 2	-11 9	-8 8	$\boxed{4}$ 6	-7
$\boxed{1}$ 6	-6 11	$\boxed{5}$ 7	$\boxed{2}$ 13	0
6	5	7	13	

Calculate all net evaluations corresponding to the non-basic cells in the table 3 with the assumption $u_2 = 0$. All net evaluations are non-positive. Hence the solution is optimal and the optimal solution is $x_{11} = 8, x_{12} = 7, x_{24} = 4, x_{31} = 1, x_{33} = 5, x_{34} = 2$ and the minimum cost of transportation is 159.

Solution to a degenerate problem

Degeneracy may occur at the initial stage or at any subsequent iteration. Here we shall discuss a problem where the initial BFS is degenerate and only one basic variable is zero. The problem can be solved similarly if more than one basic variables are zero. Allocate a quantity $\epsilon > 0$ (very small) instead of the basic variable 0 in the cell and readjust all basic variables in the cells such that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = M$$

is satisfied and it may be assumed that $x_{ij} + \epsilon = x_{ij}$. Now solve the problem treating it as a non-degenerate problem. You may drop ϵ at any subsequent iteration if it has been found that the solution at the stage will be non-degenerate on the assumption that $\epsilon = 0$. Otherwise put $\epsilon = 0$ after finding the optimal solution.

Example: Find the initial BFS to the following transportation problem using North-West corner rule and prove that the optimal solution is non-degenerate though the initial solution is degenerate.

	D_1	D_2	D_3	D_4	
O_1	9	8	5	7	12
O_2	4	6	8	7	14
O_3	5	8	9	5	16
	8	18	13	3	

Solution: Initial BFS is given in the table 1. Solution is degenerate with $x_{23} = 0$. Calculate the net evaluations corresponding to the non-basic cells with the assumption $u_1 = 0$. Net evaluations are 5, 3 and 3 corresponding to the non-basic cells (1,3), (2, 1) and (3, 1) respectively. Hence the solution given is not optimal.

					u_i
	8 9	4 $-\theta$ 8	5 θ 5	-1 7	0
	3 4	14 $+\theta$ 6	0 $\epsilon-\theta$ 8	-3 7	-2
	3 5	-1 8	13 9	3 5	-1
v_j	9	8	10	6	

Put $\epsilon > 0$ (very small +ve quantity) in the cell (2, 3) instead of zero. As the net evaluation 5 in the cell (1, 3) is the positive maximum, then the cell (1, 3) will be the new basic cell. Construct a loop as given I the table 1. Put $\theta > 0$ in the cell (1, 3) and readjust the basic variables in the cells containing the loop. Now the maximum value of θ will be ϵ such that basic variable in the cell (2, 3) will be zero and all other variables remain positive. Thus the cell (2, 3) will leave the set of basic cells. With the value of $\theta = \epsilon$, construct the table 2 and again calculate the net evaluations corresponding to the non-basic cells. All net evaluations are not non-positive. Thus the solution is not optimal.

					u_i
	$\boxed{8}$ $-\theta$ 9	$\boxed{4}$ 8	$\boxed{\epsilon}$ $+\theta$ 5	-6 7	0
	3 4	$\boxed{14+\epsilon}$ 6	-5 8	-8 7	-2
	8 θ 5	4 8	$\boxed{13}$ $-\theta$ 9	$\boxed{3}$ 5	4
v_j	9	8	5	1	

Net evaluation corresponding to the cell (3, 1) is 8 which is the positive maximum. Thus the cell (3, 1) will be the new basic cell. Proceed accordingly as given in the table 2. Maximum value of θ will be 8 and cell (1, 1) will leave the set of basic cells. With the value $\theta = 8$ construct the table 3. Now it is interesting to note that the solution will remain non-degenerate if we put $\epsilon = 0$ at this stage. Thus we put $\epsilon = 0$ and get the basic variables in the table 3.

					u_i
	-8 9	$\boxed{4}$ $-\theta$ 8	$\boxed{8}$ $+\theta$ 5	-6 7	-4
	-5 4	$\boxed{14}$ 6	-5 8	-8 7	-6
	$\boxed{8}$ 5	4 θ 8	$\boxed{5}$ $-\theta$ 9	$\boxed{3}$ 5	0
	5	12	9	5	

Now proceed step by step and in the table 4, we get the optimal solution because all net evaluations corresponding to the non-basic cells are non-positive. The optimal solution is $x_{13} = 12, x_{22} = 14, x_{31} = 8, x_{32} = 4, x_{33} = 1, x_{34} = 3$ and transportation cost is 240.

					u_i
	-8 9	-4 8	$\boxed{12}$ 5	-6 7	-4
	-1 4	$\boxed{14}$ 6	-1 8	-4 7	-2
	$\boxed{8}$ 5	$\boxed{4}$ 8	$\boxed{1}$ 9	$\boxed{3}$ 5	0
			9	5	
v_j	5	8			

Unbalanced Transportation problem

There are two type of unbalanced TP.

1. When $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, i.e., the total available capacities of m-origins are greater than the total demands of n-destinations. But this problem can be converted into a balanced transportation problem using the following device:
   (a) Imagine a fictitious or fake $(n+1)$th destination D_{n+1}.
   (b) Assume that the demand of the destination $D_{n+1}$is $b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$.
   (c) Assume that the cost component $c_{i,n+1} = 0$ for all i.

   With this assumption, the TP will be a balanced problem having m origins and $(n+1)$ destinations. Due to the assumption (c) the minimum cost of transportation remains unaffected and the capacities of the origin will not be exhausted completely.

2. When $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$, i.e., the total demands of n destinations are greater than the total available capacities of $m - origins. Here\ also\ the\ problem\ can\ be\ converted\ into$ an ordinary balanced transportation problem but the total demands on n-destinations will not be satisfied completely. Though we cannot satisfy all demands, we can still allocate the

materials available at the origins to the destinations in such a way that minimizes the cost of transportation.

To convert it into a balanced transportation problem:

(a) Imagine a fictitious or fake $(m+1)$th origin O_{n+1}.

(b) Assume that the capacity of the origin $O_{n+1}$is $a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$.

(c) Assume that the cost component $c_{m+1,j} = 0$ for all j.

The problem will be ultimately a transportation problem having $(m+1)$ origins and n destinations. Now the problem can be solved as in the previous case.