Optimization Techniques

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Introduction

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- The three major aspects of optimization are: Theory, Algorithms and Applications. It has been possible to use optimization in real life applications because of the availability of the efficient algorithms, and these algorithms have been developed because of certain interesting research in optimization theory.

Historical Development

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- Simulated annealing, genetic algorithms, and neural network methods represent a new class of mathematical programming techniques that have come into prominence during the last decade.

Engineering applications of optimization

- Design of civil engineering structures such as frames, foundations, bridges, towers, chimneys and dams for minimum cost.
- Design of water resources systems for obtaining maximum benefit.
- Design of aircraft and aerospace structure for minimum weight.
- Finding the optimal trajectories of space vehicles.
- Optimum design of electrical networks.
- Selection of a site for an industry
- Planning of maintenance and replacement of equipment to reduce operating costs.
- Inventory control.
- Designing the shortest route to be taken by a salesperson to visit various cities in a single tour.
- Machine Learning.

Optimization Problem

Example: With the limited available resources (i.e., raw materials, manpower, capital, power, technical appliance, etc.) the main objective of an industry is to produce different products in such a way that maximum profit may be earned by selling them at market price.

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There are Three basic components of an optimization problem: **1. Objective Function, 2. Unknowns and 3. Constraints.**

Optimization Problem

An **objective function** expresses the main aim of the model which is either to be minimized or maximized.

For example:

- In a manufacturing process, the aim may be to maximize the profit or minimize the cost.
- In comparing the data prescribed by a user-defined model with the observed data, the aim is minimizing the total deviation of the predictions based on the model from the observed data.
- In designing a bridge, the goal is to maximize the strength and minimize size.

Optimization Problem

A set of **unknowns or variables** control the value of the objective function.

For example:

- In the manufacturing problem, the variables may include the amounts of different resources used or the time spent on each activity.
- In fitting-the-data problem, the unknowns are the parameters of the model.
- In the bridge design problem, the variables are the shape and dimensions of the pier.

Optimization Problem

A set of **constraints** are those which allow the unknowns to take on certain values but exclude others.

For example:

- In the manufacturing problem, one cannot spend negative amount of time on any activity, so one constraint is that the "time" variables are to be non-negative.
- In the bridge design problem, one would probably want to limit the breadth of the base and to constrain its size.

The optimization problem is then to find values of the variables that minimize or maximize the objective function while satisfying the constraints.

Statement of an optimization problem

An optimization or a mathematical programming problem can be stated as follows:

Find $\mathbf{X} = (x_1, x_2, \cdots, x_n)^T$ which

$$\textit{Minimizes / Maximizes } f(\mathbf{X})$$

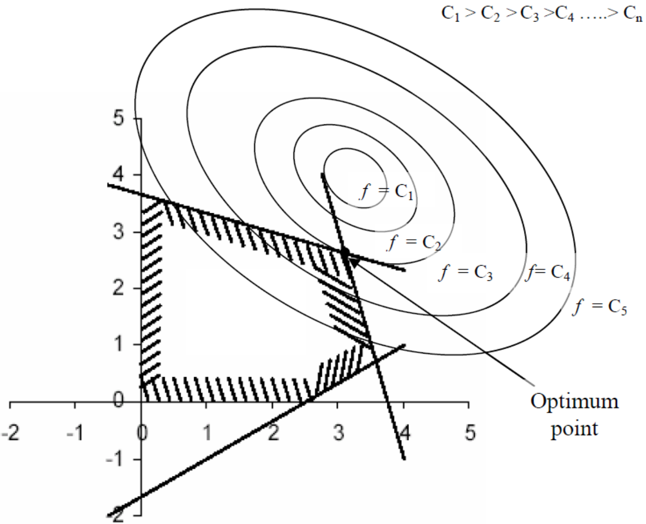
subjects to the constraints

$$g_i(\mathbf{X}) \leq 0, \quad i = 1, 2, \ldots, m$$

$$h_j(\mathbf{X}) = 0, \quad j = 1, 2, \ldots, p$$

where $\mathbf{X}$ is an $n-$ dimensional vector called the **design vector**, $f(\mathbf{X})$ is called the **objective function**, and $g_i(\mathbf{X})$ and $h_j(\mathbf{X})$ are known as **inequality and equality constraints**, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. This type problem is called a **constrained optimization problem.**

Statement of an optimization problem



Statement of an optimization problem

Optimization problems can be defined without any constraints as well.

Find $\mathbf{X} = (x_1, x_2, \cdots, x_n)^T$ which

$$\textit{Minimizes} \ / \ \textit{Maximizes} \ f(\mathbf{X}).$$

Such problems are called **unconstrained optimization problems**. The field of unconstrained optimization is quite a large and prominent one, for which a lot of algorithms and software are available.

Classification of Optimization Problems

Classification based on existence of constraints:

Under this category optimizations problems can be classified into two groups as follows:

- **Constrained optimization problems:** which are subject to one or more constraints.
- **Unconstrained optimization problems:** in which no constraints exist.

Classification of Optimization Problems

Classification based on the nature of the equations involved:

Based on the nature of equations for the objective function and the constraints, optimization problems can be classified as linear, nonlinear, geometric and quadratic programming problems. The classification is very useful from a computational point of view since many predefined special methods are available for effective solution of a particular type of problem.

- **Linear programming problem:** If the objective function and all the constraints are 'linear' functions of the design variables, the optimization problem is called a linear programming problem (LPP).

  Find $\mathbf{X} = (x_1, x_2, \cdots, x_n)^T$ which

$$Minimizes \ / \ Maximizes \ f(\mathbf{X}) = \sum_{i=1}^{n} c_i x_i$$

  subjects to the constraints

$$\sum_{i=1}^{n} a_{ij} x_i \leq (= \ \ or \ \ \geq) b_j, \quad j = 1, 2, \ldots, m$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, n$$

  where c_i, a_{ij}, and b_j are constants.

Classification of Optimization Problems

Classification based on the nature of the equations involved:

- **Nonlinear programming problem:** If any of the functions among the objectives and constraint functions is nonlinear, the problem is called a nonlinear programming (NLP) problem. This is the most general form of a programming problem and all other problems can be considered as special cases of the NLP problem.

Classification of Optimization Problems

Classification based on the nature of the equations involved:

- **Geometric programming problem:** A geometric programming (GMP) problem is one in which the objective function and constraints are expressed as polynomials in $\mathbf{X}$.

  A function $h(\mathbf{X})$ is called a polynomial (with m terms) if h can be expressed as

  $$h(\mathbf{X}) = c_1 X_1^{a_{11}} X_2^{a_{21}} \cdots X_n^{a_{n1}} + c_2 X_1^{a_{12}} X_2^{a_{22}} \cdots X_n^{a_{n2}} + \cdots + c_m X_1^{a_{1m}} X_2^{a_{2m}} \cdots X_n^{a_{nm}}$$

  where $c_j (j = 1, \cdots, m)$ and $a_{ij} (i = 1, \cdots, n \; and \; j = 1, \cdots, m)$ are constants with $c_j \geq 0$ and $x_j \geq 0$.

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  where $c_j (j = 1, \cdots, m)$ and $a_{ij} (i = 1, \cdots, n \textit{ and } j = 1, \cdots, m)$ are constants with $c_j \geq 0$ and $x_j \geq 0$.

  Thus GMP problems can be posed as follows: Find $\mathbf{X}$ which minimize

  $$f(\mathbf{X}) = \sum_{j=1}^{N_0} c_j \left(\prod_{i=1}^{n} x_i^{a_{ij}} \right), \quad c_j > 0, x_i > 0$$

  subject to

  $$g_k(\mathbf{X}) = \sum_{j=1}^{N_k} a_{jk} \left(\prod_{i=1}^{n} x_i^{q_{ijk}} \right) > 0, \quad \textit{where } a_{jk} > 0, x_j > 0, k = 1, 2, \cdots, m$$

  where N_0 and N_k denote the number of terms in the objective function and in the $k-$th constraint function, respectively.

Classification of Optimization Problems

Classification based on the nature of the equations involved:

- **Quadratic programming problem:** A quadratic programming problem is the best behaved nonlinear programming problem with a quadratic objective function and linear constraints and is concave (for maximization problems). It can be solved by suitably modifying the linear programming techniques. It is usually formulated as follows:

$$Minimize/Maximize\ f(\mathbf{X}) = c + \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij} x_i x_j$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i = b_j \quad j = 1, 2 \ldots, m$$

$$x_i \geq 0 \quad i = 1, 2 \ldots, n$$

where c, q_i, Q_{ij}, a_{ij}, and b_j are constants.

Classification of Optimization Problems

Classification based on the permissible values of the decision variables: Under this classification, objective functions can be classified as integer and real-valued programming problems.

- **Integer programming problem:** If some or all of the design variables of an optimization problem are restricted to take only integer (or discrete) values, the problem is called an integer programming problem.
- **Real-valued programming problem:** A real-valued problem is that in which it is sought to minimize or maximize a real function by systematically choosing the values of real variables from within an allowed set. When the allowed set contains only real values, it is called a real-valued programming problem.

Classification of Optimization Problems

Classification based on the number of objective functions: Under this classification, objective functions can be classified as single-objective and multiobjective programming problems.

- **Single-objective programming problem:** in which there is only a single objective function.
- **Multi-objective programming problem:** A multiobjective programming problem can be stated as follows:

  Find $\mathbf{X}$ which

  $$\textit{minimize } f_1(\mathbf{X}) f_2(\mathbf{X}) \cdots f_k(\mathbf{X})$$

  subject to

  $$g_j(\mathbf{X}) \leq 0 \ \ j = 1, 2 \cdots, m$$

  where $f_1, f_2, \cdots, f_k$ denote the objective functions to be minimized simultaneously.

  For example in some design problems one might have to minimize the cost and weight of the structural member for economy and, at the same time, maximize the load carrying capacity under the given constraints.