Quantum Physics of Light-Matter Interactions

FAU - Summer semester 2019

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Introduction

This is a summary of topics covered during lectures at FAU in the summer semesters 2018 and 2019. A first goal of the class is to introduce the student to the physics of fundamental processes in light-matter interactions such as:

- stimulated emission/absorption, spontaneous emission
- motional effects of light onto atoms, ions and mechanical resonators (optomechanics)
- electron-vibrations-light coupling in molecules

The main purpose of the class is to build a toolbox of methods useful for tackling real applications such as cooling, lasing, etc. Some of the general models and methods introduced here and used throughout the notes are:

- quantum master equations
- the Jaynes (Tavis)-Cummings Hamiltonian
- the Holstein Hamiltonian
- the radiation pressure Hamiltonian
- quantum Langevin equations
- the polaron transformation

Among others, some of the applications presented either within the main course or as exercises cover aspects of:

- laser theory
- Doppler cooling, ion trap cooling, cavity optomechanics with mechanical resonators
- electromagnetically induced transparency
- optical bistability
- applications of subradiant and superradiant collective states of quantum emitters

Here is a short list of relevant textbooks.

- Milburn
- Gardiner and Zoller
- Zubairy

Other references to books and articles are listed in the bibliography section at the end of the course notes.

As some numerical methods are always useful here is a list of useful programs and platforms for numerics in quantum optics:

- Mathematica
- QuantumOptics.jl - A Julia Framework for Open Quantum Dynamics
- QuTiP - Quantum Toolbox in Python

The exam will consist of a 90 mins written part and 15 mins oral examination.

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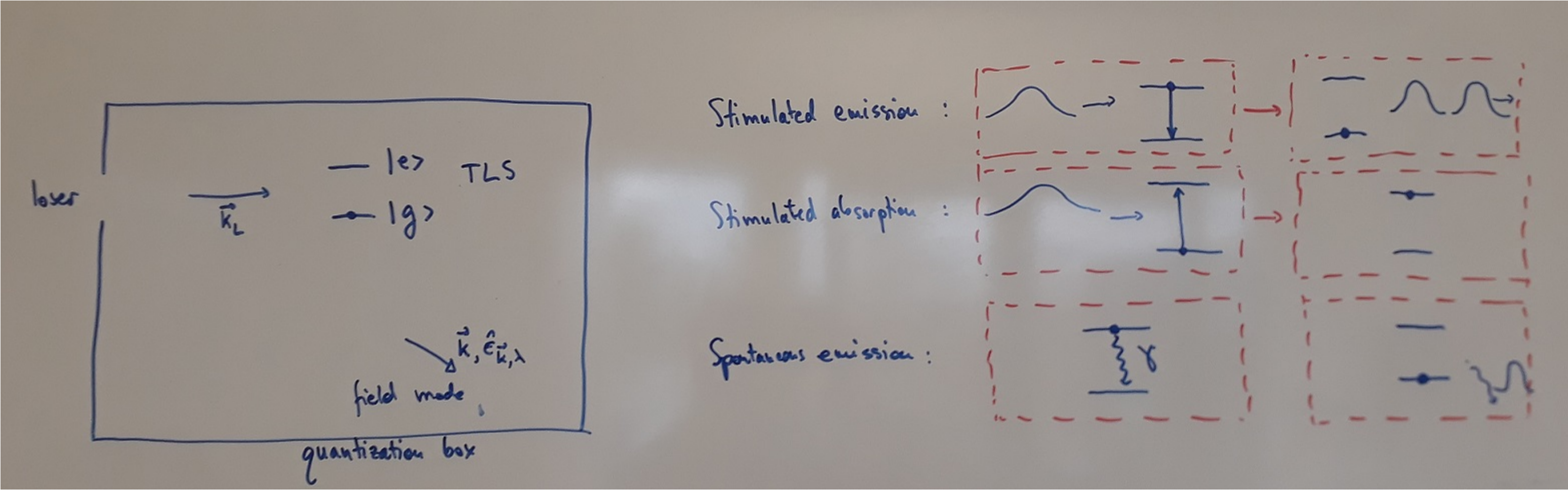

1. Quantum light and the two level system

The purpose of this chapter to introduce the simplest model that can describe light-matter interactions at the quantum level and specifically explain phenomena such as stimulated emission/absorption and spontaneous emission. On the light side, we proceed by quantizing the electromagnetic field in a big box by introducing photon creation and annihilation operators. The action of a creation operator is to produce an excitation of a given plane wave mode while the annihilation operator does the opposite. We analyze the states of light in the number (Fock) basis and show how to construct coherent, thermal and squeezed states. We then describe the matter system as an atom with two relevant levels between which a transition dipole moment exists and can be driven by a light mode. A minimal coupling Hamiltonian in the dipole approximation between the two level system and the quantum light in the box then describes emission and absorption processes.

1.1 Light in a box

Let us consider a finite box of dimensions $L \times L \times L$ consisting of our whole system. Later we will eventually take the limit of $L \to \infty$. The box is empty (no charges or currents). We can write then the following Maxwell equations:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c}\partial_t \mathbf{E}, \tag{1.2}$$

where the speed of light emerges as $c = 1/\sqrt{\varepsilon_0 \mu_0}$. We can also connect both the electric and magnetic field amplitudes to a vector potential

$$\mathbf{E} = -\partial_t \mathbf{A} \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{1.3}$$

which fulfills the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0. \tag{1.4}$$

We immediately obtain a wave equation for the vector potential (of course we could have already

$$\nabla^2 \mathbf{A}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} \tag{1.5}$$

with solutions that can be separated into positive and negative components.

$$\mathbf{A}^{(+)}(\mathbf{r},t) = i\sum_{\mathbf{k}} c_{\mathbf{k}}\mathbf{u}_{\mathbf{k}}(\mathbf{r})e^{-i\omega_k t}, \tag{1.6}$$

and $\mathbf{A}^{(+)}(\mathbf{r},t) = [\mathbf{A}^{(-)}(\mathbf{r},t)]^*$. Notice that $\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$. We have separated c-numbers, spatial dependence and time dependence. Moreover, the index $\mathbf{k}$ generally can include two specifications: i) the direction and amplitude of the wavevector $\mathbf{k}$ and ii) the associated polarization of the light mode contained in the unit vector $\hat{\varepsilon}_{\mathbf{k}}^{\lambda}$ where $\lambda = 1,2$ denotes the two possible transverse polarizations (one can choose as basis either linear of circular polarizations). For simplicity we will stick to the linear polarization choice such $\hat{\varepsilon}_{\mathbf{k}}$ is real. As we are in free space the dispersion relation is simply $\omega_k = ck$, relating the frequency of the mode only to the absolute value of the wavevector. Plugging these type of solutions into the wave equation we find that the set of vector mode functions have to satisfy

$$\left[\nabla^2 + \frac{\omega_k^2}{c^2}\right]\mathbf{u}_{\mathbf{k}}(\mathbf{r}) = 0. \tag{1.7}$$

Remember that the Coulomb gauge condition requires that the divergence of individual spatial mode functions is vanishing

$$\nabla\cdot\mathbf{u}_{\mathbf{k}}(\mathbf{r}) = 0. \tag{1.8}$$

Since these are eigenvectors of the differential operator above they form a complete orthonormal set:

$$\int_V d\mathbf{r}\mathbf{u}_{\mathbf{k}}^*(\mathbf{r})\mathbf{u}_{\mathbf{k}'}(\mathbf{r}) = \delta_{\mathbf{k}\mathbf{k}'}. \tag{1.9}$$

Imposing periodic boundary conditions, the proper solutions for the above equation are traveling waves:

$$\mathbf{u}_{\mathbf{k}}(\mathbf{r}) = \frac{1}{L^{3/2}}e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\varepsilon}_{\mathbf{k}}. \tag{1.10}$$

We could have as well chosen different boundary conditions which would have given standing waves for the mode functions. The allowed wave-vectors have components $(n_x,n_y,n_z)2\pi/L$ where the indexes are positive integers. Finally we can write

$$\mathbf{A}(\mathbf{r},t) = i\sum_{\mathbf{k}}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}[c_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\omega_k t} - c_{\mathbf{k}}^*e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\omega_k t}]\hat{\varepsilon}_{\mathbf{k}}, \tag{1.11}$$

and derive the expression for the electric field as

$$\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k}}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}[c_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\omega_k t} + c_{\mathbf{k}}^*e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\omega_k t}]\hat{\varepsilon}_{\mathbf{k}}. \tag{1.12}$$

The quantization is straightforward and consists in replacing the c-number amplitudes with operators satisfying the following relations: $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$. Let's write the expression for the quantized electric field (which we will extensively use in this class) as a sum of negative and positive frequency components:

$$\hat{\mathbf{E}}(\mathbf{r}) = \sum_{\mathbf{k}} \mathscr{E}_k \left[a_{\mathbf{k}} e^{i\mathbf{kr}} + a_{\mathbf{k}}^{\dagger} e^{-i\mathbf{kr}} \right] = \hat{\mathbf{E}}^{(+)} + \hat{\mathbf{E}}^{(-)}, \tag{1.13}$$

where $\mathscr{E}_k = \sqrt{\hbar\omega_k/(2\varepsilon_0 V)}$ is the zero-point amplitude of the electric field. Notice that the electric field is written now in the Schrödinger picture where the time dependence is not explicit. Instead, when writing equations of motion for the field operators we will recover time dependence in the time evolution of the creation and annihilation operators as we will shortly see in the next section. Starting now from the Hamiltonian of the electromagnetic field as an integral over the energy density over the volume of the box, we obtain a diagonal representation as a sum over an infinite number of quantum harmonic oscillator free Hamiltonians:

$$H = \frac{1}{2} \int_V d\mathbf{r} \left(\varepsilon_0 \hat{\mathbf{E}}^2(\mathbf{r}) + \frac{1}{\mu_0} \hat{\mathbf{B}}^2(\mathbf{r}) \right) = \sum_{\mathbf{k}} \hbar\omega_k \left(a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \frac{1}{2} \right). \tag{1.14}$$

There is a first observation that the sum over the $1/2$ term might diverge: this is the starting point for the derivation of effects such as the vacuum-induced Casimir force. We will not deal with it in this course but instead remove the term in the following as it does not play any role in our intended derivations. The second observation is that free evolution of operators in the Heisenberg picture

$$\frac{d}{dt} a_{\mathbf{k}}(t) = \frac{i}{\hbar} [H_0, a_{\mathbf{k}}(t)] = -i\omega_k a_{\mathbf{k}}(t), \tag{1.15}$$

directly gives us the expected time evolution of the expectation value of the electric field operator

$$\langle \hat{\mathbf{E}} \rangle (\mathbf{r}, t) = \sum_{\mathbf{k}} \mathscr{E}_k \left[\langle a_{\mathbf{k}} \rangle e^{i\mathbf{kr}} e^{-i\omega_k t} + \langle a_{\mathbf{k}}^{\dagger} \rangle e^{-i\mathbf{kr}} e^{i\omega_k t} \right], \tag{1.16}$$

1.2 Quantum states of light

Until now we dealt with operators. We found that the electric field operator and total free Hamiltonian for a quantum electromagnetic field inside the box can be expressed as an expansion in plane waves and with coefficients which are creation and annihilation operators. Now we will focus a bit on the possible states of light. We will especially describe fundamental differences between properties of coherent and thermal states. For this we use the properties of a the single mode harmonic oscillator detailed in Appendix. 1.6.1. For a given mode $\mathbf{k}$ we will then use the Fock basis $|n_{\mathbf{k}}\rangle$ where the index goes from 0 to ∞. The collective basis is then expressed as $\prod_{\mathbf{k}} |n_{\mathbf{k}}\rangle$ where all indexes go from 0 to ∞ and a tensor product over all allowed wave-vectors and polarizations is performed.

Thermal light

Let us assume that the box is in contact and thermalizes with a heat bath at constant temperature T. This means each of the modes of the box is in thermal equilibrium and this described by a diagonal density operator at some initial time $t = 0$ given by:

$$\rho_F(0) = \frac{e^{-H_F/(k_B T)}}{\mathrm{Tr}_F[e^{-H_F/(k_B T)}]} = \prod_{\mathbf{k}} \rho_F^{(\mathbf{k})} = \sum_{\mathbf{k}} \sum_{n_{\mathbf{k}}} P(n_{\mathbf{k}}) |n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|, \tag{1.17}$$

where the occupancy probability is given by

$$P(n_{\mathbf{k}}) = \frac{e^{-n_{\mathbf{k}}\hbar\omega_{\mathbf{k}}/(k_B T)}}{1 - e^{-\hbar\omega_{\mathbf{k}}/(k_B T)}}. \tag{1.18}$$

Each mode defined by a given wavevector is in a thermal state with some average occupancy set by the temperature T. To evaluate the electric field at some position $\mathbf{r}$ at a later time t we use Eq. 1.16. From the single mode calculations in Appendix. 1.6.1 we know that for each mode the thermal distribution predicts zero average for the creation and annihilation operators. This means that the elctric field amplitude is on average actually zero in a thermal state. On the other hand, one can compute the variance of the field which is equal to the expectation value of the photon number.

Coherent light

Let us now imagine that the box has a very little hole where an antenna emits coherent light. This means that a specific mode with wave-vector $\mathbf{k}_0$ (and frequency $\omega_0 = ck_0$) is constantly externally driven into a coherent state of amplitude $\alpha_{\mathbf{k}_0}$. With all other modes in the vacuum state we can write the initial density operator as:

$$\rho_F(0) = |\alpha_{\mathbf{k}_0}\rangle\langle\alpha_{\mathbf{k}_0}| \otimes |0\rangle\langle 0|. \tag{1.19}$$

We can now again evaluate the electric field expectation value at some time t (and $\mathbf{r} = 0$)

$$\langle\hat{\mathbf{E}}\rangle(\mathbf{r},t) = \sum_{\mathbf{k}} \mathscr{E}_k[\langle a_{\mathbf{k}}\rangle e^{-i\omega_k t} + \langle a_{\mathbf{k}}^\dagger\rangle e^{i\omega_k t}]\hat{\varepsilon}_{\mathbf{k}} = \mathscr{E}_{k_0}[\alpha_\ell e^{-i\omega_0 t} + \alpha_\ell^* e^{i\omega_0 t}]\hat{\varepsilon}. \tag{1.20}$$

As opposed to thermal light, the coherent state as a non-zero average electric field amplitude $\mathscr{E}_{k_0}\alpha_{\mathbf{k}_0}$.

The Mollow transformation

Let us again consider the case described above where an antenna continuously drives a given mode in the coherent state $|\alpha_{\mathbf{k}_0}\rangle$. This mode evolves in time at its natural frequency so the state can be written as: $|\alpha_{\mathbf{k}_0}(t)\rangle = |\alpha_{\mathbf{k}_0} e^{-i\omega_{k_0} t}\rangle$. We would like to remove the coherent state from the vacuum which we can do by performing an inverse displacement transformation:

$$\tilde{\rho}_F = D^\dagger_{\alpha_{\mathbf{k}_0}} |\alpha_{\mathbf{k}_0}\rangle\langle\alpha_{\mathbf{k}_0}| D_{\alpha_{\mathbf{k}_0}} \otimes |0\rangle\langle 0| = |0\rangle\langle 0|. \tag{1.21}$$

This transformation simply displaces the coherent state back into the vacuum. Let's see what happens to the Hamiltonian:

$$\tilde{H}_F = D^\dagger_{\alpha_{\mathbf{k}_0}} \left[\sum_{\mathbf{k}} \hbar\omega_k \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}\right)\right] D_{\alpha_{\mathbf{k}_0}} = \sum_{\mathbf{k}} \hbar\omega_k \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}\right) + |\alpha_{\mathbf{k}_0}|^2 + \hbar\left(\alpha^*_{\mathbf{k}_0} a_{\mathbf{k}_0} + \alpha_{\mathbf{k}_0} a^\dagger_{\mathbf{k}_0}\right) \tag{1.22}$$

The first constant term is a simple constant energy shift of the Hamiltonian which can be ignored. The next term shows how a Hamiltonian for driving the vacuum into the coherent state should be written. Let us also notice that the electric field operator has now a different term:

$$\hat{\tilde{\mathbf{E}}}(\mathbf{r}) = D^\dagger_{\alpha_{\mathbf{k}_0}}\left[\hat{\mathbf{E}}(\mathbf{r})\right] D_{\alpha_{\mathbf{k}_0}} = \sum_{\mathbf{k}} \mathscr{E}_k \left[a_{\mathbf{k}} e^{i\mathbf{kr}} + a_{\mathbf{k}}^\dagger e^{-i\mathbf{kr}}\right] + \mathscr{E}_{k_0}\left[\alpha_{\mathbf{k}_0} e^{-i\omega_{k_0} t} e^{i\mathbf{kr}} + \alpha^*_{\mathbf{k}_0} e^{i\omega_{k_0} t} e^{-i\mathbf{kr}}\right] \tag{1.23}$$

The last part is what we will refer to in the future as a classical field in a coherent state (as produced by a laser).

1.3 Light-matter interactions: the dipole approximation

Let us consider an atom (nucleus static positioned at the origin and an electron with a position $\mathbf{r}$) in the presence of external electromagnetic fields described by scalar potential $\Phi(\mathbf{r},t)$ and vector potential $\mathbf{A}(\mathbf{r},t)$. From classical electrodynamics (see Jackson) we know that the Hamiltonian of a charged particle in the presence of external fields is modified (the conjugate variable of the position is no-longer the momentum but the generalized momentum)

$$H(\mathbf{r},t) = \frac{1}{2m}(\mathbf{p}+e\mathbf{A}(\mathbf{r},t))^2 - e\Phi(\mathbf{r},t) + V(r). \tag{1.24}$$

The term $V(r)$ is the spherically symmetric Coulomb potential. In the absence of an external field the solutions to the above Hamiltonian are therefore simply the Hydrogen atom wavefunctions. We impose the Coulomb gauge $\Phi(\mathbf{r},t) = 0$ under the observation that even if this gauge is not relativistically invariant, most of quantum optics phenomena is non-relativistic anyway. Next we make an observation which will be pretty important and therefore we list it in a special box.

Important 1.3.1 — The dipolar approximation. As the size of the electronic orbital (on a Bohr radius length scale of 10^{-10} m) is much smaller than a typical optical wavelength (around microns - 10^{-6} m) we can estimate that the vector potential is practically position independent $\mathbf{A}(\mathbf{r},t) = \mathbf{A}(t)$

This allows us to write a simplified Hamiltonian:

$$H(\mathbf{r},t) = \frac{\mathbf{p}^2}{2m} + \frac{e}{m}\mathbf{A}(t)\cdot\mathbf{p} + \frac{e^2}{2m}\mathbf{A}^2(t) + V(r). \tag{1.25}$$

We now perform a gauge transformation to a length gauge via the following function:

$$\chi(\mathbf{r},t) = -\mathbf{A}(t)\cdot\mathbf{r}. \tag{1.26}$$

which results in the new scalar and vector fields:

$$\Phi'(\mathbf{r},t) = -\frac{\partial\chi(\mathbf{r},t)}{\partial t} = -\frac{\partial\mathbf{A}(t)}{\partial t}\mathbf{r} = \mathbf{r}\cdot\mathbf{E}(t). \tag{1.27}$$

and

$$\mathbf{A}'(\mathbf{r},t) = \mathbf{A}(t) + \nabla\chi(\mathbf{r},t) = \mathbf{A}(t) - \nabla(\mathbf{A}(t)\cdot\mathbf{r}) = 0. \tag{1.28}$$

so we end up with a transformed Hamiltonian

$$H' = \frac{\mathbf{p}^2}{2m} + V(r) - \mathbf{d}\cdot\mathbf{E}(t), \tag{1.29}$$

where the dipole moment is defined as $\mathbf{d} = -e\mathbf{r}$. From here on one can proceed with a semiclassical picture where only the dipole is quantized so that the interaction is time dependent $-\hat{\mathbf{d}}\cdot\mathbf{E}(t)$ or a fully quantum picture where also the electromagnetic modes are quantized with time-independent interaction Hamiltonian $-\hat{\mathbf{d}}\cdot\hat{\mathbf{E}}$.

1.4 The two level system

We consider now a two level system with ground state $|g\rangle$ and some excited state $|e\rangle$ which can be reached from the ground state as a dipole-allowed transition. In the Appendix. 1.6.2 we show in detail what we mean by ground and excited on the easy to understand particular case of the Hydrogen atom. Excluding possibilities that the electron can go anywhere else except these two levels the basis is then complete meaning that $|g\rangle\langle g| + |e\rangle\langle e| = I_2$ (I_2 is the identity in the 2-dimensional Hilbert space). The two terms can be though of as projectors or either the ground or excited state. We then define ladder operators that take the system up and down

$$\sigma = |g\rangle\langle e| \qquad \text{and} \qquad \sigma^\dagger = |e\rangle\langle g|. \tag{1.30}$$

According to the Appendix. 1.6.2, the dipole moment operator does not have matrix elements on the individual orbitals such that $\langle g|\hat{\mathbf{d}}|g\rangle = 0$ and $\langle e|\hat{\mathbf{d}}|e\rangle = 0$ We then can write the basis decomposition of the dipole moment operator as

$$\hat{\mathbf{d}} = |g\rangle\, \mathbf{d}_{ge}\, \langle e| + |e\rangle\, \mathbf{d}^*_{ge}\, \langle g| \qquad \text{where} \qquad \mathbf{d}_{ge} = \langle g|\hat{\mathbf{d}}|e\rangle\,. \tag{1.31}$$

Finally we can now write the total Hamiltonian in terms of ladder operators as:

$$H = \hbar\omega_e\sigma^\dagger\sigma + \hbar\omega_g(1-\sigma^\dagger\sigma) + \left[\sigma\mathbf{d}_{eg}\cdot\mathbf{E}(t) + \sigma^\dagger\mathbf{d}_{ge}\cdot\mathbf{E}(t)\right] \tag{1.32}$$

As we are free to substract a constant energy from the system, we will substract the term $\hbar\omega_g I_2$ and denote the energy diference $\omega_e - \omega_g = \omega_0$. Moreover, in some cases (see Appendix. 1.6.2) the transition dipole moment element can be real: we will therefore, for simplicity, set $\mathbf{d}_{eg} = \mathbf{d}^*_{eg} = \mathbf{d}$. We can now write the final form of the free Hamiltonian plus the interaction with the field as

$$H = \hbar\omega_0\sigma^\dagger\sigma - \mathbf{d}\cdot\mathbf{E}(t)\left[\sigma + \sigma^\dagger\right] \tag{1.33}$$

Next we will introduce another crucial approximation. To this end first we consider the case of a classically driven TLS (as we mentioned before as a semiclassical picture). For an atom in the origin a classical drive at frequency ω_ℓ is

$$E(t) = \mathscr{E}_\ell\cos(\omega_L t)\hat{\varepsilon} = \frac{\mathscr{E}_\ell}{2}(e^{-i\omega_\ell t} + e^{i\omega_\ell t})\hat{\varepsilon}. \tag{1.34}$$

The semiclassical interaction then can be expressed as a sum of four terms:

$$H_{int} = \hbar\frac{(\mathbf{d}\cdot\varepsilon)\mathscr{E}_\ell}{2\hbar}\left(\sigma e^{i\omega_\ell t} + \sigma^\dagger e^{-i\omega_\ell t} + \sigma e^{-i\omega_\ell t} + \sigma^\dagger e^{i\omega_\ell t}\right). \tag{1.35}$$

First, we will denote the frequency $(\mathbf{d}\cdot\varepsilon)\mathscr{E}_\ell/2\hbar$ as the Rabi frequency. Most importantly, we will perform a transformation into the Heisenberg picture (with the free Hamiltonian as shown in Appendix. 1.6.3):

$$H^{HP}_{int} = \hbar\frac{(\mathbf{d}\cdot\varepsilon)\mathscr{E}_\ell}{2\hbar}\left(\sigma e^{i(\omega_\ell-\omega_0)t} + \sigma^\dagger e^{-i(\omega_\ell-\omega_0)t} + \sigma e^{-i(\omega_\ell+\omega_0)t} + \sigma^\dagger e^{i(\omega_\ell+\omega_0)t}\right). \tag{1.36}$$

Notice that some terms are very quickly oscillating in time which means that their effect averages to zero over any small interval much larger than $1/\omega_0$. This allows us to make the following statement:

Important 1.4.1 — Rotating wave approximation (RWA). One can neglect the quickly oscillating terms in the dipole Hamiltonian and only keep terms showing detunings (frequency differences). The interaction Hamiltonian in the RWA then becomes

$$H_{int} = \hbar\Omega_\ell\left(\sigma e^{i\omega_\ell t} + \sigma^\dagger e^{-i\omega_\ell t}\right). \qquad \text{where} \qquad \Omega_\ell = \frac{(\mathbf{d}\cdot\varepsilon)\mathscr{E}_\ell}{2\hbar}. \tag{1.37}$$

The same arguments apply to the fully quantum light-matter Hamiltonian. We can directly write it as

$$H_{int} = \hbar\omega_0\sigma^\dagger\sigma + \sum_{\mathbf{k}}\hbar\omega_k a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \sum_{\mathbf{k}}\hbar g_{\mathbf{k}}\left(a_{\mathbf{k}}\sigma^\dagger + \sigma a^\dagger_{\mathbf{k}}\right) \tag{1.38}$$

where the coupling is

$$g_{\mathbf{k}} = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\frac{\mathbf{d}\cdot\hat{\varepsilon}_{\mathbf{k}}}{\hbar} = \frac{\mathscr{E}_k\mathbf{d}\cdot\hat{\varepsilon}_{\mathbf{k}}}{\hbar}. \tag{1.39}$$

Here the rotating wave approximation has a more direct interpretation. Terms like $a_{\mathbf{k}}\sigma^\dagger$ should be read as photon destroyed accompanied by an excitation of the TLS while its hermitian conjugate $\sigma a_{\mathbf{k}}^\dagger$ shows the creation of a photon accompanying the de-excitation of the TLS. These are terms conserving energy. The terms which are equivalent with the counter-rotating terms discussed above are $a_{\mathbf{k}}^\dagger\sigma^\dagger$ with its hermitian conjugate. These terms are energy-non-conserving (creating photons while exciting the TLS).

1.5 Spontaneous and stimulated emission, stimulated absorption

Let us now analyze the fundamental processes through which a TLS inside the big quantization box can make transitions between the ground and excited states. First, let us assume an atom in the excited state while the whole box is in the vacuum state except for a given specific mode which is already occupied by a photon (with a given direction and polarization specified by a $\mathbf{k}_0$), i.e. we start with the state $|e\rangle \otimes |0...1_{\mathbf{k}_0}...0\rangle$. The interaction Hamiltonian will then produce a sum of two states

$$H\,|e\rangle \otimes |0...1_{\mathbf{k}_0}...0\rangle = \hbar g_{\mathbf{k}_0}\sqrt{2}\,|e\rangle \otimes |0...2_{\mathbf{k}_0}...0\rangle + \sum_{\mathbf{k}\neq\mathbf{k}_0} \hbar g_{\mathbf{k}}\,|e\rangle \otimes |0...1_{\mathbf{k}_0}....1_{\mathbf{k}}..0\rangle\,. \tag{1.40}$$

The first one is a *stimulated emission* process where the first photon stimulates the emission of a second photon in exactly the same direction and with the same polarization as the first one. The second term is a *spontaneous emission* event where a photon is emitted in a random direction with any polarization. which means that the atom gets de-excited while a photon is emitted into a random direction with any polarization.

Let us now start with an initially ground state TLS and a given specific one photon state, i.e. in the state $|g\rangle \otimes |0...1_{\mathbf{k}_0}...0\rangle$. The action of the Hamiltonian is then:

$$H\,|g\rangle \otimes |0...1_{\mathbf{k}_0}...0\rangle = \hbar g_{\mathbf{k}_0}\,|e\rangle \otimes |0...0...0\rangle\,. \tag{1.41}$$

The only possible process now is a *stimulated absorption* of the laser photon and subsequent excitation of the atom.

Notice that there are factors multiplying the transition probabilities for the stimulated processes, i.e. if we start with an arbitrary Fock state $|0...n_{\mathbf{k}_0}...0\rangle$ the action of creation and annihilation operators will lead to an extra $\sqrt{n_{\mathbf{k}_0}}$ or $\sqrt{n_{\mathbf{k}_0}+1}$ factor. More generally, let us assume that the filled mode is in a coherent state $\alpha_{\mathbf{k}_0}$ which will see that both the stimulated absorption and emission will have collectively enhanced rates $g_{\mathbf{k}_0}\alpha_{\mathbf{k}_0}$

1.6 Appendix

1.6.1 Appendix A: The quantum harmonic oscillator

The one dimensional quantum harmonic oscillator is described by the following Hamiltonian

$$H = \frac{1}{2}\hbar\omega(\hat{P}^2 + \hat{Q}^2), \tag{1.42}$$

where ω is the oscillation frequency while the two canonically conjugated operators fulfill the following commutation relation $[\hat{Q}, \hat{P}] = i\hbar$. For the electromagnetic field the dimensionless $\hat{Q}, \hat{P}$ operators have the meaning of quadratures associated with the electric and magnetic fields. For a finite mass system, they are dimensionless position and momentum operators obtained via the following transformations from the real momentum and position:

$$\hat{p} = \hat{P}\sqrt{m\hbar\omega} = p_{\text{zpm}}\hat{P} \qquad \text{and} \qquad \hat{x} = \hat{Q}\sqrt{\frac{\hbar}{m\omega}} = x_{\text{zpm}}\hat{Q} \tag{1.43}$$

From here one can go further to define (non-hermitian) creation and annihilation operators:

$$a = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}) \qquad \text{and} \qquad a^\dagger = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}), \tag{1.44}$$

where $[a, a^\dagger] = 1$ and the Hamiltonian becomes $H = \hbar\omega(a^\dagger a + \frac{1}{2})$. Of course the inverse transformations are

$$\hat{Q} = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad \text{and} \qquad \hat{P} = \frac{i}{\sqrt{2}}(a^\dagger - a) \tag{1.45}$$

Number (Fock) basis. Displacement operator. Coherent states.

The ground state is defined as the empty state $|0\rangle$ such that $a|0\rangle = 0$. Fock (number states) are created by the continuous application of the creation operator onto the vacuum:

$$|n\rangle = \sqrt{\frac{1}{n!}}a^{\dagger n}|0\rangle, \qquad a|n\rangle = \sqrt{n}|n-1\rangle \qquad \text{and} \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{1.46}$$

A special category of states, coherent states, are obtained by displacing the vacuum $|\alpha\rangle = D_\alpha|0\rangle$ via the following operator

$$D_\alpha = e^{\alpha a^\dagger - \alpha^* a} = e^{-|\alpha|^2}e^{\alpha a^\dagger}e^{-\alpha^* a}. \tag{1.47}$$

We have applied above the Baker-Hausdorff-Campbell formula

$$e^{A+B} = e^A e^B e^{-[A,B]/2} \qquad \text{when} \qquad [A,[A,B]] = 0 \qquad \text{and} \qquad [B,[A,B]] = 0. \tag{1.48}$$

One can then generally express the coherent state as a sum over the number states with Poissonian coefficients

$$|\alpha\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{1.49}$$

Notice a couple of useful properties such as

$$a|\alpha\rangle = \alpha|\alpha\rangle, \qquad \langle\alpha|a^\dagger = \alpha^*\langle\alpha|, \qquad D_\alpha^\dagger a D_\alpha = a + \alpha, \qquad D_\alpha^\dagger a^\dagger D_\alpha = a^\dagger + \alpha^*. \tag{1.50}$$

Assuming that one wants to characterize the signal-to-noise ratio in a coherent state, the following two expressions come in handy:

$$\bar{n} = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2 \qquad \text{and} \qquad \Delta n = \sqrt{\bar{n^2} - \bar{n}^2} = \sqrt{\bar{n}} = |\alpha|. \tag{1.51}$$

For large amplitudes the average is much larger then the variance which means the signal-to-noise ratio $\bar{n}/\Delta n = 1/\sqrt{\bar{n}} = 1/|\alpha|$ is extremely high. For quadratures notice that

$$\bar{Q} = \frac{1}{\sqrt{2}}(\alpha + \alpha^*), \qquad \bar{Q} = \frac{i}{\sqrt{2}}(\alpha^* - \alpha), \qquad \Delta\hat{Q} = \Delta\hat{P} = 1/2. \tag{1.52}$$

Thermal states

A system in a pure state can be represented by a single ket and satisfies the Schrödinger equation $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$. One can as well rewrite the equation for the following quantity, the density operator or density matrix $\rho = |\psi\rangle\langle\psi|$ with the following von Neumann equation of motion: $i\hbar\partial_t\rho = [H,\rho]$. More generally however, the system is in a mixed state where the density matrix is expressed as $\rho = \sum_\psi p_\psi |\psi\rangle\langle\psi|$ which also satisfy the von-Neumann equation of motion.
An example of a mixed state is the thermal state, where the density operator can be written as

$$\rho_{\text{th}} = \frac{e^{-H/k_BT}}{\text{Tr}[e^{-H/k_BT}]} = \frac{1}{1-e^{-\hbar\omega/k_BT}} e^{-\hbar\omega a^\dagger a/k_BT}. \tag{1.53}$$

As a reminder, the trace is obtained by summing all the diagonal terms of the density matrix in the Fock basis $\text{Tr}[O] = \sum_{n=0}^\infty \langle n|O|n\rangle$. Let's compute the average occupancy in such a state (we will make the notation $\beta = \hbar\omega/(k_BT)$:

$$\bar{n} = Tr(\rho_{\text{th}} a^\dagger a) = (1-e^{-\beta})\text{Tr}[e^{-\beta a^\dagger a} a^\dagger a] = (1-e^{-\beta})\sum_{n=0}^\infty n e^{-\beta n}. \tag{1.54}$$

We have used the property that the exponential of the Hamiltonian is diagonal in the number basis. In the above equation we can immediately see that the term $(1-e^{-\beta})\sum_{n=0}^\infty n e^{-\beta n}$ plays the role of a probability distribution $p(n)$. We will evaluate it after finding the expression for the average. Notice that the sum above is the derivative of the following sum.

$$\sum_{n=0}^\infty n e^{-\beta n} = -\frac{d}{d\beta}\sum_{n=0}^\infty e^{-\beta n} = -\frac{d}{d\beta}\frac{1}{1-e^{-\beta}} = \frac{e^{-\beta}}{(1-e^{-\beta})^2}. \tag{1.55}$$

This readily gives us the expected result of a Planck distribution average number and distribution

$$\bar{n} = \frac{1}{e^\beta - 1} = \frac{1}{e^{\hbar\omega/(k_BT)} - 1} \qquad \text{and} \qquad p_n = \frac{e^{-\beta n}}{1-e^{-\beta}} = \frac{1}{1-\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n \tag{1.56}$$

We can also compute (exercise) the variance in a thermal state

$$[\Delta n]_{\text{th}} = \sqrt{\bar{n}+\bar{n}^2}. \tag{1.57}$$

and remark that for large $\bar{n}$ this is at the level $[\Delta n]_{\text{th}} \simeq \bar{n}$ which is much larger than the variance of the coherent state of $[\Delta n]_{\text{coh}} = \sqrt{\bar{n}}$. Let us also remark $\hbar/k_BT$ is around 10^{-11}s at $T = 1$K, which means that the exponent can very well be approximated ($\beta \ll 1 \to e^\beta \simeq 1+\beta$ and the average number is given by

$$\bar{n} \simeq \frac{\hbar\omega}{k_BT}. \tag{1.58}$$

This is valid especially for small frequencies such as vibrations of membranes/mirrors, of ions in a trap, or acoustic phonons in a bulk solid. For high frequencies (optical frequencies of a mode in an optical cavity, molecular vibrations etc) we have $\beta \ll 1$ and the average occupancy is estimated by

$$\bar{n} = e^{-\hbar\omega/(k_BT)}. \tag{1.59}$$

Notice that we can also re-express the density operator as

$$\rho_{\text{th}} = \frac{1}{1+\bar{n}}\sum_{n=0}^\infty \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|. \tag{1.60}$$

1.6.2 Appendix B: The hydrogen atom. Dipole allowed transitions.

In general operator notations the problem of an electron (momentum and position operators $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$)orbiting around a (fixed - in a first, simplifying approximation) nucleus can be solved by solving the following Schrödinger equation

$$i\hbar\partial_t |\psi\rangle = H |\psi\rangle, \qquad \text{with} \qquad H = \frac{\hat{\mathbf{p}}^2}{2\mu} + V(\hat{\mathbf{r}}). \tag{1.61}$$

Generally one separates time dependence from spatial dependence and then solves a time independent Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ to get the solutions for the eigenvectors $|nlm\rangle$ indexed by (n,l,m). The indexes satisfy the following inequalities $1 \leq n < \infty, 0 \leq \ell \leq n-1$ and $-\ell \leq m \leq \ell$).

Denoting the eigenvalues of the Hamiltonian by E_{nlm} the diagonal representation is Hamiltonian is then

$$H = \sum E_{nlm} |nlm\rangle\langle nlm|. \tag{1.62}$$

The standard procedure is to turn the Schrödinger equation into a second order differential equation. This is done by writing it in the position representation where the position operator is replaced by the position parameter while the momentum is replaced by $-i\hbar\nabla$. One then has to solve the following differential equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r,t)\right]\psi(\mathbf{r},t). \tag{1.63}$$

Notice that the potential is assumed spherically symetric as is the case for the Coulomb interaction. Without further details we now state the well known results obtained for the Hydrogen atom (reducing the nucleus to a single proton). The eigenvalue and eigenvectors are

$$E_n = -\left[\frac{\mu e^4}{32\pi^2\varepsilon_0^2\hbar^2}\right]\frac{1}{n^2} \qquad \text{and} \qquad \psi_{nlm_l}(r,\theta,\phi) = \langle \mathbf{r}|nlm\rangle = R_{nl}(r)Y_{l,m}(\theta,\phi). \tag{1.64}$$

The energies only depend on the principal quantum number n (which will not be the case anylonger when one considers spin-orbit interactions, relativistic corrections etc). The radial part of the wavefunction for the Hydrogen atom (in reduced coordinate $\rho = 2r/(na)$ is

$$R_{nl}(r) = -\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}\rho^l L_{n+l}^{2l+1}(\rho)e^{-\rho/2}. \tag{1.65}$$

where the Bohr radius is $a = 4\pi\varepsilon_0\hbar^2/(\mu e^2)$ ($\mu$is the reduced mass roughly equal to the electron mass) and is around $0.529 \times 10^{-10}m$.

The average orbital radius can be computed to give the result:

$$\bar{r}_{nl} = n^2 a_0 \left[1 + \frac{1}{2}\left(1 - \frac{l(l+1)}{n^2}\right)\right] \tag{1.66}$$

An atom is normally found in the electronic ground state in the absence of light-induced excitations. For the Hydrogen atom the ground state is the 1s state. We will denote it by the ket $|g\rangle$. The states closest in energy are lying at energy differences close to $\hbar \times 10^{15}$ Hz. This means that for resonant excitation we will need electromagnetic modes at optical frequencies around 10^{15} Hz. Now the atom is embedded in the vacuum which might be thermal. The vacuum is made by a collection of

harmonic oscillators in thermal states and therefore with average occupancy $\bar{n} = e^{-\hbar\omega/k_B T}$. Let's remember the constants $\hbar = 1.0545 \times 10^{-34}\mathrm{m^2kg/s}$ and $k_B = 1.3806 \times 10^{-23}\mathrm{m^2kgs^{-2}K^{-1}}$. We can compute a useful quantity in $k_B/\hbar = 1.309 \times 10^{11}\mathrm{K^{-1}s^{-1}}$. Now we see that even at room temperature with $T = 300\mathrm{K}$ the value of $\bar{n}$ is pretty small as $(10^{15}\mathrm{Hz}/(k_B T/\hbar) \simeq 10 \div 100$. In conclusion, thermal effects at such high frequencies do not play any role.
Now we assume that we can somehow couple to state 2p in either of the degenerate sublevels indexed by $m = -1, 0, 1$. Let's select a single possible excited state and denote it by the ket $|e\rangle$. We can evaluate now the matrix elements of the dipole moment operator $\hat{\mathbf{d}} = -e\hat{\mathbf{r}}$ as follows

$$\langle g|\hat{\mathbf{r}}|e\rangle = \int d\mathbf{r} \int d\mathbf{r}' \langle g|\mathbf{r}\rangle\langle\mathbf{r}|\hat{\mathbf{r}}|\mathbf{r}\rangle\langle\mathbf{r}|e\rangle = \int d\mathbf{r}\psi_g^*(\mathbf{r})\mathbf{r}\psi_e(\mathbf{r}). \tag{1.67}$$

Explicitly writing the integral in spherical coordinates for an unspecified m we get any of the sublevels of 2p

$$\int\int\int drd\theta d\phi(r^2\sin\theta)\left[2\frac{1}{a^{3/2}}e^{-r/2a}\frac{1}{\sqrt{4\pi}}\right](x\hat{x}+y\hat{y}+z\hat{z})\left[\frac{1}{8\sqrt{3}}\frac{1}{a^{3/2}}\frac{2r}{2a}e^{-r/2a}Y_{1,m}(\theta,\phi)\right] = \tag{1.68}$$

$$\frac{1}{\sqrt{4\pi}}\frac{1}{4\sqrt{3}}\frac{1}{a^4}\int dr(r^4)e^{-r/a}\int\int d\theta d\phi(\sin\theta)(\sin\theta\cos\phi\hat{x}+\sin\theta\sin\phi\hat{y}+\cos\theta\hat{z})Y_{1,m}(\theta,\phi).$$

The m=0 case: For $m = 0$ the last integral over angles becomes

$$\sqrt{\frac{3}{4\pi}}\int\int d\theta d\phi(\sin^2\theta\cos\phi\hat{x}+\sin^2\theta\sin\phi\hat{y}+\sin\theta\cos\theta\hat{z})\cos\theta. \tag{1.69}$$

We immediately see that the ϕ integration kills the contributions on x and y. The result is

$$2\pi\sqrt{\frac{3}{4\pi}}\int d\theta\sin\theta\cos\theta^2\hat{z} = 2\pi\sqrt{\frac{3}{4\pi}}\frac{2}{3} = \sqrt{\frac{4\pi}{3}}. \tag{1.70}$$

Putting it together

$$\langle 1s|\hat{\mathbf{r}}|2p_z\rangle = \frac{1}{12a^4}.\int dr(r^4)e^{-r/a}\hat{z} = \frac{1}{12a^4}.(24a^5)\hat{z} = 2a\hat{z}. \tag{1.71}$$

The m= - 1 case: For $m = -1$ the last integral over angles becomes

$$\sqrt{\frac{3}{8\pi}}\int\int d\theta d\phi(\sin^2\theta\cos\phi\hat{x}+\sin^2\theta\sin\phi\hat{y}+\sin\theta\cos\theta\hat{z})\sin\theta e^{-i\phi}. \tag{1.72}$$

We immediately see that the ϕ integration kills the contribution on z. We then use $\int d\phi\cos^2\phi = \int d\phi\sin^2\phi = \pi$ and $\int d\phi\sin\phi\cos\phi = 0$. The integral above becomes\pi

$$\pi\sqrt{\frac{3}{8\pi}}\left[\int d\theta\sin^3\theta(\hat{x}-i\hat{y})\right] = \sqrt{\frac{3}{4\pi}}\frac{4\pi}{3}(\hat{x}-i\hat{y}) = \sqrt{\frac{4\pi}{3}}(\hat{x}-i\hat{y}). \tag{1.73}$$

n	l	m	Orbital	$R_{nl}(\rho)$	$Y_{l,m}(\theta,\phi)$
1	0	0	1s	$2\frac{1}{a^{3/2}}e^{-\rho/2}$	$\frac{1}{\sqrt{4\pi}}$
2	0	0	2s	$\frac{1}{2\sqrt{2}}\frac{1}{a^{3/2}}(2-\rho)e^{-\rho/2}$	$\frac{1}{\sqrt{4\pi}}$
	1	-1	2p	$\frac{1}{8\sqrt{3}}\frac{1}{a^{3/2}}\rho e^{-\rho/2}$	$\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$
	1	0	2p	$\frac{1}{8\sqrt{3}}\frac{1}{a^{3/2}}\rho e^{-\rho/2}$	$\sqrt{\frac{3}{4\pi}}\cos\theta$
	1	1	2p	$\frac{1}{8\sqrt{3}}\frac{1}{a^{3/2}}\rho e^{-\rho/2}$	$-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$

Table 1.1: Energy levels and orbitals for the Hydrogen atom.

Putting it together

$$\langle 1s|\hat{\mathbf{r}}|2p_{-1}\rangle = 2a(\hat{x} - i\hat{y}). \tag{1.74}$$

The m= +1 case:
For $m = 1$, similarly we get

$$\langle 1s|\hat{\mathbf{r}}|2p_{+1}\rangle = 2a(\hat{x} + i\hat{y}). \tag{1.75}$$

1.6.3 Appendix C: Changing of picture (Interaction picture).

Let us see how a change of picture is performed. Suppose we start with the Schrödinger equation:

$$i\hbar\partial_t |\psi\rangle = H|\psi\rangle, \qquad \text{with} \qquad H = \hbar\omega_0\sigma^\dagger\sigma + \hbar\Omega_\ell\left(\sigma e^{i\omega_\ell t} + \sigma^\dagger e^{-i\omega_\ell t}\right) \tag{1.76}$$

and aim at removing the time dependence in the driving part of the Hamiltonian. To this end we transform into a different picture by a time dependent operator unitary $U(t)$ such that $|\psi\rangle_{IP} = U(t)|\psi\rangle$. Doing the proper transformations we end up with

$$i\hbar\partial_t |\psi\rangle_{IP} = \left[U(t)HU^\dagger(t) - i\hbar U(t)\partial_t U^\dagger(t)\right]|\psi\rangle_{IP}. \tag{1.77}$$

So we simply rewrote the Schrödinger equation in a different picture where the Hamiltonian is properly modified as indicated above. Let us try our luck with the following choice for $U(t) = e^{-i\omega_\ell t\sigma^\dagger\sigma}$. The last term leads to a modification of the free Hamiltonian from $\hbar\omega_0\sigma^\dagger\sigma$ to $\hbar(\omega_0 - \omega_\ell)\sigma^\dagger\sigma$. The transformation of operators is a bit more complicated:

$$\sigma_{IP} = U(t)\sigma U^\dagger(t) \tag{1.78}$$

$$= \left[1 + \frac{-i\omega_\ell t}{1!}\sigma^\dagger\sigma + \frac{(-i\omega_\ell t)^2}{2!}\sigma^\dagger\sigma + \ldots\right]\sigma\left[1 + \frac{i\omega_\ell t}{1!}\sigma^\dagger\sigma + \frac{(i\omega_\ell t)^2}{2!}\sigma^\dagger\sigma + \ldots\right] = \tag{1.79}$$

$$= \sigma\left[1 + \frac{i\omega_\ell t}{1!}\sigma^\dagger\sigma + \frac{(i\omega_\ell t)^2}{2!}\sigma^\dagger\sigma + \ldots\right] = \sigma\left[1 + \frac{i\omega_\ell t}{1!} + \frac{(i\omega_\ell t)^2}{2!} + \ldots\right] = \sigma e^{-i\omega_\ell t} \tag{1.80}$$

We have used the properties $\sigma\sigma = 0$ and $\sigma\sigma^\dagger\sigma = \sigma$. Check them out! In the end, the transformation to the IP removes the fast time dependence in the driving Hamiltonian and one can write the following Schrödinger equation: the Schrödinger equation for the transformed $|\psi\rangle_{IP} = U(t)|\psi\rangle$ (with $U(t) = e^{-iH_0t/\hbar}$)

$$i\hbar\partial_t |\psi\rangle_{IP} = H_{IP}|\psi\rangle_{IP} \qquad \text{with} \qquad H_{IP} = \hbar(\omega_0 - \omega_\ell)\sigma^\dagger\sigma + \hbar\Omega_\ell\left(\sigma + \sigma^\dagger\right) \tag{1.81}$$

2. The driven, decaying two level system

In the last chapter we discussed the quantization of light inside a box containing an infinite number of modes (which we treated as plane waves) with wave vectors pointing in any direction. We then added a TLS and derived the light-matter interaction Hamiltonian. The dynamics of the system then is unitary as excitations can flow from the field to the TLS and back. However, in reality we would like to take the limit of an infinite box where a photon emitted by the TLS will practically never return to it. To this end we perform an elimination of the box modes and derive the dynamics only in the 2-dimensional Hilbert space of the TLS. All other information about the electromagnetic modes is then uninteresting and the TLS dynamics becomes irreversible.

2.1 The quantum master equation for spontaneous emission

We start with an initial state density operator in the full Hilbert space of the TLS plus the infinitely many electromagnetic modes. We assume that the initial state at some time t is separable and write

$$\rho(t) = \rho_A(t) \otimes \rho_F(t). \tag{2.1}$$

This could be the case for example by starting with the atom in state $|e\rangle$ and field in vacuum state at zero temperature such that $\rho_F(t) = |0\rangle\langle 0|$ (the thermal bath case will be discussed later on). Also, the case where a coherent driving is present can be reduced to the vacuum case plus a classical term according to the Mollow transformation introduced in the previous chapter. More on this detail will be presented later when we describe the Bloch equations.
At some time t the total density operator fulfills the following equation of motion

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)]. \tag{2.2}$$

where the Hamiltonian is the one derived in the previous chapter

$$H = \hbar\omega_0\sigma^\dagger\sigma + \sum_{\mathbf{k}} \hbar\omega_k a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \sum_{\mathbf{k}} \hbar g_{\mathbf{k}} \left(a_{\mathbf{k}}\sigma^\dagger + \sigma a_{\mathbf{k}}^\dagger \right), \tag{2.3}$$

which one can break down into a sum of free evolution Hamiltonians H_A and H_F and the interaction part. Remember the scaling of the coupling coefficients of each mode $\mathbf{k}$ with the two level system incorporating the zero point electric field amplitude, the dipole transition element and the angle with respect to the polarization of the respective electromagnetic mode:

$$g_{\mathbf{k}} = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\frac{\mathbf{d}\cdot\hat{\varepsilon}_{\mathbf{k}}}{\hbar} = \frac{\mathscr{E}_k\mathbf{d}\cdot\hat{\varepsilon}_{\mathbf{k}}}{\hbar}. \tag{2.4}$$

Let us now remove the free evolution part by moving into an interaction picture with a unitary operator $U(t) = e^{-i(H_A+H_F)t/\hbar}$ (see Appendix 1.6.3) which transforms $\rho_I = U^\dagger\rho U$ and

$$H_I = \sum_{\mathbf{k}}\hbar g_{\mathbf{k}}\left[a_{\mathbf{k}}\sigma^\dagger e^{i(\omega_0-\omega_k)t} + \sigma a_{\mathbf{k}}^\dagger e^{-i(\omega_0-\omega_k)t}\right] = \hbar\hat{F}^\dagger(t)\sigma + \hbar\hat{F}(t)\sigma^\dagger. \tag{2.5}$$

Notice that we have compactly written time dependent operators acting only on the photon states and with the following properties

$$\hat{F}(t) = \sum_{\mathbf{k}} g_{\mathbf{k}}a_{\mathbf{k}}e^{i(\omega_0-\omega_k)t}, \qquad \hat{F}(t)\left|0\right\rangle = 0 \qquad \text{and} \qquad \left\langle 0\right|\hat{F}^\dagger(t) = 0. \tag{2.6}$$

We can now write the von Neumann equation in the IP which looks exactly as before except that the Hamiltonian is the one above written in the IP. For simplicity of notation we will not write the index I but still remember that until the end of the derivation we stay in the IP. Let us then proceed by formally integrating the equation of motion for the density operator in a small interval Δt (we will clarify towards the end of the derivation how small the interval actually is - compared to other timescales involved in the problem):

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H_I(t),\rho(t)] \qquad \to \qquad \rho(t+\Delta t) = \rho(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} dt_1[H_I(t_1),\rho(t_1)], \tag{2.7}$$

For any moment in time such that $t < t_1 < t+\Delta t$ we can again write the formal solution as $\rho(t_1) = \rho(t) - \int_t^{t_1} dt_2[H_I(t_2),\rho(t_2)]$ and plug it back in the above expression to get

$$\rho(t+\Delta t) = \rho(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} dt_1[H_I(t_1),\rho(t)] - \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt_1\int_t^{t_1} t_2[H_I(t_1),[H_I(t_2),\rho(t_2)]]. \tag{2.8}$$

This is still exact. We can now continue dividing the interval even further in the following ordered sequence $t < ...t_3 < t_2 < t_1 < t+\Delta t$ and obtain more and more terms in the expansion above. However, assuming that the field-TLS couplings are small compared to the energies of the TLS or the field modes, a truncation at the second order level suffices as a perturbative approach. This is equivalent to replacing $\rho(t_2)$ with $\rho(t)$ in the above formula and obtain the following equation

$$\rho(t+\Delta t) = \rho(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} dt_1\frac{i}{\hbar}[H_I(t_1),\rho(t)] - \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt_1\int_t^{t_1} dt_2[H_I(t_1),[H_I(t_2),\rho(t)]]. \tag{2.9}$$

Since we are only interested in the state of the TLS and not of the bath we would like to re-write the above equation into an effective equation for $\rho_A = Tr_F[\rho]$. This means we will have to perform traces like $\mathrm{Tr}_F[H_I(t_1),\rho_A(t)\otimes\rho_F(t)]$ and $\mathrm{Tr}_F[H_I(t_1),[H_I(t_2),\rho_A(t)\otimes\rho_F(t)]$.

First order traces

Remember that $\rho_F(t) = \left|0\right\rangle\left\langle 0\right|$ which means that in the following trace $\mathrm{Tr}_F[H_I(t_1),\rho_A(t)\otimes\rho_F(t)]$ only the vacuum state survives. This is easily seen by explicitly writing the trace: $\mathrm{Tr}_F[H_I(t_1),\rho_A(t)\otimes\rho_F(t)] = \sum_n \left\langle n\right|[H_I(t_1),\rho_A(t)\otimes\rho_F(t)]\left|n\right\rangle$. Using $\left\langle n|0\right\rangle = \delta_{n,0}$ we end up with

$$\mathrm{Tr}_F[H_I(t_1),\rho_A(t)\otimes\rho_F(t)] = \left\langle 0|H_I(t_1)|0\right\rangle\rho_A(t) - \rho_A(t)\left\langle 0|H_I(t_1)|0\right\rangle = 0. \tag{2.10}$$

Second order traces

Let's explicitly separate the second order terms coming from the double commutator $[H_I(t_1),[H_I(t_2),\rho(t)]$ into 4 distinct parts:

$$T_1 = \mathrm{Tr}_F\left[H_I(t_1)H_I(t_2)\rho_A(t)\otimes\rho_F(t)\right], \tag{2.11}$$
$$T_2 = -\mathrm{Tr}_F\left[H_I(t_1)\rho_A(t)\otimes\rho_F(t)H_I(t_2)\right], \tag{2.12}$$
$$T_3 = -\mathrm{Tr}_F\left[H_I(t_2)\rho_A(t)\otimes\rho_F(t)H_I(t_1)\right], \tag{2.13}$$
$$T_4 = \mathrm{Tr}_F\left[\rho_A(t)\otimes\rho_F(t)H_I(t_2)H_I(t_1)\right]. \tag{2.14}$$

Before evaluating the terms above we can observe a few simplifying rules. With the results $F(t)|0\rangle = 0$ and $\langle 0|F^\dagger(t) = 0$ it means that we can reduce some terms like $H_I\rho_F = \hbar\sigma F^\dagger$ and $\rho_F H_I = \hbar\sigma F$. Moreover the field operators do not act on ρ_A so we can commute them. Let's then work out the first term T_1 according to these rules

$$\begin{aligned} T_1 &= \hbar^2\mathrm{Tr}_F[\left(F^\dagger(t_1)\sigma + F(t_1)\sigma^\dagger\right)\sigma F^\dagger(t_2)\rho_A(t)\otimes\rho_F(t)] && (2.15)\\ &= \hbar^2\sigma^\dagger\sigma\rho_A\mathrm{Tr}_F\left[F(t_1)F^\dagger(t_2)\rho_F(t)\right]. && (2.16)\end{aligned}$$

Similarly we can find:

$$T_2 = -\hbar^2\sigma\rho_A\sigma^\dagger\mathrm{Tr}_F[\hat{F}(t_1)\hat{F}^\dagger(t_2)\rho_F(t)], \tag{2.17a}$$
$$T_3 = -\hbar^2\sigma\rho_A\sigma^\dagger\mathrm{Tr}_F[\hat{F}(t_2)\hat{F}^\dagger(t_1)\rho_F(t)], \tag{2.17b}$$
$$T_4 = \hbar^2\rho_A\sigma^\dagger\sigma\mathrm{Tr}_F[\hat{F}(t_2)\hat{F}^\dagger(t_1)\rho_F(t)]. \tag{2.17c}$$

We see that in the end we are left with the task of evaluating bath correlations at different times and then integrate over them. For example, the term coming from T_1 will be

$$\mathscr{B} = \int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2\mathrm{Tr}_F[\hat{F}(t_1)\hat{F}^\dagger(t_2)\rho_F(t)]. \tag{2.18}$$

Also notice that inversing the time ordering has the effect of a complex conjugate: $\mathrm{Tr}_F[\hat{F}(t_2)\hat{F}^\dagger(t_1)\rho_F(t)] = \mathrm{Tr}_F[\hat{F}(t_1)\hat{F}^\dagger(t_2)\rho_F(t)]^*$. Adding everything up we can find a compact expression connecting the reduced density operator at two different times:

$$\rho_A(t+\Delta t) - \rho_A(t) = (\mathscr{B}+\mathscr{B}^*)\sigma\rho_A(t)\sigma^\dagger - \sigma^\dagger\sigma\rho_A(t)\mathscr{B} - \rho_A(t)\sigma^\dagger\sigma\mathscr{B}^*. \tag{2.19}$$

A closer inspection of the terms inside the $\mathscr{B}$ coefficient show that they are not too hard to evaluate and understand. Writing in detail

$$\begin{aligned} \mathrm{Tr}_F\left[\hat{F}(t_1)\hat{F}^\dagger(t_2)\rho_F(t)\right] &= \langle 0|\sum_{\mathbf{k}}\sum_{\mathbf{k}'} g_{\mathbf{k}}g^*_{\mathbf{k}'}a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}e^{i(\omega_0-\omega_k)t_1}e^{-i(\omega_0-\omega'_k)t_2}|0\rangle && (2.20)\\ &= \sum_{\mathbf{k}}|g_{\mathbf{k}}|^2 e^{i(\omega_0-\omega_k)(t_1-t_2)}. && (2.21)\end{aligned}$$

We have used the fact that $a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}|0\rangle = \delta_{\mathbf{k}\mathbf{k}'}|0\rangle$ to reduce to a single sum. Putting it all together we again see that the task is to evaluate the following quantity

$$\mathscr{B} = \sum_{\mathbf{k}}|g_{\mathbf{k}}|^2\int_t^{t+\Delta t} dt_1\int_t^{t_1} dt_2 e^{i(\omega_0-\omega_k)(t_1-t_2)}. \tag{2.22}$$

This we can do in two steps: first summing over all k-vectors and polarizations and then performing the time integral.

Summing over k-vectors and polarizations

To evaluate the sum we will write it as an integral. The general rule for turning a sum into an integral is

$$\sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 ... = \sum_{\lambda} \int d\mathbf{k} |g_{\mathbf{k}}|^2 D(\mathbf{k}) ..., \tag{2.23}$$

where $D(\mathbf{k})$ is a function also called density of states, which should verify that we count everything correctly. It is obvious that for the sum to integral transformation to be valid it should satisfy the following condition within a given volume in k-space

$$\sum_{\mathbf{k}} 1 = \sum_{\lambda} \int d\mathbf{k} D(\mathbf{k}), \tag{2.24}$$

Let us notice that as we quantized the field in a box of dimensions $L \times L \times L$ the allowed k-vector components on a given axis are separated by $2\pi/L$. Therefore we will only find one k-vector with a given polarization within a volume $(2\pi/L)(2\pi/L)(2\pi/L) \times 8 * \pi^3/V$. Writing the above condition then results in $2 = D(\mathbf{k}) 8 * \pi^3/V$ which readily gives

$$D(\mathbf{k}) = \frac{V}{4\pi^3}. \tag{2.25}$$

The term stemming from the coupling is expressed as

$$|g_{\mathbf{k}}|^2 = \frac{\omega_k}{2\hbar\varepsilon_0 V} (\mathbf{d} \cdot \hat{\varepsilon}_{\mathbf{k}})^2. \tag{2.26}$$

For any given direction defined the unit vector $\hat{k}$, the three unit vectors $\hat{\varepsilon}_{\mathbf{k}}^{(1)}$, $\hat{\varepsilon}_{\mathbf{k}}^{(2)}$and $\hat{k}$ are orthonormal and thus constitute a basis. Therefore we can write $\mathbf{d} = (\mathbf{d} \cdot \hat{\varepsilon}_{\mathbf{k}}^{(1)})\hat{\varepsilon}_{\mathbf{k}}^{(1)} + (\mathbf{d} \cdot \hat{\varepsilon}_{\mathbf{k}}^{(2)})\hat{\varepsilon}_{\mathbf{k}}^{(2)} + (\mathbf{d} \cdot \hat{k})\hat{k}$ and consequently the amplitude squared $|\mathbf{d}|^2 = (\mathbf{d} \cdot \hat{\varepsilon}_{\mathbf{k}}^{(1)})^2 + (\mathbf{d} \cdot \hat{\varepsilon}_{\mathbf{k}}^{(2)})^2 + (\mathbf{d} \cdot \hat{k})^2$. We can then express

$$\sum_{\lambda} |g_{\mathbf{k}}|^2 = \frac{\omega_k}{2\hbar\varepsilon_0 V} \left[|\mathbf{d}|^2 - (\mathbf{d} \cdot \hat{k})^2 \right]. \tag{2.27}$$

Assuming the $\mathbf{d}$ points out in the z direction, we effectively have $|\mathbf{d}|^2 - (\mathbf{d} \cdot \hat{k})^2 = |\mathbf{d}|^2(1 - \cos^2\theta)$. Now we can write the integral

$$\sum_{\lambda} \int d\mathbf{k} |g_{\mathbf{k}}|^2 D(\mathbf{k}) ... = \frac{V}{4\pi^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta (1 - \cos^2\theta) \int_0^{\infty} dk k^2 \frac{\omega_k |\mathbf{d}|^2}{2\hbar\varepsilon_0 V} ... \tag{2.28}$$

$$= \frac{|\mathbf{d}|^2}{8\pi^3\hbar\varepsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta (1 - \cos^2\theta) \int_0^{\infty} dk k^2 \omega_k ..., \tag{2.29}$$

and already perform the angle integral $\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta (1 - \cos^2\theta) = 2\pi(4/3) = 8\pi/3$ leading to

$$\sum_{\lambda} \int d\mathbf{k} |g_{\mathbf{k}}|^2 D(\mathbf{k}) ... = \frac{|\mathbf{d}|^2}{8\pi^3\hbar\varepsilon_0} \frac{8\pi}{3} \int dk k^2 \omega_k ... = \frac{|\mathbf{d}|^2}{3\pi^2 c^3 \hbar\varepsilon_0} \int d\omega_k \omega_k^3 .... \tag{2.30}$$

Performing the time integral

We finally get to the time dependant part as we can state

$$\mathscr{B} = \frac{|\mathbf{d}|^2}{3\pi^2 c^3 \hbar\varepsilon_0} \int_0^{\infty} d\omega_k \int_t^{t+\Delta t} dt_1 \int_t^{t_1} t_2 e^{-i(\omega_k - \omega_0)(t_1 - t_2)} \omega_k^3. \tag{2.31}$$

Let us rewrite it for clarity with the transformation $\omega_k - \omega_0 = x$ (and noticing that as ω_0 is a very high frequency we can extend the integration over to $-\infty$)

$$\mathscr{B} = \frac{|\mathbf{d}|^2}{3\pi^2 c^3 \hbar \varepsilon_0} \int_{-\infty}^{\infty} dx \int_0^{\Delta t} dt_1 e^{-ixt_1} \int_0^{t_1} t_2 e^{ixt_2} (x+\omega_0)^3 = \frac{|\mathbf{d}|^2}{3\pi^2 c^3 \hbar \varepsilon_0} \mathscr{I}(\Delta t), \tag{2.32}$$

We also removed the lower bound t to zero as the initial time is irrelevant in the integral. Notice that the exponential oscillates and averages the polynomial to zero unless x is around the origin. proceed with the first integral, where caution has to be taken as a singularity appears at $x = 0$. We will deal with this by the following trick:

$$\int_0^{t_1} t_2 e^{ixt_2} (x+\omega_0)^3 = \lim_{\varepsilon \to 0} \int_0^{t_1} t_2 e^{ixt_2 - \varepsilon t_2} (x+\omega_0)^3 = \lim_{\varepsilon \to 0} \frac{(x+\omega_0)^3}{ix - \varepsilon} \left(e^{ixt_1 - \varepsilon t_1} - 1 \right) = \tag{2.33}$$

Plugging it back into the integral

$$\mathscr{I}(\Delta t) = \int_{-\infty}^{\infty} dx \int_0^{\Delta t} dt_1 \lim_{\varepsilon \to 0} \frac{(x+\omega_0)^3}{ix - \varepsilon} \left(e^{-\varepsilon t_1} - e^{-ixt_1} \right). \tag{2.34}$$

If Δt is very large (can be taken to infinity), the last contribution is averaged to zero (the one from e^{-ixt_1}). The first contribution can be rewritten:

$$\lim_{\varepsilon \to 0} \frac{-(ix+\varepsilon)(x+\omega_0)^3}{x^2+\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{-(ix+\varepsilon)(x+\omega_0)^3}{x^2+\varepsilon^2} = \tag{2.35}$$

$$= -\lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon^2} (x+\omega_0)^3 - i \lim_{\varepsilon \to 0} \frac{x}{x^2+\varepsilon^2} (x+\omega_0)^3. \tag{2.36}$$

The limits make sense as distributions. The real part becomes a delta function

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon^2} = \pi \delta(x), \tag{2.37}$$

while the imaginary part becomes the principal value distribution

$$\lim_{\varepsilon \to \infty} \frac{x}{x^2+\varepsilon^2} = \mathrm{P}(\frac{1}{x}), \tag{2.38}$$

defined (in the sense of a distribution thus acting on test functions) as

$$\int_{-\infty}^{\infty} dx \mathrm{P}(\frac{1}{x}) \mathrm{f(x)} = \lim_{\varepsilon \to 0} [\int_{-\infty}^{-\varepsilon} dx + \int_{\varepsilon}^{\infty} dx] \frac{f(x)}{x}. \tag{2.39}$$

The principal part (as one can for example check numerically for safety) makes very little contribution. Keeping the delta function only one obtains:

$$\mathscr{I}(\Delta t) = \pi \omega_0^3 \Delta t. \tag{2.40}$$

Finally we can put everything together and find that the coefficient $\mathscr{B}$ in the density operator evolution is proportional to the small time increment

$$\mathscr{B} = \left[\frac{|\mathbf{d}|^2 \omega_0^3}{3\pi c^3 \hbar \varepsilon_0} \right] \Delta t. \tag{2.41}$$

Final result

Finally, as the terms in the right side of the master equation are real and proportional to Δt, we can write we can express the master equation as $\rho_A(t+\Delta t)-\rho_A(t)=\partial_t\rho_A(t)\Delta t$. Moreover, we will now transform back from the interaction picture and write a very simple form for the master equation (written below in the Schrödinger picture)

Important 2.1.1 — The master equation for spontaneous emission.

$$\frac{d}{dt}\rho_A=-\frac{i}{\hbar}[H_A,\rho_A]+\gamma\left\{2\sigma\rho_A\sigma^\dagger-\rho_A\sigma^\dagger\sigma-\sigma^\dagger\sigma\rho_A\right\}. \tag{2.42}$$

A few words on this. Without the box, the atom simply evolves in a coherent, deterministic way governed by the free Hamiltonian H_A. The interaction with the box brings along an infinity of terms describing the coupling of the TLS with any of the electromagnetic modes supported by the box. The trace over the bath modes leads to an irreversible dynamics contained in the super-operator Linbdlad term that describes the non-trivial action of the collapse operator σ at rate γ on the density operator. The action is non-trivial as it cannot be represented as a matrix multiplication. In simplified notation we can also write

$$\frac{d}{dt}\rho_A=-\frac{i}{\hbar}[H_A,\rho_A]+\gamma\mathscr{D}[\sigma,\rho_A] \qquad \text{with} \qquad \mathscr{D}[\sigma,\rho_A]=2\sigma\rho_A\sigma^\dagger-\left\{\rho_A,\sigma^\dagger\sigma\right\}_+. \tag{2.43}$$

where the brackets indexed by a plus sign show the anticommutator.
The quantity in the brackets is the spontaneous emission rate of a two-level system into the electromagnetic vacuum modes:

$$\gamma=\frac{|\mathbf{d}|^2\omega_0^3}{3\pi c^3\hbar\varepsilon_0}. \tag{2.44}$$

Let's check out the value of the decay rate for the Hydrogen 1s to 2p transition. The constants are: $\hbar=1.055\times10^{-34}m^2s^{-1}kg$, $c=3\times10^8 m/s$, $\varepsilon_0=8.85\times10^{-12}A^2s^4kg^{-1}m^{-3}$. The 1s energy is -13.6eV while the 2p energy is roughly $-13.6/4$ eV. We can equate $\hbar\omega_0=3/4\times13.6\times1.602\times10^{-19}J$ to obtain $\omega_0=1.139\times10^{15}$ Hz. Remembering from the first lecture that for the z-polarized transition, the dipole matrix element is $2a_0e$ where the Bohr radius is $a_0=0.53\times10^{-10}$ m, we can compute $\gamma=1.79\times10^6$ Hz, so on the order of MHz.

Master equation in a thermal bath

The calculation performed on an initial vacuum state of the bath can be easily extended to a thermally occupied bath at temperature T and with an average photon number occupancy $\bar{n}(\omega_0)$. For such a bath we can write the initial density operator

$$\rho_F(t)=\frac{e^{-H_F/(k_BT)}}{\mathrm{Tr}_F\left[e^{-H_F/(k_BT)}\right]}=\prod_{\mathbf{k}}\frac{e^{-n_{\mathbf{k}}\hbar\omega_{\mathbf{k}}/(k_BT)}}{1-e^{-\hbar\omega_{\mathbf{k}}/(k_BT)}}|n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|. \tag{2.45}$$

Without going into the details of the derivation notice that the only step where a difference occurs is the calculation of the correlations of the bath. These correlations will give rise to a modified emission rate and an extra term showing the possibility that the bath can induce absorption. In short one obtains:

$$\frac{d}{dt}\rho_A=-\frac{i}{\hbar}[H_A,\rho_A]+\gamma\left(\bar{n}(\omega_0)+1\right)\mathscr{D}[\sigma,\rho_A]+\gamma\bar{n}(\omega_0)\mathscr{D}[\sigma^\dagger,\rho_A]. \tag{2.46}$$

The Linblad term with collapse operator σ contains the expected spontaneous emission term to which a thermally activated stimulated emission is added. The second Linblad term has a collapse operator $\sigma^\dagger$ showing inverse decay from the ground to the excited state, i.e. stimulated absorption from the thermal bath. As we mentioned before, typically we will deal with optical transition where even at room temperature $\bar{n}(\omega_0) \ll 1$ such that we will not make too much use of the above expression.

Alternative derivation of the decay rate

One can use a variety of methods to derive the spontaneous emission rate. For example, some textbook are using the so-called Wigner-Weisskopf derivation. We can sketch another simple intuitive derivation by making use of the Fermi's golden rule for the computation of the transition probability between states $|e,0\rangle$ and $|g,1_{\mathbf{k}}\rangle$ for any possible direction and polarization of the emitted photon. We use

$$\begin{aligned} w_{e\to g} &= \frac{2\pi}{\hbar^2}\sum_{\mathbf{k}} |\langle e,0|H_{int}|g,1_{\mathbf{k}}\rangle|^2\delta(\omega_k-\omega_0) = \\ &= 2\pi\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\sum_{\mathbf{k}'}\langle e,0|H_{int}|g,1_{\mathbf{k}}\rangle\langle g,1_{\mathbf{k}}|H_{int}|e,0\rangle\delta(\omega_k-\omega_0) = \\ &= 2\pi\sum_{\mathbf{k}}\langle e,0|g_{\mathbf{k}'}a_{\mathbf{k}'}\sigma^\dagger|g,1_{\mathbf{k}}\rangle\langle g,1_{\mathbf{k}}|g_{\mathbf{k}''}a^\dagger_{\mathbf{k}''}\sigma|e,0\rangle\delta(\omega_k-\omega_0) = \\ &= 2\pi\sum_{\mathbf{k}}|g_{\mathbf{k}}|^2\delta(\omega_k-\omega_0) = \frac{2\pi|\mathbf{d}|^2}{3\pi^2c^3\hbar\varepsilon_0}\int d\omega_k\omega_k^3\delta(\omega_k-\omega_0) = \\ &= \frac{2|\mathbf{d}|^2\omega_0^3}{3\pi c^3\hbar\varepsilon_0} = \gamma. \end{aligned} \tag{2.47}$$

Notice that the rate obtained at which the system emits spontaneously is exactly twice the rate we computed before via the master equation derivation.

2.2 Bloch equations (in the Schrödinger picture)

We now know how the effect of the interaction with the vacuum on a TLS can be mathematically described. The resulting dynamics in the reduced Hilbert space of dimension 2 is irreversible and characterized by the action of a super-operator onto the density operator. Let us now add a coherent drive and check out the resulting driven-dissipative dynamics. This can be done as described in the first chapter by assuming one of the field modes to be in a coherent state characterized by a given direction $\mathbf{k}_0$ including a given polarization $\varepsilon_{\mathbf{k}_0}$. After performing the Mollow transformation one ends up with the effect of the drive as a semiclassical Hamiltonian (H_ℓ) added to the free evolution Hamiltonian (H_0):

$$H = \hbar\omega_0\sigma^\dagger\sigma + \hbar\Omega\left(\sigma e^{i\omega_\ell t} + \sigma^\dagger e^{-i\omega_\ell t}\right). \tag{2.48}$$

The Rabi frequency depends on the amplitude of the coherent states as well as on the transition dipole moment and its overlap with the light mode polarization vector:

$$\Omega = \frac{1}{\hbar}\mathscr{E}_{k_0}\alpha_{\mathbf{k}_0}\mathbf{d}\cdot\hat{\varepsilon}_{\mathbf{k}_0} \tag{2.49}$$

We now can follow the evolution of the system in the Schrödinger picture by solving the complete equation of motion for the density operator (we drop in the following the A subscript but remember we are always working in the 2-dimensional Hilbert space of the TLS):

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \gamma\left\{2\sigma\rho\sigma^\dagger - \rho\sigma^\dagger\sigma - \sigma^\dagger\sigma\rho\right\}. \tag{2.50}$$

Derivation starting from the density operator

Notice that in the Hilbert space spanned by the two basis states $|g\rangle$ and $|e\rangle$ the density operator has four matrix elements. The elements are computed the usual way as sandwiches between bras and ket, for example: $\rho_{eg} = \langle e|\rho|g\rangle$. Let us as an example compute the evolution of ρ_{eg}. To this end we sandwich the master equation above between $\langle e|$ and $|g\rangle$:

$$\frac{d}{dt}\rho_{eg} = -\frac{i}{\hbar}\langle e|[H,\rho]|g\rangle + \gamma\left\{2\langle e|\sigma\rho\sigma^\dagger|g\rangle - \langle e|\rho\sigma^\dagger\sigma|g\rangle - \langle e|\sigma^\dagger\sigma\rho|g\rangle\right\}. \tag{2.51}$$

We now remember the action of operators on the ladder operators on states $\sigma|e\rangle = |g\rangle$, $\sigma|g\rangle = 0$, $\sigma^\dagger|e\rangle = 0$, $\sigma^\dagger|g\rangle = |e\rangle$ and so on. Let's break the terms down in 3 parts. Free evolution gives

$$-\frac{i}{\hbar}\langle e|[H_0,\rho]|g\rangle = -i\omega_0\langle e|\left(\sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma\right)|g\rangle = -i\omega_0\rho_{eg}. \tag{2.52}$$

The drive terms leads to the following contribution:

$$\begin{aligned} -\frac{i}{\hbar}\langle e|[H_\ell,\rho]|g\rangle &= -i\Omega\langle e|(\sigma e^{i\omega_\ell t} + \sigma^\dagger e^{-i\omega_\ell t})\rho - \rho(\sigma e^{i\omega_\ell t} + \sigma^\dagger e^{-i\omega_\ell t})|g\rangle \\ &= -i\Omega\langle g|e^{-i\omega_\ell t}\rho|g\rangle + i\Omega\langle e|e^{-i\omega_\ell t}\rho|e\rangle = i\Omega(\rho_{ee} - \rho_{gg})e^{-i\omega_\ell t} \end{aligned} \tag{2.53}$$

Finally, the spontaneous emission gives rise to

$$\gamma\left\{2\langle e|\sigma\rho\sigma^\dagger|g\rangle - \langle e|\rho\sigma^\dagger\sigma|g\rangle - \langle e|\sigma^\dagger\sigma\rho|g\rangle\right\} = -\gamma\rho_{eg}. \tag{2.54}$$

Putting it all together we obtain an equation of motion for the 'coherence' ρ_{eg} showing free evolution at frequency ω_0, driving with strength Ω and frequency ω_ℓ and decay at amplitude decay rate γ

$$\frac{d}{dt}\rho_{eg} = -\gamma\rho_{eg} - i\omega_0\rho_{eg} + i\Omega(\rho_{ee} - \rho_{gg})e^{-i\omega_\ell t}. \tag{2.55}$$

Notice that the drive depends on the population difference between excited state ρ_{ee} and ground state ρ_{gg}. One can continue in deriving the equations for the other elements. Notice that since the trace of the density operator is conserved then $\rho_{ee} + \rho_{gg} = 1$. This basically means that the population can only be in one of the two states of the system. Also notice that $\rho_{ge} = \rho_{eg}^*$. In effect we only have to compute the excited state population evolution equation which we list below:

$$\frac{d}{dt}\rho_{ee} = -2\gamma\rho_{ee} + i\Omega(\rho_{eg}e^{i\omega_\ell t} - \rho_{ge}e^{-i\omega_\ell t}). \tag{2.56}$$

Derivation on the level of the density matrix

A more direct way to obtain all the equations is to make use of the matrix representation of the density operator, i.e. the density matrix formalism. We start by representing the states as vectors and properly writing the density operator as a matrix in this basis:

$$|g\rangle \to \begin{bmatrix}0\\1\end{bmatrix} \qquad |e\rangle \to \begin{bmatrix}1\\0\end{bmatrix} \qquad \text{and} \qquad \rho = \begin{bmatrix}\rho_{ee} & \rho_{eg}\\ \rho_{ge} & \rho_g\end{bmatrix}. \tag{2.57}$$

It is the straightforward to write the ladder operators as well as the projectors into the excited and ground state as matrices as well:

$$\sigma = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \qquad \sigma^\dagger = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \qquad \sigma^\dagger\sigma = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \qquad \sigma\sigma^\dagger = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \tag{2.58}$$

and check that they perform the required tasks by multiplying them with the state vectors. The free and driving Hamiltonians are then written as

$$H_0 = \begin{bmatrix} \hbar\omega_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad H_\ell = \begin{bmatrix} 0 & \hbar\Omega e^{-i\omega_\ell t} \\ \hbar\Omega e^{i\omega_\ell t} & 0 \end{bmatrix}, \tag{2.59}$$

from which one can derive the Hamiltonian part of the master equation from matrix multiplications.

$$-\frac{i}{\hbar}[H,\rho] = \begin{bmatrix} i\Omega(\rho_{eg}e^{i\omega_\ell t} - \rho_{ge}e^{-i\omega_\ell t}) & -i\omega_0\rho_{eg} + i\Omega e^{-i\omega_\ell t}(\rho_{ee} - \rho_{gg})) \\ -i\omega_0\rho_{ge} - +i\Omega e^{i\omega_\ell t}(\rho_{ee} - \rho_{gg})) & -i\Omega(\rho_{eg}e^{i\omega_\ell t} - \rho_{ge}e^{-i\omega_\ell t}) \end{bmatrix} \tag{2.60}$$

Notice that the Lindblad part gives rise to the following matrix on the rhs of the master equation:

$$\gamma\mathscr{D}[\sigma,\rho] = \begin{bmatrix} -2\gamma\rho_{ee} & -\gamma\rho_{eg} \\ -\gamma\rho_{ge} & 2\gamma\rho ee \end{bmatrix} \tag{2.61}$$

Important 2.2.1 — Bloch equations. Putting it all together we get a set of equations describing the free evolution, effect of driving and spontaneous emission of a single TLS:

$$\partial_t\rho_{ee} = -2\gamma\rho_{ee} + i\Omega(t)(\rho_{eg}e^{i\omega_\ell t} - \rho_{ge}e^{-i\omega_\ell t}), \tag{2.62a}$$
$$\partial_t\rho_{eg} = -\gamma\rho_{eg} - i\omega_0\rho_{eg} + i\Omega(t)(\rho_{ee} - \rho_{gg})e^{-i\omega_\ell t}. \tag{2.62b}$$

Remember that the other two matrix elements are derived from $\rho_{ee} + \rho_{gg} = 1$ and $\rho_{ge} = \rho_{eg}^*$.

We can easily remove the time dependence by moving into a rotating frame, procedure which is equivalent of saying that we write equations only for the slowly varying envelopes $\rho_{eg} = \tilde{\rho}_{eg}e^{-i\omega_\ell t}$. Notice that $\partial_t\rho_{eg} = \partial_t\tilde{\rho}_{eg}e^{-i\omega_\ell t} - i\omega_\ell\tilde{\rho}_{eg}$. We end up with rewriting equations of motion in a rotating frame and with defined detuning: $\Delta = \omega_0 - \omega_\ell$. I'll drop the tilde since it takes forever to type it in latex :)

$$\partial_t\rho_{ee} = -2\gamma\rho_{ee} + i\Omega(t)\left[\rho_{eg} - \rho_{ge}\right], \tag{2.63a}$$
$$\partial_t\rho_{eg} = -\gamma\rho_{eg} - i\Delta\rho_{eg} + i\Omega(t)\left(\rho_{gg} - \rho_{ee}\right). \tag{2.63b}$$

2.3 Bloch equations (in the Heisenberg picture)

An equivalent procedure is to solve not for the density operator time evolution but instead to derive equations of motion for single or more operator averages. For example let us compute the evolution for $\langle\sigma\rangle(t) = \mathrm{Tr}[\sigma\rho(t)]$. We notice that $\partial_t\langle\sigma\rangle(t) = \partial_t\mathrm{Tr}[\sigma(t)\rho(0)] = \partial_t\mathrm{Tr}[\sigma\rho(t)] = \mathrm{Tr}[\sigma\partial_t\rho(t)]$ which means we can now use the master equation to write

$$\begin{aligned}\partial_t\langle\sigma(t)\rangle &= \mathrm{Tr}\left[\sigma(-\frac{i}{\hbar}[H,\rho] + \gamma\{2\sigma\rho_A\sigma^\dagger - \rho\sigma^\dagger\sigma - \sigma^\dagger\sigma\rho\})\right] \\ &= -i\omega_0\langle\sigma(t)\rangle - \gamma\langle\sigma(t)\rangle + i\Omega e^{-i\omega_\ell t}\langle\sigma^\dagger\sigma - \sigma\sigma^\dagger\rangle.\end{aligned} \tag{2.64}$$

As one can observe that $\langle\sigma\rangle(t) = \mathrm{Tr}[|g\rangle\langle e|\rho(t)] = \langle e|\rho(t)|g\rangle = \rho_{eg}(t)$ the equation above is no surprise.

2.4 Time dynamics of the driven-dissipative TLS

We will now analyze the time dynamics imposed by the above equations under different conditions. We will mainly distinguish between resonant versus off-resonant driving, coherent versus incoherent evolution, transient versus steady state dynamics and linear versus non-linear regimes.

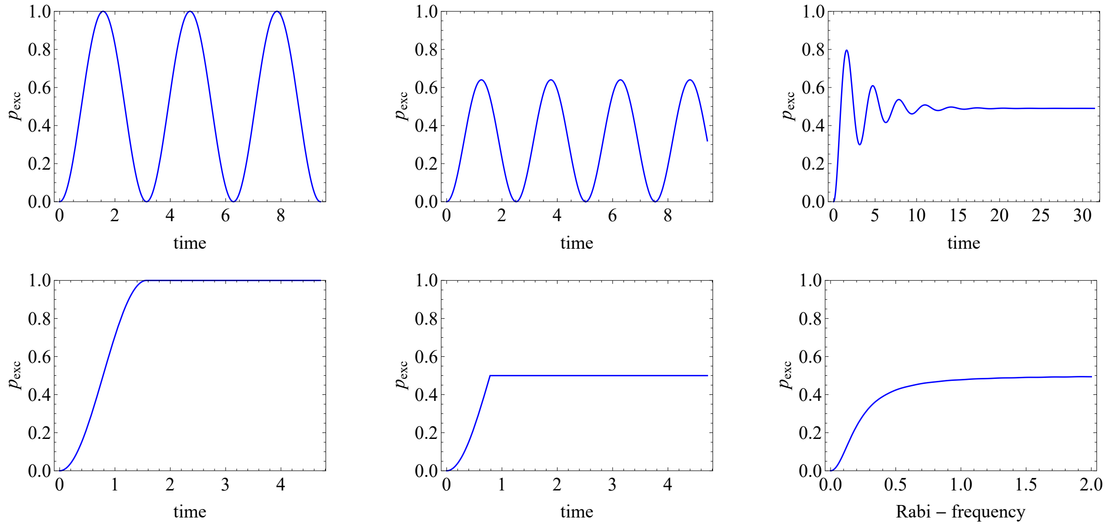

Figure 2.1: Response of a TLS under driving and decay.

Rabi oscillations

We now focus on the purely coherent dynamics of a drive system (no dissipation included). We start with the equations of motion after the removal of the fast optical oscillations such that

$$\partial_t \rho_{ee} = i\Omega(t)\left[\rho_{eg} - \rho_{ge}\right], \tag{2.65a}$$
$$\partial_t \rho_{eg} = -i\Delta\rho_{eg} + i\Omega(t)\left(\rho_{gg} - \rho_{ee}\right), \tag{2.65b}$$
$$\partial_t \rho_{ge} = i\Delta\rho_{ge} - i\Omega(t)\left(\rho_{gg} - \rho_{ee}\right). \tag{2.65c}$$

We can simplify things a bit by the following notations: $X = \rho_{eg} + \rho_{ge}$, $Y = i(\rho_{eg} - \rho_{ge})$ and $Z = \rho_{ee} - \rho_{gg}$. In terms of these three (real) values one can write

$$\partial_t Z = 2\Omega(t)Y, \tag{2.66a}$$
$$\partial_t X = -\Delta Y \tag{2.66b}$$
$$\partial_t Y = \Delta X + \Omega(t)Z. \tag{2.66c}$$

$$\frac{d}{dt}\rho_{ge} = -\frac{\gamma}{2}\rho_{ge} + i\Delta\rho_{ge} - i\Omega(t)\left(\rho_{gg} - \rho_{ee}\right), \tag{2.67}$$

allows us to derive the real and imaginary parts evolving as

$$\frac{d}{dt}(\rho_{ge} + \rho_{eg}) = -\frac{\gamma}{2}(\rho_{ge} + \rho_{eg}) + i\Delta(\rho_{ge} - \rho_{eg}) \tag{2.68}$$

$$\frac{d}{dt}(\rho_{ge} - \rho_{eg}) = -\frac{\gamma}{2}(\rho_{ge} - \rho_{eg}) + i\Delta(\rho_{ge} + \rho_{eg}) - 2i\Omega(t)\left(1 - 2\rho_{ee}\right). \tag{2.69}$$

For a simple solution we now focus on resonance where only two equations are consequently coupled.

$$\frac{d}{dt}\rho_{ee} = -\gamma\rho_{ee} - i\Omega(t)\left[\rho_{ge} - \rho_{eg}\right], \tag{2.70}$$

$$\frac{d}{dt}(\rho_{ge}-\rho_{eg}) = -\frac{\gamma}{2}(\rho_{ge}-\rho_{eg}) - 2i\Omega(t)\,(1-2\rho_{ee})\,. \tag{2.71}$$

In vector form one can write:

$$\frac{d}{dt}\begin{bmatrix}\rho_{ee}\\ \rho_{ge}-\rho_{eg}\end{bmatrix} = \begin{bmatrix}-\gamma & -i\Omega(t)\\ 4i\Omega(t) & -\gamma/2\end{bmatrix}\begin{bmatrix}\rho_{ee}\\ \rho_{ge}-\rho_{eg}\end{bmatrix} + \begin{bmatrix}0\\ -2i\Omega\rho_{ee}\end{bmatrix}. \tag{2.72}$$

The solutions from above are pretty complicated. The physics is pretty straighfoward though. We will first set γ to zero, set the Rabi driving time independent and check the existense of Rabi oscillations and their period. Consider again:

$$\frac{d}{dt}(\rho_{gg}-\rho_{ee}) = 2i\Omega\,[\rho_{ge}-\rho_{eg}]\,, \tag{2.73}$$

$$\frac{d}{dt}(\rho_{ge}-\rho_{eg}) = -2i\Omega\,(\rho_{gg}-\rho_{ee})\,. \tag{2.74}$$

Taking the double derivative of the first expression we get

$$\frac{d^2}{dt^2}(\rho_{gg}-\rho_{ee}) = 2i\Omega\,[-2i\Omega\,(\rho_{gg}-\rho_{ee})]\,, \tag{2.75}$$

which turns into a driven harmonic oscillator problem:

$$\frac{d^2}{dt^2}(\rho_{gg}-\rho_{ee}) + 4\Omega^2(\rho_{gg}-\rho_{ee}) = 0, \tag{2.76}$$

with solutions $(\rho_{gg}-\rho_{ee})(t) = A\cos 2\Omega t + B\sin 2\Omega t$. Let's consider an initial ground state atom such that $(\rho_{gg}-\rho_{ee})(0) = -1$. Also the coherences are vanishing at zero time meaning that the B term is zero. The following evolution will be

$$(\rho_{gg}-\rho_{ee})(t) = -\cos 2\Omega t. \tag{2.77}$$

A so-called pi pulsecan be achieved when $2\Omega t = \pi$ such that $(\rho_{gg}-\rho_{ee})(t=\pi/(2\Omega) = -1)$ and the population has been completely transferred in the excited state. For $2\Omega t = \pi/2$ a pi/2 pulse is realized where t $(\rho_{gg}-\rho_{ee})(t=\pi/(4\Omega) = 0)$ but the coherence between the levels is maximal.

π and $\pi/2$ pulses

Rate equations and steady state solutions

We now include spontaneous emission and look at the dynamics on a timescale larger than γ^{-1} where we can in a first step we can eliminate the dynamics of the coherence ρ_{eg} by setting its derivative to zero. We then obtain(for simplicity we assume constant driving)

$$\partial_t\rho_{ee} = -2\gamma\rho_{ee} - i\Omega\,[\rho_{ge}-\rho_{eg}]\,, \tag{2.78a}$$

$$\rho_{eg} = \frac{i\Omega}{\gamma+i\Delta}\,(\rho_{gg}-\rho_{ee}) \tag{2.78b}$$

After a few steps one can check that the population equation becomes

$$\frac{d}{dt}\rho_{ee} = -\gamma\left[1+\frac{2\Omega^2}{\gamma^2+\Delta^2}\right]\rho_{ee} + \frac{\gamma\Omega^2}{\gamma^2+\Delta^2}, \tag{2.79}$$

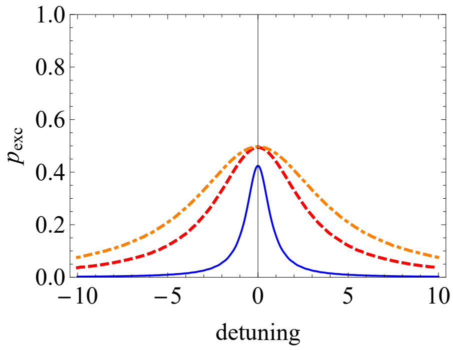

Figure 2.2: Power broadening effect shown as a modification of the Lorentzian lineshape for increasing Rabi frequencies. Black is the γ-limited linear response ($\Omega_\ell \ll \gamma$) while red and orange are for $\Omega_\ell = 3\gamma$ and $\Omega_\ell = 10\gamma$

.

where we have replaced $\rho_{gg} = 1 - \rho_{ee}$. For longer times the system will then settle in a steady state with a final excited state occupancy

$$\rho_{ee}^{ss} = \frac{\Omega^2}{\gamma^2 + \Delta^2 + 2\Omega^2}. \tag{2.80}$$

One can then compute the corresponding steady state value of the coherence

$$\rho_{eg} = -\frac{\gamma\Omega}{\gamma^2 + \Delta^2 + 2\Omega^2(t)} - i\frac{\Delta\Omega}{\gamma^2 + \Delta^2 + 2\Omega^2} \tag{2.81}$$

Linear response

Under the assumption of weak driving such that $\Omega \ll \gamma$ one gets a linear response of the TLS where its coherence is proportional to the applied field (by ignoring the $2\Omega^2$ term in the denominator):

$$\rho_{eg} = -\frac{\gamma - i\Delta}{\gamma^2 + \Delta^2}\Omega \tag{2.82}$$

Nonlinear response. Power broadening.

In the strong pump regime one can reexpress the Lorentzian profile as:

$$\rho_{eg} = -\frac{\gamma - i\Delta}{\sqrt{\gamma^2 + \Omega^2}^2 + \Delta^2}\Omega, \tag{2.83}$$

which describes a Lorentzian with modified linewidth $\gamma \to \sqrt{\gamma^2 + \Omega^2}$. The effect is dubbed *power broadening* as it shows that a given transition line can be modified under strong driving conditions. The excited state population is similarly broadened

$$\rho_{ee} = \frac{\Omega^2}{\sqrt{\gamma^2 + \Omega^2}^2 + \Delta^2}. \tag{2.84}$$

A Taylor expansion up to third order in Ω shows the next order nonlinearity called *Kerr nonlinearity*:

$$\rho_{eg} = -\frac{\gamma - i\Delta}{\sqrt{\gamma^2 + \Omega^2}^2 + \Delta^2}\Omega, \tag{2.85}$$

2.5 Application: population inversion in three level systems

The previous section has already shown that a population inversion cannot be established in a two level system under steady state conditions. However we can consider a more complicated situation where driving is performed indirectly via an intermediate level $|i\rangle$ aimed to provide inversion between the main $|g\rangle$ and $|e\rangle$ states (see Fig). We now make use of the formalism developed for TLS to apply it for each pair of levels. The Hamiltonian of the system is:

$$H = \hbar\omega_0 |e\rangle\langle e| + \hbar(\omega_0 + \nu)|i\rangle\langle i| + \hbar\Omega\left[|g\rangle\langle i| e^{i\omega_\ell t} + |i\rangle\langle g| e^{-i\omega_\ell t}\right]. \tag{2.86}$$

To this we add all the damping processes as usual Lindblad terms with the proper rate and collapse operator specifications

$$\partial_t \rho = -\frac{i}{\hbar}[H,\rho] + \gamma_i \mathscr{D}[|g\rangle\langle i|,\rho] + \gamma\mathscr{D}[|g\rangle\langle e|,\rho] + \Gamma\mathscr{D}[|e\rangle\langle i|,\rho] \tag{2.87}$$

One can proceed with deriving the following set of Bloch equations by using the rules we have derived for the closed two level system case:

$$\partial_t \rho_{ee} = -2\gamma\rho_{ee} + \Gamma\rho_{ii}, \tag{2.88}$$

$$\partial_t \rho_{eg} = i\delta\rho_{eg} - \frac{\gamma_e}{2}\rho_{eg} + i\Omega\rho_{e1}, \tag{2.89}$$

$$\partial_t \rho_{gg} = \gamma_e\rho_{ee} + \gamma_1\rho_{11} + i\Omega[\rho_{g1} - \rho_{1g}], \tag{2.90}$$

$$\partial_t \rho_{11} = -\gamma_1\rho_{11} - \gamma_{\mathrm{nr}}\rho_{11} - i\Omega[\rho_{g1} - \rho_{1g}], \tag{2.91}$$

$$\partial_t \rho_{1g} = -\frac{\gamma_{\mathrm{nr}} + \gamma_1}{2}\rho_{1g} - i\Omega(\rho_{gg} - \rho_{11}), \tag{2.92}$$

$$\partial_t \rho_{e1} = i\delta\rho_{e1} - \frac{\gamma_1 + \gamma_e + \gamma_{\mathrm{nr}}}{2}\rho_{e1} + i\Omega\rho_{eg}. \tag{2.93}$$

We want to close the equations for the e-g system under the condition that $\gamma_{\mathrm{nr}} \gg \Omega, \gamma_{1g}, \gamma_{eg}$. Notice that we can transform both ρ_{eg} and ρ_{e1} with $\delta$and eliminate

$$\rho_{e1} = \frac{2i\Omega}{\gamma_1 + \gamma_e + \gamma_{\mathrm{nr}}}\rho_{eg}. \tag{2.94}$$

In a rotating frame we get

$$\frac{d}{dt}\rho_{eg} = i\delta\rho_{eg} - \frac{\gamma_e}{2}\rho_{eg} - \frac{2\Omega^2}{\gamma_1 + \gamma_e + \gamma_{\mathrm{nr}}}\rho_{eg}. \tag{2.95}$$

Now we eliminate

$$\rho_{1g} = \frac{2i\Omega}{\gamma_{\mathrm{nr}} + \gamma_1}(\rho_{gg} - \rho_{11}), \tag{2.96}$$

leading to

$$i\Omega[\rho_{g1} - \rho_{1g}] = \frac{4\Omega^2}{\gamma_{\mathrm{nr}} + \gamma_1}(\rho_{gg} - \rho_{11}), \tag{2.97}$$

and subsequently

$$\rho_{11} = \frac{4\Omega^2}{(\gamma_{\mathrm{nr}} + \gamma_1)^2}(\rho_{gg} - \rho_{11}), \tag{2.98}$$

which results in

$$\rho_{11} = \frac{4\Omega^2}{4\Omega^2 + (\gamma_{\mathrm{nr}} + \gamma_1)^2}\rho_{gg}. \tag{2.99}$$

The effective two-level model

Plugging all this back into the equations we get:

$$\frac{d}{dt}\rho_{ee} = -\gamma_{eg}\rho_{ee} + \frac{4\Omega^2\gamma_{\mathrm{nr}}}{4\Omega^2 + (\gamma_{\mathrm{nr}} + \gamma_{1g})^2}\rho_{gg}, \tag{2.100}$$

$$\frac{d}{dt}\rho_{eg} = -\left(\frac{\gamma_e}{2} + \frac{2\Omega^2}{\gamma_1 + \gamma_e + \gamma_{\mathrm{nr}}}\right)\rho_{eg}, \tag{2.101}$$

$$\frac{d}{dt}\rho_{gg} = \gamma_{eg}\rho_{ee} - \frac{4\Omega^2\gamma_{\mathrm{nr}}}{4\Omega^2 + (\gamma_{\mathrm{nr}} + \gamma_{1g})^2}\rho_{gg}. \tag{2.102}$$

Under typical conditions, $\gamma_{\mathrm{nr}} \gg \gamma_e, \gamma_1$ and $\Omega \ll \gamma_{\mathrm{nr}}$, we can simplify to a common pump rate

$$\Gamma \approx \frac{4\Omega^2}{\gamma_{\mathrm{nr}}}, \tag{2.103}$$

and write the following equations: Plugging all this back into the equations we get:

$$\frac{d}{dt}\rho_{ee} = -2\gamma_{eg}\rho_{ee} + 2\Gamma\rho_{gg}, \tag{2.104}$$

$$\frac{d}{dt}\rho_{eg} = -(\gamma_e + \Gamma)\rho_{eg}. \tag{2.105}$$

$$\frac{d}{dt}\rho_{gg} = \gamma_{eg}\rho_{ee} - \Gamma\rho_{gg}. \tag{2.106}$$

This is exactly a model for incoherent pumping (or inverse decay). We can rewrite it as:

$$\dot{\rho} = -\frac{i}{\hbar}[H_0, \rho] + \gamma_e D[|g\rangle\langle e|, \rho] + \Gamma D[|e\rangle\langle g|, \rho]. \tag{2.107}$$

Let's check out if this is true:

$$\dot{\rho}_{eg} = \langle e|\Gamma\left[|2e\rangle\langle g|\rho|g\rangle\langle e| - |g\rangle\langle g|\rho - \rho|g\rangle\langle g|\right]|g\rangle = -\Gamma\rho_{eg}. \tag{2.108}$$

2.6 Appendix: The Bloch sphere

Anticipating the discussion in Chapter 7 on qubit operations in ion traps let us encode the 0 and 1 qubits in the internal levels $|0\rangle \equiv |g\rangle$ and $|1\rangle \equiv |e\rangle$ of a two level system. We will in the following follow a matrix approach where we denote:

$$|0\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } |1\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.109}$$

For any pure state we can form the most general superposition as:

$$|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + e^{i\phi}\sin\frac{\theta}{2}|0\rangle. \tag{2.110}$$

This can be vizualized on a unit sphere surface as a point described by a Bloch vector:

$$\mathbf{a} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = (X,Y,Z). \tag{2.111}$$

More generally, states can be mixed and then they are described by density operators. A density operator in a 2×2 space can be written uniquely as a combination of the identity matrix and three independent Pauli matrices (as they form a complete basis in this Hilbert space)

$$\rho = \frac{1}{2}(I_2 + \mathbf{a}\cdot\sigma) = \frac{1}{2}\begin{bmatrix} 1+Z & X-iY \\ X+iY & 1-Z \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{bmatrix}, \tag{2.112}$$

where by definition

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma + \sigma^\dagger, \tag{2.113a}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i(\sigma - \sigma^\dagger), \tag{2.113b}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma^\dagger\sigma - \sigma\sigma^\dagger. \tag{2.113c}$$

Now we have 3 independent parameters translatable to two angles and a radius: the corresponding Bloch vector can now be anywhere inside the Bloch sphere. For pure states this reduces to the state vector representation shown above. Let us define rotations around the axes as:

$$R_{x,y,z}(\zeta) = e^{-i\zeta/2\sigma_{x,y,z}}. \tag{2.114}$$

For all Pauli matrices, as they satisfy $\sigma_{x,y,z}^2 = I$, one can show that (as usual we expand the exponential and use the property listed above):

$$R_{x,y,z} = I_2\cos\frac{\zeta}{2} - i\sigma_{x,y,z}\sin\frac{\zeta}{2}. \tag{2.115}$$

Let's write them in matrix form:

$$R_x(\zeta) = \begin{bmatrix} \cos\frac{\zeta}{2} & -i\sin\frac{\zeta}{2} \\ -i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} \end{bmatrix}, \tag{2.116a}$$

$$R_y(\zeta) = \begin{bmatrix} \cos\frac{\zeta}{2} & -\sin\frac{\zeta}{2} \\ \sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} \end{bmatrix}, \tag{2.116b}$$

$$R_z(\zeta) = \begin{bmatrix} e^{-i\zeta/2} & 0 \\ 0 & e^{-i\zeta/2} \end{bmatrix}. \tag{2.116c}$$

Notice a few properties:

$$R_x(\pi) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \qquad R_x(\pi/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}, \qquad R_x(2\pi) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2, \tag{2.117}$$

and

$$R_y(\pi) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad R_y(\pi/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad R_y(2\pi) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2, \tag{2.118}$$

Single qubit gate: Hadamard gate

To perform a rotation from an initially zero state qubit into superpositions we apply the following transformation:

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = R_y(\frac{\pi}{2})|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle). \tag{2.119}$$

Also one can check:

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = R_y(\frac{\pi}{2})||1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \tag{2.120}$$

Single qubit gate: Pauli-X gate (NOT gate)

To negate a qubit is equivalent to turn it from 0 to 1 and the other way around. For this one can check that:

$$iR_x(\pi) = i \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.121}$$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} |0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |1\rangle, \tag{2.122}$$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} |1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |0\rangle, \tag{2.123}$$

3. Cavity quantum electrodynamics

An arrangement of two highly reflective mirrors (either dielectric or metallic) placed parallel to each other at a small distance ℓ can provide a high density of electromagnetic modes in the space in between. Such an arrangement defines an optical cavity and is widely used as a platform that can amplify the typically small light-matter interaction in free space. While in free space a single photon sent onto a TLS would interact once and then depart, in an optical cavity the photon bounces back and forth many time therefore increasing the chance to interact with the TLS. We will first proceed in providing a classical description of the optical cavity properties such as longitudinal modes, loss rate, finesse etc. We then introduce the quantum model for a single cavity mode and derive two equivalent formalisms: the quantum master equation and the Langevin equations. Then we describe the quantum model for a single cavity mode interacting with a TLS known as the Jaynes-Cummings model and introduce the strong coupling regime where hybrid light-matter states known as polaritons occur. Finally we list a few applications of cavity QED such as optical bistability, the Purcell effect (modification of the decay rate of an atom) and photon blockade.

3.1 Optical cavity - classical treatment

We assume a quasi 1D geometry where light can only propagate in the z direction and two highly reflective boundaries are placed at $z = 0$ and $z = \ell$. From the 1D Helmoltz equation we derive longitudinal modes of light inside the cavity and show that they have a Lorentzian profile owing to the tunneling of light through the mirrors. To obtain these characteristics we solve the Helmoltz equation in a very straightforward transfer matrix formalism.

Longitudinal modes

We assume that the boundaries are perfectly reflective such that the tangential electric field component will vanish at $z = 0$ and $z = \ell$. We make the simplification that the electric field has an $\hat{x}$ polarization direction. The configuration assumed will be referred to in the following by the term optical cavity or optical resonator. We then have to solve a 1D wave equation

$$\partial_{zz}E(z,t) + c^{-2}\partial_{tt}E(z,t) = 0, \tag{3.1}$$

in the region between 0 and ℓ with boundary conditions $E(0,t) = E(\ell,t) = 0$. Writing the solutions $E(z,t) = \mathscr{E}(z)f(t)$ we have

$$\mathscr{E}^{-1}(z)\partial_{zz}\mathscr{E}(z) = -c^{-2}f(t)^{-1}(t)\partial_{tt}f(t) = -k^2, \tag{3.2}$$

where we have applied the usual technique of separation of variables to derive two equations:

$$f(t) + (ck)^2\partial_{tt}f(t) = 0, \tag{3.3a}$$
$$\partial_{zz}\mathscr{E}(z) + k^2\mathscr{E}(z) = 0, \tag{3.3b}$$

with $E(0) = E(\ell) = 0$. The second equation with the imposed boundary conditions $\mathscr{E}(0) = \mathscr{E}(\ell) = 0$ leads to solutions:

$$\mathscr{E}(z) = N\sin kz, \tag{3.4}$$

where the allowed values of k are

$$k = \frac{m\pi}{\ell}\ , \tag{3.5}$$

which expressed in terms of wavelengths is

$$\lambda = \frac{2\ell}{m}. \tag{3.6}$$

Notice that the fundamental mode for $m = 1$ implies that the cavity is a half wavelength $\ell = \lambda/2$. The orthonormality requires

$$N^2\int_0^\ell dz\sin kz\sin k'z = N^2\delta_{kk'}\frac{\ell}{2}. \tag{3.7}$$

so that the normalization constant is: $N = \sqrt{2/\ell}$. In the following we will assume that there is a transverse area where light is confined and denote $S\ell$ as a quantization volume.

Lossy cavities: a transfer matrix approach

However, mirrors are not perfect so that some tunneling between the cavity mode and the continuum of modes outside the cavity is always present. We will first take a classical approach based on multiplications of transfer matrices to derive the transmission properties of an optical cavity as well as the shape of the cavity modes.

Assuming a scatterer (mirror, membrane, atom etc) in a fix position, we denote the waves on its left by Ae^{-ikx} (left propagating) and Be^{ikx} (right propagating) and on the right of it as Ce^{-ikx} (left propagating) and De^{ikx} (right propagating). The scatterer is assumed to have a reflectivity (complex) r and transmissivity t. The two are actually connected as

$$t = 1 + r, \tag{3.8}$$

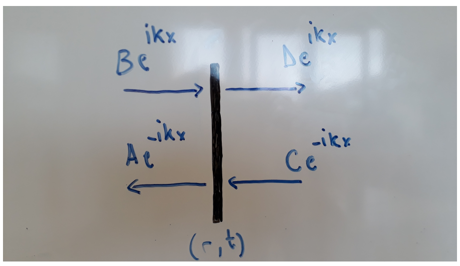

Figure 3.1: Transfer matrix of a single scatterer.

and notice that in the absence of absorption we have

$$|r|^2 + |t|^2 = 1. \tag{3.9}$$

One can relate the outgoing fields to the incoming fields as

$$D = tB + rC, \tag{3.10a}$$
$$A = rB + tD, \tag{3.10b}$$

and rewrite the conditions connecting the amplitudes on the right side with the ones on the left side of the beamsplitter:

$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{t} \begin{bmatrix} 1 & -r \\ r & t^2 - r^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 - i\zeta & -i\zeta \\ i\zeta & 1 + i\zeta \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}. \tag{3.11}$$

An important aspect is that the parametrization is done with a real polarizability ζ based on the following transformation:

$$\zeta = -\frac{ir}{t} = -\frac{ir}{1-r}. \tag{3.12}$$

We can inverse this to obtain

$$r = \frac{-\zeta}{\zeta - i}. \tag{3.13}$$

Notice that the intensity reflectivity is then given by

$$|r|^2 = \frac{\zeta^2}{\zeta^2 + 1}. \tag{3.14}$$

which further allows to express

$$\zeta^2 = \frac{|r|^2}{1 - |r|^2}. \tag{3.15}$$

For large polarizabilities one gets a close to unity reflectivity. Rewriting we get

$$r = |r| \left[\frac{\zeta}{\sqrt{1+\zeta^2}} - i \frac{1}{\sqrt{1+\zeta^2}} \right] = |r| e^{i\phi}, \tag{3.16}$$

with

$$\sin\phi = -\frac{1}{\sqrt{1+\zeta^2}}, \qquad \text{and} \qquad \cos\phi = \frac{\zeta}{\sqrt{1+\zeta^2}}. \tag{3.17}$$

We can also inverse this transformation:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{t} \begin{bmatrix} t^2 - r^2 & r \\ -r & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}. \tag{3.18}$$

The free space accumulation of phase is easily written as

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} e^{-ik\ell} & 0 \\ 0 & e^{ik\ell} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}. \tag{3.19}$$

A two mirror arrangement

Let's now assume an arrangement of two identical mirrors at $z=0$ and $z=\ell$ and no input from the right side ($C=0$) while the left side has a unit propagating field amplitude ($B=1$). We can then write

$$\begin{bmatrix} r_c \\ 1 \end{bmatrix} = \frac{1}{t}\begin{bmatrix} t^2-r^2 & r \\ -r & 1 \end{bmatrix}\begin{bmatrix} e^{-ik\ell} & 0 \\ 0 & e^{ik\ell} \end{bmatrix}\frac{1}{t}\begin{bmatrix} t^2-r^2 & r \\ -r & 1 \end{bmatrix}\begin{bmatrix} 0 \\ t_c \end{bmatrix}, \tag{3.20}$$

where now the D and A components become the transmission and reflection of the compound object i.e. the optical cavity. With a bit of math one finds

$$\begin{aligned} &\frac{1}{t^2}\begin{bmatrix} t^2-r^2 & r \\ -r & 1 \end{bmatrix}\begin{bmatrix} e^{-ik\ell} & 0 \\ 0 & e^{ik\ell} \end{bmatrix}\begin{bmatrix} t^2-r^2 & r \\ -r & 1 \end{bmatrix} = \\ &= \frac{1}{t^2}\begin{bmatrix} t^2-r^2 & r \\ -r & 1 \end{bmatrix}\begin{bmatrix} e^{-ik\ell}(t^2-r^2) & re^{-ik\ell} \\ -re^{ik\ell} & e^{ik\ell} \end{bmatrix} = \\ &= \frac{1}{t^2}\begin{bmatrix} e^{-ik\ell}(t^2-r^2)^2+r^2e^{ik\ell} & (t^2-r^2)re^{-ik\ell}+re^{ik\ell} \\ -(t^2-r^2)re^{-ik\ell}-re^{ik\ell} & -r^2e^{-ik\ell}+e^{ik\ell} \end{bmatrix}. \end{aligned} \tag{3.21}$$

Denoting the whole transfer matrix by M one can easily notice that

$$t_c = \frac{1}{M_{22}} = \frac{t^2}{e^{ik\ell}-r^2e^{-ik\ell}} = \frac{t^2e^{ik\ell}}{e^{2ik\ell}-r^2}. \tag{3.22}$$

Let's analyze the intensity transmission

$$T_c = |t_c|^2 = \frac{|t|^2}{|e^{2ik\ell}-r^2|^2} = \frac{|t|^2}{|e^{2i(k\ell-\phi)}-|r|^2|^2} = \frac{|t|^2}{|e^{2i(k\ell-\phi)}-1+|t|^2|^2}. \tag{3.23}$$

A maximum is possible at unity. Resonances are reached around the values we expected minus a small contribution:

$$2k_m\ell = 2\pi m + 2\phi \to k_m = \frac{\pi m}{\ell} - \frac{\phi}{\ell}. \tag{3.24}$$

For large polarizability

$$\phi \simeq -\frac{1}{\sqrt{1+\zeta^2}} \simeq \zeta^{-1}. \tag{3.25}$$

Cavity linewidth, cavity decay rate and finesse.

Expanding around the resonance condition:$e^{-2ik\ell} = 1-2i(\delta k)\ell = 1-2i(k-k_m)$, we obtain a Lorentzian shape of the transmission function around some resonance k_m,

$$T_c^m(k) = \frac{1}{|1+2i\zeta^2\ell(k-k_m)|^2}. \tag{3.26}$$

Notice that $t=-i/\zeta$ so that The linewidth of this Lorentzian is given by the condition of $T_c^m(k_m+\delta k) = 1/2$ which leads to

$$\delta k = \frac{1}{2\zeta^2\ell}. \tag{3.27}$$

Expressed as a cavity decay rate (for units of frequency) one can define

$$\kappa = c\delta k = \frac{c}{2\zeta^2\ell}. \tag{3.28}$$

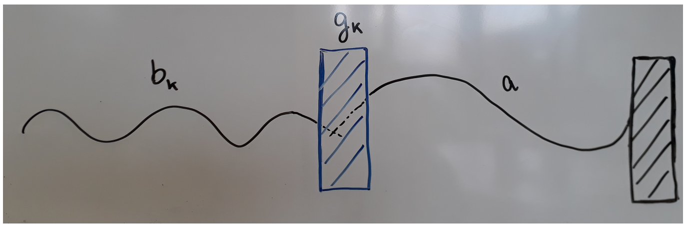

Figure 3.2: Exchange interaction of cavity modes with outside modes for a single-ended cavity. The left mirror allows for both inside (mode a) and outside modes (denoted by b_k) to penetrate through.

Reexpressing in terms of wavelengths:

$$\delta\lambda = 2\pi\frac{\delta k}{k_m^2} = \frac{\pi}{k_m^2\zeta^2\ell} = \frac{\pi}{(\frac{2\pi}{\lambda_m})^2\zeta^2\ell} = \frac{\lambda_m^2}{4\pi\zeta^2\ell}. \tag{3.29}$$

An important quantity is the finesse of a cavity defined as:

$$\mathscr{F} = \frac{\Delta\lambda}{2\delta\lambda}, \tag{3.30}$$

where $\Delta\lambda$ is the free spectral range expressed in wavelengths. This is easy to compute: $\Delta\lambda = \lambda_m - \lambda_{m+1} = \frac{2}{m+1}\ell - \frac{2}{m}\ell \simeq \lambda_m^2/(2\ell)$. Finally

$$\mathscr{F} = \pi\zeta^2 = \frac{\pi|r|^2}{1-|r|^2}. \tag{3.31}$$

Roughly speaking, the finesses represents the number of round trips before the field intensity inside the cavity decays considerably (to e^{-2}) of the initial value. One can also define the cavity decay rate as:

$$\kappa = \frac{\pi c}{2\mathscr{F}\ell}, \tag{3.32}$$

and a quality factor:

$$Q = \frac{\omega_m}{\kappa} = \frac{2\mathscr{F}\ell}{\lambda_m}. \tag{3.33}$$

3.2 Optical cavity: quantum Langevin equations

Even the simplest example of an optical cavity that we employ based on two parallel highly reflective mirrors can support many (longitudinal) modes. Real cavities are designed with curved mirrors and add to these modes additional, transverse ones (eventually degenerate). As in the first chapter we can proceed with quantizing all these modes and introducing a total Hamiltonian as a sum over many quantum harmonic oscillators, with creation/annihilation operators for each mode. However, there are good arguments for simplifying the treatment to only one optical mode. Notice that a typical free spectral range is about THz. This is much larger than the decay rate of an atom which is of the level of 10 MHz or so. In consequence, for a given TLS transition only one cavity mode can considerably interact with it. So, let us reduce the field dynamics to a single mode with operator a (referring to a standing wave with frequency ω_c) and express the field as:

$$\hat{E}(z) = \mathscr{E}_0\left(a + a^\dagger\right)\sin kz, \qquad \text{where} \qquad \mathscr{E}_0 = \left[\frac{\hbar\omega}{\varepsilon_0\ell S}\right]^{1/2}. \tag{3.34}$$

We have introduced the mode area S and grouped terms inside the zero point electric field amplitude such that the integration of the Hamiltonian volume density over the whole volume leads to

$$H = \hbar\omega_c \left[a^\dagger a + \frac{1}{2}\right]. \tag{3.35}$$

As before we will disregard the constant energy shift. As the cavity mirrors are not perfect the intra-cavity field can permeate through the dielectric material and tunnel to the outside. At the same time the outside field can also tunnel inside the cavity. If this were not the case than the cavity would be useless as no driving would be possible. Having learned what the classical treatment predicts for a cavity resonance transmission linewidth, we are in the position of writing a quantum model for the coupling of a quantized cavity mode to the continuum of modes outside. To this end we consider the Hamiltonian for the cavity mode plus the infinity of outside modes and their interaction (modelled a an excitation exchange term)

$$H = \hbar\omega_c a^\dagger a + \sum_k \hbar\omega_k b_k^\dagger b_k + \hbar\sum_k g_k \left[b_k^\dagger a + a^\dagger b_k\right]. \tag{3.36}$$

From here we can derive a set of equations of motion Let's write the equations of motion:

$$\dot{a} = \frac{i}{\hbar}[H,a] = -i\omega_c a - i\sum_k g_k b_k, \tag{3.37a}$$

$$\dot{b}_k = \frac{i}{\hbar}[H,b_k] = -i\omega_k b_k - ig_k a. \tag{3.37b}$$

The plan is to formally integrate the equation of motion for the outside modes and plug the solutions back into the cavity mode evolution equation. This will result in a differential-integro equation. In a first step we have

$$b_k(t) = b_k(0)e^{-i\omega_k t} - ig_k \int_0^t dt' a(t') e^{-i\omega_k(t-t')}. \tag{3.38}$$

Replacing this expression into the cavity mode equation of motion we obtain:

$$\dot{a} = -i\omega_c a - \sum_k g_k^2 \int_0^t dt' a(t') e^{-i\omega_k(t-t')} - i\sum_k g_k b_k(0) e^{-i\omega_k t}. \tag{3.39}$$

The decay rate

The first term, similarly to the summation performed in the case of the spontaneous emission master equation for atoms, gives a delta function selecting only the coupling terms at the cavity frequency. The terms turns into a decay term

$$-\sum_k g_k^2 \int_0^t dt' a(t') e^{-i\omega_k(t-t')} = -\kappa a(t), \tag{3.40}$$

at a rate which one can prove is the classically derived term in the previous section.

The input noise

The second term is an infinite sum over the input modes which we will denote by

$$F(t) = -i\sum_k g_k b_k(0) e^{i-\omega_k t}. \tag{3.41}$$

Going back in the original picture we can write a Langevin equation:

$$\dot{a} = -i\omega_c a(t) - \kappa a(t) + F(t). \tag{3.42}$$

Notice that in the vacuum

$$\langle 0|F(t)|0\rangle = -i\sum_k g_k \langle 0|b_k(0)|0\rangle e^{-i\omega_k t} = 0. \tag{3.43}$$

Looking at the correlations of the input term assuming the modes outside in the vacuum, all but one term vanish:

$$\langle 0|F(t)F^\dagger(t')|0\rangle = \sum_k g_k^2 e^{-i\omega_k(t-t')} = 2\kappa\delta(t-t'). \tag{3.44}$$

Finally we can write a standard Langevin equation for the cavity mode under the following form:

$$\dot{a} = -i\omega_c a(t) - \kappa a(t) + \sqrt{2\kappa}A_{\text{in}}, \tag{3.45a}$$

$$\sqrt{2\kappa}A_{\text{in}}(t) = \eta + \sqrt{2\kappa}a_{\text{in}}(t), \tag{3.45b}$$

For vacuum states the input noise is delta normalized such that $\langle A_{\text{in}}(t)A_{\text{in}}^\dagger(t')\rangle = \delta(t-t')$.

The driving term

Let's now assume that one of the incoming modes at frequency $ck_L$is populated by a coherent state $\alpha_L$while all the other modes are in the vacuum. We can then evaluate

$$\langle \alpha_L|F(t)|\alpha_L\rangle = -i\sum_k g_k \langle \alpha_L|b_k(0)|\alpha_L\rangle e^{i\omega_k t} = -i\alpha_L e^{-i\omega_L t}. \tag{3.46}$$

We then separate the input noise into a classical part and a zero-average quantum part:

$$F(t) = \sqrt{2\kappa}A_{\text{in}} = -i\alpha_L e^{-i\omega_L t} + \sqrt{2\kappa}a_{\text{in}}. \tag{3.47}$$

Generally, in terms of zero-average noise and classical drive, in a rotating frame at the laser frequency one can write the following Langevin equation:

$$\boxed{\dot{a} = -\kappa a - i(\omega_L - \omega_c)a + \eta + \sqrt{2\kappa}a_{\text{in}}}, \tag{3.48}$$

where we made the notation $\eta = -i\alpha_L$.

Input-output relations

Similarly with the procedure above we can integrate the equation for the outside modes from t to infinity.

$$\tilde{b}_k(t) = \tilde{b}_k(\infty) + i\int_t^\infty dt' g_k \tilde{a}(t') e^{-i(\omega-\omega_k)t'}. \tag{3.49}$$

A similar calculation leads to

$$\boxed{\dot{a} = -i\omega_c a(t) + \kappa a(t) - \sqrt{2\kappa}a_{\text{out}}}, \tag{3.50}$$

where a_{out}stems from the new noise term

$$\tilde{F}(t) = i\sum_k g_k \tilde{b}_k(\infty) e^{i(\omega_c-\omega_k)t}. \tag{3.51}$$

This is a convoluted way of assuming that for an interaction that lasts for a limited time, the initial conditions are obtained by evaluating the fields outside at time 0 (or equivalently $-\infty$) while the final state is obtained by evaluating the fields after the interaction at $t = \infty$. Finally we get the input output relations:

$$\boxed{A_{\text{out}} + A_{in} = \sqrt{2\kappa}a}. \tag{3.52}$$

Two sided cavities. Transmission and reflection.

The input-output relations are valid at each interface. Let us assume a two-sided cavity with equal reflectivities and thus equal decay rates on each mirror $\kappa/2$. Driving occurs from the left and is included in the A_{in} term such that $\langle A_{\text{in}}\rangle = \eta/\sqrt{\kappa}$. The input field (in zero average state) from the right is denoted by b_{in}. We then have:

$$A_{\text{in}} + B_{\text{out}} = \sqrt{\kappa} a, \tag{3.53a}$$
$$b_{\text{in}} + A_{\text{out}} = \sqrt{\kappa} a. \tag{3.53b}$$

We can define transmission and reflection coefficients as: $t_c = \langle A_{\text{out}}\rangle/\langle A_{\text{in}}\rangle$and $r_c = \langle B_{\text{out}}\rangle/\langle A_{\text{in}}\rangle$. Taking an average over the equations above we have

$$1 + r_c = \frac{\kappa}{\eta}\langle a\rangle, \tag{3.54a}$$
$$t_c = \frac{\kappa}{\eta}\langle a\rangle, \tag{3.54b}$$

which gives the usual relation as we used in the transfer matrix $t_c = r_c + 1$.The Langevin equations with both input noises are:

$$\dot{a} = -\kappa a - i(\omega_L - \omega_c)a + \eta + \sqrt{\kappa} a_{\text{in}} + \sqrt{\kappa} b_{\text{in}}, \tag{3.55}$$

and a classical average in steady state (assuming times longer than κ^{-1}) gives:

$$\langle a\rangle = \frac{\eta}{\kappa + i(\omega_L - \omega_c)}. \tag{3.56}$$

Consequently the transmission of intensity is given by:

$$T_c = |t_c|^2 = \frac{\kappa^2}{\kappa^2 + (\omega_L - \omega_c)^2}. \tag{3.57}$$

3.3 Optical cavity: master equation

Following the lines of the derivation carried out for spontaneous emission, we can show that the following master equation for a driven cavity undergoing decay can be obtained:

$$\boxed{\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \kappa\left\{a\rho a^\dagger - \frac{1}{2}\left[\{a^\dagger a\rho + \rho a^\dagger a\right]\right\},} \tag{3.58}$$

with

$$H = \hbar\omega_c a^\dagger a + i\hbar\eta\left(a e^{i\omega_L t} - a^\dagger e^{-i\omega_L t}\right). \tag{3.59}$$

3.4 The Jaynes-Cummings Hamiltonian.

Let's now place an atom within the cavity. We assume that the atom is closely resonant with a given cavity resonance and far away from any other resonances. The dipole-electric field coupling Hamiltonian will now be written as:

$$H_{JC} = \hbar g\left[a^\dagger\sigma + a\sigma^\dagger\right]. \tag{3.60}$$

The coupling strength is the usual:

$$g = \frac{1}{\hbar}\mathscr{E}_0 d_{eg} = \left[\frac{\omega}{\hbar\varepsilon_0 \ell S}\right]^{1/2} d_{eg}. \tag{3.61}$$

Let us inspect the properties of a coupled atom-cavity field system without any incoherent processes. The full Hamiltonian is:

$$H = \hbar\omega_c a^\dagger a + \hbar\omega_a \sigma^\dagger\sigma + \hbar g\left[a^\dagger\sigma + a\sigma^\dagger\right]. \tag{3.62}$$

Let's consider the complete basis in the full Hilbert space spanned by $|g,n\rangle$ and $|e,n\rangle$. All the possible matrix elements are:

$$\langle g,n|H|g,m\rangle = \hbar\omega_c n\delta_{nm}, \tag{3.63a}$$
$$\langle e,n|H|e,m\rangle = \hbar\omega_c n\delta_{nm} + \hbar\omega_a\delta_{nm}, \tag{3.63b}$$
$$\langle g,n|H|e,m\rangle = \hbar g\sqrt{n+1}\delta_{nm+1}, \tag{3.63c}$$
$$\langle e,n|H|g,m\rangle = \hbar g\sqrt{n}\delta_{nm-1}. \tag{3.63d}$$

Let's write it in matrix form agreeing that we start with vectors ordered as: $|g,0\rangle$, $|g,1\rangle$, $|e,0\rangle$, $|g,2\rangle$, $|e,1\rangle$....

$$\hbar\begin{bmatrix} \omega_c+\omega_a & g\sqrt{2} & 0 & 0 & 0\\ g\sqrt{2} & 2\omega_c & 0 & 0 & 0\\ 0 & 0 & \omega_a & g & 0\\ 0 & 0 & g & \omega_c & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.64}$$

Notice that for a given n only states $|g,n+1\rangle$and $|e,n\rangle$are coupled. One can then block diagonalize the whole Hamiltonian in excitation subspaces spanned by a constant number of excitations.

We see that the Hamiltonian matrix contains blocks which are not coupled. Let's first focus in the single excitation block with state vectors $|g,1\rangle, |e,0\rangle$. To bring this block into diagonal form we have to diagonalize the following matrix:

$$\begin{bmatrix} \omega_a & g\\ g & \omega_c \end{bmatrix}. \tag{3.65}$$

Eigenvalues are given by the equation: $(\omega_a-\lambda)(\omega_c-\lambda) = g^2$. Let's rewrite this as: $(\lambda-\omega_c+\omega_c-\omega_a)(\lambda-\omega_c) = g^2$ and make notations $\omega_c-\omega_a = \Delta$.Then we have to solve: $x^2+\Delta x - g^2 = 0$ leading to

$$\lambda = \omega_c - \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + g^2} = \frac{\omega_c+\omega_a}{2} \pm \sqrt{\left(\frac{\omega_c-\omega_a}{2}\right)^2 + g^2}. \tag{3.66}$$

For zero detuning $\omega_c = \omega_a = \omega$ the two eigenstates are located at

$$\mathscr{E}^{(1)} = \hbar(\omega \pm g), \tag{3.67}$$

in energy and are combinations

$$|p_\pm\rangle = \frac{|e,0\rangle \pm, |g,1\rangle}{\sqrt{2}}. \tag{3.68}$$

In higher excitation subspaces, the result is (by diagonalizing the blocks with n photons):

$$\mathscr{E}^{(n)} = \hbar\left(\omega \pm g\sqrt{n}\right). \tag{3.69}$$

The polariton transformation

For the resonant case, we will perform a transformation from the bare atom and cavity operators to polariton operators:

$$u = \sqrt{\frac{1}{2}}(a+\sigma), \tag{3.70a}$$
$$d = \sqrt{\frac{1}{2}}(a-\sigma). \tag{3.70b}$$

The inverse transformations:

$$a = \sqrt{\frac{1}{2}}(u+d), \tag{3.71a}$$
$$\sigma = \sqrt{\frac{1}{2}}(u-d). \tag{3.71b}$$

Notice that the application of the polariton operators to the vacuum creates the previously derived states:

$$u^\dagger |g,0\rangle = \sqrt{\frac{1}{2}}(a^\dagger + \sigma^\dagger)|g,0\rangle = \frac{|e,0\rangle + |g,1\rangle}{\sqrt{2}}. \tag{3.72}$$

The Hamiltonian contains terms as:

$$a^\dagger a = \frac{1}{2}(u^\dagger + d^\dagger)(u+d) = \frac{1}{2}\left(u^\dagger u + d^\dagger d + u^\dagger d + d^\dagger u\right), \tag{3.73a}$$
$$\sigma^\dagger \sigma = \frac{1}{2}(u^\dagger - d^\dagger)(u-d) = \frac{1}{2}\left(u^\dagger u + d^\dagger d - u^\dagger d - d^\dagger u\right), \tag{3.73b}$$
$$\sigma^\dagger a = \frac{1}{2}(u^\dagger - d^\dagger)(u+d) = \frac{1}{2}\left(u^\dagger u - d^\dagger d + u^\dagger d - d^\dagger u\right), \tag{3.73c}$$
$$a^\dagger \sigma = \frac{1}{2}(u^\dagger + d^\dagger)(u-d) = \frac{1}{2}\left(u^\dagger u - d^\dagger d - u^\dagger d + d^\dagger u\right). \tag{3.73d}$$

Adding everything we immediately find what we expected which is that the Hamiltonian can be diagonalized via this transformation:

$$H = \hbar(\omega - g)d^\dagger d + \hbar(\omega + g)u^\dagger u. \tag{3.74}$$

The strong coupling regime

Introducing decay we see that the polariton transformation already diagonalizes the Lindblad term. Let's apply the transformation above to the Lindblad terms:

$$\mathscr{L}_\gamma = \gamma\left[2\sigma\rho\sigma^\dagger - \left(\sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma\right)\right], \tag{3.75}$$
$$\mathscr{L}_\kappa = \kappa\left[2a\rho a^\dagger - \left(a^\dagger a\rho - \rho a^\dagger a\right)\right], \tag{3.76}$$

Working it out:

$$2\sigma\rho\sigma^\dagger = (u-d)\rho(u^\dagger - d^\dagger) = u\rho u^\dagger + d\rho d^\dagger - u\rho d^\dagger - d\rho u^\dagger, \tag{3.77a}$$
$$\sigma^\dagger\sigma\rho = \frac{1}{2}(u^\dagger - d^\dagger)(u-d)\rho = u^\dagger u\rho + d^\dagger d\rho - u^\dagger d\rho - d^\dagger u\rho, \tag{3.77b}$$
$$\rho\sigma^\dagger\sigma = \frac{1}{2}\rho(u^\dagger - d^\dagger)(u-d) = \rho u^\dagger u + \rho d^\dagger d - \rho u^\dagger d - \rho d^\dagger u, \tag{3.77c}$$

Also

$$2a\rho a^\dagger = (u+d)\rho(u^\dagger + d^\dagger) = u\rho u^\dagger + d\rho d^\dagger + u\rho d^\dagger + d\rho u^\dagger, \tag{3.78a}$$

$$a^\dagger a\rho = \frac{1}{2}(u^\dagger + d^\dagger)(u+d)\rho = u^\dagger u\rho + d^\dagger d\rho + u^\dagger d\rho + d^\dagger u\rho, \tag{3.78b}$$

$$\rho a^\dagger a = \frac{1}{2}\rho(u^\dagger + d^\dagger)(u+d) = \rho u^\dagger u + \rho d^\dagger d + \rho u^\dagger d + \rho d^\dagger u. \tag{3.78c}$$

Adding everything we find a simple result:

$$\mathscr{L}[\rho] = \mathscr{L}_{(\gamma+\kappa)/2}[d,\rho] + \mathscr{L}_{(\gamma+\kappa)/2}[u,\rho], \tag{3.79}$$

which states that the two polariton operators make up two equally decaying channels at rates $(\gamma+\kappa)/2$.

The Purcell effect

Let's first investigate such interactions in the regime where the cavity field is very quickly decaying $\kappa \gg \gamma$. We also assume the atom being resonant to the cavity mode both at ω. We can go into a rotating frame where both a and σrotate at the ω frequency. Thus, we have to solve the dynamics of a system with the following complete dynamics:

$$\boxed{\frac{d}{dt}\rho = -ig\left[a^\dagger\sigma + a\sigma^\dagger, \rho\right] + \kappa\mathscr{D}[a,\rho] + \gamma\mathscr{D}[\sigma,\rho].} \tag{3.80}$$

Let's look at the evolution of the field operator:

$$\frac{d}{dt}a = -\kappa a - ig\sigma + \sqrt{2\kappa}a_{\text{in}}. \tag{3.81}$$

We can go into a rotating frame where both a and σrotate at the ω frequency. Formal integration will lead to

$$a(t) = a(0)e^{-\kappa t} - ig\int_0^t dt'\sigma(t')e^{-\kappa(t-t')} - \kappa a - ig\sigma + \sqrt{2\kappa}\int_0^t dt' a_{\text{in}}(t')e^{-\kappa(t-t')}. \tag{3.82}$$

The first term decays fast and will vanish for $t \gg \kappa^{-1}$. The second term we will integrate by parts:

$$\frac{1}{\kappa}\int_0^t dt'\sigma(t')\left[e^{\kappa t'}\right]' = \frac{1}{\kappa}\sigma(t')e^{\kappa t'}|_0^t - \frac{1}{\kappa}\int_0^t dt'\sigma'(t')e^{\kappa t'}. \tag{3.83}$$

Putting it together we get:

$$-ig\int_0^t dt'\sigma(t')e^{-\kappa(t-t')} = -\frac{ig}{\kappa}\left(\sigma(t) - \sigma(0)e^{-\kappa t}\right) + \frac{ig}{\kappa}\int_0^t dt'\sigma'(t')e^{-\kappa(t-t')}. \tag{3.84}$$

The transient term will vanish again in the same long time limit. The second term can be again integrated by parts and slowly we get smaller and smaller terms containing κ^{-2}, κ^{-3}etc. We then conclude that:

$$a(t) \simeq -\frac{ig}{\kappa}\sigma(t) + \bar{a}_{\text{in}}. \tag{3.85}$$

Now let's replace the first term in the expression in the Lindblad term for the cavity field:

$$\kappa\left\{a\rho a^\dagger-\frac{1}{2}\left[\{aa\rho+\rho a^\dagger a\right]\right\}=\frac{g^2}{\kappa}\left\{\sigma\rho\sigma^\dagger-\frac{1}{2}\left[\{\sigma^\dagger\sigma\rho+\rho\sigma^\dagger\sigma\right]\right\}. \tag{3.86}$$

We see that the cavity contributes to the decay of the atom with a rate g^2/κ. The total decay rate of the atom will therefore be:

$$\gamma+\frac{g^2}{\kappa}=\gamma(1+C), \tag{3.87}$$

where we will denote the cooperativity parameter by

$$C=\frac{g^2}{\gamma\kappa}. \tag{3.88}$$

Let us remember what all these parameters are:

$$g^2=\left[\frac{\omega}{\hbar\varepsilon_0\ell S}\right]d_{eg}^2 \tag{3.89a}$$

$$\gamma=\frac{d_{eg}^2\omega^3}{3\pi c^3\hbar\varepsilon_0}, \tag{3.89b}$$

$$\kappa=\frac{\pi c}{2\mathscr{F}\ell}. \tag{3.89c}$$

Putting it all together:

$$C=\frac{\omega}{\hbar\varepsilon_0\ell S}d_{eg}^2\frac{3\pi c^3\hbar\varepsilon_0}{d_{eg}^2\omega^3}\frac{2\mathscr{F}\ell}{\pi c}=\frac{6\mathscr{F}c^2}{S\omega^2}. \tag{3.90}$$

Also, replacing $\omega/c=k=2\pi/\lambda$ we have $6c^2/\omega^2=6\lambda^2/(4\pi^2)$ so that

$$C=\mathscr{F}\frac{\frac{3\lambda^2}{2\pi^2}}{S}. \tag{3.91}$$

One can then only improve this quantity by designing better mirrors and focusing down the cavity mode area to small values.

3.5 Optical bistability

We will now add decay into the problem at ratesγ for the atom and κ for the cavity field. We also assume a weak pump coming through the left mirror and analyze the transmission properties of the system. We can derive the equations of motion for the field and atomic operators:

$$\frac{d}{dt}\langle a\rangle=-\kappa\langle a\rangle+i\Delta\langle a\rangle-ig\langle\sigma\rangle+\eta, \tag{3.92a}$$

$$\frac{d}{dt}\langle\sigma\rangle=-\gamma\langle\sigma\rangle+i\Delta\langle\sigma\rangle+ig\langle a\left(\sigma^\dagger\sigma-\sigma\sigma^\dagger\right)\rangle. \tag{3.92b}$$

We assumed atoms resonant to the cavity mode and the detuning is $\Delta=\omega_L-\omega_c$. The equations are non-linear as the last term contains correlations between the field and the atom dipole. However, under the weak pumping condition: $\eta\ll\kappa$, the cavity field in steady state will have much less than one photon and the atom will pretty much stay close to the ground state. We therefore replace $\sigma^\dagger\sigma - \sigma\sigma^\dagger$ with -1 and analyze the set of linearized equations

$$\frac{d}{dt}\langle a\rangle = -\kappa\langle a\rangle + i\Delta\langle a\rangle - ig\langle\sigma\rangle + \eta, \tag{3.93a}$$

$$\frac{d}{dt}\langle\sigma\rangle = -\gamma\langle\sigma\rangle + i\Delta\langle\sigma\rangle - ig\langle a\rangle. \tag{3.93b}$$

Looking at steady state we have:

$$\langle\sigma\rangle = -\frac{ig}{\gamma - i\Delta}\langle a\rangle, \tag{3.94}$$

such that

$$\langle a\rangle\left[\kappa - i\Delta + \frac{g^2}{\gamma - i\Delta}\right] = \eta. \tag{3.95}$$

Rewriting the last equation:

$$\langle a\rangle\left[\kappa + \frac{g^2\gamma}{\gamma^2+\Delta^2} - i\Delta\left(1 - \frac{g^2}{\gamma^2+\Delta^2}\right)\right] = \eta. \tag{3.96}$$

Remember from last lecture that the transmission of the cavity is:

$$T_c = \left|\frac{\sqrt{\kappa}\langle a\rangle}{\eta/\sqrt{\kappa}}\right|^2 = \frac{\kappa^2}{\left|\kappa + \frac{g^2\gamma}{\gamma^2+\Delta^2} - i\Delta\left(1 - \frac{g^2}{\gamma^2+\Delta^2}\right)\right|^2}. \tag{3.97}$$

One observation is that we see that $\Delta = 0$ gives the maximum so the minimum of the transmission at

$$T_c(\Delta = 0) = \frac{\kappa^2}{\left|\kappa + \frac{g^2}{\gamma}\right|^2} = \frac{1}{(1+C)^2}, \tag{3.98}$$

where the cooperativity $C = g^2/(\kappa\gamma)$ (just as before in the limit of bad-cavity we defined it proportional to the Purcell factor) plays a major role. One can also see the effect in the next order approximation for the equation of motion when more significant population in the excited state is taken into account. Let's rewrite

$$\frac{d}{dt}\langle a\rangle = -\kappa\langle a\rangle + i\Delta\langle a\rangle - ig\langle\sigma\rangle + \eta, \tag{3.99}$$

$$\frac{d}{dt}\langle\sigma\rangle = -\gamma\langle\sigma\rangle + i\Delta\langle\sigma\rangle + ig\langle a\left(\sigma^\dagger\sigma - \sigma\sigma^\dagger\right)\rangle. \tag{3.100}$$

$$\frac{d}{dt}\langle\sigma^\dagger\sigma - \sigma\sigma^\dagger\rangle = -2\gamma\langle\sigma^\dagger\sigma - \sigma\sigma^\dagger\rangle + ig\langle a^\dagger\sigma - a\sigma^\dagger\rangle - 2\gamma. \tag{3.101}$$

We perform two factorizations$\langle a\left(\sigma^\dagger\sigma - \sigma\sigma^\dagger\right)\rangle = \langle\sigma^\dagger\sigma - \sigma\sigma^\dagger\rangle\langle a\rangle$ and $\langle a^\dagger\sigma - a\sigma^\dagger\rangle = \langle a^\dagger\rangle\langle\sigma\rangle - \langle a\rangle\langle\sigma^\dagger\rangle$ and evaluate in steady state:

$$\langle\sigma\rangle(\gamma - i\Delta) = ig\langle a\rangle\langle\sigma^\dagger\sigma - \sigma\sigma^\dagger\rangle, \tag{3.102}$$

$$\langle\sigma\rangle^*(\gamma + i\Delta) = -ig\langle a\rangle^*\langle\sigma^\dagger\sigma - \sigma\sigma^\dagger\rangle. \tag{3.103}$$

This immediatily leads to

$$ig\langle a\rangle^*\langle\sigma\rangle(\gamma - i\Delta) = -g^2|\langle a\rangle|^2\langle\sigma^\dagger\sigma - \sigma\sigma^\dagger\rangle, \tag{3.104}$$

$$-ig\langle a\rangle\langle\sigma\rangle^*(\gamma + i\Delta) = -g^2|\langle a\rangle|^2\langle\sigma^\dagger\sigma - \sigma\sigma^\dagger\rangle, \tag{3.105}$$

so that

$$ig\langle a^\dagger\sigma - a\sigma^\dagger\rangle = -\frac{2\gamma g^2}{\gamma^2+\Delta^2}|\langle a\rangle|^2\langle\sigma^\dagger\sigma-\sigma\sigma^\dagger\rangle \tag{3.106}$$

The equation for the population difference then gives

$$\langle\sigma^\dagger\sigma-\sigma\sigma^\dagger\rangle = -1-\frac{2g^2}{\gamma^2+\Delta^2}|\langle a\rangle|^2\langle\sigma^\dagger\sigma-\sigma\sigma^\dagger\rangle \to \langle\sigma^\dagger\sigma-\sigma\sigma^\dagger\rangle = \frac{-1}{1+\frac{2g^2}{\gamma^2+\Delta^2}|\langle a\rangle|^2}. \tag{3.107}$$

Now we directly have

$$\langle\sigma\rangle = \frac{-ig}{\gamma-i\Delta}\frac{\langle a\rangle}{1+\frac{2g^2}{\gamma^2+\Delta^2}|\langle a\rangle|^2}, \tag{3.108}$$

which leads to the cavity field amplitude in steady state:

$$\langle a\rangle(\kappa-i\Delta) = \frac{-g^2}{\gamma-i\Delta}\frac{\langle a\rangle}{1+\frac{2g^2}{\gamma^2+\Delta^2}|\langle a\rangle|^2}+\eta, \tag{3.109}$$

or

$$\langle a\rangle\left[(\kappa-i\Delta)+\frac{1}{1+\frac{2g^2}{\gamma^2+\Delta^2}|\langle a\rangle|^2}\frac{g^2}{\gamma-i\Delta}\right] = \eta. \tag{3.110}$$

The expression above shows that the cavity resonances are shifted for larger input fields as the intracavity field intensity depends nonlinearly on the input field. Taking the absolute value squared for the above expression leads to

$$I_c\left|(\kappa-i\Delta)+\frac{1}{1+\frac{2g^2}{\gamma^2+\Delta^2}I_c}\frac{g^2}{\gamma-i\Delta}\right|^2 = I_p, \tag{3.111}$$

which is an equation with more than one solution (three) for give ranges of the input For example, at $\Delta = 0$,

$$I_c\kappa^2\left(1+\frac{C^2}{\left(1+\frac{2g^2}{\gamma^2}I_c\right)^2}\right) = I_p, \tag{3.112}$$

leading to

$$I_c\kappa^2\left(\left(1+\frac{2g^2}{\gamma^2}I_c\right)^2+C^2\right) = I_p\left(1+\frac{2g^2}{\gamma^2}I_c\right)^2. \tag{3.113}$$

3.6 Photon blockade

One notices that in the first and second excitation manifolds nonlinearities appear due to the progressive $g\sqrt{n}$ shifts of levels. Looking at the resonance condition for single cavity input photon one notices two resonances at the lower (at $\omega - g$) and at the upper $(\omega+g)$ polaritons. Assuming the driving frequency at the lower polariton, one can also notice that the 2 photon resonance condition is not fulfilled as a detuning $(2\omega-\sqrt{2}g)-2(\omega-g) = (2-\sqrt{2})g$ appears. In the limit that this detuning is very large, the presence of the atom strongly coupled to the cavity effectively suppresses the entering of a second photon, thus providing a photon blockade mechanism.

3.7 Appendix: Langevin equations in the Fourier domain.

A Fourier transformation can simplify our task of finding operator averages and correlations (at equal or different times) as it transforms the set of coupled differential equations into an ordinary set of equations. Notice however that this is only valid around steady state. When one needs to take the initial conditions into account, the Laplace transform can be used instead.

Let us introduce the Fourier transform of the time dependent operators:

$$a(\omega) = \mathscr{F}[a(t)] = \frac{1}{\sqrt{2\pi}}\int dta(t)e^{+i\omega t}, \tag{3.114}$$

$$a(t) = \frac{1}{\sqrt{2\pi}}\int d\omega a(\omega)e^{-i\omega t}. \tag{3.115}$$

Notice two properties. First the Fourier transform of a derivative is:

$$\mathscr{F}[\dot{a}(t)] = -i\omega a(\omega), \tag{3.116}$$

Second, we use:

$$a^\dagger(\omega) = \mathscr{F}[a^\dagger(t)] = \frac{1}{\sqrt{2\pi}}\int dta^\dagger(t)e^{+i\omega t}, \tag{3.117}$$

which leads to:

$$[a(\omega)]^\dagger = \frac{1}{\sqrt{2\pi}}\int dta^\dagger(t)e^{-i\omega t} = a^\dagger(-\omega). \tag{3.118}$$

For delta-correlated input noise (of zero average) then we have:

$$\langle a(\omega)a^\dagger(\omega')\rangle = \frac{1}{2\pi}\int dt\int dt'\langle a(t)a^\dagger(t')\rangle e^{+i\omega t}e^{+i\omega' t'} = \tag{3.119}$$

$$= \frac{1}{2\pi}\int dt\int dt'\delta(t-t')e^{+i\omega t}e^{+i\omega' t'} = \frac{1}{2\pi}\int dte^{+i(\omega+\omega')t} = \delta(\omega+\omega'). \tag{3.120}$$

We consider a driven cavity

$$\dot{a} = -\kappa a + i\Delta a + \eta + \sqrt{\kappa}\left(a_{in} + b_{in}\right), \tag{3.121}$$

and first linearize around steady state:

$$\dot{\delta a} = -\kappa\delta a + i\Delta\delta a + \sqrt{2\kappa}d_{in}, \tag{3.122}$$

while the average is as above. We also wrote $d_{in} = (a_{in} + b_{in})/\sqrt{2}$ with the combined input noise being delta correlated in time. Taking the Fourier transform:

$$\delta a(\omega)\left[\kappa - i(\Delta+\omega)\right] = \sqrt{2\kappa}d_{in}(\omega), \tag{3.123}$$

together with its couterpart

$$\delta a^\dagger(\omega)\left[\kappa + i(\Delta-\omega)\right] = \sqrt{2\kappa}d_{in}^\dagger(\omega). \tag{3.124}$$

One can express these in terms of the susceptibilities:

$$\delta a(\omega) = \varepsilon(\omega)\sqrt{2\kappa}d_{in}(\omega), \tag{3.125}$$

$$\delta a^\dagger(\omega) = \varepsilon^*(-\omega)\sqrt{2\kappa}d_{in}^\dagger(\omega), \tag{3.126}$$

The intracavity correlations are:

$$\langle \delta a(\omega)\delta a^\dagger(\omega')\rangle = \varepsilon(\omega)\varepsilon^*(-\omega')2\kappa\langle d_{in}(\omega)d_{in}^\dagger(\omega')\rangle = \tag{3.127}$$

$$= 2\kappa\left|\varepsilon(\omega)\right|^2\delta(\omega+\omega') = \frac{2\kappa}{\kappa^2+(\Delta+\omega)^2}\delta(\omega+\omega'). \tag{3.128}$$

The linearized input-output relations (for zero-average operators) transform straighforwardly into the Fourier space:

$$a_{out}(\omega) = \sqrt{\kappa}\delta a(\omega) - b_{in}(\omega), \tag{3.129}$$

$$b_{out}(\omega) = \sqrt{\kappa}\delta a(\omega) - a_{in}(\omega), \tag{3.130}$$

and

$$a_{out}^\dagger(\omega) = \sqrt{\kappa}\delta a^\dagger(\omega) - b_{in}^\dagger(\omega), \tag{3.131}$$

$$b_{out}^\dagger(\omega) = \sqrt{\kappa}\delta a^\dagger(\omega) - a_{in}^\dagger(\omega). \tag{3.132}$$

One can connect output noise to input noise directly:

$$a_{out}(\omega) = \varepsilon(\omega)\kappa a_{in}(\omega) - b_{in}(\omega)\left[1-\varepsilon(\omega)\kappa\right], \tag{3.133}$$

$$a_{out}^\dagger(\omega) = \varepsilon^*(-\omega)\kappa a_{in}^\dagger(\omega) - b_{in}^\dagger(\omega)\left[1-\varepsilon^*(-\omega)\kappa\right], \tag{3.134}$$

Now we can express for example:

$$\langle a_{out}(\omega)a_{out}^\dagger(\omega')\rangle = \varepsilon(\omega)\varepsilon^*(-\omega')\kappa^2\langle a_{in}(\omega)a_{in}^\dagger(\omega')\rangle + \left[1-\varepsilon(\omega)\kappa\right]\left[1-\varepsilon^*(-\omega')\kappa\right]\langle b_{in}(\omega)b_{in}^\dagger(\omega')\rangle = \tag{3.135}$$

$$= \left[1+2\kappa^2|\varepsilon(\omega)|^2 - 2\kappa Re[\varepsilon(\omega)]\right]\delta(\omega+\omega') = \tag{3.136}$$

$$= \left[1+\frac{2\kappa^2}{\kappa^2+(\Delta+\omega)^2} - 2\kappa\frac{\kappa}{\kappa^2+(\Delta+\omega)^2}\right]\delta(\omega+\omega') = \delta(\omega+\omega') \tag{3.137}$$

4. Fundamentals of laser theory

We will now make use of the techniques developed in the previous chapters to describe a few characteristics of a lasing cavity filled with a gain medium. Our goals are modest as we only aim at providing a simplistic treatment to reveal two main characteristic of lasing action: i) the existence of a threshold for lasing connected to the achieved population inversion, and ii) the statistics of photons exiting a laser cavity (ideally close to a Poisson distribution characteristic of a coherent state). We will employ two slightly different models to derive these crucial aspects. First, we assume a collection of effective two level systems incoherently pumped such that population inversion can be achieved. We show that in the macroscopic limit of large number of gain medium atoms the cavity can sustain the build-up of a non-zero amplitude field past a certain pumping threshold. We then follow a standard approach (Milburn, Scully) of four-level systems passing through a cavity to arrive at a simplified form for the photon number distribution and prove that past the threhold a laser exhibits Poissonian statistics. Finally, we analyze the laser linewidth by well above threshold and connect it to the diffusion process that the cavity field undergoes during spontaneous emission events.

4.1 Laser threshold

We assume N identical two level systems equally coupled to a cavity mode. The Hamiltonian of the system is comprised of

$$H_0 = \hbar\omega a^\dagger a + \sum_j \hbar\omega\sigma_z^{(j)}, \tag{4.1a}$$

$$H_{JC} = \sum_j \hbar g \left[a^\dagger \sigma_j + a\sigma_j^\dagger \right]. \tag{4.1b}$$

The Lindblad terms contain the natural spontaneous emission at rate γ, cavity decay at rate κ as well as the engineered pump rate (described as an artificial inverse decay terms as derived in Chapter 2) at rate Γ

$$\mathscr{L} = \gamma \sum_j \mathscr{D}[\sigma_j, \rho] + \Gamma \sum_j \mathscr{D}[\sigma_j^\dagger, \rho] + \kappa \sum_j \mathscr{D}[a, \rho]. \tag{4.2}$$

Of course we can immediately get rid of the free evolution by moving into the interaction picture with respect to H_0. We can write equations of motion for the averages:

$$\frac{d}{dt}\langle a\rangle = -\kappa\langle a\rangle - ig\sum_j\langle\sigma_j\rangle, \tag{4.3a}$$

$$\frac{d}{dt}\langle\sigma_j\rangle = -\frac{\Gamma+\gamma}{2}\langle\sigma_j\rangle + 2ig\langle a\sigma_z^{(j)}\rangle, \tag{4.3b}$$

$$\frac{d}{dt}\langle\sigma_z^{(j)}\rangle = -(\Gamma+\gamma)\langle\sigma_z^{(j)}\rangle + ig\langle a^\dagger\sigma_j - a\sigma_j^\dagger\rangle + \frac{\Gamma-\gamma}{2}. \tag{4.3c}$$

Let's quickly check that this is correct. In the absence of coupling to the cavity starting with the ground state we get a steady state with

$$\langle\sigma_z^{(j)}\rangle = \frac{\Gamma-\gamma}{2(\Gamma+\gamma)} = d_0. \tag{4.4}$$

This is the expected population inversion when pumping the system. If $\Gamma \gg \gamma$, we get d_0 close to $1/2$ meaning most of the population is in the excited state. The above equations are coupled and nonlinear and thus not easy to solve. First we notice that we can perform sums such that:

$$\frac{d}{dt}\langle a\rangle = -\kappa\langle a\rangle - ig\langle S\rangle, \tag{4.5a}$$

$$\frac{d}{dt}\langle S\rangle = -\frac{\Gamma+\gamma}{2}\langle S\rangle + 2ig\langle aS_z\rangle, \tag{4.5b}$$

$$\frac{d}{dt}\langle S_z\rangle = -(\Gamma+\gamma)\langle S_z\rangle + ig\langle a^\dagger S - aS^\dagger\rangle + N\frac{\Gamma-\gamma}{2}. \tag{4.5c}$$

where the collective operators are:

$$S = \sum_j \sigma_j \qquad \text{and} \qquad S_z = \sum_j \sigma_z^{(j)}. \tag{4.6}$$

In steady state without cavity we will now have $\langle S_z\rangle = Nd_0 = D_0$.In the large system expansion when $N \gg 1$, one can analyze the classical equations for this system by factorizing (we split each operator into an average and a zero average fluctuation):

$$\langle aS_z\rangle = \langle(\alpha+\delta a)(s_z+\delta S_z)\rangle = \alpha s_z + \langle\delta a\delta S_z\rangle \simeq \alpha s_z, \tag{4.7a}$$

$$\langle a^\dagger S\rangle = \langle\left(\alpha^*+\delta a^\dagger\right)(s+\delta S)\rangle = \alpha^* s + \langle\delta a^\dagger\delta S\rangle \simeq \alpha^* s, \tag{4.7b}$$

$$\langle aS^\dagger\rangle = \langle(\alpha+\delta a)\left(s^*+\delta S^\dagger\right)\rangle = \alpha s^* + \langle\delta a\delta S^\dagger\rangle \simeq \alpha s^*. \tag{4.7c}$$

This should be valid as long as the averages of fluctuation products are small compared to the averages.

Classical equations. Threshold.

Let's proceed with the system to be solved:

$$\frac{d\alpha}{dt} = -\kappa\alpha - igs, \tag{4.8a}$$

$$\frac{ds}{dt} = -\frac{\Gamma+\gamma}{2}s + 2ig\alpha s_z, \tag{4.8b}$$

$$\frac{ds_z}{dt} = -(\Gamma+\gamma)s_z + ig\left(\alpha^* s - \alpha s^*\right) + N\frac{\Gamma-\gamma}{2}. \tag{4.8c}$$

We set steady state conditions and notice that

$$s = \frac{4ig}{\Gamma+\gamma}\alpha s_z. \tag{4.9}$$

Replacing in equation 3 we get:

$$\left[(\Gamma+\gamma)+\frac{8g^2}{\Gamma+\gamma}|\alpha|^2\right]s_z = N\frac{\Gamma-\gamma}{2}. \tag{4.10}$$

In simplified notation:

$$s_z = \frac{D_0}{1+\frac{8g^2}{(\Gamma+\gamma)^2}|\alpha|^2}. \tag{4.11}$$

Notice that as the cavity field is turned on the population inversion starts going down. We can now compute:

$$s = \frac{4ig}{\Gamma+\gamma}\frac{D_0}{1+\frac{8g^2}{(\Gamma+\gamma)^2}|\alpha|^2}\alpha, \tag{4.12}$$

which leads to an effective equation of motion for the cavity field amplitude:

$$\frac{d\alpha}{dt} = -\kappa\alpha + \frac{4g^2}{\Gamma+\gamma}\frac{D_0}{1+\frac{8g^2}{(\Gamma+\gamma)^2}|\alpha|^2}\alpha. \tag{4.13}$$

With more notations:

$$n_0 = \frac{(\Gamma+\gamma)^2}{8g^2}, \qquad C = \frac{4g^2 D_0}{\kappa(\Gamma+\gamma)} = \frac{2g^2\,(\Gamma-\gamma)}{\kappa(\Gamma+\gamma)^2} \simeq \frac{2g^2}{\kappa\Gamma} \tag{4.14}$$

we can rewrite:

$$\frac{d\alpha}{dt} = \kappa\left[-1+\frac{C}{1+|\alpha|^2/n_0}\right]\alpha. \tag{4.15}$$

It is easy to see that the steady state is zero unless the gain surpasses the cavity loss. We then find a thresold at $C = 1$ and compute the cavity field intensity as:

$$1 = \frac{C}{1+|\alpha|^2/n_0}, \qquad \text{leading to} \qquad |\alpha|^2 = n_0\,(C-1)\,. \tag{4.16}$$

Well above threshold

Consider $C \gg 1$ and look again at the population inversion

$$s_z = \frac{D_0}{C} = \frac{\kappa\Gamma}{4g^2,} = \frac{\Gamma^2}{8g^2}\frac{\kappa}{\Gamma}. \tag{4.17}$$

Notice that as the cavity field is turned on the population inversion starts going down. We can now compute:

$$s = \frac{4ig}{\Gamma+\gamma}\frac{D_0}{1+\frac{8g^2}{(\Gamma+\gamma)^2}|\alpha|^2}\alpha, \tag{4.18}$$

4.2 Photon statistics of a laser

We now move to a more standard model for deriving equations for the reduced density operator of the cavity field alone. The goal is to find the photon number distribution of cavity photons below and above threshold and see the transition from thermal light to coherent light in the cavity output. The model is used extensively in Quantum Optics textbooks (Milburn, Scully, etc). It consists of a collection of four level atoms where the important ones are levels $|2\rangle$ (excited) and $|1\rangle$(ground) both of them decaying to some other levels with γ_2 (from $|2\rangle$to $|4\rangle$) and with γ_1(from $|1\rangle$to $|3\rangle$). There is no spontaneous emission (or negligible) between the lasing levels, and the transition is coupled resonantly to a cavity field.

Model assumptions

The atoms are prepared in the excited state and sent through the cavity spending a time τ inside which is much larger than the time it takes the atoms to reach steady state (roughly given by $\gamma_{1,2}^{-1}$). The rate of which atoms are pushed into the cavity is r (understood as number of atoms per unit time). The probability of an atom entering the cavity during a small time interval Δt would then be $r\Delta t$ while the opposite event has the probability $(1-r\Delta t)$. We now assume that at time t the cavity field is in state $\rho(t)$. After the passing of an atom through the cavity the reduced cavity field density operator becomes

$$\rho_F(t+\Delta t) = \mathscr{P}(\Delta t)\rho_F(t). \tag{4.19}$$

Summing together probabilities that either an atom passed or not through the cavity we will have a final density operator for the cavity:

$$\rho_F(t+\Delta t) = r\Delta t\mathscr{P}(\Delta t)\rho_F(t) + (1-r\Delta t)\rho_F(t). \tag{4.20}$$

In the small Δt limit, we will have an effective reduced master equation:

$$\frac{d\rho_F(t)}{dt} = r(\mathscr{P}-1)\rho_F(t) + \kappa\mathscr{D}_a[\rho_F], \tag{4.21}$$

after writing the difference as a derivative and adding the decay Lindblad term for the cavity field.

The action of a single atom on the cavity field

Let us now focus on deriving the action $\mathscr{P}(\Delta t)$. We set the initial time to zero for simplicity and consider an initial density matrix generally expressed as:

$$\rho_F = \sum_{n,m=0}^{\infty} \rho_{nm}^F(0)|n\rangle\langle m|. \tag{4.22}$$

Of course the atom initial density matrix is $|2\rangle\langle 2|$. We will show that

$$\rho_F(\Delta t) = \sum_{n,m=0}^{\infty} \rho_{nm}^F(0)\left[A_{nm}|n\rangle\langle m| + B_{nm}|n+1\rangle\langle m+1|\right]. \tag{4.23}$$

The master equation to be solved during the Δt time interval evolution is:

$$\frac{d\rho(t)}{dt} = \frac{i}{\hbar}[H,\rho] + \gamma\mathscr{D}_{\sigma_1}[\rho] + \gamma\mathscr{D}_{\sigma_2}[\rho], \tag{4.24}$$

where $\sigma_1 = |3\rangle\langle 1|$ and $\sigma_2 = |4\rangle\langle 2|$ and the $\mathscr{D}$ superoperators acts as usual as Lindblad terms describing decay. Notice that one can rewrite the master equation in the following form:

$$\frac{d\rho(t)}{dt} = \frac{i}{\hbar}\left[H_{eff}\rho - \rho H_{eff}^{\dagger}\right] + R[\rho], \tag{4.25}$$

grouping recycling terms:

$$R[\rho] = 2\gamma\left[\sigma_1\rho\sigma_1^\dagger + \sigma_2\rho\sigma_2^\dagger\right], \tag{4.26}$$

and adding the rest in an effective Hamiltonian (careful that this is an effective non-hermitian evolution operator)

$$H_{eff} = H - i\gamma\left(\sigma_1^\dagger\sigma_1 + \sigma_2^\dagger\sigma_2\right). \tag{4.27}$$

A zeroth order solution can be found by taking the derivatives of the following function:

$$\rho_0(t) = B(t)\rho(0)B^\dagger(t) = S(t)[\rho(0)], \tag{4.28}$$

satisfying the equation:

$$\frac{d\rho_0(t)}{dt} = \frac{i}{\hbar}\left[H_{eff}\rho_0(t) - \rho_0(t)H_{eff}^\dagger\right]. \tag{4.29}$$

Now we look for the next order solution of the form:

$$\rho(t) = \rho_0(t) + B(t)\left[\rho_1(t)\right]B^\dagger(t) = S(t)[\rho(0)] + S(t)[\rho_1(t)]. \tag{4.30}$$

One can check that the master equation turns into:

$$\begin{aligned} &\frac{d\rho_0(t)}{dt} + \dot{B}(t)\left[\rho_1(t)\right]B^\dagger(t) + \dot{B}(t)\left[\rho_1(t)\right]B^\dagger(t) + \dot{B}(t)\left[\dot{\rho}_1(t)\right]\dot{B}^\dagger(t) = \\ &= \frac{i}{\hbar}\left[H_{eff}\rho_0 - \rho_0 H_{eff}^\dagger\right] + \frac{i}{\hbar}\left[H_{eff}B(t)\left[\rho_1(t)\right]B^\dagger(t) - B(t)\left[\rho_1(t)\right]B^\dagger(t)H_{eff}^\dagger\right] + \\ &R[\rho_0 + B(t)\left[R[\rho_1(t)]\right]B^\dagger(t). \end{aligned} \tag{4.31}$$

After simplifications one arrives at (ignoring the last term):

$$\frac{d\rho_1(t)}{dt} = B(t)\left[R[\rho_0(t)]\right]B^\dagger(t) = S(t)\left[R\left[S(t)\left[\rho(0)\right]\right]\right], \tag{4.32}$$

and integration gives:

$$\rho_1(\Delta t) = \int_0^{\Delta t} dt S(t)\left[R\left[S(t)\left[\rho(0)\right]\right]\right]. \tag{4.33}$$

Finally, one can put it all together:

$$\rho(\Delta t) = S(\Delta t)\left[\rho(0)\right] + \int_0^{\Delta t} dt S(t)\left[R\left[S(t)\left[\rho(0)\right]\right]\right] + \dots. \tag{4.34}$$

After painful calculations one can derive the next order terms and show that they are vanishing. Let's therefore focus on the zeroth and first order term. We are left with evaluating terms like:

$$B(\Delta t)|n\rangle|2\rangle = c_n^+(\Delta t)|n,+\rangle + c_n^-(\Delta t)|n,-\rangle, \tag{4.35}$$

where the states defined as

$$|n_\pm\rangle = \frac{1}{\sqrt{2}}\left[|n,2\rangle \pm |n,1\rangle\right] \tag{4.36}$$

are the polariton states in the $n+1$ excitation manifold. One can find a simple expression (based on the assumptions $\gamma_1 = \gamma_2$):

$$c_n^+(\Delta t) = \frac{1}{2\sqrt{2}} e^{-ig\sqrt{n+1}t} e^{-\gamma\Delta t}, \tag{4.37a}$$

$$c_n^-(\Delta t) = \frac{1}{2\sqrt{2}} e^{ig\sqrt{n+1}t} e^{-\gamma\Delta t}. \tag{4.37b}$$

We can find the first action:

$$S(t)[\rho(0)] = B(t)\rho(0)B^\dagger(t) = \left[c_n^+(t)|n_+\rangle + c_n^-(t)|n_-\rangle\right]\left[c_n^{+*}(t)\langle n_+| + c_n^{-*}(t)\langle n_-|\right]. \tag{4.38}$$

When plugged in to replace the zeroth order solution the results is zero in the limit $\Delta t \gg \gamma^{-1}$. The next term however will not vanish. Let's first calculate one term to get an idea what the recycling term does.

$$\begin{aligned} R\left[|n_+\rangle\langle n_+|\right] &= 2\gamma\left[\sigma_1|n_+\rangle\langle n_+|\sigma_1^\dagger + \sigma_2|n_+\rangle\langle n_+|\sigma_2^\dagger\right] = \\ &= \gamma\left[|n+1,3\rangle\langle n+1,3| + |n,4\rangle\langle n,4|\right]. \end{aligned} \tag{4.39}$$

Then we can work out all the other terms and trace over the atomic states to obtain:

$$\begin{aligned} \mathrm{Tr}_A\left[S(t)\left[R\left[S(t)\left[\rho(0)\right]\right]\right]\right] &= \gamma\left[c_n^+(t) + c_n^-(t)\right]\left[c_m^{+*}(t) + c_m^{-*}(t)\right]|n\rangle\langle m| + \\ &+ \gamma\left[c_n^+(t) - c_n^-(t)\right]\left[c_m^{+*}(t) - c_m^{-*}(t)\right]|n+1\rangle\langle m+1|. \end{aligned} \tag{4.40}$$

Finally, performing the integral over the first order solution one obtaines:

$$\rho_F(\Delta t) = \sum_{n,m=0}^{\infty} \rho_{nm}^F(0)\left[A_{nm}|n\rangle\langle m| + B_{nm}|n+1\rangle\langle m+1|\right], \tag{4.41}$$

where

$$A_{nm} = \int_0^{\Delta t} dt\left[c_n^+(t) + c_n^-(t)\right]\left[c_m^{+*}(t) + c_m^{-*}(t)\right], \tag{4.42a}$$

$$B_{nm} = \int_0^{\Delta t} dt\left[c_n^+(t) - c_n^-(t)\right]\left[c_m^{+*}(t) - c_m^{-*}(t)\right]. \tag{4.42b}$$

For the diagonal elements one can check that the result is:

$$A_{nn} = \frac{g^2(n+1) + 2\gamma^2}{2g^2(n+1) + \gamma^2}, \tag{4.43a}$$

$$B_{nn} = 1 - A_{nn}. \tag{4.43b}$$

The reduced master equation and photon statistics

Putting together the results of the last two subsections we get an effective master equations written as:

$$\begin{aligned} \frac{d\rho_{nm}}{dt} = G&\left[\frac{\sqrt{nm}}{1+(n+m)/(2n_s)}\rho_{n-1,m-1} - \frac{(m+n+2)/2 + (m-n)^2/(8n_s)}{1+(n+m+2)/(2n_s)}\rho_{nm}\right] + \\ &+ \kappa\left[2\sqrt{(n+1)(m+1)}\rho_{n+1,m+1} - (n+m)\rho_{nm}\right]. \end{aligned} \tag{4.44}$$

The gain is derived as:

$$G = \frac{r}{2n_s}, \qquad \text{where} \qquad n_s = \frac{\gamma^2}{g^2}. \tag{4.45}$$

Focusing now on the photon number distribution, let's consider the diagonal elements ($p_n = \rho_{nn}$):

$$\frac{dp_n}{dt} = -G\left[\frac{n+1}{1+(n+1)/n_s}p_n - \frac{n}{1+n/n_s}p_{n-1}\right] + 2\kappa\left[(n+1)p_{n+1} - \kappa n p_n\right]. \tag{4.46}$$

We will analyze some simplified cases.

Thermal statistics (below threshold)

The first one assumes $n_s \gg 1$ such that the average photon number is much smaller than n_s. This allows the approximation leading to:

$$\frac{dp_n}{dt} = -G\left[(n+1)p_n - np_{n-1}\right] + 2\kappa\left[(n+1)p_{n+1} - \kappa n p_n\right]. \tag{4.47}$$

In steady state one can show that the solution to this recursive equation is:

$$p_n = p_0\left(\frac{G}{2\kappa}\right)^n. \tag{4.48}$$

Together with the normalization condition $\sum p_n = 1$, one can deduce

$$p_n = \left(1 - \frac{G}{2\kappa}\right)^{-1}\left(\frac{G}{2\kappa}\right)^n. \tag{4.49}$$

This can be casted in terms of a black-body radiation distribution with the identification:

$$\frac{G}{2\kappa} = e^{-\frac{\hbar\omega}{k_B T}}, \tag{4.50}$$

where one computes the effective temperature for values lower than the threshold. Notice that the average number of photons in the cavity (in thermal state) is

$$\bar{n} = \frac{G}{2\kappa - G}. \tag{4.51}$$

As a remark, the amplitude of the field is zero; characteristically of the thermal state, the average photon number instead is non-zero and equal to the fluctuations.

Poisson statistics (well-above threshold)

We now assume that we are well beyond threshold such that $\bar{n} \gg n_s$ and only look at photon numbers around the average (since we expect a Poisson distribution quite narrow far from the threshold). We can approximate again:

$$\frac{dp_n}{dt} = -G\left[n_s p_n - n_s p_{n-1}\right] + 2\kappa\left[(n+1)p_{n+1} - \kappa n p_n\right]. \tag{4.52}$$

Setting steady state condition, the recursive equation leads to the following solution:

$$p_n = e^{-\bar{n}}\frac{\bar{n}^n}{n!}, \tag{4.53}$$

with an average photon number

$$\bar{n} = \frac{Gn_s}{2\kappa}. \tag{4.54}$$

This is a Poisson distribution made up by the contribution of many coherent states with different phases.

Four-level system: Laser linewidth.

The phase of the output light is undergoing a continuous phase diffusion owing to the spontaneous emission events accompanying the coherent exchanges between excited atoms and the cavity field. While the Poisson nature indicates that the amplitude of the cavity field stays pretty much stable, we can derive an equation for the damping of the phase. Let's first write:

$$\langle a\rangle = \mathrm{Tr}\left[a\rho_F\right] = \sum_{n=0}^{\infty}\langle n|a\rho_F|n\rangle = \sum_{n=1}^{\infty}\sqrt{n}\rho_{n,n-1}. \tag{4.55}$$

and its derivative

$$\frac{d}{dt}\langle a\rangle = \sum_{n=1}^{\infty}\sqrt{n}\frac{d}{dt}\rho_{n,n-1}. \tag{4.56}$$

Using the general equation derived above for the laser, one can show (see Scully) that:

$$\frac{d}{dt}\langle a\rangle = -\frac{G}{2}\sum_{n=1}^{\infty}\frac{1/(4n_s)-1}{1+(2n+1)/(2n_s)}\rho_{n,n-1} \simeq -\frac{G}{8\bar{n}}\langle a\rangle. \tag{4.57}$$

We can also look at the laser field spectrum defined as (and using a quantum regression theorem):

$$S(\omega) = \frac{1}{\sqrt{2\pi}}\int d\tau\langle a^\dagger(\tau)a(0)\rangle e^{-i\omega\tau} = \mathscr{F}[\bar{n}e^{-\frac{G}{8\bar{n}}\tau}] = \frac{\bar{n}}{\omega^2+\left(\frac{G}{8\bar{n}}\right)^2}. \tag{4.58}$$

The conclusion is that the laser spectrum is a Lorentzian with a linewidth of

$$\Delta\omega = \frac{G}{8\bar{n}}. \tag{4.59}$$

5. Cooling of atoms

Up to here we examined fundamental processes between quantized light (photons) and *static* two (or more than two) level systems. However, as a photon is emitted or absorbed by an atom a transfer of momentum occurs modifying its initial motional state. In order to properly account for this we will supplement the starting Hamiltonian of a TLS in a box containing quantized light with the free particle Hamiltonian. This allows us then to derive the effect of stimulated absorption/emission as well as that of spontaneous emission on the atom. The fact that light can have an impact on the motion of atoms also means that manipulation of its state means of lasers is possible. In a semiclassical approximation we then exemplify a simple derivation of the cooling force of a TLS in plane and standing waves. This is known as Doppler cooling. We describe its particularities including the final achievable temperature. We also shortly review dipole forces (arising from motion in focused light beams used in optical tweezers) and describe the mechanism for polarization gradient cooling.

5.1 Light-induced forces.

Let us consider an atom moving in one dimension only (axis y) and quantize its motion (that of the nucleus) by imposing the commutation relations $[\hat{y},\hat{p}] = i\hbar$. In the absence of any confining potentials or interaction its evolution is described by the kinetic energy operator $\hat{p}^2/2M$. One can go either in the position or momentum representation to describe the system's evolution. We choose (for a good reason) the momentum representation where the basis is given by $|p\rangle$ such that $\hat{p}\,|p\rangle = p\,|p\rangle$ and $\hat{p}\,|p\rangle = i\hbar\partial_p\,|p\rangle$. Notice that in this reciprocal space representation the action the following action is simply a translation (the proof involves a Taylor expansion)

$$e^{ik\hat{y}}\,|p\rangle = |p+\hbar k\rangle\,. \tag{5.1}$$

Of course the atom, as it has an internal electronic structure will also have the following free Hamiltonian $\hbar\omega\sigma^{\dagger}\sigma$. The basis for the Hilbert space just increased as it contains $|g,p\rangle$ or $|e,p\rangle$. On top of this the atom is always in contact with the electromagnetic environment. We also assume a laser drive, which after the Mollow transformation (presented in the first chapter) is included as a

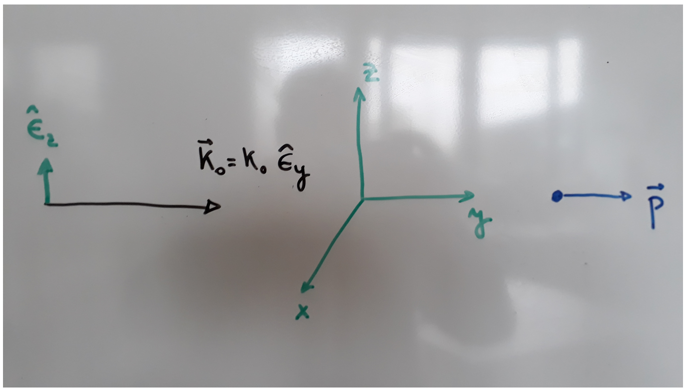

Figure 5.1: An atom is assumed to move in one (quantized) direction y and can be driven with z-polarized fields. The laser drive is in the positive y direction with momentum k_0. Photons can be spontaneously emitted in all directions meaning that the recoil kicks suffered by the atom can be in any direction. However, we only quantify the mechanical effect in the direction of interest.

semiclassical term. We fix the direction and magnitude of the laser drive as $k_0\hat{\epsilon}_y$, Rabi frequency Ω_ℓ and its frequency ω_ℓ. Let's write the total Hamiltonian:

$$H = \frac{\hat{p}^2}{2M} + \hbar\omega_0\sigma^\dagger\sigma + \sum_{\mathbf{k}} \hbar\omega_k a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \hbar\Omega_\ell\left(\sigma e^{i\omega_\ell t} e^{-ik_0\hat{y}} + \sigma^\dagger e^{-i\omega_\ell t} e^{ik_0\hat{y}}\right) \\ + \sum_{\mathbf{k}} \hbar g_{\mathbf{k}}\left(\sigma a_{\mathbf{k}}^\dagger e^{-ik_y\hat{y}} e^{-i(k_x x + k_z z)} + a_{\mathbf{k}}\sigma^\dagger e^{ik_y\hat{y}} e^{i(k_x x + k_z z)}\right) \tag{5.2}$$

Notice that as we only quantized the motion along the y-axis we write the other two coordinates without a hat. Also we have assumed that the electronic transition can be excited only by a z-polarized light field. The total Hilbert space now spans all the momentum states, the ground or excited for internal electronic states and all the states of the photon field with occupancies from 0 to infinity. However the interesting states are limited to initial states of the form: $|p_0\rangle \otimes |g\rangle \otimes |0\rangle$ or $|p_0\rangle \otimes |e\rangle \otimes |0\rangle$. We basically reproduce the derivation of fundamental light-matter scattering processes described in the first chapter but considering now the change in the particle's motion. First we take the atom in the ground state and compute the final possible states:

$$H|p_0\rangle \otimes |g\rangle \otimes |0\rangle = \frac{p_0^2}{2M}|p_0\rangle \otimes |g\rangle \otimes |0\rangle + \hbar\Omega_\ell e^{-i\omega_\ell t}|p_0 - \hbar k_0\rangle \otimes |e\rangle \otimes |0\rangle \tag{5.3}$$

We have just deduced that the stimulated absorption process is actually accompanied by a change of the state of the atom by the photon transferred momentum. Notice that, as expected, the vacuum states do not play a role in this process. However, when starting in the excited state we will have a different expression:

$$H|p_0\rangle \otimes |e\rangle \otimes |0\rangle = \left(\frac{p_0^2}{2M} + \hbar\omega_0\right)|p_0\rangle \otimes |g\rangle \otimes |0\rangle + \hbar\Omega_\ell e^{i\omega_\ell t}|p_0 + \hbar k_0\rangle \otimes |g\rangle \otimes |0\rangle \\ + \sum_{\mathbf{k}} \hbar g_{\mathbf{k}} e^{-ik_y\hat{y}} e^{-i(k_x x + k_z z)}|p_0 - \hbar k_y\rangle \otimes |g\rangle \otimes |0\rangle\,. \tag{5.4}$$

The above expression includes terms associated with the stimulated emission showing a change in the momentum by $\hbar k_0$ (opposite in sign to the change acquired with stimulated absorption) and changes in momentum with the projection on the y-axis of the momentum of the spontaneously emitted photon $\hbar k_y$. Let us now proceed by deriving a fully quantum expression for the force acting onto the atom by making use of the Heisenberg equations of motion:

$$\begin{aligned}\partial_t \hat{p} =& i\Omega_\ell \left(\sigma e^{i\omega_\ell t}[e^{-ik_0\hat{y}},\hat{p}] + \sigma^\dagger e^{-i\omega_\ell t}[e^{ik_0\hat{y}},\hat{p}]\right) \\ &+ \sum_{\mathbf{k}} i g_{\mathbf{k}} \left(\sigma a^\dagger_{\mathbf{k}}[e^{-ik_y\hat{y}},\hat{p}]e^{-i(k_x x+k_z z)} + a_{\mathbf{k}}\sigma^\dagger[e^{ik_y\hat{y}},\hat{p}]e^{i(k_x x+k_z z)}\right).\end{aligned} \tag{5.5}$$

We see that we need to evaluate commutators like $[e^{\pm ik\hat{y}},\hat{p}]$. This we can do by expanding the exponential

$$[e^{ik\hat{y}},\hat{p}] = \sum_{j=0}^{\infty}\frac{(ik)^j}{j!}[\hat{y}^j,\hat{p}] = 0 + \frac{(ik)^1}{1!}(i\hbar) + \frac{(ik)^2}{2!}2i\hbar\hat{y} + \frac{(ik)^3}{3!}3i\hbar\hat{y}^2 + ... = -\hbar k e^{ik\hat{y}}. \tag{5.6}$$

Above we have used the general rule that $[\hat{y}^j,\hat{p}] = [\hat{y}\hat{y}^{j-1},\hat{p}] = \hat{y}[\hat{y}^{j-1},\hat{p}] + [\hat{y},\hat{p}]\hat{y}^{j-1}$. For the sign change in the exponential we obtain a simple sign change in the momentum: $[e^{-ik\hat{y}},\hat{p}] = \hbar k e^{ik\hat{y}}$. Finally we can properly write the equation of motion.

$$\begin{aligned}\partial_t \hat{p} =& i\hbar k_0\Omega_\ell \left(\sigma e^{i\omega_\ell t}e^{-ik_0\hat{y}} - \sigma^\dagger e^{-i\omega_\ell t}e^{ik_0\hat{y}}\right) \\ &+ \sum_{\mathbf{k}} i\hbar k_y g_{\mathbf{k}} \left(\sigma a^\dagger_{\mathbf{k}}e^{-ik_y\hat{y}}e^{-i(k_x x+k_z z)} - a_{\mathbf{k}}\sigma^\dagger e^{ik_y\hat{y}}e^{i(k_x x+k_z z)}\right).\end{aligned} \tag{5.7}$$

We obtain two contributions, one from the laser drive and a second one from the vacuum field.

5.2 Doppler cooling

Any light emission or absorption event by an atom is associated with a transfer of momentum $\Delta p = \hbar k = \hbar\omega/c$. If absorption is followed by stimulated emission then the momentum change at absorption will cancel the momentum change at emission and no net momentum is transfered to the atom. If absorption is followed by spontaneous emission, as the spontaneously emitted atom, on average a net momentum change will be obtained. Assuming an atom initially at rest, the recoil energy can be computed from the acquired kinetic energy $(\Delta p)^2/2m$. One can then define a recoil frequency:

$$\hbar\omega_{rec} = \frac{(\Delta p)^2}{2m} = \frac{(\hbar k)^2}{2m}. \tag{5.8}$$

For atoms, the value is typically in the range of kHz. To derive the effect of a light field onto the motion of an atom we will first consider a semiclassical formulation where the atom position and momentum are classical parameters undergoing some trajectory described by $R(t)$ and $P(t)$. We will the write a Hamiltonian that has these variables as parameters (so it is a time dependent Hamiltonian):

$$H_{A+F} = H_A + H_V + H_{A-L} + H_{A-V}, \tag{5.9}$$

separated into free atom Hamiltonian and free vacuum field Hamiltonian,

$$H_A + H_V = \hbar\omega_{eg}\sigma^\dagger\sigma + \sum_k \hbar\omega_k a^\dagger_k a_k. \tag{5.10}$$

classical laser-atom interaction (already in the rotating wave approximation)

$$H_{A+L} = \hbar\left[\Omega(R)\sigma e^{i\omega_L t} + \Omega^*(R)\sigma^\dagger e^{-i\omega_L t}\right], \tag{5.11}$$

and quantum vacuum field-atom interaction (also already in the rotating wave approximation)

$$H_{A+V} = \hbar \sum_k (g_k a_k^\dagger \sigma e^{-ik\cdot R} + g_k^* a_k \sigma^\dagger e^{ik\cdot R}). \tag{5.12}$$

We can derive a semiclassical force operator:

$$F(R) = -\langle \nabla H \rangle = -\langle \nabla H_{A-L} \rangle - \langle \nabla H_{A-V} \rangle, \tag{5.13}$$

stemming from the effect of the laser onto the atom and of the vacuum onto the atom as well. The vacuum force is a zero average force as one can derive from applying the gradient: $\nabla e^{ik\cdot R} = k e^{ik\cdot R}$and summing over alll directions. Let us analyze the laser driving induced force which is

$$-\langle \nabla H_{A-L} \rangle = -\hbar \left[\nabla \Omega(R)\right] \langle \sigma \rangle e^{i\omega_L t} - \hbar \left[\nabla \Omega^*(R)\right] \langle \sigma^\dagger \rangle e^{-i\omega_L t}. \tag{5.14}$$

The averages that we have to compute are simply the density matrix elements ρ_{eg}, ρ_{ge} (coherences) in the fast rotating frame. We can reexpress

$$F(R) = -2\hbar \Re \left[\nabla \Omega(R) \rho_{eg} e^{i\omega_L t}\right]. \tag{5.15}$$

Let us assume that the atom is driven by a plane wave with $k = k\hat{z}$ propagating from left to right. The electric field assumed polarized in the $\hat{x}$ direction is a sum of positive and negative frequency terms: a plane wave with the electric field:

$$E(R,t) = E^{(+)}(R,t) + E^{(-)}(R,t) = E_0 e^{-i\omega_L t} e^{ik\cdot R} \hat{x} + E_0 e^{i\omega_L t} e^{-ik\cdot R} \hat{x} = \tag{5.16}$$

$$= 2E_0 \hat{x} \cos(k \cdot R - \omega_L t) = 2E_0 \cos(kz - \omega_L t). \tag{5.17}$$

We aim at computing the semiclassical force expression from the expression above with $\Omega(R)$ replaced by Ωe^{ikz}. We immediatily see that

$$F(z) = -2\hbar k \Omega \Re \left[\rho_{eg} e^{ikz} e^{i\omega_L t}\right] \hat{z} \tag{5.18}$$

5.2.1 Modified Bloch equations

We have previously derived a set of Bloch equations for static atoms. Now the driving Hamiltonian has a position dependence on the$R$ function:

$$H_{A+L} = \hbar \Omega \left[\sigma e^{-ikz} e^{i\omega_L t} + \sigma^\dagger e^{ikz} e^{-i\omega_L t}\right], \tag{5.19}$$

It is easy to compute the modified Bloch equations in the original frame (making no transformations yet)

$$\frac{d}{dt} \rho_{ee} = -\gamma \rho_{ee} - i\Omega \left[e^{-i\omega_L t} e^{ikz} \rho_{ge} - e^{i\omega_L t} e^{-ikz} \rho_{eg}\right], \tag{5.20}$$

$$\frac{d}{dt} \rho_{eg} = -\frac{\gamma}{2} \rho_{eg} - i\omega_{eg} \rho_{eg} + i\Omega e^{ikz} e^{-i\omega_L t} \left(\rho_{gg} - \rho_{ee}\right). \tag{5.21}$$

We see that in order to eliminate the time dependence we will have to transform to a slow moving frame by removing the laser frequency: $\rho_{eg} = \tilde{\rho}_{eg} e^{-i\omega_L t}$. With$\Delta = \omega_{eg} - \omega_L$ we have

$$\frac{d}{dt} \tilde{\rho}_{ee} = -\gamma \tilde{\rho}_{ee} - i\Omega \left[e^{ikz} \tilde{\rho}_{ge} - e^{-ikz} \tilde{\rho}_{eg}\right], \tag{5.22}$$

$$\frac{d}{dt}\tilde{\rho}_{eg} = -\frac{\gamma}{2}\tilde{\rho}_{eg} - i\Delta\tilde{\rho}_{eg} + i\Omega e^{-ikz}\left(\tilde{\rho}_{gg} - \tilde{\rho}_{ee}\right). \quad (5.23)$$

We will now assume that the atom is instantaneously performing a linear motion with constant velocity such that $z = vt$. The terms $e^{ikz} = e^{ikvt}$ give time dependence which we again can eliminate via the usual transformation $\tilde{\rho}_{eg} = \bar{\rho}_{eg}e^{ikvt}$ so that $\rho_{eg} = \bar{\rho}_{eg}e^{-ikvt}e^{-i\omega_L t}$ or inversely $\bar{\rho}_{eg} = \rho_{eg}e^{ikvt}e^{i\omega_L t}$

$$\frac{d}{dt}\bar{\rho}_{ee} = -\gamma\bar{\rho}_{ee} - i\Omega\left[\bar{\rho}_{ge} - \bar{\rho}_{eg}\right], \quad (5.24)$$

$$\frac{d}{dt}\bar{\rho}_{eg} = -\frac{\gamma}{2}\bar{\rho}_{eg} - i\left(\Delta - kv\right)\bar{\rho}_{eg} + i\Omega e^{-ikz}\left(\bar{\rho}_{gg} - \bar{\rho}_{ee}\right). \quad (5.25)$$

If we look at the new detuning appearing above $\Delta - kv = \omega_{eg} - \omega_L - kv$ we notice that this simply shows a Doppler shifting effect. In other words, when the atom is moving away from the laser, it looks as if the laser has a lower frequency by the quantity kv.

5.2.2 Steady state

Let us look at the steady state of the system. We set the derivatives to zero and remember that $\bar{\rho}_{gg} + \bar{\rho}_{ee} = 1$. We then get (see Lecture 3)

$$\bar{\rho}_{eg} = -\frac{\gamma\Omega/2}{\gamma^2/4 + (\Delta - kv)^2 + 2\Omega^2} - i\frac{\Delta\Omega}{\gamma^2/4 + (\Delta - kv)^2 + 2\Omega^2}. \quad (5.26)$$

We only need the real part of this expression and obtain the optically induced force as:

$$F(z) = \hbar k\frac{\gamma\Omega^2}{\gamma^2/4 + (\Delta - kv)^2 + 2\Omega^2}\hat{z}. \quad (5.27)$$

Notice that for an atom at rest the force is positive and has a maximum value of $2\hbar k/\gamma$ (in the limit of high pumping power). The origin of the force is in absorption-spontaneous emission cycles where a momentum $\hbar k$ can be imparted to the atom and not compensated by the stimulated emission $-\hbar k$ imparted momentum. In the limit $\Omega \ll \gamma$ and $kv \ll \Delta$we can expand the force:

$$F(z) = \hbar k\frac{\gamma\Omega^2}{\gamma^2/4 + (\Delta - kv)^2}\hat{z} \simeq \left\{\hbar k\frac{\gamma\Omega^2}{\gamma^2/4 + \Delta^2} + \hbar k\frac{\gamma\Omega^2 k\Delta}{\left[\gamma^2/4 + \Delta^2\right]^2}v\right\}\hat{z} = (F_0 + \beta v)\hat{z}. \quad (5.28)$$

The first one is a constant force while the second gives rise to friction as it is proportional to the velocity and it changes sign as the detuning is changed. Notice that changing the direction of the laser propagation corresponds to a sign flip of k resulting in:

$$F(z) = (-F_0 + \beta v)\hat{z}. \quad (5.29)$$

5.3 Force experienced in standing waves

We now assume two counterpropagating waves making up a standing wave. We can then write at the position of the atom (for a sum of waves coming from the left and from the right):

$$E(R,t) = E_L^{(+)}(R,t) + E_R^{(+)}(R,t) + E_L^{(-)}(R,t) + E_R^{(-)}(R,t) = \quad (5.30)$$

$$= E_0 e^{-i\omega_L t} e^{ik\cdot R} + E_0 e^{-i\omega_L t} e^{-ik\cdot R} + E_0 e^{i\omega_L t} e^{ik\cdot R} + E_0 e^{i\omega_L t} e^{-ik\cdot R} = 2E_0 \cos(k\cdot R) e^{-i\omega_L t} + 2E_0 \cos(k\cdot R) e^{i\omega_L t} = \tag{5.31}$$

$$= 4E_0 \cos(k\cdot R)\sin(\omega_L t). \tag{5.32}$$

As above we restrict to the z direction and replace $\Omega(R)$ replaced by $\Omega\cos(kz)$. We immediatily see that

$$F(z) = -2\hbar k\Omega\Re\left[\rho_{eg}\sin(kz)e^{i\omega_L t}\right]\hat{z}. \tag{5.33}$$

One has therefore to rederive the Bloch equations and their solution for the two waves. However, as intuition also dictates, in the limit of low intensity the interference between the two waves vanishes and the total force is simply the sum of the forces of two counterpropagating waves:

$$F_{standing}(z) = (F_0 + \beta v)\hat{z} + (-F_0 + \beta v)\hat{z} = 2\beta v\hat{z}. \tag{5.34}$$

The net effect is that an atom can be cooled with a damping rate

$$\beta = \hbar k^2 \frac{\gamma\Omega^2\Delta}{[\gamma^2/4+\Delta^2]^2}. \tag{5.35}$$

The maximum cooling rate (for negative detunings so that the rate is negative) occurs at $\Delta = -\gamma/(2\sqrt{3})$ and equals:

$$\beta_{max} = -\hbar k^2 \frac{6\sqrt{3}\Omega^2}{\gamma^2}. \tag{5.36}$$

We can write an equation of motion for the momentum:

$$\dot{p} = -\frac{2\beta}{m}p. \tag{5.37}$$

The solution is an exponential decay of the momentum to zero:

$$p(t) = p(0)e^{-\frac{2\beta}{m}t}. \tag{5.38}$$

The cooling rate is:

$$\frac{2\beta}{m} = \frac{\hbar k^2}{2m}\frac{4\gamma\Omega^2\Delta}{[\gamma^2/4+\Delta^2]^2} = \omega_{rec}\frac{4\gamma\Omega^2\Delta}{[\gamma^2/4+\Delta^2]^2}. \tag{5.39}$$

Remembering that we asked for the pump to be weak with respect to the decay rate, even at optimal detuning, the cooling rate is limited by the recoil frequency, thus in the range of kHz.

5.3.1 Final temperature

To obtain the final temperature of the cooled atoms one has to model the random kicks obtained from spontaneous emission. They lead to a diffusion process which limits the final achievable temperature or in other words leaves the atom in a state with momentum uncertainty characterized by $(\Delta p)^2/(2m) = k_B T/2$. For a simple justification of the origin of the diffusion process let us go back to the starting point of our derivation and start again with the full Hamiltonian

$$H_{A+F} = H_A + H_V + H_{A-L} + H_{A-V}, \tag{5.40}$$

but this time we add the kinetic energy operator in the z direction

$$H_A + H_V = \frac{\hat{p}^2}{2m} + \hbar\omega_{eg}\sigma^\dagger\sigma + \sum_k \hbar\omega_k a_k^\dagger a_k. \tag{5.41}$$

We have assumed a single direction of interest z with a quantized momentum p. Now we don't take the classical limit but directly derive Heisenberg equations of motion for the momentum operator:

$$\dot{\hat{p}} = \frac{i}{\hbar}[H,\hat{p}] = \frac{i}{\hbar}[H_{A-L},\hat{p}] + \frac{i}{\hbar}[H_{A-V},\hat{p}]. \tag{5.42}$$

The first term will lead to commutators like

$$\frac{i}{\hbar}[H_{A-L},\hat{p}] = i\,[\Omega(z),\hat{p}]\,\sigma e^{i\omega_L t} + i\,[\Omega^*(z),\hat{p}]\,\sigma^\dagger e^{-i\omega_L t}. \tag{5.43}$$

Let us assume that the driving is with a plane wave. Then, using

$$\left[e^{\pm ik_L z},\hat{p}\right] = \pm\hbar k e^{\pm ik_L z}, \tag{5.44}$$

the first component of the force will lead to:

$$\frac{i}{\hbar}[H_{A-L},p] = -\hbar k_L \Omega\sigma e^{-ik_L z}e^{i\omega_L t} - \hbar k_L \sigma^\dagger e^{+ik_L z}e^{-i\omega_L t}. \tag{5.45}$$

which is quantum version of the classically derived force in the first section. The last term and quantum vacuum field-atom interaction (also already in the rotating wave approximation)

$$\frac{i}{\hbar}[H_{A-V},p] = i\sum_k (g_k a_k^\dagger \sigma\left[e^{-ik_z z},p\right] + g_k^* a_k \sigma^\dagger\left[e^{ik_z z},p\right]) = -\sum_k \hbar k_z (g_k a_k^\dagger \sigma e^{-ik_z z} + g_k^* a_k \sigma^\dagger e^{ik_z z}). \tag{5.46}$$

The final equation is a Langevin type which can be written as:

$$\dot{p} = -f(p) + \zeta(t), \tag{5.47}$$

with

$$p_{in} = -\sum_k \hbar k_z (g_k a_k^\dagger \sigma e^{-ik_z z} + g_k^* a_k \sigma^\dagger e^{ik_z z}), \tag{5.48}$$

a zero-average stochastic operator with correlations:

$$\langle\zeta(t)\zeta(t')\rangle = \langle\sum_k \hbar k_z (g_k a_k^\dagger \sigma e^{-ik_z z} + g_k^* a_k \sigma^\dagger e^{ik_z z})\sum_k \hbar k'_z (g_{k'} a_{k'}^\dagger \sigma(t') e^{-ik'_z z} + g_k^* a_k \sigma^\dagger(t') e^{ik'_z z})\rangle = \tag{5.49}$$

$$= \sum_k \hbar^2 k_z^2 |g_k|^2 \langle\sigma^\dagger(t)\sigma(t')\rangle. \tag{5.50}$$

One has to perform the sum over all directions of the emitted photon and then obtain a result:

$$\langle\zeta(t)\zeta(t')\rangle = D\delta(t-t'). \tag{5.51}$$

In general D depends on a few factors especially on the excited state population. However, the expression can be greatly simplified in the optimal cooling regime:

$$D = \frac{\hbar\omega_{rec}\gamma}{2}. \tag{5.52}$$

We will now use the previously derived result and write an effective equation for

$$\langle \zeta(t)\zeta(t')\rangle = D\delta(t-t'). \tag{5.53}$$

The equation of motion for the momentum has now damping and diffusion

$$\dot{p}(t) = -\frac{2\beta}{m}p(t) + \zeta(t), \tag{5.54}$$

and we will consider in the following the scenario where the damping is optimal such that $-\frac{2\beta}{m} \simeq \omega_{rec}$.The formal solution is

$$p(t) = p(0)e^{-\omega_{rec}t} + \int_0^t dt' e^{-\omega_{rec}(t-t')}\zeta(t'). \tag{5.55}$$

We will only look in the long time limit $t \gg \omega_{rec}^{-1}$ where $\langle p(t)\rangle = 0$ and

$$\langle p^2(t)\rangle = \int_0^t dt' \int_0^t dt'' e^{-\omega_{rec}(t-t')}e^{-\omega_{rec}(t-t'')}\langle \zeta(t')\zeta(t'')\rangle = \tag{5.56}$$

$$= D\int_0^t dt' e^{-2\omega_{rec}(t-t')} = \left[De^{-2\omega_{rec}t}\right]\frac{1-e^{2\omega_{rec}t}}{2\omega_{rec}} \simeq \frac{D}{2\omega_{rec}}. \tag{5.57}$$

In consequence the final variance of kinetic energy will be

$$\langle \frac{p^2}{2m}\rangle = \frac{D}{2\omega_{rec}} = \frac{\hbar\gamma}{4}. \tag{5.58}$$

This can be expressed as an effective temperature by assuming that the equipartition theorem applies and the system is in thermal equilibrium $\langle \frac{p^2}{2m}\rangle = k_B T_{eff}/2$ such that

$$T_{eff} = \frac{\hbar\gamma}{2k_B}. \tag{5.59}$$

5.4 Semiclassical gradient forces (trapping in focused beams)

Let us consider now a focused light beam propagating into the z direction and with a strong variation along the transverse (x,y) coordinates. We can compute the force in the transverse direction as:

$$F(x,y) = -2\hbar\Re\left[\nabla_\perp\Omega(x,y)\rho_{eg}e^{i\omega_L t}\right]. \tag{5.60}$$

We can use the result computed above

$$\bar{\rho}_{eg} = -\frac{\gamma\Omega(x,y)/2}{\gamma^2/4 + (\Delta - kv)^2 + 2\Omega(x,y)^2} - i\frac{\Delta\Omega(x,y)}{\gamma^2/4 + (\Delta - kv)^2 + 2\Omega(x,y)^2}, \tag{5.61}$$

in the limit $\Omega(x,y) \ll \gamma$ and assuming small velocities, we can derive

$$F(x,y) = -\frac{2\hbar\Delta}{\gamma^2/4+\Delta^2}\nabla_\perp\Omega^2(x,y). \tag{5.62}$$

Let's see what happens for a Gaussian beam with a minimum at $x = 0, y = 0$:

$$\Omega(x,y) = \Omega_0 e^{-(x^2+y^2)/w^2}. \tag{5.63}$$

The gradient gives:

$$\nabla_\perp\Omega^2(x,y) = \frac{-2}{w^2}\Omega_0^2 e^{-(x^2+y^2)/w^2}(x\hat{x}+y\hat{y}) = \frac{-2}{w^2}\Omega^2(x,y) \tag{5.64}$$

so that the force is

$$F(x,y) = \frac{2\hbar\Delta}{\gamma^2/4+\Delta^2}\frac{-2}{w^2}\Omega^2(x,y)(x\hat{x}+y\hat{y}). \tag{5.65}$$

By adjusting the sign of the detuning, the force can push the incoming atoms towards the center of the focused beam thus providing a means for trapping them (assuming some loss of energy allowing for trapping occurse at the same time).

5.5 Polarization gradient cooling

We have see above that the limit in Doppler cooling is the spontaneous emission rate. This is typically in the MHz regime. One can however, go below that, to the recoil limit by assuming that the two counterpropagating waves have oposite polarizations and that the atom is a three level lambda-system where each transition can only be driven by a given polarization.

5.6 Other cooling techniques

5.7 Exercises

6. Cavity optomechanics

We start by deriving the relations between the noise spectrum and the damping rate and occupancy number for a good macroscopic mechanical oscillator (large quality factor) in the presence of a quantized thermal bath. We then describe the coupling of a cavity mode to the vibrations of a mirror or membrane in a standard optomechanical scenario. Classical considerations based on the transfer matrix approach are used to derive the modification of the cavity resonances in the presence of small resonator displacements. The full quantum model is then analyzed in a quantum Langevin equations picture and fundamental concepts such as optomechanical cooling in the resolved sideband limit derived and discussed.

6.1 A macroscopic quantum oscillator in thermal equilibrium

Assume a macroscopic quantum oscillator of frequency ω_m in thermal equilibrium with the environment at some temperature T. We can write a set of Langevin equations to describe the dynamics of the oscillator (in terms of dimensionless quadratures):

$$\dot{q} = \omega_m p, \qquad \dot{p} = -\gamma_m p - \omega_m q + \zeta(t). \tag{6.1}$$

Remember from exercise 1 that the dimensionless quadratures are defined by the following renormalizations $p_x = p\sqrt{m\hbar\omega} = pp_{\text{zpm}}$ and $x = q\sqrt{\frac{\hbar}{m\omega}} = qx_{\text{zpm.}}$ The zero point motion x_{zpm} simply gives the spatial extent of the wavepacket in the ground state (around 10^{-15}m for micro-to nanogram oscillators), while the zero-point momentum is $p_{\text{zpm}} = \hbar/x_{\text{zpm}}$. The model includes a noise terms responsible for thermalization and damping at rate γ_m. The correlations (in the time domain) are:

$$\langle\zeta(t)\zeta(t')\rangle = \frac{\gamma_m}{\omega_m}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}\omega\left[\coth\frac{\hbar\omega}{2k_BT}+1\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}S_{th}(\omega), \tag{6.2}$$

where the thermal spectrum of the stochastic force has been denoted by $S_{th}(\omega)$. We will assume in the following two things: 1) the oscillator is a very good one such that $\gamma_m \ll \omega_m$ and 2) $\hbar\omega_m \ll k_BT$

such that $x = \hbar\omega_m/(2k_BT) \ll 1$. One can then make the following approximation:

$$\coth\frac{\hbar\omega}{2k_BT} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1} \simeq \frac{2+2x}{2x} = 1+\frac{1}{x}. \tag{6.3}$$

Now notice that the average occupancy is

$$\bar{n} = \frac{1}{e^{2x}-1} \simeq \frac{1}{2x}, \tag{6.4}$$

such that we can approximate the thermal noise spectrum at $\pm\omega_m$ as:

$$S_{th}(\omega_m) = \gamma_m\frac{\omega_m}{\omega_m}\left[\coth\frac{\hbar\omega_m}{2k_BT}+1\right] = \gamma_m\frac{\omega_m}{\omega_m}\left[1+\frac{1}{2x}+1\right] = 2\gamma_m(\bar{n}+1), \tag{6.5a}$$

$$S_{th}(-\omega_m) = \gamma_m\frac{-\omega_m}{\omega_m}\left[\coth\frac{-\hbar\omega_m}{2k_BT}+1\right] = -\gamma_m\left[-1-\frac{1}{2x}+1\right] = 2\gamma_m\bar{n}. \tag{6.5b}$$

Notice that both the decay rate and the thermal occupancy can be derived from the the thermal spectrum: First observation is that the spectrum difference at sideband frequencies indicates the decay rate:

$$\gamma_m = \frac{S_{th}(+\omega_m)-S_{th}(-\omega_m)}{2} \qquad \text{and} \qquad \bar{n} = \frac{S_{th}(-\omega_m)}{2\gamma_m}. \tag{6.6}$$

Now let's see what this means. We use the Fourier domain transformations we have introduced in Lecture 6.

$$q(\omega) = \mathscr{F}[q(t)] = \frac{1}{\sqrt{2\pi}}\int dt q(t)e^{+i\omega t}, \qquad q(t) = \frac{1}{\sqrt{2\pi}}\int d\omega q(\omega)e^{-i\omega t}. \tag{6.7}$$

First we will analyze the properties of the noise in the Fourier domain:

$$\langle\zeta(\omega)\zeta(\omega')\rangle = \frac{1}{(2\pi)^2}\int dt\int dt'\int_{-\infty}^{\infty} d\omega'' e^{-i\omega''(t-t')}S_{th}(\omega'')e^{+i\omega t}e^{+i\omega' t'} = S_{th}(\omega)\delta(\omega+\omega'). \tag{6.8}$$

We will use the property that the Fourier transform of a derivative is $\mathscr{F}[\dot{q}(t)] = -i\omega q(\omega)$ and correspondingly transform the equations of motion:

$$-i\omega q(\omega) = \omega_m p(\omega), \tag{6.9a}$$

$$-i\omega p(\omega) = -\gamma_m p(\omega) - \omega_m q(\omega) + \zeta(\omega). \tag{6.9b}$$

We can the write an effective equation for the position quadrature in terms of the mechanical susceptibility

$$q(\omega) = \varepsilon(\omega)\zeta(\omega), \qquad \text{with} \qquad \varepsilon(\omega) = \frac{\omega_m}{\omega_m^2-\omega^2-i\gamma_m\omega}. \tag{6.10}$$

One can now compute the variance of the quadrature in steady state:

$$\langle q^2(t)\rangle = \frac{1}{2\pi}\int d\omega\int d\omega'\langle q(\omega)q(\omega')\rangle e^{-i\omega t}e^{-i\omega' t} = \frac{1}{2\pi}\int d\omega|\varepsilon(\omega)|^2 S_{th}(\omega). \tag{6.11}$$

Notice that the susceptibility function is sharply peaked at $\pm\omega_m$and also that $\int_0^\infty d\omega|\varepsilon(\omega)|^2 = \pi/(2\gamma_m)$. This allows us to simply estimate the slowly varying function of the spectrum at the $\pm\omega_m$ leading to

$$\langle q^2(t)\rangle = \frac{1}{4\gamma_m}\left(S_{th}(-\omega_m) + S_{th}(+\omega_m)\right) = \bar{n} + \frac{1}{2}. \tag{6.12}$$

Similarly one has

$$\langle p^2(t)\rangle = \bar{n} + \frac{1}{2}, \tag{6.13}$$

fulfilling the equipartition theorem. Of course this is what we would have guessed from the beginning imposing that in steady state, under thermal equilibrium, the equipartitition theorem holds and one would have $\langle q\rangle = \langle p\rangle = 0$ and $1/2\left[\langle q^2(t)\rangle + \langle p^2(t)\rangle\right] = \bar{n} + 1/2$.

6.2 Transfer matrix approach to optomechanics

A given resonance of a cavity of length ℓ is given by

$$\omega_0 = \frac{2\pi c}{2\ell/m} = \frac{\pi c}{\ell}m, \tag{6.14}$$

and its variations with respect to small cavity changes would lead to

$$\omega(\delta x) = \frac{\pi c}{\ell + \delta x}m \simeq \frac{\pi c}{\ell}m(1 - \frac{\delta x}{\ell}) = \omega_0(1 - \frac{\delta x}{\ell}). \tag{6.15}$$

A direct naive Hamiltonian for the optomechanical interaction can be directly derived as:

$$H = \hbar\omega_0(1 - \frac{\delta x}{\ell})a^\dagger a + \frac{\hbar\omega_m}{2}(q^2 + p^2). \tag{6.16}$$

Writing $\delta x = x_{\text{zpm}}q$ the effective optomechanical coupling is

$$g_{\text{OM}} = \hbar\frac{\omega_0 x_{\text{zpm}}}{\ell}. \tag{6.17}$$

The full radiation pressure optomechanical Hamiltonian becomes then:

$$H = \hbar\omega_0 a^\dagger a + \frac{1}{2}(q^2 + p^2) + \hbar g_{\text{OM}} a^\dagger a q. \tag{6.18}$$

6.2.1 Transfer matrix for end-mirror

The naive approach can be supplemented by a more rigourous approach based on cascaded transfer matrix multiplications

$$M(k,\delta x) = \begin{bmatrix} 1+i\zeta & i\zeta \\ -i\zeta & 1-i\zeta \end{bmatrix} \begin{bmatrix} e^{ik(\ell+\delta x)} & 0 \\ 0 & e^{-ik(\ell+\delta x)} \end{bmatrix} \begin{bmatrix} 1+i\zeta & i\zeta \\ -i\zeta & 1-i\zeta \end{bmatrix}. \tag{6.19}$$

Looking at the matrix element giving the amplitude transmission coefficient we get:

$$t(k,\delta x) = \frac{1}{\zeta^2 e^{ik(\ell+\delta x)} + (1-i\zeta)^2 e^{-ik(\ell+\delta x)}}. \tag{6.20}$$

Imposing the resonance condition for the static case with $\delta x = 0$ we find the system resonances where the denominator is exactly unity. We can then write

$$\zeta^2 e^{ik_0\ell} + (1-i\zeta)^2 e^{-ik_0\ell} = 1. \tag{6.21}$$

Neglecting terms involving the product of small variations $\delta k \delta x$, we can expand

$$e^{ik(\ell+\delta x)} = e^{ik_0\ell} e^{i\ell\delta k} e^{ik_0\delta x} \simeq e^{ik_0\ell}(1+i\ell\delta k)(1+ik_0\delta x) \simeq e^{ik_0\ell}\left[1+i(\ell\delta k + k_0\delta x)\right]. \tag{6.22}$$

Lets rewrite the denominator using this expansion

$$t^{-1}(k,\delta x) = \zeta^2 e^{ik_0\ell} + (1-i\zeta)^2 e^{-ik_0\ell} + i\left[\zeta^2 - (1-i\zeta)^2\right](\ell\delta k + k_0\delta x). \tag{6.23}$$

The first terms gives unity and we ask for the second term to vanish in order to reach the modified resonance. This gives a very simple result:

$$\frac{\delta k}{\delta x} = -\frac{k_0}{\ell}, \tag{6.24}$$

which one immediately sees to coincide with the prediction of the naive approach previously used.

6.2.2 Transfer matrix for membrane-in-the-middle approach

The naive approach however does not help us for deriving the proper optomechanical interaction of a membrane-in-the-middle setup. We have to properly write the transfer matrices and compute the resonance changes with respect to small variations of the membrane's position around the equilibrium. Let us write the whole matrix as:

$$M(k,x) = \begin{bmatrix} 1+i\zeta & i\zeta \\ -i\zeta & 1-i\zeta \end{bmatrix} \begin{bmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_1} \end{bmatrix} \begin{bmatrix} 1+i\zeta_m & i\zeta_m \\ -i\zeta_m & 1-i\zeta_m \end{bmatrix} \begin{bmatrix} e^{i\phi_2} & 0 \\ 0 & e^{-i\phi_2} \end{bmatrix} \begin{bmatrix} 1+i\zeta & i\zeta \\ -i\zeta & 1-i\zeta \end{bmatrix}. \tag{6.25}$$

We have assumed the membrane to have a susceptibility ζ_m and the two phases accumulated during the evolution of the field between the left mirror and the membrane and the membrane to the right mirror are $\phi_{1,2}$such that $\phi_1 + \phi_2 = k\ell$ and $\phi_1 - \phi_2 = 2kx$. Multiplying we obtain:

$$t(k,x) = \frac{1}{(1+i\zeta_m)\zeta^2 e^{ik\ell} + 2\zeta\zeta_m(1-i\zeta)\cos(2kx) + (1-i\zeta_m)(1-i\zeta)^2 e^{-ik\ell}}. \tag{6.26}$$

The resonance depend on the positioning of the membrane inside the cavity. Let us assume a fixed position and variations around it $x_0 + \delta x$ leading to changes in the resonance $k_0 + \delta k$. Rewriting the denominator

$$\begin{aligned} t^{-1}(\delta k, \delta x) = {} & (1+i\zeta_m)\zeta^2 e^{ik_0\ell} + 2\zeta\zeta_m(1-i\zeta)\cos(2k_0x_0) + (1-i\zeta_m)(1-i\zeta)^2 e^{-ik_0\ell} \\ & + e^{ik_0\ell}(1+i\zeta_m)\zeta^2(i\ell\delta k) + 2\zeta\zeta_m(1-i\zeta)(2\sin k_0x_0)(x_0\delta k + k_0\delta x) \\ & + e^{-ik_0\ell}(1-i\zeta_m)(1-i\zeta)^2(-i\ell\delta k). \end{aligned} \tag{6.27}$$

We now set the first part (the unperturbed resonance condition) to unity and ask for the second part to vanish.

$$\left[(1+i\zeta_m)\zeta^2 e^{ik_0\ell} - (1-i\zeta_m)(1-i\zeta)^2 e^{-ik_0\ell}\right] i\ell\delta k + \tag{6.28}$$

$$2\zeta\zeta_m(1-i\zeta)(2\sin k_0x_0)(x_0\delta k + k_0\delta x) = 0. \tag{6.29}$$

Working it out a bit we get

$$\frac{\delta k}{\delta x} = \frac{2\zeta_m k_0 \sin 2k_0x_0}{\ell(\cos k_0\ell - \zeta_m \sin k_0\ell) - 2x_0\zeta_m \sin 2k_0x_0}. \tag{6.30}$$

6.3 Quantum cavity optomechanics

Having derived a model for the optomechanical interaction, let us see the effect of a driven cavity onto the mechanical resonator. We start with the Hamiltonian

$$H = \hbar\omega_0 a^\dagger a + \hbar\frac{\omega_m}{2}(q^2+p^2) - \hbar g_{\mathrm{OM}} a^\dagger a q - i\hbar\eta(ae^{i\omega_L t} - a^\dagger e^{-i\omega_L t}), \tag{6.31}$$

to which we add dissipation as a Lindblad term at rate κ for the optical mode and via the correlations described above for the mechanical resonator. We will work the dynamics in the Langevin equations formalism. We get the following equations of motion including noise in the momentum and the cavity field operator:

$$\dot{q} = \omega_m p, \tag{6.32a}$$
$$\dot{p} = -\gamma_m p - \omega_m q + g_{\mathrm{OM}} a^\dagger a + \zeta(t). \tag{6.32b}$$
$$\dot{a} = -(\kappa + i\Delta_0)a + ig_{\mathrm{OM}} aq + \eta + \sqrt{2\kappa}a_{\mathrm{in}}, \tag{6.32c}$$

where $\Delta_0 = \omega_0 - \omega_L$.

Steady state solution

We can first solve the steady state problem by expanding $a = \alpha + \delta a$, $q = q_s + \delta q$ and $p = p_s + \delta p$. From the steady state equations (of course we took an average that canceled all the zero-average noises and fluctuation terms)

$$0 = \omega_m p_s, \tag{6.33a}$$
$$0 = -\gamma_m p_s - \omega_m q_s + g_{\mathrm{OM}}|\alpha|^2, \tag{6.33b}$$
$$0 = -(\kappa + i\Delta_0)\alpha + ig_{\mathrm{OM}}\alpha q_s + \eta, \tag{6.33c}$$

we derive

$$p_s = 0, \tag{6.34a}$$
$$q_s = \frac{g_{\mathrm{OM}}}{\omega_m}|\alpha|^2, \tag{6.34b}$$
$$\alpha\left[\kappa + i\left(\Delta_0 - \frac{g_{\mathrm{OM}}^2}{\omega_m}|\alpha|^2\right)\right] = \eta. \tag{6.34c}$$

The last equation can have more than one solution. It gives rise to bistability. Let us assume a steady state solution exists with α and $q_s$non-zero and we compactly write the detuning

$$\Delta = \Delta_0 - \frac{g_{\mathrm{OM}}^2}{\omega_m}|\alpha|^2. \tag{6.35}$$

Linearized Langevin equations (for fluctuation operators)

We can now write equations for the fluctuations

$$\dot{\delta q} = \omega_m \delta p, \tag{6.36a}$$
$$\dot{\delta p} = -\gamma_m \delta p - \omega_m \delta q + G(\delta a^\dagger + \delta a) + \zeta(t). \tag{6.36b}$$
$$\dot{\delta a} = -(\kappa + i\Delta)\delta a + iG\delta q + \sqrt{2\kappa}a_{\mathrm{in}}, \tag{6.36c}$$
$$\dot{\delta a}^\dagger = -(\kappa - i\Delta)\delta a^\dagger - iG\delta q + \sqrt{2\kappa}a_{\mathrm{in}}^\dagger. \tag{6.36d}$$

One way to proceed with solving the set of equations is to transform them into the Fourier domain:

$$-i\omega\delta q = \omega_m \delta p, \tag{6.37a}$$

$$-i\omega\delta p = -\gamma_m \delta p - \omega_m \delta q + G(\delta a^\dagger + \delta a) + \zeta(\omega), \tag{6.37b}$$

$$-i\omega\delta a = -(\kappa + i\Delta)\delta a + iG\delta q + \sqrt{2\kappa} a_{\text{in}}, \tag{6.37c}$$

$$-i\omega\delta a^\dagger = -(\kappa - i\Delta)\delta a^\dagger - iG\delta q + \sqrt{2\kappa} a_{\text{in}}^\dagger. \tag{6.37d}$$

More compactly, we have a modified version of the Fourier domain response of the mechanical position operator:

$$\delta q(\omega) = \varepsilon_m(\omega)\left[G(\delta a^\dagger + \delta a) + \zeta(\omega)\right], \tag{6.38}$$

with the mechanical susceptibility

$$\varepsilon_m(\omega) = \frac{\omega_m}{\omega_m^2 - \omega^2 - i\gamma_m \omega}. \tag{6.39}$$

One should understand the equation above as a competition between two noise terms: i) from the thermal environment and ii) via the cavity mode. Expressing the cavity operators:

$$[(\kappa + i\Delta) - i\omega]\,\delta a = iG\delta q + \sqrt{2\kappa} a_{\text{in}}, \tag{6.40a}$$

$$[(\kappa - i\Delta) - i\omega]\,\delta a^\dagger = -iG\delta q + \sqrt{2\kappa} a_{\text{in}}^\dagger, \tag{6.40b}$$

and with notation:

$$\varepsilon_f(\omega) = \frac{1}{i(\Delta - \omega) + \kappa}, \tag{6.41}$$

we can write in simplified fashion:

$$\delta a(\omega) = \varepsilon_f(\omega)\left[iG\delta q + \sqrt{2\kappa} a_{\text{in}}\right], \tag{6.42a}$$

$$\delta a(\omega) = \varepsilon_f^*(-\omega)\left[-iG\delta q + \sqrt{2\kappa} a_{\text{in}}^\dagger\right]. \tag{6.42b}$$

Plugging it back in the equation for the mechanical position:

$$\left\{\varepsilon_m^{-1}(\omega) - iG^2\left[\varepsilon_f(\omega) - \varepsilon_f^*(-\omega)\right]\right\}\delta q = \sqrt{2\kappa} G\left[\varepsilon_f(\omega) a_{\text{in}} + \varepsilon^*(-\omega) a_{\text{in}}^\dagger\right] + \zeta(\omega). \tag{6.43}$$

We have an effective modified susceptibility and an effective input noise. Let us analyze the susceptibility first:

$$\begin{aligned}
\bar{\varepsilon}_m^{-1} &= \varepsilon_m^{-1}(\omega) - iG^2\left[\varepsilon_f(\omega) - \varepsilon_f^*(-\omega)\right] \\
&= \frac{1}{\omega_m}\left[\omega_m^2 - \omega^2 - i\gamma_m\omega - iG^2\omega_m\left[\frac{1}{i(\Delta-\omega)+\kappa} - \frac{1}{-i(\Delta+\omega)+\kappa}\right]\right] \\
&= \frac{1}{\omega_m}\left[\omega_m^2 - \omega^2 - i\gamma_m\omega - iG^2\omega_m\left[\frac{i(\Delta-\omega)+\kappa}{(\Delta-\omega)^2+\kappa^2} - \frac{-i(\Delta+\omega)+\kappa}{(\Delta+\omega)^2+\kappa^2}\right]\right] \\
&= \frac{1}{\omega_m}\left[\omega_m^2 - \omega^2 - i\gamma_m\omega - iG^2\kappa\omega_m\left[\frac{1}{(\Delta-\omega)^2+\kappa^2} - \frac{1}{(\Delta+\omega)^2+\kappa^2}\right]\right] + \\
&\quad + G^2\omega_m\left[\frac{(\Delta-\omega)}{(\Delta-\omega)^2+\kappa^2} + \frac{(\Delta+\omega)}{(\Delta+\omega)^2+\kappa^2}\right].
\end{aligned} \tag{6.44}$$

The imaginary term gives a renormalization of the damping rate. Evaluating at the mechanical frequency we get:

$$\Gamma_{\text{opt}} = G^2\kappa\left[\frac{1}{(\Delta-\omega_m)^2+\kappa^2} - \frac{1}{(\Delta+\omega_m)^2+\kappa^2}\right]. \tag{6.45}$$

The real term is a renormalization of the mechanical frequency (optical spring effect) and it's typically small unless the oscillator is slow.

Perturbative approach

Taking the approach sketched in the first section, we can simply find all the properties of the oscillator in steady state by analyzing the correlations of the noise. Let us denote by

$$F_{\rm opt}(\omega) = \sqrt{2\kappa}G\left[\varepsilon_f(\omega)a_{\rm in} + \varepsilon^*(-\omega)a^\dagger_{\rm in}\right], \tag{6.46}$$

the optical zero average Langevin force stemming from the optical field action onto the resonator. The correlations are straightforward to compute

$$\langle F_{\rm opt}(\omega)F_{\rm opt}(\omega')\rangle = 2\kappa G^2\varepsilon_f(\omega)\varepsilon^*(-\omega')\langle a_{\rm in}(\omega)a^\dagger_{\rm in}(\omega')\rangle = 2\kappa G^2|\varepsilon_f(\omega)|^2\delta(\omega+\omega'). \tag{6.47}$$

We then simply have to compute the spectrum at the sidebands:

$$S_{\rm opt}(\pm\omega_m) = 2\kappa G^2|\varepsilon_f(\pm\omega_m)|^2, \tag{6.48}$$

and we can write the optical damping term as:

$$\Gamma_{\rm opt} = \frac{S_{\rm opt}(\omega_m) - S_{\rm opt}(-\omega_m)}{2} = \frac{\kappa G^2}{\kappa^2 + (\omega_m-\Delta)^2} - \frac{\kappa G^2}{\kappa^2 + (\omega_m+\Delta)^2}. \tag{6.49}$$

The final temperature can then be written as:

$$n_{\rm eff} = \frac{S_{\rm opt}(-\omega_m) + S_{\rm th}(-\omega_m)}{2(\Gamma_{\rm opt}+\gamma_m).}. \tag{6.50}$$

The resolved sideband regime

Let us assume $\omega_m \gg \kappa$. We can estimate that the largest damping rate occurs at $\Delta = \omega_m$ such that

$$\Gamma_{\rm opt} = \frac{G^2}{\kappa} - \frac{\kappa G^2}{\kappa^2 + 4\omega_m^2} \simeq \frac{G^2}{\kappa}. \tag{6.51}$$

In the limit that $\Gamma_{\rm opt} \gg \gamma_m$, the final occupancy is given by

$$n_{\rm eff} = \frac{\gamma_m \bar{n}}{\Gamma_{\rm opt}} = \frac{\bar{n}}{C_{\rm OM}}, \tag{6.52}$$

where we can define an optomechanical cooperativity

$$C_{\rm OM} = \frac{G^2}{\kappa\gamma_m} = \frac{g_{\rm OM}^2}{\kappa\gamma_m}|\alpha|^2. \tag{6.53}$$

6.4 Exercises

7. Cooling of trapped ions

We describe dynamics of ions in linear Paul traps. The interaction of internal levels with the center-of-mass motion allows for indirect cooling of the motion to the vibrational quantum ground state via a properly detuned laser driving. We extend the treatment to two ions where two normal modes (center of mass versus relative motion) occur in the trap owing to the intrinsic coupling via the Coulomb force.

7.1 Radio frequency traps (short summary)

Let us assume a combination of electrodes that gives rise to an effective quadratic potential (in all directions) of the form:

$$\Phi(x,y,z) = \frac{U}{2}(\alpha x^2 + \beta y^2 + \gamma z^2). \tag{7.1}$$

Imposing the Laplace equation $\triangle\Phi = 0$ requires a constraint $\alpha + \beta + \gamma = 0$ which cannot be fulfilled. This cannot with only positive coefficients $\alpha, \beta, \gamma > 0$ (which is the condition for having harmonic trapping in all three directions). Therefore one should consider an additional time-dependent potential. Typically one assumes a rf driving (frequencies ω_{rf} of the order of MHz) of the following form:

$$\Phi_{\text{rf}}(x,y,z) = \frac{\bar{U}}{2}(\alpha' x^2 + \beta' y^2 + \gamma' z^2)\cos(\omega_{\text{rf}} t). \tag{7.2}$$

Let us consider the motion in the x direction (along the trap axis):

$$\ddot{x} = -\frac{Q}{M}\frac{\partial\Phi}{\partial x} = -\frac{Q}{M}\left(\alpha U + \alpha' \bar{U}\cos(\omega_{\text{rf}} t)\right)x. \tag{7.3}$$

We make the following notations:

$$\zeta = \frac{\omega_{\text{rf}} t}{2}, \qquad a_x = \frac{4QU\alpha}{M\omega_{\text{rf}}^2}, \qquad q_x = \frac{2Q\bar{U}\alpha'}{M\omega_{\text{rf}}^2}, \tag{7.4}$$

with which we can express the equation of motion as a Mathieu differential equation:

$$\frac{d^2x(\zeta)}{d\zeta^2}+[a_x-2q_x\cos(\omega_{\rm rf}t)]x(\zeta)=0. \tag{7.5}$$

This is an equation with periodic coefficients that allows the following general solution:

$$x(\zeta)=Ae^{i\beta_x\zeta}\sum_{n=-\infty}^{\infty}C_{2n}e^{i2n\zeta}+Be^{-i\beta_x\zeta}\sum_{n=-\infty}^{\infty}C_{2n}e^{-i2n\zeta}=0. \tag{7.6}$$

For a linear Paul trap one sets $q_z=0$ and $q_x=-q_y$ such that the rf potential is hyperbolic:

$$\Phi_{\rm rf}(x,y,z)=\frac{\bar{U}}{2}(x^2-y^2)\cos(\omega_{\rm rf}t). \tag{7.7}$$

In the lowest order approximation one assumes $|a_x|,q_x^2\ll 1$ and finds an approximate solution:

$$x(t)\approx 2AC_0\cos\left(\beta_x\frac{\omega_{\rm rf}t}{2}\right)\left[1-\frac{q_x}{2}\cos(\omega_{\rm rf}t)\right]. \tag{7.8}$$

This is periodic motion at a slow frequency:

$$\omega_x=\beta_x\frac{\omega_{\rm rf}}{2}=\sqrt{a_x+\frac{q_x^2}{2}}\frac{\omega_{\rm rf}}{2}, \tag{7.9}$$

with a amplitude modulated by the rf frequency.

7.2 Laser cooling

Let us assume that we have perfect harmonic motion in the x direction and we quantize this motion via the usual procedure. On top of this we add the internal degrees of freedom for the ion consisting of two levels resonant to the laser frequency. The laser direction is assumed in the x axis with wavevector $k\hat{x}$ and polarization in the z direction (assuming the dipole is also along z polarized. We can write the usual coupling by stating the complete Hamiltonian in the Schrödinger picture:

$$H=\frac{p_x^2}{2M}+\frac{1}{2}M\omega_x^2x^2+\hbar\omega_0\sigma^\dagger\sigma+\hbar\Omega\left[e^{-ikx}\sigma e^{i\omega_Lt}+e^{ikx}\sigma^\dagger e^{-i\omega_Lt}\right]. \tag{7.10}$$

Let us already write the Hamiltonian of the motion in terms of center-of-mass vibrations operators for creation and destruction of quanta: b and $b^\dagger$and express

$$p_x=i(b-b^\dagger)\sqrt{\frac{m\hbar\omega}{2}}=p_{\rm zpm}i\frac{b-b^\dagger}{\sqrt{2}}, \tag{7.11}$$

and

$$x=(b-b^\dagger)\sqrt{\frac{\hbar}{2m\omega}}=\frac{x_{\rm zpm.}}{\sqrt{2}}\left(b+b^\dagger\right). \tag{7.12}$$

The Lamb-Dicke limit

Moreover, we will asumme that the motional state is not too highly excited meaning that

$$kx\ll 1. \tag{7.13}$$

This would basically mean that we ask for the Lamb-Dicke parameter

$$\eta=kx_{\rm zpm}/\sqrt{2}=\sqrt{2}\pi x_{\rm zpm}/\lambda\ll 1 \tag{7.14}$$

and $\eta \langle b^\dagger b\rangle \ll 1$. In such a case we can expand the exponent

$$e^{\pm ikx} \simeq 1 \pm kx = 1 \pm \eta (b + b^\dagger). \tag{7.15}$$

The total Hamiltonian (now written in a frame rotating at the laser frequency such that $\Delta = \omega_L - \omega_0$) is now:

$$H = \hbar\omega_x b^\dagger b + \hbar\omega_0 \sigma^\dagger \sigma + \hbar\Omega \left[\sigma + \sigma^\dagger\right] - i\hbar\eta\Omega \left[\sigma - \sigma^\dagger\right] \left[b + b^\dagger\right]. \tag{7.16}$$

We'll denote $\omega_0 - \omega_L = \Delta$. Let us go back to our previous lecture formulation in terms of position and momentum and derive an equation with Langevin forces:

$$\dot{q} = \omega_x p, \tag{7.17a}$$
$$\dot{p} = -\omega_x q + i\eta\Omega \left[\sigma - \sigma^\dagger\right], \tag{7.17b}$$
$$\dot{\sigma} = -(\gamma + i\Delta)\sigma + i\Omega\sigma_z + \sigma_{\text{in}}, \tag{7.17c}$$
$$\dot{\sigma}^\dagger = -(\gamma - i\Delta)\sigma^\dagger - i\Omega\sigma_z + \sigma_{\text{in}}^\dagger. \tag{7.17d}$$

We will make a simplification of low driving and replace σ_z with -1 while considering the noise terms to be delta correlated: $\langle \sigma_{\text{in}}(t)\sigma_{\text{in}}^\dagger(t')\rangle = \delta(t - t')$. In steady state we will have

$$0 = \omega_x p_s, \tag{7.18a}$$
$$0 = -\omega_x q_s + i\eta\Omega [\beta - \beta^*], \tag{7.18b}$$
$$0 = -(\gamma + i\Delta)\beta - i\Omega, \tag{7.18c}$$
$$0 = -(\gamma - i\Delta)\beta + i\Omega. \tag{7.18d}$$

One can now compute

$$\beta - \beta^* = \frac{-i\Omega}{\gamma + i\Delta} - \frac{i\Omega}{\gamma - i\Delta} = -i\frac{2\gamma\Omega}{\gamma^2 + \Delta^2}, \tag{7.19}$$

This leads to

$$q_s = \frac{2\gamma\eta\Omega^2}{\omega_x(\gamma^2 + \Delta^2)}. \tag{7.20}$$

Now we can again compute the spectrum of the force $F_{\text{opt}} = \eta\Omega \left[\delta\sigma - \delta\sigma^\dagger\right]$ by first going into the Fourier domain:

$$\delta q(\omega^2 - \omega_x^2) = i\eta\Omega \left[\delta\sigma - \delta\sigma^\dagger\right], \tag{7.21a}$$
$$\delta\sigma(\omega) [\gamma + i(\Delta - \omega)] = \sigma_{\text{in}}, \tag{7.21b}$$
$$\delta\sigma^\dagger(\omega) [\gamma - i(\Delta - \omega)] = \sigma_{\text{in}}^\dagger. \tag{7.21c}$$

As in the previous lecture we compute

$$\langle F_{\text{opt}}(\omega)F_{\text{opt}}(\omega')\rangle = \eta^2\Omega^2\varepsilon_{\text{ion}}(\omega)\varepsilon^*(-\omega')\langle\sigma_{\text{in}}(\omega)\sigma_{\text{in}}^\dagger(\omega')\rangle \tag{7.22}$$
$$= 2\gamma\eta^2\Omega^2|\varepsilon_{\text{ion}}(\omega)|^2\delta(\omega + \omega'), \tag{7.23}$$

so that we can state that the spectrum of the Langevin force is:

$$S_{\text{opt}}(\omega) = 2\gamma\eta^2\Omega^2|\varepsilon_{\text{ion}}(\omega)|^2 = \frac{2\gamma\eta^2\Omega^2}{\gamma^2 + (\Delta - \omega)^2}. \tag{7.24}$$

Evaluating the sidebands we find:

$$\Gamma_{\text{opt}} = \gamma\eta^2\Omega^2 \left[\frac{1}{\gamma^2 + (\Delta - \omega_x)^2} - \frac{1}{\gamma^2 + (\Delta + \omega_x)^2}\right]. \tag{7.25}$$

The final temperature is given by:

$$\bar{n} = \frac{S_{\text{opt}}(-\omega_x)}{2\Gamma_{\text{opt}}}. \tag{7.26}$$

In the resolved sideband limit, where $\gamma \ll \omega_x$, we can assume that $S_{\text{opt}}(\omega_x) \gg S_{\text{opt}}(-\omega_x)$ and

$$\Gamma_{\text{opt}} \simeq \frac{\eta^2\Omega^2}{\gamma}. \tag{7.27}$$

The final temperature is then

$$\bar{n} \simeq \frac{\gamma^2}{\gamma^2 + 4\omega_x^2} \simeq \left(\frac{\gamma}{2\omega_x}\right)^2. \tag{7.28}$$

7.3 Collective ion trap modes

Let's consider more than one ion inside a trap. On top of the external trapping potential, the ions also experience a mutual Coulomb force depending on the distance. In the absence of driving (and for the moment we don't consider the internal structure) we can therefore write a Hamiltonian (at the moment classical) for the two ions in the trap:

$$H_{1+2} = \frac{p_1^2}{2M} + \frac{p_2^2}{2M} + \frac{1}{2}M\omega_x^2(x_1^2 + x_2^2) + \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{|x_1 - x_2|}. \tag{7.29}$$

Equilibrium

The average distance we will denote by $\ell = x_1^{(0)} - x_2^{(0)}$ where the two positions are written with respect to the origin fixed at the center of the trap. We set the condition for equilibrium by canceling the force on each ions:

$$0 = 2M\omega_x^2 x_1^{(0)} - \frac{1}{4\pi\varepsilon_0}\frac{2Q^2}{\ell^2}, \tag{7.30a}$$

$$0 = 2M\omega_x^2 x_2^{(0)} + \frac{1}{4\pi\varepsilon_0}\frac{2Q^2}{\ell^2}. \tag{7.30b}$$

We find as a solution:

$$x_1^{(0)} = -x_2^{(0)} = \sqrt[3]{\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{4M\omega_x^2}}. \tag{7.31}$$

Typically this is of the order of a few microns, providing enough separation that ions can be excited and imaged independently from each other. Notice an useful expression:

$$\frac{Q^2}{4\pi\varepsilon_0\ell^3} = \frac{1}{2}M\omega_x^2. \tag{7.32}$$

Normal modes of oscillation

Now we assume small variations of the equilibrium position with $x_1 = -\ell/2 + \delta x_1$ and $x_2 = \ell/2 + \delta x_2$. The harmonic term leads to:

$$\frac{1}{2}M\omega_x^2(x_1^2+x_2^2) = \frac{1}{8}M\omega_x^2\ell^2 + \frac{1}{2}M\omega_x^2(\delta x_1^2+\delta x_2^2) + \frac{1}{2}M\omega_x^2\ell(\delta x_2 - \delta x_1). \tag{7.33}$$

The Coulomb term can be expanded in terms of the small parameter $((\delta x_1 - \delta x_2)/\ell)$

$$\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{|x_1-x_2|} = \frac{Q^2}{4\pi\varepsilon_0\ell}\frac{1}{1+(\delta x_2-\delta x_1)/\ell} = \frac{Q^2}{4\pi\varepsilon_0\ell}\left[1-\frac{\delta x_2-\delta x_1}{\ell}+\left(\frac{\delta x_2-\delta x_1}{\ell}\right)^2\right]. \tag{7.34}$$

Notice that the linear term can be reexpressed:

$$\frac{Q^2}{4\pi\varepsilon_0\ell}\frac{\delta x_2-\delta x_1}{\ell} = \frac{1}{2}M\omega_x^2\ell^3\frac{1}{\ell}\frac{\delta x_2-\delta x_1}{\ell} = \frac{1}{2}M\omega_x^2\ell(\delta x_2-\delta x_1), \tag{7.35}$$

cancelling the term previously obtained from the purely harmonic trap. Apart from the trivial energy shifts one gets:

$$\begin{aligned} H_{1+2} &= \frac{p_1^2}{2M}+\frac{p_2^2}{2M}+\frac{1}{2}M\omega_x^2\left[\delta x_1^2+\delta x_2^2+(\delta x_2-\delta x_1)^2\right] \\ &= \frac{p_1^2}{2M}+\frac{p_2^2}{2M}+M\omega_x^2\left[\delta x_1^2+\delta x_2^2-\delta x_2\delta x_1\right]. \end{aligned} \tag{7.36}$$

Writing equations of motion

$$\dot{x}_1 = \frac{p_1}{M}, \tag{7.37a}$$
$$\dot{x}_2 = \frac{p_2}{M}, \tag{7.37b}$$
$$\dot{p}_1 = -M\omega_x^2(2\delta x_1-\delta x_2), \tag{7.37c}$$
$$\dot{p}_2 = -M\omega_x^2(2\delta x_2-\delta x_1). \tag{7.37d}$$

We can simplify to two coupled second order differential equations:

$$\ddot{\delta x}_1 = -\omega_x^2(2\delta x_1-\delta x_2), \tag{7.38a}$$
$$\ddot{\delta x}_2 = -\omega_x^2(2\delta x_2-\delta x_1). \tag{7.38b}$$

Assuming solutions of the form $A_{1,2}e^{i\omega t}$ we are left with the following diagonalization:

$$(2\omega_x^2-\omega^2)A_1-\omega_x^2A_2 = 0, \tag{7.39a}$$
$$-\omega_x^2A_1+(2\omega_x^2-\omega^2)A_2 = 0. \tag{7.39b}$$

The system has nontrivial solutions only if the multiplying matrix has a zero determinant. In other words, we have to find the eigenvalues of a simple matrix $[\{2,-1\},\{-1,2\}]$ which are 1 and 3. Finally, we have the two modes of vibration (center-of-mass versus relative motion)

$$\omega = \omega_x \qquad \text{and} \qquad \omega = \sqrt{3}\omega_x \tag{7.40}$$

and

$$\delta x_{\mathrm{CM}} = \frac{1}{\sqrt{2}}(\delta x_2+\delta x_1), \qquad \text{and} \qquad \delta x_{\mathrm{RM}} = \frac{1}{\sqrt{2}}(\delta x_2-\delta x_1). \tag{7.41}$$

We could've realized from the start, of course, that one can write:

$$\delta x_1^2 + \delta x_2^2 + (\delta x_2 - \delta x_1)^2 = \frac{1}{2}\left[(\delta x_2 + \delta x_1)^2 + 3(\delta x_2 - \delta x_1)^2\right], \tag{7.42}$$

which would have given the diagonalized Hamiltonian in terms of CM and RM:

$$H_{1+2}^{\text{harm}} = \frac{p_{\text{CM}}^2}{2M} + \frac{p_{\text{RM}}^2}{2M} + \frac{1}{2}M\omega_x^2 \delta x_{\text{CM}}^2 + \frac{1}{2}M(\sqrt{3}\omega_x)^2 \delta x_{\text{RM}}^2. \tag{7.43}$$

7.4 Ion trap logic

Red and blue sideband addressing

Let's look again on a single ion addressing but now in an interation picture:

$$H = \hbar\Omega\left[\sigma e^{-i\Delta t} + \sigma^\dagger e^{i\Delta t}\right] - i\hbar\eta\Omega\left[\sigma e^{-i\Delta t} - \sigma^\dagger e^{i\Delta t}\right]\left[b e^{i\omega_x t} + b^\dagger e^{-i\omega_x t}\right]. \tag{7.44}$$

We can set the resonance of some processes by playing with the laser frequency. For example, the blue sideband addressing where $\Delta = \omega_x$ will lead to the following process being resonant

$$H_{red} = -i\hbar\eta\Omega\left[\sigma b - h.c\right]. \tag{7.45}$$

while for $\Delta = -\omega_x$ we would have

$$H_{blue} = -i\hbar\eta\Omega\left[\sigma b^\dagger - h.c\right]. \tag{7.46}$$

Let us assume we are in a state $|g,n\rangle$ (see Figure above). From here we either cycle up to $|e,n\rangle$ and back down to $|g,n\rangle$ via the driving which is uncoupled to motion or up to $|e,n-1\rangle$ via H_{blue} or up to $|e,n+1\rangle$ via H_{red}. The uncoupled driving could be followed by decay back into $|g,n\rangle$ which means nothing happened except some recoil kick. The driving to $|e,n-1\rangle$ followed by decay into $|g,n-1\rangle$mean effective cooling of the motion. This is the process which we try to optimize via detuning of the laser below the atomic frequency.

7.4.1 Two qubit gate: Controlled NOT gate (CNOT gate)

This is a gate acting on 2 qubits placing one in control of the state of the other one. If the first qubit is in 0 nothing happens to the other one. If the first qubit is in state 1 then a negation of the second one occurs. In matrix form one writes it as:

$$\text{CNOT} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.47}$$

Let's first write the two qubit state as:

$$|00\rangle \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } |01\rangle \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } |10\rangle \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } |11\rangle \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{7.48}$$

and check the action of the gate:

$$\text{CNOT}|00\rangle = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |00\rangle, \tag{7.49}$$

$$\mathrm{CNOT}|01\rangle = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |01\rangle, \tag{7.50}$$

$$\mathrm{CNOT}|10\rangle = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |11\rangle, \tag{7.51}$$

$$\mathrm{CNOT}|11\rangle = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |10\rangle. \tag{7.52}$$

7.4.2 Two qubit gate with trapped ions

The conditional operation is allowed by the common interaction of the two qubits with the phonon bus. One prepares the phonon is the ground state $|0\rangle$and applies three steps:

(1) map the internal state of one ion to the motion of an ion string,
(2) to flip the state of the target ion conditioned on the motion of the ion string,
(3) to map the motion of the ion string back onto the original ion.

$$\mathrm{CNOT} = R(\frac{\pi}{2},0)R_{\text{blue}}\,(\pi,\frac{\pi}{2})R_{\text{blue}}\,(\pi/\sqrt{2},0)R_{\text{blue}}\,(\pi,\frac{\pi}{2})R_{\text{blue}}\,(\pi/\sqrt{2},0)R_{\text{blue}}\,(\pi/2,\pi) \tag{7.53}$$

7.5 Exercises

8. Collective effects:super/subradiance

9. Interaction of light with molecules

9.1 The Holstein Hamiltonian for vibronic coupling

Let us justify the form of the Holstein Hamiltonian by following a first-principle derivation for a single nuclear coordinate R of effective mass μ. We assume that, along the nuclear coordinate, the equilibria for ground (coordinate R_g, state vector $|g\rangle$) and excited (coordinate R_e and state vector $|e\rangle$) electronic orbitals are different. Notice that, for the simplest case of a homonuclear diatomic molecule made of nuclei each with mass m, a single vibrational mode exists corresponding to the relative motion mode with effective mass $\mu = m/2$; the equilibria coordinates R_g and R_e correspond to the bond length in ground/excited states. Assuming that both the ground and the excited state have the same parabolic shape around the minima (with second derivative leading to a vibrational frequency ν) one can write the total Hamiltonian:

$$H = \left[\omega_e g a_e + \frac{p^2}{2\mu} + \frac{1}{2}\mu\nu^2 (R-R_e)^2\right]\sigma^\dagger\sigma + \left[\frac{p^2}{2\mu} + \frac{1}{2}\mu\nu^2 (R-R_g)^2\right]\sigma\sigma^\dagger. \tag{9.1}$$

Notice that the Hamiltonian is written in a Hilbert space spanning both electronic dynamics (via the Pauli operators) and mechanical dynamics. Introducing small oscillations around the equilibria $Q = R - R_g$ and subsequently $R - R_e = Q + R_g - R_e =: Q - R_{ge}$ we obtain

$$H = \frac{P^2}{2\mu} + \frac{1}{2}\mu\nu^2 Q^2 + \omega_e g a_e \sigma^\dagger\sigma - \mu\nu^2 Q R_{ge}\sigma^\dagger\sigma + \frac{1}{2}\mu\nu^2 R_{ge}^2\sigma^\dagger\sigma. \tag{9.2}$$

The last term is a renormalization of the bare electronic transition frequency energy (absent in some theoretical treatments) which will naturally go away when diagonalizing the Hamiltonian via the polaron transformation resulting in $\omega_e g a_e$ as the natural electronic transition. We can now rewrite the momentum and position operators in terms of bosonic operators $Q = r_{\text{zpm}}(b^\dagger + b)$, $P = i p_{\text{zpm}}(b^\dagger - b)$ by introducing the zero-point-motion $r_{\text{zpm}} = 1/\sqrt{2\mu\nu}$ and $p_{\text{zpm}} = \sqrt{\mu\nu/2}$. Reexpressing the terms above via the Huang-Rhys factor $\lambda = \mu\nu R_{ge} r_{\text{zpm}}$ yields the Holstein-Hamiltonian

$$H = \nu b^\dagger b + (\omega_e + \lambda^2\nu)\sigma^\dagger\sigma - \lambda\nu(b^\dagger + b)\sigma^\dagger\sigma. \tag{9.3}$$

One can bring this Hamiltonian into diagonal form $\nu b^\dagger b + \omega_e \sigma^\dagger \sigma$ via the polaron transformation $U^\dagger = \mathscr{D}^{\sigma^\dagger \sigma}$ where the displacement is defined as $\mathscr{D} = \exp(-i\sqrt{2}\lambda p) = \exp[\lambda(b^\dagger - b)]$ with the dimensionless momentum quadrature $p = i(b^\dagger - b)/\sqrt{2}$ as generator (we also define the dimensionless position quadrature $q = (b^\dagger + b)/\sqrt{2}$). Notice that the polaron transformation also removes the vibronic shift of the excited state.

9.1.1 The polaron transformation

This Hamiltonian is not diagonal. Let's compute it's matrix elements:

$$\langle e, m|H_0|g, n\rangle = 0, \tag{9.4}$$

but

$$\langle g, m|H_0|g, n\rangle = \hbar\nu n\delta_{mn} \tag{9.5}$$

and

$$\langle e, m|H_0|e, n\rangle = \hbar\nu n\delta_{mn} + \hbar\omega_{eg}a_e\delta_{mn} + \hbar\lambda\nu\left[\sqrt{n}\delta_{mn-1} + \sqrt{n+1}\delta_{mn+1}\right]. \tag{9.6}$$

So it can mix vibrational levels in the electronic excited orbital. We will perform a polaron transformation with

$$S = \lambda(b^\dagger - b)\sigma^\dagger\sigma \tag{9.7}$$

such that $S^\dagger = -S$ and the operation e^S is unitary. Notice that this is simply a coherent state displacement operator with the particularity that it only displaces vibrational states in the excited electronic manifold. Of course, the displacement is also an operator instead of a c-number. First notice that it comutes with the population operator so that it leaves the following term unchanged:

$$\hbar\omega_{eg}a_e e^S\sigma^\dagger\sigma e^{-S} = \hbar\omega_{eg}a_e\sigma^\dagger\sigma. \tag{9.8}$$

Using the displacement rules that we have listed in Exercises 1 when looking at the coherent states, we have

$$e^S b^\dagger e^{-S} = b^\dagger - \lambda\sigma^\dagger\sigma \text{ and } e^S b e^{-S} = b - \lambda\sigma^\dagger\sigma. \tag{9.9}$$

Then we can work out

$$\hbar\nu e^S b^\dagger b e^{-S} = \hbar\nu(b^\dagger - \lambda\sigma^\dagger\sigma)(b - \lambda\sigma^\dagger\sigma) = \hbar\nu b^\dagger b + \lambda^2\hbar\nu\sigma^\dagger\sigma - \lambda\hbar\nu(b + b^\dagger)\sigma^\dagger\sigma \tag{9.10}$$

$$\hbar\lambda\nu e^S(b + b^\dagger)\sigma^\dagger\sigma e^{-S} = \hbar\lambda\nu(b + b^\dagger - 2\lambda\sigma^\dagger\sigma)\sigma^\dagger\sigma = \lambda\hbar\nu(b + b^\dagger)\sigma^\dagger\sigma - 2\lambda^2\hbar\nu\sigma^\dagger\sigma \tag{9.11}$$

Putting it all together we get the Hamiltonian

$$\bar{H} = e^S H_0 e^{-S} = \hbar(\omega_{eg}a_e - \lambda^2\nu)\sigma^\dagger\sigma + \hbar\nu b^\dagger b, \tag{9.12}$$

in diagonal form in the new picture.

9.1.2 Transition strengths

We would like to know how the absorption and emission spectra of such a simple diatomic molecule look like. To this end we add a driving term:

$$H_d = \hbar\eta(\sigma e^{i\omega_e g a_L t} + \sigma^\dagger e^{-i\omega_e g a_L t}). \tag{9.13}$$

and see how it looks like in the new picture (let's assume already an interaction picture where the new energy levels are $\omega_e g a_e - \omega_e g a_L - \lambda^2\nu$)

$$\boxed{\bar{H}_d = \hbar\eta(e^S\sigma e^{-S} + e^S\sigma^\dagger e^S) = \hbar\eta\sigma e^{\lambda(b-b^\dagger)} + \hbar\eta\sigma^\dagger e^{-\lambda(b-b^\dagger)}.} \tag{9.14}$$

We can now ask a few questions. For example, assuming a molecule in the ground state $|g,0\rangle$driven by a laser what transitions are possible. For this let's write:

$$\langle n,e|\bar{H}|g,0\rangle = \hbar\eta\langle n|e^{-\lambda(b-b^\dagger)}|0\rangle = \hbar\eta\langle n||\lambda\rangle_{coh} = \hbar\eta e^{-\lambda^2/2}\frac{\lambda^n}{\sqrt{n!}}. \tag{9.15}$$

We used the following relation: $e^{-\lambda(b-b^\dagger)} = e^{\lambda b^\dagger}e^{-\lambda b}e^{-\lambda^2/2}$. Also notice that the operator we used for the transformation is simply a displacement operator creating a coherent state from the vacuum. The conclusion is straightforward i.e. the efficiency of the laser excitation depends very much on the Frank-Condon overlap of the wavefunctions in the ground and excited states. For example, the zero-phonon line is weakened by a factor $e^{-\lambda^2/2}$. The maximum of the absorption spectrum is then obtained by finding the Fock state with most probability in the Poissonian distribution. For the emission spectrum let's analyze

$$\langle n,g|\bar{H}|e,0\rangle = \hbar\eta\langle n|e^{\lambda(b-b^\dagger)}|0\rangle = \hbar\eta\langle n|-\lambda\rangle_{coh} = \hbar\eta e^{-\lambda^2/2}\frac{(-\lambda)^n}{\sqrt{n!}}. \tag{9.16}$$

9.1.3 The branching of the radiative decay rates.

We now transform the Lindblad term

$$\mathscr{D}[\sigma,\rho] = \sigma\rho\sigma^\dagger - \frac{1}{2}\left(\rho\sigma^\dagger\sigma + \sigma^\dagger\sigma\rho\right) \tag{9.17}$$

with the polaron transformation. This leads to

$$e^S\mathscr{D}[\sigma,\rho]e^{-S} = \sigma e^{\lambda(b-b^\dagger)}\tilde{\rho}\sigma^\dagger e^{-\lambda(b-b^\dagger)} - \frac{1}{2}\left(\tilde{\rho}\sigma^\dagger\sigma + \sigma^\dagger\sigma\tilde{\rho}\right) \tag{9.18}$$

Now let us assume non-radiative transitions coming from a Lindblad term:

$$\mathscr{D}[b,\rho] = b\rho b^\dagger - \frac{1}{2}\left(\rho b^\dagger b + b^\dagger b\rho\right) \tag{9.19}$$

The transformation leads to:

$$e^S\mathscr{D}[b,\rho]e^{-S} = (b-\lambda\sigma^\dagger\sigma)\tilde{\rho}(b^\dagger-\lambda\sigma^\dagger\sigma) - \frac{1}{2}\left(\tilde{\rho}(b^\dagger-\lambda\sigma^\dagger\sigma)(b-\lambda\sigma^\dagger\sigma) + (b^\dagger-\lambda\sigma^\dagger\sigma)(b-\lambda\sigma^\dagger\sigma)\tilde{\rho}\right) = \tag{9.20}$$

$$= b\tilde{\rho}b^\dagger - \frac{1}{2}\left[\tilde{\rho}b^\dagger b + b^\dagger b\tilde{\rho}\right] + \tag{9.21}$$

$$\lambda^2\left[\sigma^\dagger\sigma\tilde{\rho}\sigma^\dagger\sigma - \frac{1}{2}\left[\tilde{\rho}\sigma^\dagger\sigma + \sigma^\dagger\sigma\tilde{\rho}\right]\right] + \tag{9.22}$$

$$-\lambda\left[b\tilde{\rho}\sigma^\dagger\sigma + \sigma^\dagger\sigma\tilde{\rho}b^\dagger\right] + \frac{\lambda}{2}\left[\tilde{\rho}(b^\dagger+b)\sigma^\dagger\sigma + (b^\dagger+b)\sigma^\dagger\sigma\tilde{\rho}\right]. \tag{9.23}$$

9.2 Molecular spectroscopy

9.3 Exercises

10. Bibliography

Books

Articles