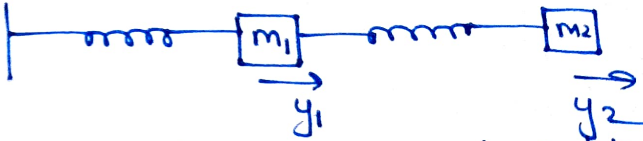
Schematic Diagram



Given Eq^n

$$m_1 \ddot{y}_1 = -k_1 y_1 + k_2 (y_2 - y_1)$$

$$m_2 \ddot{y}_2 = -k_2 (y_2 - y_1)$$

Using two physics conc.

$$F = Ma = m\frac{d^2 y}{dt^2}$$

$$F = -kx = -ky$$

We will move ahead and write the eq^n in form of $\ddot{Y} = AY$

$$\ddot{y}_1 = \left(\frac{-k_1 - k_2}{m_1}\right) y_1 + \left(\frac{k_2}{m_1}\right) y_2$$

$$\ddot{y}_2 = \left(\frac{k_2}{m_2}\right) y_1 - \left(\frac{k_2}{m_2}\right) y_2$$

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\ddot{y} = Ay$$

$$y'' + 5y = 0 \quad (\text{format})$$

we will move on with our calculation using Gilbert Strang Method,

$$y(t) = ve^{rt}$$

$$y''(t) = r^2 ve^{rt}$$

$$r^2 ve^{rt} = Ave^{rt} \qquad \lambda = r^2$$

$$\lambda v = Av \qquad \sqrt{\lambda} = r$$

Eigen form

Part A

$\rightarrow \det(A - \lambda I)$ will give us Eigen Values

$m_1 = m_2 = 1$

$k_1 = 3, k_2 = 2$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\det \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$\lambda = -6, -1$$

$r_1^2 = -6$ $\quad \omega_1 = \sqrt{6}$

$r_2^2 = -1$ $\quad \omega_2 = \sqrt{1}$

Eigen vectors for $\lambda = -1$

$$\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigen vectors for $\lambda = -6$ $\quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t))\, v_1 + (A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t))\, v_2$$

we have written principle of superposition in order to get eq^n of y.

Idea is taken from reference book of Gilbert Strang – Differential Eq^n (Page 372)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left(A_1 \cos(\text{[redacted]}\sqrt{6}t) + B_1 \sin(\text{[redacted]}\sqrt{6}t)\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \left(A_2 \cos(\text{[redacted]}t) + B_2 \sin(\text{[redacted]}t)\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Now using initial condition given in Q we will get values of A_1, B_1, A_2, B_2.

$A_i \rightarrow y(0) \quad B_i \rightarrow y'(0)$

$$A_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + A_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$-2A_1 + A_2 = 1$$

$$A_2 = 1 + 2A_1$$

$$A_1 + 2A_2 = 2$$

$$A_1 + 2(1 + 2A_1) = 2$$

$$5A_1 = 0$$

$$\left\{ \begin{aligned} A_1 &= 0 \\ A_2 &= 1 \end{aligned} \right.$$

$$5A_1 = 0$$

$$A_1 = 0$$

$$A_2 = 1$$

$$B_1\begin{bmatrix}-2\\1\end{bmatrix} + B_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-2\sqrt{6}\\ \sqrt{6}\end{bmatrix}$$

$$-2B_1 + B_2 = -2\sqrt{6}$$

$$B_1 + 2B_2 = \sqrt{6}$$

$$B_1 + 2(2B_1 - 2\sqrt{6}) = \sqrt{6}$$

$$4B_1 + B_1 = 5\sqrt{6}$$

$$B_1 = \sqrt{6}$$

$$B_2 = 0$$

After putting the values in eq^n

$\begin{pmatrix}y_1\\y_2\end{pmatrix} = \sqrt{6}\sin($ [redacted] $\sqrt{6}t)\begin{bmatrix}-2\\1\end{bmatrix} + \cos($ [redacted] $t)\begin{bmatrix}1\\2\end{bmatrix} \leftarrow Eq^n$

Part B $m_1 = m_2 = 1$ $\quad \dot{y}_1(0) = \sqrt{2}$ $\quad y_1(0) = y_2(0) = 0$

$k_1 = 6 \quad k_2 = 4$ $\quad \dot{y}_2(0) = 2\sqrt{2}$

$$\ddot{y} = Ay$$

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$A = \begin{bmatrix} -10 & 4 \\ 4 & -4 \end{bmatrix}$$

Finding Eigenvalue by

$$\det(A - \lambda I) = 0$$

$$((10+\lambda)(4+\lambda) - 16) = 0$$

$$(\lambda+12)(\lambda+2) = 0$$

$$\lambda = -12, -2$$

$$\begin{pmatrix} -10+12 & 4 \\ 4 & -4+12 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -(1)$$

$$\begin{pmatrix} -10+2 & 4 \\ 4 & -4+2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -(2)$$

$$\begin{pmatrix} -10+12 & 4 \\ 4 & -4+12 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad - (1)$$

$$\begin{pmatrix} -10+2 & 4 \\ 4 & -4+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad - (2)$$

$\lambda_1 = -12 = r_1^2$

$\lambda_2 = -2 = r_2^2$

$\omega_1 = 2\sqrt{3}$

$\omega_2 = \sqrt{2}$

Eigen vector for -2

Eigen vector for -12

for -12 Eigen value $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

for -2 Eigen value $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Now using ~~Gilbert~~ Strang method

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left(A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)\right) v_1 + \left(A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)\right) v_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left(A_1 \cos(2\sqrt{3}t) + B_1 \sin(2\sqrt{3}t)\right) \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \left(A_2 \cos(\sqrt{2}t) + B_2 \sin \sqrt{2}t\right) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

we will be calculating A_1, A_2, B_1 and B_2

$A_i \rightarrow y(0)$ $\quad$ $B_i \rightarrow y'(0)$ (Initial condition)

$$A_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + A_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2A_1 + A_2 = 0$

$A_2 = -2A_1$

$-A_1 + 2A_2 = 0$

$-A_1 + 2(-2A_1) = 0$

$A_1 = 0$

$A_2 = 0$

$A_2 = -2A_1$

$-A_1 + 2A_2 = 0$

$-A_1 + 2(-2A_1) = 0$

$A_1 = 0$

$A_2 = 0$

$$B_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + B_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$$

$2B_1 + B_2 = \sqrt{2}$

$-B_1 + 2B_2 = 2\sqrt{2}$

$B_2 = \sqrt{2} - 2B_1$

$-B_1 + 2(\sqrt{2} - 2B_1) = 2\sqrt{2}$

$-B_1 + 2\sqrt{2} - 4B_1 = 2\sqrt{2}$

$B_1 = 0$

$B_2 = \sqrt{2}$

After putting the values we get,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \sqrt{2}\sin(\sqrt{2}t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \leftarrow \text{Reqd Eq}^n$$