

AI1103-Assignment-4

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Download all python codes from

<https://github.com/Vikhyath-vec/AI1103/tree/main/Assignment-4/codes>

and latex-tikz codes from

<https://github.com/Vikhyath-vec/AI1103/blob/main/Assignment-4/Assignment-4.tex>

A random variable converges, in probability, to a value if it converges, in mean square, to the same particular value by Markov's Inequality.

$$Y_n \xrightarrow{\mu_s} c \Rightarrow Y_n \xrightarrow{p} c \quad (0.0.3)$$

Proof for (0.0.3): For any $\epsilon > 0$

$$\Pr(|Y_n - Y| \geq \epsilon) = \Pr(|Y_n - Y|^2 \geq \epsilon^2) \quad (0.0.4)$$

$$\Pr(|Y_n - Y| \geq \epsilon) \leq \frac{E|Y_n - Y|^2}{\epsilon^2} \quad (\text{by Markov's Inequality}) \quad (0.0.5)$$

$$\lim_{n \rightarrow \infty} E(|Y_n - Y|^2) = 0 \quad (0.0.6)$$

$$0 \leq \lim_{n \rightarrow \infty} \Pr(|Y_n - Y| \geq \epsilon) \leq \frac{0}{\epsilon^2} \quad (0.0.7)$$

$$\lim_{n \rightarrow \infty} \Pr(|Y_n - Y| \geq \epsilon) = 0 \quad \forall \epsilon > 0 \quad (0.0.8)$$

Given in the question that $\{X_j\}$ is a sequence of random variables with

$$\Pr(X_j = 1) = \frac{1}{4} \quad (0.0.9)$$

$$\Pr(X_j = 0) + \Pr(X_j = 1) = 1 \quad (0.0.10)$$

$$\Pr(X_j = 0) = 1 - \frac{1}{4} = \frac{3}{4} \quad (0.0.11)$$

$$X_j \in \{0, 1\} \quad (0.0.12)$$

Since $0^2 = 0$ and $1^2 = 1$,

$$X_j^2 = X_j \quad \forall j \in \{1, 2, \dots, n\} \quad (0.0.13)$$

QUESTION

Let $\{X_j\}$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = \frac{1}{4}$ and let $Y_n = \frac{1}{n} \sum_{j=1}^n X_j^2$. Then Y_n converges, in probability, to _____.

SOLUTION

A sequence of random variables $Y_1, Y_2, Y_3 \dots$ converges, in probability, to a random variable Y if

$$\lim_{n \rightarrow \infty} \Pr(|Y_n - Y| \geq \epsilon) = 0 \quad \forall \epsilon > 0 \quad (0.0.1)$$

Similarly, a sequence of random variables $Y_1, Y_2, Y_3 \dots$ converges, in mean square, to a random variable Y if

$$\lim_{n \rightarrow \infty} E(|Y_n - Y|^2) = 0 \quad (0.0.2)$$

Thus,

$$Y_n = \frac{1}{n} \sum_{j=1}^n X_j^2 \quad (0.0.14)$$

$$= \frac{1}{n} \sum_{j=1}^n X_j \quad (0.0.15)$$

$$\Pr(Y_n = y) = {}^nC_{ny} \left(\frac{1}{4}\right)^{ny} \left(\frac{3}{4}\right)^{n-ny} \quad (0.0.16)$$

Let us assume

$$k = ny \quad (0.0.17)$$

$$k \in \{0, 1, 2, \dots, n-1, n\} \quad (0.0.18)$$

$$\Pr(Y_n = \frac{k}{n}) = {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.19)$$

$$E\left(|Y_n - \frac{1}{4}|^2\right) = E\left(Y_n^2 - \frac{1}{2}Y_n + \frac{1}{16}\right) \quad (0.0.20)$$

$$= E(Y_n^2) - \frac{1}{2}E(Y_n) + \frac{1}{16} \quad (0.0.21)$$

$$E(Y_n^2) = \sum_{k=0}^n \left(\frac{k}{n}\right)^2 \Pr\left(Y_n = \frac{k}{n}\right) \quad (0.0.22)$$

$$= \sum_{k=0}^n \left(\frac{k^2}{n^2}\right) {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.23)$$

$$E(Y_n^2) = 0 + \frac{1}{n^2} \times n \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{n-1} + \sum_{k=2}^n \left(\frac{k}{n}\right)^2 \times \frac{n(n-1)}{k(k-1)} \times {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.24)$$

$$E(Y_n^2) = \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \times \sum_{k=2}^n \left(\frac{k}{k-1}\right)^{n-2} C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.25)$$

$$E(Y_n^2) = \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \left(\sum_{k=2}^n {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \right) + \frac{n-1}{n} \left(\sum_{k=2}^n \frac{1}{k-1} {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \right) \quad (0.0.26)$$

$$E(Y_n^2) = \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \times \frac{1}{16} \left(\sum_{k=2}^n {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^{k-2} \left(\frac{3}{4}\right)^{(n-2)-(k-2)} \right) + \frac{1}{n} \left(\sum_{k=2}^n \frac{n-1}{k-1} {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \right) \quad (0.0.27)$$

$$E(Y_n^2) = \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} \left(\sum_{j=0}^{n-2} {}^{n-2}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-2)-j} \right) + \frac{1}{4n} \left(\sum_{k=2}^n {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^{k-1} \left(\frac{3}{4}\right)^{(n-1)-(k-1)} \right) \quad (0.0.28)$$

$$E(Y_n^2) = \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} \left(\frac{1}{4} + \frac{3}{4}\right)^{n-2} + \frac{1}{4n} \left(\sum_{j=1}^{n-1} {}^{n-1}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-1)-j} \right) \quad (0.0.29)$$

$$E(Y_n^2) = \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n} \left(\left(\frac{1}{4} + \frac{3}{4}\right)^{n-1} - \left(\frac{3}{4}\right)^{n-1} \right) \quad (0.0.30)$$

$$E(Y_n^2) = \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n} - \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} \quad (0.0.31)$$

$$= \frac{1}{16} + \frac{3}{16n} \quad (0.0.32)$$

$$E(Y_n) = \sum_{k=0}^n \frac{k}{n} \Pr\left(Y_n = \frac{k}{n}\right) \quad (0.0.33)$$

$$= \sum_{k=0}^n \left(\frac{k}{n}\right)^n C_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.34)$$

$$= 0 + \sum_{k=1}^n \frac{k}{n} \times \frac{n}{k} \times {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.35)$$

$$= \frac{1}{4} \sum_{j=0}^{n-1} {}^{n-1}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-1)-j} \quad (0.0.36)$$

$$= \frac{1}{4} \left(\frac{1}{4} + \frac{3}{4}\right)^{n-1} \quad (0.0.37)$$

$$= \frac{1}{4} \quad (0.0.38)$$

Using equations (0.0.32) and (0.0.38) in (0.0.21),

$$E\left(\left|Y_n - \frac{1}{4}\right|^2\right) = \frac{1}{16} + \frac{3}{16n} - \frac{1}{2} \times \frac{1}{4} + \frac{1}{16} \quad (0.0.39)$$

$$= \frac{3}{16n} \quad (0.0.40)$$

$$\lim_{n \rightarrow \infty} E\left(\left|Y_n - \frac{1}{4}\right|^2\right) = \lim_{n \rightarrow \infty} \frac{3}{16n} \quad (0.0.41)$$

$$= \frac{3}{16} \lim_{n \rightarrow \infty} \frac{1}{n} \quad (0.0.42)$$

$$= 0 \quad (0.0.43)$$

Thus, Y_n converges, in mean square, to $\frac{1}{4}$ and hence Y_n converges, in probability, to $\frac{1}{4}$.

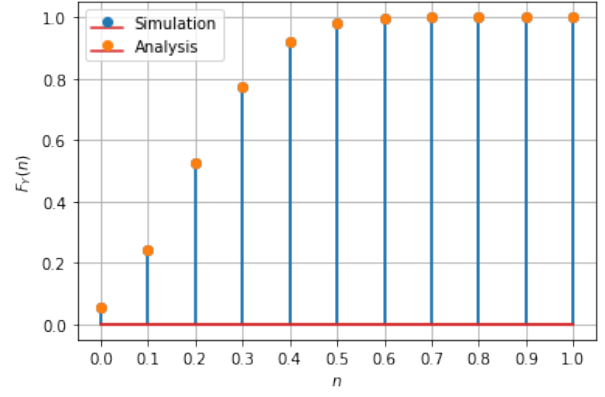


Fig. 0: The CDF distribution of Y_n for $n=10$

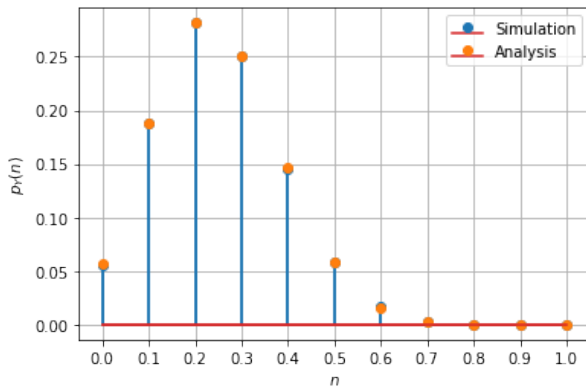


Fig. 0: The PMF distribution of Y_n for $n=10$