AI1103-Assignment-7

Name: Vikhyath Sai Kothamasu, Roll Number: CS20BTECH11056

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Download all python codes from

```
https://github.com/Vikhyath−vec/AI1103/tree/main/
    Assignment−7/codes
```

and latex-tikz codes from

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https://github.com/Vikhyath−vec/AI1103/blob/main
    /Assignment−7/Assignment−7.tex
```

Question

A fair die is thrown two times independently. Let X, Y be the outcomes of these two throws and $Z = X + Y$. Let U be the remainder obtained when Z is divided by 6. Then which of the following statement(s) is/are true?

1) X and Z are independent
2) X and U are independent
3) Z and U are independent
4) Y and Z are not independent

Solution

Let $X \in \{1, 2, 3, 4, 5, 6\}$ represent the random variable which represents the outcome of the first throw of a dice. Similarly, $Y \in \{1, 2, 3, 4, 5, 6\}$ represents the random variable which represents the outcome of the second throw of a dice.

$$n(X = i) = 1, \quad i \in \{1, 2, 3, 4, 5, 6\} \tag{0.0.1}$$

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

Similarly,

$$\Pr(Y = i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.3}$$

$$Z = X + Y \tag{0.0.4}$$

$$\text{Let } z \in \{1, 2, \ldots, 11, 12\} \tag{0.0.5}$$

$$\Pr(Z = z) = \Pr(X + Y = z) \tag{0.0.6}$$

$$= \sum_{x=0}^{z} \Pr(X = x) \Pr(Y = z - x) \tag{0.0.7}$$

$$= (6 - |z - 7|) \times \frac{1}{6} \times \frac{1}{6} \tag{0.0.8}$$

$$= \frac{6 - |z - 7|}{36} \tag{0.0.9}$$

$$\Pr(Z = z) = \begin{cases} \frac{6-|z-7|}{36} & z \in \{1, 2, \ldots, 11, 12\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.10}$$

U is the remainder obtained when Z is divided by 6.

$$\text{Let } u \in \{0,1,2,3,4,5\} \tag{0.0.11}$$

$$\Pr(U=u) = \sum_{k=0}^{2} \Pr(Z = 6k+u) \tag{0.0.12}$$

$$\Pr(U=0) = \Pr(Z=0) + \Pr(Z=6) + \Pr(Z=12) \tag{0.0.13}$$

$$= 0 + \frac{5}{36} + \frac{1}{36} = \frac{1}{6} \tag{0.0.14}$$

$$\text{for } u \in \{1,2,3,4,5\} \tag{0.0.15}$$

$$\Pr(U=u) = \Pr(Z = 0+u) + \Pr(Z = 6+u) \tag{0.0.16}$$

$$= \frac{6-|u-7|}{36} + \frac{6-|6+u-7|}{36} \tag{0.0.17}$$

$$= \frac{6-(7-u)}{36} + \frac{6-(u-1)}{36} \tag{0.0.18}$$

$$= \frac{u-1+7-u}{36} = \frac{6}{36} \tag{0.0.19}$$

$$= \frac{1}{6} \tag{0.0.20}$$

$$\Pr(U=u) = \begin{cases} \frac{1}{6} & u \in \{0,1,2,3,4,5\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.21}$$

Now, for checking each option,

1) Checking if X and Z are independent

$$p_1 = \Pr(Z=z, X=x) \tag{0.0.22}$$

$$= \Pr(Y = z-x, X=x) \tag{0.0.23}$$

$$= \Pr(Y = z-x) \times \Pr(X=x) \tag{0.0.24}$$

$$= \begin{cases} \frac{1}{36} & z-x \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.25}$$

$$\Pr(Z=z) \times \Pr(X=x) = \frac{6-|z-7|}{36} \times \frac{1}{6} \tag{0.0.26}$$

$$= \frac{6-|z-7|}{216} \tag{0.0.27}$$

$$\Pr(Z=z)\Pr(X=x) \neq \Pr(Z=z, X=x) \tag{0.0.28}$$

X and Z are not independent from (0.0.28) and hence option (1) is false.

2) Checking if X and U are independent

$$p_2 = \Pr(U=u, X=x) \tag{0.0.29}$$

$$p_2 = \Pr((Z=u) + (Z=6+u) + (Z=12+u), X=x) \tag{0.0.30}$$

$$p_2 = \Pr((Y=u-x) + (Y=6+u-x) + (Y=12+u-x), X=x) \tag{0.0.31}$$

$$p_2 = \frac{1}{6} \times \frac{1}{6} \tag{0.0.32}$$

$$= \frac{1}{36} \tag{0.0.33}$$

$$\Pr(U=u) \times \Pr(X=x) = \frac{1}{6} \times \frac{1}{6} \tag{0.0.34}$$

$$= \frac{1}{36} \tag{0.0.35}$$

$$\Pr(U=u)\Pr(X=x) = \Pr(U=u, X=x) \tag{0.0.36}$$

X and U are independent from (0.0.36) and hence option (2) is true.

3) Checking if Z and U are independent

$$p_3 = \Pr(Z=z|U=u) \tag{0.0.37}$$

$$p_3 = \begin{cases} 1 & u=1 \text{ and } z=7 \\ \frac{1}{2} & u=0 \text{ and } z \in \{6,12\} \\ \frac{1}{2} & u \in \{2,3,4,5\} \text{ and} \\ & z=u \text{ or } z=6+u \\ 0 & \text{otherwise} \end{cases} \tag{0.0.38}$$

$$\Pr(Z=z) = \frac{6-|z-7|}{36} \tag{0.0.39}$$

If Z and U are independent, then

$$\Pr(Z=z|U=u) = \frac{\Pr(Z=z, U=u)}{\Pr(U=u)} \tag{0.0.40}$$

$$= \frac{\Pr(Z=z)\Pr(U=u)}{\Pr(U=u)} \tag{0.0.41}$$

$$= \Pr(Z=z) \tag{0.0.42}$$

But,

$$\Pr(Z=z|U=u) \neq \Pr(Z=z) \tag{0.0.43}$$

X and U are not independent from (0.0.43) and hence option (3) is false.

4) Checking if Y and Z are independent

$$p_1 = \Pr(Z = z, Y = y) \tag{0.0.44}$$
$$= \Pr(X = z - y, Y = y) \tag{0.0.45}$$
$$= \Pr(X = z - y) \times \Pr(Y = y) \tag{0.0.46}$$
$$= \begin{cases} \frac{1}{36} & z - y \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{0.0.47}$$

$$\Pr(Z = z) \times \Pr(Y = y) = \frac{6 - |z - 7|}{36} \times \frac{1}{6} \tag{0.0.48}$$
$$= \frac{6 - |z - 7|}{216} \tag{0.0.49}$$
$$\Pr(Z = z) \Pr(Y = y) \neq \Pr(Z = z, Y = y) \tag{0.0.50}$$

X and Z are not independent from (0.0.50) and hence option (4) is true.

Thus, options (2) and (4) are true.