Convergence

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Question

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Let $\{X_j\}$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = \frac{1}{4}$ and let $Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$. Then Y_n converges, in probability, to ___________ .

Various Convergences

In probability theory, there exist several different notions of convergence of random variables.

1. Convergence in distribution or Converge weakly

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \forall x \in \mathbb{R} \tag{1}$$

2. Convergence in probability

$$\lim_{n\to\infty} \Pr\left(|X_n - X| \geq \epsilon\right) = 0 \tag{2}$$

3. Almost sure convergence

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1 \tag{3}$$

More convergences

4. Sure convergence or pointwise convergence

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall\omega \in \Omega \tag{4}$$

where Ω is the sample space of the underlying probability space over which the random variables are defined.

5. Convergence in mean. Given a real number $r \geq 1$

$$\lim_{n\to\infty} E(|X_n - X|^r) = 0 \tag{5}$$

Properties

There are several properties involving different types of convergence of which few are listed below

1. Almost sure convergence implies convergence in probability
2. Convergence in probability implies convergence in distribution
3. If X_n converges in distribution to a constant c, then X_n converges in probability to c
4. Convergence in r-th order mean implies convergence in lower order mean, assuming that both orders are greater than or equal to one
5. Convergence in the r-th mean, for $r \geq 1$, implies convergence in probability

For more properties, refer to the properties section in https://en.wikipedia.org/wiki/Convergence_of_random_variables

Proof of property (5)

Proof

$$\text{For any } \epsilon > 0, \tag{6}$$

$$\Pr\left(|Y_n - Y| \geq \epsilon\right) = \Pr\left(|Y_n - Y|^2 \geq \epsilon^2\right) \tag{7}$$

$$\Pr\left(|Y_n - Y| \geq \epsilon\right) \leq \frac{E|Y_n - Y|^2}{\epsilon^2} \text{ (by Markov's Inequality)} \tag{8}$$

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{9}$$

$$0 \leq \lim_{n\to\infty} \Pr\left(|Y_n - Y| \geq \epsilon\right) \leq \frac{0}{\epsilon^2} \tag{10}$$

$$\lim_{n\to\infty} \Pr\left(|Y_n - Y| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0 \tag{11}$$

Markov's Inequality

If X is a non-negative random variable and $a > 0$, then the probability that X is at least a is at most the expectation of X divided by a:

$$\Pr(X \geq a) \leq \frac{E(X)}{a} \tag{12}$$

Solution

$$\Pr\left(X_j = 1\right) = \frac{1}{4} \tag{13}$$

$$\Pr\left(X_j = 0\right) = 1 - \frac{1}{4} = \frac{3}{4} \tag{14}$$

$$Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2 \tag{15}$$

$$= \frac{1}{n}\sum_{j=1}^{n} X_j \tag{16}$$

$$\Pr\left(Y_n = y\right) = {}^{n}\mathrm{C}_{ny}\left(\frac{1}{4}\right)^{ny}\left(\frac{3}{4}\right)^{n-ny} \tag{17}$$

$$\Pr\left(Y_n = \frac{k}{n}\right) = {}^{n}\mathrm{C}_{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \tag{18}$$

Solution contd.

$$E\left(|Y_n - \frac{1}{4}|^2\right) = E\left(Y_n^2 - \frac{1}{2}Y_n + \frac{1}{16}\right) \tag{19}$$

$$= E(Y_n^2) - \frac{1}{2}E(Y_n) + \frac{1}{16} \tag{20}$$

$$E(Y_n^2) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \Pr\left(Y_n = \frac{k}{n}\right) \tag{21}$$

$$= \sum_{k=0}^{n} \left(\frac{k^2}{n^2}\right) {}^n\mathrm{C}_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \tag{22}$$

$$= \frac{1}{16} + \frac{3}{16n} \tag{23}$$

Solution contd.

$$E(Y_n) = \sum_{k=0}^{n} \frac{k}{n} \Pr\left(Y_n = \frac{k}{n}\right) \tag{24}$$

$$= \sum_{k=0}^{n} \left(\frac{k}{n}\right) {}^{n}\mathrm{C}_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \tag{25}$$

$$= \frac{1}{4} \tag{26}$$

Using equation (23),

$$E\left(|Y_n - \frac{1}{4}|^2\right) = \frac{1}{16} + \frac{3}{16n} - \frac{1}{2} \times \frac{1}{4} + \frac{1}{16} \tag{27}$$

$$= \frac{3}{16n} \tag{28}$$

Solution contd.

$$\lim_{n\to\infty} E\left(|Y_n - \frac{1}{4}|^2\right) = \lim_{n\to\infty} \frac{3}{16n} \tag{29}$$

$$= \frac{3}{16} \lim_{n\to\infty} \frac{1}{n} \tag{30}$$

$$= 0 \tag{31}$$

Thus, Y_n converges, in mean square, to $\frac{1}{4}$ and hence Y_n converges, in probability, to $\frac{1}{4}$.

Figures

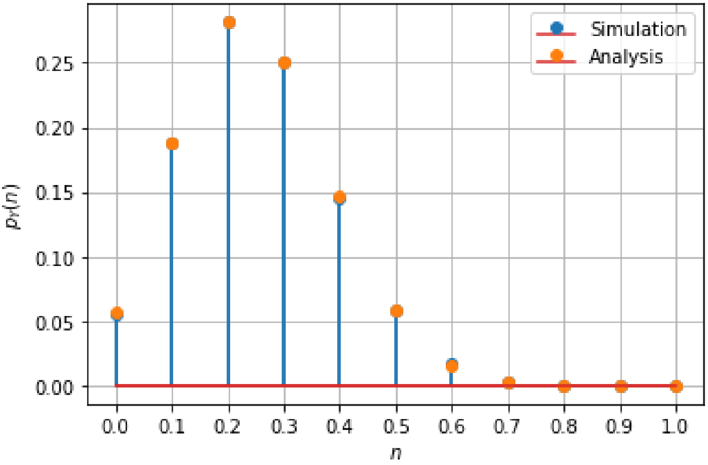

Figure: The PMF distribution of Y_n for n=10

Figures

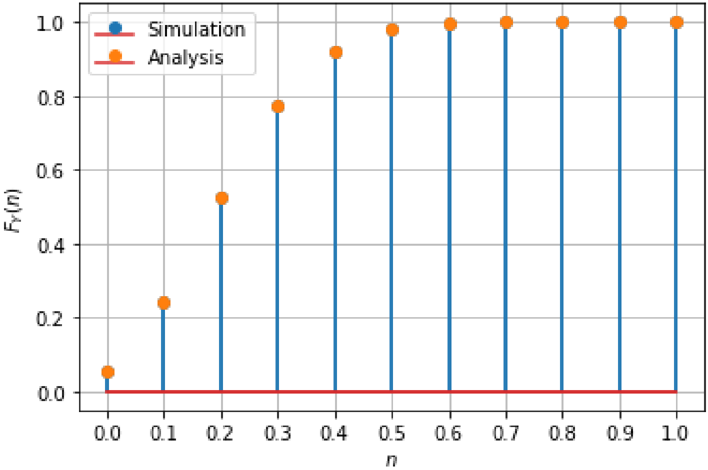

Figure: The CDF distribution of Y_n for n=10