AI1103-Assignment-4

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Download all python codes from

https://github.com/Vikhyath−vec/AI1103/tree/main/Assignment−4/codes

and latex-tikz codes from

https://github.com/Vikhyath−vec/AI1103/blob/main/Assignment−4/Assignment−4.tex

Question

Let $\{X_j\}$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = \frac{1}{4}$ and let $Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$. Then Y_n converges, in probability, to __________ .

Solution

A sequence of random variables $Y_1, Y_2, Y_3 \dots$ converges, in probability, to a random variable Y if

$$\lim_{n\to\infty} \Pr\left(|Y_n - Y| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0 \tag{0.0.1}$$

Similarly, a sequence of random variables $Y_1, Y_2, Y_3 \dots$ converges, in mean square, to a random variable Y if

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{0.0.2}$$

A random variable converges, in probability, to a value if it converges, in mean square, to the same particular value by Markov's Inequality. Proof for this is: For any $\epsilon > 0$

$$\Pr\left(|Y_n - Y| \geq \epsilon\right) = \Pr\left(|Y_n - Y|^2 \geq \epsilon^2\right) \tag{0.0.3}$$

$$\Pr\left(|Y_n - Y| \geq \epsilon\right) \leq \frac{E|Y_n - Y|^2}{\epsilon^2} \quad \text{(by Markov's Inequality)} \tag{0.0.4}$$

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{0.0.5}$$

$$0 \leq \lim_{n\to\infty} \Pr\left(|Y_n - Y| \geq \epsilon\right) \leq \frac{0}{\epsilon^2} \tag{0.0.6}$$

$$\lim_{n\to\infty} \Pr\left(|Y_n - Y| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0 \tag{0.0.7}$$

Given in the question that $\{X_j\}$ is a sequence of random variables with

$$\Pr\left(X_j = 1\right) = \frac{1}{4} \tag{0.0.8}$$

$$\Pr\left(X_j = 0\right) + \Pr\left(X_j = 1\right) = 1 \tag{0.0.9}$$

$$\Pr\left(X_j = 0\right) = 1 - \frac{1}{4} = \frac{3}{4} \tag{0.0.10}$$

$$X_j \in \{0, 1\} \tag{0.0.11}$$

Since $0^2 = 0$ and $1^2 = 1$,

$$X_j^2 = X_j \quad \forall j \in \{1, 2, \dots, n\} \tag{0.0.12}$$

Thus,

$$Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2 \tag{0.0.13}$$

$$= \frac{1}{n}\sum_{j=1}^{n} X_j \tag{0.0.14}$$

$$\Pr\left(Y_n = y\right) = {}^{n}C_{ny}\left(\frac{1}{4}\right)^{ny}\left(\frac{3}{4}\right)^{n-ny} \tag{0.0.15}$$

Let us assume

$$k = ny \tag{0.0.16}$$

$$k \in \{0, 1, 2, \dots, n-1, n\} \tag{0.0.17}$$

$$\Pr(Y_n = \frac{k}{n}) = {}^{n}C_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \tag{0.0.18}$$

$$E\left(|Y_n - \frac{1}{4}|^2\right) = E\left(Y_n^2 - \frac{1}{2}Y_n + \frac{1}{16}\right) \tag{0.0.19}$$

$$= E(Y_n^2) - \frac{1}{2}E(Y_n) + \frac{1}{16} \tag{0.0.20}$$

$$E(Y_n^2) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \Pr\left(Y_n = \frac{k}{n}\right) \tag{0.0.21}$$

$$= \sum_{k=0}^{n} \left(\frac{k^2}{n^2}\right) {}^{n}C_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \tag{0.0.22}$$

$$E(Y_n^2) = 0 + \frac{1}{n^2} \times n \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{n-1} + \sum_{k=2}^{n} \left(\frac{k}{n}\right)^2 \times \frac{n(n-1)}{k(k-1)} \times {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \tag{0.0.23}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \times \sum_{k=2}^{n} \left(\frac{k}{k-1}\right) {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \tag{0.0.24}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n}\left(\sum_{k=2}^{n} {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k}\right) + \frac{n-1}{n}\left(\sum_{k=2}^{n} \frac{1}{k-1}\, {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k}\right) \tag{0.0.25}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \times \frac{1}{16}\left(\sum_{k=2}^{n} {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^{k-2} \left(\frac{3}{4}\right)^{(n-2)-(k-2)}\right) + \frac{1}{n}\left(\sum_{k=2}^{n} \frac{n-1}{k-1}\, {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k}\right) \tag{0.0.26}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n}\left(\sum_{j=0}^{n-2} {}^{n-2}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-2)-j}\right) + \frac{1}{4n}\left(\sum_{k=2}^{n} {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^{k-1} \left(\frac{3}{4}\right)^{(n-1)-(k-1)}\right) \tag{0.0.27}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n}\left(\frac{1}{4} + \frac{3}{4}\right)^{n-2} + \frac{1}{4n}\left(\sum_{j=1}^{n-1} {}^{n-1}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-1)-j}\right) \tag{0.0.28}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n}\left(\left(\frac{1}{4} + \frac{3}{4}\right)^{n-1} - \left(\frac{3}{4}\right)^{n-1}\right) \tag{0.0.29}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n} - \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} \tag{0.0.30}$$

$$= \frac{1}{16} + \frac{3}{16n} \tag{0.0.31}$$

$$E(Y_n) = \sum_{k=0}^{n} \frac{k}{n} \Pr\left(Y_n = \frac{k}{n}\right) \tag{0.0.32}$$

$$= \sum_{k=0}^{n} \left(\frac{k}{n}\right) {}^{n}C_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \tag{0.0.33}$$

$$= 0 + \sum_{k=1}^{n} \frac{k}{n} \times \frac{n}{k} \times {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \tag{0.0.34}$$

$$= \frac{1}{4} \sum_{j=0}^{n-1} {}^{n-1}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-1)-j} \tag{0.0.35}$$

$$= \frac{1}{4}\left(\frac{1}{4} + \frac{3}{4}\right)^{n-1} \tag{0.0.36}$$

$$= \frac{1}{4} \tag{0.0.37}$$

Using equations (0.0.31) and (0.0.37) in (0.0.20),

$$E\left(|Y_n - \frac{1}{4}|^2\right) = \frac{1}{16} + \frac{3}{16n} - \frac{1}{2} \times \frac{1}{4} + \frac{1}{16} \quad (0.0.38)$$

$$= \frac{3}{16n} \quad (0.0.39)$$

$$\lim_{n\to\infty} E\left(|Y_n - \frac{1}{4}|^2\right) = \lim_{n\to\infty} \frac{3}{16n} \quad (0.0.40)$$

$$= \frac{3}{16} \lim_{n\to\infty} \frac{1}{n} \quad (0.0.41)$$

$$= 0 \quad (0.0.42)$$

Thus, Y_n converges, in mean square, to $\frac{1}{4}$ and hence Y_n converges, in probability, to $\frac{1}{4}$.

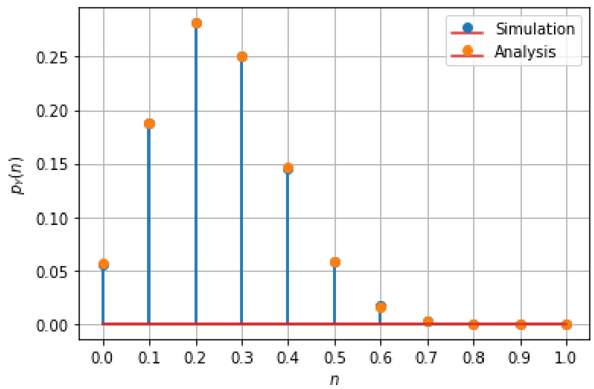

Fig. 0: The PMF distribution of Y_n for n=10

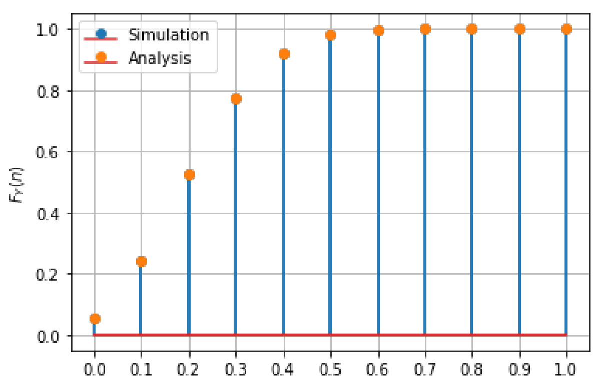

Fig. 0: The CDF distribution of Y_n for n=10