AI1103-Assignment-4

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Download all python codes from

```
https://github.com/Vikhyath−vec/AI1103/tree/main/
    Assignment−4/codes
```

and latex-tikz codes from

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https://github.com/Vikhyath−vec/AI1103/blob/main
    /Assignment−4/Assignment−4.tex
```

Question

Let $\{X_j\}$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = \frac{1}{4}$ and let $Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$. Then Y_n converges, in probability, to ________ .

Solution

A sequence of random variables $Y_1, Y_2, Y_3 \ldots$ converges, in probability, to a random variable Y if

$$\lim_{n\to\infty} \Pr\left(|Y_n - Y| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0 \tag{0.0.1}$$

Similarly, a sequence of random variables $Y_1, Y_2, Y_3 \ldots$ converges, in mean square, to a random variable Y if

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{0.0.2}$$

A random variable converges, in probability, to a value if it converges, in mean square, to the same particular value by Markov's Inequality.

$$\mathrm{Y}_n \xrightarrow{\mu_s} \mathrm{c} \Rightarrow \mathrm{Y}_n \xrightarrow{\mathrm{p}} \mathrm{c} \tag{0.0.3}$$

Proof for (0.0.3): For any $\epsilon > 0$

$$\Pr\left(|Y_n - Y| \geq \epsilon\right) = \Pr\left(|Y_n - Y|^2 \geq \epsilon^2\right) \tag{0.0.4}$$

$$\Pr\left(|Y_n - Y| \geq \epsilon\right) \leq \frac{E|Y_n - Y|^2}{\epsilon^2} \quad \text{(by Markov's Inequality)} \tag{0.0.5}$$

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{0.0.6}$$

$$0 \leq \lim_{n\to\infty} \Pr\left(|Y_n - Y| \geq \epsilon\right) \leq \frac{0}{\epsilon^2} \tag{0.0.7}$$

$$\lim_{n\to\infty} \Pr\left(|Y_n - Y| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0 \tag{0.0.8}$$

Given in the question that $\{X_j\}$ is a sequence of random variables with

$$\Pr\left(X_j = 1\right) = \frac{1}{4} \tag{0.0.9}$$

$$\Pr\left(X_j = 0\right) + \Pr\left(X_j = 1\right) = 1 \tag{0.0.10}$$

$$\Pr\left(X_j = 0\right) = 1 - \frac{1}{4} = \frac{3}{4} \tag{0.0.11}$$

$$X_j \in \{0, 1\} \tag{0.0.12}$$

Since $0^2 = 0$ and $1^2 = 1$,

$$X_j^2 = X_j \quad \forall j \in \{1, 2, \ldots, n\} \tag{0.0.13}$$

Thus,

$$Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2 \tag{0.0.14}$$

$$= \frac{1}{n}\sum_{j=1}^{n} X_j \tag{0.0.15}$$

$$= \frac{X_1 + X_2 + \dots X_n}{n} \tag{0.0.16}$$

For $\Pr(Y_n = y)$,

$$\frac{X_1 + X_2 + \dots X_n}{n} = y \tag{0.0.17}$$

$$X_1 + X_2 + \dots X_n = ny \tag{0.0.18}$$

$$ny \in \{0, 1, 2, \dots, n-1, n\} \tag{0.0.19}$$

$$y \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\} \tag{0.0.20}$$

For equation (0.0.18), the number of possible combinations is

$$= {}^{n}C_{ny} \tag{0.0.21}$$

Then,

$$\Pr(Y_n = y) = \sum_{x_1, x_2, \dots, x_n = 0}^{y} \Pr(X_1 = x_1, X_2 = x_2, \dots$$
$$X_{n-1} = x_{n-1}, X_n = y - x_1 - x_2 - \dots - x_{n-1}) \tag{0.0.22}$$

$$\Pr(Y_n = y) = {}^{n}C_{ny}\left(\frac{1}{4}\right)^{ny}\left(\frac{3}{4}\right)^{n-ny} \tag{0.0.23}$$

Let us assume

$$k = ny \tag{0.0.24}$$

$$k \in \{0, 1, 2, \dots, n-1, n\} \tag{0.0.25}$$

$$\Pr(Y_n = \frac{k}{n}) = {}^{n}C_{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \tag{0.0.26}$$

$$E(|Y_n - \frac{1}{4}|^2) = \sum_{k=0}^{n}\left(\frac{k}{n} - \frac{1}{4}\right)^2 \Pr(Y_n = \frac{k}{n}) \tag{0.0.27}$$

$$\begin{aligned} E(|Y_n - \frac{1}{4}|^2) &= \sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right){}^{n}C_{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \\ &- \sum_{k=0}^{n}\left(\frac{k}{2n}\right){}^{n}C_{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \\ &+ \sum_{k=0}^{n}\left(\frac{1}{16}\right){}^{n}C_{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \end{aligned} \tag{0.0.28}$$

$$\begin{aligned} E(|Y_n - \frac{1}{4}|^2) &= 0 + \frac{1}{n^2} \times n\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{n-1} \\ &+ \sum_{k=2}^{n}\left(\frac{k}{n}\right)^2 \times \frac{n(n-1)}{k(k-1)} \times {}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \\ &- 0 - \frac{1}{2}\sum_{k=1}^{n}\frac{k}{n} \times \frac{n}{k} \times {}^{n-1}C_{k-1}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \\ &+ \frac{1}{16}\left(\frac{1}{4} + \frac{3}{4}\right)^{n} \end{aligned} \tag{0.0.29}$$

$$\begin{aligned} E(|Y_n - \frac{1}{4}|^2) &= \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \\ &\times \sum_{k=2}^{n}\left(\frac{k}{k-1}\right){}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \\ &- \frac{1}{2} \times \frac{1}{4}\sum_{j=0}^{n-1}{}^{n-1}C_{j}\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{(n-1)-j} + \frac{1}{16} \end{aligned} \tag{0.0.30}$$

$$\begin{aligned} E(|Y_n - \frac{1}{4}|^2) = E(Y_n^2) &= \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} \\ &+ \frac{n-1}{n}\left(\sum_{k=2}^{n}{}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) \\ &+ \frac{n-1}{n}\left(\sum_{k=2}^{n}\frac{1}{k-1}\,{}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) \\ &- \frac{1}{8}\left(\frac{1}{4} + \frac{3}{4}\right)^{n-1} + \frac{1}{16} \end{aligned} \tag{0.0.31}$$

$$\begin{aligned} E(|Y_n - \frac{1}{4}|^2) &= \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} \\ &+ \frac{n-1}{n} \times \frac{1}{16}\left(\sum_{k=2}^{n}{}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k-2}\left(\frac{3}{4}\right)^{(n-2)-(k-2)}\right) \\ &+ \frac{1}{n}\left(\sum_{k=2}^{n}\frac{n-1}{k-1}\,{}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) - \frac{1}{16} \end{aligned} \tag{0.0.32}$$

$$E(|Y_n - \frac{1}{4}|^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n}\left(\sum_{j=0}^{n-2} {}^{n-2}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-2)-j}\right) + \frac{1}{4n}\left(\sum_{k=2}^{n} {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^{k-1} \left(\frac{3}{4}\right)^{(n-1)-(k-1)}\right) - \frac{1}{16} \tag{0.0.33}$$

$$E(|Y_n - \frac{1}{4}|^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n}\left(\frac{1}{4}+\frac{3}{4}\right)^{n-2} + \frac{1}{4n}\left(\sum_{j=1}^{n-1} {}^{n-1}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-1)-j}\right) - \frac{1}{16} \tag{0.0.34}$$

$$E(|Y_n - \frac{1}{4}|^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n}\left(\left((\frac{1}{4}+\frac{3}{4})\right)^{n-1} - \left(\frac{3}{4}\right)^{n-1}\right) - \frac{1}{16} \tag{0.0.35}$$

$$E(|Y_n - \frac{1}{4}|^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n} - \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} - \frac{1}{16} \tag{0.0.36}$$

$$E(|Y_n - \frac{1}{4}|^2) = \frac{n-1}{16n} + \frac{1}{4n} - \frac{1}{16} \tag{0.0.37}$$

$$= \frac{1}{16} + \frac{3}{16n} - \frac{1}{16} \tag{0.0.38}$$

$$= \frac{3}{16n} \tag{0.0.39}$$

$$\lim_{n\to\infty} E(|Y_n - \frac{1}{4}|^2) = \lim_{n\to\infty} \frac{3}{16n} \tag{0.0.40}$$

$$= \frac{3}{16}\lim_{n\to\infty}\frac{1}{n} \tag{0.0.41}$$

$$= 0 \tag{0.0.42}$$

Thus, Y_n converges, in mean square, to $\frac{1}{4}$ and hence Y_n converges, in probability, to $\frac{1}{4}$ using (0.0.3).

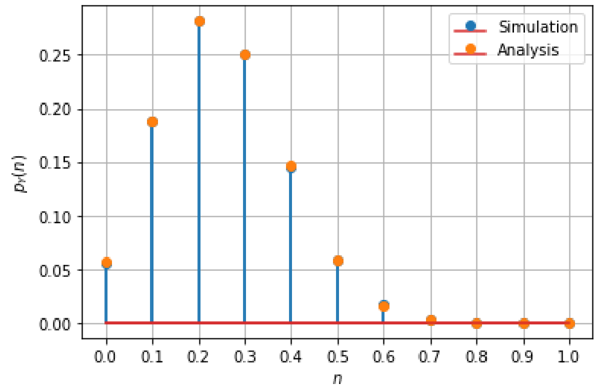

Fig. 0: The PMF distribution of Y_n for n=10

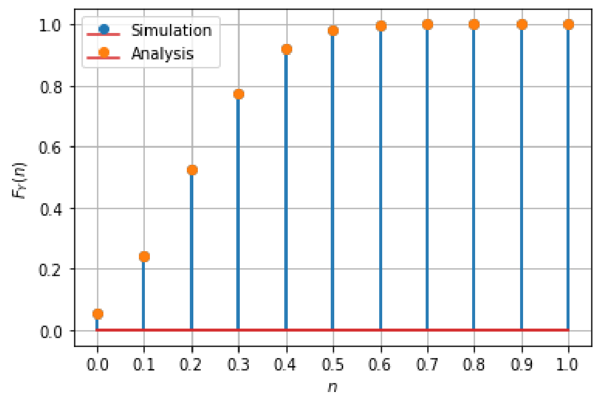

Fig. 0: The CDF distribution of Y_n for n=10