AI1103-Assignment-8

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https://github.com/Vikhyath−vec/AI1103/blob/main/Assignment−8/Assignment−8.tex

Question

Suppose X_1 and X_2 are independent and identically distributed random variables each following an exponential distribution with mean θ, i.e., the common pdf is given by $f_\theta(x) = \frac{1}{\theta}e^{\frac{-x}{\theta}}, 0 < x < \infty, 0 < \theta < \infty$. Then which of the following is true? Conditional distribution of X_2 given $X_1 + X_2 = t$ is

1) exponential with mean $\frac{t}{2}$ and hence $X_1 + X_2$ is sufficient for θ
2) exponential with mean $\frac{t\theta}{2}$ and hence $X_1 + X_2$ is not sufficient for θ
3) uniform$(0, t)$ and hence $X_1 + X_2$ is sufficient for θ
4) uniform$(0, t\theta)$ and hence $X_1 + X_2$ is not sufficient for θ

Solution

Let $f_{X_1,X_2}(x_1, x_2)$ denote the joint probability distribution of random variables X_1 and X_2. Let Z be another random variable such that $Z = X_1 + X_2$. Let $\Phi_{X_1}(\omega)$ and $\Phi_Z(\omega)$ be the characteristic functions of the probability density functions $f_{X_1}(x)$ and $f_Z(x)$ respectively. The conditional probability density function of X_2 can be defined by:

$$f_{X_2|(X_1+X_2=t)}(x_2) = \begin{cases} \frac{f_{X_1,X_2}(x_1,x_2)}{f_{(X_1+X_2)}(t)} & \text{if } x_1 + x_2 = t \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

$$x_1 + x_2 = t \tag{0.0.2}$$

$$0 < x_1, x_2 < \infty \tag{0.0.3}$$

$$x_1 = t - x_2 \tag{0.0.4}$$

$$t - x_2 > 0 \tag{0.0.5}$$

$$x_2 < t \tag{0.0.6}$$

From equations (0.0.3) and (0.0.6), we can conclude that $x_2 \in (0, t)$ if $x_1 + x_2 = t$. Also, given in the question,

$$0 < \theta < \infty \tag{0.0.7}$$

$$f_{X_1}(x_1) = \frac{1}{\theta}e^{\frac{-x_1}{\theta}}, 0 < x_1 < \infty \tag{0.0.8}$$

$$f_{X_2}(x_2) = \frac{1}{\theta}e^{\frac{-x_2}{\theta}}, 0 < x_2 < \infty \tag{0.0.9}$$

Since X_1 and X_2 are independent,

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) \times f_{X_2}(x_2) \tag{0.0.10}$$
$$= \frac{1}{\theta} e^{\frac{-x_1}{\theta}} \times \frac{1}{\theta} e^{\frac{-x_2}{\theta}} \tag{0.0.11}$$
$$= \frac{1}{\theta^2} e^{\frac{-(x_1+x_2)}{\theta}} \tag{0.0.12}$$
$$\Phi_{X_1}(\omega) = \frac{1}{\theta} \int_0^\infty e^{i\omega x} e^{\frac{-x}{\theta}} \, dx \tag{0.0.13}$$
$$= \frac{1}{\theta} \times \frac{1}{i\omega - \frac{1}{\theta}} \left(e^{x(i\omega - \frac{1}{\theta})} \right) \Big|_0^\infty \tag{0.0.14}$$
$$= \frac{1}{1 - i\omega\theta} - \frac{\lim_{x\to\infty} \left(e^{x(i\omega - \frac{1}{\theta})} \right)}{1 - i\omega\theta} \tag{0.0.15}$$
$$= \frac{1}{1 - i\omega\theta} - 0 = \frac{1}{1 - i\omega\theta} \tag{0.0.16}$$
$$\Phi_Z(\omega) = \left(\frac{1}{1 - i\omega\theta} \right)^2 \tag{0.0.17}$$
$$f_Z(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{\left(\frac{1}{1 - i\omega\theta} \right)^2} \, d\omega \tag{0.0.18}$$

The equation (0.0.18) is the characteristic function expression of a gamma random variable with k=2. Thus,

$$f_Z(x) = \frac{x^{k-1} e^{\frac{-x}{\theta}}}{\Gamma(k)\theta^k} \tag{0.0.19}$$
$$= \frac{x^{2-1} e^{\frac{-x}{\theta}}}{\Gamma(2)\theta^2} \tag{0.0.20}$$
$$= \frac{x e^{\frac{-x}{\theta}}}{\theta^2} \tag{0.0.21}$$

$$f_{X_2|(X_1+X_2=t)}(x_2) = \begin{cases} \frac{f_{X_1,X_2}(x_1,x_2)}{f_Z(t)} & x_2 \in (0,t) \\ 0 & \text{otherwise} \end{cases} \tag{0.0.22}$$

Let $x_2 \in (0,t)$.

$$f_{X_2|(X_1+X_2=t)}(x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_Z(t)} \tag{0.0.23}$$
$$= \frac{\frac{1}{\theta^2} e^{\frac{-(x_1+x_2)}{\theta}}}{\frac{1}{\theta^2} e^{\frac{-t}{\theta}} t} \tag{0.0.24}$$
$$= \frac{e^{\frac{-(t)}{\theta}}}{e^{\frac{-t}{\theta}} t} \tag{0.0.25}$$
$$= \frac{1}{t} \quad \forall x_2 \in (0,t) \tag{0.0.26}$$

The obtained pdf is uniform$(0,t)$. Any distribution is sufficient for underlying parameter θ if the conditional probability distribution of the data does not depend on the parameter θ. And since the conditional distribution of X_2 does not depend on θ for any value of t, $X_1 + X_2$ is sufficient for θ. Verifying the pdf,

$$\text{total probability} = \int_0^t f_{X_2|(X_1+X_2=t)}(x_2) \, dx_2 \tag{0.0.27}$$
$$= \int_0^t \frac{1}{t} \, dx_2 \tag{0.0.28}$$
$$= 1 \tag{0.0.29}$$

Hence, the correct answer is option (3)