# AI1103-Assignment-4

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Download all python codes from

https://github.com/Vikhyath−vec/AI1103/tree/main/Assignment−4/codes

and latex-tikz codes from

https://github.com/Vikhyath−vec/AI1103/blob/main/Assignment−4/Assignment−4.tex

## Question

Let $\{X_j\}$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = \frac{1}{4}$ and let $Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$. Then $Y_n$ converges, in probability, to ________ .

## Solution

A random variable converges, in probability, to a value if it converges, in mean square, to the same particular value by Markov's Inequality.

$$\mathrm{Y}_n \xrightarrow{\mu_s} \mathrm{c} \Rightarrow \mathrm{Y}_n \xrightarrow{\mathrm{p}} \mathrm{c} \tag{0.0.1}$$

Proof for (0.0.1): For any $\epsilon > 0$

$$\Pr(|Y_n - Y| \geq \epsilon) = \Pr(|Y_n - Y|^2 \geq \epsilon^2) \tag{0.0.2}$$

$$\Pr(|Y_n - Y| \geq \epsilon) \leq \frac{E|Y_n - Y|^2}{\epsilon^2}$$
$$\text{(by Markov's Inequality)} \tag{0.0.3}$$

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{0.0.4}$$

$$0 \leq \lim_{n\to\infty} \Pr(|Y_n - Y| \geq \epsilon) \leq \frac{0}{\epsilon^2} \tag{0.0.5}$$

$$\lim_{n\to\infty} \Pr(|Y_n - Y| \geq \epsilon) = 0 \quad \forall \epsilon > 0 \tag{0.0.6}$$

And for a random variable to converge, in mean square, the required conditions are:

$$\lim_{n\to\infty} E(Y_n) = c \text{ for some constant c} \in \mathbb{R} \tag{0.0.7}$$

$$\lim_{n\to\infty} \sigma^2 = 0 \tag{0.0.8}$$

Given in the question that $\{X_j\}$ is a sequence of random variables with

$$\Pr(X_j = 1) = \frac{1}{4} \tag{0.0.9}$$

$$\Pr(X_j = 0) + \Pr(X_j = 1) = 1 \tag{0.0.10}$$

$$\Pr(X_j = 0) = 1 - \frac{1}{4} = \frac{3}{4} \tag{0.0.11}$$

$$X_j \in \{0, 1\} \tag{0.0.12}$$

Since $0^2 = 0$ and $1^2 = 1$,

$$X_j^2 = X_j \quad \forall j \in \{1, 2, \ldots, n\} \tag{0.0.13}$$

Thus,

$$Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2 \tag{0.0.14}$$

$$= \frac{1}{n}\sum_{j=1}^{n} X_j \tag{0.0.15}$$

$$= \frac{X_1 + X_2 + \ldots X_n}{n} \tag{0.0.16}$$

For $\Pr(Y_n = y)$,

$$\frac{X_1 + X_2 + \dots X_n}{n} = y \tag{0.0.17}$$
$$X_1 + X_2 + \dots X_n = ny \tag{0.0.18}$$
$$ny \in \{0, 1, 2, \dots, n-1, n\} \tag{0.0.19}$$
$$y \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\} \tag{0.0.20}$$

For equation (0.0.18), the number of possible combinations is

$$= {}^nC_{ny} \tag{0.0.21}$$

Then,

$$\Pr(Y_n = y) = \sum_{x_1, x_2, \dots, x_n = 0}^{y} \Pr(X_1 = x_1, X_2 = x_2, \dots$$
$$X_{n-1} = x_{n-1}, X_n = y - x_1 - x_2 - \dots - x_{n-1}) \tag{0.0.22}$$

$$\Pr(Y_n = y) = {}^nC_{ny} \left(\frac{1}{4}\right)^{ny} \left(\frac{3}{4}\right)^{n-ny} \tag{0.0.23}$$

Let us assume

$$k = ny \Rightarrow k \in \{0, 1, 2, \dots, n-1, n\} \tag{0.0.24}$$

$$E(Y_n) = \sum_{y=0}^{1} y \times \Pr(Y_n = y) \tag{0.0.25}$$
$$= \sum_{y=0}^{1} y \times {}^nC_{ny} \left(\frac{1}{4}\right)^{ny} \left(\frac{3}{4}\right)^{n-ny} \tag{0.0.26}$$
$$= \sum_{k=0}^{n} \frac{k}{n} \times {}^nC_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{0.0.27}$$
$$= 0 + \sum_{k=1}^{n} \frac{k}{n} \times \frac{n}{k} \times {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{0.0.28}$$
$$= \sum_{k=1}^{n} {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{0.0.29}$$
$$= \frac{1}{4} \sum_{k=1}^{n} {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^{k-1} \left(\frac{3}{4}\right)^{(n-1)-(k-1)} \tag{0.0.30}$$
$$= \frac{1}{4} \sum_{j=0}^{n-1} {}^{n-1}C_{j} \left(\frac{1}{4}\right)^{j} \left(\frac{3}{4}\right)^{(n-1)-j} \tag{0.0.31}$$
$$= \frac{1}{4} \left(\frac{1}{4} + \frac{3}{4}\right)^{n-1} = \frac{1}{4} \tag{0.0.32}$$

$$\lim_{n\to\infty} E(Y_n) = \lim_{n\to\infty} \frac{1}{4} \tag{0.0.33}$$
$$= \frac{1}{4} \tag{0.0.34}$$

$$\sigma^2 = \sum_{y=0}^{1} (y - E(Y_n))^2 \times \Pr(Y_n = y) \tag{0.0.35}$$
$$= \sum_{y=0}^{1} (y - \frac{1}{4})^2 \times {}^nC_{ny} \left(\frac{1}{4}\right)^{ny} \left(\frac{3}{4}\right)^{n-ny} \tag{0.0.36}$$
$$= \sum_{k=0}^{n} (\frac{k}{n} - \frac{1}{4})^2 \times {}^nC_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{0.0.37}$$

$$\sigma^2 = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \times {}^nC_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k}$$
$$- \frac{1}{2} \sum_{k=0}^{n} \frac{k}{n} \times {}^nC_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k}$$
$$+ \frac{1}{16} \sum_{k=0}^{n} {}^nC_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{0.0.38}$$

$$\sigma^2 = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \times {}^nC_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} - \frac{1}{2} \times \frac{1}{4} + \frac{1}{16} \tag{0.0.39}$$

$$\sigma^2 = 0 + \left(\frac{1}{n}\right)^2 \times {}^nC_1 \left(\frac{1}{4}\right)^{1} \left(\frac{3}{4}\right)^{n-1}$$
$$+ \sum_{k=2}^{n} \left(\frac{k}{n}\right)^2 \times {}^nC_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} - \frac{1}{8} \tag{0.0.40}$$

$$\sigma^2 = \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} - \frac{1}{8}$$
$$+ \sum_{k=2}^{n} \left(\frac{k}{n}\right)^2 \times \frac{n(n-1)}{k(k-1)} {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{0.0.41}$$

$$\lim_{n\to\infty} \sigma^2 = \lim_{n\to\infty} \left(\frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} - \frac{1}{8}\right)$$
$$+ \lim_{n\to\infty} \left(\sum_{k=2}^{n} \left(\frac{k}{n}\right) \times \frac{(n-1)}{(k-1)} {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k}\right) \tag{0.0.42}$$

$$\lim_{n\to\infty} \sigma^2 = 0 - \frac{1}{8} + \frac{1}{8} \tag{0.0.43}$$
$$= 0 \tag{0.0.44}$$

To summarise,

$$\lim_{n\to\infty} E(Y_n) = \frac{1}{4} \tag{0.0.45}$$
$$\lim_{n\to\infty} \sigma^2 = 0 \tag{0.0.46}$$

Thus, $Y_n$ converges, in probability, to 0.25.

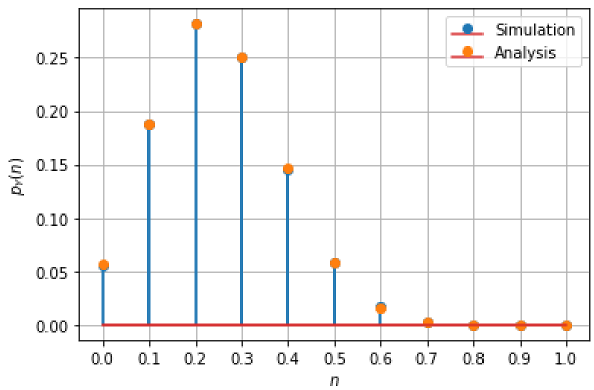


Fig. 0: The PMF distribution of $Y_n$ for n=10

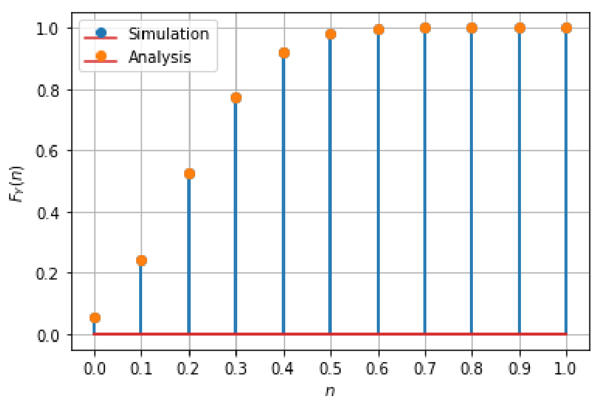


Fig. 0: The CDF distribution of $Y_n$ for n=10