# AI1103-Assignment-8

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### QUESTION

Suppose $X_1$ and $X_2$ are independent and identically distributed random variables each following an exponential distribution with mean $\theta$, i.e., the common pdf is given by $f_\theta(x) = \frac{1}{\theta}e^{\frac{-x}{\theta}}, 0 < x < \infty, 0 < \theta < \infty$. Then which of the following is true? Conditional distribution of $X_2$ given $X_1 + X_2 = t$ is

1) exponential with mean $\frac{t}{2}$ and hence $X_1 + X_2$ is sufficient for $\theta$
2) exponential with mean $\frac{t\theta}{2}$ and hence $X_1 + X_2$ is not sufficient for $\theta$
3) uniform$(0, t)$ and hence $X_1 + X_2$ is sufficient for $\theta$
4) uniform$(0, t\theta)$ and hence $X_1 + X_2$ is not sufficient for $\theta$

### SOLUTION

Let $f_{X_1,X_2}(x_1, x_2)$ denote the joint probability distribution of random variables $X_1$ and $X_2$. Let $Z$ be another random variable such that $Z = X_1 + X_2$. Also, given in the question,

$$0 < \theta < \infty \tag{0.0.1}$$

$$f_{X_1}(x_1) = \frac{1}{\theta}e^{\frac{-x_1}{\theta}}, 0 < x_1 < \infty \tag{0.0.2}$$

$$f_{X_2}(x_2) = \frac{1}{\theta}e^{\frac{-x_2}{\theta}}, 0 < x_2 < \infty \tag{0.0.3}$$

Since $X_1$ and $X_2$ are independent,

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) \times f_{X_2}(x_2) \tag{0.0.4}$$

$$= \frac{1}{\theta}e^{\frac{-x_1}{\theta}} \times \frac{1}{\theta}e^{\frac{-x_2}{\theta}} \tag{0.0.5}$$

$$= \frac{1}{\theta^2}e^{\frac{-(x_1+x_2)}{\theta}} \tag{0.0.6}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(z - x_2)f_{X_2}(x_2)\, dx_2 \tag{0.0.7}$$

$$= \int_{0}^{z} f_{X_1}(z - x_2)f_{X_2}(x_2)\, dx_2 \tag{0.0.8}$$

$$= \int_{0}^{z} f_{X_1,X_2}(z - x_2, x_2)\, dx_2 \tag{0.0.9}$$

$$= \int_{0}^{z} \frac{1}{\theta^2}e^{\frac{-(z-x_2+x_2)}{\theta}}\, dx_2 \tag{0.0.10}$$

$$= \frac{1}{\theta^2} \int_{0}^{z} e^{\frac{-z}{\theta}}\, dx_2 \tag{0.0.11}$$

$$= \frac{1}{\theta^2}e^{\frac{-z}{\theta}}z \tag{0.0.12}$$

$$f_{X_2|(X_1+X_2=t)}(x_2) = \begin{cases} \frac{f_{X_1,X_2}(x_1,x_2)}{f_Z(t)} & x_2 \in [0, t] \\ 0 & \text{otherwise} \end{cases} \tag{0.0.13}$$

Let $x_2 \in [0, t]$.

$$f_{X_2|(X_1+X_2=t)}(x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_Z(t)} \tag{0.0.14}$$

$$= \frac{\frac{1}{\theta^2}e^{\frac{-(x_1+x_2)}{\theta}}}{\frac{1}{\theta^2}e^{\frac{-t}{\theta}}t} \tag{0.0.15}$$

$$= \frac{e^{\frac{-(t)}{\theta}}}{e^{\frac{-t}{\theta}}t} \tag{0.0.16}$$

$$= \frac{1}{t} \quad \forall x_2 \in [0, t] \tag{0.0.17}$$

The obtained pdf is uniform$(0, t)$. And since the conditional distribution of $X_2$ does not depend on $\theta$ for any value of $t$, $X_1 + X_2$ is sufficient for $\theta$. Verifying the pdf,

$$\text{total probability} = \int_0^t f_{X_2|(X_1+X_2=t)}(x_2)\, dx_2 \tag{0.0.18}$$

$$= \int_0^t \frac{1}{t}\, dx_2 \tag{0.0.19}$$

$$= 1 \tag{0.0.20}$$

Hence, the correct answer is option (3)