

AI1103-Assignment-4

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Download all python codes from

<https://github.com/Vikhyath-vec/AI1103/tree/main/Assignment-4/codes>

and latex-tikz codes from

<https://github.com/Vikhyath-vec/AI1103/blob/main/Assignment-4/Assignment-4.tex>

QUESTION

Let $\{X_j\}$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = \frac{1}{4}$ and let $Y_n = \frac{1}{n} \sum_{j=1}^n X_j^2$. Then Y_n converges, in probability, to _____.

SOLUTION

Given in the question that $\{X_j\}$ is a sequence of random variables with

$$\Pr(X_j = 1) = \frac{1}{4} \quad (0.0.1)$$

$$\Pr(X_j = 0) + \Pr(X_j = 1) = 1 \quad (0.0.2)$$

$$\Pr(X_j = 0) = 1 - \frac{1}{4} = \frac{3}{4} \quad (0.0.3)$$

$$X_j \in \{0, 1\} \quad (0.0.4)$$

Since $0^2 = 0$ and $1^2 = 1$,

$$X_j^2 = X_j \quad \forall j \in \{1, 2, \dots, n\} \quad (0.0.5)$$

Thus,

$$Y_n = \frac{1}{n} \sum_{j=1}^n X_j^2 \quad (0.0.6)$$

$$= \frac{1}{n} \sum_{j=1}^n X_j \quad (0.0.7)$$

$$= \frac{X_1 + X_2 + \dots + X_n}{n} \quad (0.0.8)$$

For $\Pr(Y_n = y)$,

$$\frac{X_1 + X_2 + \dots + X_n}{n} = y \quad (0.0.9)$$

$$X_1 + X_2 + \dots + X_n = ny \quad (0.0.10)$$

$$ny \in \{0, 1, 2, \dots, n-1, n\} \quad (0.0.11)$$

$$y \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\} \quad (0.0.12)$$

For equation 0.0.10,

the number of possible combinations is =

$${}^nC_{ny} \quad (0.0.13)$$

Then,

$$\Pr(Y_n = y) = \sum_{x_1, x_2, \dots, x_n=0}^y \Pr(X_1 = x_1, X_2 = x_2, \dots$$

$$X_{n-1} = x_{n-1}, X_n = y - x_1 - x_2 - \dots - x_{n-1}) \quad (0.0.14)$$

$$\Pr(Y_n = y) = {}^nC_{ny} \left(\frac{1}{4}\right)^{ny} \left(\frac{3}{4}\right)^{n-ny} \quad (0.0.15)$$

Let us assume

$$k = ny \quad (0.0.16)$$

$$k \in \{0, 1, 2, \dots, n-1, n\} \quad (0.0.17)$$

$$E(Y_n) = \sum_{y=0}^1 y \times \Pr(Y_n = y) \quad (0.0.18)$$

$$= \sum_{y=0}^1 y \times {}^nC_{ny} \left(\frac{1}{4}\right)^{ny} \left(\frac{3}{4}\right)^{n-ny} \quad (0.0.19)$$

$$= \sum_{k=0}^n \frac{k}{n} \times {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.20)$$

$$= 0 + \sum_{k=1}^n \frac{k}{n} \times \frac{n}{k} \times {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.21)$$

$$= \sum_{k=1}^n {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.22)$$

$$= \frac{1}{4} \sum_{k=1}^n {}^{n-1}C_{k-1} \left(\frac{1}{4}\right)^{k-1} \left(\frac{3}{4}\right)^{(n-1)-(k-1)} \quad (0.0.23)$$

$$= \frac{1}{4} \sum_{j=0}^{n-1} {}^{n-1}C_j \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{(n-1)-j} \quad (0.0.24)$$

$$= \frac{1}{4} \left(\frac{1}{4} + \frac{3}{4}\right)^{n-1} \quad (0.0.25)$$

$$= \frac{1}{4} \quad (0.0.26)$$

$$\lim_{n \rightarrow \infty} E(Y_n) = \lim_{n \rightarrow \infty} \frac{1}{4} \quad (0.0.27)$$

$$= \frac{1}{4} \quad (0.0.28)$$

$$\sigma^2 = \sum_{y=0}^1 (y - E(Y_n))^2 \times \Pr(Y_n = y) \quad (0.0.29)$$

$$= \sum_{y=0}^1 (y - \frac{1}{4})^2 \times {}^nC_{ny} \left(\frac{1}{4}\right)^{ny} \left(\frac{3}{4}\right)^{n-ny} \quad (0.0.30)$$

$$= \sum_{k=0}^n \left(\frac{k}{n} - \frac{1}{4}\right)^2 \times {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \quad (0.0.31)$$

$$\begin{aligned} \sigma^2 &= \sum_{k=0}^n \left(\frac{k}{n}\right)^2 \times {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \\ &\quad - \frac{1}{2} \sum_{k=0}^n \frac{k}{n} \times {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \\ &\quad + \frac{1}{16} \sum_{k=0}^n {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \end{aligned} \quad (0.0.32)$$

$$\sigma^2 = \sum_{k=0}^n \left(\frac{k}{n}\right)^2 \times {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} - \frac{1}{2} \times \frac{1}{4} + \frac{1}{16} \quad (0.0.33)$$

$$\begin{aligned} \sigma^2 &= 0 + \left(\frac{1}{n}\right)^2 \times {}^nC_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{n-1} \\ &\quad + \sum_{k=2}^n \left(\frac{k}{n}\right)^2 \times {}^nC_k \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} - \frac{1}{8} \end{aligned} \quad (0.0.34)$$

$$\begin{aligned} \sigma^2 &= \frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} - \frac{1}{8} \\ &\quad + \sum_{k=2}^n \left(\frac{k}{n}\right)^2 \times \frac{n(n-1)}{k(k-1)} {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \end{aligned} \quad (0.0.35)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sigma^2 &= \lim_{n \rightarrow \infty} \left(\frac{1}{4n} \left(\frac{3}{4}\right)^{n-1} - \frac{1}{8} \right) \\ &\quad + \lim_{n \rightarrow \infty} \left(\sum_{k=2}^n \left(\frac{k}{n}\right)^2 \times \frac{(n-1)}{(k-1)} {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} \right) \end{aligned} \quad (0.0.36)$$

$$\lim_{n \rightarrow \infty} \sigma^2 = 0 - \frac{1}{8} + \frac{1}{8} \quad (0.0.37)$$

$$= 0 \quad (0.0.38)$$

To summarise,

$$\lim_{n \rightarrow \infty} E(Y_n) = \frac{1}{4} \quad (0.0.39)$$

$$\lim_{n \rightarrow \infty} \sigma^2 = 0 \quad (0.0.40)$$

Thus, Y_n converges, in probability, to 0.25.

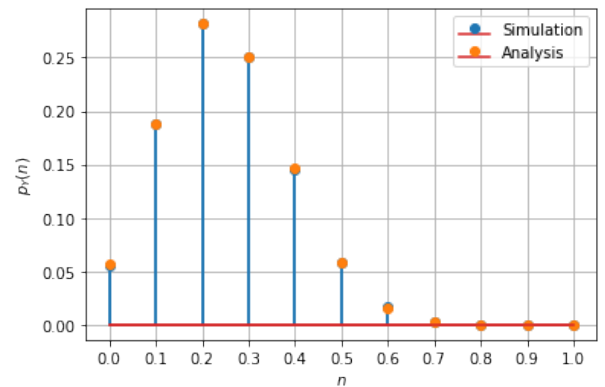


Fig. 0: The PMF distribution of Y_n for $n=10$

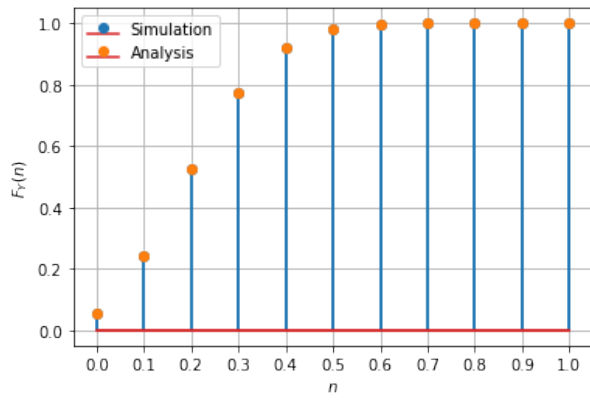


Fig. 0: The CDF distribution of Y_n for $n=10$