AI1103-Assignment-8

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Download all latex-tikz codes from

https://github.com/Vikhyath−vec/AI1103/blob/main/Assignment−8/Assignment−8.tex

Question

Suppose X_1 and X_2 are independent and identically distributed random variables each following an exponential distribution with mean θ, i.e., the common pdf is given by $f_\theta(x) = \frac{1}{\theta}e^{\frac{-x}{\theta}}, 0 < x < \infty, 0 < \theta < \infty$. Then which of the following is true? Conditional distribution of X_2 given $X_1 + X_2 = t$ is

1) exponential with mean $\frac{t}{2}$ and hence $X_1 + X_2$ is sufficient for θ
2) exponential with mean $\frac{t\theta}{2}$ and hence $X_1 + X_2$ is not sufficient for θ
3) uniform$(0, t)$ and hence $X_1 + X_2$ is sufficient for θ
4) uniform$(0, t\theta)$ and hence $X_1 + X_2$ is not sufficient for θ

Solution

Let $f_{X_1,X_2}(x1, x2)$ denote the joint probability distribution of random variables X_1 and X_2. Let Z be another random variable such that $Z = X_1 + X_2$. Also, given in the question,

$$0 < \theta < \infty \tag{0.0.1}$$
$$f_{X_1}(x_1) = \frac{1}{\theta}e^{\frac{-x_1}{\theta}}, 0 < x_1 < \infty \tag{0.0.2}$$
$$f_{X_2}(x_2) = \frac{1}{\theta}e^{\frac{-x_2}{\theta}}, 0 < x_2 < \infty \tag{0.0.3}$$

Since X_1 and X_2 are independent,

$$\begin{aligned}
f_{X_1,X_2}(x1, x2) &= f_{X_1}(x_1) \times f_{X_2}(x_2) && (0.0.4)\\
&= \frac{1}{\theta}e^{\frac{-x_1}{\theta}} \times \frac{1}{\theta}e^{\frac{-x_2}{\theta}} && (0.0.5)\\
&= \frac{1}{\theta^2}e^{\frac{-(x_1+x_2)}{\theta}} && (0.0.6)\\
f_Z(z) &= \int_{-\infty}^{\infty} f_{X_1}(z - x_2) f_{X_2}(x_2)\, dx_2 && (0.0.7)\\
&= \int_0^z f_{X_1}(z - x_2) f_{X_2}(x_2)\, dx_2 && (0.0.8)\\
&= \int_0^z f_{X_1,X_2}(z - x_2, x_2)\, dx_2 && (0.0.9)\\
&= \int_0^z \frac{1}{\theta^2}e^{\frac{-(z-x_2+x_2)}{\theta}}\, dx_2 && (0.0.10)\\
&= \frac{1}{\theta^2}\int_0^z e^{\frac{-z}{\theta}}\, dx_2 && (0.0.11)\\
&= \frac{1}{\theta^2}e^{\frac{-z}{\theta}} z && (0.0.12)
\end{aligned}$$

$$f_{X_2|(X_1+X_2=t)}(x_2) = \begin{cases} \frac{f_{X_1,X_2}(x1,x2)}{f_Z(t)} & x_2 \in [0, t] \\ 0 & \text{otherwise} \end{cases} \tag{0.0.13}$$

Let $x_2 \in [0, t]$.

$$f_{X_2|(X_1+X_2=t)}(x_2) = \frac{f_{X_1,X_2}(x1, x2)}{f_Z(t)} \tag{0.0.14}$$

$$= \frac{\frac{1}{\theta^2} e^{\frac{-(x_1+x_2)}{\theta}}}{\frac{1}{\theta^2} e^{\frac{-t}{\theta}} t} \tag{0.0.15}$$

$$= \frac{e^{\frac{-(t)}{\theta}}}{e^{\frac{-t}{\theta}} t} \tag{0.0.16}$$

$$= \frac{1}{t} \quad \forall x_2 \in [0, t] \tag{0.0.17}$$

The obtained pdf is uniform$(0, t)$. And since the conditional distribution of X_2 does not depend on θ for any value of t, $X_1 + X_2$ is sufficient for θ. Verifying the pdf,

$$\text{total probability} = \int_0^t f_{X_2|(X_1+X_2=t)}(x_2)\, dx_2 \tag{0.0.18}$$

$$= \int_0^t \frac{1}{t}\, dx_2 \tag{0.0.19}$$

$$= 1 \tag{0.0.20}$$

Hence, the correct answer is option (3)