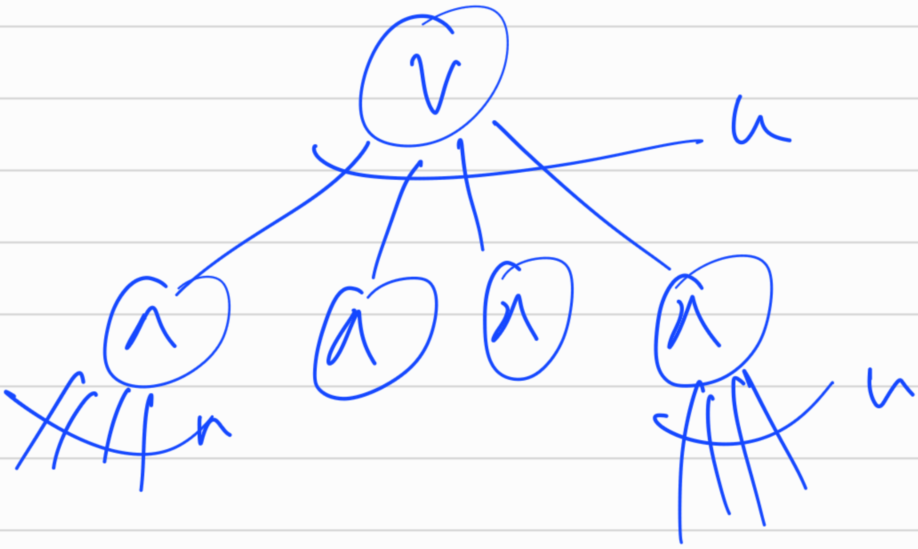
28/08/23

Thm :-

$$C(f) \le D(f) \le C^0(f)\cdot C^1(f) \le C(f)^2 .$$

tight.

Example : $\text{Tribes}_{n,n}$



Sensitivity

Remember the hypercube graph.

$f : \{0,1\}^n \to \{0,1\}$

labelling of hypercube graph with 0 and 1.

Sensitivity of f at an input $x \in \{0,1\}^n$ is

$$\left| \left\{ y \;\middle|\; \begin{array}{l} f(y) \neq f(x) \\ \text{and } y \text{ is a nbr of } x \\ \text{in the graph} \end{array} \right\} \right|.$$

denoted $S(f,x)$

$$s(f,x) = \left| \{ i \in [n] \mid f(x) \neq f(x^i) \} \right|$$

where x^i is x with i-th bit flipped.

Sensitivity of f, $s(f)$,

$$s(f) := \max_x s(f,x)$$

Example:

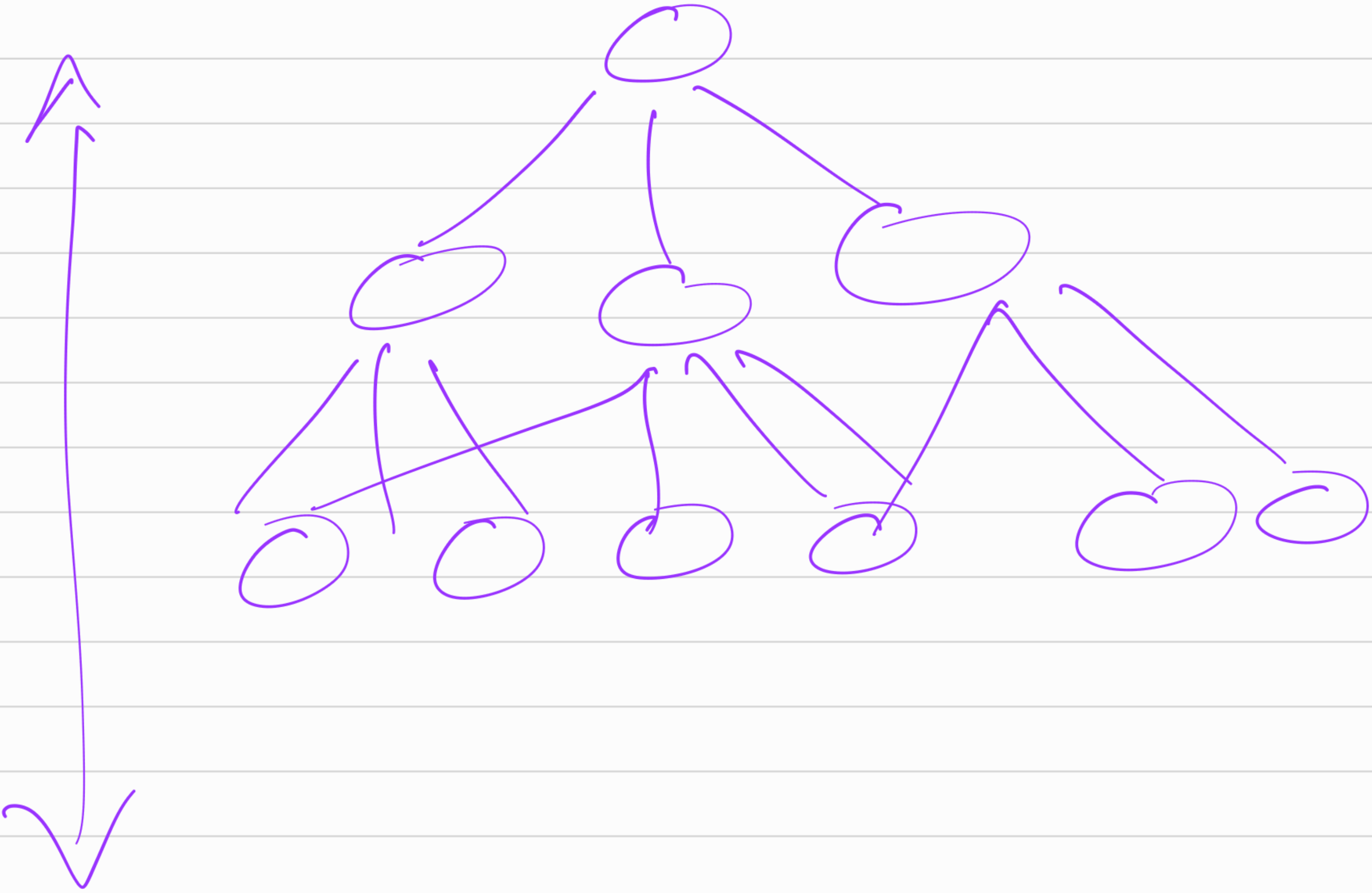
$$s(AND_n) = s(AND_n, 1^n)$$

$$= n.$$

$$s(OR_n) = n$$

$$s(\oplus_n) = n$$

"CREW PRAM"



block sensitivity

a block $B \subseteq [n]$ is

a sensitive block of

the input x if

$$f(x) \neq f(x^B)$$

where x^B represents the input x with indices in B flipped.

$\{1\}$ is a sensitive block for OR_n at 0^n.

$bs(f, x) :=$ the largest number t s.t. there are disjoint blocks $B_1, B_2, \ldots, B_t$

s.t. $f(x) \neq f(x^{B_i})$ $\forall i$.

Prop:- $s(f,x) \leq bs(f,x)$.

$\blacksquare$

Prop:- $s(f,x) \leq bs(f,x) \leq C(f,x)$

Claim:- $s(f,x) \leq C(f,x)$.

Proof:- We will in fact show that any certificate queries the set of sensitive bits.

Suppose not.

then $\exists$ a sensitive bit

$i \in [n]$ s.t. this certificate leaves i free.

Let us call this certificate (S, α).

$\Rightarrow$ x and x^i both are consistent with (S, α).

But $f(x) \neq f(x^i)$.

$\Rightarrow$ (S, α) is not a certificate

Hence a contradiction.

Prop :- $bs(f, x) \leq C(f, x)$.

Proof :- certificate should query at least one bit from each sensitive block.

$\blacksquare$

Thm :- $s(f, x) \leq bs(f, x) \leq C(f, x)$
$\leq D(f)$

Thm :- Consider the set S of variables that appear in some sensitive block at x w.r.t. f.

Then S is a certificate for f at x.

Claim:- Any input y consistent with S at x must satisfy $f(y) = f(x)$.

Proof:- Suppose not. Then $\exists$ another input y' that is consistent x over S

$\Rightarrow \exists$ a block B' disjoint from S.

s.t. $x^{B'} = y'$

and $f(x^{B'}) \neq f(y')$.

$\Rightarrow$ a contradiction to the way you built the set S.

$\blacksquare$

Q:- What is the $|S|$?

<u>Claim</u>:- $|S| \leq s(f) \cdot bs(f)$?

$\leq s(f,x) \cdot bs(f,x)$?

$\underline{Maj_n}$: $\{0,1\}^n \to \{0,1\}$.

sensitive blocks of Maj_n at 0^n. $\geq \binom{n}{\frac{n}{2}} \approx \frac{2^n}{\sqrt{n}}$

$\underline{\text{Prop}}$:- $C(f,x) \leq s(f) \cdot bs(f,x)$.

$\underline{\text{proof}}$:-

Consider the largest t

s.t. $B_1, \ldots, B_t$ are disjoint

minimal-
sensitive blocks at x.

$t = bs(f, x)$.

Let $S = \bigcup_{i=1}^{t} B_i$.

Then, S is a certificate

for x.

Suppose not.

$\Rightarrow \exists\, y \neq x$ s.t. $f(y) \neq f(x)$

but y is consistent

with x over $\bigcup_{i=1}^{t} B_i$

$\Rightarrow$ $\exists$ B_{t+1} disjoint from $\bigcup_{i=1}^{t} B_i$

s.t. $x^{B_{t+1}} = y$

and since $f(y) \neq f(x)$.

we obtain $bs(f,x) \geq t+1$.

Hence a contradiction.

Claim :- $|B_i| \leq s(f)$.

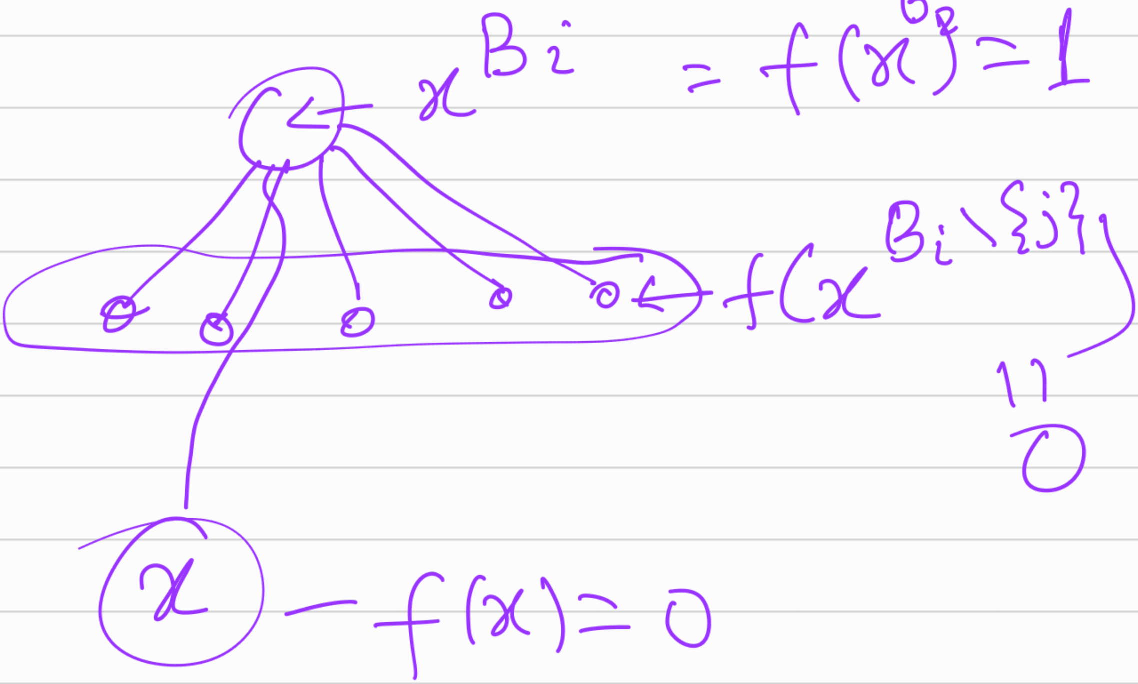
where B_i's are minimal

sensitive block.

A block B is said to be minimal if $\forall$ subset $T \subsetneq B$

$$f(x) = f(x^T).$$

<u>Claim'</u>: $|B_i| \leq s(f, x^{B_i}) \leq s(f)$



$$s(f, x^{B_i}) \geq |B_i|$$

Thm :- $C(f,x) \le s(f) \cdot bs(f,x)$

$$\le bs(f)^2$$

$\boxed{\text{OPEN :- } \exists ?\ f \text{ s.t. } C(f) = \Omega(bs(f)^2)}$