07/09/23

$$AND_n(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i$$

$$OR_n(x_1, \dots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i)$$

monomials = $2^n - 1$.

deg of a Boolean function f

is the deg of the unique polynomial

that represents f.

deg of a polynomial :=

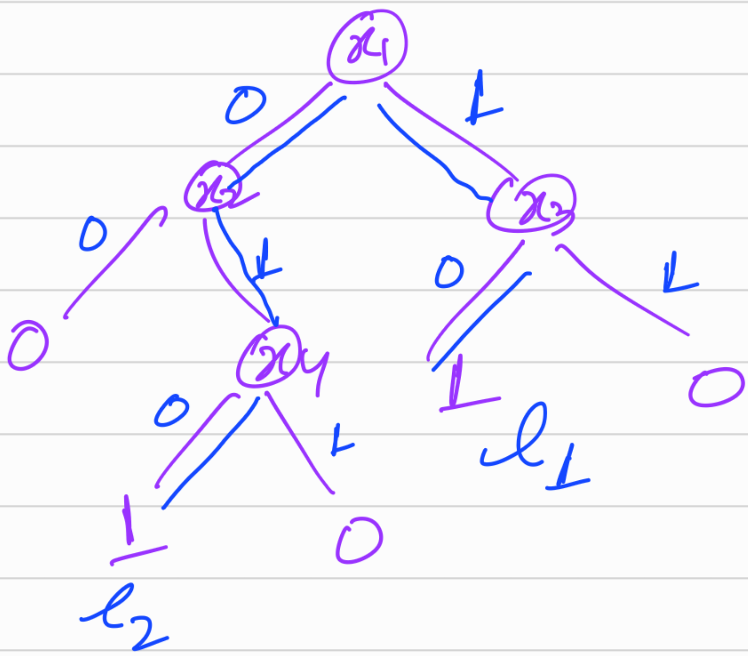
max degree of a monomial
with non-zero coefficients.

$\deg(AND_n) = n$

$\deg(OR_n) = n$

Prop : $\deg(f) \le D(f)$

Proof :-



$$p(x_1, \ldots, x_4) = \sum \boxed{\text{Indicator that input } x \text{ reaches } l}$$

l: leaves of decision tree s.t. label at the leaf $= 1$

[Indicator for l_1] $x_1 \cdot (1-x_3)$

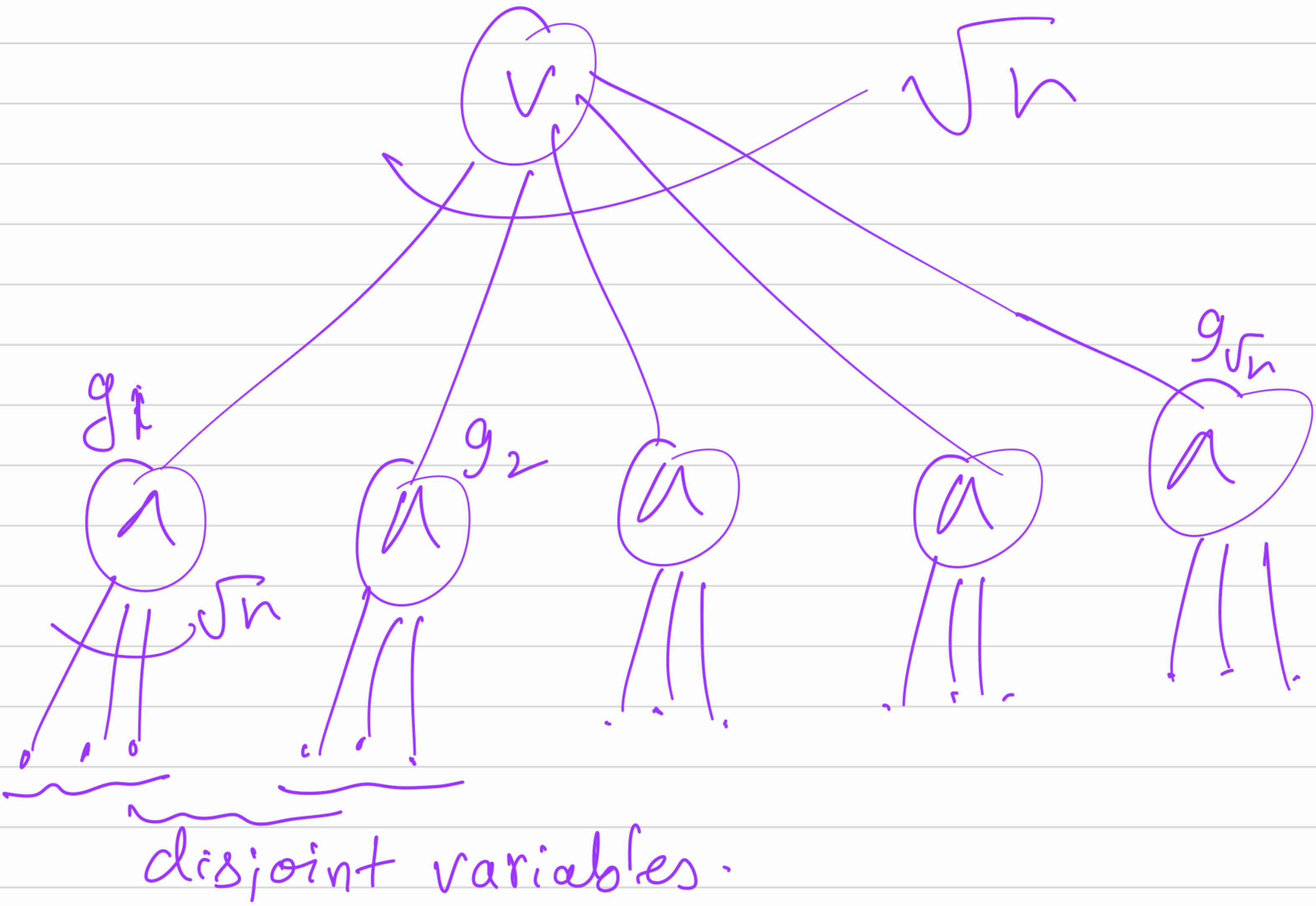
[Indicator for l_2] $(1-x_1) \cdot x_2 \cdot (1-x_4)$

$$x_1 \cdot (1-x_3) + (1-x_1) \cdot x_2 \cdot (1-x_4)$$

In this way we obtain a polynomial p that represent the f with $\deg \leq D(f)$.

$\blacksquare$

Example :- $\text{Tribes}_{\sqrt{n},\sqrt{n}}$



$$\deg\left(\text{Tribes}_{\sqrt{n},\sqrt{n}}\right) \leq n$$

$$OR_{\sqrt{n}}(y_1, \ldots, y_{\sqrt{n}}) = 1 - \prod_{i=1}^{\sqrt{n}} (1 - y_i)$$

$$AND_{\sqrt{n}}(z_1, \ldots, z_{\sqrt{n}}) = \prod_{i=1}^{\sqrt{n}} z_i$$

Let g_i be the polynomial for i-th AND gate.

Then $p(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{\sqrt{n}} (1 - g_i)$

$\deg = n$.

<u>Obs</u>:- the product of highest degree monomials from g_i has non-zero coeff.

$s(f) = bs(f) = C(f) = \sqrt{n}$.

$D(f) = n$.

Example 2 :- degree can be much lower than s, bs, C, and D.

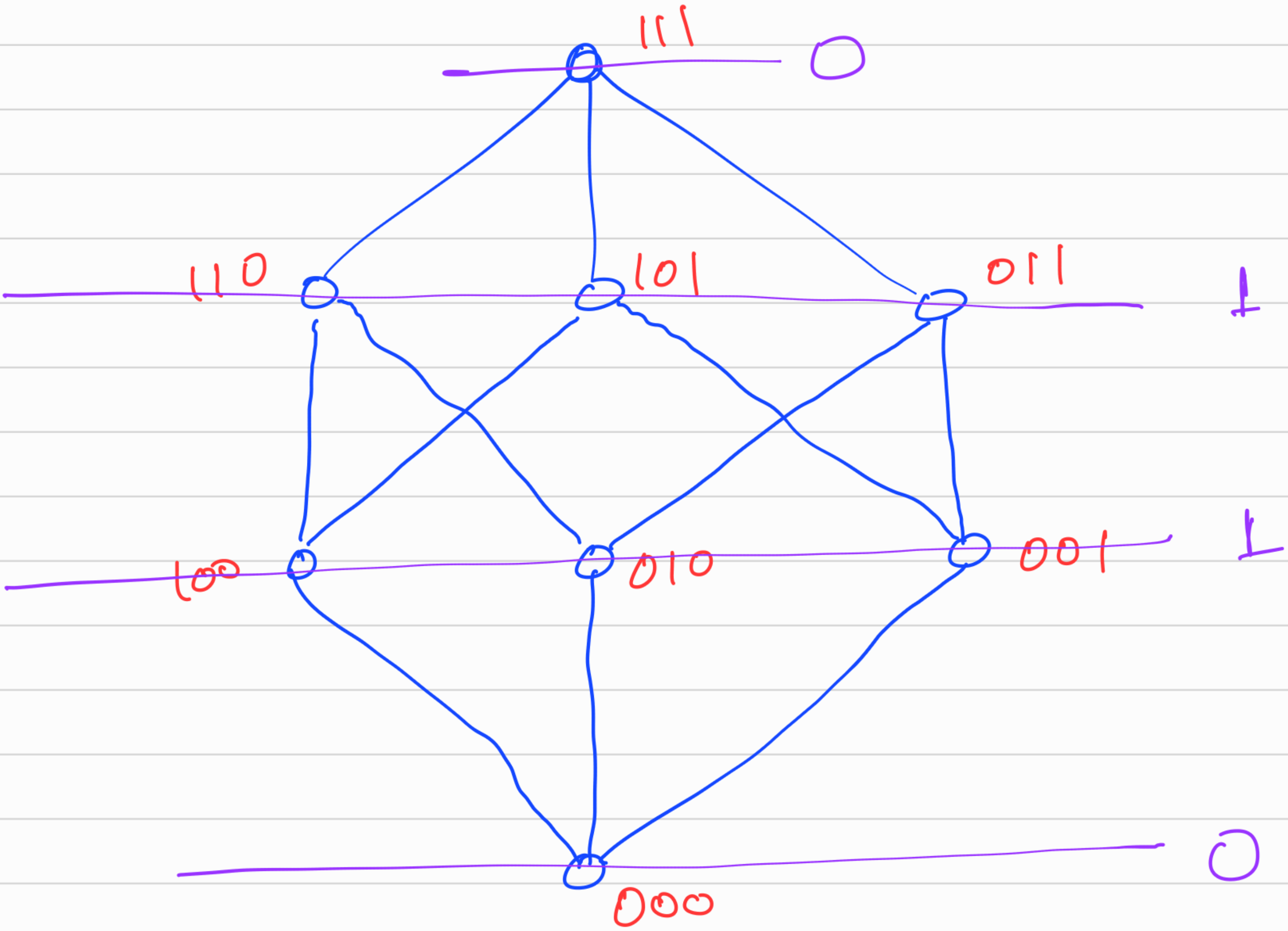
$$NAE : \{0,1\}^3 \to \{0,1\}$$

$$NAE(x_1, x_2, x_3) = \begin{cases} 1 & \text{if not all inputs are equal} \\ 0 & \text{o/w.} \end{cases}$$

$$p_{NAE}(x_1, x_2, x_3) = x_1 + x_2 + x_3 - x_1 x_2 - x_1 x_3 - x_2 x_3$$

$$\deg(p_{NAE}) = 2$$

$D(NAE) = 3$

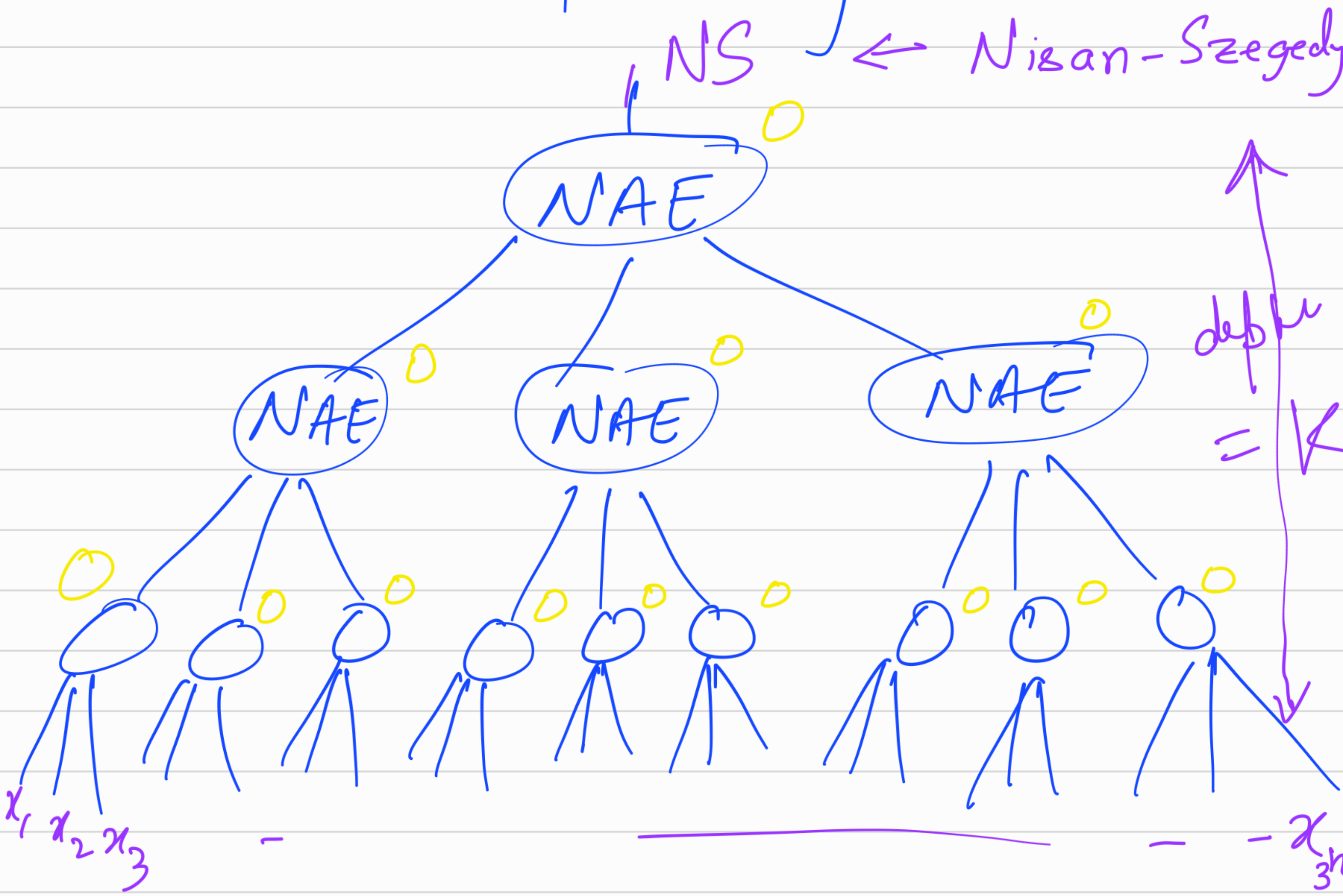


$s(NAE) = 3 = s(NAE, 0^3)$

$= s(NAE, 1^3)$

$s(NAE) = bs(NAE) = C(NAE) = 3$

Consider the following function



leaves in this tree $= 3^K$.

leaves are labelled by distinct variables.

internal nodes are labelled by NAE_3.

This gives a function.

$$NS : \{0,1\}^{3^k} \to \{0,1\}$$

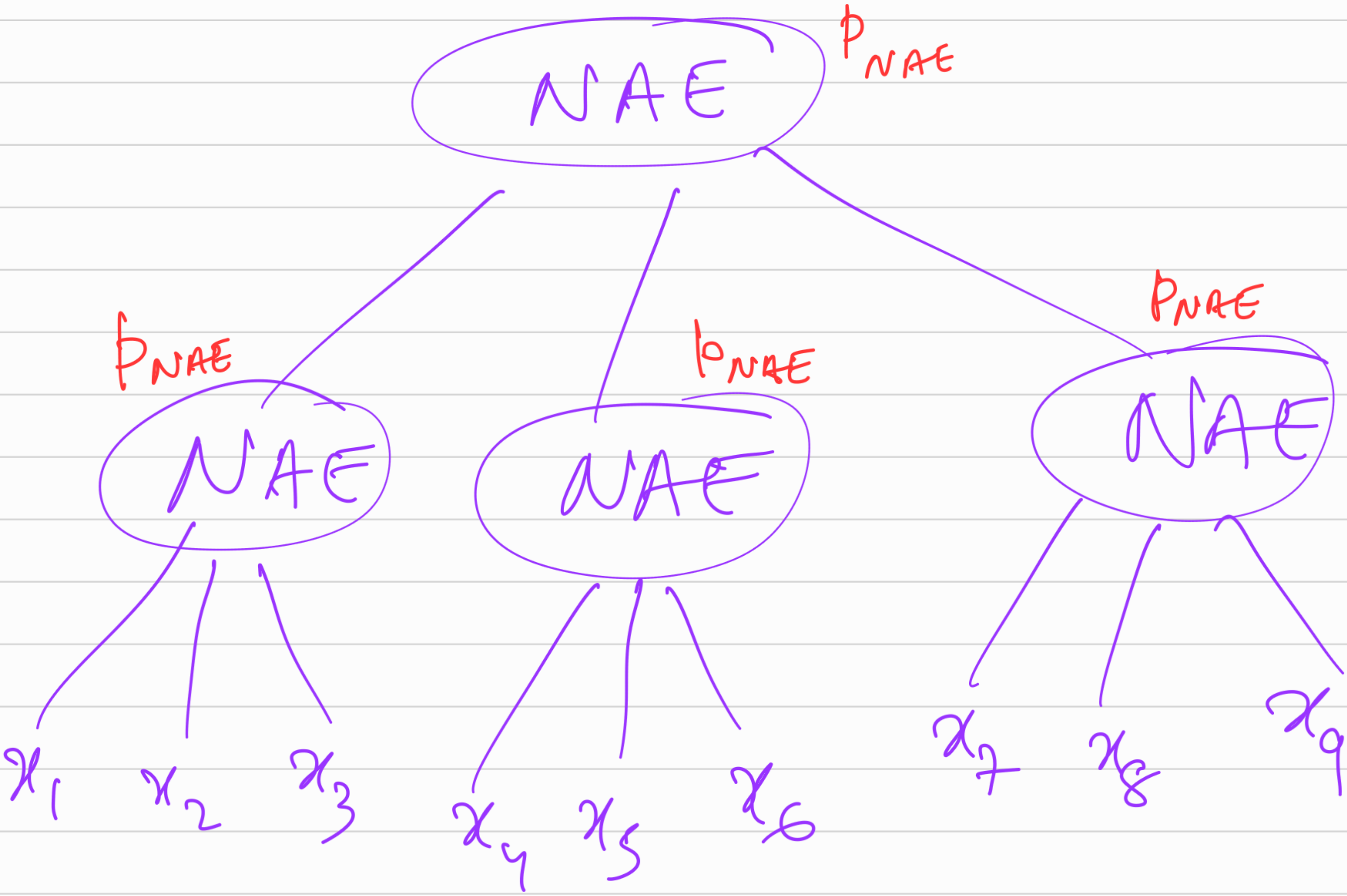
$$s(NS, 0^{3^k}) = 3^k$$

$$NS(0^{3^k}) = 0$$

$$NS(1\,0^{3^k-1}) = 1$$

$$s(NS) = bs(NS) = C(NS) = 3^k$$

$$\deg(NS) \leq 2^k.$$



$$p_{NAE}\Big(p_{NAE}(x_1, x_2, x_3),\ p_{NAE}(x_4, x_5, x_6),\ p_{NAE}(x_7, x_8, x_9)\Big)$$

degree $\leq$ 4

$$\underset{\substack{\uparrow \\ p_{NAE}}}{x_1} \cdot \underset{\substack{\uparrow \\ p_{NAE}}}{x_2}$$

$\deg(NAE) \leq 2^k = n^{\log_3 2}$
$= n^{0.63\ldots}$

$s(f) = bs(f) = C(f) = D(f) = 3^k$.
$= n$

→ We know $\exists$ $f: \{0,1\}^n \to \{0,1\}$

s.t. $s(f) = bs(f) = C(f) = D(f) = n$

But $\deg(f) = n^{\log_3 2} = n^{0.63\ldots}$

Defn:
→ We say that a function

f depends on variable

x_i

iff $\exists$ an input $a \in \{0,1\}^n$

s.t. $f(a) \neq f(a^i)$

$a^i = a \oplus e_i$:= flip the i^{th} variable.

$\Rightarrow$ $f: \{0,1\}^3 \to \{0,1\}$

$f(x_1, x_2, x_3) = x_1 \cdot x_2$.

Does f depends on all variable? NO.! x_3.

$\Rightarrow$ How low can the degree of a Boolean function be when it depends on all variables?

$\rightarrow$ $0 \leq deg(f) \leq n$.

$\Rightarrow$ $1 \leq deg(f) \leq n$.

Example:

$$MUX(x_{n-1}, \ldots, x_0; y_0, \ldots, y_{2^n-1}) := y_{bin(x_{n-1}, \ldots, x_0)}$$

where $bin(x_1, \ldots, x_n) := \sum_{i=0}^{n-1} 2^i x_i$

$$MUX : \{0,1\}^{n+2^n} \to \{0,1\}$$

Also called as Indexing function,

Storage Access function,

Addressing function.

$$MUX(x; y) = \sum_{a \in \{0,1\}^n} \underbrace{\prod_{i:a_i=1} x_i \prod_{i:a_i=0} (1-x_i)}\, \underline{y_{bin(a)}}$$

$\deg(MUX) = n+1$

MUX also depends on every variable..

$\#$ of Variables $= n + 2^n := \underline{N}$

$\deg = n+1 := \Theta(\log N)$

$\Rightarrow$ Wrt $\#$ of Variable degree can be really low, i.e., as low as $\log(\#\text{Vars})$.

→ How low can degree be wrt sensitivity, bs, or certificate?

Thm:- [Nisan-Szegedy]

$$\deg(f) \geq \Omega\left(\sqrt{bs(f)}\right).$$

Thm:- if $f:\{0,1\}^n \to \{0,1\}$ that depends on all its variables then

$$\deg(f) = \Omega(\log n).$$