12/10/23

$$f : \{\pm 1\}^n \to \{\pm 1\}$$

$-1 \equiv$ True $\equiv 1$

$1 \equiv$ False $\equiv 0$

$$g : \{0,1\}^n \to \{0,1\}$$

$$g^{\pm}(x) = 1 - 2g\left(\frac{1-x_1}{2}, \frac{1-x_2}{2}, \ldots, \frac{1-x_n}{2}\right)$$

$$\chi_S : \{\pm 1\}^n \to \{\pm 1\}$$

$$\forall S \subseteq [n]$$

$$\langle f, g \rangle := \mathbb{E}_{x \sim_u \{\pm 1\}^n}[f(x) \cdot g(x)]$$

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if } S \neq T. \end{cases}$$

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i \leftarrow \chi_S(x)$$

$$\langle f(x), \chi_S(x) \rangle = \mathbb{E}_x[f(x) \cdot \chi_S(x)]$$

$$= \hat{f}(S).$$

$$\langle f, f \rangle := \mathbb{E}_x[f(x)^2] = 1$$

$$\parallel$$

$$\sum_{S \subseteq [n]} \hat{f}(S)^2$$

Fourier transform / Fourier Spectrum

of f : $\sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i$

$$\|\hat{f}\|_1 := \sum_{S \subseteq [n]} |\hat{f}(S)|$$

Thm :- $f : \{\pm 1\}^n \to \{\pm 1\}$

Then $L(f) \geq \|\hat{f}\|_1$

$IP(x, y) : \{\pm 1\}^n \to \{\pm 1\}$

$\sum_{i=1}^{n/2} x_i y_i$ on $\{0, 1\}$ (mod 2)

$$|\hat{IP}(S)| = \frac{1}{2^{n/2}}$$

$\forall S \subseteq [n]$

$$\|\hat{IP}\|_1 = 2^{n/2}.$$

Prop 1 $\quad f, g$

$$\|\widehat{f+g}\|_1 \leq \|\hat{f}\|_1 + \|\hat{g}\|_1$$

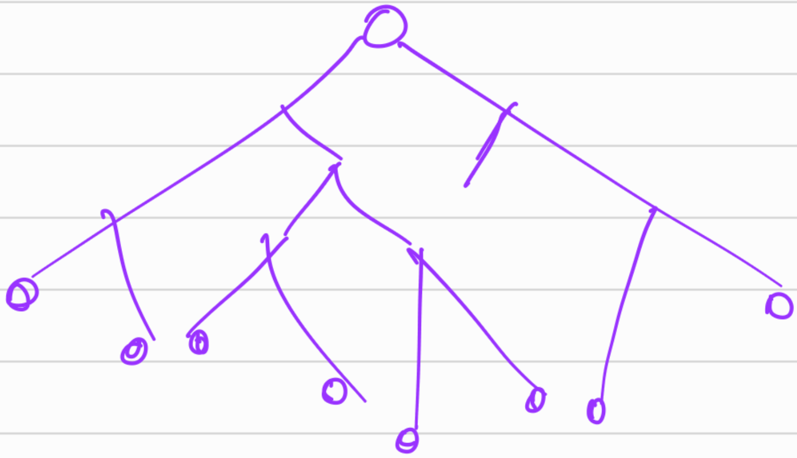
$$\|\widehat{f \cdot g}\|_1 \leq \|\hat{f}\|_1 \cdot \|\hat{g}\|_1$$

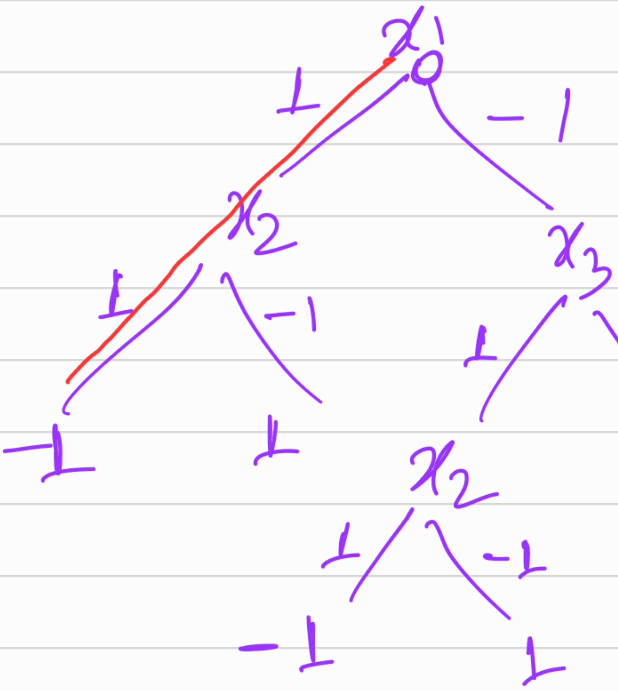
$$\left\| \left(\sum \hat{f}(S) \chi_S\right)\left(\sum \hat{g}(T) \chi_T\right) \right\|_1$$

Thm :- $\|\hat{f}\|_1 \leq$ size of a DT computing f.

Pf :-

T





$$\sum f(\ell) \cdot \left(\frac{1+x_1}{2}\right)\left(\frac{1+x_2}{2}\right)$$

ℓ: leaves of T

Indicator for ℓ.

$$(-1) \cdot \left(\frac{1+x_1}{2}\right)\left(\frac{1+x_2}{2}\right) +$$

$$(1) \cdot \left(\frac{1+x_1}{2}\right)\left(\frac{1-x_2}{2}\right) +$$

$$(-1) \left(\frac{1-x_1}{2}\right)\left(\frac{1+x_3}{2}\right)\left(\frac{1+x_2}{2}\right) +$$

$$f(x) = \sum_{\ell:\text{ leaves of } T} f(\ell) \cdot \mathbb{1}_\ell .$$

$$\|\hat{f}\|_1 = \left\| \sum_\ell f(\ell) \cdot \mathbb{1}_\ell \right\|_1$$

$$\leq \sum_{\ell} \left\| \widehat{f(\ell) \cdot \mathbb{1}_\ell} \right\|_1$$

$$= \sum_{\ell} \left\| \hat{\mathbb{1}}_\ell \right\|_1$$

$$\mathbb{1}_\ell = \left(\frac{1 + x_{i_1}}{2} \right) \left(\frac{1 - x_{i_2}}{2} \right) \cdots \left(\frac{1 \pm x_{i_k}}{2} \right)$$

$$\left| \pm \frac{1}{2^k} \right| \cdot 2^k = 1$$

$$\left\| \hat{\mathbb{1}}_\ell \right\| \leq 1.$$

$$\sum_{\ell:\text{ leaves of } T} \|\hat{1}_\ell\|_1$$

$$= \#\text{ leaves in } T$$

<u>Thm</u>:- $L(f) \geq \|\hat{f}\|_1$.

$\blacksquare$

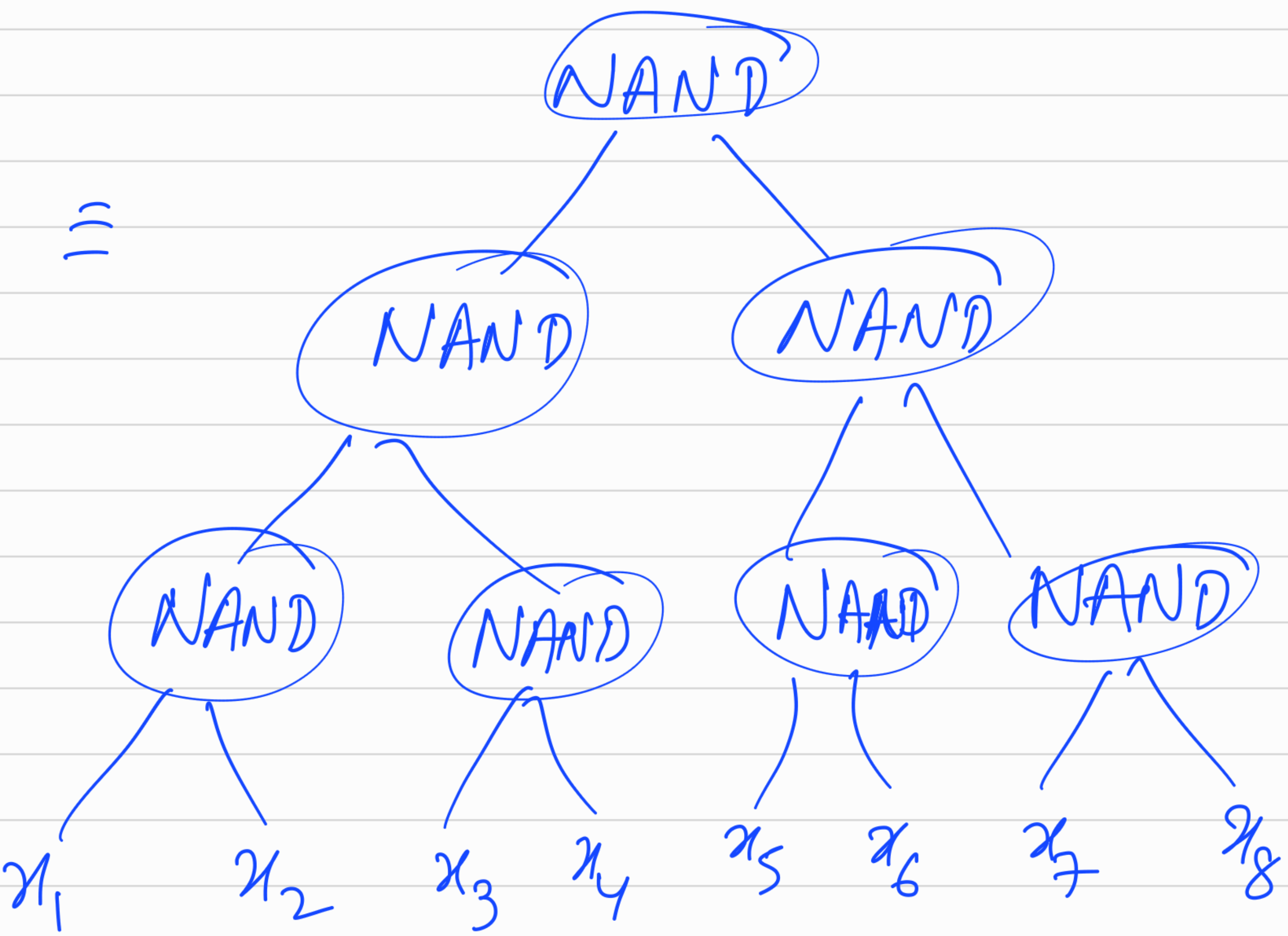
<u>Thm</u>:- $\exists\, f : \{\pm 1\}^n \to \{\pm 1\}$

s.t. $N = \text{DNF-size}(f) + \text{CNF-size}(f)$

Then $L(f) = N^{\Omega(\log N)}$

<u>Thm'</u>: Consider

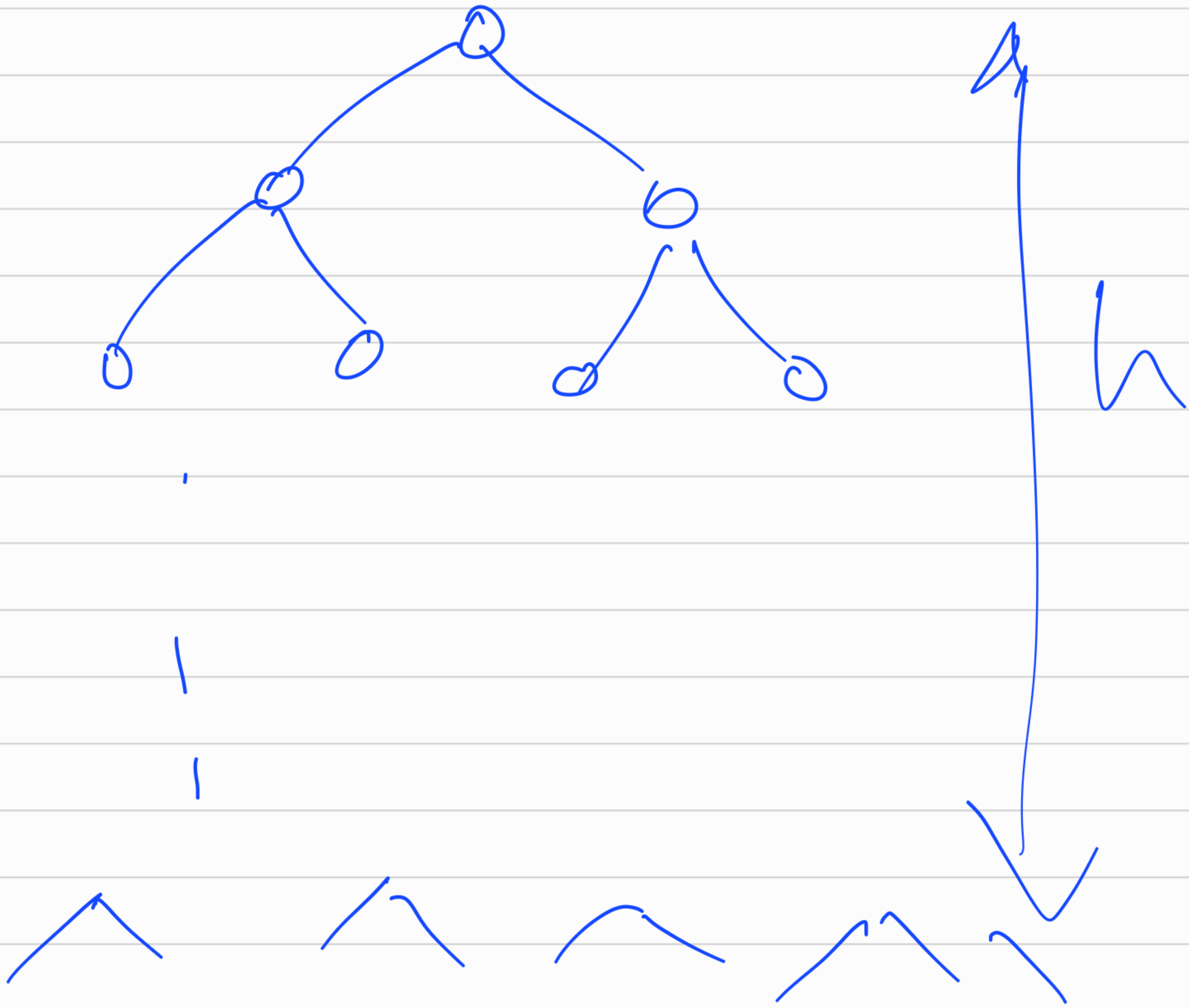
$f_3 =$



$$NAND(x,y) = \neg(x \wedge y)$$

$$f_0 = x$$

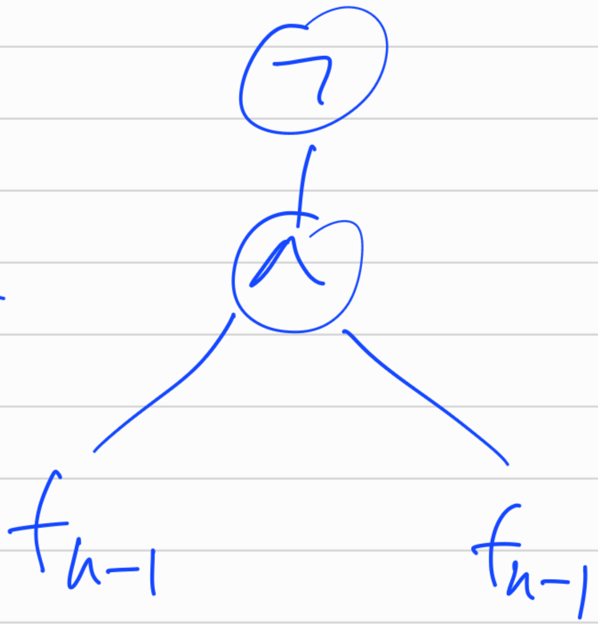
$$f_n = NAND(f_{n-1}, f_{n-1})$$



$$\#Vars(f_n) = 2^n$$

$$\text{DNF-size}(f_n) \leq 2 \cdot \text{DNF size}(\neg f_{n-1})$$

$f_n \equiv$



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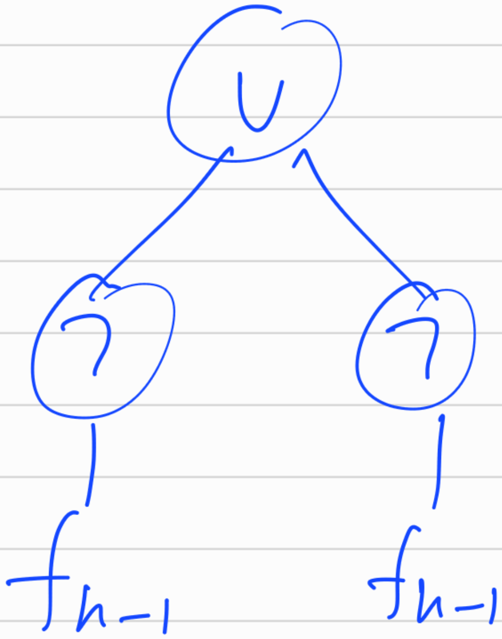
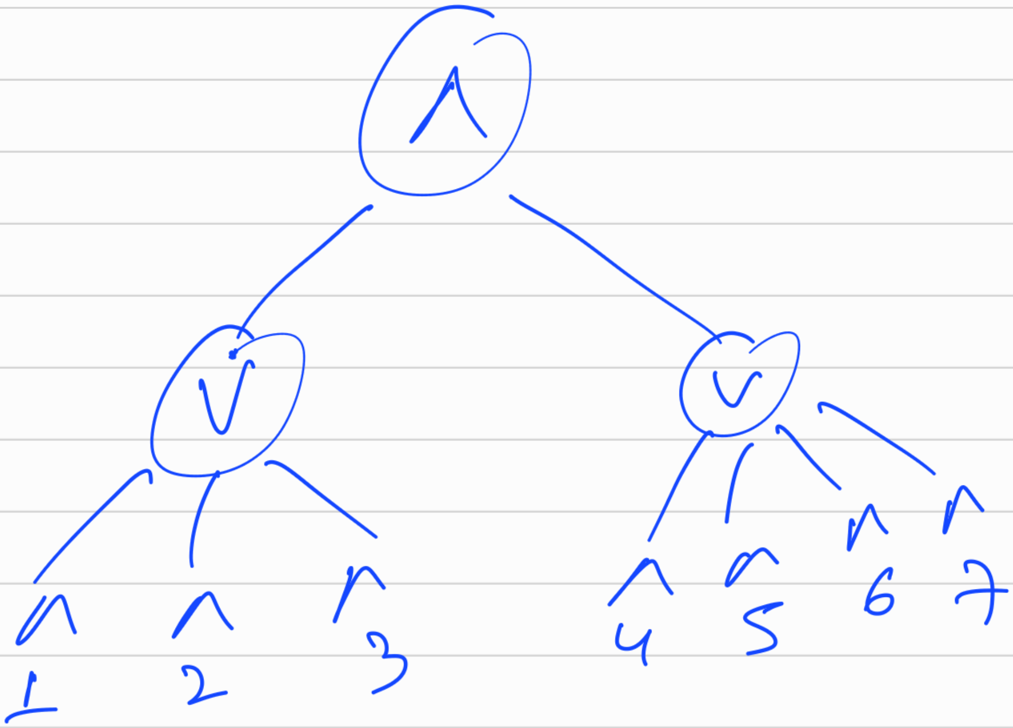
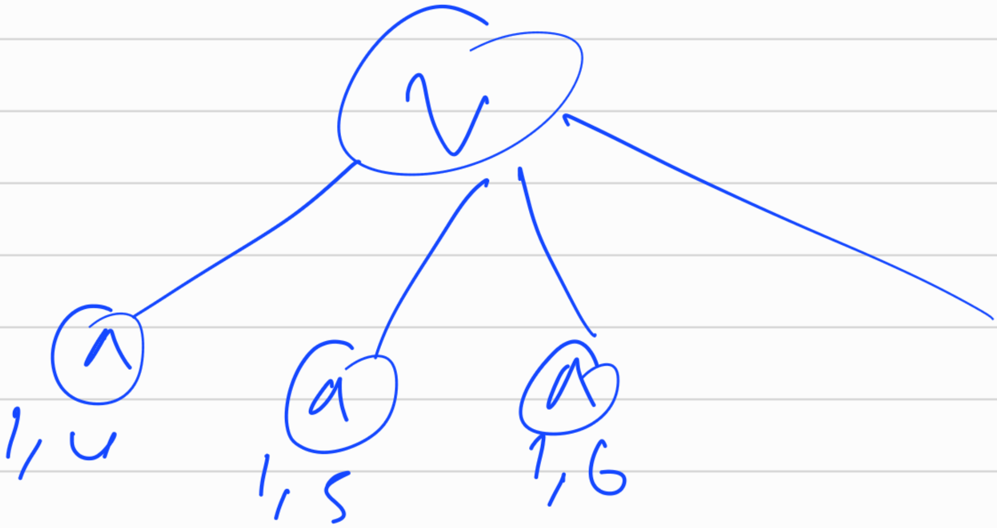


$$\text{DNF-size}(\neg f_n) \leq \text{DNF-size}(f_{n-1})$$

$\neg f_n \equiv$





$$\mathrm{dnf}(f_n) \leq 2 \cdot \mathrm{dnf}(\neg f_{n-1})$$

$$\mathrm{dnf}(\neg f_n) \leq \mathrm{dnf}(f_{n-1})^2$$

$$\Rightarrow \mathrm{dnf}(f_n) \leq 2 \cdot \mathrm{dnf}(f_{n-2})^2$$

$$\Rightarrow \mathrm{dnf}(f_h) \leq 2^{2^{\frac{h}{2}+o(1)}}$$

$$\mathrm{dnf}(\neg f_h) \leq 2^{2^{\frac{h}{2}+o(1)}}$$

$$n = 2^h$$

$$N = 2^{O(\sqrt{n})}$$

$$N^{\log N} = \left(2^{O(\sqrt{n})}\right)^{\sqrt{n}}$$

$$= 2^{\Omega(n)}$$

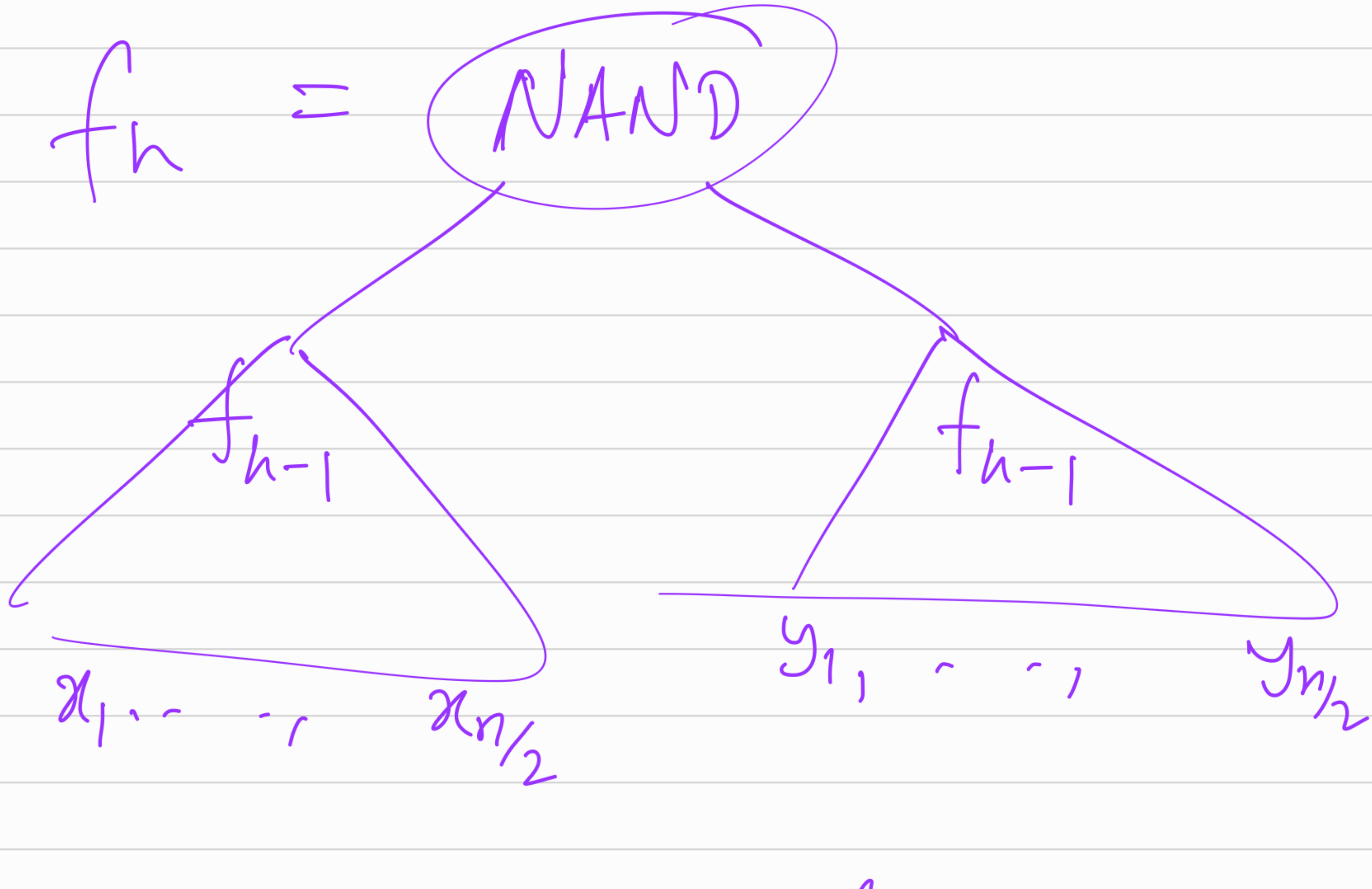
$$\|\hat{f}_h\|_1 = 2^{\Omega(2^h)}$$

$$\mathrm{NAND}(x,y) \equiv \neg(x \wedge y)$$

$$= -\left[1 - 2\left(\frac{1-x}{2}\right)\left(\frac{1-y}{2}\right)\right]$$

$$= \frac{xy}{2} - \frac{x}{2} - \frac{y}{2} - \frac{1}{2}.$$

$f_n =$



$$f_n(x,y) = \frac{f_{n-1}(x) \cdot f_{n-1}(y)}{2}$$

$$- \frac{f_{n-1}(x)}{2} - \frac{f_{n-1}(y)}{2} - \frac{1}{2}$$

$$\hat{f}_n(\emptyset) = \frac{\hat{f}_{n-1}(\emptyset) \cdot \hat{f}_{n-1}(\emptyset)}{2}$$

$$- \frac{\hat{f}_{n-1}(\emptyset)}{2} - \frac{\hat{f}_{n-1}(\emptyset)}{2} - \frac{1}{2}$$

$$\hat{f}_n(S, T) = \frac{\hat{f}_{n-1}(S) \cdot \hat{f}_{n-1}(T)}{2}$$

when $S, T \neq \emptyset$.

$$\hat{f}_n(S) = \frac{\hat{f}_{n-1}(S) \cdot \hat{f}_{n-1}(\emptyset)}{2} - \frac{\hat{f}_{n-1}(S)}{2}$$

when $S \neq \emptyset$ $\quad S \subseteq \{x_1, \dots, x_{n/2}\}$

$$= \frac{1}{2}\left(\hat{f}_{n-1}(\emptyset) - 1\right) \cdot \hat{f}_{n-1}(S)$$

$$\hat{f}_n(T) = \frac{\hat{f}_{n-1}(\emptyset) \cdot \hat{f}_{n-1}(T)}{2} - \frac{\hat{f}_{n-1}(T)}{2}.$$

$$= \frac{1}{2} \cdot \left(\hat{f}_{n-1}(\emptyset) - 1\right) \cdot \hat{f}_{n-1}(T)$$

$$\|\hat{f}_n\|_1 - |\hat{f}_n(\emptyset)|$$

$$:= \|\hat{f}_n\|_1^{\neq \emptyset}$$

$$= \frac{1}{2} \cdot \left(\| \hat{f}_{n-1} \|_1 - |\hat{f}_{n-1}(\emptyset)| \right) \cdot \left(\| \hat{f}_{n-1} \|_1 - |\hat{f}_{n-1}(\emptyset)| \right)$$

$$+ \frac{1}{2} \cdot \left(1 - |\hat{f}_{n-1}(\emptyset)| \right) \cdot \left(\| \hat{f}_{n-1} \|_1 - |\hat{f}_{n-1}(\emptyset)| \right)$$

$$\sum_{\emptyset \neq S, T \neq \emptyset} |\hat{f}_n(S, T)|$$

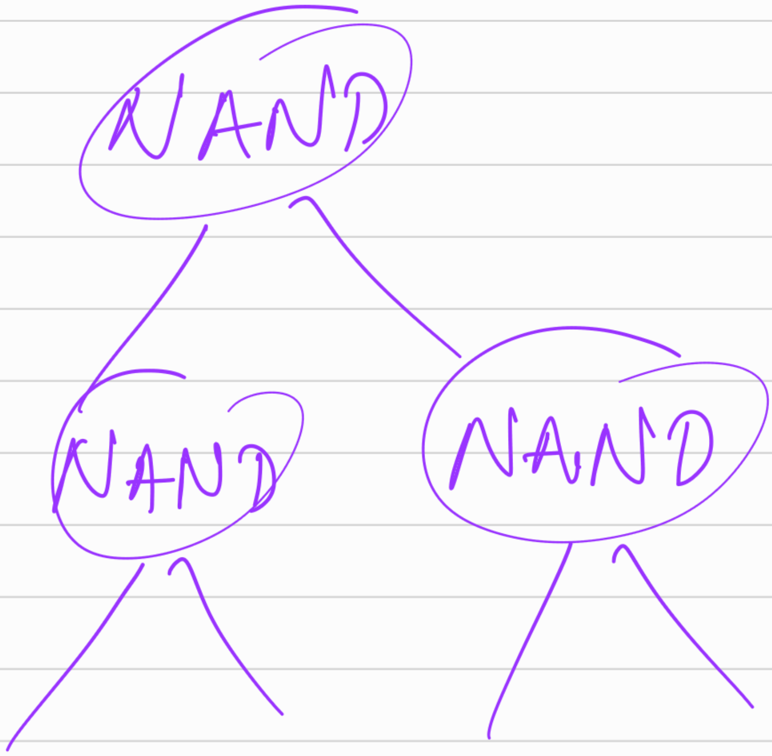
$$\|\hat{f}_n\|_1^{\neq\emptyset} \geq \frac{1}{2} \cdot \left(\|\hat{f}_{n-1}\|_1^{\neq\emptyset}\right)^2$$

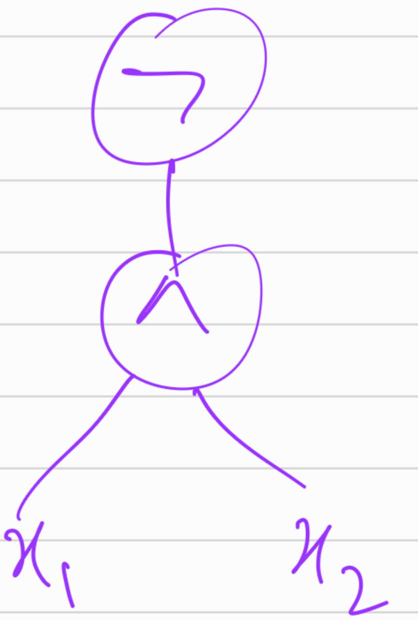
$$T(n) \geq \frac{1}{2}\left(T(n-1)^2\right)$$

$$\Rightarrow T(n) \geq 2^{c \cdot 2^n}$$

$$= 2^{\Omega(2^n)}$$

$$= 2^{\Omega(n)}$$





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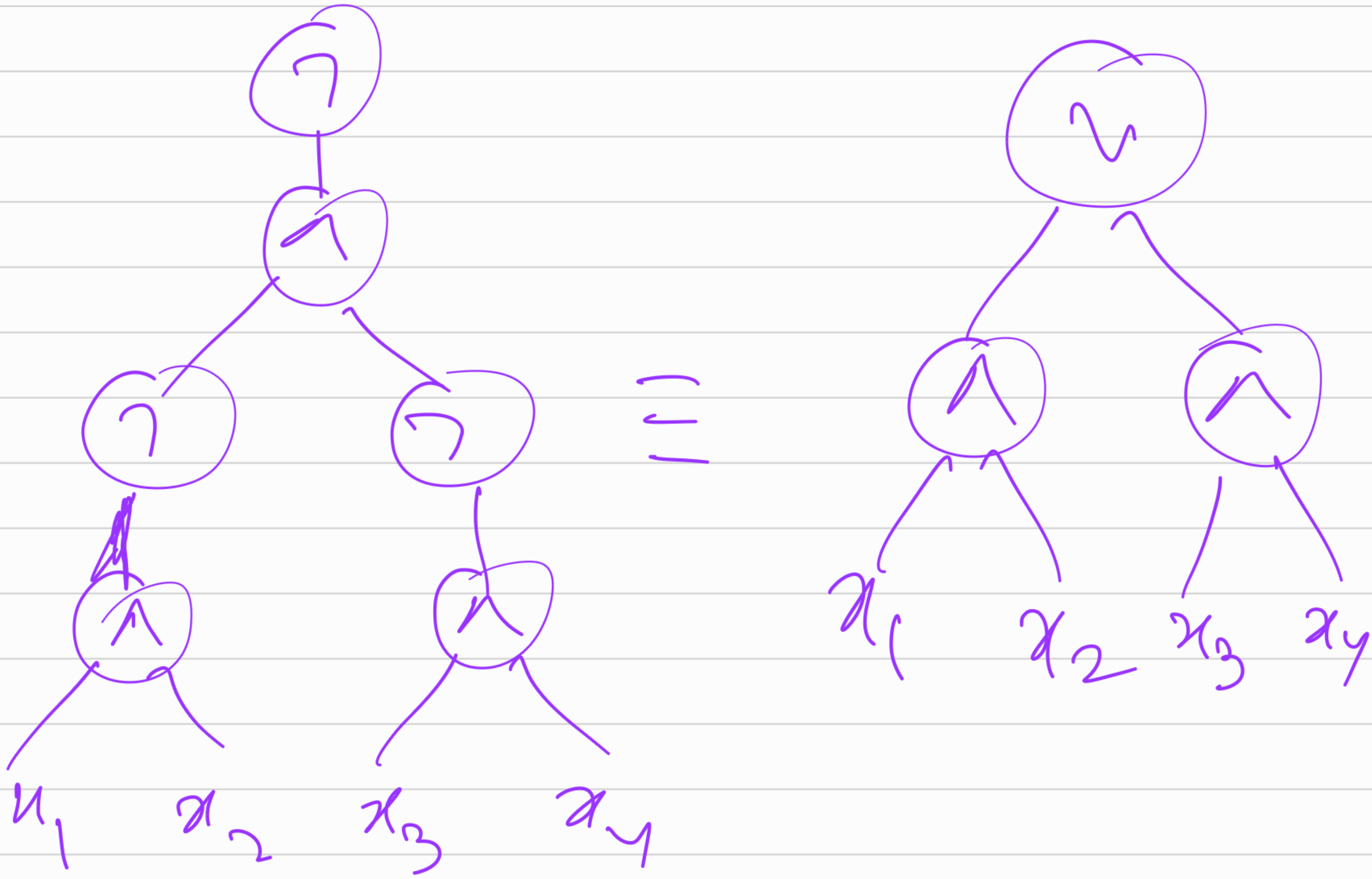

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