30/10/23

$$f : \{\pm 1\}^n \to \{\pm 1\}$$

two candidates

$+1$ $\qquad$ -1

n voters.

$$x = (1, -1, 1 \ldots)$$

$f(x)$ = Outputs a winner.

Ex: f = Majority.

Q: How many voters you need to bribe to "almost fix" the election?

properties of a social choice function

1) f should depend on every voter.

2) $f(1, 1, -1, 1) = 1$

$(1, 1, 1, 1) = 1$

f should be monotone.

3) $f(1, \ldots, 1) = 1$

$f(-1, \ldots, -1) = -1$

$f(1^n) = -1$

$f(-1^n) = 1$

4) fair, unbiased election,

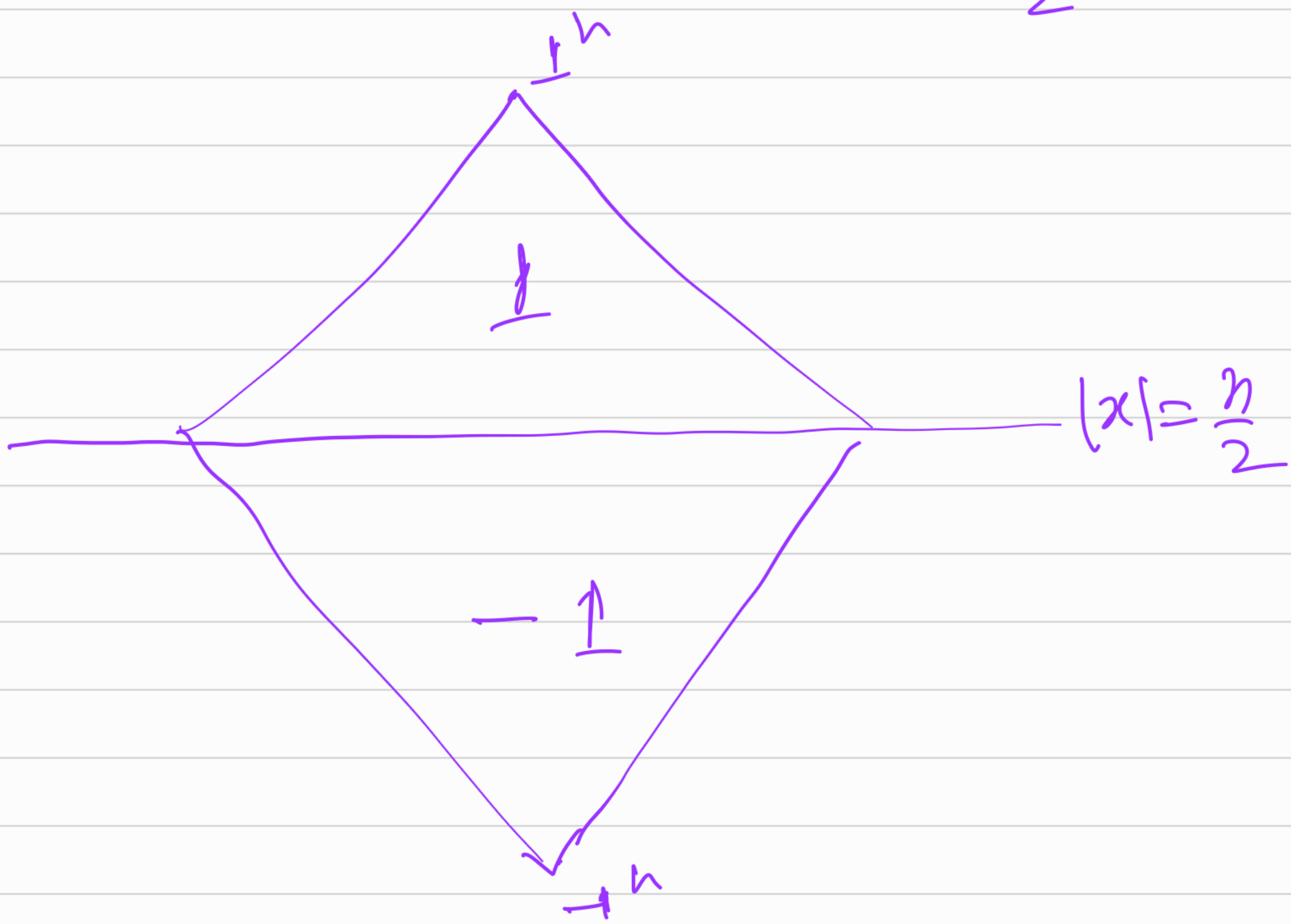
Assumption: each voter is equally likely to vote for each candidate,

$$\Pr[x_i = 1] = \frac{1}{2}.$$

<u>Given f.</u>

$$\Pr_x[f(x) = 1] = \frac{1}{2}$$

$$\Pr_x[f(x) = -1] = \frac{1}{2}$$



$\mathrm{Maj} : \{\pm 1\}^n \to \{\pm 1\}$

$$\left[\sum_{i=1}^{n} x_i \geq 0\right]$$

$\mathrm{Maj}(x_1, \ldots, x_n) = \mathrm{sgn}(\sum x_i)$

Suggestion :- $\frac{8n}{10}$ voters are set to 1

$\left.\begin{array}{l}\text{Pr of } 1 \text{ winning} \\ \text{the election}\end{array}\right\} \geq \frac{9}{10}$

Answer :- bribe $O(\sqrt{n})$ voters to make the prob $\geq 2/3$.

Set Up :-

bribe voters to vote for 1.

Pr[1 wins the election] ≥ ?

$Pr[\sum x_i \geq 0]$

$$\underbrace{x_1, \dots, x_n}_{c\sqrt{n}}$$

$$x_1, \dots, x_{n-c\sqrt{n}}, \underbrace{1, \dots, 1}_{c\sqrt{n}}$$

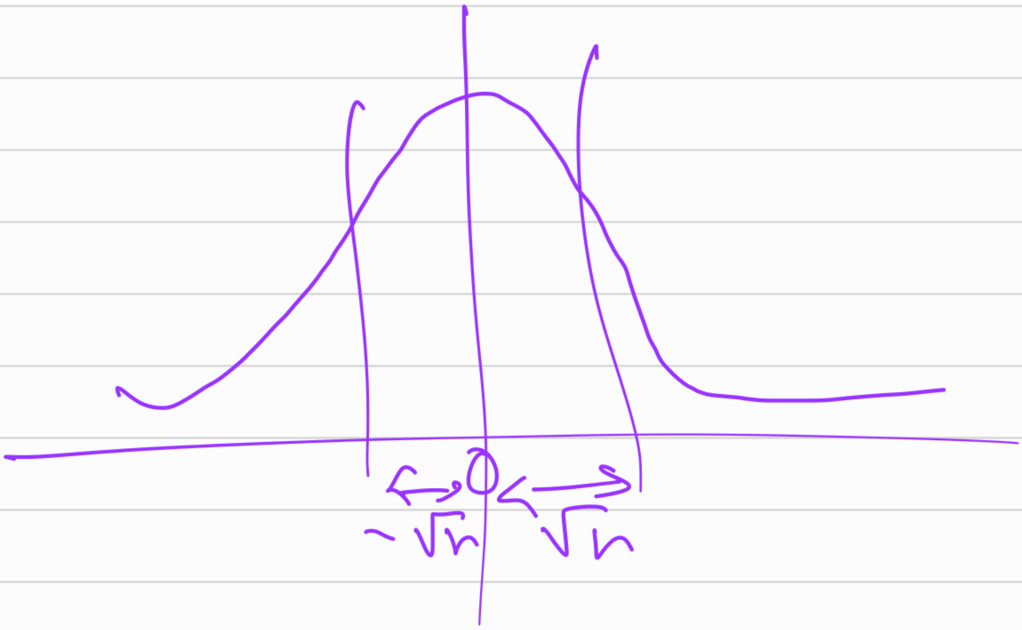
uniform distribution $\{x_1, \dots, x_n\}$

mean $= 0$

$E[x_i] = 0$

$E[\sum x_i] = 0$

Standard deviation $= \sqrt{n}$



Consider the case

$x_1, \ldots, x_{n-c\sqrt{n}}, 1, \ldots, 1,$

$E[\sum x_i] = c\sqrt{n}$

$\sqrt{n - c\sqrt{n}} \leq \sqrt{n}$

$$\Pr\left[\sum_{i=1}^{n-c\sqrt{n}} x_i + c\sqrt{n} \geq 0\right]$$

$$\Pr\left[\sum_{i=1}^{n-c\sqrt{n}} x_i \geq -c\sqrt{n}\right]$$

$$\Pr\left[\sum_{i=1}^{n-c\sqrt{n}} x_i \geq -c\right]$$

$$\geq 1 - O\left(\frac{1}{2^c}\right)$$

Ajtai - Linial : constructed

a function s.t.

you need to bribe $\Omega\left(\frac{n}{\log^2 n}\right)$

voters to get a constant prob. of biasing the election.

→ **Kahn-Kalai-Linial Lemma**

Consider $f : \{\pm 1\}^n \to \{\pm 1\}$ that is balanced then

$\exists\ i \in [n]$, s.t. $\mathrm{Inf}_i(f) \geq \Omega\left(\frac{\log n}{n}\right)$.

Balanced function: $E[f] = 0$

(generalize $E[f] = \delta$ when $\delta > 0$ is a fixed constant)

When you bribe a voter, How does the probability of the winner changes?

$\Rightarrow$

$$\begin{array}{cccccc} x_1 & x_2 & \cdots & x_i & \cdots & x_n \\ 1 & -1 & & x_i & & -1 \end{array}$$

$$\left(x_{[n]\setminus\{i\}},\ x_i\right)$$

$$= (y,\ x_i)$$

fix $y \in \{\pm 1\}^{n-1}$

$$f(y,1) = f(y,-1)$$

then bribing 'i' doesn't help.

it is helpful when $f(y,1) \neq f(y,-1)$

$$\Pr_{x_i}[f(y,x_i)=1] = \frac{1}{2}$$

$$\underline{\Pr_{x_i=1}[f(x)=1]} = \underline{\Pr[f(x)=1]} + \underline{\frac{1}{2}\cdot \mathrm{Inf}_i(f)}$$

After bribing each voter

Prob. estimate increases

additively by $\Omega\left(\frac{\log n}{n}\right)$

$\Rightarrow$ in $O\left(\frac{n}{\log n}\right)$

Steps you have

a prob. estimate of

outputting $t \geq 1 - \frac{1}{100}$

$$\mathrm{Inf}(f) = \sum_{S \subseteq [n]} |S| \hat{f}(S)^2$$

$$\geq \sum_{S \subseteq [n]} \hat{f}(S)^2$$

$$E[f] = 0 = \hat{f}(\emptyset)$$

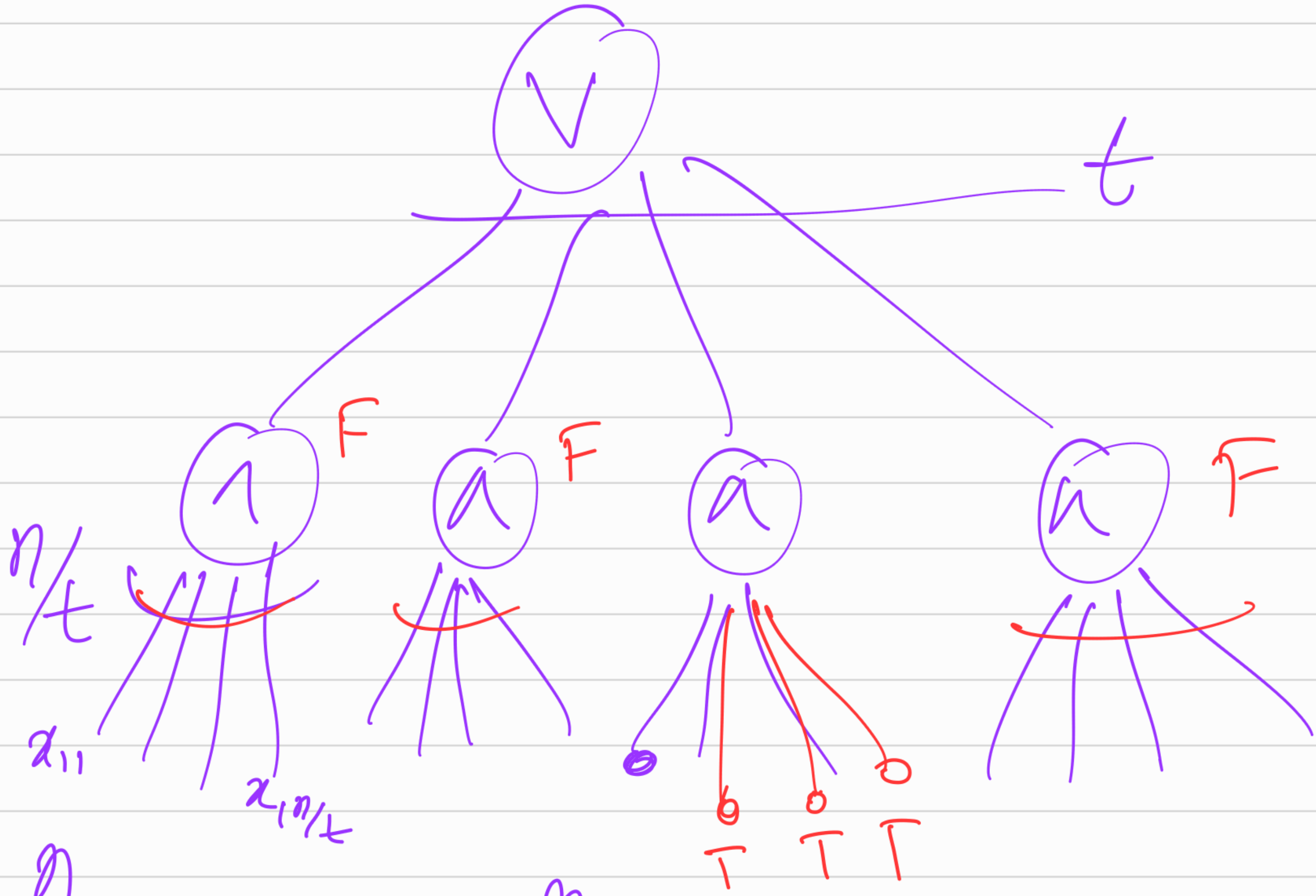
$$\mathrm{Inf}(f) \geq \underbrace{1}\ ?$$

$n \quad ||$

$$\sum_{i=1} \mathrm{Inf}_i(f)$$

$$\Rightarrow \max_{i=1}^{n} \left\{ \mathrm{Inf}_i(f) \right\} \geq \frac{1}{n}.$$

Ben-Or & Linial



$$\frac{n}{t} = \log \frac{n}{\log n} = \omega$$

$$\Pr[\text{Tribes}_t \text{ outputs } 1] \approx \frac{1}{e}$$

$$\mathrm{Inf}_{ij}(\text{Tribes}) =$$

$$\frac{1}{2^{w-1}} \cdot \left(1 - \frac{1}{2^w}\right)^{\frac{n}{w} - 1}$$

$$= \frac{2 \log n}{n} e^{-\frac{n}{2^w \cdot w}}$$

$$\left(\log \frac{n}{\log n}\right) \frac{n}{\log n} \approx n.$$

$$= \Omega\left(\frac{\log n}{n}\right)$$

$\Rightarrow$ Tribes shows that KKL lemma is tight.