The local lemma

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[Erdos - Lovasz 1975] $K(n,k)$

Lemma: [The Local Lemma, General case] Let $A_1, A_2, \ldots, A_n$ be n events in an arbitrary probability space. A directed graph $D=(V,E)$ with $V=\{1,2,\ldots,n\}$ is called a <u>dependency graph</u> for the events $A_1, A_2, \ldots, A_n$ if for each A_i, $1 \le i \le n$, A_i is mutually independent of all the events $\{A_j : (i,j) \notin E\}$. Suppose that $D=(V,E)$ is a dependency digraph of the above events and suppose there are $x_1, x_2, \ldots, x_n$ such that $0 \le x_i \le 1$ and $\Pr[A_i] \le x_i \prod_{(i,j)\in E} (1-x_j)$ for all $1 \le i \le n$. Then,

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \geq \prod_{i=1}^{n} (1-x_i).$$

$A_1 \downarrow$ Coin 1 giving H

$A_2 \downarrow$ Coin 2 giving H

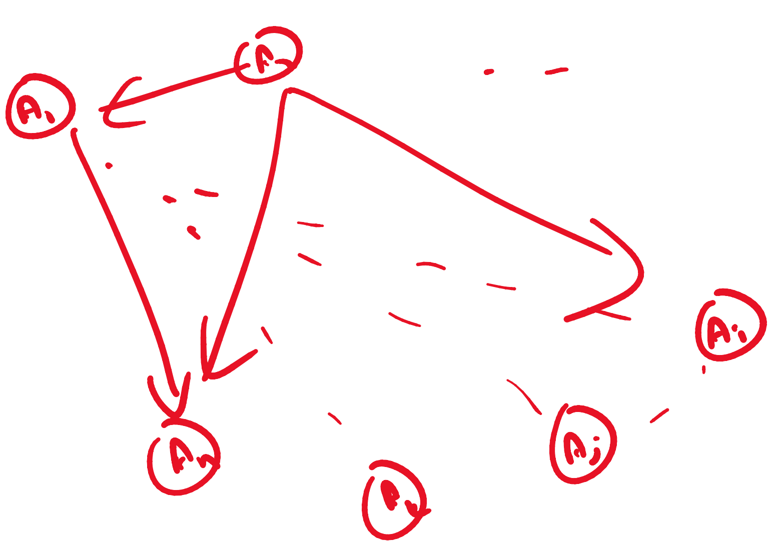
$E \downarrow$ both Coin 1 & Coin 2 give the same outcome

H H coin 2 same outcome

E_1 is independent of A_1

E_1 " " " A_2

E_1 is not mutually indep of A_1 & A_2



$Pr(A_1 | A_i, A_j, A_k)$

$= Pr(A_1)$

Corollary 2: [Symmetric case]

Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each A_i is mutually independent of all but at most d other events and $Pr[A_i] \le p$ $\forall i \in [n]$. If

$$e \cdot p \cdot (d+1) \le 1$$

remembering $d \le n-1$

OR

0

$$4 \cdot p \cdot d \leq 1$$

then $\Pr\left[\bigwedge_{i=0}^{n} \overline{A_i}\right] > 0.$

Proof: put $x_i = \frac{1}{d+1}$, $\forall i$, in the above theorem.

$\square$

Without Local lemma,

$$\Pr[A_1 \vee A_2 \vee \dots \vee A_n] \leq \Pr(A_1) + \Pr(A_2) + \dots + \Pr(A_n)$$

$$\leq np$$

If $np < 1$, then $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$

if A_1 & A_2 are independent.

$$\Pr[A_1 \vee A_2] = \Pr(A_1) + \Pr(A_2) - \Pr[A_1] \cdot \Pr[A_2]$$

if A_1 and A_2 are mutually exclusive

$$\Pr[A_1 \vee A_2] = \Pr[A_1] + \Pr[A_2]$$

Applications.

Thm: Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph. If $|E| < 2^{k-1}$, then $\exists$ a coloring of vertices in V with 2 colors such that every hyperedge $e \in E$ sees two colors.

$$V = \{\overset{G}{1}, \overset{G}{2}, \overset{R}{3}, \overset{R}{4}, \overset{R}{5}\}$$

$\mathcal{H} = (V, E)$ is 3-uniform

$$E = \{\{\overset{G}{1}, \overset{G}{2}, \overset{R}{3}\}, \{\overset{G}{1}, \overset{R}{3}, \overset{R}{4}\}, \{\overset{G}{2}, \overset{R}{3}, \overset{R}{5}\}\}$$

Local lemma can prove something stronger:

Theorem Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph where every hyperedge overlaps at most 2^{k-3} other hyperedges.

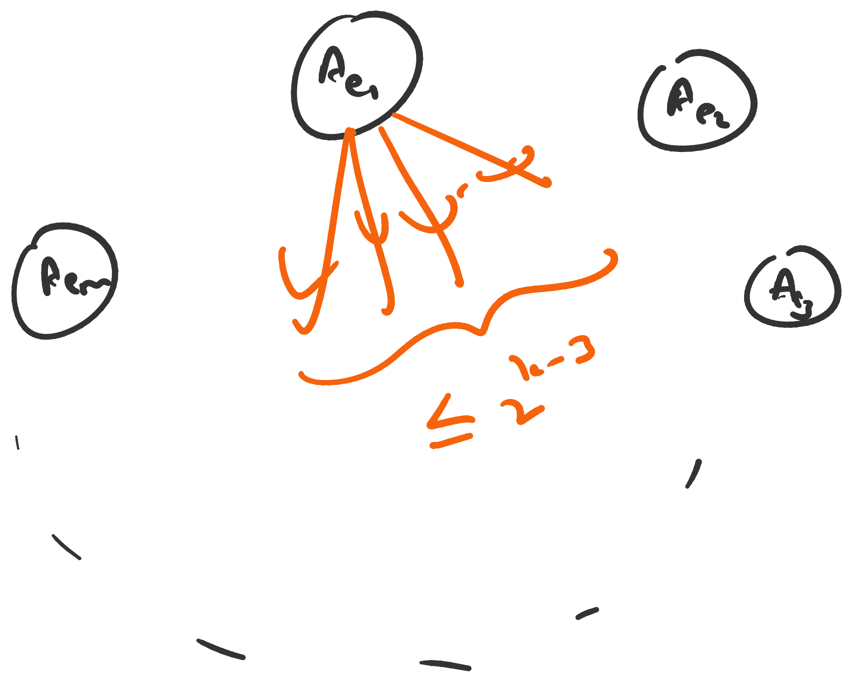
Then, there $\exists$ a coloring of vertices in V with 2 colors s.t. every hyperedge sees both the colors.

Proof: For each vertex $v \in V$, independently and uniformly at random assign it a color from the set $\{RED, BLUE\}$.

For each hyperedge $e \in E$, let A_e denote the 'bad' event that e is monochromatic under the coloring.

$$\Pr[A_e] = \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}}$$

Let $E = \{e_1, e_2, \ldots, e_m\}$



Dependency graph

$d = 2^{k-3}$

$$4 \cdot p \cdot d = 4 \cdot \frac{1}{2^{k-3}} \cdot 2^{k-3} \leq 1.$$

$$4 \cdot p^{\cdots} = 4 \cdot \frac{1}{2^{k-3}} \cdot 2^{\cdots} \leq 1.$$

Hence, by Local lemma, $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0.$

$\square$

Satisfiability Problem

Let $x_1, x_2, \ldots, x_n$ be Boolean variables.

A literal is a Boolean variable in its original form or in the negated form.

$$x_i \text{ or } \overline{x_i}.$$

A Conjunctive Normal Form (CNF) is a Boolean formula which is AND of many clauses. A clause is an OR of many literals.

A CNF is called a k-CNF if every clause has exactly k literals.

Example:

$$\phi = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\overset{1}{\bar{x}_1} \vee \overset{0}{x_2} \vee \overset{0}{x_3}) \wedge (x_2 \vee x_3 \vee \bar{x}_1)$$

↳ a 3-CNF.

For each clause C, let A_C denote the event that the clause C is not satisfied.

$$\Pr[A_C] = \frac{1}{2^k}$$

two clauses **overlap** when they share the same variable.

Theorem: Let ϕ be a k-CNF Boolean formula where every clause overlaps with at most 2^{k-2} other clauses. Then, ϕ is satisfiable.

Proof: Let $x_1, x_2, \ldots, x_n$ be the variables on which ϕ is defined.

Independently and uniformly at random assign each x_i a value from the set

$\{ \ \ \}$

$\{0,1\}$.

Consider a clause C.

Let A_C denote the 'bad' event that C is not satisfied in this random assignment.

$$\Pr[A_C] = \frac{1}{2^k}.$$

$$d = 2^{k-2}. \text{ (given)}.$$

Since $4 \cdot p \cdot d = 4 \cdot \frac{1}{2^k} \cdot 2^{k-2} \leq 1$,

by Local lemma, $\Pr\left[\bigwedge_{\substack{C \text{ is} \\ \text{a clause} \\ \text{in } \phi}} \overline{A_C}\right] > 0$.

So ϕ is satisfiable.

$\Box$.

<u>Non-trivial</u> <u>applications of the local lemma</u>

<u>Theorem</u>. Every <u>k-regular</u> <u>digraph</u> has a

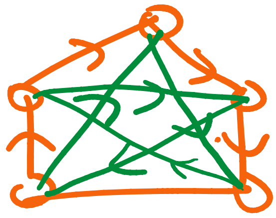
in-degree and out-degree of every vertex is k

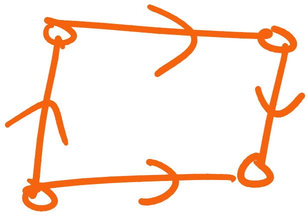
directed graph with no multiedges

collection of $\lfloor \frac{k}{\ } \rfloor$ vertex disjoint ...

collection of $\left\lfloor \frac{k}{3\ln k} \right\rfloor$ vertex disjoint cycles.

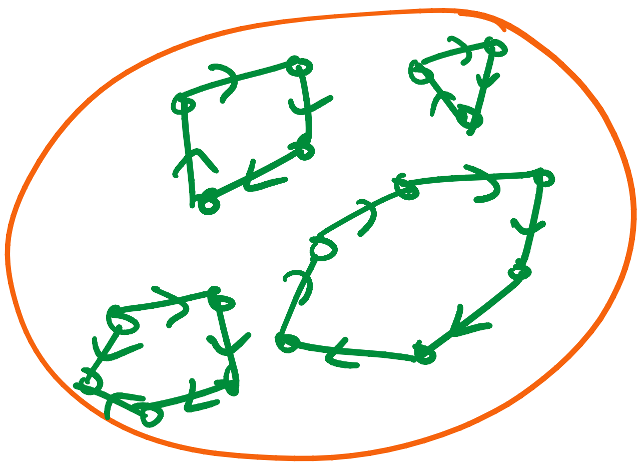
Proof: Let G be a k-regular digraph.





2-regular digraph

1-regular digraph



Let $r = \left\lfloor \frac{k}{3\ln k} \right\rfloor$.

Idea: Color the vertices of G with r colors (doesn't have to be proper coloring) such that every vertex sees all the r colors in its out-neighborhood.

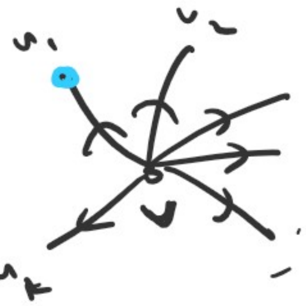
Let S be a set of r colors.
Choose a color of each vertex independently and uniformly at random from S.

For a vertex v in G,

A_v: is the 'bad' event that some color color in S does not appear in the out-neighborhood of v.

$$p := \Pr[A_v] \le \underbrace{r\left(1-\frac{1}{r}\right)^k}_{\text{bad prob that blue color is not present in out-neighborhood of } v.} \le r\cdot e^{-\frac{k}{r}} \quad \left(\text{since } 1+x \le e^x\right) \quad (1)$$

$$\le r\cdot\frac{1}{e^{\frac{3\cdot\cancel{k}\cdot\ln k}{\cancel{k}}}} = \frac{r}{k^3} \le \frac{1}{3k^2\ln k}$$



To apply Local lemma,

let $\{v_1, v_2, \dots, v_n\} := V(G)$.

$A_{v_1}, A_{v_2}, \dots, A_{v_n}$

Dependency graph

A_{v_i}

How many vertices v_i $(i \neq 1)$ have their out-neighborhood overlap with the out-neighborhood of v_1?   ANS: $\leq k(k-1)$.

This is the degree d of the dependency graph.

Local lemma says, if

$$4 \cdot p \cdot d \leq 1, \text{ then } \Pr\Big[\bigwedge_{i=1}^{n} \bar{A}_{v_i}\Big] > 0.$$

Since $4 \cdot \frac{1}{3k^2 \ln k} \cdot k(k-1) \leq 1$, we are done.

$\square$

Can you improve this bound?

Say, can you set $\left\lfloor \frac{k}{3 \ln \ln k} \right\rfloor$?

Theorem Let k be sufficiently large. Let $\mathcal{F}$ be a k-uniform family of sets and suppose that no point belongs to more than k sets that no point belongs to more than k sets of $\mathcal{F}$. Then it is possible to color the points in $r = \lfloor \frac{k}{\log k} \rfloor$ colors so that every member of $\mathcal{F}$ has at most $v = \lceil 2e \log k \rceil$ points of the same color.

Proof: Let A be a set of r $= \lfloor \frac{k}{\log k} \rfloor$ colors.

For each point, indep & uniformly at random choose a color from A and assign that color to it.

For a set S in $\mathcal{F}$ (k-sized) and for a color $i \in A$, we define a 'bad' event

$E(S, i)$: the event of S containing more than $\lceil 2e \log k \rceil$ points of color i under the random coloring described above.

$$p := \Pr[E(S, i)] \le \binom{k}{v}\left(\frac{1}{r}\right)^{v}$$

$$\leq \left(\frac{ek}{v}\right)^v \cdot \frac{1}{r^v}$$

$$= \left(\frac{ek}{vr}\right)^v$$

$v = \left\lfloor \frac{k}{\log k} \right\rfloor$

$r = \lceil 2e \log k \rceil$

$$\leq \left(\frac{e\cancel{k}}{2e\cancel{\log k} \cdot \left(\frac{\cancel{k}}{\cancel{\log k}}\right)}\right)^v$$

$$\leq \frac{1}{2^v} = \frac{1}{2^{2e\log k}} \leq \frac{1}{k^{2e}} \quad \text{— (1)}$$

Construct the dependency graph with vertices $E(S, i)$, S is a subset of $\mathcal{F}$, $i \in A$.

Degree of the dependency graph d:

$$d \leq k(k-1)\frac{k}{\log k} \leq k^3 \quad \text{— (2)}$$

Clearly, from (1) & (2),

$$4 \cdot p \cdot d \leq 4 \cdot \frac{1}{k^{2e}} \cdot k^3 \leq 1.$$

Hence, by local lemma, $\Pr\left[\bigwedge_{\substack{i\in A,\\ S\in \mathcal{F}}} \overline{E(S,i)}\right] > 0.$

$\square$