Derivative of Piola mapping

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Abstract

We present the derivatives of the Piola mapping.

1. First-order derivatives

If we have a mapping $(\xi, \eta) \to (x, y)$, then its Jacobian is defined as

$$J = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \quad , \quad |J| = x_\xi y_\eta - x_\eta y_\xi \tag{1}$$

From the general inversion formula, we get

$$J^{-1} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} \tag{2}$$

From the chain rule, we have an universal formula for the total derivative:

$$f_x = f_\xi \xi_x + f_\eta \eta_x \quad , \quad f_y = f_\xi \xi_y + f_\eta \eta_y \tag{3}$$

The Piola mapping is defined as $J/|J|$. Its first-order derivatives become

$$\frac{\partial}{\partial x}\left[\frac{1}{|J|}J\right] = -\frac{|J|_x}{|J|^2}J + \frac{1}{|J|}J_x \quad , \quad \frac{\partial}{\partial y}\left[\frac{1}{|J|}J\right] = -\frac{|J|_y}{|J|^2}J + \frac{1}{|J|}J_y \tag{4}$$

Further computations yield two similar formulas:

$$|J|_x = \xi_x \widetilde{J}_1 + \eta_x \widetilde{J}_2 \quad , \quad |J|_y = \xi_y \widetilde{J}_1 + \eta_y \widetilde{J}_2 \tag{5}$$

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In this setting, the auxiliary functions are

$$\widetilde{J}_1 = x_{\xi\xi}y_\eta + x_\xi y_{\xi\eta} - x_{\xi\eta}y_\xi - x_\eta y_{\xi\xi} \tag{6a}$$

$$\widetilde{J}_2 = x_{\xi\eta}y_\eta + x_\xi y_{\eta\eta} - x_{\eta\eta}y_\xi - x_\eta y_{\xi\eta} \tag{6b}$$

Likewise, we get the following matrix derivatives:

$$J_x = \xi_x \begin{bmatrix} x_{\xi\xi} & x_{\xi\eta} \\ y_{\xi\xi} & y_{\xi\eta} \end{bmatrix} + \eta_x \begin{bmatrix} x_{\xi\eta} & x_{\eta\eta} \\ y_{\xi\eta} & y_{\eta\eta} \end{bmatrix} \tag{7a}$$

$$J_y = \xi_y \begin{bmatrix} x_{\xi\xi} & x_{\xi\eta} \\ y_{\xi\xi} & y_{\xi\eta} \end{bmatrix} + \eta_y \begin{bmatrix} x_{\xi\eta} & x_{\eta\eta} \\ y_{\xi\eta} & y_{\eta\eta} \end{bmatrix} \tag{7b}$$

Imagine now that we have a vector function on the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{|J|} J \begin{bmatrix} U \\ V \end{bmatrix}$$

By using the product rule, we obtain

$$\begin{bmatrix} u \\ v \end{bmatrix}_x = \left(-\frac{|J|_x}{|J|^2} J + \frac{1}{|J|} J_x \right) \begin{bmatrix} U \\ V \end{bmatrix} + \frac{1}{|J|} J \begin{bmatrix} U_\xi \xi_x + U_\eta \eta_x \\ V_\xi \xi_x + V_\eta \eta_x \end{bmatrix} \tag{8a}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_y = \left(-\frac{|J|_y}{|J|^2} J + \frac{1}{|J|} J_y \right) \begin{bmatrix} U \\ V \end{bmatrix} + \frac{1}{|J|} J \begin{bmatrix} U_\xi \xi_y + U_\eta \eta_y \\ V_\xi \xi_y + V_\eta \eta_y \end{bmatrix} \tag{8b}$$

2. Second-order derivatives

From the double product rule, we have

$$\left[\frac{1}{|J|} J \right]_{xx} = \left[\frac{1}{|J|} \right]_{xx} J + 2 \left[\frac{1}{|J|} \right]_x J_x + \frac{1}{|J|} J_{xx} \tag{9a}$$

$$\left[\frac{1}{|J|} J \right]_{yy} = \left[\frac{1}{|J|} \right]_{yy} J + 2 \left[\frac{1}{|J|} \right]_y J_y + \frac{1}{|J|} J_{yy} \tag{9b}$$

Applying the same rule on $1/|J|$ yields

$$\left[\frac{1}{|J|} \right]_{xx} = -\frac{|J|_{xx}}{|J|^2} + \frac{|J|_x^2}{|J|^3} \tag{10a}$$

$$\left[\frac{1}{|J|} \right]_{yy} = -\frac{|J|_{yy}}{|J|^2} + \frac{|J|_y^2}{|J|^3} \tag{10b}$$

As we see from these formulas, the real objective is finding $|J|_{xx}$, $|J|_{yy}$, J_{xx} and J_{yy}. We start with J_{xx} and use the total derivative rule:

$$
\begin{aligned}
J_{xx} &= \xi_{xx} \begin{bmatrix} x_{\xi\xi} & x_{\xi\eta} \\ y_{\xi\xi} & y_{\xi\eta} \end{bmatrix}_x + \eta_{xx} \begin{bmatrix} x_{\xi\eta} & x_{\eta\eta} \\ y_{\xi\eta} & y_{\eta\eta} \end{bmatrix}_x \\
&= \xi_{xx} \begin{bmatrix} x_{\xi\xi\xi}\xi_x + x_{\xi\xi\eta}\eta_x & x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x \\ y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x & y_{\xi\xi\eta}\xi_x + y_{\xi\eta\eta}\eta_x \end{bmatrix} + \eta_{xx} \begin{bmatrix} x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x & x_{\xi\eta\eta}\xi_x + x_{\eta\eta\eta}\eta_x \\ y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x & y_{\xi\eta\eta}\xi_x + y_{\eta\eta\eta}\eta_x \end{bmatrix}
\end{aligned}
$$

Summarizing, we have arrived at

$$
J_{xx} = \xi_{xx} \begin{bmatrix} x_{\xi\xi\xi}\xi_x + x_{\xi\xi\eta}\eta_x & x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x \\ y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x & y_{\xi\xi\eta}\xi_x + y_{\xi\eta\eta}\eta_x \end{bmatrix} + \eta_{xx} \begin{bmatrix} x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x & x_{\xi\eta\eta}\xi_x + x_{\eta\eta\eta}\eta_x \\ y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x & y_{\xi\eta\eta}\xi_x + y_{\eta\eta\eta}\eta_x \end{bmatrix} \tag{11a}
$$

$$
J_{yy} = \xi_{yy} \begin{bmatrix} x_{\xi\xi\xi}\xi_y + x_{\xi\xi\eta}\eta_y & x_{\xi\xi\eta}\xi_y + x_{\xi\eta\eta}\eta_y \\ y_{\xi\xi\xi}\xi_y + y_{\xi\xi\eta}\eta_y & y_{\xi\xi\eta}\xi_y + y_{\xi\eta\eta}\eta_y \end{bmatrix} + \eta_{xx} \begin{bmatrix} x_{\xi\xi\eta}\xi_y + x_{\xi\eta\eta}\eta_y & x_{\xi\eta\eta}\xi_y + x_{\eta\eta\eta}\eta_y \\ y_{\xi\xi\xi}\xi_y + y_{\xi\xi\eta}\eta_y & y_{\xi\eta\eta}\xi_y + y_{\eta\eta\eta}\eta_y \end{bmatrix} \tag{11b}
$$

A similar derivation is valid for U and V: