Derivative of Piola mapping

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Abstract

We present the derivatives of the Piola mapping.

1. First-order derivatives

If we have a mapping $(\xi, \eta) \to (x, y)$, then its Jacobian is defined as

$$J = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \qquad , \qquad |J| = x_\xi y_\eta - x_\eta y_\xi \tag{1}$$

From the general inversion formula, we get

$$J^{-1} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} \tag{2}$$

From the chain rule, we have an universal formula for the total derivative:

$$f_x = f_\xi \xi_x + f_\eta \eta_x \qquad , \qquad f_y = f_\xi \xi_y + f_\eta \eta_y \tag{3}$$

The Piola mapping is defined as $J/|J|$. Its first-order derivatives become

$$\frac{\partial}{\partial x}\left[\frac{1}{|J|}J\right] = -\frac{|J|_x}{|J|^2}J + \frac{1}{|J|}J_x \tag{4a}$$

$$\frac{\partial}{\partial y}\left[\frac{1}{|J|}J\right] = -\frac{|J|_y}{|J|^2}J + \frac{1}{|J|}J_y \tag{4b}$$

Further computations yield two similar formulas:

$$|J|_x = \xi_x \widetilde{J}_1 + \eta_x \widetilde{J}_2 \tag{5a}$$

$$|J|_y = \xi_y \widetilde{J}_1 + \eta_y \widetilde{J}_2 \tag{5b}$$

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In this setting, the auxiliary functions are

$$\widetilde{J}_1 = x_{\xi\xi}y_\eta + x_\xi y_{\xi\eta} - x_{\xi\eta}y_\xi - x_\eta y_{\xi\xi} \tag{6a}$$

$$\widetilde{J}_2 = x_{\xi\eta}y_\eta + x_\xi y_{\eta\eta} - x_{\eta\eta}y_\xi - x_\eta y_{\xi\eta} \tag{6b}$$

Likewise, we get the following matrix derivatives:

$$J_x = \xi_x \begin{bmatrix} x_{\xi\xi} & x_{\xi\eta} \\ y_{\xi\xi} & y_{\xi\eta} \end{bmatrix} + \eta_x \begin{bmatrix} x_{\xi\eta} & x_{\eta\eta} \\ y_{\xi\eta} & y_{\eta\eta} \end{bmatrix} \tag{7a}$$

$$J_y = \xi_y \begin{bmatrix} x_{\xi\xi} & x_{\xi\eta} \\ y_{\xi\xi} & y_{\xi\eta} \end{bmatrix} + \eta_y \begin{bmatrix} x_{\xi\eta} & x_{\eta\eta} \\ y_{\xi\eta} & y_{\eta\eta} \end{bmatrix} \tag{7b}$$

Imagine now that we have a vector function on the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{|J|} J \begin{bmatrix} U \\ V \end{bmatrix}$$

By using the product rule, we obtain

$$\begin{bmatrix} u \\ v \end{bmatrix}_x = \left(-\frac{|J|_x}{|J|^2} J + \frac{1}{|J|} J_x \right) \begin{bmatrix} U \\ V \end{bmatrix} + \frac{1}{|J|} J \begin{bmatrix} U_\xi \xi_x + U_\eta \eta_x \\ V_\xi \xi_x + V_\eta \eta_x \end{bmatrix} \tag{8a}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}_y = \left(-\frac{|J|_y}{|J|^2} J + \frac{1}{|J|} J_y \right) \begin{bmatrix} U \\ V \end{bmatrix} + \frac{1}{|J|} J \begin{bmatrix} U_\xi \xi_y + U_\eta \eta_y \\ V_\xi \xi_y + V_\eta \eta_y \end{bmatrix} \tag{8b}$$

2. Second-order derivatives

From the double product rule, we have

$$\left[\frac{1}{|J|} J \right]_{xx} = \left[\frac{1}{|J|} \right]_{xx} J + 2 \left[\frac{1}{|J|} \right]_x J_x + \frac{1}{|J|} J_{xx} \tag{9a}$$

$$\left[\frac{1}{|J|} J \right]_{yy} = \left[\frac{1}{|J|} \right]_{yy} J + 2 \left[\frac{1}{|J|} \right]_y J_y + \frac{1}{|J|} J_{yy} \tag{9b}$$

Applying the same rule on $1/|J|$ yields

$$\left[\frac{1}{|J|} \right]_{xx} = -\frac{|J|_{xx}}{|J|^2} + \frac{|J|_x^2}{|J|^3} \tag{10a}$$

$$\left[\frac{1}{|J|} \right]_{yy} = -\frac{|J|_{yy}}{|J|^2} + \frac{|J|_y^2}{|J|^3} \tag{10b}$$

As we see from these formulas, the real objective is finding $|J|_{xx}$, $|J|_{yy}$, J_{xx} and J_{yy}. We start with J_{xx} and use the total derivative rule:

$$\begin{aligned}
J_{xx} &= \xi_{xx} \begin{bmatrix} x_{\xi\xi} & x_{\xi\eta} \\ y_{\xi\xi} & y_{\xi\eta} \end{bmatrix}_x + \eta_{xx} \begin{bmatrix} x_{\xi\eta} & x_{\eta\eta} \\ y_{\xi\eta} & y_{\eta\eta} \end{bmatrix}_x \\
&= \xi_{xx} \begin{bmatrix} x_{\xi\xi\xi}\xi_x + x_{\xi\xi\eta}\eta_x & x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x \\ y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x & y_{\xi\xi\eta}\xi_x + y_{\xi\eta\eta}\eta_x \end{bmatrix} + \eta_{xx} \begin{bmatrix} x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x & x_{\xi\eta\eta}\xi_x + x_{\eta\eta\eta}\eta_x \\ y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x & y_{\xi\eta\eta}\xi_x + y_{\eta\eta\eta}\eta_x \end{bmatrix}
\end{aligned}$$

Summarizing, we have arrived at

$$J_{xx} = \xi_{xx} \begin{bmatrix} x_{\xi\xi\xi}\xi_x + x_{\xi\xi\eta}\eta_x & x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x \\ y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x & y_{\xi\xi\eta}\xi_x + y_{\xi\eta\eta}\eta_x \end{bmatrix} + \eta_{xx} \begin{bmatrix} x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x & x_{\xi\eta\eta}\xi_x + x_{\eta\eta\eta}\eta_x \\ y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x & y_{\xi\eta\eta}\xi_x + y_{\eta\eta\eta}\eta_x \end{bmatrix} \tag{11a}$$

$$J_{yy} = \xi_{yy} \begin{bmatrix} x_{\xi\xi\xi}\xi_y + x_{\xi\xi\eta}\eta_y & x_{\xi\xi\eta}\xi_y + x_{\xi\eta\eta}\eta_y \\ y_{\xi\xi\xi}\xi_y + y_{\xi\xi\eta}\eta_y & y_{\xi\xi\eta}\xi_y + y_{\xi\eta\eta}\eta_y \end{bmatrix} + \eta_{xx} \begin{bmatrix} x_{\xi\xi\eta}\xi_y + x_{\xi\eta\eta}\eta_y & x_{\xi\eta\eta}\xi_y + x_{\eta\eta\eta}\eta_y \\ y_{\xi\xi\xi}\xi_y + y_{\xi\xi\eta}\eta_y & y_{\xi\eta\eta}\xi_y + y_{\eta\eta\eta}\eta_y \end{bmatrix} \tag{11b}$$

The product rule implies that

$$\begin{aligned}
|J|_{xx} &= \xi_{xx}\widetilde{J}_1 + \xi_x(\partial_x\widetilde{J}_1) + \eta_{xx}\widetilde{J}_2 + \eta_x(\partial_x\widetilde{J}_2) \\
|J|_{yy} &= \xi_{yy}\widetilde{J}_1 + \xi_y(\partial_y\widetilde{J}_1) + \eta_{yy}\widetilde{J}_2 + \eta_y(\partial_y\widetilde{J}_2)
\end{aligned}$$

The real challenge is differentiating $\widetilde{J}_1$ and $\widetilde{J}_2$. This requires that we must introduce 16 auxiliary functions in total:

$$\begin{aligned}
k_{1x} &= (x_{\xi\xi\xi}\xi_x + x_{\xi\xi\eta}\eta_x)y_\eta + x_{\xi\xi}(y_{\xi\eta}\xi_x + y_{\eta\eta}\eta_x) \\
k_{2x} &= (x_{\xi\xi}\xi_x + x_{\xi\eta}\eta_x)y_{\xi\eta} + x_\xi(y_{\xi\xi\eta}\xi_x + y_{\xi\eta\eta}\eta_x) \\
k_{3x} &= (x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x)y_\xi + x_{\xi\eta}(y_{\xi\xi}\xi_x + y_{\xi\eta}\eta_x) \\
k_{4x} &= (x_{\xi\eta}\eta_x + x_{\eta\eta}\eta_x)y_{\xi\xi} + x_\eta(y_{\xi\xi\xi}\xi_x + y_{\xi\xi\eta}\eta_x) \\
k_{5x} &= (x_{\xi\xi\eta}\xi_x + x_{\xi\eta\eta}\eta_x)y_\eta + x_{\xi\eta}(y_{\xi\eta}\xi_x + y_{\eta\eta}\eta_x) \\
k_{6x} &= (x_{\xi\xi}\xi_x + x_{\xi\eta}\eta_x)y_{\eta\eta} + x_\xi(y_{\xi\eta\eta}\xi_x + y_{\eta\eta\eta}\eta_x) \\
k_{7x} &= (x_{\xi\eta\eta}\xi_x + x_{\eta\eta\eta}\eta_x)y_\xi + x_{\eta\eta}(y_{\xi\xi}\xi_x + y_{\xi\eta}\eta_x) \\
k_{8x} &= (x_{\xi\eta}\eta_x + x_{\eta\eta}\eta_x)y_{\xi\eta} + x_\eta(y_{\xi\xi\eta}\xi_x + y_{\xi\eta\eta}\eta_x) \\
k_{1y} &= (x_{\xi\xi\xi}\xi_y + x_{\xi\xi\eta}\eta_y)y_\eta + x_{\xi\xi}(y_{\xi\eta}\xi_y + y_{\eta\eta}\eta_y) \\
k_{2y} &= (x_{\xi\xi}\xi_y + x_{\xi\eta}\eta_y)y_{\xi\eta} + x_\xi(y_{\xi\xi\eta}\xi_y + y_{\xi\eta\eta}\eta_y) \\
k_{3y} &= (x_{\xi\xi\eta}\xi_y + x_{\xi\eta\eta}\eta_y)y_\xi + x_{\xi\eta}(y_{\xi\xi}\xi_y + y_{\xi\eta}\eta_y) \\
k_{4y} &= (x_{\xi\eta}\eta_y + x_{\eta\eta}\eta_y)y_{\xi\xi} + x_\eta(y_{\xi\xi\xi}\xi_y + y_{\xi\xi\eta}\eta_y) \\
k_{5y} &= (x_{\xi\xi\eta}\xi_y + x_{\xi\eta\eta}\eta_y)y_\eta + x_{\xi\eta}(y_{\xi\eta}\xi_y + y_{\eta\eta}\eta_y) \\
k_{6y} &= (x_{\xi\xi}\xi_y + x_{\xi\eta}\eta_y)y_{\eta\eta} + x_\xi(y_{\xi\eta\eta}\xi_y + y_{\eta\eta\eta}\eta_y) \\
k_{7y} &= (x_{\xi\eta\eta}\xi_y + x_{\eta\eta\eta}\eta_y)y_\xi + x_{\eta\eta}(y_{\xi\xi}\xi_y + y_{\xi\eta}\eta_y) \\
k_{8y} &= (x_{\xi\eta}\eta_y + x_{\eta\eta}\eta_y)y_{\xi\eta} + x_\eta(y_{\xi\xi\eta}\xi_y + y_{\xi\eta\eta}\eta_y)
\end{aligned}$$

Thus, we can define the four essential derivatives as

$$\partial_x \widetilde{J}_1 = k_{1x} + k_{2x} - k_{3x} - k_{4x} \tag{12a}$$

$$\partial_y \widetilde{J}_1 = k_{1y} + k_{2y} - k_{3y} - k_{4y} \tag{12b}$$

$$\partial_x \widetilde{J}_2 = k_{5x} + k_{6x} - k_{7x} - k_{8x} \tag{12c}$$

$$\partial_y \widetilde{J}_2 = k_{5y} + k_{6y} - k_{7y} - k_{8y} \tag{12d}$$

A similar easier derivation is valid for U:

$$\begin{aligned} U_{xx} &= \begin{bmatrix} (U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x)\xi_x + U_\xi\xi_{xx} + (U_{\xi\eta}\xi_x + U_{\eta\eta}\eta_x)\eta_x + U_\eta\eta_{xx} \\ (V_{\xi\xi}\xi_x + V_{\xi\eta}\eta_x)\xi_x + V_\xi\xi_{xx} + (V_{\xi\eta}\xi_x + V_{\eta\eta}\eta_x)\eta_x + V_\eta\eta_{xx} \end{bmatrix} \\ &= \begin{bmatrix} U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x & U_{\xi\eta}\xi_x + U_{\eta\eta}\eta_x \\ V_{\xi\xi}\xi_x + V_{\xi\eta}\eta_x & V_{\xi\eta}\xi_x + V_{\eta\eta}\eta_x \end{bmatrix} \begin{bmatrix} \xi_x \\ \eta_x \end{bmatrix} + \begin{bmatrix} U_\xi & U_\eta \\ V_\xi & V_\eta \end{bmatrix} \begin{bmatrix} \xi_{xx} \\ \eta_{xx} \end{bmatrix} \end{aligned}$$

Hence, the double-derivatives of U and V are given by

$$\begin{bmatrix} U \\ V \end{bmatrix}_{xx} = \begin{bmatrix} U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x & U_{\xi\eta}\xi_x + U_{\eta\eta}\eta_x \\ V_{\xi\xi}\xi_x + V_{\xi\eta}\eta_x & V_{\xi\eta}\xi_x + V_{\eta\eta}\eta_x \end{bmatrix} \begin{bmatrix} \xi_x \\ \eta_x \end{bmatrix} + \begin{bmatrix} U_\xi & U_\eta \\ V_\xi & V_\eta \end{bmatrix} \begin{bmatrix} \xi_{xx} \\ \eta_{xx} \end{bmatrix} \tag{13a}$$

$$\begin{bmatrix} U \\ V \end{bmatrix}_{yy} = \begin{bmatrix} U_{\xi\xi}\xi_y + U_{\xi\eta}\eta_y & U_{\xi\eta}\xi_y + U_{\eta\eta}\eta_y \\ V_{\xi\xi}\xi_y + V_{\xi\eta}\eta_y & V_{\xi\eta}\xi_y + V_{\eta\eta}\eta_y \end{bmatrix} \begin{bmatrix} \xi_y \\ \eta_y \end{bmatrix} + \begin{bmatrix} U_\xi & U_\eta \\ V_\xi & V_\eta \end{bmatrix} \begin{bmatrix} \xi_{yy} \\ \eta_{yy} \end{bmatrix} \tag{13b}$$