Contents

1 Jan 13, Monday

- Administrative details:
  - **Grading**: $20 + 20 + 60$
  - **Class timings**: Keep all four slots open (M,T,Th,Fr); This week we will meet on M,T,Th
  - **Pop quiz**: Attendance
- Motivation
- Frequentist and Bayesian interpretation of probability
- Randomness: (i) Inherently random; (ii) Lack of information
- Sets: Forms the basis over which we will define probability; Sample space; Loosely speaking is a collection of objects
- Defining a set: (i) Explicitly listing out the elements; (ii) Defining through a property
- Finite set
- Infinite set:
  - Countably infinite set: A bijection from the set of natural numbers to the set exists;
    Examples: $\mathbb{N}$, $\mathbb{Z}$, Set of even integers, Set of prime numbers, $\mathbb{Q}$
  - Uncountably infinite set (or uncountable set): No bijection from the set of natural numbers to the set exists;
    Examples: $\mathbb{R}$, the interval $(0, 1)$, any non-empty interval on $\mathbb{R}$
  - When we define random variables, we will define probability mass function (in the case of discrete random variables) and probability density function (in the case of continuous random variables) depending on whether the underlying sample space is countable or uncountable.

2 Jan 14, Tuesday

- Set: Union, Intersection, Complement, Infinite union, infinite intersection; Examples
- Two sets are disjoint if their intersection is empty.
- A collection of sets is a partition of a set S if their union is S and each pair is mutually disjoint.
- Venn Diagram
- Some trivial properties:
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
  - $A \cup (B \cup C) = (A \cup B) \cup (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cup C) = (A \cup B) \cup (A \cup C)$
  - De Morgan's law:
  $$(\cup_n S_n)^c = \cap_n S_n^c$$
  $$(\cap_n S_n)^c = \cup_n S_n^c$$
- **Probability**: To define a probability model, we need the following ingredients.
  - **Experiment**: Process of measurement or observation
  - **Trial**: A single performace of an experiment
  - **Outcome**: Result of a trial
  - **Sample space**: Set of all possible distinct outcomes of an experiment
  - **Event**: Some collection of distinct outcomes, i.e., a subset of the sample space

- **Probability law**: A number tagged to *certain events*. If the probability of an event is defined, then they have to satisfy certain constraints.
  * $0 \leq \mathbb{P}(A) \leq 1$ where $A \subseteq \Omega$
  * $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ whenever A and B are disjoint
  * $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \displaystyle\sum_{i=1}^{n} \mathbb{P}(A_i)$, whenever A_i's are mutually disjoint
  * Normalization: $\mathbb{P}(\Omega) = 1$
  * $\mathbb{P}(\emptyset) = 0$
  * $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$

  Not all constraints above are independent. Also, note that it is **okay** not to define probability for all subsets of Ω.

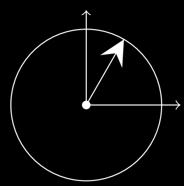
3 Jan 16, Thursday

- **Probability law**: A number tagged to *certain events* of the sample space satisfying the following independent constraints:
  * $0 \leq \mathbb{P}(A)$ for every $A \subseteq \Omega$
  * $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \displaystyle\sum_{i=1}^{n} \mathbb{P}(A_i)$, whenever A_i's are mutually disjoint
  * Normalization: $\mathbb{P}(\Omega) = 1$

  The other constraints can be derived from these.
- For example, consider rolling a six-sided die.
  * Sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$
  * One could defined $\mathbb{P}(\{1, 3\}) = 1/2$.
  * From this, we obtain that $\mathbb{P}(\{2, 4, 5, 6\}) = 1/2$.
  * We also have $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$.

  Note that if one asks the probability of the event of obtaining an odd number, we cannot give an answer. However, the above is a valid probability function, since it doesn't violate the definition of probability.
- If A and B are not disjoint, then
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$
- Probability of a finite set: If $A = \{a_1, a_2, \ldots, a_n\}$ is an n-element set, and if $\mathbb{P}(a_i) = p_i$, then
$$\mathbb{P}(A) = \sum_{i=1}^{n} p_i$$
- Probability of a countably infinite set: If $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ is a countably infinite set, and if $\mathbb{P}(a_i) = p_i$, then
$$\mathbb{P}(A) = \sum_{i=1}^{\infty} p_i$$
- Probability law on a finite sample space $\Omega = \{s_1, s_2, \ldots, s_n\}$ is said to be discrete uniform probability if
$$\mathbb{P}(s_1) = \mathbb{P}(s_2) = \cdots = \mathbb{P}(s_n)$$
- Under the discrete uniform probability law, if $A \subseteq \Omega$,
$$\mathbb{P}(A) = \frac{\text{Number of elements in } A}{\text{Number of elements in } \Omega}$$
- Note that discrete uniform probability doesn't exist for countably infinite set.
- Rolling a pair of six-sided fair dice and noting the sum of their top faces.
  * Experiment is rolling two dices and noting the sum of their top faces.

  * Trial is rolling two dices once.
  * Outcome is sum of the top faces on a single trial.
  * Sample space $\Omega = \{2, 3, 4, \ldots, 12\}$.
  * $\mathbb{P}(\{i\}) = \dfrac{6 - |7 - i|}{36}$ for $i \in \{2, 3, \ldots, 12\}$.
- Spinning an arrow on a horizontal surface. Angle the arrow makes with X axis lies "uniformly" in the interval $[0, 2\pi)$.



- Note that $\mathbb{P}(\Theta = \pi/2) = 0$. In fact, probability of Θ taking any specific value is zero.
- However, $\mathbb{P}(\Theta \in [0, 2\pi]) = 1$. Intuitively, we can also see that $\mathbb{P}(\Theta \in [a, b]) = \dfrac{b - a}{2\pi}$, where $0 \leq a \leq b \leq 2\pi$.
- **Paradox(?)**: We have

$$1 = \mathbb{P}([0, 2\pi]) = \mathbb{P}\left(\cup_{x \in [0,2\pi]} \{x\}\right) = \sum_{x \in [0,2\pi]} \mathbb{P}(\{x\}) = 0$$

  The fallacy in the above statement lies in the third equality. Probability of a countable union of disjoint sets is sum of their individual probabilities. However, in this case, the union is an uncountable union.