Contents

1 Jan 14, Tuesday

- Administrative details:
  - **Grading**: $30 + 15 + 15 + 40$
  - **Class timings**: Keep all four slots open (T,W,Th,Fr); This week we will be travelling on Thursday, Friday.
  - **Pop quiz**: Attendance
- Optimization in nature:
  - Principle of least action
  - Angle of reflection equals angle of incidence since light minimizes the time taken to travel between two points.
- **Generic optimization problem**:

$$\boxed{\min_{x \in \mathbb{R}^n} f(x) \text{ subject to } x \in S \qquad (\spadesuit)}$$

  x is the unknowns/parameters/variables, $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a real valued function, and S is a set of values that x can take. f is referred to as the objective function, S is referred to as the set of feasible values. Note that a maximisation problem can be reformulated into a minimisation problem, since

$$\max f(x) = -\min(-f(x))$$

- Optimization is also termed as mathematical programming.
- Form of optimization problems:
  - **Continuous, unconstrained optimization**: Minimize $x_1^2 - x_1x_2 + x_2^2 + 4x_1 - 2x_2 + 7$, where $x_1, x_2 \in \mathbb{R}$.
  - **Continuous, constrained optimization**: We want to buy x_1 kilos of onions (vegetable 1), x_2 kilos of potato (vegetable 2) and x_3 kilos of carrot (vegetable 3), where the cost per kilo being c_1, c_2 and c_3 respectively. Each of these vegetables provide 4 nutrients n_1, n_2, n_3 and n_4. Let n_{ij} is the per kilo contribution of vegetable j to nutrient i. We need to ensure that for a balanced diet we need at-least N_i units of nutrient n_i. A natural optimization problem would be to minimize the cost of purchasing the vegetables subject to the nutrient constraint. This can be posed as

$$\text{Minimize } c_1x_1 + c_2x_2 + c_3x_3$$

    subject to

$$\sum_{j=1}^{3} n_{ij}x_j \geq N_i \ \ \forall i \in \{1,2,3,4\}$$

$$x_j \geq 0, x_j \in \mathbb{R}, \ \ \forall j \in \{1,2,3\}$$

    Note that the feasible set is determined by the constraint and is an uncountably infinite set.
  - **Discrete, constrained optimization**: The same problem above could be converted to a discrete problem if instead of cost per kilo and nutrient per kilo, we have cost per piece of vegetable and nutrient per piece of vegetable. The optimization problem now becomes

$$\text{Minimize } c_1x_1 + c_2x_2 + c_3x_3$$

    subject to

$$\sum_{j=1}^{3} n_{ij}x_j \geq N_i \ \ \forall i \in \{1,2,3,4\}$$

$$x_j \in \mathbb{N} \cup \{0\}, \ \ \forall j \in \{1,2,3\}$$

    where the only difference is x_j's now belong to $\mathbb{N} \cup \{0\}$ than $\mathbb{R}$. Note that, as opposed to the previous case, the feasible set is a countably infinite set.
  - **Discrete/Combinatorial optimization**: It is proposed to lay roads between four cities so that the four cities are connected. The cost of laying road between the cities is indicated in Table 1. The goal is to devise a construction strategy with minimum cost. Note that the feasible set in this case is a finite set, since there are only finitely many choices.

Table 1: Cost of road construction in crores

	1	2	3	4
1	–	60	10	100
2	60	–	115	70
3	10	115	–	75
4	100	70	75	–

– **Stochastic optimization**: Consider the problem of buying vegetables again. Here we want to follow a set pattern over the next one month, i.e., every day we need to purchase x_i kilos of vegetable i. However, the cost of the vegetable i varies daily and its distribution is given by $\mathcal{N}\left(c_i, \sigma_i^2\right)$. Further, the nutrient content per kilo corresponding to nutrient i from vegetable j, also varies daily with a distribution given by $\mathcal{N}\left(n_{ij}, s_{ij}^2\right)$. Given this devise a strategy that minimizes the cost on these vegetables over a month.

– In this course, we are only going to look at

$$\boxed{\text{Deterministic, continuous (constrained and unconstrained) optimization problems}}$$

– When we formulate an optimization problem as in (♠), we need to ask couple of questions:

  * Does a solution exist?
  * Is the solution unique?

– In this course, we only look at optimization problems, where f is continuous and the feasible set S is compact (in our case since we are going to work in $\mathbb{R}^n$, hence the set S is closed and bounded.) This guarantees that the minimum exists.

– Typically, the feasible set S is specified by constraints of the form

$$c_i(x) \geq 0, \quad i \in \{1, 2, \ldots, m\}$$

Note that constraints of the form $c_i(x) \leq 0$ can be reformulated as $-c_i(x) \geq 0$. Similarly, equality constraints can be obtained by enforcing $c_i(x) \geq 0$ and $c_i(x) \leq 0$.

– **Convex programming**: Objective function f is a convex function and if the set S is a convex set

– **Linear programming**: Special case of convex programming where the objective f and the constraints c_i's are linear