Discrete Structures

IIIT Hyderabad

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Tutorial 8

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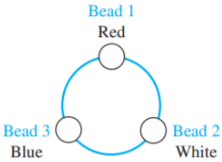
Introduction

Question 0

Each bead on a bracelet with three beads is either red, white or blue:



Define the relation R between bracelets as: (B_1, B_2) where B_1 and B_2 are bracelets, belongs to R if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

- Show that R is an equivalence relation
- What are the equivalence classes for R

Question 0 solution

Given $A =$ Set of bracelets with beads being red, white or blue
$R = \{(B_1, B_2) | B_2 \text{ is } B_1 \text{ rotated or } B_2 \text{ is } B_1 \text{ rotated plus reflected}\}$
Reflexive Let $B \in A$:
Bracelet B is obtained by rotating B by 360°. Thus $(B, B) \in R$. Thus R is reflexive.
Symmetric Let $(B_1, B_2) \in R$
If we obtained B_2 by rotating by x°, then we can obtain B_1 from B_2 by rotating x° in the opposite direction. If we have obtained B_2 by rotating by x° and then reflecting, then we can obtain B_1 by rotating x° in the opposite direction and then reflecting. Thus $(B_2, B_1) \in R$. Thus R is symmetric.
Transitive Let $(B_1, B_2), (B_2, B_3) \in R$
If B_2 is obtained by rotation by x° and B_3 is obtained by rotation by y°. Then to obtain B_3 from B_1 we rotate by $(x + y)^\circ$ and then applying the respective reflections. Thus R is transitive.
R is an equivalence relation

Question 0 solution - 2

Each bead can take on three colors: W White, R Red, B Blue.
The equivalence classes from bracelet B contain all bracelets that can be obtained by rotating/reflecting bracelet B, which means the bracelets contain the same colors but the ordering will be different.

$[RRR]_R = \{RRR\}$
$[WWW]_R = \{WWW\}$
$[BBB]_R = \{BBB\}$
$[BBR]_R = \{BBR, BRB, RBB\}$
$[BBW]_R = \{BBW, BWB, WBB\}$
$[WWR]_R = \{WWR, WRW, RWW\}$
$[WWB]_R = \{WWB, WBW, BWW\}$
$[RRB]_R = \{RRB, RBR, BRR\}$
$[RRW]_R = \{RRW, RWR, WRR\}$
$[BRW]_R = \{BRW, BWR, RWB, RBW, WRB, WBR\}$

Question 1

1.1 Let R be the relation $\{(a, b)|a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?
1.2 Let R be the relation $\{(a, b)|a \neq b\}$ on the set of integers. What is the reflexive closure of R?
1.3 Let R be the relation $\{(a, b)|a > b\}$ on the set of positive integers. Find the smallest relation containing R that is both reflexive and symmetric?

Question 1 Solutions

1.1 To form the symmetric closure of R we need to add all pairs (b, a) such that (a, b) is in R. In this case, that means that we need to include pairs (b, a) such that a divides b, which is equivalent to saying that we need to include all the pairs (a, b) such that b divides a. Thus the closure is $\{(a, b)|a \text{ divides } b \text{ or } b \text{ divides } a\}$

1.2 To form the reflexive closure we have to get $Rcup\triangle$ i.e. $R \cup \{(a, b)|a = b\}$. Thus we will get $R = \{(a, b)|a \neq b \vee a = b\}$

1.3 The reflexive closure is $R = \{(a, b)|a \geq b\}$. The symmetric closure is $R = \{(a, b)|a \neq b\}$. The smallest possible is when $a, b \in \mathbb{Z}^+$. Thus the answer is $\mathbb{Z}^+ \times \mathbb{Z}^+$

Question 2

2.1 Determine which of the following relations f are functions from set X to set Y:

1. $X = \{[-2, 1, 0, 1, 2\},\ Y = \{-3, 4, 5\}$ and $f = \{(-2, -3), (1, 4), (2, 5)\}$
2. $X = Y = \{-3, 1, 0, 2\}$ and $f = \{(-3, -1), (0, 2), (2, -1), (-3, 0), (-1, 2)\}$
3. $X = Y =$ the set of all integers, and $f = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | b = \sqrt{a}\}$
4. $X = Y =$ the set of all integers, and $f = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | b = a + 1\}$

2.2 Determine which of the following functions are one-one, onto or both?

1. $f : \mathbb{N} \to \mathbb{Z} - \{0\}$ defined by $f(n) = -n$ for all $n \in N$
2. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x| + x$ for all $x \in \mathbb{R}$
3. $f : \mathbb{C} \to \mathbb{R}$ defined by $f(z) = |z|$ for all $z \in \mathbb{C}$
4. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ for all $x \in \mathbb{R}$

Question 2 solutions

2.1

- The domain of f, $D(f) = \{-2, 1, 2\} \neq X$. Hence f is not a function X.
- We have $(-3, 1)$ and $(-3, 0)$, so f is not a function
- We have $\sqrt{2} \neq in\mathbb{Z}$. Thus f is not a function.
- It is a function

2.2

- f is one-one, but not onto.
- f is not one-one, because for negative numbers this is 0. f is not onto because no negative.
- f is not onto because it gives only $R^+ \cup \{0\}$. f is not one-one.
- f is not one-one. f is not onto.

Question 3

3.1 Let f be a function from the set $\mathbb{N}$ into the set $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ defined by $f(x) = x(\text{mod } 7)$ for $x \in \mathbb{N}$. Find $Im(f)$. Is f onto X? Is f one-one?

3.2 For what domain and co-domain will the function $f(x) = \lfloor \sqrt{n} \rfloor$ be one-one and onto?

Question 3 solutions

3.1 We get $n = 7t + r$, $0 \leq r \leq 7$. Thus the image of f is $\{0, 1, 2, 3, 4, 5, 6\}$. Also f is not one-one because many numbers give the same remainder.
3.2 If we have domain as $S = \{x|x \text{ is a perfect square}\}$. If we have codomain as $\mathbb{Z}$ then we will have co-domain $=$ range.

BONUS

NOTE:

The following content is additional, and present to round out the discussion on closures. No need to invest time to this as you will learn this properly in Graph Theory. This was not taught in the lectures.

Relations as Matrices

Definition

Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then the relation R can be represented by a matrix $\boldsymbol{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \textbf{if } (a_i, b_j) \notin R \end{cases}$$

Such representation depends upon the ordering of A and B
Such representations make sense when both A and B are finite.

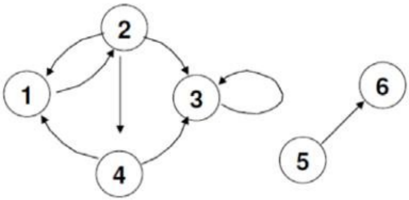
Relations as Matrices - 2

The matrix of a relation on a set, which is a square matrix is very useful to determine certain properties of the relation.

- If R is reflexive, then $\boldsymbol{M_{ii}} = 1$ for all $i = 1, 2, 3, \ldots, n$
- if R is symmetric, then $\boldsymbol{M_R} = (\boldsymbol{M_R})^{\boldsymbol{t}}$
- $\boldsymbol{M_{R_1 \cup R_2}} = \boldsymbol{M_{R_1}} \vee \boldsymbol{M_{R_2}}$
- $\boldsymbol{M_{R_1 \cap R_2}} = \boldsymbol{M_{R_1}} \wedge \boldsymbol{M_{R_2}}$
- $\boldsymbol{M_{S \circ R}} = \boldsymbol{M_R} \odot \boldsymbol{M_S}$
- $\boldsymbol{M_{R^n}} = (\boldsymbol{M_R})^{\boldsymbol{n}}$

Relationship between matrix and DGs

The matrix representation and the graph representation are interchangeable, and the matrix representation is called the **Adjacency Matrix** of the directed graph.



	1	2	3	4	5	6
1	0	1	0	0	0	0
2	1	0	1	1	0	0
3	0	0	1	0	0	0
4	1	0	1	0	0	0
5	0	0	0	0	0	1
6	0	0	0	0	0	0

Figure: An example of matrix and graph relation

Paths in a directed graph

A path between a and b is a sequence of edges that are followed starting from a to finally reach b. The path length is given by the number of edges traversed to reach b. We are generally interested in minimum path lengths. It is possible that no path may exist between a and b.

Theorem

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b iff $(a, b) \in R^n$.

Connectivity Relation and Transitive Closure

Definition

Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Theorem

The transitive closure of a relation R equals the connectivity relation R^*

How to find closures for relations on finite set

If we have a relation R such that **we can construct a square matrix $\boldsymbol{M_R}$** for the relation, then:

- Reflexive Closure:

$$\boldsymbol{M_{R\cup\triangle} = M_R \vee I_n}$$

- Symmetric Closure:

$$\boldsymbol{M_{R\cup R^{-1}} = M_R \vee (M_R)^t}$$

- Transitive Closure:

$$\boldsymbol{M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}}$$

Bonus Question

Using the method shown above, find the reflexive, symmetric and transitive closures for the relations on $\{1, 2, 3, 4\}$

1. $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$
2. $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$