Solutions

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1.1)

①

(+)	EVEN	ODD
EVEN	EVEN	ODD
ODD	ODD	EVEN

②

(x)	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

1.2)

$Z_9^* = \{1, 2, 4, 5, 7, 8\}$

① 1 is the identity (e)

as

$\forall a \in Z_9^*$

$a \times 1 = 1 \times a = a \pmod 9$

②

(x)	1	2	4	5	7	8
4	4	8	7	2	1	5

$4 \times 7 = 1 \pmod 9$,

$\therefore$ 7 is the inverse of 4.

③ $5 \times 8 = 4 \pmod 9$,

④ $g \rightarrow$ generator, $g = 2$

$$\begin{tikzcd} 1 \arrow[r, "\times 2"] & 2 \arrow[r, "\times 2"] & 4 \arrow[r, "\times 2"] & 8 \arrow[r, "\times 2"] & 7 \arrow[r, "\times 2"] & 5 \arrow[lllll, bend left, "\times 2"] \end{tikzcd}$$

we have

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 7$

$2^5 = 5$

$2^6 = 1.$

$\therefore$ we have '2' as generator,

and order of $g = 6$ [as $g^6 = 1$]

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2)

1. We know

$(a*b)^{-1} * (a*b) = e$ .... [property of inverse]

$\therefore (a*b)^{-1} * (a*b) * b^{-1} = e * b^{-1}$ ..... [right multiplying b^{-1}].

$\therefore (a*b)^{-1} * a * (b^{-1} * b) = b^{-1}$ ... [using associativity and that $e*a = a$].

$\therefore (a*b)^{-1} * a * e = b^{-1}$ ... [$aa^{-1} = a^{-1}a = e$]

$\therefore (a*b)^{-1} * a = b^{-1}$ ... {$a*e = e*a = a$ [property of identity]}

$\therefore (a*b)^{-1} * a * a^{-1} = b^{-1} * a^{-1}$ ... [right multiplying a^{-1}]

$\therefore (a*b)^{-1} * e = b^{-1} * a^{-1}$ ... {$aa^{-1} = a^{-1}a = e$ [property of inverse]}

$\therefore \boxed{(a*b)^{-1} = b^{-1} * a^{-1}}$ ... {$e*a = a, a*e = a$ [property of identity]}

2. ($\Rightarrow$ part) let $G = \langle S, * \rangle$ be a group.

We need to prove that, for any 2 elements $a, b \in S$,

$a*b = b*a$.

we have,

$(a*b)^{-1} = a^{-1} * b^{-1}$ ...... {given}

$\rightarrow (a*b) * (a*b)^{-1} = (a*b) * a^{-1} * b^{-1}$

$e = (a*b) * a^{-1} * b^{-1}$ .... {$a^{-1}a = aa^{-1} = e$, property of inverse}

$e * b = (a*b) * a^{-1} * b^{-1} * b$ ..... {right multiplying by b}

$b = (a*b) * a^{-1} * b^{-1} * b$ ..... {$ae = ea = a$, property of identity}

$b = (a*b) * a^{-1} * e$ ..... {$a^{-1}a = aa^{-1} = e$, property of inverse.}

$b = (a*b) * a^{-1}$ ..... {$ae = ea = a$, property of identity}

$b*a = (a*b) * a^{-1} * a$ ..... {right multiplying by a}

$b*a = (a*b) * e$ ..... {$aa^{-1} = a^{-1}a = e$, property of inverse}

$\boxed{b*a = (a*b)}$ ..... {$ae = ea = a$, property of identity}.

Hence proved.

$\boxed{b * a = (a * b)}$ - - - - - $\{ae = ea = a$, property of identity$\}$.

Hence proved.

③ Semi-group has properties of closure & associativity.

$G = \langle S, * \rangle$,

$\forall x \in S$, $\exists u, v$, such that

$\rightarrow \underline{a * u = v * a = x}$. [a is any fixed element]

we need to prove, - - (1)

$\forall a \in S$,

$\exists e \in S$, such that

$\underline{a * e = e * a = a}$.

We have,

$a * u = v * a = x$,

for a, $\exists u', v'$ such that

$a * u' = v' * a = a$ - - - (2)

thus we get,

$v' * a * u' = (v' * a) * u'$

$= a * u'$

$= a$

for x, we write using (1) & (2) $\rightarrow$

$x = a * u = v' * a * u = v' * x$.

and

$x = v * a = v * a * u' = x * u'$

$\therefore \quad \underline{\vartheta' * x = x * u' = x}$

$\cdot\ \vartheta' \rightarrow$ left identity, $u' \rightarrow$ right identity.

Thus left identity and right identity exist and thus they must be equal.

$$\boxed{\vartheta' = u' = e}.$$

3)

For the given sets and operations (Opn), mark the correct ticks. For all the cyclic groups, find the generators. -

Set	Opn	1	2	3	4	5	6	7	8
$\mathbb{Z}_7$	$+$	✓	✓	✓	✓	✓	✓	✓	✓
$\mathbb{Z}$	-	✓	✗	✗	✗	✗	✗	✗	✗
$\mathbb{Z}_5^*$	^	✓	✗	✗	✗	✗	✗	✗	✗
$\mathbb{Q} - \{0\}$	$\div$	✓	✗	✗	✗	✗	✗	✗	✗
$\mathbb{Z}_{11}^*$	$\times$	✓	✓	✓	✓	✓	✓	✓	✓
$\mathbb{Z}_{12}^*$	$\times$	✓	✓	✗	✓	✗	✓	✗	✗
2×2 Matrices	$\times$	✓	✓	✗	✓	✗	✗	✗	✗
2×2 Matrices	$+$	✓	✓	✗	✓	✗	✓	✗	✗
$\mathbb{N}$	$\div$	✗	✗	✗	✗	✗	✗	✗	✗
[*] Reflection lines in n-gon	Composition								