Discrete Structures (MA5.101)

Instructor: Dr. Ashok Kumar Das **Assignment 4 Solutions**
IIIT Hyderabad **Total Marks**: 70

Problem 1 **Base Case :** $n = 1$. It is trivially true.

$$A \cup (B_1) = (A \cup B_1)$$

Induction Hypothesis : Assume it to be true for $n = k$. That is

$$A \cup (B_1 \cap B_2 \dots B_k) = (A \cup B_1) \cap (A \cup B_2) \dots (A \cup B_k)$$

We shall prove it for $n = k + 1$.

$$\begin{aligned} A \cup (B_1 \cap B_2 \dots B_{k+1}) &= \big(A \cup (B_1 \cap B_2 \dots B_k)\big) \cap \big(A \cap B_{k+1}\big) && \dots\{\text{ by distributive law}\} \\ &= \big((A \cup B_1) \cap (A \cup B_2) \dots (A \cup B_k)\big) \cap (A \cup B_{k+1}) && \dots\{\text{ by hypothesis}\} \\ &= (A \cup B_1) \cap (A \cup B_2) \dots (A \cup B_k) \cap (A \cup B_{k+1}) && \dots\{\text{ by associative law}\} \end{aligned}$$

Problem 2 Let $f = (1\ i), g = (1\ 2\ 3 \dots\ n)$. We have 2 cases -

1. $i \leq n$
   Then we have

$$f \circ g(k) = \begin{cases} k+1 & \text{when } k \notin \{i+1, n\} \\ 1 & \text{when } k = i-1 \\ i & \text{when } k = n \end{cases}$$

   Thus we have 2 independent cycles here $(1\ 2\ 3\ \dots i-1)$ and $(i\ i+1\ \dots n)$, and our final permutation is a composition of these two -

$$(1\ 2\ 3\ \dots i-1)(i\ i+1\ \dots n)$$

2. $i > n$
   Then we have

$$f \circ g(k) = \begin{cases} k+1 & \text{when } k \notin \{i, n\} \\ 1 & \text{when } k = i-1 \\ i & \text{when } k = n \end{cases}$$

   Thus we can express our permutation as

$$(1\ 2\ 3\ \dots n\ i)$$

Problem 3 Let $S = \{n_1, n_2 \dots n_{16}\}$. The primes ≤ 7 are 2,3,5,7. Thus each of them is expressable as

$$n_i = 2^{x_1} 3^{x_2} 5^{x_3} 7^{x_4}$$

We construct a mapping from $f : S \to B_4$, where B_4 is the set of binary tuples of length 4, such that -

$$f(n_i) = ((x_1 \text{ mod} 2), (x_2 \text{ mod} 2), (x_3 \text{ mod} 2), (x_4 \text{ mod} 2))$$

As we have $|B| = 16$, we have 2 cases -

1. The mapping f is onto. Every tuple in B_4 has a pre-image. In this case the tuple $(0, 0, 0, 0)$ should also be there, which means $f^{-1}(0, 0, 0, 0) = 2^{x_1} 3^{x_2} 5^{x_3} 7^{x_4}$ such that x_1, x_2, x_3, x_4 are divisible by 2. We can write each of them as $x_i = 2.k_i$ where $k_i \in \mathbb{N}$. Thus we have

$$\begin{aligned} n_i &= 2^{x_1} 3^{x_2} 5^{x_3} 7^{x_4} \\ &= 2^{2k_1} 3^{2k_2} 5^{2k_3} 7^{2k_4} \\ &= \left(2^{k_1} 3^{k_2} 5^{k_3} 7^{k_4}\right)^2 \end{aligned}$$

   Thus an element is a perfect square.

2. If f is not onto, then we must have, by pigeon hole principle since $|Range(f)| < |S|$, at least one element must have 2 pre-images. Here the images are our holes and the set S contains our pigeons. Therefore, let the common element be (y_1, y_2, y_3, y_4) (where each $y_i < 2$), which are pre-images of n_l and n_m. Thus we have $f(n_l) = f(n_m) = (y_1, y_2, y_3, y_4)$. Let $x_{li} = 2.k_{li} + y_1$ and $x_{mi} = 2.k_{mi} + y_1$. We have

$$\begin{aligned} n_l n_m &= \left(2^{x_{l1}} 3^{x_{l2}} 5^{x_{l3}} 7^{x_{l4}}\right)\left(2^{x_{m1}} 3^{x_{m2}} 5^{x_{m3}} 7^{x_{m4}}\right) \\ &= \left(2^{2.k_{l1}+y_1} 3^{2.k_{l2}+y_2} 5^{2.k_{l3}+y_3} 7^{2.k_{l4}+y_4}\right)\left(2^{2.k_{m1}+y_2} 3^{2.k_{m2}+y_3} 5^{2.k_{m3}+y_4} 7^{2.k_{m4}+y_4}\right) \\ &= \left(2^{k_{l1}+k_{m1}+y_1} 3^{k_{l2}+k_{m2}+y_2} 5^{k_{l3}+k_{m3}+y_3} 7^{k_{l4}+k_{m4}+y_4}\right)^2 \end{aligned}$$

   and thus is a perfect square. Thus product of 2 elements is a perfect square.

Problem 4 Take our pigeonholes as $\{1, 12\}, \{2, 11\}, \{3, 10\}, \{4, 9\}, \{5, 8\}, \{6, 7\}$. If we take 7 numbers (pigeons) from them, since there are 6 holes, we must select 2 pigeons from the same hole, thus by PHP, we have at least 1 pair must sum to 13.

Problem 5,6,7 [Explained in Tut]