Discrete Structures

IIIT Hyderabad

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Tutorial 4

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Introduction

1. Questions
   - Question 1
   - Question 2
   - Question 3

Question 1

Prove the following -

1. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

   Sol:

$$
\begin{aligned}
&(a, b) \in (R \cap S)^{-1} \\
&\implies (b, a) \in (R \cap S) \\
&\implies ((b, a) \in R) \land ((b, a) \in S) \\
&\implies ((a, b) \in R^{-1}) \land ((a, b) \in S^{-1}) \\
&\implies (a, b) \in (R^{-1} \cap S^{-1})
\end{aligned}
$$

$$
\begin{aligned}
&(a, b) \in (R^{-1} \cap S^{-1}) \\
&\implies ((a, b) \in R^{-1}) \land ((a, b) \in S^{-1}) \\
&\implies ((b, a) \in R) \land ((b, a) \in S) \\
&\implies (b, a) \in (R \cap S) \\
&(a, b) \in (R \cap S)^{-1}
\end{aligned}
$$

2. R, S are symmetric $\implies R \cap S$ is symmetric.
Sol:

$$
\begin{aligned}
&(R \cap S)^{-1} \\
&= R^{-1} \cap S^{-1} \\
&= R \cap S
\end{aligned}
$$

which implies that $R \cap S$ is symmetric.

3. R is transitive $\implies R^{-1}$ is transitive.
Sol: Suppose if not, then $\exists (a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$. But $(a, c) \notin R^{-1}$. We take inverse of them, $(b, a) \in R$ and $(c, b) \in R$, but now we have $(c, a) \in R$, which means $(a, c) \in R$ which is contradiction. Hence it is transitive.

4. * [R, S are transitive $\implies R \cap S$ is transitive.]

Question 2

To answer the questions visit: tinyurl.com/dstut4

2.1: State true or false

1. If R and S are transitive, $R \cup S$ not always transitive.
   Sol: True, take $R = \{(a, b), (b, c), (a, c)\}$ and $S = \{(c, d), (d, a), (c, a)\}$ defined on $A = \{a, b, c, d\}$. Since $(a, c) \in (R \cup S)$ and $(c, d) \in (R \cup S)$ but $(d, a)(R \cup S)$
   If it is symmetric, then $(a, b) \in R$ and $(b, a) \in$R, which means $(a, a) \in R$ by transitive rule. Thus reflexive and equivalent.
2. Every relation must either be symmetric or anti-symmetric.
   Sol: False
   Take the relation on set $S = \{a, b, c\}$, $R = \{(a, b), (b, a), (b, c)\}$.
   This is not symmetric as $(c, b) \notin R$ and not anti-symmetric as $(b, a) \in R$ and $(a, b) \in R$.

2.2: Mark as Reflexive, Symmetric, Anti-symmetric and/or Transitive.

1. $S = \mathbb{C},\ {}_xR_y \iff x^2 + y^2 = 1$
   Sol: Symmetric.
2. $S = \mathbb{R}^2,\ {}_{(a,b)}R_{(c,d)} \iff a + d = b + c$
   Sol: Reflexive, Symmetric, Transitive.
3. $S =$ The set of all lines the plane $\mathbb{R} \times \mathbb{R},\ {}_lR_m \iff l \parallel m$
   Sol: Reflexive, Symmetric, Transitive.
4. $S =$ The powerset of $\{1,2,3 \ldots 10\}$. ${}_AR_B \iff A \subseteq B$
   Sol: Reflexive, Anti-Symmetric and Transitive.

Explanations -

1. Not reflexive as $(0,1) \in R$, but $(0,0) \notin R$.
$(x,y) \in R \implies x^2 + y^2 = 1 \iff y^2 + x^2 = 1 \implies (y,x) \in R$.
Not transitive as $(0,1) \in R, (1,0) \in R$ but $(0,0) \notin R$.
2. ${}_{(a,b)}R_{(a,b)}$ as $a + b = a + b \forall (a,b) \in \mathbb{R}^2$.
${}_{(a,b)}R_{(c,d)} \implies a + d = b + c \iff {}_{(c,d)}R_{(a,b)}$.
${}_{(a,b)}R_{(c,d)} \wedge_{(c,d)} R_{(e,f)} \implies ((a + d = c + b) \wedge (c + f = d + e)) \implies ((a - b = c - d) \wedge (c - d = e - f)) \implies (a + f = b + e) \implies {}_{(a,b)}R_{(e,f)}$.
3. $l \parallel l$ by definition.
$l \parallel m \iff m \parallel l$ by definition.
$(l \parallel m) \wedge (m \parallel n) \implies (l \parallel n)$.
4. $A \subseteq A$ by definition.
$A \subset B$, then $B \not\subset A$. Thus it has to be anti-symmetric.
$A \subseteq B, B \subseteq C$, then we have $(\forall x)(x \in A) \implies (x \in B)$ and $(\forall x)(x \in B) \implies (x \in C)$, thus we have, $(\forall x)(x \in A) \implies (x \in C)$ which means $A \subseteq C$.

2.2: A set S has 3 elements. Find -

1. Number of binary relations.
   Sol: $2^{3\times3} = 512$
2. Number of anti-symmetric relations.
   Sol: $2^3 3^3 = 216$.
3. Number of equivalent relations.
   Sol: $S(3,1) + S(3,2) + S(3,3) = 2 + S(2,1) + 2 \cdot S(2,2) = 5$
4. Number of relations neither symmetric nor antisymmetric.
   Sol: $512 - (216 + 64) = 232$

Question 3

1. Let R be a symmetric and transitive relation on a set A. Show that if for every a in A there exists b in A such that (a, b) is in R, then R is an equivalence relation.
   Sol: Since R is symmetric, if $(a, b) \in R \implies (b, a) \in R$ and since R is transitive, $(a, b) \in R, (b, a) \in R \implies (a, a) \in R$ and this argument is true $\forall a \in A$. Therefore R is reflexive.
   Hence R is an equivalence relation.