Discrete Structures

IIIT Hyderabad

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Tutorial 18

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Introduction

1. Questions
   - Question 1
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Question 1

1.1 Find the minimum Hamming distance of the set of code words. Find the number of combinations of errors that can be detected. Also find the number of combinations of errors that can be corrected.

$$C = \{< 1000101010, 1100101111, 0000101101, 0111101101 >\}$$

1.2 Find the bit-wise XOR $\bigoplus$ between all the codes in C, and verify $H_C(x, y) = H_C(x \oplus y, 0) = \min w(x \oplus y)\ \forall x, y \in C$.

Question 1 Solutions

1.1 Sol:

- (1, 2) - 0100000101 - w(1, 2) = 3
- (1, 3) - 1000000111 - w(1, 3) = 4
- (1, 4) - 1111000111 - w(1, 4) = 7
- (2, 3) - 1100000010 - w(2, 3) = 3
- (2, 4) - 1011000010 - w(2, 4) = 4
- (3, 4) - 111000000 - w(2, 4) = 3

Reference - https://faculty.uml.edu/klevasseur/ads/s-coding-theory-groups.html Min distance is 3. (3, 4) is one pair. (1, 2) is another pair.
Number of combinations detect = 2, $(d + 1)$
Number of combinations corrected = 1, $(2t + 1)$
1.2 Sol: Calculate XOR and count the number of 1s.

Question 2

2.1 Let the matrix H be:

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(A matrix of the form $G = [I_R|P]$ is called the generator matrix of the code.)

1. Verify that this gives a valid parity check matrix
2. Encode the message (1, 0, 0) and (1, 0, 1)
3. Decode the following messages and correct the errors if possible
   a. (1, 0, 0, 0, 1, 1)
   b. (0, 1, 1, 1, 1, 0)
   c. (1, 0, 0, 0, 0, 1)
   d. (1, 0, 0, 1, 0, 0)

Question 2 Solutions

2.1 Sol:

1. It is a valid matrix for $d = 3$ because:
   a. No column is completely 0
   b. No 2 columns are same
   c. Sum of 3 columns gives the 0 code-word. Indeed, $c_1 \bigoplus c_2 \bigoplus c_3 = (0, 0, 0)$.

2. We have y should be 6 bits long out of which first 3 bits are from the message itself. To get the remaining, we take the j^{th} row of P and multiply element wise to y and add it. (Essentially we calculate $m.P^T$)
   a. $y_1 = 0, y_2 = 0, y_3 = 0.$

   $$(1,0,0).\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}^T$$
   $$= (1,1,0)$$

   Thus the message is (1, 0, 0, 1, 1, 0).
   b. $y_1 = 1, y_2 = 0, y_3 = 1.$

   $$(1,0,1).\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}^T$$
   $$= (1,0,1)$$

   Thus the message is (1, 0, 1, 1, 0, 1).

3. Calculate $x.H^t$ with the operation of $\bigoplus$ instead of $+$. This is the syndrome, convert that to digits using binary to decimal conversion.
    a. Syndrome (1, 0, 1), which gives 5th place. Thus the corrected code is (1, 0, 0, 0, 1, 0). The original message is (1, 0, 0)
    b. Syndrome (0, 0, 0). No error. Thus the original message is (0, 1, 1)
    c. Syndrome (1, 1, 1). This syndrome occurs only when 2 bits have error. Thus this cannot be fixed.
    d. Syndrome (0, 1, 0), which gives the 2nd place. Thus the corrected code is (1, 1, 0, 1, 0, 0). The original message is (1, 1, 0).