Discrete Structures

IIIT Hyderabad

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Tutorial 11

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Introduction

1. Questions
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Question 0

0.1: f, g are bijective. $\implies$ $f \circ g$ is bijective.
Sol:
Assume $f : A \to B, g : C \to A$
Proof that it is Injective: Say if $f \circ g(x_1) = g \circ f(x_2)$. Now since f is injective, we must have that $g(x_1) = g(x_2)$ and since g is injective, we must have $x_1 = x_2$. Thus we have proved that $f \circ g(x_1) = g \circ f(x_2) \implies x_1 = x_2$.
Proof that it is Surjective: We need to prove that

$$\forall y \in B, \exists x \in C \text{ such that } f(x) = y$$

Similarly, we say that since f is surjective, we have that

$$\forall y \in B, \exists z \in A \text{ such that } f(z) = y$$

We first use the fact that g is surjective, due to which we have that

$$\forall z \in A, \exists x \in C \text{ such that } g(x) = z$$

We combine the above 2 to get -

$$\forall y \in B, \exists x \in C \text{ such that } f \circ g(x) = y$$

0.2: $f^{-1}(A-B)=f^{-1}(A)-f^{-1}(B)$.
Sol:

$$\begin{aligned} & x \in f^{-1}(A-B) \\ \implies & f(x) \in (A-B) \\ \implies & (f(x) \in A) \wedge (f(x) \notin B) \\ \implies & (x \in f^{-1}(A)) \wedge (x \notin f^{-1}(B)) \\ \implies & x \in f^{-1}(A) - f^{-1}(B) \end{aligned}$$

Thus $f^{-1}(A-B) \subseteq f^{-1}(A)-f^{-1}(B)$

$$\begin{aligned} & x \in f^{-1}(A)-f^{-1}(B) x \in f^{-1}(A-B) \\ \implies & (x \in f^{-1}(A)) \wedge (x \notin f^{-1}(B)) \\ \implies & (f(x) \in A) \wedge (f(x) \notin B) \\ \implies & (f(x) \in A) \wedge (f(x) \notin B) \\ \implies & x \in f^{-1}(A) - f^{-1}(B) \end{aligned}$$

Thus $f^{-1}(A)-f^{-1}(B) \subseteq f^{-1}(A-B)$

0.3: Prove that the statement

$$h(f(x)) = k(f(x)) \implies h = k$$

implies f is onto.

Sol: Assume that f is not onto and that $h(f(x)) = k(f(x))$. Now we should be able to deduce that $h(y) = k(y) \forall y \in B$. We construct h, k such that

$$h(y) = c_0 \forall y \in B$$
$$k(y) = \begin{cases} c_0 \text{ if } y \in Range(f) \\ c_1 \text{ otherwise} \end{cases}$$

(where c_0 and c_1 are 2 distinct). By this construction, we have $h(f(x)) = k(f(x))$, but $h \neq k$. Which means our assumption should have been wrong.

0.4: Let $f, g : \mathbb{N} \rightarrow \mathbb{N}, f(x) = x^2, g(x) = x^3$. Prove that the sets $Range(f)$ and $Range(g)$ have same cardinality.

Sol: Construct a mapping h from $Range(f) \rightarrow Range(g)$ such that if $x \in Range(f)$, then $h(x) = x^{\frac{3}{2}}$. It is easy to show that $h(x) \in Range(g)$. Now we prove that this mapping is injective and surjective.

For injective, if $h(x_1) = h(x_2)$, we have $x_1^{\frac{3}{2}} = x_2^{\frac{3}{2}}$, from which we get $x_1^3 = x_2^3$ which implies either $x_1 = x_2$ or $x_1^2 + x_2^2 + x_1x_2 = 0$. We know that $x_1^2 + x_2^2 + x_1x_2 > 0$, as they are in $\mathbb{N}$, thus $x_1 = x_2$.

For surjective, for any $y \in Range(g)$, we have it's pre-image as $y^{\frac{2}{3}}$ as $h(y^{\frac{2}{3}}) = (y^{\frac{2}{3}})^{\frac{3}{2}} = y$.

Thus the mapping is bijective and the cardinalities are same.

Question 1

Let a permutation p be :-

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 6 & 7 & 3 & 4 \end{pmatrix}$$

1. Let q be defined as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 4 & 3 & 1 & 2 & 6 \end{pmatrix}$$

Find the permutation $q \circ p$.
Sol:

$$q \circ p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 1 & 2 & 6 & 4 & 3 \end{pmatrix}$$

2. Identify all the cycles in p.
Sol:

$$p = (1,2)(3,5,7,4,6)$$

3. How many transpositions are there in p ? p an odd or even permutation ?

$$\begin{aligned} p &= (1,2)(3,5,7,4,6) \\ &= (1,2)(3,6)(3,5,7,4) \\ &= (1,2)(3,6)(3,4)(3,5,7) \\ &= (1,2)(3,6)(3,4)(3,7)(3,5) \end{aligned}$$

Odd permutation (5 transpositions)

* Can you give a general formula for number of transpositions ?
Sol: The number of transpositions in a k-cycle in generally k - 1.

Question 2

2.1: Show that the sets $S = \{x \in \mathbb{C}, |x| = 1\}$ and $\mathbb{R}$ have same cardinality.
Sol: For each element $x \in S$, we can have it as -

$$x = e^{i\theta}$$

For each θ, have a function f which maps it as follows -

$$f(\theta) = tan\left(\frac{\theta}{2}\right)$$

The function above is injective and surjective. Note that the point -1 will not be representable. From the construction, we have cardinality of $S - \{-1\}$ equal to that of $\mathbb{R}$. The cardinality of $S - \{-1\}$ is equal to that of S, as adding a finite set to an infinite set does not change the cardinality. Thus we have cardinality of S equal to that of $\mathbb{R}$.

Question 3

Let $A = \{x \in \mathbb{R} | x \in [0, 1]\}$
$B = \{x \in \mathbb{N} | x \text{ is a perfect square}\}$
$C = \{x \in \mathbb{N} | x < 10\}$
Which of the following are countable ?

1. $B \cap C$
   Sol: $B \cap C = \{1, 4, 9\}$ which is finite and hence countable.
2. $B \cup C$
   Sol: Construct f such that $f : \mathbb{N} \rightarrow \mathbb{N}$,
   $f(2) = 1, f(3) = 2, f(5) = 3, f(6) = 4, f(7) = 5, f(8) = 6$ and $\forall n \in B, f(n) = \sqrt{n} + 6$. We notice that f is injective as it maps uniquely to each $n \in N$, and surjective as well since we are denumerating the elements.

3. $A \cup B$

   Sol: (In this proof I am assuming B is uncountable. Please refer to the slides for the proof.)
   Assume $A \cup B$ is countable. Then $\exists f : (A \cup B) \rightarrow \mathbb{N}$. But from here, we are able to construct a function $g : B \rightarrow X$ such that $\forall b \in B, g(b) = f(b)$, and that $X \subseteq \mathbb{N}$. Since $\mathbb{N}$ is countable, we must have $X \subseteq \mathbb{N}$ also as countable. Cardinality of B is same as X as g is a bijective map from B to X, and so B must also be countable . This is a contradiction and hence $A \cup B$ must be uncountable.

4. $A \cap B$

   Sol: $A \cap B = \{1\}$ which is finite and hence countable.

5. $\mathcal{P}(B)$

Sol: We use Cantor's diagonalization argument over here. Let us define a new representation of a subset S_i of B such that

$$S_i = x_{i1}x_{i2}x_{i3}x_{i4}\dots$$

where $x_{ij} = e_{S_i}(j^2)$. For example, if $A = \{4, 9\}$. We have representation as

$$repr(A) = 01100000\dots$$

Now since $\mathcal{P}(B)$ is countable, we should be able to list each of the subsets as -

$$\begin{aligned}
repr(S_1) &= x_{11}x_{12}x_{13}x_{14}\dots \\
repr(S_2) &= x_{21}x_{22}x_{23}x_{24}\dots \\
repr(S_3) &= x_{31}x_{32}x_{33}x_{34}\dots \\
repr(S_4) &= x_{41}x_{42}x_{43}x_{44}\dots \\
&\vdots
\end{aligned}$$

Now construct

$$S = \{\overline{x}_{11}\overline{x}_{22}\overline{x}_{33}\dots\}$$

We notice that S is not in any of the $S_1, S_2, S_3\dots$ we listed. Thus our assumption must be wrong that $\mathcal{P}(B)$ is countable. Hence it must be uncountable.

Question 4

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4.1: Find left and right inverses of each of them (wherever exist) -

1. $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n + 3$
   Sol: Since $2 \notin$ Range(f), it is not onto. It is one-one. Left-inverse could be - $g(n) = n - 3$.

2. $f : \mathbb{Z} \to \mathbb{E}^*, f(x) = |x| + x$
   Sol: Since f is not one-one but onto, the right inverse could be $g(x) = \frac{|x|}{2}$.

4.2: Which of the following is/are projections -

1. $f(x) = e^x, f : \mathbb{R} \to \mathbb{R}$ **No**, $e^{e^x} \neq e^x$.
2. $||x||$, $f : \mathbb{C} \to \mathbb{C}$ **Yes**, $||(||x||)|| = ||x||$.
3. $\lfloor x \rfloor$, $f : \mathbb{Z} \to \mathbb{Z}$ **Yes** $\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor$.

4.3: Find $\sum_{j=1}^{j=100} e_S(j)$ on $U = \mathbb{Z}$, when $f(x) = x^2, S = Range(f(x)), f : \mathbb{R} \to \mathbb{R}$.
Sol: We have this as

$$\sum_{i=1}^{i=100} e_S(k)k$$

$$= 10$$

Note that $e_s(j) = 1$ only for $j \in \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$.