Discrete Structures (MA5.101)

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Assignment 2 Solutions
Total Marks: 70

Problem 1

$$\begin{aligned}
&\forall a \in A, \exists b \text{ such that } {}_aR_b && \ldots\{\text{given}\}\\
&\forall a \in A, {}_aR_b \implies {}_bR_a && \ldots\{\text{since symmetric}\}\\
&\forall a \in A, ({}_aR_b) \wedge ({}_bR_a) \implies ({}_aR_a) && \ldots\{\text{since transitive}\}
\end{aligned}$$

Thus we get that R is reflexive too, which is why it is equivalent.

You must write the $\forall$ quantifier, while mentioning both the reason for it being reflexive, and when you are saying $(a,b) \in R$.

Problem 2

Given $N = |A| = 10$,

1. $2^{N^2} = 2^{100}$
2. $2^{N(N-1)} = 2^{90}$
3. $2^N 3^{\frac{N(N-1)}{2}} = 2^{10}3^{45}$
4. $2^{\frac{N(N-1)}{2}} = 2^{45}$
5. $115,975$

Problem 3

a.

$$\begin{aligned}
&(a,b) \in (R_1 \cup R_2)^{-1}\\
&\implies (b,a) \in (R_1 \cup R_2)\\
&\implies ((b,a) \in R_1) \vee ((b,a) \in R_2)\\
&\implies ((a,b) \in (R_1)^{-1}) \vee ((a,b) \in R_2^{-1})\\
&\implies (a,b) \in (R_1^{-1} \cup R_2^{-1})\\
&(R_1 \cup R_2)^{-1} \subseteq (R_1^{-1} \cup R_2^{-1})
\end{aligned}$$

$$\begin{aligned}
&(a,b) \in (R_1^{-1} \cup R_2^{-1})\\
&\implies ((a,b) \in (R_1)^{-1}) \vee ((a,b) \in R_2^{-1})\\
&\implies ((b,a) \in R_1) \vee ((b,a) \in R_2)\\
&\implies (b,a) \in (R_1 \cup R_2)\\
&\implies (a,b) \in (R_1 \cup R_2)^{-1}\\
&(R_1^{-1} \cup R_2^{-1}) \subseteq (R_1 \cup R_2)^{-1}
\end{aligned}$$

b. i. If $\forall a \in A, (a, a) \in R$, then by definition, so should $\forall a \in A, (a, a) \in R^{-1}$. Thus R^{-1} should also be reflexive.

ii. We assume $(a, b) \in R^{-1}$. We have to prove that $(b, a) \in R^{-1}$.

$$\begin{aligned}
&(b, a) \in R^{-1} \\
\implies &(a, b) \in R \ldots \{\text{ by definition}\} \\
\implies &(b, a) \in R \ldots \{\text{ by symmetric property}\} \\
\implies &(a, b) \in R^{-1} \ldots \{\text{ by definition}\}
\end{aligned}$$

iii. We assume $(a, b), (b, c) \in R^{-1}$. We have to prove that $(a, c) \in R^{-1}$.

$$\begin{aligned}
&((a, b) \in R^{-1}) \wedge ((b, c) \in R^{-1}) \\
\implies &((b, a) \in R) \wedge ((c, b) \in R) \ldots \{\text{ by definition}\} \\
\implies &((c, b) \in R) \wedge ((b, a) \in R) \ldots \{\text{ commutativity of } \wedge\} \\
\implies &(c, a) \in R \ldots \{\text{ by transitive property}\} \\
\implies &(a, c) \in R \ldots \{\text{ by definition}\}
\end{aligned}$$

For ii and iii, you had to start with a tuple in R^{-1} and not R, since you have to generalize over R^{-1}, we have cut marks if you have not.

Problem 4

a. $\rho \subseteq R^2$ such that ${}_{(a,b)}\rho_{(c,d)}$ means that (a, b) and (c, d) lie on the same curve $4x+5y = k$ for some $k \in \mathbb{R}$.

i. **Reflexive:** (a, b) and (a, b) obviously lie on the same curve $4a + 5b = k$. Thus the relation ρ is reflexive.

ii. **Symmetric:** Let ${}_{(a,b)}\rho_{(c,d)}$ be true for some (a, b) and (c, d), i.e. (a, b) and (c, d) lie on the same curve $4x + 5y \implies (c, d)$ and (a, b) lie on the same curve. $4a + 5b = 4c + 5d = k$. The relation ρ is symmetric.

iii. **Transitive:** Let us assume that ${}_{(a,b)}\rho_{(c,d)}$ and ${}_{(c,d)}\rho_{(e,f)}$, i.e

$$\begin{aligned}
4a + 5b = k_1 = 4c + 5d \\
4c + 5d = k_2 = 4e + 5f
\end{aligned}$$

Thus we have $k_1 = k_2$, thus $4a + 5b = 4e + 5f$. Thus (a, b) and (e, f) lie on the same curve. Thus ${}_{(a,b)}\rho_{(e,f)}$ holds. The relation ρ is transitive.

Thus the relation ρ is an equivalent relation. The equivalence class for the relation ρ is given by:

$$[(a, b)]_\rho = \{(x, y) \mid 4x + 5y = 4a + 5b\}$$

(Source of mistake): Most have written the equivalence class as:

$$[(a, b)]_\rho = \{(x, y) \mid 4x + 5y = k\} \text{ for some } k \in \mathbb{R}$$

This is incorrect as this is the relation ρ itself and not an equivalence class of (a, b)

b. $\psi \subseteq R^2$ such that ${}_{(a,b)}\psi_{(c,d)}$ means that (a,b) and (c,d) lie on the same curve $9x^2 + 16y^2 = k^2$ for some $k \in \mathbb{R}$.

   i. **Reflexive:** (a,b) and (a,b) obviously lie on the same curve $9x^2+16y^2 = k^2$. Thus the relation ψ is reflexive.

   ii. **Symmetric:** Let ${}_{(a,b)}\psi_{(c,d)}$ be true for some (a,b) and (c,d), i.e. (a,b) and (c,d) lie on the same curve $9x^2 + 16y^2 \implies (c,d)$ and (a,b) lie on the same curve. $9a^2 + 16b^2 = 9c^2 + 16d^2 = k^2$. The relation ψ is symmetric.

   iii. **Transitive:** Let us assume that ${}_{(a,b)}\psi_{(c,d)}$ and ${}_{(c,d)}\psi_{(e,f)}$, i.e

$$9a^2 + 16b^2 = k_1^2 = 9c^2 + 16d^2$$
$$9c^2 + 16d^2 = k_2^2 = 9e^2 + 16f^2$$

   Thus we have $k_1^2 = k_2^2$, thus $9a^2 + 16b^2 = 9e^2 + 16f^2$. Thus (a,b) and (e,f) lie on the same curve. Thus ${}_{(a,b)}\psi_{(e,f)}$ holds. The relation ψ is transitive.

Thus the relation ψ is an equivalent relation. The equivalence class for the relation ψ is given by: (Try to guess the geometric curve formed as well)

$$[(a,b)]_\psi = \{(x,y) \mid 9x^2 + 16y^2 = 9a^2 + 16b^2\}$$

(Source of mistake): Most have written the equivalence class as:

$$[(a,b)]_\psi = \{(x,y) \mid 9x^2 + 16y^2 = k^2\} \text{ for some } k \in \mathbb{R}$$

This is incorrect as this is the relation ψ itself and not an equivalence class of (a,b)

Problem 5

a. We need to show the double implication, $[a] = [b] \iff (a,b) \in R$ where R is the equivalent relation.

   i. Let us assume $[a] = [b]$. We need to show $(a,b) \in R$. Now we have:

$$[a]_R = \{x | (a,x) \in R\}; \quad [b]_R = \{x | (b,x) \in R\}$$

   Now we begin:

$$\begin{aligned} R \text{ is reflexive} \implies (a,b) &\implies a \in [a]_R \ldots (\text{By defn. of } [a]) \\ &\implies a \in [b]_R \ldots ([a]_R = [b]_R) \\ &\implies (b,a) \in R \ldots (\text{By defn of } [b]_R) \\ &\implies (a,b) \in R \ldots (R \text{ is symmetric}) \end{aligned}$$

   Thus $[a]_R = [b]_R \implies (a,b) \in R$

   ii. Let us assume that $(a,b) \in R$ and show that $[a]_R = [b]_R$. For this we neted to first show $[a]_R \subseteq [b]_R$ and then $[b]_R \subseteq [a]_R$.

(a) $[a]_R \subset [b]_R$: Let $x \in [a]_R$

$$\begin{aligned}
&\implies (a,x) \in R \ldots (\text{By defn}) \\
&\implies (x,a) \in R \ldots (R \text{ is symmetric}) \\
&\implies (x,b) \in R \ldots (\text{Using } (a,b) \in R \text{ and } R \text{ is transitive}) \\
&\implies (b,x) \in R \ldots (R \text{ is symmetric}) \\
&\implies x \in [b]_R \ldots (\text{By defn})
\end{aligned}$$

Thus we have $[a]_R \subseteq [b]_R$.

(b) $[b]_R \subset [a]_R$: Let $x \in [b]_R$

$$\begin{aligned}
&\implies (b,x) \in R \ldots (\text{By defn}) \\
&\implies (x,b) \in R \ldots (R \text{ is symmetric}) \\
&\implies (x,a) \in R \ldots (\text{Using } (a,b) \in R \text{ and } R \text{ is symmetric and transitive}) \\
&\implies (a,x) \in R \ldots (R \text{ is symmetric}) \\
&\implies x \in [a]_R \ldots (\text{By defn})
\end{aligned}$$

Thus we have $[b]_R \subseteq [a]_R$.

Thus from *(a)* and *(b)* we can say that $(a,b) \in R \implies [a]_R = [b]_R$

Thus from *(i)* and *(ii)* we have shown that the double implication holds.

b. We need to show that either $[a] \cap [b] = \phi$ or $[a] = [b]$, We will prove one part by using proof by contradiction.

Let us assume the contrary that $[a] \cap [b] \neq \phi$. Then

$$\exists x \ \ (x \in [a]) \wedge (x \in [b])$$

That is,

$$\begin{aligned}
x \in [a] &\implies (a,x) \in R \ldots (\text{By defn}) \\
x \in [b] &\implies (b,x) \in R \ldots (\text{By defn}) \\
&\implies (x,b) \in R \ldots (R \text{ is symmetric}) \\
&\implies (a,b) \in R \ldots (\text{Using the above two, } R \text{ is transitive})
\end{aligned}$$

Thus we have $[a] = [b]$ (Using part a.) which is contradiction to our assumption, which means that our assumption was incorrect and $[a]$ and $[b]$ are not disjoint under certain scenarios, rather $[a] = [b]$.

If we have $(a,b) \notin R$ then, $[a] \cup [b]$ will have to be disjoint because of the above proof. Thus we have only the two cases.

Problem 6 Many examples exist. The most common one among all submissions is the following:

The set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is a totally ordered set with respect to the relation $R = \{(x, y) \in \mathbb{Z}^2 | x \leq y\}$

a. **Reflexive:** For any integer $a \in \mathbb{Z}$ $a \leq a$. Thus the relation is reflexive.

b. **Anti-Symmetric:** For any arbitrary integers $a, b \in \mathbb{Z}$, if $aleqb$ does not necessarily imply $b \leq a$. Thus the relation is anti-symmetric.

c. **Transitive:** Let us assume that the relation R holds for the pair (a, b) and (b, c). Then we have $a \leq b \leq c$. Thus (a, c) is also holds. Thus the relation is transitive.

Problem 7 We are given that $A_i \cup A_j = \phi$. So all we need to show that $\bigcup_{i=1}^{k} A_i = A$. We have to show this by showing both sides are subsets of each other:

a. $\bigcup_{i=1}^{k} A_i \subseteq A$: We have $A_i \subseteq A$. Therefore we will definitely have $\bigcup_{i=1}^{k} A_i \subseteq A$

b. $A \subset \bigcup_{i=1}^{k} A_i$: We can prove by contradiction. Let us assume the contrary, i.e.

$$\exists x \in A \;\; x \notin A_i \forall i = 1, 2, \ldots, k$$

Now we know that $(x, x) \in R$ because R is a reflexive relation. Thus $x \in A_i$ which is a contradiction to our assumption. Thus our assumption was incorrect, and we have that for all $x \in A \implies x \in A_i$ for some $i = 1, 2, \ldots, k$. Thus $A \subset \bigcup_{i=1}^{k} A_i$.

Therefore from $a.$ and $b.$, we have $A = \bigcup_{i=1}^{k} A_i$.

Thus we have our result that A_is are partitions of the set A with respect to the equivalent relation R.