Discrete Structures

IIIT Hyderabad

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Tutorial 10

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Introduction

1. Questions
   - Question 1
   - Question 2

Prove the following -

1. A function f has left inverse $\iff$ it is injective.
   Sol: Say $f : A \to B$
   $\implies$ part :
   Suppose f has a left inverse, we need to show that it is injective, that is

$$f(x_1) = f(x_2) \implies x_1 = x_2 \tag{1}$$

Assume $f(x_1) = f(x_2)$. We shall prove $x_1 = x_2$. Say, if g is the left inverse of it, then we have

$$\begin{align}
g(f(x_1)) &= x_1 && \ldots \{\text{definition of left inverse}\} \tag{2}\\
g(f(x_2)) &= x_2 && \ldots \{\text{definition of left inverse}\} \tag{3}\\
g(f(x_1)) &= g(f(x_2)) && \ldots \{\text{as } f(x_1) = f(x_2)\} \tag{4}\\
x_1 &= x_2 && \ldots \{\text{from (2) and (3)}\} \tag{5}
\end{align}$$

$\Longleftarrow$ part: Construct $g : B \to A$ such that -

$$g(y) = \begin{cases} x \text{ if } \exists x, f(x) = y \\ c \text{ otherwise} \end{cases}$$

Here c is an arbitary fixed element $\in A$. We need to prove that it is a function, in order to do that, we need to prove that it is

1. Exhaustive over domain. That is $\forall y \in B, g(y)$ is defined. This is true by our construction (either an element will have it's pre-image or will be mapped to the element c).
2. $\forall y \in B, (g(y) = x_1) \wedge (g(y) = x_2) \implies x_1 = x_2$. Say if this were not true, we have that $g(y) = x_1$ and $g(y) = x_2$ but $x_1 \neq x_2$. Either of the 2 cases can happen -
   1. y doesn't have a pre-image in A. In which case both x_1, x_2 must be equal to c. That is, $x_1 = x_2 = c$.
   2. y has a pre-image in A, thus x_1 and x_2 are 2 distinct pre-images of y. Or $f(x_1) = f(x_2) = y$. but since f is injective, we must have $x_1 = x_2$.

Thus g should be a function.Now we have

$$\begin{aligned} &(\forall x \in A)\ g \circ f(x) \\ &= g(y) && \ldots\{\text{where } y = f(x)\} \\ &= x && \ldots\{\text{ by definition}\} \end{aligned}$$

Thus we have g is indeed a left-inverse function.

2. A function f has right inverse $\iff$ it is surjective.

Sol:

$\implies$ part:

Suppose f has a right inverse, say g. For surjectivity, we need to show that

$$(\forall y \in B)(\exists x \in A, f(x) = y)$$

For any $y \in B$, take $x = g(y)$ (this we can take since g is a function), indeed we have

$$\begin{aligned} f(x) &= f \circ g(y) \\ &= y \qquad \ldots \{ \text{ by definition of right inverse.} \} \end{aligned}$$

Thus for any y, we have it's pre-image as $x = g(y)$. Thus the function f is surjective.

$\Longleftarrow$ part:

Now if, f is surjective, construct a mapping $g : B \to A$. Since f is surjective, and for each y, we have atleast one pre-images. Let's say we map it to only one of them -

$$\forall y \in B, g(y) = x \text{ such that } f(x) = y$$

The function g is

1. Exhaustive over domain. That is $\forall y \in B, g(y)$ is defined. This is true by our construction (we are defining it over all $y \in B$).
2. $\forall y \in B, (g(y) = x_1) \land (g(y) = x_2) \implies x_1 = x_2$. Since we are choosing only one of the pre-images, an element will never be mapped to more than 1 distinct pre-image.

Thus g is a function, indeed -

$$\begin{aligned}
& f \circ g(y) \\
& = f(x) && \ldots \{ \text{ where } f(x) = y, \text{ by construction } \} \\
& = y
\end{aligned}$$

Thus our function is right inverse.

3. Prove that f is onto $\iff$ $h \circ f(x) = k \circ f(x)$ implies $h = k$.

Sol:

$\implies$ part: Since f is onto,

$$\forall y \in B, \exists x \in A \to f(x) = y$$

Thus we have,

$$\begin{aligned} & h(y) \\ & \quad = h(f(x)) && \ldots\{\text{ pre-image } x \text{ exists}\} \\ & \quad = k(f(x)) && \ldots\{\text{ assumption}\} \\ & \quad = k(y) && \ldots\{\text{ y} = f(x)\} \end{aligned}$$

Thus we get, $\forall y \in B, h(y) = k(y)$. Hence they are same.

Question 2

2.1: Prove that the set of of all odd numbers are countable.
Sol: We construct a mapping -

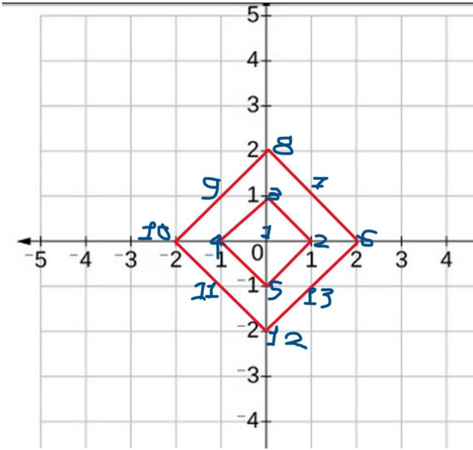
$$
\begin{gathered}
1 \to 1 \\
3 \to 2 \\
5 \to 3 \\
\vdots
\end{gathered}
$$

The function can be written as $f(n) = \frac{n+1}{2}$. It can be seen that this is a injective function as $f(n_1) = f(n_2) \implies n_1 = n_2$. And that this function is surjective, as we are able to list out each and every $n \in \mathbb{N}$ in the set. Thus it is bijective.

2.2: Prove that $\mathbb{Z} \times \mathbb{Z}$ is countable.

Sol: We construct a mapping -

$$
\begin{aligned}
(0,0) &\to 1 \\
(1,0) &\to 2 \\
(0,1) &\to 3 \\
(-1,0) &\to 4 \\
(0,-1) &\to 5 \\
(2,0) &\to 6 \\
&\vdots
\end{aligned}
$$



We claim this mapping is injective, as it map's to all elements uniquely (no natural number is repeated). This mapping is surjective as we are listing out all the natural numbers one by one, and hence all numbers are listed.