1 Introduction

2 Void statistics

3 Weiner reconstruction

Let's say we have an underlying signal that is a realization of a Gaussian random field with zero mean. The observed signal is a noisy estimate of the underlying signal at any given location.

$$\mathbf{y} = \mathbf{s}_d + \mathbf{e} \tag{1}$$

The aim is to design a operator $\mathbf{G}$, such that the linear estimate of the signal at any given location $\hat{\mathbf{s}}_m = \mathbf{G}\ \mathbf{y}$, minimizes the objective function:

$$\epsilon = \mathbb{E}[|\hat{\mathbf{s}}_m - \mathbf{s}_m|^2] \tag{2}$$

Writing all the one-dimensional vectors as a column vector, we have

$$\begin{aligned}\epsilon &= \mathbb{E}[\mathrm{Tr}[(\hat{\mathbf{s}}_m - \mathbf{s}_m)(\hat{\mathbf{s}}_m - \mathbf{s}_m)^T]] \\ &= \mathrm{Tr}[\mathbb{E}(\hat{\mathbf{s}}_m\ \hat{\mathbf{s}}_m^T) - \mathbb{E}(\hat{\mathbf{s}}_m\ \mathbf{s}_m^T) - \mathbb{E}(\mathbf{s}_m\ \hat{\mathbf{s}}_m^T) + \mathbb{E}(\mathbf{s}_m\ \mathbf{s}_m^T)]\end{aligned}$$

The last term is nothing but the covariance matrix of the signal at the mapped locations, denoted as $\mathbf{C}_{mm}$. The second term can be written as:

$$\mathbb{E}[\hat{\mathbf{s}}_m\ \mathbf{s}_m^T] = \mathbb{E}[\mathbf{G}\ \mathbf{y}\ \mathbf{s}_m^T] = \mathbb{E}[\mathbf{G}\ (\mathbf{s}_d + \mathbf{e})\ \mathbf{s}_m^T] = \mathbf{G}\mathbb{E}[\mathbf{s}\ \mathbf{s}_m^T] = \mathbf{G}\ \mathbf{C}_{dm}.$$

It is assumed that the signal and the error processes are uncorrelated. Similarly, the third term reduces to $\mathbf{C}_{md}\mathbf{G}^T$. The first term can be simplified as:

$$\begin{aligned}\mathbb{E}(\hat{\mathbf{s}}_m\ \hat{\mathbf{s}}_m^T) &= \mathbb{E}[\mathbf{G}\ \mathbf{y}\mathbf{y}^T\ \mathbf{G}^T] \\ &= \mathbb{E}[\mathbf{G}\ (\mathbf{s}_d + \mathbf{e})(\mathbf{s}_d^T + \mathbf{e}^T)\ \mathbf{G}^T] \\ &= \mathbf{G}\ (\mathbf{C}_{dd} + \mathbf{N})\ \mathbf{G}^T\end{aligned}$$

The error term then becomes:

$$\epsilon = \mathrm{Tr}[\mathbf{G}\ (\mathbf{C}_{dd} + \mathbf{N})\ \mathbf{G}^T] - 2\ \mathrm{Tr}[\mathbf{G}\ \mathbf{C}_{dm}] + \mathrm{Tr}[\mathbf{C}_{mm}]$$

Minimizing this term w.r.t $\mathbf{G}$ by setting the first derivative to 0, the optimal estimator is:

$$\mathbf{G} = \mathbf{C}_{md}[\mathbf{C}_{dd} + \mathbf{N}]^{-1} \tag{3}$$

The Weiner reconstructions thus provides the unbiased, minimum least-square error estimate of the true underlying signal, provided that the signal is governed by a Gaussian process.

4 Persistent homology

5 Results and discussion