(1) equating posteriors de.B can be found

$$p(\omega_1|x) = p(\omega_2|x)$$

$$\Rightarrow p(x|\omega_1)\cdot p(\omega_1) = p(x|\omega_2)\cdot p(\omega_2)$$

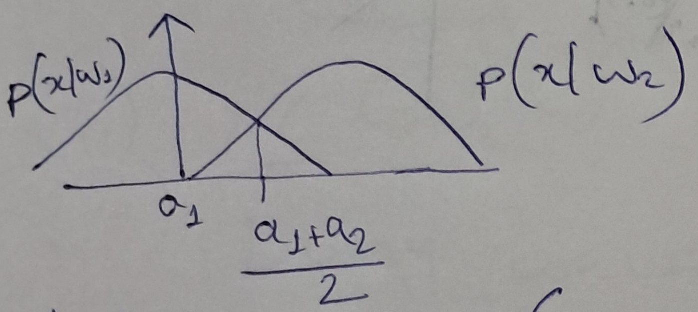
$$p(\omega_1) = p(\omega_2) = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\pi b\left(1+\frac{x-a_1}{b}\right)^2} = \frac{1}{\pi b\left(1+\frac{x-a_2}{b}\right)^2}$$

$$\Rightarrow \pm\frac{x-a_1}{b} = \pm\frac{x-a_2}{b}$$

$$x = \frac{a_1+a_2}{2}$$

1/2



total error $p(\text{error}) = \int p(\text{error}|x)\,p(x)\,d$

$$p[\text{error}|x] = \min\left[p(\omega_1|x),\ p(\omega_2|x)\right]$$

$$p(\text{error}) = \int_{-\infty}^{a_1+a_2/2} p(x|\omega_2)\,p(\omega_2)\,dx + \int_{a_1+a_2/2}^{+\infty} p(x|\omega_1)\,p(\omega_1)\,d$$

1/4

$$= \frac{1}{2}\left[\int_{-\infty}^{a_1+a_2/2} \frac{1}{\pi b\left(1+\frac{x-a_2}{b}\right)^2}\,dx + \int_{\frac{a_1+a_2}{2}}^{+\infty} \frac{1}{\pi b\left(1+\frac{x-a_1}{b}\right)^2}\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{a_2-a_1}{2b}\right)$$

1/4

Q.2.) $X = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad X \in \mathbb{R}^{d \times N} \Rightarrow d = 3$

$N = 3$

$$\mu = \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \end{bmatrix}. \quad \text{So } \sum_i X_i - \mu = \begin{bmatrix} 1-\frac{1}{3} & 0-1/3 & 0-1/3 \\ -1-0 & 0-0 & 1-0 \\ 0-2/3 & 1-2/3 & 1-2/3 \end{bmatrix}$$

$$\Rightarrow \sum_i X_i - \mu = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1 & 0 & 1 \\ -2/3 & 1/3 & 1/3 \end{bmatrix}.$$

Unbiased covariance matrix is $Cov(X) = \frac{1}{N-1} \sum_i (X_i - \mu) \cdot (X_i - \mu)^T$

$$\Rightarrow Cov(X) = \frac{1}{3-1} \cdot \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1 & 0 & 1 \\ -2/3 & 1/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 & -1 & -2/3 \\ -1/3 & 0 & 1/3 \\ -1/3 & 1 & 1/3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2/3 & -1 & -2/3 \\ -1 & 2 & 1 \\ -2/3 & 1 & 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 & -1/3 \\ -1/2 & 1 & 1/2 \\ -1/3 & 1/2 & 1/3 \end{bmatrix}$$

$$\therefore Cov(X) = \begin{pmatrix} 1/3 & -1/2 & -1/3 \\ -1/2 & 1 & 1/2 \\ -1/3 & 1/2 & 1/3 \end{pmatrix}.$$

0.5

SML A-1 Q3 solution

3 categories : w_1, w_2, w_3

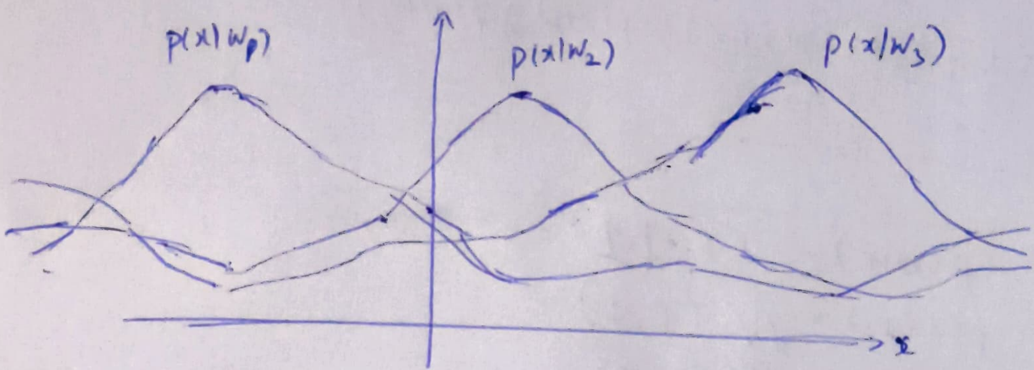
→ Constraint : $p(w_1|x) + p(w_2|x) + p(w_3|x) = 1.$

→ $p(w_i|x) = \dfrac{p(x|w_i)\, p(w_i)}{p(x)}$ $\Rightarrow$ $\boxed{p(x|w_i)\, p(w_i) = p(w_i|x).\ p(x)}$.

[Bayes' theorem]

~~$p(w_i|x) \propto p(x|w_i)$~~

$\boxed{p(x|w_i)\, p(w_i) \propto p(w_i|x)}$



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$$p(\text{error}) = \int_x p(\text{error}|x)\, p(x)\, dx.$$

$$p(\text{error}|x) = 1 - p(\text{correct}|x)$$
$$= 1 - \max\left(p(w_1|x),\ p(w_2|x),\ p(w_3|x)\right)$$

Since we assume equiprobable priors, we also have

$$p(w_1) = p(w_2) = p(w_3)$$

and $p(w_i) = \frac{1}{3}\ \forall\, i \in \{1, 2, 3\}$.

$$p(\text{error}) = \int_x \left[1 - \max\left(p(w_1|x),\ p(w_2|x),\ p(w_3|x)\right)\right] p(x)\, dx$$

[probability from x=-∞ to ∞]

$$= \int_x \cancel{p(x)\, dx}^{\,1} - \int_x \max\left(p(w_1|x),\ p(w_2|x),\ p(w_3|x)\right) p(x)\, dx$$

$$= 1 - \int_x \max\left(p(w_1|x),\ p(w_2|x),\ p(w_3|x)\right) p(x)\, dx.$$

$$= 1 - \int_x \max\left(\frac{p(x|w_1)\,p(w_1)}{p(x)}, \frac{p(x|w_2)\cdot p(w_2)}{p(x)}, \frac{p(x|w_3)\,p(w_3)}{p(x)} \right) p(x)\,dx$$

[Bayes' theorem]

$$= 1 - \int_x \max\left[p(x|w_1)p(w_1), p(x|w_2)p(w_2), p(x|w_3)p(w_3) \right] dx$$

$\because$ equiprobable priors :

$$P(\text{error}) = 1 - \int_x \max\left(p(x|w_1)\cdot\frac{1}{3}, \frac{p(x|w_2)}{3}, \frac{p(x|w_3)}{3} \right) dx$$

$$\boxed{P(\text{error}) = 1 - \frac{1}{3}\int_x \max\left[p(x|w_1), p(x|w_2), p(x|w_3) \right] dx.}$$

1/2

(ii) We now have $p(x|w_1) \sim \mathcal{N}(-1, 1)$

$p(x|w_2) \sim \mathcal{N}(0, 1)$

$p(x|w_3) \sim \mathcal{N}(1, 1)$

- Find points of intersection between ~~two~~ $p(x|w_i)$, $p(x|w_{i+1})$ $\forall\ i \in \{1, 2\}$.

a) $p(x|w_1)$, $p(x|w_2)$

$$p(x|w_1) = p(x|w_2)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Rightarrow (x+1)^2 = x^2$$

$$\Rightarrow \boxed{x = -\frac{1}{2}}$$

1/4

b) $p(x|w_2)$, $p(x|w_3)$

$$p(x|w_2) = p(x|w_3)$$

$$\rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$$

$$\rightarrow x^2 = (x-1)^2$$

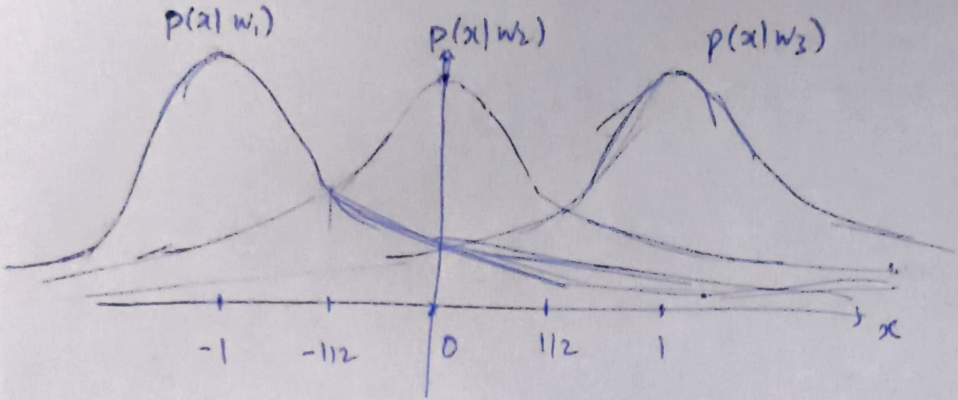
$$\Rightarrow \boxed{x = \frac{1}{2}}$$

1/4

$\therefore$ for $x \in \left(-\infty, -\frac{1}{2}\right]$, $p(x|w_1)$ is max

for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, $p(x|w_2)$ is max

for $x \in \left(\frac{1}{2}, \infty\right)$, $p(x|w_3)$ is max.



$p(w|x_i) \propto p(x|w_i)$

$\because$ equiprobable priors.

$$P(\text{error}) = 1 - \frac{1}{3}\int_x \max\left(p(x|w_1), p(x|w_2), p(x|w_3)\right) dx$$

$$= 1 - \frac{1}{3}\left[\int_{-\infty}^{-1/2} p(x|w_1)\, dx + \int_{-1/2}^{1/2} p(x|w_2)\, dx + \int_{+1/2}^{\infty} p(x|w_3)\, dx\right]$$

a) $$\int_{-\infty}^{-1/2} p(x|w_1)\, dx = \int_{-\infty}^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}}\, dx$$

1/4

Let $x+1 = t$;

$$\int_{-\infty}^{1/2} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt = \Phi\left(\frac{1}{2}\right). \quad -(1)$$

b) $$\int_{-1/2}^{1/2} p(x|w_2)\, dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \int_{-\infty}^{1/2} \frac{1}{2} e^{-\frac{x^2}{2}}\, dx - \int_{-\infty}^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

1/4

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(\frac{-1}{2}\right) \quad \rightarrow (2)$$

c) $$\int_{1/2}^{\infty} p(x|w_3)\, dx = \int_{1/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}\, dx - \int_{-\infty}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}\, dx$$

$$= 1 - \int_{-\infty}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}\, dx$$

1/4

(Let $(x-1) = t$), $$= 1 - \int_{-\infty}^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt = 1 - \Phi\left(-\frac{1}{2}\right) \quad \rightarrow (3)$$

$$P(\text{error}) = 1 - \frac{1}{3}\left((1) + (2) + (3) \right)$$

$$= 1 - \frac{1}{3}\left[\phi\left(\frac{1}{2}\right) + \phi\left(\frac{1}{2}\right) - \phi\left(\frac{-1}{2}\right) + 1 - \phi\left(\frac{-1}{2}\right) \right]$$

$$= \frac{2}{3} - \frac{2}{3}\phi\left(\frac{1}{2}\right) + \frac{2}{3}\phi\left(-\frac{1}{2}\right)$$

$$= \frac{2}{3}\left[1 - \phi\left(\frac{1}{2}\right) + \phi\left(-\frac{1}{2}\right) \right]$$

$$= \frac{2}{3}\left[1 - \phi\left(\frac{1}{2}\right) + 1 - \phi\left(\frac{1}{2}\right) \right] \because \left[\phi(-x) = 1 - \phi(x) \right]$$

$$= \boxed{\frac{4}{3}\left[1 - \phi\left(\frac{1}{2}\right) \right] = \frac{4}{3}\phi\left(-\frac{1}{2}\right)}$$

Q4) $$\frac{p(x|w_1)}{p(x|w_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \frac{p(w_2)}{p(w_1)}$$

Given $\Rightarrow$ 0-1 loss, $p(w_1) = p(w_2)$

$$\frac{p(x|w_1)}{p(x|w_2)} > 1$$

$$\frac{\dfrac{1}{\pi b\left[1+\left(\frac{x-a_1}{b}\right)^2\right]}}{\dfrac{1}{\pi b\left[1+\left(\frac{x-a_2}{b}\right)^2\right]}} > 1$$

$$\frac{1+\left(\frac{x-a_2}{b}\right)^2}{1+\left(\frac{x-a_1}{b}\right)^2} > 1$$

$$\cancel{1}+\left(\frac{x-a_2}{\cancel{b}}\right)^2 > \cancel{1}+\left(\frac{x-a_1}{\cancel{b}}\right)^2$$

$$\cancel{x^2}+a_2^2-2a_2x > \cancel{x^2}+a_1^2-2a_1x$$

$$a_2^2-a_1^2 > 2(a_2-a_1)x$$

$$\boxed{x < \frac{a_1+a_2}{2}}$$

1/2

1/2 if x satisfies this, it will mapped to class 1