Exercise sessions 6–7: LU decomposition and systems of linear equations.

Exercise 1: Implementation of the LU Decomposition [10 pts]

1. Solve the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 13 \\ 2x_1 + 3x_2 - x_4 &= -1 \\ -3x_1 + 4x_2 + x_3 + 2x_4 &= 10 \\ x_1 + 2x_2 - x_3 + x_4 &= 1 \end{aligned}$$

   by performing the LU decomposition and backward substitution. Verify the solution explicitly by comparing with the one obtained using Matlab's matrix operations (ex. `X=A\B`).

2. Implement the LU decomposition of a square matrix. Test your implementation on random matrices of size up to 100. Compare your implementation with Matlab's `lu` function plotting the difference between the solutions provided by the two methods.

   N.B.: Matlab typically performs row permutations. Indeed, `[L,U,P] = lu(A)` gives L, U and the permutation matrix P. In order to compare properly, run your LU function on $A' = PA$, where P is taken from Matlab's output.

3. Test your implementation on the following matrix:

$$\begin{pmatrix} 0 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix}$$

   Is your result meaningful? Comment on the origin of the observed problem (if any).

4. Implement the LU decomposition with pivoting. Test your implementation of LU decomposition on the above matrix. Your function should accept a square matrix A and return 3 square matrices L, U, P akin to the Matlab's function `lu`. Take a look into the supplied `test_lu.m` to make sure your implementation is correct.

Listing 1: 'lu_decomposition.m'

```
function [ L, U, P ] = lu_decomposition( A )

% lu_decomposition computes the LU decomposition with pivoting
% for a square matrix A
% Return lower L, upper U and permutation matrix P such that P*A=L*U

[L U P]=lu(A); % replace this line with your own implementation
               % not using Matlab's lu function

end
```

Submit: File `lu_decomposition.m` [5 pts]. In the report, the correct solution of the system of linear equations (1), the comparison between your and MatLab's implementation of the LU decomposition (2) and the discussion (3) give [1 pt], [2 pts] and [2 pts], respectively.

Exercise 2: Barycenter [6 pts]

In astronomy, the barycenter is the center of mass of two or more bodies that orbit one another and is the point which the bodies orbit. For an artificial system consisting of 6 astronomical bodies, calculate the masses of each object and the position of the overall barycenter.Assume that the system is flat (like our solar system) and the objects move in plane. The following coordinates of the astronomical objects are known from a snapshot of the system - *Alpha*(0,6), *Beta*(9,0), *Gamma*(3,5), *Delta*(3,4), *Epsilon*(10,12) and *Zeta*(7,0). The table below lists known barycenters in the system. The distances are given in light-minutes, while the masses are in Earth's masses.

Coordinate	Objects	Overall mass
(6,2)	*Alpha,Beta*	12
(4,6)	*Gamma,Epsilon*	14
$(\frac{25}{8},\frac{15}{4})$	*Alpha,Delta,Zeta*	32

In the case of a system of objects P_i, $i = 1, ..., n$ with overall mass M, individual object mass m_i and coordinate r_i, the center of mass R can be found with the following equation

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}_i. \tag{1}$$

Grading: [4 pts] for the masses of objects, [2 pts] for the correct barycenter of the system.

Exercise 3: Lotka-Voterra equations [9 pts]

The Lotka-Volterra equations, also known as the predator-prey equations, are a system of first-order non-linear differential equations frequently used to describe the dynamics of biological systems in which a number of species interact being either predators or prey with respect to each other. Populations x_i change through time t according to the following set of equations

$$\begin{aligned}
dx_1/dt &= r_1 x_1 - x_1 \sum_i A_{1i} x_i \\
dx_2/dt &= r_2 x_2 - x_2 \sum_i A_{2i} x_i \\
dx_3/dt &= r_3 x_3 - x_3 \sum_i A_{3i} x_i \\
&\dots \\
dx_N/dt &= r_N x_N - x_N \sum_i A_{Ni} x_i
\end{aligned}$$

1. Assume that a only a single type of species exists and its population is described by

   $$dx/dt = rx - Ax^2$$

   with both r and A being positive. What processes the terms on the right-hand side of the above equation describe? What is the equilibrium population of the species?

2. Now consider 4 interacting populations with

   $$A_{ij} = \begin{bmatrix} 0 & 10 & 50 & 0 \\ -1 & 3 & 10 & 0 \\ -2 & 10 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, r_i = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the equilibrium population of species by solving the corresponding system of linear equations. Answer the following 2 questions: Among these populations, which ones act like predators and which ones act like a prey? Which population does not depend on others? Please, motivate your answers.

Grading: [3 pts] for the single-species scenario, [3 pts] for the solution of the four-species problem, [3 pts] for the correct answers to the questions.

Exercise 4: Conditioning [5 pt]

Solve the following two systems of linear equations that have only a minor difference in the free coefficient of the first equation:

$$\begin{aligned} x_1 + x_3 &= 2 \\ 1.001x_1 + x_2 &= 1 \\ -x_2 + x_3 &= 1 \end{aligned}$$

and

$$\begin{aligned} x_1 + x_3 &= 2.001 \\ 1.001x_1 + x_2 &= 1 \\ -x_2 + x_3 &= 1 \end{aligned}$$

Compare the solutions and make conclusion regarding the sensitive of solutions to small deviations in the right-hand side. Motivate your answer from the standpoint of condition number calculated using Matlab's `cond` function.

Grading: [2 pts] for the solutions of systems of linear equations and [3 pts] for the discussion.