TECHNICAL UNIVERSITY BERLIN

LECTURE NOTES

Differential Equations II A

Theory of weak solutions for stationary differential equations

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Primer in functional analysis

DEFINITION 0.0.1 (DUALITY AND SEPARABILITY)

Let $(X, \| \cdot \|)$ be a normed space. We call

$$X^* := \big\{ f : X \to \mathbb{R} : f \text{ linear and bounded} \big\}$$

the dual space of X and equip it with the dual norm $\qquad$ dual space

$$\| \cdot \|_* : X^* \to \mathbb{R}, \ f \mapsto \sup_{\substack{x \in X \\ x \neq 0}} \frac{|\langle f, x \rangle_{X^* \times X}|}{\|x\|}.$$

Then $(X^*, \| \cdot \|_*)$ is a BANACH-Space, where $f(x) := \langle f, x \rangle_{X^* \times X}$ denotes the dual
pairing. $\qquad$ dual pairing

X is separable if it contains a dense at most countable subset. $\qquad$ separable

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Example 0.0.2 The space $L^p(I)$ is separable only for $p \in [1, \infty)$.

Proof. In the appendix. $\qquad \square$

DEFINITION 0.0.3 (BIDUAL SPACE, CANONICAL EMBEDDING)

We call $X^{**} := (X^*)^*$ the bidual space of X and $\hat{\imath} : X \to X^{**}$ defined by $\qquad$ bidual space
$\langle \hat{\imath}(x), f \rangle_{X^{**} \times X} := \langle f, x \rangle_{X^* \times X}$. the canonical embedding, where $x \in X$ and $f \in X^{**}$. $\qquad$ canonical
embedding

Corollary 0.0.4 (of the HAHN-BANACH theorem)

The canonical embedding is linear, isometric and injective.

DEFINITION 0.0.5 (REFLEXIVE SPACE)

$(X, \| \cdot \|)$ is called reflexive if the canonical embedding is surjective. $\qquad$ reflexive

Example 0.0.6 Every finite dimensional BANACH space is reflexive, and by the FRÉCHET–RIESZ
representation theorem so is every HILBERT space. A space which is not separable is ℓ^∞,
which contains the uncountable subset $\{0, 1\}^{\mathbb{N}}$.

In the following, let $I \subset \mathbb{R}$ be an open interval.

Lemma 0.0.7 (Dual space of L^p)

*We have $(L^p(I))^* \cong L^q(I)$, where q is the HÖLDER-conjugate to $p \in [1, \infty)$, but
$(L^\infty(I))^* \supset L^1(I)$.*

Corollary 0.0.8

For $p \in (1, \infty)$ the space $L^p(I)$ is reflexive.

Lemma 0.0.9 (Continuity in the p-mean)

Let $p \in [1, \infty)$ and $u \in L^p(I)$ be a function. Then we have

$$\forall \varepsilon > 0 \ \exists \delta > 0 : |h| < \delta \implies \left(\int_a^b |u(x + h) - u(x)|^p \right)^{\frac{1}{p}} < \varepsilon$$

> *where, outside of I, u is continued with 0.*

Proof. In the appendix. □

We will now introduce two function spaces which, to some extent, lie on opposite sides of the regularity spectrum.

DEFINITION 0.0.10 (LOCALLY INTEGRABLE FUNCTIONS)

The space $L_{\mathrm{loc}}^p(I)$ defined by

$$\left\{ u : I \to \mathbb{R} \text{ measurable} : u|_K \in L^p(K) \; \forall K \subset I \text{ compact} \right\}$$

is not a normed space.

Example 0.0.11 (locally but not globally integrable function)
Consider $I := (0, 1)$ and $u(x) := \frac{1}{x}$ or alternatively $I := (0, \pi/2)$ and $u(x) := \tan(x)$. Since the compact domain is bounded away from the critically point, we have $u \in L_{\mathrm{loc}}^\infty(I) \backslash L^1(I)$.

DEFINITION 0.0.12 (COMPACTLY SUPPORTED FUNCTIONS)

We define $\mathcal{C}_0^\infty(I) := \left\{ u \in \mathcal{C}^\infty(I) : \mathrm{supp}(u) \subset I \text{ compact} \right\}$.

Example 0.0.13 ($\mathcal{C} \nsubseteq \mathcal{C}_0^\infty$)
The function $\varphi : (0, \pi)$, $x \mapsto \sin(x)$ is in $\mathcal{C}^\infty((0, \pi); \mathbb{R})$ but not even in $\mathcal{C}_0((0, \pi); \mathbb{R})$ and thus not in $\mathcal{C}_0^\infty((0, \pi); \mathbb{R})$, either .

Example 0.0.14 Consider the function

$$J : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} \exp\left(\frac{1}{x^2 - 1} \right), & \text{for } |x| < 1, \\ 0, & \text{otherwise} \end{cases}$$

Then, $\mathrm{supp}(J) = [-1, 1]$. For $\varepsilon > 0$ define $J_\varepsilon(x) := J(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}$.
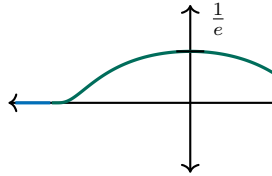

Figure 1: The test function $J \in \mathcal{C}_0^\infty$.

1 Generalised Derivatives and Regularisation in One Dimension

1.1 The weak derivative

09.04.2019

Example 1.1.1 (Why do we need weak solutions in real life?)
To understand why we would want to allow non-continuous coefficient functions $c(x)$ and $d(x)$, we revisit the first example of a stationary partial differential equation from the very first chapter in DGL I and allow the bean to consist of different material.

Alternatively, consider the elliptical POISSON's equation

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Figure 2: The bending of a beam.

Its solutions are hard to find, therefore we want to find a generalised definition of solutions. If we want to $u \in \mathcal{C}^2(\Omega)$, we have to require $f \in \mathcal{C}(\Omega)$, which might be unrealistic.

Example 1.1.2 (V, Not at all classically differentiable function)
Consider

$$u(x) := \begin{cases} x + 1, & \text{if } x \in (-1, 0), \\ \frac{6}{5}, & \text{if } x = 0, \\ 1 - x & \text{if } x \in (0, 1), \\ 0, & \text{else.} \end{cases}$$

Then, f is neither continuous in zero nor differentiable in $-1, 0$ or 1 but still has a weak derivative

$$v(x) := \begin{cases} -1, & \text{if } x \in (-1, 0), \\ 1 & \text{if } x \in (0, 1), \\ 0, & \text{else.} \end{cases}$$

Figure 3: The mentioned functions u and ν.

which is in $W^{1,p}(-1, 1)$ (see section 2).

From now on, let $I := (a, b)$ be an open real interval with $a < b$, but we will later see that most of the following theory holds true for any open subset of $\mathbb{R}^d$.

Example 1.1.3 (Weak formulation for a BVP)
Consider the following boundary value problem with homogeneous NEUMANN boundary conditions

$$\begin{cases} -u''(x) = f(x), & x \in I, \\ u(a) = u(b) = 0. \end{cases} \tag{1}$$
$$\tag{2}$$
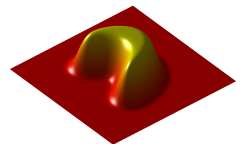
Figure 4: A test function $v \in \mathcal{C}_0^\infty(I)$.

① Multiply (1) with a suitable test function v satisfying (2).

test function

② Integrating over the domain yields

$$-\int_a^b u''(x)v(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x.$$

③ Integration by parts yields

$$\int_a^b u'(x)v'(x)\,\mathrm{d}x + \underbrace{\left[u'(x)v(x)\right]_{x=a}^b}_{=0} = \int_a^b f(x)v(x)\,\mathrm{d}x.$$

Instead of (1) we can consider it's variational formulation

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$$\int_a^b u'(x)v'(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x. \tag{3}$$

</div>

variational formulation

for suitable v with $v(a) = v(b) = 0$, where $fv, u'v' \in L^1(I)$ (alternative: $u, u', v \in L^2(I;\mathbb{R})$) and u' and v' are weak derivatives.

weak derivative

definition 1.1.4 (weak derivative (Sobolev, Schwartz))

Let $u, v \in L^1_{\mathrm{loc}}(I)$. If the equation

$$\int_a^b u(x)\varphi'(x)\,\mathrm{d}x = -\int_a^b v(x)\varphi(x)\,\mathrm{d}x, \tag{4}$$

holds for all $\varphi \in \mathcal{C}_0^\infty(I;\mathbb{R})$, we call u weakly differentiable with the weak derivative v.

The above integrals are well defined as for $\varphi \in \mathcal{C}_0^\infty(I;\mathbb{R})$ we have

$$\int_a^b v(x)\varphi(x)\,\mathrm{d}x = \int_{\mathrm{supp}(\varphi)} v(x)\varphi(x)\,\mathrm{d}x \leqslant \max_{x \in \mathrm{supp}(\varphi)} |\varphi(x)| \cdot \|v\|_{L^1(\mathrm{supp}(\varphi))} < \infty.$$

Remark 1.1.5 Notice that this is not a pointwise definition. The weak derivative is unique (up to null sets), which will be proven later. The weak derivative is linear, that is, the weak derivative of a linear combination of functions is the linear combinations of its weak derivatives.

Lemma 1.1.6 (Fundamental Lemma for continuous functions)
Let $u \in \mathcal{C}([a,b];\mathbb{R})$ be chosen such that $\int_a^b u(x)\varphi(x)\,\mathrm{d}x = 0$ for all $\varphi \in \mathcal{C}_0^\infty((a,b);\mathbb{R})$. Then $u|_{[a,b]} \equiv 0$.

Proof. Assume that there exists a $x_0 \in (a,b)$ so that $u(x_0) \neq 0$, without loss of generality $u(x_0) > 0$. Because u is continuous there exists an interval $(\alpha,\beta) \subset (a,b)$ containing x_0 so that $u|_{(\alpha,\beta)} > 0$. Now, define

$$\varphi \colon \mathbb{R} \to \mathbb{R}, \ x \mapsto \begin{cases} \exp\left(\frac{1}{(x-\alpha)(x-\beta)}\right), & x \in (\alpha,\beta), \\ 0, & \text{elsewhere.} \end{cases}$$

Then, we have $\varphi \in \mathcal{C}_0^\infty((a,b);\mathbb{R})$, $\mathrm{supp}(\varphi) = [\alpha,\beta]$ and

$$0 = \int_a^b u(x)\varphi(x)\,\mathrm{d}x = \int_\alpha^\beta \underbrace{u(x)\varphi(x)}_{>0}\,\mathrm{d}x > 0,$$

which is a contradiction. $\qquad\qquad\square$

Lemma 1.1.7 (Classical and weak derivatives)

   ① *Let $u \in \mathcal{C}^1([a,b]; \mathbb{R})$. Then the weak derivative of u coincides with its classic derivative.*

   ② *Let u' be the weak derivative of u on (a,b). Then for all intervals $(\alpha, \beta) \subset (a,b)$ it holds that $u'|_{(\alpha, \beta)}$ is also the weak derivative of $u|_{(\alpha, \beta)}$ on (α, β).*

Proof. ① Follows directly from the formula for integration and $\operatorname{supp}(\varphi) \subset (a,b)$. The uniqueness of the weak derivative will be proven later.

 ② Let $(\alpha, \beta) \subset (a,b)$ and $\varphi \in \mathcal{C}_0^\infty(\alpha, \beta)$ and define the trivial extension of φ by $\tilde{\varphi} \in \mathcal{C}_0^\infty(a,b)$. Then, we conclude

$$\int_\alpha^\beta u\varphi' \,\mathrm{d}x = \int_a^b u\tilde{\varphi}' \,\mathrm{d}x = -\int_a^b u'\tilde{\varphi} \,\mathrm{d}x = -\int_\alpha^\beta u'\varphi \,\mathrm{d}x,$$

which implies the proposition.   □

Example 1.1.8 For $i \in \{1, 2\}$ define the two functions

$$u_i(x) := \begin{cases} x, & \text{if } x \in (0,1], \\ i, & \text{if } x \in (1,2) \end{cases}$$

From Lemma 1.1.7 we know that their weak derivative coincide almost everywhere with the function

$$u'(x) := \begin{cases} 1, & \text{if } x \in (0,1], \\ 0, & \text{if } x \in (1,2) \end{cases}$$

Using the Definition of weak differentiability, for all $\varphi \in \mathcal{C}_0^\infty(0,2)$ we obtain

$$\int_0^2 u_1(x)\varphi'(x) \,\mathrm{d}x = \int_0^1 x\varphi'(x) \,\mathrm{d}x + \int_1^2 \varphi'(x) \,\mathrm{d}x$$

$$= \cancel{\varphi(1)} - \int_0^1 \varphi(x) \,\mathrm{d}x \cancel{-\varphi(1)} = -\int_0^2 u'(x)\varphi(x) \,\mathrm{d}x.$$

Now we choose an $\varphi \in \mathcal{C}_0^\infty(0,2)$ so that $\varphi(1) \neq 0$, then we obtain

$$\int_0^2 u_2(x)\varphi'(x) \,\mathrm{d}x = \varphi(1) - \int_0^1 \varphi(x) \,\mathrm{d}x - 2\varphi(1)$$

$$= -\int_0^1 \varphi(x) \,\mathrm{d}x - \varphi(1) \neq -\int_0^2 u'(x)\varphi(x) \,\mathrm{d}x.$$

Therefore, u_1 is weakly differentiable with weak derivative u' but u_2 is not.

Counterexample 1.1.9

The function

$$f \colon (-1,1) \to \mathbb{R}, \qquad x \mapsto \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is continuous as

$$0 = \lim_{x \to 0} -x^2 \leqslant \lim_{x \to 0} f(x) \leqslant \lim_{x \to 0} x^2 = 0$$

and differentiable everywhere except in the origin: for $x \neq 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x} x^2 \sin\left(\frac{1}{x}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

The function

$$v\colon (-1,1) \to \mathbb{R}, \qquad x \mapsto \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is not continuous in zero, but integrable over $(-1,1)$.

Thus the weak derivative of f **TODO**

Figure 5: The function $x \mapsto x^2 \sin\left(\frac{1}{x}\right)$.

The following corollary shows that weak derivatives generalise classical derivatives.

Corollary 1.1.10 (Link to classical derivatives)

    Let $u\colon I \to \mathbb{R}$ be absolutely continuous. Then (shown in DGL I) u is classically differentiable almost everywhere and $u' \in L^1(I)$. Therefore, u is also weakly differentiable with weak derivative u', which exists almost everywhere.

Corollary 1.1.11 (V)

    f continuous and weakly differentiable $\iff$ f absolutely continuous.

Corollary 1.1.12 (V)

    If g is the weak derivative of f, then the Fundamental Theorem of Calculus holds:

$$f(b) - f(a) = \int_a^b g(x)\,\mathrm{d}x$$

TODOund die Produktregel gilt auch!

Example 1.1.13 (Weak derivative of the absolute value)

Consider some open interval $(-a,a)$ for $a \in (0,\infty]$ and the functions $u(x) := |x|$ and $v(x) := \mathrm{sgn}(x)$. Then v is the weak derivative of u: For all test functions $\varphi \in \mathcal{C}_0^\infty((-a,a))$ we have

$$\int_{-a}^a u(x)\varphi'(t)\,\mathrm{d}x = \int_0^a x\varphi'(x)\,\mathrm{d}x - \int_{-a}^0 x\varphi'(x)\,\mathrm{d}x$$

$$= \underbrace{\left[-x\varphi(x)\right]_{x=-a}^0}_{=0} + \int_{-a}^0 \varphi(x)\,\mathrm{d}x$$

$$+ \underbrace{\left[x\varphi(x)\right]_{x=0}^a}_{=0} - \int_0^a \varphi(x)\,\mathrm{d}x$$

$$= -\int_{-a}^a v(x)\varphi(x)\,\mathrm{d}x.$$

Example 1.1.14 (Heaviside function has no weak derivative)

Consider the Heaviside function

$$H\colon (-1,1) \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Assume it had a weak derivative $v \in L^1((-1,1);\mathbb{R})$, then this implies

$$\varphi(0) = -\int_0^1 \varphi'(x)\,\mathrm{d}x = -\int_{-1}^1 H(x)\varphi'(x)\,\mathrm{d}x = \int_{-1}^1 v(x)\varphi(x)\,\mathrm{d}x$$

for all $\varphi \in \mathcal{C}_0^\infty((-1,1);\mathbb{R})$. Now, choose $\Phi(x) := J_\varepsilon(x) := J\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon \in (0,1)$. Then we have $\Phi \in \mathcal{C}_0^\infty(-1,1)$ for all $\varepsilon \in (0,1)$ and thus

Figure 6: The test function J_ε for $\varepsilon \in \left\{\frac{1}{2}, \frac{1}{4}\right\}$.

$$\frac{1}{e} = J_\varepsilon(0) = \int_{-1}^1 v(x)J_\varepsilon(x)\,\mathrm{d}x = \int_{-\varepsilon}^\varepsilon v(x) \underbrace{J_\varepsilon(x)}_{\leq \frac{1}{e}}\,\mathrm{d}x$$
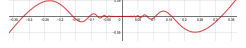
$$\leqslant \frac{1}{e}\int_{-\varepsilon}^{\varepsilon}|v(x)|\,\mathrm{d}x \xrightarrow[v\in L^1((-1,1);\mathbb{R})]{\varepsilon\searrow 0} 0$$

holds by the Dominated Convergence Theorem, which is a contradiction.

Lemma 1.1.15 (HW 1.3)

Let $(u_n)_{n\in\mathbb{N}} \subset L^1((a,b);\mathbb{R})$ be a sequence of functions that converges to some $u \in L^1((a,b);\mathbb{R})$ with regard to the $L^1((a,b);\mathbb{R})$-norm. Furthermore the weak derivative u_n' of u_n exists for each $n \in \mathbb{N}$ as a function in $L^1((a,b);\mathbb{R})$ and the sequence $(u_n')_{n\in\mathbb{N}}$ also converges to some $v \in L^1((a,b);\mathbb{R})$ with regard to the $L^1((a,b);\mathbb{R})$-norm. Then the weak derivative of u exists and coincides with v.

Proof. As $u_n \xrightarrow{n\to\infty} u$, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}$ such that $u_{n_k}(x) \xrightarrow{k\to\infty} u(x)$ for almost all $x \in (a,b)$. As $u_{n_k}' \xrightarrow{k\to\infty} v$, there exists a subsequence $(u_{n_{k_j}}')_{j\in\mathbb{N}} \subset (u_{n_k}')_{k\in\mathbb{N}}$ such that $u_{n_{k_j}}(x) \xrightarrow{j\to\infty} v(x)$ for almost all $x \in (a,b)$.

For $\varphi \in \mathcal{C}_0^\infty((a,b);\mathbb{R})$ we have by the weak differentiability of the u_{n_k} and Dominated Convergence Theorem (the functions u_{n_k} and $u_8 n_{k_j}'$ are in L^1 and both φ and φ' are bounded functions)

$$\int_a^b u(x)\varphi'(x)\,\mathrm{d}x = \int_a^b \lim_{k\to\infty} u_{n_k}(x)\varphi'(x)\,\mathrm{d}x = \lim_{k\to\infty}\int_a^b u_{n_k}(x)\varphi'(x)\,\mathrm{d}x$$

$$= \lim_{k\to\infty} -\int_a^b u_{n_k}'(x)\varphi(x)\,\mathrm{d}x = -\int_a^b \lim_{k\to\infty} u_{n_k}'(x)\varphi(x)\,\mathrm{d}x$$

$$= -\int_a^b \lim_{j\to\infty} u_{n_{k_j}}'(x)\varphi(x)\,\mathrm{d}x = -\int_a^b v(x)\varphi(x)\,\mathrm{d}x,$$

so v is the weak derivative of u. $\qquad\square$

1.2 The Fundamental Theorem & mollifiers

To prove the uniqueness of the weak derivative (up to null sets), we first show the following Theorem.

> **THEOREM 1.2.1: FUNDAMENTAL THEOREM OF THE CALCULUS OF VARIATIONS**
>
> Let $u \in L^1_{\text{loc}}(I)$ be a function such that
>
> $$\int_a^b u(x)\varphi(x) = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(I). \tag{5}$$
>
> Then, $u|_I \equiv 0$ almost everywhere.

💡 Proof idea

If $\varphi(x) := \text{sign}(u(x))$ were in $\mathcal{C}_0^\infty(I)$, we could test with it:

$$0 = \int_a^b u(x)\varphi(x)\,\mathrm{d}x = \int_a^b |u(x)|\,\mathrm{d}x = \|u\|_1 \implies u \equiv 0 \text{ a.e.}$$

But φ is neither smooth nor compactly supported. We can modify φ so that it is compactly supported: consider $\psi := \varphi \cdot \mathbb{1}_{[c,d]}$ for $a < c < d < b$. To "make ψ smooth" we will convolve it with a $\mathcal{C}_0^\infty$ function, a so called mollifier.

To prove this theorem, we need to smoothen the sign function with mollifiers (dt.: *Glättungskern*, also called smoothing operators / kernels). Set $\mathcal{J}_\varepsilon(x) := c_\varepsilon \cdot J_\varepsilon(x)$, where $c_\varepsilon > 0$ is a constant chosen such that $\int_{\mathbb{R}} \mathcal{J}_\varepsilon(x)\,\mathrm{d}x = 1$, that is, $\frac{1}{c_\varepsilon} = \int_{\mathbb{R}} J_\varepsilon(x)\,\mathrm{d}x$. By the substitution $u = \frac{x}{\varepsilon}$, we have $c_\varepsilon = \frac{c}{\varepsilon}$, where $\frac{1}{c} = \int_{\mathbb{R}} J(x)\,\mathrm{d}x$. Then, $\mathcal{J}_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R})$ is a nonnegative function with $\text{supp}(\mathcal{J}_\varepsilon) \subset [-\varepsilon, \varepsilon]$ and $\mathcal{J}_\varepsilon(x) = \frac{1}{\varepsilon}\mathcal{J}_1\left(\frac{x}{\varepsilon}\right)$. This is sometimes called "FRIEDRICHSsche Glättungsfunktion". ·mollifiers

> **DEFINITION 1.2.1 (MOLLIFIER)**
>
> A function $J_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R})$ is called mollifier if
>
> $$\bullet \int_{\mathbb{R}} \mathcal{J}_\varepsilon(x)\,\mathrm{d}x = 1 \quad \bullet \; \mathcal{J}_\varepsilon(\mathbb{R}) \subset \left[0, \frac{1}{\varepsilon}\right] \quad \bullet \; \text{supp}(\mathcal{J}_\varepsilon) \subset [-\varepsilon, \varepsilon].$$

> **DEFINITION 1.2.2 (REGULARISATION / MOLLIFICATION)**
>
> Let $u: I \to \mathbb{R}$ be a function extended by zero outside of I. For $\varepsilon > 0$ its regularisation is $u_\varepsilon(x) := (\mathcal{J}_\varepsilon * u)(x) = \int_{\mathbb{R}} \mathcal{J}_\varepsilon(y)u(x-y)\,\mathrm{d}y$.

The value of $u_\varepsilon(x)$ is a weighted mean over the interval $[x - \varepsilon, x + \varepsilon]$: we have

$$u_\varepsilon(x) = \int_{x-\varepsilon}^{x+\varepsilon} \mathcal{J}_\varepsilon(x-y)u(y)\,\mathrm{d}y.$$

We will now see that the mollification of u inherits the differentiability of J_ε and can be as similar to u as we want it to be:
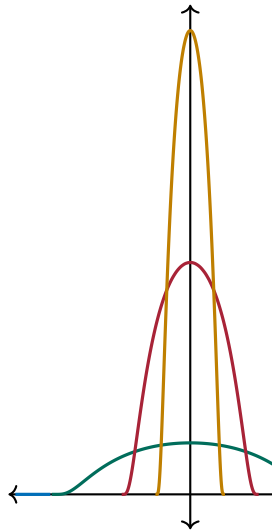
Figure 7: The mollifier $\mathcal{J}_\varepsilon$ for $\varepsilon \in \{\frac{1}{4}, \frac{1}{2}, 1\}$.

> **THEOREM 1.2.2: PROPERTIES OF THE MOLLIFIER**
>
> Let $u \in L^p(I; \mathbb{R})$ and $p \in [1, \infty)$. Then u_ε is well defined and
> (1) we have $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ and for all $k \in \mathbb{N}$ we have
>
> $$u_\varepsilon^{(k)}(x) = \int_{\mathbb{R}} \mathcal{J}_\varepsilon^{(k)}(x - y) u(y) \, \mathrm{d}y.$$
>
> (2) If $\operatorname{supp}(u) \subset I$ and $\varepsilon < \operatorname{dist}(\operatorname{supp}(u), \delta I)$, then $\operatorname{supp}(u_\varepsilon) \subset I$ and therefore, $u_\varepsilon \in \mathcal{C}_0^\infty(I)$.
> (3) $\|u - u_\varepsilon\|_p \xrightarrow{\varepsilon \searrow 0} 0$.
> (4) $\|u_\varepsilon\|_p \leqslant \|u\|_p$ (also holds for $p = \infty$).
> (5) $u_\varepsilon(x) \xrightarrow{\varepsilon \searrow 0} u(x)$ for almost all $x \in I$.
> (6) $\|u_\varepsilon - u\|_{\mathcal{C}(K)} \xrightarrow{\varepsilon \searrow 0} 0$ for compact subsets $K \subset I$ if $u \in \mathcal{C}(I)$.

Proof. Since $\mathcal{J}_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R})$ we have $\mathcal{J}_\varepsilon \in L^q(I; \mathbb{R})$, where $q \in (1, \infty]$ is the HÖLDER conjugate of p. Since $u \in L^p(I; \mathbb{R})$, it follows from HÖLDER's inequality that $\mathcal{J}_\varepsilon * u \in L^1(I; \mathbb{R})$ hence the convolution is well defined.

(1) We show the classical differentiability of u_ε. The claim then follows iteratively. For $x \in \mathbb{R}$ and $h \neq 0$ we have

$$\underbrace{\frac{u_\varepsilon(x + h) - u_\varepsilon(x)}{h}}_{\xrightarrow{h \to 0} u_\varepsilon'(x)} = \int_{\mathbb{R}} \underbrace{\frac{\mathcal{J}_\varepsilon(x + h - y) - \mathcal{J}_\varepsilon(x - y)}{h}}_{\substack{\xrightarrow{h \to 0} \mathcal{J}_\varepsilon(x-y) \in \mathcal{C}_0^\infty \\ \mathcal{J}_\varepsilon \text{ continuous}}} \cdot u(y) \, \mathrm{d}y$$

$$\xrightarrow[\text{LEBESGUE}]{h \to 0} \int_{\mathbb{R}} \mathcal{J}_\varepsilon'(x - y) u(y) \, \mathrm{d}y.$$

(2) We have $u_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) u(x - y) \, \mathrm{d}y$. Thus if $\operatorname{supp}(u) \subset [c, d] \subset (a, b)$, then $\operatorname{supp}(u_\varepsilon) \subset [c - \varepsilon, d + \varepsilon]$, which is precisely the case if $\varepsilon > 0$ is chosen like in the Theorem.

(3) For $p = 1$ the claim is trivial. Let $p \in (1, \infty)$. Using that $\int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) \, \mathrm{d}y = 1$, the HÖLDER-inequality ($\star$) (as detailed in (4)) and FUBINI's theorem ($\ddagger$), we have

$$\|u_\varepsilon - u\|_p^p = \int_a^b \left| \int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) u(x - y) - u(x) \, \mathrm{d}y \right|^p \mathrm{d}x$$

$$\leqslant \int_a^b \left(\int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) |u(x - y) - u(x)| \, \mathrm{d}y \right)^p \mathrm{d}x$$

$$\overset{(\star)}{\leqslant} \int_a^b \int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) |u(x - y) - u(x)|^p \, \mathrm{d}y \, \mathrm{d}x$$

$$\overset{(\ddagger)}{=} \int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) \int_a^b |u(x - y) - u(x)|^p \, \mathrm{d}x \, \mathrm{d}y$$

$$\leqslant \sup_{|y| < \varepsilon} \int_a^b |u(x - y) - u(x)|^p \, \mathrm{d}x \underbrace{\int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) \, \mathrm{d}y}_{= 1} \xrightarrow[\text{L } 0.0.9]{\varepsilon \searrow 0} 0.$$

(4) Additionally using the the translational invariance of the LEBESGUE integral ($\dagger$), we have

$$\|u_\varepsilon\|_p^p = \int_a^b |u_\varepsilon(x)|^p \, \mathrm{d}x \leqslant \int_a^b \left(\int_{\mathbb{R}} \mathcal{J}_\varepsilon(x - y)^{\frac{1}{p} + \frac{1}{q}} |u(y)| \, \mathrm{d}y \right)^p \mathrm{d}x$$

Detailed argument on why we can exchange integral and limit: We want to use the Mean Value Theorem. $\mathcal{J}_\varepsilon'$ is also bounded and compactly supported, therefore, $\|\mathcal{J}_\varepsilon'\|_\infty$ exists. We now find an integrable majorant g by building a "box" around $\mathcal{J}_\varepsilon'$ and multiplying with u: $g := u \cdot \|\mathcal{J}_\varepsilon'\|_\infty \cdot \mathbb{1}_{[x - h - \varepsilon, x + h + \varepsilon]}$. Note that $u \cdot \|\mathcal{J}_\varepsilon'\|_\infty$ doesn't have to be in L^1 since u is only in L^p. With the Dominated Convergence Theorem we obtain for an $\xi \in [0, h]$

$$\lim_{h \to 0} \int_{\mathbb{R}} \frac{\mathcal{J}_\varepsilon(x + h - y) - \mathcal{J}_\varepsilon(}{h}$$

$$= \lim_{h \to 0} \int_{\mathbb{R}} \mathcal{J}_\varepsilon'(x - y + \xi) \cdot u(y$$

$$= \int_{\mathbb{R}} u(y) \cdot \lim_{h \to 0} \mathcal{J}_\varepsilon'(x - y + \xi$$

$$= \int_{\mathbb{R}} \mathcal{J}_\varepsilon'(x - y) \cdot u(y) \, \mathrm{d}y.$$

$$\overset{(\star)}{\leqslant} \int_a^b \left(\int_{\mathbb{R}} \mathcal{J}_\varepsilon(x-y)^{\frac{p}{p}} |u(y)|^p \, \mathrm{d}y \right)^{\frac{p}{p}} \underbrace{\left(\int_{\mathbb{R}} \mathcal{J}_\varepsilon(x-y)^{\frac{q}{q}} \, \mathrm{d}y \right)^{\frac{p}{q}}}_{=1} \, \mathrm{d}x$$

$$\overset{(\dagger)}{=} \int_a^b \left(\int_{\mathbb{R}} \mathcal{J}_\varepsilon(x-y) |u(y)|^p \, \mathrm{d}y \right) \, \mathrm{d}x$$

$$\overset{(\ddagger)}{=} \int_{\mathbb{R}} |u(y)|^p \underbrace{\int_a^b \mathcal{J}_\varepsilon(x-y) \, \mathrm{d}y}_{=1} = \int_{\mathbb{R}} |u(y)|^p \, \mathrm{d}y$$

$$= \int_a^b |u(y)|^p \, \mathrm{d}y = \|u\|_p^p.$$

⑤ We have

$$|u_\varepsilon - u| \leqslant \int_{\mathbb{R}} \mathcal{J}_\varepsilon(y) |u(x-y) - u(x)| \, \mathrm{d}y$$

$$= \int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) |u(x-y) - u(x)| \, \mathrm{d}y$$

$$= \underbrace{\|\mathcal{J}_\varepsilon\|_\infty}_{=\frac{c}{\varepsilon} \cdot \frac{1}{\varepsilon}} \int_{-\varepsilon}^{\varepsilon} |u(x-y) - u(x)| \, \mathrm{d}y$$

$$= \frac{2c}{\varepsilon} \cdot \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |u(x-y) - u(x)| \, \mathrm{d}y,$$

which converges to 0 almost everywhere as $u \in L^p$ and hence almost all points are Lebesgue points (cf. DGL I).

⑥ Let $K \subset (a,b)$ be compact. Then u is uniformly continuous on a compact interval $[c,d] \subset (a,b)$ chosen such that $x - y \in [c,d]$ for all $x \in K$ and for all $|y| < \varepsilon$ for some small enough $\varepsilon > 0$. Then, for some $\eta > 0$ we have $|u(x-y) - (x)| < \eta$ and, therefore,

$$\sup_{x \in K} |u_\varepsilon(x) - u(x)| \leqslant \sup_{x \in K} \int_{-\varepsilon}^{\varepsilon} \mathcal{J}_\varepsilon(y) \underbrace{|u(x-y) - u(x)|}_{<\eta} \, \mathrm{d}y < \eta. \qquad \square$$
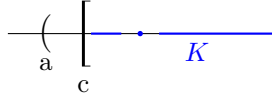
Figure 8: Unnamed figure

23.04.19

Proof. (Fundamental theorem) Let $u \in L^1_{\mathrm{loc}}((a,b); \mathbb{R})$ and $[c,d] \subset (a,b)$. Define $w = \mathrm{sgn}(u) \mathbb{1}_{[c,d]}$. Then we have $w \in L^1_{\mathrm{loc}}((a,b); \mathbb{R})$ and $\mathrm{supp}(w) \subset [c,d]$. We define $w_\varepsilon := \mathcal{J}_\varepsilon * w$. Then, $w_\varepsilon \to w$ almost everywhere on (a,b) and $\mathrm{supp}(w_\varepsilon) \subset [c - \varepsilon, d + \varepsilon]$, hence $w_\varepsilon \in \mathcal{C}_0^\infty((a,b); \mathbb{R})$ if ε is small enough by Theorem 1.2.2.

We test (5) with $\varphi = w_\varepsilon \in \mathcal{C}_0^\infty((a,b); \mathbb{R})$, obtaining

$$0 = \int_a^b \underbrace{u(x) w_\varepsilon(x)}_{\overset{\text{a.e.}}{\longrightarrow} u(x)w(x)} \, \mathrm{d}x = \int_{c-\varepsilon}^{d+\varepsilon} u(x) w_\varepsilon(x) \, \mathrm{d}x$$

$$= \int_a^b u(x) \mathbb{1}_{[c-\varepsilon, d+\varepsilon]}(x) w_\varepsilon(x) \, \mathrm{d}x.$$

We have

$$|w_\varepsilon(x)| \leqslant \int_a^b \mathcal{J}_\varepsilon(x-y) \underbrace{|w(y)|}_{\leqslant 1} \, \mathrm{d}y \leqslant 1.$$

For $\varepsilon_0 < \min(c - a, b - d)$ and all $\varepsilon < \varepsilon_0$ we get

$$|u(x) w_\varepsilon(x)| \leqslant |u(x)| \mathbb{1}_{[c-\varepsilon_0, d+\varepsilon_0]}(x)$$

This function is integrable on (a, b). LEBESGUE's Theorem shows

$$0 = \int_a^b u(x)w(x)\,\mathrm{d}x = \int_c^d |u(x)|\,\mathrm{d}x,$$

hence $u \equiv 0$ almost everywhere on $[c, d]$. As $[c, d] \subset (a, b)$ was chosen arbitrarily, this yields the claim. □

Corollary 1.2.3 (HW 1.5)

Let $u \in L^1_{\mathrm{loc}}((a, b); \mathbb{R})$ be a function such that

$$\int_a^b u(x)\varphi'(x)\,\mathrm{d}x = 0 \ \forall \varphi \in \mathcal{C}_0^\infty((a, b); \mathbb{R}). \tag{6}$$

Then there exists an $c \in \mathbb{R}$ so that $u \equiv c$ almost everywhere on (a, b).

Proof. From (6) we can deduce

$$\int_a^b \big(u(x) - c\big)\varphi'(x)\,\mathrm{d}x = 0 \ \forall \varphi \in \mathcal{C}_0^\infty((a, b); \mathbb{R}),$$

where $c := \frac{1}{b-a}\int_a^b u(x)\,\mathrm{d}x \in \mathbb{R}$, as

$$\int_a^b c\varphi'(x)\,\mathrm{d}x = c\,(\varphi(b) - \varphi(a)) = 0$$

for all $\varphi \in \mathcal{C}_0^\infty((a, b); \mathbb{R})$. □

Corollary 1.2.4 (Uniqueness)

If the weak derivative exists, it is unique.

Proof. Assume $v, w \in L^1_{\mathrm{loc}}((a, b); \mathbb{R})$ were weak derivatives of $u \in L^1_{\mathrm{loc}}((a, b); \mathbb{R})$. This implies that for all $\varphi \in \mathcal{C}_0^\infty((a, b); \mathbb{R})$

$$\int_a^b (v - w)(x)\varphi(x)\,\mathrm{d}x = \int_a^b v(x)\varphi(x)\,\mathrm{d}x - \int_a^b w(x)\varphi(x)\,\mathrm{d}x$$

$$= -\int_a^b u(x)\varphi'(x)\,\mathrm{d}x + \int_a^b u(x)\varphi'(x)\,\mathrm{d}x = 0.$$

The Fundamental Theorem implies $v - w \equiv 0$ almost everywhere on $(a.b)$. □

Example 1.2.5 Let

$$M_1 := \{u\colon [a, b] \to \mathbb{R} : \exists f\colon [a, b] \to \mathbb{R} \text{ continuous} : f \equiv u \text{ a. e.}\}$$

and

$$M_2 := \{u\colon [a, b] \to \mathbb{R} : u \text{ is continuous almost everywhere}\}$$

Then we have $M_1 \nsubseteq M_2$ and $M_2 \nsubseteq M_1$: Consider $f_1 := \mathbb{1}_{\mathbb{R}\setminus\mathbb{Q}} \in M_1$ but $f_1 \notin M_2$ and $f_2^{(\varepsilon)} := \mathbb{1}_{[a+\varepsilon, b-\varepsilon]} \in M_2$ but $f_2^{(\varepsilon)} \notin M_1$ or the heaviside function.

1.3 Weak differentiability and absolute continuity

The following shows $W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$ (cf. Chapter 2).

THEOREM 1.3.1: $W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$

Let $u \in L^1((a,b);\mathbb{R})$ be weakly differentiable with $u' \in L^1((a,b);\mathbb{R})$. Then u coincides almost everywhere with a function, which is absolutely continuous on (a,b) and which can then be extended to an absolutely continuous function on $[a,b]$ ("u is absolutely continuous"). We have

$$\|u\|_\infty := \|u\|_{\mathcal{C}([a,b];\mathbb{R})} \leqslant \frac{\max(1, b-a)}{b-a} \left(\|u\|_1 + \|u'\|_1 \right)$$

Alternatively: there exists an absolutely continuous function on $[a,b]$ whose restriction to (a,b) is in the equivalence class of u.

This is generalises the fact that continuously differentiable functions are absolutely continuous.

Proof. Set $v(x) := \int_a^x u'(y)\,\mathrm{d}y$. As $u' \in L^1((a,b);\mathbb{R})$, v is absolutely continuous and $v' = u'$ almost everywhere on (a,b). Therefore, we obtain

$$\int_a^b u\varphi'\,\mathrm{d}x = -\int_a^b u'\varphi\,\mathrm{d}x = -\int_a^b v'\varphi\,\mathrm{d}x = \int_a^b v\varphi'\,\mathrm{d}x$$

for all $\varphi \in \mathcal{C}_0^\infty((a,b);\mathbb{R})$ and hence by Corollary 1.2.3 $u \equiv v + c$ for some $c \in \mathbb{R}$ almost everywhere on (a,b), so u is almost everywhere equal to an absolutely continuous function, which we will call u, too $(\diamond)$.

By the Integral Mean Value Theorem $(\star)$ there exists a $x_0 \in [a,b]$ so that $\int_a^b u(x)\,\mathrm{d}x = u(x_0)(b-a)$. This implies

$$|u(x)| \overset{(\diamond)}{\underset{\triangle \neq}{\leqslant}} |u(x_0)| + \left| \int_{x_0}^x u'(x)\,\mathrm{d}x \right| \overset{(\star)}{\underset{\triangle \neq}{\leqslant}} \frac{1}{b-a} \int_a^b |u(x)|\,\mathrm{d}x + \int_a^b |u'(x)|\,\mathrm{d}x. \qquad \square$$

Remark 1.3.1 This doesn't hold in higher dimensions, u must not even by continuous. (cf. SOBOLEV Embedding Theorem. We only have $W^{1,1}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \leqslant \frac{d}{d-1} \in (1,2]$, where $\Omega \subset \mathbb{R}^d$ is a bounded LIPSCHITZ domain) **TODO** example needed!

DEFINITION 1.3.2 (HIGHER WEAK DERIVATIVES)

Let $u,v \in L^1_{\mathrm{loc}}((a,b);\mathbb{R})$. Then v is the n-th weak derivative of u if

$$\boxed{\int_a^b u(x)\varphi^{(n)}\,\mathrm{d}x = (-1)^n \int_a^b v(x)\varphi(x)\,\mathrm{d}x}$$

holds for all $\varphi \in \mathcal{C}_0^\infty((a,b);\mathbb{R})$.

Remark 1.3.3 (Higher order derivatives) We could also define the n-th weak derivative iteratively. In one dimension, this yields the definition as above, in multiple dimensions it does not. More precisely: If $u \in L^1(I)$ and $v \in L^1_{\mathrm{loc}}(I;\mathbb{R})$ is the n-th weak derivative of u, we have $v \in L^1(I;\mathbb{R})$ the k-th weak derivatives of u exist for all $k \in \{1, \ldots, n-1\}$.

> ### THEOREM 1.3.2: IN BETWEEN WEAK DERIVATIVES
>
> Let $u \in L^1((a,b); \mathbb{R})$ so that the n-th weak derivative $u^{(n)} \in L^1((a,b); \mathbb{R})$ exists. Then, for all $k \leqslant n-1$ the weak derivative $u^{(k)}$ exists and is absolutely continuous.

Proof. It suffices to consider $n = 2$. Let $u'' \in L^1((a,b); \mathbb{R})$. Then $v_1(x) := \int_a^x u''(y)\,\mathrm{d}y$ is absolutely continuous with $v_1' = u''$ almost everywhere in (a,b).

We set $v_0(x) := \int_a^x v_1(y)\,\mathrm{d}y$. Then v_0 is absolutely continuous with $v_0' = v_1$. Then, we have

$$
\begin{aligned}
(-1)^2 \int_a^b u(x)\varphi''(x)\,\mathrm{d}x &= \int_a^b u''(x)\varphi(x)\,\mathrm{d}x = \int_a^b v_1'(x)\varphi(x)\,\mathrm{d}x \\
&= -\int_a^b v_1(x)\varphi'(x)\,\mathrm{d}x = -\int_a^b v_0'(x)\varphi'(x)\,\mathrm{d}x \\
&= \int_a^b v_0(x)\varphi''(x)\,\mathrm{d}x.
\end{aligned}
$$

Hence $\int_a^b (u - v_0)(x)\varphi''(x)\,\mathrm{d}x = 0$. Similar to the Fundamental Theorem this implies that $u \equiv v_0 + p$, where p is a polynomial of degree one. Hence, u' exists and coincides with v_1 plus an polynomial of degree zero (and u'' coincides with v_0). $\square$

Remark 1.3.4 This is not true for $d > 1$. **TODO**(TODO:example needed!).

2 SOBOLEV spaces

2.1 First definitions and properties

We now aim to combine the notion of weak derivatives and LEBESGUE norms.

Later, we will see that the SOBOLEV spaces have "nice geometry" because they are uniformly convex and thus reflexive, which in turn gives concrete representation of linear functionals, enabling reformulation of problems using duality and weak compactness of bounded sets, leading the way to calculus of variations.

uniformly convex
weak compactness
calculus of variations

Also, the smooth functions are dense, therefore one can prove statements for them first and then extend to the whole space by density, see Lemma 1.1.15, which is used in Lemma 2.1.5.

DEFINITION 2.1.1 (SOBOLEV SPACE $W^{k,p}$ (SOBOLEW))

Let $k \in \mathbb{N}_{>0}$ and $p \in [1, \infty]$. We call

$$W^{k,p}((a,b); \mathbb{R}) := \{u \in L^p((a,b); \mathbb{R}) : \text{weak derivative } u^{(l)} \in L^p((a,b); \mathbb{R}) \; \forall \ell \leqslant k\}$$

a SOBOLEV space and equip it with the SOBOLEV norm

SOBOLEV space

$$\|u\|_{k,p} := \left(\sum_{\ell=0}^{k} \|u^{(\ell)}\|_p^p \right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_{k,\infty} = \max_{\ell=0}^{k} \|u^{(\ell)}\|_\infty.$$

A seminorm on $W^{k,p}$ is $|u|_{k,p} := \|u^{(k)}\|_p$.

The SOBOLEV norm measures both regularity and size of a function.

DEFINITION 2.1.2 (SOBOLEV INNER PRODUCT SPACE)

We set $H^k((a,b); \mathbb{R}) := W^{k,2}((a,b); \mathbb{R})$ and equip it with the inner product

$$\langle u, v \rangle_{k,2} := \sum_{\ell=0}^{k} \langle u^{(\ell)}, v^{(\ell)} \rangle_2,$$

where $\langle \cdot, \cdot \rangle_2$ is the $L^2((a,b); \mathbb{R})$ inner product.

We have $W^{0,p} = L^p$ and $H^0 = L^2$.

Remark 2.1.3 (Wiki, todo proofs)
- The norm $\|f\|_p + \|f^{(k)}\|_p$ is equivalent to the norm above.
- $W^{1,\infty}(a,b)$ is the space of the LIPSCHITZ continuous functions.
- $W_0^{2,1}(a,b) \subset L^2(a,b)$ is dense. proof. **TODO**

THEOREM 2.1.1: PROPERTIES OF SOBOLEV SPACES

① $W^{k,p}((a,b); \mathbb{R})$ is a BANACH space.
② $W^{k,p}((a,b); \mathbb{R})$ is separable for $p \in [1, \infty)$.
③ $W^{k,p}((a,b); \mathbb{R})$ is reflexive for $p \in (1, \infty)$.
④ $H^k((a,b); \mathbb{R})$ is a HILBERT space.

We have $W^{k,p}((a,b);\mathbb{R}) \hookrightarrow W^{k,q}((a,b);\mathbb{R})$ for $q \leqslant p$ and $W^{k,p}((a,b);\mathbb{R}) \hookrightarrow W^{j,p}((a,b);\mathbb{R})$ for $j \leqslant k$. The embedding $W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$ (cf. Theorem 1.3.1) is not compact. We have $W^{1,p}((a,b);\mathbb{R}) \hookrightarrow W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$ continuously, and thus $W^{1,p}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$

Proof. ($k = 1$) ① Let $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}$ be a CAUCHY sequence. Then the sequences $(u_n)_{n\in\mathbb{N}}$ and $(u'_n)_{n\in\mathbb{N}}$ are CAUCHY sequences in $L^p(a,b)$. Hence there exists functions $u,v \in L^p(a,b)$ with $u_n \to u$ and $u'_n \to v$.

As $p \geqslant 1$ we can use Lemma 1.1.15 to show that $u' = v$.

② Define
$$T\colon W^{1,p}(a,b) \to L^p(a,b)^2, \ u \mapsto \langle u, u' \rangle.$$

Then, T is well defined. Further, we have

$$\|Tu\|_{L^p(a,b)^2} = \left(\|u\|_{L^p((a,b);\mathbb{R})}^p + \|u'\|_{L^p((a,b);\mathbb{R})}^p \right)^{\frac{1}{p}} = \|u\|_{1,p}.$$

Hence $W^{1,p}(a,b)$ isometrically coincides with a subspace of $(L^p(a,b))^2$. This subspace is closed as $W^{1,p}(a,b)$ is complete. As $L^p(a,b)$ is separable, so is $(L^p(a,b))^2$ and hence the closed subspace, and hence $W^{1,p}(a,b)$.

For $k > 1$, show that

$$T\colon W^{k,p}((a,b);\mathbb{R}) \to (L^p((a,b);\mathbb{R}))^{k+1}, \qquad u \mapsto (u, u', \dots, u^{(k)}) \qquad \square$$

is an isomorphism.

③ **TODO**

④ **TODO**

Counterexample 2.1.4 ($W^{1,\infty}(a,b)$ is not reflexive)
$W^{1,\infty}(a,b)$ is isomorphic to $\mathbb{R} \times L^\infty(a,b)$ via $v \mapsto (v(a), v')$ but $L^\infty(a,b)$ is not reflexive.

Lemma 2.1.5 (Classical rules for H^1)

Let $u, v \in H^1(a,b)$. Then the product rule $(uv)' = uv' + u'v$ holds and the mean value theorem

$$u(x) - v(x) = (x - y) \int_0^1 u'(y + t(x - y)) \, \mathrm{d}t$$

holds, where pointwise evaluation of u is defined via its absolutely continuous representative.

Proof. <u>Product rule.</u> Since $\mathcal{C}^\infty([a,b]) \overset{\mathrm{d}}{\hookrightarrow} H^1(a,b) \overset{\mathrm{d}}{\hookrightarrow} \mathcal{C}([a,b])$, we can find sequences $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \subset \mathcal{C}^\infty([a,b])$ so that $u_n \to u$ and $v_n \to v$ in $H^1(a,b)$. **(BUT WE HAVEN'T DEFINED EMBEDDINGS YET!!)** **TODO** Because of the dense embeddings we have

$$\|uv\|_{0,2} \leqslant \|u\|_{\mathcal{C}} \|v\|_{\mathcal{C}} < \infty, \ \|u'v\|_{0,2} \leqslant \|u'\|_{0,2} \|v\|_{\mathcal{C}}, \ \|v'u\|_{0,2} \leqslant \|u\|_{\mathcal{C}} \|v\|_{0,2}.$$

Therefore, we have $uv, u'v + uv' \in L^2$ and

$$\|u_n v_n - uv\|_{0,1} \overset{\triangle\neq}{\leqslant} \|u_n(v_n - v)\|_{0,1} + \|(u_n - u)v\|_{0,1}$$

$$\overset{(\mathrm{H})}{\leqslant} \underbrace{\|u_n\|_{0,2}}_{\leqslant C} \underbrace{\|v_n - v\|_{0,2}}_{\xrightarrow{n\to\infty} 0} + \underbrace{\|u_n - u\|_{0,2} \|v\|_{0,2}}_{\xrightarrow{n\to\infty} 0}.$$

and

$$\|u'_n v_n - u'v\|_{0,1} \leqslant \|u'_n\|_{0,2} \|v_n - v\|_{0,2} \leqslant \text{???}$$

and, analogously, $\|u_n v_n' - uv'\| \xrightarrow{n\to\infty} 0$.

Putting this together, we have

$$(u_n v_n)' = u_n v_n' + u_n' v_n \xrightarrow[L^1(a,b)]{n\to\infty} uv' + u'v \quad \text{and} \quad u_n v_n \xrightarrow[L^1(a,b)]{n\to\infty} uv.$$

With Lemma 1.1.15 the proposition follows.

<u>Mean value theorem.</u> Analogously to the above, we can choose a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^\infty([a,b])$ so that $u_n \xrightarrow[W^{1,1}(a,b)]{n\to\infty} u$.

Because of $W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$ we know that $\|u_n - u\|_{\mathcal{C}} \to 0$ hence also $u_n(x) \to u(x)$ for all $x \in [a,b]$.

We conclude

$$\left| \int_0^1 u_n'(y + t(x-y))\,dt - \int_0^1 u'(y + t(x-y))\,dt \right| = \left| \int_x^y u_n'(\xi) - u'(\xi)\,d\xi \right|$$

$$\leq \int_a^b \left| u_n'(\xi) - u(\xi) \right| d\xi$$

$$= \|u_n' - u'\|_{0,1} \xrightarrow[u_n \xrightarrow{W^{1,1}(a,b)} u]{} 0.$$

Using the mean value theorem for $\mathcal{C}^\infty$, we have

$$u(x) - u(y) = \lim_{n\to\infty} u_n(x) - u_n(y) = \lim_{n\to\infty} (x-y) \int_0^1 u_n'(y + t(x-y))\,dt. \qquad \square$$

Lemma 2.1.6 (Chain rule in H^1 (HW 2.2))

Let $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ with $f(0) = 0$ be such that there exists a $M > 0$ with $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Then for $u \in H^1((a,b);\mathbb{R})$ we have $f \circ u \in H^1((a,b);\mathbb{R})$ and

$$(f \circ u)' = (f' \circ u)u'.$$

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1((a,b);\mathbb{R})$ converge to $u \in H^1((a,b);\mathbb{R})$. For $n \in \mathbb{N}$ we have

$$\|(f' \circ u_n)u_n' - (f' \circ u)u'\|_{0,2}^2 = \int_a^b |f'(u_n(x))u_n'(x) - f'(u(x))u'(x)|^2\,dx$$

$$= \int_a^b |f'(u_n(x))u_n'(x) - f'(u_n(x))u'(x) + f'(u_n(x))u'(x) - f'(u(x))u'(x)|^2\,dx$$

$$= \int_a^b |f'(u_n(x))u_n'(x) - f'(u_n(x))u'(x)|^2\,dx$$

$$+ 2\int_a^b f'(u_n(x))(u_n'(x) - u'(x))(f'(u_n(x)) - f'(u(x)))u'(x)\,dx$$

$$+ \int_a^b |f'(u_n(x))u'(x) - f'(u(x))u'(x)|^2\,dx$$

$$\overset{(\star)}{\leq} \|f'\|_\infty^2 \|u_n' - u'\|_{0,2}^2 + \int_a^b |f'(u_n(x)) - f'(u(x))|^2|u'(x)|^2\,dx$$

$$+ \int_a^b |f'(u_n(x))|^2|u_n'(x) - u'(x)|^2 + |f'(u_n(x)) - f'(u(x))|^2|u'(x)|^2\,dx$$

$$\leq 2\|f'\|_\infty^2 \|u_n - u'\|_{0,2}^2 + 2\int_a^b |f'(u_n(x)) - f'(u(x))|^2|u'(x)|^2\,dx.$$

using $2cd \leq c^2 + d^2$ in $(\star)$. Up to a subsequence, which we will again call $(u_n)_{n\in\mathbb{N}}$, we have $u_n \to u$ pointwise almost everywhere and by the continuity of f we have $f(u_n(x)) \to$

$f(u(x))$ almost everywhere. Hence the second integral converges to zero by the Dominated Convergence Theorem. Hence we have

$$\|(f \circ u_n)' - (f' \circ u)u'\|_{0,2} = \|(f' \circ u_n)u_n' - (f' \circ u)u'\|_{0,2} \to 0. \qquad \square$$

Lemma 2.1.7

Let $f, g \in L^1((a,b); \mathbb{R})$ such that

$$- \int_a^b \varphi'(x)g(x) \, \mathrm{d}x \leqslant \int_a^b \varphi(x)f(x) \, \mathrm{d}x$$

for all $\varphi \in \mathcal{C}_0^\infty((a,b); \mathbb{R}_{\geqslant 0})$. Then for almost all $s, t \in (a,b)$ we have

$$g(t) - g(s) \leqslant \int_s^t f(x) \, \mathrm{d}x.$$

Proof. Let $s, t \in (a,b)$ and $w := \mathbb{1}_{[s,t]}$ with $s < t$ with $s, t \in (a,b)$ and $w_\varepsilon := J_\varepsilon * w$. We have

$$w_\varepsilon(x) = \int_{\mathbb{R}} J_\varepsilon(x-y)w(y) \, \mathrm{d}y = \int_s^t J_\varepsilon(x-y) \, \mathrm{d}y = \int_{x-t}^{x-s} J_\varepsilon(y) \, \mathrm{d}y = \int_{-\varepsilon}^\varepsilon J_\varepsilon(y) \, \mathrm{d}y = 1$$

for all x with $[-\varepsilon, \varepsilon] \subset [x-s, x-t]$. We have $w_\varepsilon \in \mathcal{C}_0^\infty((a,b); \mathbb{R})$ by Theorem ... and for all $x \in (a,b)$

$$w_\varepsilon'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{x-t}^{x-s} J_\varepsilon(y) \, \mathrm{d}y = J_\varepsilon(x-s) - J_\varepsilon(x-t)$$

and thus for most x

$$\int_a^b (J_\varepsilon(x-t) - J_\varepsilon(x-s)) \, g(x) \, \mathrm{d}x \leqslant \int_a^b f(x) \, \mathrm{d}x. \qquad \square$$

Example 2.1.8 (Sign and HEAVISIDE function in fractional SOBOLEV spaces)
For $\sigma \in (0,1)$ we define the fractional order SOBOLEV space

$$H^\sigma(a,b) := \{u \in L^2(a,b) : |u|_\sigma < \infty\}$$

with the SLOBODECKIJ seminorm

$$|u|_\sigma := \left(\int_a^b \int_a^b \frac{|u(x) - u(y)|^2}{|x-y|^{1+2\sigma}} \, dx dy \right)^{\frac{1}{2}}.$$

For which choice of σ are the HEAVISIDE function and sign function an element of $H^\sigma(-1,1)$?

Let $u(x) = \mathrm{sign}(x)$. We have

$$|u|_\sigma^2 = \int_{-1}^1 \int_{-1}^1 \frac{|u(x) - u(y)|^2}{|x-y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y$$

$$= \underbrace{\int_{-1}^0 \int_{-1}^0 \frac{|-1-(-1)|^2}{|x-y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y}_{=0} + \int_{-1}^0 \int_0^1 \frac{|1-(-1)|^2}{|x-y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y$$

$$+ \int_0^1 \int_{-1}^0 \frac{|-1-1|^2}{|x-y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y + \underbrace{\int_0^1 \int_0^1 \frac{|1-1|^2}{|x-y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y}_{=0}$$

$$
= \int_{-1}^{0} \int_{0}^{1} \frac{|1 - (-1)|^2}{|x - y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y + \int_{0}^{1} \int_{-1}^{0} \frac{|-1 - 1|^2}{|x - y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y
$$

$$
= 8 \int_{0}^{1} \int_{-1}^{0} \frac{1}{|x - y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y = 8 \int_{0}^{1} \int_{0}^{1} \frac{1}{(x + y)^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y
$$

$$
= \frac{8}{2\sigma} \frac{2^{1-2\sigma} - 2}{2\sigma - 1} = 8 \frac{2^{-2\sigma} - 1}{\sigma(2\sigma - 1)},
$$

where the evaluation of the integral of is only valid $2\sigma + 1 < 2$, i.e. for $\sigma < \frac{1}{2}$, otherwise the integral diverges **TODO**

Now let h be the HEAVISIDE function. Then we have

$$
|h|_{\sigma}^2 = \int_{-1}^{1} \int_{-1}^{1} \frac{|h(x) - h(y)|^2}{|x - y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y = 2 \int_{-1}^{0} \int_{0}^{1} \frac{1}{|x - y|^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y
$$

$$
= 2 \int_{0}^{1} \int_{0}^{1} \frac{1}{(x + y)^{1+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y
$$

as before so we have $h \in H^{\sigma}(-1, 1)$ only for $\sigma < \frac{1}{2}$.

2.2 Embedding theorems

DEFINITION 2.2.1 (EMBEDDINGS)
Let X and Y be normed spaces.

① X is embedded into Y if and only if there exists an a injective linear function   embedded
$\iota \colon X \to Y$ and X can identified with a subspace of Y.

② X is continuously / compactly embedded into Y and we write $X \hookrightarrow Y$ / $X \overset{c}{\hookrightarrow} Y$ if
ι is continuous / compact.

③ X is densely embedded into Y and we write $X \overset{d}{\hookrightarrow} Y$ if $\iota(X)$ is dense in Y with
respect to $\| \cdot \|_Y$.

Remark 2.2.2 (Embeddings) In the case of ②, there exists an $c > 0$ such that $\|\iota(x)\|_Y \leq c\|x\|_X$ for all $x \in X$. Mostly, $\iota \equiv \mathrm{id}$ and therefore, $\|x\|_Y \leq c\|x\|_X$ for all $x \in X$. If $X \overset{c}{\hookrightarrow} Y$, then any bounded sequence in X admits a subsequence converging with respect to the Y norm.

Lemma 2.2.3 (Continuous noncompact $L^q \hookrightarrow L^p$ for $p \leq q$)
For a bounded interval $(a, b) \subset \mathbb{R}$ and $1 \leq p \leq q \leq \infty$ we have $L^q((a,b); \mathbb{R}) \hookrightarrow L^p((a,b); \mathbb{R})$ but the embedding is not compact.

Proof. Set $\hat{\iota} \colon L^q(a,b) \to L^p(a,b), \ u \mapsto u$. For $q < \infty$ we have

$$\|\hat{\iota} u\|_p^p = \int_a^b 1 |u|^p \, \mathrm{d}x \overset{(\mathrm{H})}{\leq} \left(\int_a^b 1^{\frac{q}{q-p}} \, \mathrm{d}x \right)^{\frac{q-p}{p}} \left(\int_a^b |u|^q \, \mathrm{d}x \right)^{\frac{p}{q}} = (b-a)^{\frac{q-p}{q}} \|u\|_q^p.$$

where for the HÖLDER inequality (H) uses $r := \frac{q}{p}$ and $s := \frac{q}{q-p}$ as conjugated exponents. For $q = \infty$ we have

$$\|\hat{\iota} u\|_p = \left(\int_a^b |u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq (b-a)^{\frac{1}{p}} \|u\|_\infty.$$

The sequence $(f_n(x) := \sin(nx))_{n \in \mathbb{N}} \subset L^q((a,b); \mathbb{R})$ is bounded, as for all $n \in \mathbb{N}$ we have

$$\|f_n\|_q^q = \int_a^b \underbrace{|\sin(nx)|^q}_{\leq 1} \, \mathrm{d}x \leq b - a.$$

But the sequence $(f_n)_{n \in \mathbb{N}}$ does not contain any L^p-convergent subsequences, as it doesn't even contain L^p CAUCHY subsequences: suppose there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p \xrightarrow{k \to \infty} 0.$$

By HÖLDER's inequality we have

$$\begin{aligned}
\|f_{n_{k+1}} - f_{n_k}\|_2^2 &\overset{(\mathrm{H})}{\leq} \|f_{n_{k+1}} - f_{n_k}\|_p \|f_{n_{k+1}} - f_{n_k}\|_q \\
&\leq \|f_{n_{k+1}} - f_{n_k}\|_p (\|f_{n_{k+1}}\|_q + \|f_{n_k}\|_q) \\
&\leq 2(b-a)^{\frac{1}{q}} \|f_{n_{k+1}} - f_{n_k}\|_p \xrightarrow{k \to \infty} 0,
\end{aligned}$$

but the left hand side is constant and equal to $b - a > 0$, which is a contradiction. $\qquad \square$

TODO We have $\mathcal{C}([a,b]; \mathbb{R}) \hookrightarrow L^p((a,b); \mathbb{R})$ but the embedding is not compact $(\sin(nx))$ With ARZELÁ-ASCOLI we get $\mathcal{C}^1([a,b]; \mathbb{R}) \overset{c}{\hookrightarrow} \mathcal{C}([a,b]; \mathbb{R})$.

In the following Theorem we "spend" one degree of differentiability and "gain compactness".

> ### Theorem 2.2.1: $W^{1,p}((a,b);\mathbb{R}) \overset{c}{\hookrightarrow} \mathcal{C}([a,b])$
>
> If $p > 1$ then $W^{1,p}((a,b);\mathbb{R}) \overset{c}{\hookrightarrow} \mathcal{C}([a,b])$.

Proof. As $L^p((a,b);\mathbb{R}) \hookrightarrow L^1((a,b);\mathbb{R})$ (as $p > 1$) we have

$$W^{1,p}((a,b);\mathbb{R}) \hookrightarrow W^{1,1}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R}).$$

Let $A \subset W^{1,p}((a,b);\mathbb{R})$ be bounded. Then there exists an $M \geqslant 0$ such that $\|u\|_{1,p} \leqslant M$ for all $u \in A$. As $W^{1,p}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b])$, there exists a $c > 0$ such that $\|u\|_\infty \leqslant c\|u\|_{1,p} \leqslant cM$ for all $u \in A$.

We now show that A is equicontinuous. For $u \in A$ and $x_1, x_2 \in [a,b]$ we get

From now on, we use $\max(m,n) := m \vee n$ and $\min(m,n) := m \wedge n$.

$$
\begin{aligned}
|u(x_1) - u(x_2)| = \left| \int_{x_1}^{x_2} u'(t)\,\mathrm{d}t \right| &\overset{\triangle \neq}{\leqslant} \int_{x_1 \wedge x_2}^{x_1 \vee x_2} |u'(t)|\,\mathrm{d}t \\
&\overset{(\mathrm{H})}{\leqslant} \left(\int_{x_1 \wedge x_2}^{x_1 \vee x_2} |u'(t)|^p\,\mathrm{d}t \right)^{\frac{1}{p}} \left(\int_{x_1 \wedge x_2}^{x_1 \vee x_2} 1^q\,\mathrm{d}t \right)^{\frac{1}{q}} \\
&\leqslant \|u\|_{1,p} |x_1 - x_2|^{\frac{1}{q}} \leqslant M|x_1 - x_2|^{\frac{1}{q}},
\end{aligned}
$$

where $q \in [1,\infty)$ is the Hölder conjugate to p. The Theorem of Arzelá-Ascoli yields the claim since the identity maps bounded set to relatively compact sets and therefore is compact. $\qquad\square$

Corollary 2.2.4

  We have $H^1((a,b);\mathbb{R}) \overset{c}{\hookrightarrow} L^2((a,b);\mathbb{R})$.

Proof. By Theorem 2.2.1 we have

$$H^1((a,b);\mathbb{R}) = W^{1,2}((a,b);\mathbb{R}) \overset{c}{\hookrightarrow} \mathcal{C}([a,b];\mathbb{R}) \hookrightarrow L^2((a,b);\mathbb{R})$$

and the composition of a continuous and a compact map is compact. $\qquad\square$

Counterexample 2.2.5 (R)
$W^{1,1}(a,b)$ is continuously (cf. theorem 1.3.1) but not compactly embedded in $\mathcal{C}([a,b])$: consider $f_n \colon [0,1] \to \mathbb{R}$, $x \mapsto 2n(1 - nx)$ for $n \in \mathbb{N}$. The sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{C}([0,1])$ is bounded in $W^{1,1}(0,1)$

$$\|f_n\|_{1,1} = \|f_n\|_{0,1} + \|f_n'\|_{0,1} = 2n \int_0^{\frac{1}{n}} 1 - nx\,\mathrm{d}x + \int_0^{\frac{1}{n}} (-2n^2)\,\mathrm{d}x = 1 - 2n \leqslant 1.$$

But there exists no convergent subsequence of $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{C}([0,1])$.

We will now see that $u \in W^{k,p}$ can be approximated by smooth functions. This fact often allows us to translate properties of smooth functions to Sobolev functions.

> ### Theorem 2.2.2: Meyer-Serrin
>
> The space $\mathcal{C}^\infty([a,b];\mathbb{R}) \subset W^{1,p}((a,b);\mathbb{R})$ is dense for $p \in [1,\infty)$.

Geht auch für alle anderen k und alle offenen Teilmenge $\Omega \subset \mathbb{R}^n$! TODOsame proof but with induction?

Remark 2.2.6
- $\mathcal{C}^\infty([a,b]) = \{u \in \mathcal{C}^\infty((a,b)) : u^{(k)}$ are uniformly continuous on (a,b) for all $k \in \mathbb{N}\}$.

- Then, $C^\infty([a,b])$ is a subset of $W^{1,p}(a,b)$, but $\mathcal{C}^\infty(a,b)$ is not.

- Something similar holds for Lipschitz domain in $\mathbb{R}^d$.

Lemma 2.2.7 (Auxiliary lemma: local approximation)
For $u \in W^{1,p}(a,b)$ and $\varepsilon_0 \in \left(0, \frac{b-a}{2}\right)$, $\|u_\varepsilon - u\|_{W^{1,p}(a+\varepsilon_0, b-\varepsilon_0)} \xrightarrow{\varepsilon \searrow 0} 0$.

Alternative formulation $u \in H^1(a,b)$, then for any compact subinterval $K \subset (a,b)$ we have $\|u - u_\varepsilon\|_{H^1(K)} \xrightarrow{\varepsilon \searrow 0} 0$.

Proof. Let $x \in K$. For $\varepsilon > 0$ small enough $(\varepsilon < \operatorname{dist}(K, \partial(a,b)))$ we have $J_\varepsilon(x - \cdot) \in \mathcal{C}_0^\infty((a,b); \mathbb{R})$. Then

$$
\begin{aligned}
(u')_\varepsilon(x) &= \int_a^b \mathcal{J}_\varepsilon(x-y) u'(y)\,\mathrm{d}y = -\int_a^b \frac{d}{\mathrm{d}y} \mathcal{J}_\varepsilon'(x-y) u(y)\,\mathrm{d}y \\
&= \int_a^b \frac{d}{\mathrm{d}x} J_\varepsilon(x-y) u(y)\,\mathrm{d}y = \frac{d}{\mathrm{d}x} \int_a^b J_\varepsilon(x-y) u(y)\,\mathrm{d}y \\
&= \frac{d}{\mathrm{d}x} u_\varepsilon(x) = (u_\varepsilon)'(x).
\end{aligned}
$$

Proof. We know $u_\varepsilon \to u$ in $L^p(a,b)$, hence $L^p(a+\varepsilon_0, b-\varepsilon_0)$ $(u')_\varepsilon \to u'$ in $L^p(a,b)$ hence $L^p(a+\varepsilon_0, b-\varepsilon_0)$. (??)

For $x \in [a+\varepsilon_0, b-\varepsilon_0]$ we have $(u')_\varepsilon(x) = (u_\varepsilon)'(x)$. For sufficiently small $\varepsilon < \varepsilon_0$ the function $y \mapsto \mathcal{J}_\varepsilon(x-y)$ is in $\mathcal{C}_0^\infty(a,b)$. Hence,

$$
\begin{aligned}
(u')_\varepsilon(x) &= \int_a^b \mathcal{J}_\varepsilon(x-y) u'(y)\,\mathrm{d}y = -\int_a^b \frac{d}{\mathrm{d}y} \mathcal{J}_\varepsilon'(x-y) u(y)\,\mathrm{d}y \\
&= \int_a^b \frac{d}{\mathrm{d}x} J_\varepsilon(x-y) u(y)\,\mathrm{d}y = u_\varepsilon'(x).
\end{aligned}
$$

Altogether we have $u_\varepsilon \to u$ in $L^p(a+\varepsilon_0, b-\varepsilon_0)$ and $(u_\varepsilon)' = (u')_\varepsilon \to u'$ in $L^p(a+\varepsilon_0, b-\varepsilon_0)$. This yields $u_\varepsilon \to u$ in $W^{1,p}(a+\varepsilon_0, b-\varepsilon_0)$. $\qquad\square$

Proof. (of the theorem) Let $u \in W^{1,p}(a,b)$ and $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \subset \mathbb{R}$ be open intervals such that

$$
a \in \mathcal{I}_1, \quad b \in \mathcal{I}_3, \quad \mathcal{I}_2 \subset (a,b) \quad \text{and} \quad [a,b] \subset \bigcup_{k=1}^3 \mathcal{I}_k \,.
$$

Let $(\Psi_k)_{k=1}^3$ be a corresponding partition of unity, i.e.

$$
\Psi_k \in \mathcal{C}_0^\infty(\mathbb{R}),\ \operatorname{supp}(\Psi_k) \subset \mathcal{I}_k\ \forall k \in \{1,2,3\} \quad \text{and} \quad \sum_{k=1}^3 \Psi_k|_{(a,b)} \equiv 1.
$$

We set $u_k := u \cdot \Psi_k \in W^{1,p}(a,b)$ with $u_k' = u'\Psi_k + u\Psi_k'$.

(2) As $u_2 \in W^{1,p}(a,b)$ and $\operatorname{dist}(\mathcal{I}_2, \partial(a,b)) > 0$ the lemma shows $(u_2)_\varepsilon \to u_2$ in $W^{1,p}(\mathcal{I}_2)$.

(1) For a sufficiently small $\delta > 0$ we set $v_1(x) := u_1(x+\delta)$, as $n \notin \operatorname{supp}(v_1)$ then $v_1 \in W^{1,p}(a-\delta, b+\delta)$. The lemma shows $(v_1)_\varepsilon \to v_1$ in $W^{1,p}(??)$.

As the L^p continuity of u_1 yields that $\|u_1 - v_1\|_{0,p} \to 0$ and $\|u_1 - v_1\|_{0,p} \to 0$ for $\delta > 0$

$$
\left(\int_a^b |u_1(x) - u_1(x+\delta)|^p \right)^{\frac{1}{p}}
$$

$$
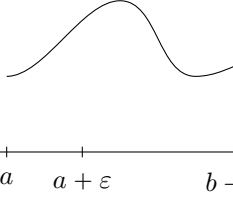\left(\int_a^b |u_1'(x) - u_1'(x+\delta)|^p \right)^{\frac{1}{p}}
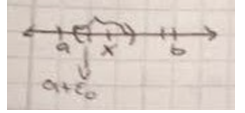$$

Figure 9: Unnamed figure
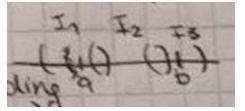

Figure 10: Unnamed figure
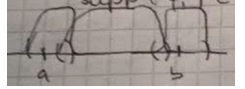

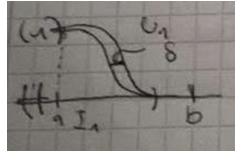Figure 11: Unnamed figure

Figure 12: Unnamed figure

Figure 13: Unnamed figure

for $\delta \searrow 0$ and hence $\|u_1 - v_1\|_{0,p} \xrightarrow{\delta \searrow 0} 0$ we get for $\eta > 0$ fix δ so that $\|u_1 - v_1\|_{1,p} < \frac{\eta}{2}$ and ε so that $\|v_1 - (v_1)_\varepsilon\|_{1,p} < \frac{\eta}{2}$.

Hence, $\|u_1 - (v_1)_\varepsilon\|_{1,p} < \eta$. Recall that $(v_1)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$.

(3) The same for u_3.

(4) We know $(u_2)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ and $(u_2)_\varepsilon \to u_2$ in $W^{1,p}(a,b)$.

We define $w := (v_1)_\varepsilon + (u_2)_\varepsilon + (v_3)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$. Hence, $w|_{(a,b)} \in \mathcal{C}^\infty([a,b])$ and

$$\|u - w\|_{1,p} \leqslant \|u_1 - (v_1)_\varepsilon\|_{1,p} + \|u_2 - (u_2)_\varepsilon\|_{1,p} + \|u_3 - (v_3)_\varepsilon\|_{1,p} < 3\eta. \qquad \square$$

Remark 2.2.8 $\mathcal{C}_0^\infty(a,b)$ is a subset of $W^{1,p}(a,b)$ but in general it is not dense. **TODO** why

Remark 2.2.9 (V, Defining Sobolev space as topological closure)
Let $C^{k,p}(a,b)$ be the space of $\mathcal{C}^\infty$ functions f so that $f^{(\ell)} \in L^p$ for all $\ell \leqslant k$. Then, we have

$$\mathcal{C}^{k,p}(a,b) \subset W^{k,p}(a,b) \subset L^p(a,b).$$

The space $C^{k,p}(a,b)$ isn't complete with respect to the norm on $W^{k,p}$; its completion <u>is</u> $W^{k,p}(\Omega)$ (Serrin-Meyer). The derivatives up to order k, being continuous operators can be uniquely continued. These continuations are precisely the weak derivatives.

DEFINITION 2.2.10 (Sobolev space w/ compact support)

We define the closed subspace

$$W_0^{1,p}((a,b);\mathbb{R}) := \overline{\mathcal{C}_0^\infty((a,b);\mathbb{R})}^{\|\cdot\|_{1,p}} \subset W^{1,p}((a,b);\mathbb{R}).$$

THEOREM 2.2.3: Characterisation of $W_0^{1,p}((a,b);\mathbb{R})$

We have $W_0^{1,p}((a,b);\mathbb{R}) = \left\{ u \in W^{1,p}((a,b);\mathbb{R}) : u(a) = u(b) = 0 \right\}$.

Remark 2.2.11
- As $W^{1,p}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b];\mathbb{R})$, this makes sense.
- This not true in $\mathbb{R}^d$ for $d > 1$ (further reading: trace operators)

Proof. "$\subset$": Let $u \in W_0^{1,p}((a,b);\mathbb{R})$ and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty((a,b);\mathbb{R})$ so that $u_n \to u$ in $W^{1,p}((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b])$. Hence,

$$\sup_{x\in[a,b]} |u_n(x) - u(x)| = \|u_n - u\|_\infty \leqslant c\|u_n - u\|_{1,p} \to 0$$

in part $0 = u_n(a) \to u(a)$, $0 = u_n(b) \to u(b)$.

"$\supset$": Let $u \in W^{1,p}((a,b);\mathbb{R})$ such that $u(a) = u(b) = 0$. Let $\eta > 0$. We construct $u_\eta \in \mathcal{C}_0^\infty((a,b);\mathbb{R})$ so that $\|u - u_\eta\| < 2\eta$.

The "cut-off function" $w := \mathbb{1}_{[-1,1]}$ is in $\mathcal{C}_0^\infty((a,b);\mathbb{R})$. First, we cut off u in a neighbourhood of a. For $\varepsilon > 0$ we define

$$w_\varepsilon(x) := w\left(\frac{x-a}{\varepsilon}\right) = \begin{cases} 0, & \text{for } |x-a| \leqslant \varepsilon, \\ 1, & \text{for } |x-a| \geqslant 2\varepsilon. \end{cases}$$
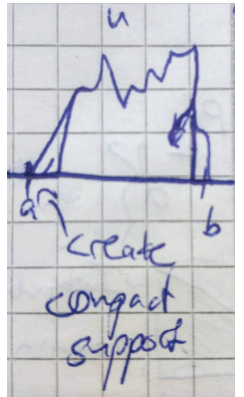

Figure 14: The mollifier u_η with compact support.

$$\text{and} \quad u_\varepsilon := u \cdot w_\varepsilon(x) = \begin{cases} 0, & \text{for } x \in [a, a+\varepsilon], \\ u(x), & \text{else.} \end{cases}.$$

Then we have $u_\varepsilon \in W^{1,p}((a,b); \mathbb{R})$ with $u'_\varepsilon = u'w_\varepsilon + uw'_\varepsilon \in L^p$ since $u, u' \in L^p$ and $w_\varepsilon \leqslant 1$ and w'_ε is also bounded (see (7)).

We now show that $\|u - u_\varepsilon\|_{1,p} \leqslant \eta$ for sufficiently small ε:

$$\begin{aligned} \|u - u_\varepsilon\|_{1,p}^p &= \int_a^b |u(x) - u_\varepsilon(x)|^p + |u'(x) - u'_\varepsilon(x)|^p \, dx \\ &\leqslant \int_a^b |u(x)|^p |1 - w_\varepsilon(x)|^p + |u'(x)|^p |1 + w_\varepsilon(x)|^p \\ &\quad + |u(x)|^p |w'_\varepsilon(x)|^p \, dx \\ &\leqslant \int_a^{a+2\varepsilon} |u(x)|^p \, dx + \int_a^{a+\varepsilon} |u'(x)|^p \, dx + 2 \int_{a+\varepsilon}^{a+2\varepsilon} |u'(x)|^p \, dx \\ &\quad + \int_{a+\varepsilon}^{a+2\varepsilon} |u(x)|^p |w'_\varepsilon(x)|^p \, dx \\ &\leqslant 2 \int_a^{a+2\varepsilon} |u(x)|^p + |u'(x)|^p \, dx + \int_a^{a+2\varepsilon} |u(x)|^p \underbrace{|w'_\varepsilon(x)|^p}_{\xrightarrow[\varepsilon \searrow 0]{} \infty} \, dx. \end{aligned}$$

As $w'_\varepsilon(x) = \frac{1}{\varepsilon} w' \left(\frac{x-a}{\varepsilon} \right)$ (chain rule) there exists an $C > 0$ so that

$$|w'_\varepsilon(x)| \leqslant \frac{1}{\varepsilon} \|w'\|_\infty \leqslant \frac{C}{\varepsilon}. \tag{7}$$

As $u(a) = 0$ we have $u(x) = \int_a^x u'(y) \, dy$. Hence,

$$\int_a^y |u(\xi)|^p \, d\xi = \int_a^y \left| \int_a^b 1 \cdot u'(t) \, dt \right|^p \, d\xi \overset{(H)}{\leqslant} \int_a^y \int_a^\xi |u'(t)|^p \, dt \cdot |\xi - a|^{\frac{p}{q}} \, d\xi. \tag{8}$$

Hence,

$$\begin{aligned} \int_{a+\varepsilon}^{a+2\varepsilon} |u(x)|^p |w'_\varepsilon(x)|^p \, dx &\leqslant \int_a^{a+2\varepsilon} |u(x)|^p |w'_\varepsilon(x)|^p \, dx \\ &\leqslant \left(\frac{C}{\varepsilon} \right)^p \int_a^{a+2\varepsilon} \int_a^\xi |u'(t)|^p \, dt \cdot |\xi - a|^{\frac{p}{q}} \, d\xi \\ &= \frac{C^p}{\varepsilon^p} \int_a^{a+2\varepsilon} \int_t^{a+2\varepsilon} |u'(t)|^p |\xi - a|^{\frac{p}{q}} \, d\xi \, dt \\ &\leqslant \frac{C^p}{\varepsilon^p} \int_a^{a+2\varepsilon} \int_a^{a+2\varepsilon} |u'(t)|^p |\xi - a|^{\frac{p}{q}} \, d\xi \, dt \\ &\overset{(F?)}{\leqslant} \frac{C^p}{\varepsilon^p} \int_a^{a+2\varepsilon} |u'(t)|^p \, dt \int_a^{a+2\varepsilon} |\underbrace{\xi - a}_{\leqslant 2\varepsilon}|^{\frac{p}{q}} \, d\xi \\ &\overset{(\star)}{\leqslant} \underbrace{C^p 2^{\frac{p}{q}}}_{=: \widetilde{C}_{p,q}^{(w)}} \int_a^{a+2\varepsilon} |u'(t)| \, dt \end{aligned}$$

Figure 15: The integration domain

where in $(\star)$ we use $-p + \frac{p}{q} + 1 = p \left(-1 + \frac{1}{q} + \frac{1}{p} \right) = 0$. We conclude

$$\|u - u_\varepsilon\|_{1,p} \leqslant \widetilde{C}_{p,q}^{(w)} \int_a^{a+2\varepsilon} |u(x)|^p + |u'(x)|^p \, dx \xrightarrow[\varepsilon \text{ suff. small}]{u,u' \in L^p} 0.$$

Proceeding analogously with the right endpoint, b, we get a function $\tilde{u}_\eta \in W^{1,p}((a,b); \mathbb{R})$ with $\|u - \tilde{u}_\eta\|_{1,p} \leqslant 2\eta$ and $\mathrm{supp}(\tilde{u}_\eta) \subset (a,b)$.

Hence $\mathcal{J}_\delta * \tilde{u}_\eta \in \mathcal{C}_0^\infty((a,b); \mathbb{R})$ for δ small enough and $\|\mathcal{J}_\delta * \tilde{u}_\eta - \tilde{u}_\eta\|_{1,p} < \eta$. $\qquad \square$
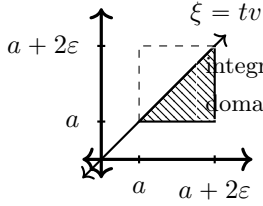
Corollary 2.2.12

$W^{1,p}((a,b); \mathbb{R}) \subset L^p((a,b); \mathbb{R})$ *is dense for $p \in [1, \infty)$.*

Proof. Exercise. Use that test functions are dense in L^p. □

The Characterisation of $W_0^{1,2}$ is also true for higher dimensions: If (a,b) is bounded the injection $W_0^{1,2}((a,b); \mathbb{R}) \to L^2$ is compact. (See the POINCARE inequality.)

THEOREM 2.2.4: POINCARÉ-FRIEDRICHS-INEQUALITY

For $u \in W_0^{1,p}((a,b); \mathbb{R})$ we have

$$\|u\|_{0,p} \leqslant (b-a)|u|_{1,p}.$$

Remark 2.2.13 This is not true for $W^{1,p}((a,b); \mathbb{R})$.

Corollary 2.2.14

On $W_0^{1,p}$, the norms $\|\cdot\|_{1,p}$ and $|\cdot|_{1,p}$ are equivalent. Thus, $(W_0^{1,p}((a,b); \mathbb{R}), |\cdot|_{1,p})$ is a closed and therefore complete subspace of $W^{1,p}((a,b); \mathbb{R})$.

Proof. By Theorem 2.2.3 we have $u(a) = 0$ and thus as in (8)

$$\|u\|_p^p = \int_a^b |u(x)|^p \, \mathrm{d}x \leqslant \int_a^b \left(\int_a^x 1 \cdot |u'(y)| \, \mathrm{d}y \right)^p \mathrm{d}x \overset{\text{(H)}}{\leqslant} \int_a^b \left(\left(\int_a^x 1^q \, \mathrm{d}y \right)^{\frac{1}{q}} \left(\int_a^x |u'(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \right)^p \mathrm{d}x$$

$$= \int_a^b \underbrace{\int_a^x |u'(y)|^p \, \mathrm{d}y}_{\leqslant \|u'\|_p^p} |x-a|^{\frac{p}{q}} \, \mathrm{d}x \leqslant |b-a|^{1+\frac{p}{q}} |u|_{1,p}^p.$$

and $\left(|b-a|^{1+\frac{p}{q}} \right)^{\frac{1}{p}} = |b-a|^{\frac{1}{p}+\frac{1}{q}} = b-a.$ □

Remark 2.2.15 For $p = 2$ we even have $\|u\|_{0,2} \leqslant \frac{b-a}{\sqrt{2}} |u|_{1,2}$, as $\int_a^b |x-a|^{\frac{2}{2}} \, \mathrm{d}x = \frac{1}{2}(b-a)^2$ and we can even instead have $\frac{b-a}{\pi}$. **TODO**

Remark 2.2.16 This is not true for unbound domains but for open subsets Ω of $\mathbb{R}^d$ we have $\|u\|_{0,2} \leqslant \left(\frac{|\Omega|}{w_d} \right)^{\frac{1}{d}} |u|_{1,2}$, where w_d is the measure of the unit ball.

Remark 2.2.17 (POINCARÉ-FRIEDRICHS-Inequality on a cone)
The POINCARÉ-FRIEDRICHS-Inequality also holds for all $u \in \{u \in H^1((a,b); \mathbb{R}) : \int_a^b u(x) = 0\} =: H_D^1((a,b); \mathbb{R}) \subset H^1((a,b); \mathbb{R})$. Suppose not, then there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset H_D^1((a,b); \mathbb{R})$ with $\|u_n\|_{0,2} \geqslant n\|u_n'\|_{0,2}$. Let $v_n := \frac{u_n}{\|u_n\|_{0,2}} \subset H_D^1((a,b); \mathbb{R})$. Then $\|v_n\|_{1,2} = \frac{\|u_n\|_{0,2}+\|u_n'\|_{0,2}}{\|u_n\|_{0,2}} \leqslant 1 + \frac{1}{n} \leqslant 2$, so $(v_n)_{n \in \mathbb{N}} \subset H^1((a,b); \mathbb{R})$ is bounded. Thus there exists a weakly convergent subsequence $(v_{n_k})_{k \in \mathbb{N}}$ with $v_{n_k} \rightharpoonup v$ with $\int_\Omega v(x) = 0$. We have $\|v_n\|_{0,2} = 1$ and $\|\nabla v_n\|_{0,2} \to 0$, so $\nabla v = 0$ and thus v is constant and thus $v = 0$ (as $\int v(x) \, \mathrm{d}x = 0$), which contradicts $\|v_n\|_{0,2} = 1$.

Dual Spaces

DEFINITION 2.2.18 (DUAL SPACE OF $W_0^{1,p}(a,b)$)

We set $W^{-1,q}(a,b) := \left(W_0^{1,p}(a,b)\right)^*$, where p and q are HÖLDER conjugates. It is equipped with the norm

$$\|f\|_{-1,q} := \sup_{\substack{u \in W_0^{1,p} \\ u \neq 0}} \frac{\langle f, u \rangle}{|u|_{1,p}}$$

Attention: $W^{-1,q}(a,b) \neq \left(W^{1,p}(a,b)\right)^*$. (but $\subset$??)

Reminder $(L^p)^* \cong L^q$ via $\langle f, u \rangle_{(L^p)^* \times L^p} = \int v_f u \, dx$, where $v_f \in L^q$ is unique and $u \in L^p$.

Lemma 2.2.19

- $L^q \hookrightarrow W^{-1,q}$.

- *for all $f \in W^{-1,q}$ there exists a not necessarily unique $u_f \in L^q(a,b)$ so that*

$$\langle f, v \rangle_{W^{-1,q} \times W_0^{1,p}} = \int u_f v' \, dx,$$

where $v \in W_0^{1,p}(a,b)$.

Proof. Exercise. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 2.2.20 We could identify the HILBERT space H^{-1} with H_0^1 by the H_0^1 inner product (RIESZ). But we won't do that and rather identify $H_0^1 \hookrightarrow L^2 \cong (L^2)^* \hookrightarrow H^{-1}$ and therefore regard H_0^1 as a subspace of H^{-1} via the L^2 inner product.

If $\tilde{f} \in L^2((a,b); \mathbb{R})$, then f, defined by

$$\langle f, u \rangle := \int_a^b f(x) u(x) \, dx,$$

where $u \in H_0^1((a,b); \mathbb{R})$ is an element of $H^{-1}((a,b); \mathbb{R})$ such that there exists a constant $C > 0$ with $\|f\|_{-1,2} \leqslant C \|\tilde{f}\|_{0,2}$. thus

$$L^2((a,b); \mathbb{R}) \hookrightarrow H^{-1}((a,b); \mathbb{R})$$

Counterexample 2.2.21 (L^p convergence $\implies W^{k,p}$ convergence)

The function family $f_n(x) := \frac{\sin(nx)}{n}$ converges in any $L^p[a,b]$ to zero but does not converge in any $W^{k,p}$.

THEOREM 2.2.5: MEYERS-SERRIN

For $p \in [1, \infty)$ the subspace $\mathcal{C}^\infty(a,b) \cap W^{k,p} \subset W^{k,p}(a,b)$ is dense.

Instead of (a,b), one can use any open subset of $\mathbb{R}^n$.

$\mathcal{C}_0^\infty((a,b); \mathbb{R}) \subset L^p((a,b); \mathbb{R})$ is dense for $p < \infty$. But $\mathcal{C}_0^\infty((a,b); \mathbb{R}) \subset W^{k,p}((a,b); \mathbb{R})$ is not dense. $\mathcal{C}^\infty([a,b]; \mathbb{R})$ is dense in $W^{k,p}((a,b); \mathbb{R})$. Formally, $\mathcal{C}^\infty([a,b]; \mathbb{R}) \nsubseteq W^{k,p}((a,b); \mathbb{R})$ but we show that

$$\mathcal{C}^\infty([a,b]; \mathbb{R}) = \{u \in \mathcal{C}^\infty((a,b); \mathbb{R}) : u^{(\ell)} \text{ is uniformly continuous } \forall \ell \in \mathbb{N}\}$$

📚 Further Reading

Generalizations of Sobolev spaces include Besov and Sobolev-Slobodeckij spaces.

Lemma 2.2.22 (Important inequalities. Tut, needs to be somewhere else)

For $p \in [1, \infty)$ and $(a_k)_{k=1}^n \geqslant 0$ we have

$$\sum_{k=1}^n a_k^p \leqslant \left(\sum_{k=1}^n a_k \right)^p \leqslant n^{p-1} \sum_{k=1}^n a_k^p$$

For $p \in (0,1)$ the inequalities are reversed.

Proof. For $p \geqslant 1$ the function $f(x) := x^p$ is convex. With Jensens inequality (J) we have

$$\left(\sum_{k=1}^n a_k \right)^p = n^p \left(\sum_{k=1}^n \frac{a_k}{n} \right)^p \overset{(J)}{\leqslant} n^p \sum_{k=1}^n \frac{a_k^p}{n} = n^{p-1} \sum_{k=1}^n a_k^p.$$

Now, let $a_j \neq 0$ for one $j \in \{1, \ldots, n\}$. Then we have

$$x_k := a_k \cdot \left(\sum_{k=1}^n a_k \right)^{-1} \leqslant 1 \implies \sum_{k=1}^n x_k^p \leqslant \sum_{k=1}^n x_k = 1,$$

which shows the claim. $\qquad\square$

The following is a L^p-Generalization of the theorem of Arzela-Ascoli:

THEOREM 2.2.6: Fréchet–Kolmogorov-Riesz

Let $(a,b) \subset \mathbb{R}$ and $(u_n)_{n \in \mathbb{N}}$ a bounded sequence in $L^p((a,b); \mathbb{R})$ where $p \in [1, \infty)$. If for all $\varepsilon > 0$ and all intervals $[\alpha, \beta] \subset (a,b)$ there is an $\delta \in (0, \min(\alpha - a, b - \beta))$ such that for all $h \in \mathbb{R}$ with $|h| < \delta$ and all $n \in \mathbb{N}$ it holds that

$$\int_\alpha^\beta |u_n(x+h) - u_n(x)|^p \, \mathrm{d}x < \varepsilon$$

and if there exists an interval $[\alpha', \beta'] \subset (a,b)$ such that for all $n \in \mathbb{N}$ it holds that

$$\int_a^{\alpha'} |u_n(x)|^p \, \mathrm{d}x + \int_{\beta'}^b |u_n(x)|^p \, \mathrm{d}x < \varepsilon$$

then there is subsequence of $(u_n)_{n \in \mathbb{N}}$ that converges in $L^p((a,b); \mathbb{R})$.

Lemma 2.2.23 (Wikipedia formulation of the above)

A bounded set $\mathcal{F} \subset L^p(\mathbb{R}^n)$, with $p \in [1, \infty)$ is relatively compact if and only if $\int_{|x|>r} |f|^p \xrightarrow{r \to \infty} 0$ and $\|\tau_a f - f\|_{L^p(\mathbb{R}^n)} \xrightarrow{a \to 0}$, both uniformly on $\mathcal{F}$.

THEOREM 2.2.7: Rellich

$H^1((a,b); \mathbb{R}) \overset{c}{\hookrightarrow} L^2([a,b]; \mathbb{R})$.

Proof. The embedding $H^1((a,b); \mathbb{R}) \hookrightarrow L^2([a,b])$ is clear since

$$\|u\|_{0,2}^2 \leqslant \|u\|_{0,2}^2 + \|u'\|_{0,2}^2 = \|u\|_{1,2}^2.$$

Let $(u_n)_{n\in\mathbb{N}} \subset H^1((a,b);\mathbb{R})$ a bounded sequence so that $\|u\|_{1,2} \leqslant M$ for all $n \in \mathbb{N}$, $\varepsilon > 0$ and $[\alpha, \beta]$ given. Choose

$$\delta := \min\left\{ \frac{\varepsilon}{M(\beta - \alpha)}, \min\{\alpha - a, b - \beta\} \right\}.$$

For $x \in (a, b)$ and $h \in \mathbb{R}$ such that $|h| < \delta$. Then we have

$$\int_\alpha^\beta |u_n(x+h) - u_n(x)|^2 \, \mathrm{d}x \stackrel{\text{MVT}}{=} \int_\alpha^\beta \left| \int_x^{x+h} h \cdot u_n'(\xi) \, \mathrm{d}\xi \right|^2 \, \mathrm{d}x$$

$$\stackrel{(\text{H})}{\underset{\triangle \neq}{\leqslant}} \int_\alpha^\beta |h| \int_x^{x+h} |u_n'(\xi)|^2 \, \mathrm{d}\xi \, \mathrm{d}x$$

$$\leqslant |h| |\beta - \alpha| \|u'\|_{0,2}^2 \leqslant \varepsilon.$$

Let $\tilde{\delta} \in \left(0, \min\left(b - a, \frac{\varepsilon}{2(cM)^2} \right) \right)$, where x is the embedding constant of $H^1((a,b);\mathbb{R}) \hookrightarrow \mathcal{C}([a,b])$. Set $[\alpha', \beta'] := [a + \tilde{\delta}, b - \tilde{\delta}]$. Then we have

$$\int_a^{\alpha'} |u_n(x)|^2 \, \mathrm{d}x + \int_{\beta'}^b |u_n(x)|^2 \, \mathrm{d}x \leqslant 2\tilde{\delta} \|u_n\|_\infty^2 \leqslant 2\tilde{\delta} c^2 \|u_n\|_{1,2}^2 < \varepsilon.$$

With the Theorem of Fréchet-Kolmogorov-Riesz we get the existence of a convergent subsequence. $\qquad\square$

DEFINITION 2.2.24 (Hölder continuity / Space)

For $\alpha \in (0,1)$ a function $u \colon [a, b] \to \mathbb{R}$ is α-Hölder continuous if

$$\exists c \geqslant 0 : |u(x) - u(y)| \leqslant c|x - y|^\alpha \ \forall x, y \in [a, b].$$

The space of Hölder continuous functions

$$\mathcal{C}^{0,\alpha}([a,b]) := \left\{ v \in \mathcal{C}([a,b]) : |u|_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \right\}$$

equipped with the norm $\|u\|_{\mathcal{C}^{0,\alpha}} := \|u\|_\infty + |u|_\alpha$ is complete.

Lemma 2.2.25 (Hölder embeddings)

① For $0 < \alpha < \beta < 1$ we have $\mathcal{C}^{0,\beta}([a,b]) \overset{c}{\hookrightarrow} \mathcal{C}^{0,\alpha}([a,b])$.

② We have $H^1((a,b);\mathbb{R}) \overset{c}{\hookrightarrow} \mathcal{C}^{0,\alpha}([a,b])$ for $\alpha \in \left(0, \frac{1}{2}\right)$

Proof. ① <u>Continuity.</u> Since

$$|u|_\alpha = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha - \beta} |x - y|^\beta} \leqslant \sup_{x \neq y} |x - y|^{\beta - \alpha} \cdot \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta}$$

$$\leqslant |b - a|^{\beta - \alpha} |u|_\beta,$$

the claim follows.

<u>Compactness.</u> Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^{0,\beta}([a,b])$ a bounded sequence, i.e. there exists a $M > 0$ such that $\|u_n\|_\beta \leqslant M$. Particularly $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded. We have

$$|u_n(x) - u_n(y)| \leqslant |u_n|_\beta |x - y|^\beta \leqslant M|x - y|^\beta.$$

Therefore $(u_n)_{n\in\mathbb{N}}$ is equicontinuous. By the theorem of Arzelá-Ascoli there exists an convergent subsequence $(u_n')_{n\in\mathbb{N}} \subset \mathcal{C}([a,b])$ converging to $u \in \mathcal{C}([a,b])$.

We now show $\|u_n' - u\|_\infty \xrightarrow{n' \to \infty} 0$.

Let $\varepsilon > 0$ and $\delta < \min\left(\left(\frac{\varepsilon}{6M} \right)^{\frac{1}{\beta - \alpha}}, 1 \right)$ and $j \leqslant n$ so that

$$\|u - u_j\|_\infty < \frac{\varepsilon}{6}\delta^k < \frac{\varepsilon}{3}.$$

and let $x \neq y$, $|x - y| < \delta$. Then,

$$
\begin{aligned}
\frac{|(u - u_j)(x) - (u - u_j)(y)|}{|x - y|^\alpha} &= \lim_{n' \to \infty} \frac{|(u_{n'} - u_j)(x) - (u_{n'} - u_j)|}{|x - y|^\beta} \cdot |x - y|^{\beta - \alpha} \\
&\leqslant \delta\beta - \alpha \sup_{n' \in \mathbb{N}} |u_{n'} - u_j|_\beta \\
&\leqslant \delta^{\beta - \alpha} \sup_{n' \in \mathbb{N}} \|u_{n'}\|_\beta \cdot 2 \leqslant \frac{\varepsilon}{3}.
\end{aligned}
$$

For $|x - y| \geqslant \delta$ we have

$$\frac{|(u - u_j)(x) - (u - u_j)|}{|x - y|^\alpha} \leqslant \delta^{-\alpha}\|u - u_j\|_\infty < \frac{\varepsilon}{3}.$$

Therefore, we have

$$\|u - u_j\|_\alpha \leqslant \sup \|u - u_j\|_\infty + |u - u_j|_\alpha < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \qquad \square$$

3 Reformulation using variational formulations and operator equations

3.1 Reformulation using variational formulations

Example 3.1.1 (Obtaining variational formulation from BVP)
Consider the linear second order boundary value problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} -u''(x) + c(x)u'(x) + d(x)u(x) = f(x), & x \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \qquad (9)$$

where $c, d \in \mathcal{C}([a,b]; \mathbb{R})$. A classical solution to (9) is a function $u \in \mathcal{C}^2((a,b); \mathbb{R}) \cap \mathcal{C}([a,b]; \mathbb{R})$.

(1) Multiply (9) with a (yet to be specified) test function v and

(2) integrate over the domain:

$$\int_a^b u''(x)v(x) + c(x)u'(x)v(x) + d(x)u(x)v(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x$$

(3) Integrate by parts in the highest order derivative.

$$\int_a^b u'(x)v'(x)\,\mathrm{d}x - u(x)v'(x)\big|_{x=a}^b + \int_a^b c(x)u'(x)v(x) + d(x)u(x)v\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x$$

If $v(a) = v(b) = 0$ we obtain

$$\boxed{\int_a^b u'(x)v'(x) + c(x)u'(x)v(x) + d(x)u(x)v(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x.} \qquad (10)$$

The equation (10) is well defined for e.g. $u, v \in H_0^1((a,b); \mathbb{R})$, $c, d \in L^\infty((a,b); \mathbb{R})$, $f \in L^2((a,b); \mathbb{R})$ (or L^1).

Instead of finding a classical solution to (9) we now search a function $u \in V := H_0^1((a,b); \mathbb{R})$ so that (10) holds for all $v \in V$.

For sake of brevity we define the bilinear form bilinear

$$\alpha\colon V \times V \to \mathbb{R}, \ (u,v) \mapsto \int_a^b u'(x)v'(x) + c(x)u'(x)v(x) + d(x)u(x)v(x)\,\mathrm{d}x.$$

and

$$\langle\, \tilde{f}, v \,\rangle := \int_a^b f(x)v(x)\,\mathrm{d}x$$

Then, $\tilde{f}$ is linear in v and bounded: With the Cauchy-Schwartz inequality (CS) and the Poincaré-Friedrichs-inequality (PF) we obtain

$$|\langle\, \tilde{f}, v \,\rangle| \overset{\text{(CS)}}{\leqslant} \|f\|_2 \|v\|_2 \overset{\text{(PF)}}{\leqslant} C\|f\|_2 |v|_{1,2}.$$

for a constant $C = \frac{b-a}{\pi} > 0$.

Similarly, for $u, v \in H_0^1((a,b); \mathbb{R})$, α fulfills

$$|\alpha(u,v)| \overset{\text{(CS)}}{\leqslant} C\left(1 + \|c\|_{0,\infty} + \|d\|_{0,\infty}\right)|u|_{1,2}|v|_{1,2}. \qquad (11)$$

for a constant $C > 0$, hence it is bounded.

> 💡 **Current Formulation of BVP (I)**
>
> Find $u \in V := H_0^1((a,b); \mathbb{R})$ such that
>
> $$\alpha(u, v) = \langle \tilde{f}, v \rangle \quad \forall v \in V.$$

Remark 3.1.2

① We want to consider the same space V for solution and test function e.g to "test with the solution" (see below).

② We will write $f \in H^{-1}((a,b); \mathbb{R})$ instead of $\tilde{f}$.

③ As we started out with a linear equation the bilinear form α is linear in u, too (it is always linear in v by construction).

④ For homogeneous DIRICHLET boundary conditions (DBCs) we always (need to) choose a "zero-space".

⑤ When finding a solution to the variational formulation we have to be aware that this might not make any sense in the classical sense.

Example 3.1.3 (Transforming inhomogeneous DIRICHLET BCs)

Consider the boundary value problem

$$\begin{cases} -u''(x) = f(x), & x \in (a,b) \\ u(a) = \alpha, \ u(b) = \beta. \end{cases}$$

To obtain the variational formulation we write

$$-\int_a^b u''(x)v(x)\,\mathrm{d}x = \int_a^b u'(x)v'(x)\,\mathrm{d}x - u'(x)v(x)\Big|_{x=a}^b.$$

We could choose

$$V := \left\{ u \in H^1((a,b); \mathbb{R}) : u(a) = \alpha, \ u(b) = \beta \right\},$$

but this is not a linear space since it does not contain the zero function.

Therefore we choose $g \in H^1((a,b); \mathbb{R})$ such that $g(a) = \alpha$ and $g(b) = \beta$ and set $\tilde{u} := u - g$. Because we are in one dimension, this g always exists and can be a line and is therefore regular enough. If $u \in H^1((a,b); \mathbb{R})$ with $u(a) = \alpha$ and $u(b) = \beta$ consider $\tilde{u} \in H_0^1((a,b); \mathbb{R})$ and vice versa.

We set $\alpha(u, v) := \int_a^b u'(x)v'(x)\,\mathrm{d}x$ and $\langle f, v \rangle = \int_a^b f(x)v(x)\,\mathrm{d}x$.

Then, $\alpha(\tilde{u}, v) = \alpha(u, v) - \alpha(g, v)$. Hence if $\alpha(u, v) = \langle f, v \rangle$ then $\alpha(\tilde{u}, v) = \langle f, v \rangle - \alpha(g, v) =: \langle \tilde{f}, v \rangle$ and vice versa.

In higher dimensions this can be problematic: Consider the unit circle U with the boundary conditions $u(x) = 0$ if $x < 0$ on δU and $u(x) = 1$ if $x \geqslant 0$ on δU. Then, h must have a very steep derivative in the neighbourhood of $x = 0$. Therefore, we require $h \in H^{\frac{1}{2}}(\delta U)$ in this case.

Our problem now reads

> 💡 **Current Formulation of BVP (II)**
>
> Find $\tilde{u} \in V := H_0^1((a,b); \mathbb{R})$ such that
>
> $$\alpha(\tilde{u}, v) = \langle \tilde{f}, v \rangle \quad \forall v \in V.$$

Example 3.1.4 (Transforming Neumann boundary condition)

Consider the boundary value problem

$$\begin{cases} -u''(x) = f(x), & x \in (a, b) \\ u'(a) = \alpha, \ u'(b) = \beta. \end{cases}$$

Analogously to above we write

$$\int_a^b u'(x)v(x)\,\mathrm{d}x + \underbrace{u'(x)v(x)\Big|_{x=a}^b}_{=\beta v(b) - \alpha v(a)} = \int_a^b f(x)v(x)\,\mathrm{d}x.$$

Thus we want to find $u \in H^1((a, b); \mathbb{R})$ (not H_0^1!) such that

$$\alpha(u, v) := \int_a^b u'(x)v'(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x + \alpha v(a) - \beta v(b)$$

for all $v \in H^1((a, b); \mathbb{R})$. We observe that the variational formulation of this problem differs from the last example by the fact that we consider $V := H^1((a, b); \mathbb{R})$ instead of $H_0^1((a, b); \mathbb{R})$.

For homogeneous Neumann boundary conditions we just search a $u \in H^1((a, b); \mathbb{R})$ such that $\int u'v' = \langle f, v \rangle$ for all $v \in H^1((a, b); \mathbb{R})$, while for homogeneous Dirichlet boundary conditions we search a $u \in H_0^1((a, b); \mathbb{R})$ such that $\int u'v' = \langle f, v \rangle$ for all $v \in H_0^1((a, b); \mathbb{R})$; only the space differs. We will later see that if f is the Heaviside function, the Neumann problem is not uniquely solvable, while the Dirichlet problem is.

Example 3.1.5 (Variational formulation of Robin BCs)

The weak formulation of the boundary value problem

$$\begin{cases} -u''(x) + c(x)u'(x) + d(x)u(x) & = f(x), & \text{on } (a, b), \\ u'(a) + c_a u(a) & = \alpha, \\ u'(b) + c_b u(b) & = \beta, \end{cases}$$

where $c, d \in L^\infty((a, b); \mathbb{R})$, $f \in L^2((a, b); \mathbb{R})$ and $c_a, c_b, \alpha, \beta \in \mathbb{R}$ can be obtained as follows:

Multiply by $v \in H^1((a, b); \mathbb{R})$ and integrate (by parts):

$$\text{LHS} = \underbrace{\int_a^b -u''(x)v(x)\,\mathrm{d}x}_{=:(\star)} + \int_a^b c(x)u'(x)v(x) + d(x)u(x)v(x)\,\mathrm{d}x,$$

where

$$(\star) = \int_a^b u'(x)v'(x)\,\mathrm{d}(x) - [u'(x)v(x)]_{x=a}^b$$

$$= \int_a^b u'(x)v'(x)\,\mathrm{d}(x) - [(\beta - c_b u(b))v(b) - (\alpha - c_a u(a))v(a)].$$

The variational formulation then is:

$$\begin{cases} \text{For } f \in L^2((a, b); \mathbb{R}) \text{ find } u \in V := H^1((a, b); \mathbb{R}) \text{ such that for all } v \in V^* \text{ we have} \\ \int_a^b u'v' + cu'v + duv'\,\mathrm{d}x + c_b u(b)v(b) - c_a u(a)v(b) = \int_a^b fv\,\mathrm{d}x + \beta v(b) - \alpha v(a). \end{cases}$$

Example 3.1.6 (Variational formulation of periodic BCs)

The weak formulation of the boundary value problem

$$
\begin{cases}
-u''(x) + c(x)u'(x) + d(x)u(x) & = f(x), \qquad \text{on } (a,b), \\
u'(a) & = u'(b), \\
u(a) & = u(b),
\end{cases}
$$

where $c, d \in L^\infty((a,b); \mathbb{R})$ and $f \in L^2((a,b); \mathbb{R})$ can be obtained as follows:

Let $V := \{v \in H^1((a,b); \mathbb{R}) : v(a) = v(b)\}$ the space of periodic $H^1((a,b); \mathbb{R})$ functions.

Taking the same steps as with Robin boundary conditions, this time we obtain

$$
[u'(x)v(x)]_{x=a}^{b} = u'(b)v(b) - u'(a)v(a) = (u'(b) - u'(a))v(a) = 0.
$$

Therefore, our variational formulation reads

For $f \in L^2((a,b); \mathbb{R})$ find $u \in V := H^1((a,b); \mathbb{R})$ such that

$$
\int_a^b u'v' + cu'v + duv' \, dx = \int_a^b fv \, dx.
$$

for all $v \in V^*$.

Remark 3.1.7 In both of the former examples we could have also chosen $f \in H^{-1}((a,b); \mathbb{R})$, weakening the requirement $f \in L^2$.

Example 3.1.8 Let H be the Heaviside function. Consider the boundary value problem

$$
\begin{cases}
-u''(x) = 2H(x) + \delta_0(x), & \text{on } (-1,1) \\
u(-1) = u(1) = 0.
\end{cases}
$$

We find the variational formulation

$$
V := H_0^1(-1,1), \qquad \alpha(u,v) := \int_{-1}^1 u'(x)v'(x) \, dx,
$$

$$
\langle f, v \rangle := 2 \int_{-1}^1 H(x)v(x) \, dx + \langle \delta_0, v \rangle = 2 \int_0^1 v(x) \, dx + v(0).
$$

Then, $f \in H^{-1}((a,b); \mathbb{R})$ is linear and we have

$$
|\langle f, v \rangle| \leqslant 2\|v\|_{0,1} + |v(0)| \leqslant 2\|v\|_{0,2} + \|v\|_{0,\infty} \overset{\text{(PF)}}{\leqslant} C|v|_{1,2},
$$

because v is absolutely continuous.

We can see that our solution is

$$
u(x) = \begin{cases}
x + 1, & \text{if } x \in [-1, 0), \\
1 - x^2, & \text{if } x \in [0, 1],
\end{cases}
$$

Figure 16: The solution u for $x \in [-1, 1]$.

Then for $v \in H_0^1(-1, 1)$ we have

$$
\int_{-1}^1 u'(x)v'(x) \, dx = \int_{-1}^0 v'(x) \, dx + (-2) \int_0^1 xv'(x) \, dx
$$

$$
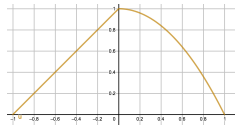= v(0) - v(-1) + 2 \int_0^1 v(x) \, dx - 2xv(x)\Big|_{x=0}^{1} = v(0) - v(-1) - 2v(-1) + 2 \int_0^1 v(x) \, dx.
$$

The solution is unique by the superposition principle. But we can also show it like this: Let $u, \tilde{u}$ be solutions and define $w := u - \tilde{u} \in H_0^1((a,b); \mathbb{R})$. Then for $v = w$ (this is the aforementioned testing with the solution). Then, we have

$$\alpha(w, v) = \alpha(u, w) - \alpha(\tilde{u}, w) = \langle f, w \rangle - \langle f, w \rangle = 0.$$

From this we obtain

$$0 = \alpha(w, v) = \int_a^b w'(x) w'(x) \, \mathrm{d}x = \int_a^b |w'(x)|^2 \, \mathrm{d}x = |w|_{1,2}^2$$

hence $w \equiv 0$ in $H_0^1((a,b); \mathbb{R})$.

For $v \in V$ we define the integral operator

$$A \colon V \to V^*, \qquad \langle Au, v \rangle := \alpha(u, v) \quad \text{or} \quad Au := \alpha(u, \cdot).$$

As we have seen in (11) we have

$$|\langle Au, v \rangle| = |\alpha(u, v)| \leqslant C |u|_{1,2} |v|_{1,2} = C \|u\|_V \|v\|_V.$$

Thus we have $Au \in V^*$ with

$$\|Au\|_* = \sup_{\substack{v \in V \\ v \not\equiv 0}} \frac{\langle Au, v \rangle}{\|v\|_V} \leqslant C \|u\|_V$$

Since α is linear in its first argument, A is linear.

Our problem now reads

> ### 💡 Final Formulation of BVP (III)
>
> For $f \in V^*$ find $u \in V$ such that
>
> $$Au = f \qquad \text{(in } V^*\text{)}.$$

Lemma 3.1.9 (Emmerich Lemma 3.4.5)

Let V be a real reflexive Banach space and $a \colon V \times V \to \mathbb{R}$ defined by $A \colon V \to V^*$. Then

- *A is linear if and only if a is bilinear.*

- *A is symmetric if and only if a is symmetric.*

- *if a is bilinear, A is bounded if and only if a is bounded.*

- *A is strongly positive if and only if a is strongly positive.*

Remark 3.1.10 (Emmerich, remark 3.4.7) The restriction onto real spaces is not necessary. If V is a complex Hilbert space, we replace bilinearity by sesquilinearity and instead of strong positivity we require $\Re(a(v,v)) \geqslant \mu \|v\|^2$ for all $v \in V$.

Example 3.1.11 (Tut, Weak solutions can be strong)
Let $u \in \mathcal{C}^2((a,b); \mathbb{R}) \cap \mathcal{C}[a,b]$ be a weak solution of

$$-u''(x) + c(x)u'(x) + d(x)u(x) = f(x) \qquad \text{in } (a,b)$$

equipped with homogeneous Dirichlet boundary conditions, where $c, d, f \in \mathcal{C}[a,b]$. Then u is already a classical solution of the boundary value problem:

Since $u \in \mathcal{C}^2((a,b); \mathbb{R}) \cap \mathcal{C}[a,b]$, the point evaluation of u, u', u'' is well defined. Since u is weak solution,

$$\int_a^b u'v' + (cu' + du)v \, \mathrm{d}x = \langle f, v \rangle$$

holds for all $v \in H_0^1((a,b); \mathbb{R}) \supset \mathcal{C}_0^\infty((a,b); \mathbb{R})$. Since u is regular enough, we can integrate by parts to obtain

$$0 = \int_a^b (-u'' + cu' + du - f)v \, \mathrm{d}x + \underbrace{\left[u'v\right]_{x=a}^b}_{=0} \qquad \forall v \in \mathcal{C}_0^\infty((a,b); \mathbb{R})$$

The Fundamental theorem of the Calculus of Variations implies that

$$-u'' + cu' + du - f \equiv 0$$

for almost all $x \in (a,b)$. But since all functions involved are continuous, the identity holds everywhere.

3.2 Linear variational problems with strongly positive bilinear form

DEFINITION 3.2.1 (OPERATOR PROPERTIES)

Let $\alpha\colon V \times V \to \mathbb{R}$ be a bilinear form. We call α

- symmetric if $\alpha(u,v) = \alpha(v,u)$ holds for all $u, v \in V$. — symmetric
- strongly positive there exists a $\mu > 0$ such that $\alpha(u,u) \geqslant \mu\|u\|^2$ for all $u \in V$. — strongly positive
- positive if $\alpha(u,u) \geqslant 0$ for all $u \in V$. — positive
- bounded if there exists a $\beta > 0$ such that $\alpha(u,v) \leqslant \beta\|u\|\|v\|$ holds for all $u, v \in V$. — bounded

Let $A\colon V \to V^*$ be a linear operator. We call A

- symmetric if $\langle Au, v \rangle = \langle Av, u \rangle$ holds for all $u, v \in V$. — symmetric
- strongly positive if there exists a $\mu > 0$ such that $\langle Au, u \rangle \geqslant \mu\|u\|^2$ for all $u \in V$. — strongly positive
- positive if $\langle Au, u \rangle \geqslant 0$ for all $u \in V$. — positive
- bounded if it maps bounded sets to bounded sets. Since A is linear, this is equivalent to requiring that there exists a $\beta > 0$ such that $\|Au\|_* \leqslant \beta\|u\|$ holds for all $u \in V$. — bounded

Remark In the literature, strong positive is also called (strong) coercivity or strong ellipticity.

Lemma 3.2.2 (Boundedness of symmetric bilinear forms)

Let V be a Banach space and $\alpha\colon V \times V \to \mathbb{R}$ a symmetric bilinear form. Then α is bounded if and only if $|\alpha(u,u)| \leqslant M\|u\|^2$ holds for some $M \geqslant 0$.

Proof. " $\Longrightarrow$ " is trivial.

" $\Longleftarrow$ ": Because α is symmetric we have

$$|\alpha(u,v)| = \left|\frac{1}{2}\alpha(u,u) + \frac{1}{2}\alpha(v,v) - \frac{1}{2}\alpha(u-v,u-v)\right|$$

$$\overset{\triangle\neq}{\leqslant} \frac{M}{2}(\underbrace{\|u-v\|^2}_{\leqslant 2(\|u\|^2+\|v\|^2)} + \|u\|^2 + \|v\|^2) \leqslant \frac{3M}{2}(\|u\|^2 + \|v\|^2).$$

For $\|u\| = 1 = \|v\|$ it follows that $|\alpha(u,v)| \leqslant 3M$ for all $u, v \in V$ implying

$$\left|\alpha\left(\frac{u}{\|u\|}, \frac{v}{\|v\|}\right)\right| \leqslant 3M \overset{\text{L}}{\Longrightarrow} |\alpha(u,v)| \leqslant 3M\|u\|\|v\|. \qquad \square$$

Example 3.2.3 (from Physics: Minimising Energy Functional) Let $a\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form. We define the corresponding energy functional

$$J\colon V \to \mathbb{R}, \qquad v \mapsto \frac{a(v,v)}{2} - \langle f, v \rangle,$$

where V is comprised of all the states v of a certain system and $J(v)$ gives its energy in that state. Our goal is find minimisers of J.

If u is a minimiser, then "$J'(u) = 0$" should hold. We aim to give meaning to that expression. Let $v \in V$. Then,

$$\langle J'(u), v \rangle = \lim_{h \to 0} \frac{1}{h}\left(J(u+hv) - J(u)\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{a(u+hv, u+hv)}{2} - \langle f, u+hv \rangle - \frac{a(u,v)}{2} + \langle f, u \rangle \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(ha(u,v) + \frac{h^2}{2} a(v,v) - h \langle f, v \rangle \right) = a(u,v) - \langle f, v \rangle \stackrel{!}{=} 0.$$

Hence the necessary condition $J'(u) = 0$ is fulfilled if u is a variational solution of our boundary value problem ($\to$ DGL II B).

THEOREM 3.2.1: LAX-MILGRAM (1954)

Let $(V, (\cdot, \cdot), \|\cdot\|)$ be a (real) HILBERT space and $A \colon V \to V^*$ a linear, strongly positive, bounded operator. Then A is bijective.

Proof. As V is a HILBERT space, there is an isometric isomorphism $\hat{\imath} \colon V^* \to V$, the RIESZ map, such that $\langle f, v \rangle = (\hat{\imath}(f), v)$ and $\|f\|_* = \|\hat{\imath}(f)\|$ for all $f \in V^*$ and all $v \in V$.

Since A is strongly positive and bounded, there exist $\mu, \beta > 0$ such that

$$\langle Au, u \rangle \geqslant \mu \|u\|^2 \quad \text{and} \quad \langle Au, v \rangle \leqslant \beta \|u\| \|v\| \quad \forall u, v \in V.$$

Fix $f \in V^*$, choose $\tau > 0$ such that $\tau < \frac{2\mu}{\beta^2}$, i.e. $1 - 2\mu\tau + \tau^2 \beta^2 < 1$ and define

$$\Phi \colon V \to V, \qquad v \mapsto v + \tau \hat{\imath}(f - Av).$$

Then $f = Au$ if and only if $\Phi(u) = u$. To use BANACH's Fixed Point Theorem it remains to verify that Φ is a contraction: for $u, v \in V$ we have

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|^2 &= \|u - v + \tau \hat{\imath}(f - Au - f + Av)\|^2 \\
&= \|u - v\|^2 + 2\tau(u - v, \hat{\imath}(A(v - u))) + \tau^2 \|\hat{\imath}(A(u - v))\|^2 \\
&= \|u - v\|^2 - 2\tau(\hat{\imath}(A(u - v)), u - v) + \tau^2 \|\hat{\imath}(A(u - v))\|^2 \\
&= \|u - v\|^2 - 2\tau \langle A(u - v), u - v \rangle + \tau^2 \|A(u - v)\|_*^2 \\
&\leqslant \|u - v\|^2 - 2\tau\mu \|u - v\|^2 + \tau^2 \beta^2 \|u - v\|^2 \\
&= \underbrace{\left(1 - 2\mu\tau + \tau^2 \beta^2\right)}_{<1} \|u - v\|^2. \qquad \square
\end{aligned}$$

Corollary 3.2.4 (Solution operator)
  Under the above conditions the bijectivity of A implies the existence of a unique solution $u \in V$ to the problem $Au = f$ for all $f \in V^*$ as well as the existence of the solution operator $A^{-1} \colon V^* \to V$, which is linear, bounded and strongly positive.

solution operator

Proof. (Left as an exercise) By the Theorem of LAX-MILGRAM A is bijective, implying the existence of the linear A^{-1}. Its boundedness i.e follows from the inverse mapping theorem but can be show with much more elementary means: For all $f \in V^*$ we have

$$\mu \|A^{-1}(f)\|_V^2 \leqslant \langle AA^{-1}f, A^{-1}f \rangle = \langle f, A^{-1}f \rangle \leqslant \|f\|_{V*} \|A^{-1}f\|_V.$$

Finally, the strict positivity follows from

$$\begin{aligned}
\|f\|_{V*}^2 = \|AA^{-1}f\|_{V*}^2 &\leqslant \beta^2 \|A^{-1}f\|_V^2 \\
&\leqslant \frac{\beta^2}{\mu} \langle AA^{-1}f, A^{-1}f \rangle = \frac{\beta^2}{\mu} \langle f, A^{-1}f \rangle. \qquad \square
\end{aligned}$$

Corollary 3.2.5 (Continuous dependence)

There exists a $c > 0$ such that $\|A^{-1}f\| \leqslant c\|f\|_*$ for all $f \in V^*$, proving stability / continuous dependence *(because A is linear even Lipschitz dependence) on the right*   *continuous* *hand side.*   *dependence*

TODO: prove Lax-Milgram with Galerkin scheme

Proof. Let $u, \tilde{u} \in V$ be solutions to the right hand sides $f, \tilde{f}$. Then, there exists a $c > 0$ such that

$$\|u - \tilde{u}\| = \|A^{-1}(f) - A^{-1}(\tilde{f})\| \overset{(\mathrm{L})}{=} \|A^{-1}(f - \tilde{f})\| \leqslant C\|f - \tilde{f}\|_* \qquad \square$$

Remark 3.2.6 (Energy Norm) Let $(V, \|\cdot\|)$ be a real Banach space and $A \colon V \to V^*$ linear, strongly positive and bounded operator.

- If A is symmetric, we define

$$(\cdot, \cdot)_A : V \times V \to \mathbb{R}, \ (u, v) \mapsto \langle Au, v \rangle,$$

which is a inner product on V. The induced norm is given by $\|u\|_A^2 := (u, u)_A := \langle Au, u \rangle$, called the energy norm. Both norms are equivalent:

$$\mu\|u\|^2 \leqslant \langle Au, u \rangle = \|u\|_A^2 \leqslant \beta\|u\|^2. \qquad (12)$$

- If A is not symmetric we consider its symmetric part:

$$(\cdot, \cdot)_A : V \times V \to \mathbb{R}, \ (u, v) \mapsto \frac{1}{2}\left(\langle Au, v \rangle + \langle Av, u \rangle\right).$$

Proof. (My idea of how to simplify proof of Lax-Milgram if A is symmetric)
As $(V, (\cdot, \cdot)_A)$ is a Hilbert space (due to the equivalence of the induced norms), there is an isometric isomorphism $\hat{\imath} \colon V^* \to V$, the Riesz map, such that $\langle f, v \rangle = (\hat{\imath}(f), v)_A = \langle A\hat{\imath}(f), v \rangle$ for all $f \in V^*$ and all $v \in V$, i.e. $A\hat{\imath} = \hat{\imath}A = \mathrm{id}$. Hence A is invertible. $\qquad \square$

Lemma 3.2.7 (Laplacian fulfills Lax-Milgram conditions on H_0^1)

Let $V := H_0^1((a, b); \mathbb{R})$. Then

$$\alpha \colon V \times V \to \mathbb{R}, \ (u, v) \mapsto \int_a^b u'(x)v'(x)\,\mathrm{d}x$$

defines a symmetric, strongly positive bounded bilinear form on V.

Proof. The symmetry and bilinearity is clear.
Strong positivity: For $u \in V$ we have

$$\alpha(u, u) = \int_a^b u'(x)u'(x)\,\mathrm{d}x = \int_a^b |u'(x)|^2\,\mathrm{d}x = |u|_{1,2}^2. \qquad (13)$$

Boundedness: For $u, v \in V$ we have

$$|\alpha(u, v)| \leqslant \int_a^b |u'||v'|\,\mathrm{d}x \overset{\mathrm{CS}}{\leqslant} \|u'\|_{0,2}\|v'\|_{0,2} = |u|_{1,2}|v|_{1,2} \qquad (14)$$

so α is bounded with $\beta = 1$. $\qquad \square$

Consider

$$\begin{cases} -u''(x) = H(x), & x \in (-1, 1), \\ u(-1) = 0 = u(1), \end{cases}$$

where H is the Heaviside function. We consider $V := H_0^1((-1, 1); \mathbb{R})$ as above and define $\langle f, v \rangle := \int_0^1 v(x) \, dx$. Then $f \in V^* = H^{-1}((-1, 1); \mathbb{R})$; the linearity is clear and the boundedness follows from

$$\langle f, v \rangle = \int_0^1 v(x) \, dx \leqslant \|v\|_{0,1} \leqslant c\|v\|_{0,2} \overset{\text{PF}}{\leqslant} C|v|_{1,2}$$

for all $v \in H_0^1((-1, 1); \mathbb{R})$, where $c > 0$ exists by the continuous embedding $L^2((-1, 1); \mathbb{R}) \hookrightarrow L^1((-1, 1); \mathbb{R})$, so the problem is uniquely solvable by the Theorem of Lax-Milgram.

Example 3.2.8 (Nonuniqueness because of Neumann boundary conditions)
Now consider the same boundary value problem with homogeneous Neumann boundary conditions

$$\begin{cases} -u''(x) = H(x), & x \in (-1, 1), \\ u'(-1) = 0 = u'(1). \end{cases}$$

We choose $V = H^{-1}((-1, 1); \mathbb{R})$ and define a and $\langle f, \cdot \rangle$ exactly as above. The only difference to the Dirichlet problem above is the different norm. The bilinear form a is again bounded: for all $u, v \in V$ we have

$$a(u, v) \overset{(14)}{\leqslant} |u|_{1,2}|v|_{1,2} \leqslant \|u\|_{1,2}\|v\|_{1,2}.$$

But a is not strongly positive as the Poincaré-Friedrichs-Inequality does not hold on $H^1((a, b); \mathbb{R})$: we have

$$a(u, u) \overset{(13)}{=} |u|_{1,2}^2,$$

which we can't bound below by $C\|u\|_{1,2}$. The bilinear form α is only positive: for $u \equiv 1 \in H^1(a, b) \backslash H_0^1(a, b)$ we have

$$\alpha(u, u) = \int_a^b |u'(x)|^2 = 0$$

and $\|u\|_{1,2} = b - a + 0 > 0$, so the inequality $\alpha(u, u) \geqslant \mu\|u\|_{1,2}^2$ is only fulfilled for $\mu = 0$. But the problem is not uniquely solvable, as **TODO**

Thus the Theorem of Lax-Milgram can not be applied, as the seminorm $|\cdot|_{1,2}$ is not a norm on $H^1((-1, 1); \mathbb{R})$. Instead of $H^1((a, b); \mathbb{R})$ we can consider $\tilde{H} := H^1((a, b); \mathbb{R})/\sim$, where $u \sim 0$ if and only if u is constant almost everywhere. Then $(\tilde{H}, |\cdot|_{1,2})$ is a normed space, as then $|\cdot|_{1,2}$ is definite.

Another approach is to add a u to the left side, that is, we consider

$$\begin{cases} -u''(x) + u(x) = 0, & x \in (a, b), \\ u'(a) = u'(b) = 0 \end{cases}$$

because then $a(u, v) = \int_a^b u'(x)v'(x) + u(x)v(x) \, dx$ and thus $|a(u, u)| = \|u\|_{1,2}^2$.

Example 3.2.9 (Applications of the Lax-Milgram Theorem) ① Consider the boundary value problem

$$\begin{cases} -u''(x) = \delta_0(x) & \text{on } (-1, 1), \\ u(-1) = u(1) = 0. \end{cases}$$

Its variational formulation is given by $V := H_0^1$, $\langle f, v \rangle = \langle \delta_0, v \rangle = v(0)$. As $H_0^1((-1,1); \mathbb{R}) \hookrightarrow \mathcal{C}([-1,1]; \mathbb{R})$, the point evaluation and hence $f \in V^*$ is well defined, bounded and linear:

$$|\langle f, v \rangle| = |v(0)| \leqslant \|v\|_\infty \leqslant C|v|_{1,2},$$

where C is the embedding constant.

Now, let $A \colon V \to V^*$ be defined by

$$\langle Au, v \rangle = \int_{-1}^1 u'(x)v'(x) \, \mathrm{d}x = \alpha(u,v)$$

for $\alpha \colon V \times V \to \mathbb{R}$, which is bilinear since A is linear. Furthermore, α is bounded: for $u, v \in H_0^{-1}(-1,1)$

$$\alpha(u,v) = \int_{-1}^1 u'(x)v'(x) \, \mathrm{d}x \leqslant \|u\|_{0,2}\|v\|_{0,2} = |u|_{1,2}|v|_{1,2}.$$

This implies A is well defined and bounded:

$$\langle Au, v \rangle \leqslant |u|_{1,2}|v|_{1,2} \implies \|Au\|_* \leqslant \|v\|_{1,2} \implies \beta = 1.$$

Furthermore, a and A are strongly positive:

$$\alpha(u,u) = \langle Au, u \rangle = \int_{-1}^1 u'(x)u'(x) \, \mathrm{d}x = \int_{-1}^1 |u'(x)|^2 \, \mathrm{d}x = |u|_{1,2}^2.$$

This implies $\mu = 1$, meaning that the energy norm is equivalent to the $\| \cdot \|_{1,2}$ norm! By the Theorem of Lax-Milgram there exists a unique solution, which is $u(x) := \frac{1}{2}(1 - |x|)$.

② For $f \in L^2(0, \pi)$ consider the linear second order imhomogeneous boundary value problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} -u''(x) - u(x) = f(x) & x \in (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases}$$

The variational formulation is $V := H_0^1(0, \pi)$, $\langle f, v \rangle := \int_0^\pi f(x)v(x) \, \mathrm{d}x$ (here we are abusing notation, again: the f on the left hand side of the equation is the functional $\tilde{f}$ and the f on the right hand side a function!). Then $f \in H^{-1}(0, \pi)$. For $u, v \in V$ define

$$a(u,v) := \int_0^\pi u'(x)v'(x) - u(x)v(x) \, \mathrm{d}x,$$

which is bilinear, well-defined and bounded:

$$|a(u,v)| \overset{\triangle \neq}{\leqslant} \int_0^\pi |u'(x)v'(x)| + |u(x)v(x)| \, \mathrm{d}x \overset{\text{CS}}{\leqslant} |u|_{1,2}|v|_{1,2} + |u|_{0,2}|v|_{0,2}$$

$$\overset{\text{PF}}{\leqslant} \left(1 + \left(\frac{\pi - 0}{\pi}\right)^2\right) |u|_{1,2}|v|_{1,2} = 2|u|_{1,2}|v|_{1,2}.$$

Due to the Poincaré-Friedrichs inequality, a is positive: for $v \in V$ we have

$$a(v,v) = |v|_{1,2}^2 - \|v\|_{0,2}^2 \overset{\text{PF}}{\geqslant} |v|_{1,2}^2 - \frac{\pi - 0}{\pi}|v|_{1,2}^2 = 0.$$

But a is not strongly positive, as for $v := \sin \in V$ we have

$$a(v,v) = \int_0^\pi \cos^2(x) - \sin^2(x) \, \mathrm{d}x = 0.$$

Indeed the problem is not uniquely solvable for $f = 0$: the family $(u(x) := c\sin(x))_{c \in \mathbb{R}}$ solves the boundary value problem. For $f \equiv 1$, there is no solution.

Lemma 3.2.10 (Linear second order inhomogeneous boundary value problem)

Consider the linear second order inhomogeneous boundary value problem with homogeneous Dirichlet *boundary conditions*

$$\begin{cases} -u''(x) + c(x)u'(x) + d(x)u(x) = f(x), & x \in (a,b), \\ u(a) = 0 = u(b) \end{cases}$$

with $f \in H^{-1}((a,b);\mathbb{R})$, $c, c', d \in L^\infty((a,b);\mathbb{R})$ *such that there is a* $D \in \mathbb{R}$ *with*

$$d(x) - \frac{1}{2}c'(x) \geqslant D > -\frac{\pi^2}{(b-a)^2}.$$

for almost all $x \in (a,b)$. *Then the problem has a unique solution.*

Proof. We have already shown that with $V := H_0^1(a,b)$ that

$$\langle Au, v \rangle := a(u,v) := \int_a^b u'(x)v'(x) + c(x)u'(x)v(x) + d(x)u(x)v(x)\,\mathrm{d}x$$

for $u, v \in V$ is linear and bounded. It remains to show the strong positivity of a. For $u \in V$ we have (with partial integration)

$$a(u,u) = |u|_{1,2}^2 + \int_a^b c(x)u'(x)u(x) + d(x)|u(x)|^2\,\mathrm{d}x$$

$$= |u|_{1,2}^2 + \int_a^b \left(d(x) - \frac{1}{2}c'(x) \right)|u(x)|^2\,\mathrm{d}x$$

$$\geqslant |u|_{1,2} + D\|u\|_{0,2}^2 \begin{cases} \geqslant |u|_{1,2}^2, & \text{if } D \geqslant 0, \\ \overset{\text{PF}}{\geqslant} \underbrace{\left(1 + D\frac{(b-a)^2}{\pi^2} \right)}_{>0} |u|_{1,2}^2, & \text{if } D < 0 \end{cases}$$

$$= \underbrace{\min\left(1, 1 + D\frac{(b-a)^2}{\pi^2} \right)}_{>0} |u|_{1,2}^2. \qquad \square$$

Example 3.2.11 (Tut, Sturm–Liouville Problem)

Let $p, q \in \mathcal{C}([a,b])$. Furthermore let there be some $\mu > 0$ such that $p(x) \geqslant \mu$ for all $x \in [a,b]$ and $\min_{x \in [a,b]} q(x) > -\frac{\pi^2 \mu}{(b-a)^2}$.

① Then for each $f \in H^{-1}(a,b)$ there exists a unique weak solution $u \in H_0^1(a,b)$ of the Sturm–Liouville problem

$$-(p(x)u'(x))' + q(x)u(x) = f(x)$$

equipped with homogeneous Dirichlet boundary conditions:

The weak formulation is

$$\begin{cases} \text{Find } u \in H_0^1(a,b) \text{ such that} \\ \int_a^b p(x)u'(x)v'(x) + q(x)u(x)v(x)\,\mathrm{d}x = \langle f, v \rangle \ \forall v \in H_0^1(a,b). \end{cases}$$

Now define $A \colon H_0^1(a,b) \to H^{-1}(a,b)$ by $\langle Au, v \rangle = \int_a^b pu'v' + quv$, which is linear.

To see that A is bounded consider

$$|\langle Au, v \rangle| \overset{\text{(H)}}{\leqslant} \|p\|_\infty |u|_{1,2}|v|_{1,2} + \|q\|_\infty \|u\|_{0,2}\|v\|_{0,2} \overset{\text{PF}}{\leqslant}$$

$$\leqslant \max\Big(\|p\|_\infty, \frac{\pi^2}{(b-a)^2}\|q\|_\infty\Big)|u|_{1,2}|v|_{1,2}$$

which implies

$$\|Au\|_{H^{-1}} = \sup_{v \neq 0} \frac{|\langle Au, v\rangle|}{|v|_{1,2}} \leqslant \max(\|p\|_\infty, \|q\|_\infty)\|u\|_{1,2}\frac{\|v\|_{1,2}}{|v|_{1,2}}$$
$$\overset{\text{(PF)}}{\leqslant} C \cdot \max(\|p\|_\infty, \|q\|_\infty)\|u\|_{1,2}$$

Furthermore, A is strongly positive:

$$\langle Au, u\rangle = \int_a^b (u')^2 p + qu^2 \,\mathrm{d}x \geqslant \mu|u|_{1,2}^2 + \min_{x\in[a,b]} q(x)\|u\|_{0,2}^2$$
$$\overset{\text{(PF)}}{\geqslant} \underbrace{\Big(\mu - \frac{(b-a)^2}{2}\min_{x\in[a,b]} q(x)\Big)}_{=:c>0} |u|_{1,2}^2 \geqslant \tilde{c}\|u\|_{1,2}^2$$

② If $p \in \mathcal{C}^1[a,b]$ and $f \in \mathcal{C}[a,b]$ then $u \in \mathcal{C}^2[a,b]$: We have

$$\int_a^b pu'v' \,\mathrm{d}x = \int_a^b fv - quv \,\mathrm{d}x$$
$$= -\int_a^b \Big(\int_a^x f(\xi) - q(\xi)u(\xi) \,\mathrm{d}\xi\Big) v'(x) \,\mathrm{d}x.$$

The corollary from the Fundamental Theorem of the Calculus of Variations implies that

$$p(x)u'(x) = \int_a^x f(\xi) - q(\xi)u(\xi) \,\mathrm{d}\xi + C$$

almost every for some C. Diving by $p() > 0$ gives that u' is continuously differentiable since $p \in \mathcal{C}^1$. Hence $u \in \mathcal{C}^2[a,b]$ and analogously to the example 3.1.11 we have show its a classical solution.

3.3 Variational problems with a strongly monotone operator

DEFINITION 3.3.1 ((STRONGLY) MONOTONE OPERATOR)

Let $(V, \|\cdot\|)$ be a real BANACH space. An operator $A\colon V \to V^*$ is

- LIPSCHITZ continuous if there exists a $\beta > 0$ such that

$$\|Au - Av\|_* \leqslant \beta\|u - v\| \ \forall u, v \in V.$$

- monotone if $\langle Au - Av, u - v \rangle \geqslant 0$ for all $u, v \in V$.

- strongly monotone if there exists a $\mu > 0$ if

$$\langle Au - Av, u - v \rangle \geqslant \mu\|u - v\|^2 \ \forall u, v \in V.$$

LIPSCHITZ
continuous

monotone

strongly monotone

Remark 3.3.2 Let $V := \mathbb{R} = \mathbb{R}^*$ and $A\colon \mathbb{R} \to \mathbb{R}$ be monotone in the above sense. For all $u, v \in \mathbb{R}$ we have

$$0 \leqslant \langle Au - Av, u - v \rangle = (A(u) - A(v))(u - v).$$

Then $u > v$ implies $A(u) - A(v) \geqslant 0$, i.e. $A(u) \geqslant A(v)$ and $u < v$ implies $A(u) - A(v) \leqslant 0$, i.e. $A(u) \leqslant A(v)$. Thus, in operators terms, (strongly) monotone is analogous to (strictly) monotonically *increasing*.

Remark 3.3.3 Let A be linear. Then (cf. DGL I)

① A LIPSCHITZ continuous $\iff$ A bounded and

② A strongly monotone $\iff$ A strongly positive

hold.

THEOREM 3.3.1: ZARANTONELLO (1960)

Let $(V, (\cdot, \cdot), \|\cdot\|)$ be a (real) HILBERT space and $A\colon V \to V^*$ LIPSCHITZ continuous and strongly monotone. Then A is bijective.

Remark 3.3.4 Let $V := \mathbb{R}$ and f be LIPSCHITZ continuous and strongly monotone as in the Theorem above. Then $f\colon V \to V^*$ is (LIPSCHITZ-)continuous and strongly monotonically increasing and thus injective. Additionally, the strong monotonicity implies (by setting $v = 0$)

$$\mu u^2 \leqslant (f(u) - f(0)) \cdot u$$

for some $\mu > 0$. A case distinction reveals

$$f(u) \begin{cases} \geqslant \mu u + f(0), & \text{if } u > 0, \\ \leqslant \mu u + f(0), & \text{if } u < 0 \end{cases}$$

implying $f(u) \xrightarrow{u \nearrow \infty} \infty$ and $f(u) \xrightarrow{u \searrow -\infty} -\infty$, implying the surjectivity and therefore the bijectivity of f.

Proof. Let $\hat{\imath}\colon V^* \to V$ be the Riesz isomorphism and define

$$\Phi\colon V \to V, \ v \mapsto v + \tau\hat{\imath}(f - Av),$$

where $\tau > 0$ is chosen such that $1 - 2\tau\mu + \tau^2\beta^2 < 1$.

Analogously to the proof of the Theorem of Lax-Milgram we only need to show that Φ is a contraction: for $u, v \in V$ we have

$$
\begin{aligned}
\|\Phi(u) - \Phi(v)\|^2 &= \|u - v + \tau\hat{\imath}(Av - Au)\|^2 \\
&= \|u - v\|^2 + 2\tau\left(\hat{\imath}(Av - Au), u - v\right) + \tau^2\|\hat{\imath}(Au - Av)\|^2 \\
&= \|u - v\|^2 - 2\tau\left\langle Au - Av, u - v\right\rangle + \tau^2\|Au - Av\|_*^2 \\
&\leqslant \underbrace{\left(1 - 2\tau\mu + \tau^2\beta^2\right)}_{<1}\|u - v\|^2. \qquad \square
\end{aligned}
$$

Example 3.3.5 (Boundary value problem in divergence form with DBCs)

Consider the divergence form of a boundary value problem

$$
\begin{cases}
-(\Psi(|u'(x)|)u'(x))' + c(x)u'(x) + d(x)u(x) = f(x) & \text{on } (a, b), \\
u(a) = u(b) = 0.
\end{cases}
\tag{15}
$$

We assume $f \in L^2(a, b)$, $c, c', d \in L^\infty(a, b)$ and that $\Psi\colon [0, \infty) \to \mathbb{R}$ is continuous and that there exists $m, M > 0$ such that

1. $m \leqslant |\Phi(t)| \leqslant M$ for all $t \in [0, \infty)$,

2. $|\Psi(t) \cdot t - \Psi(s) \cdot s| \leqslant M|t - s|$ for all $s, t \geqslant 0$ and

3. $\Psi(t) \cdot t - \Psi(s) \cdot s \geqslant m(t - s)$ for all $t \geqslant s \geqslant 0$.

Then, the function $t \mapsto \Psi(t) \cdot t$ is Lipschitz-continuous, strictly monotonically increasing and the function Ψ is bounded from above by M and from below by m (take $s = 0$ in ③).

To obtain a variational formulation we choose $V := H_0^1(a, b)$ to get

$$\int_a^b \Psi(|u'|)u'v' + cu'v + duv \, dx = \int_a^b fv \, dx$$

for $v \in V$.

> **Theorem 3.3.2: TODO**
>
> Under the above condition if the weak derivative c' exists, $c' \in L^\infty(a, b)$ holds and there exists a $\hat{d} \in \left[-\frac{\pi^2}{(b-a)^2}, d(x) - \frac{c'(x)}{2}\right]$ for almost all $x \in [a, b]$, the the problem (15) is unique solvable in $H_0^1(a, b)$.

Proof. We set

$$A\colon V \to V^*, \qquad \left\langle Au, v\right\rangle = \int_a^b \Psi(|u'(x)|)u'(x)v'(x) + c(x)u'(x)v(x) + d(x)u(x)v(x) \, dx.$$

Then, A is Lipschitz continuous since for $v, w \in V$

$$\left|\left\langle Au - Aw, v\right\rangle\right| \leqslant \int_a^b \left|\Psi(|u'|)u' - \Psi(|w'|)w'\right| |v'| + |c||u'||v| + |d||u||v| \, dx$$

$$\overset{\text{(CS)}}{\leqslant} \left(\int_a^b \left| \Psi(|u'(x)|)u'(x) - \Psi(|w'(x)|)w'(x) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \|v'\|_{0,2}$$
$$+ \|c\|_{0,\infty} \|u' - w'\|_{0,2} \|v\|_{0,2} + \|d\|_{0,\infty} \|u - w\|_{0,2} \|v\|_{0,2}.$$

holds. We now estimate the remaining integral term.

Case 1: $u'(x), w'(x) \geqslant 0$.

$$\left| \Psi(|u'|)u' - \Psi(|w'|)w' \right| \overset{\text{②}}{\leqslant} M|u'(x) - w'(x)|$$

Case 2: W.l.o.g $u'(x) \geqslant 0 \geqslant w'(x)$.

$$\left| \Psi(|u'|)u' - \Psi(|w'|)w' \right| \overset{\triangle \neq}{\leqslant} \left| \Psi(|u'|) \right| |u'| - \left| \Psi(|w'|) \right| |w'|$$
$$\overset{\text{②}}{\leqslant} Mu'(x) - Mw'(x) = M|u'(x) - w'(x)|.$$

Therefore,

$$\left| \langle Au - Aw, v \rangle \right| \leqslant M \|u' - w'\|_{0,2} \|v'\|_{0,2} + \|c\|_{0,\infty} \|u' - w'\|_{0,2} \|v\|_{0,2}$$
$$+ \|d\|_{0,\infty} \|u - w\|_{0,2} \|v\|_{0,2}$$
$$\overset{\text{(PF)}}{\leqslant} C \left(M + \|c\|_{0,\infty} + \|d\|_{0,\infty} \right) |u - v|_{1,2} |v|_{1,2}.$$

Thus

$$\|Au - Av\|_{-1,2} = \sup_{\substack{v \in H_0^1(a,b) \\ v \neq 0}} \frac{\langle Aw - Av, v \rangle}{|v|_{1,2}}$$
$$\leqslant C(M + \|c\|_\infty + \|d\|_\infty)|u - w|_{1,2},$$

implying that A is Lipschitz continuous.

It remains to show the strong monotonicity of A. For $v, w \in V$ we have

$$\langle Au - Aw, u - v \rangle = \int_a^b \left(\Psi(|u'(x)|)u'(x) - \Psi(|w'(x)|)w'(x) \right) (u'(x) - w'(x)) \, \mathrm{d}x$$
$$+ \int_a^b c(x)(u(x) - w(x))'(u(x) - w(x)) \, \mathrm{d}x + \int_a^b d(x)(u(x) - w(x))^2 \, \mathrm{d}x.$$

The chain rules gives $(u - w)'(u - w) = \frac{((u-w)^2)'}{2}$. Partial integration yields

$$\int_a^b c(u - w)'(u - w) \, \mathrm{d}x = -\frac{1}{2} \int_a^b c'(u - w)^2 \, \mathrm{d}x.$$

Therefore, with $z := \int_a^b \left(\Psi(|u'|)u' - \Psi(|w'|)w' \right)(u' - w') \, \mathrm{d}x$ we obtain

$$\langle Au - Aw, u - v \rangle = z + \int_a^b \left(d - \frac{c'}{2} \right)(u - w)^2 \, \mathrm{d}x$$
$$\geqslant z + \hat{d} \|u - w\|_{0,2}^2. \, \mathrm{d}x.$$

Now, we estimate $\left(\Psi(|u'|)u' - \Psi(|w'|)w' \right)(u' - w')$.

Case 1: $u'(x) \geqslant w'(x) \geqslant 0$. We have

$$y(x) := \left(\Psi(|u'(x)|)u'(x) - \Psi(|w'(x)|)w'(x) \right)(u'(x) - w'(x)) \geqslant m|u'(x) - w'(x)|^2.$$

The cases $w'(x) \geqslant u'(x) \geqslant 0$, $u'(x) \leqslant w'(x) \leqslant 0$ and $w'(x) \leqslant u'(x) \leqslant 0$ are analogous to case 1.

Case 2: w.l.o.g $u'(x) \leqslant 0 \leqslant w'(x)$. Since $\Psi(t) \geqslant m$ for all $t \geqslant 0$ we have

$$y = \Psi(|u'|) \underbrace{u'}_{\leqslant 0} \underbrace{(u' - w')}_{\leqslant 0} - \Psi(|w'|) \underbrace{w'}_{\geqslant 0} \underbrace{(u' - w')}_{\leqslant 0}$$

$$\geqslant mu'(u' - w') - w'(u' - w') = m(u' - w')^2.$$

In conclusion we notice that if $\hat{d}$ can be chosen non negatively, we have

$$(Au - Aw, u - w) \geqslant m|u - w|_{1,2}^2 + \underbrace{\hat{d}\|u - w\|_{0,2}^2}_{\geqslant 0} \geqslant m|u - w|_{1,2}^2.$$

If not, we have

$$(Au - Aw, u - w) \geqslant m|u - w|_{1,2}^2 + \hat{d}\|u - w\|_{0,2}^2 \overset{(\text{PF})}{\geqslant} \underbrace{\left(m + \hat{d}\frac{\pi^2}{(b-a)^2}\right)}_{>0} |u - w|_{1,2}^2. \qquad \square$$

4 GALERKIN-Schemes and Finite Elements

4.1 GALERKIN schemes and GALERKIN bases

In the following, let $(V, \|\cdot\|)$ be a BANACH space.

DEFINITION 4.1.1 (GALERKIN SCHEME AND GALERKIN BASIS)

The sequence $(V_n \subset V)_{n \in \mathbb{N}}$ of finite-dimensional subspaces is a GALERKIN scheme if it is "complete in the limit", that is, the approximation error vanishes:

$$\lim_{n \to \infty} \operatorname{dist}(V_n, v) = 0 \quad \forall v \in V.$$

GALERKIN scheme

A pairwisely linearly independent sequence $(\Phi_k)_{k \in \mathbb{N}} \subset V$ is called GALERKIN basis if $(V_n)_{n \in \mathbb{N}}$ is a GALERKIN scheme, where $V_n := \operatorname{span}((\Phi_k)_{k=1}^n)$.

GALERKIN basis

Remark 4.1.2 We have $V = \overline{\bigcup_{n \in \mathbb{N}} V_n}$. The V_n must not be nested.

Example 4.1.3 (GALERKIN scheme and GALERKIN basis)
The real polynomials of degree less than or equal to $n \in \mathbb{N}$ are a GALERKIN-Base of $\mathcal{C}(\mathbb{R}, \mathbb{R})$. The same is true for the trigonometric polynomials in L^2, where the trigonometric monomials form a GALERKIN-Base.

THEOREM 4.1.1: EXISTENCE OF GALERKIN BASIS

Every separable space has a GALERKIN basis.

Proof. Let V be a separable space. Then there exists a countable dense subset $(\Psi_i)_{i \in \mathbb{N}}$. We set $\Phi_1 := \Psi_1$ and $V_1 := \operatorname{span}(\Phi_1)$. We iteratively define $\Phi_{k+1} = \Psi_k$ with $k = \min\{\ell \in \mathbb{N} : w_\ell \notin V_n\}$ and $V_{n+1} := \operatorname{span}(V_n \cup \{\Phi_{n+1}\})$. For $v \in V$ and $\varepsilon > 0$ there exists a Ψ_i with $\|v - \Psi_i\| < \varepsilon$. For large enough m we thus have $\Psi_i \in V_m$ and hence $\operatorname{dist}(v, V_m) < \varepsilon$.

Remark 4.1.4 (Notation) As in practice the V_n arise from some discretisation process with a parameter $h \searrow 0$, we will write V_h instead of V_n.

Example 4.1.5 (Discretising bilinear form problems)
We consider an abstract problem

$$\begin{cases} \text{for } f \in V^* \text{ find } u \in V \text{ such that} \\ a(u, v) = \langle f, v \rangle \ \forall v \in V \end{cases} \tag{p}$$

The restriction $f|_{V_h} =: f_{V_h} : V_h \to \mathbb{R}$ is also linear and bounded since the norm on V_h is the norm on V. Therefore $f_{V_n} \in V_h^*$. Hence we may consider the discretized problem

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a|_{V_h \times V_h}(u_h, v_h) = \langle f_{V_h}, v_h \rangle \ \forall v_h \in V_h. \end{cases} \tag{p_D}$$

Example 4.1.6 (Discretising operator problems)

Consider the operator problem

$$\begin{cases} \text{for } f \in V^* \text{ find } u \in V \text{ such that} \\ Au = f \text{ in } V^*. \end{cases} \tag{$\tilde{p}$}$$

We have $V_h \subset V$ and thus $V^* \subset V_h^*$. Then, $P_h \colon V_h \to V$ defined by $P_h v_h = v_h$ is called prolongation operator. Its dual operator, the reconstruction operator, $P_h^* \colon V^* \to V_h^*$ is defined by

$$\langle P_h^* g, v_h \rangle := \langle g, P_h v_h \rangle, \quad g \in V^*, v_h \in V_h.$$

This means that $P_h^* g$ is the restriction of g to V_h. Then,

$$\alpha(u_h, v_h) = \langle AP_h u_h P_h, v_h \rangle = \langle P_h^* AP_h u_h, v_h \rangle$$

holds. Here the discretized problem reads

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ P_h^* AP_h u_h = P_h^* f, \text{ in } V_h^* \end{cases} \tag{$\tilde{p}_D$}$$

prolongation operator

THEOREM 4.1.2: LEMMA OF CÉA (1964)

Let V be a real HILBERT space and $V_h \subset V$ a closed subspace (e.g. a finite dimensional subspace). If $a \colon V \times V \to \mathbb{R}$ is bilinear, strongly positive and bounded, then the restriction $a|_{V_h \times V_h} \colon V_h \times V_h \to \mathbb{R}$ is, too. Let $f \in V^*$ and $u \in V$ be the solution of

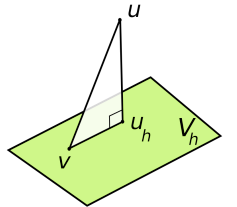$$a(u, v) = \langle f, v \rangle \qquad \forall v \in V. \tag{16}$$

Then there exists a solution $u_h \in V_h$ of

$$a(u_h, v_h) = \langle f, v_h \rangle \qquad \forall v_h \in V_h. \tag{17}$$

Then we have

$$\|u - u_h\| \leqslant \frac{\beta}{\mu} \inf_{v_h \in V_h} \|u - v_h\| = \frac{\beta}{\mu} \operatorname{dist}(u, V_h),$$

where β and μ come from Definition 3.2.1.

Proof. By the Theorem of LAX-MILGRAM both problems have unique solutions $u \in V$ and $u_h \in V_h$, respectively. For any $v_h \in V_h$ we have

$$a(u_h, v_h) = \langle f, v_h \rangle = a(u, v_h).$$

Hence $a(u - u_h, v_h) = 0$ for all $v_h \in V_h$, i.e $u - u_h \perp_a V_h$ with respect to the inner product $a(\cdot, \cdot)$ (cf. diagram). This relation is called GALERKIN orthogonality.

Hence for all $v_h \in V_h$ we have

$$\begin{aligned} \mu\|u - u_h\|^2 &\leqslant a(u - u_h, u - u_h) = a(u - u_h, u) - a(u - u_h, u_h) \\ &= a(u - u_h, u) - 0 = a(u - u_h, u) - a(u - u_h, v_h) \\ &= a(u - u_h, u - v_h) \leqslant \beta\|u - u_h\|\|u - v_h\|. \end{aligned} \qquad \square$$

Figure 17: The subspace solution u_h is the projection of u onto the subspace V_h in respect to the inner product $a(\cdot, \cdot)$. [Source: Wiki/Céa's lemma]

Lemma 4.1.7 (Better constant if a is symmetric)

If a is symmetric, then we can use $\sqrt{\frac{\beta}{\mu}}$ instead of $\frac{\beta}{\mu}$ in the above inequality.

This constant is better, as we always have $\beta \geqslant \mu$.

Proof. As a is bilinear, bounded, symmetric and strongly positive, it is an inner product on V and $\|\cdot\|_a^2 := a(\cdot, \cdot)$ is a norm on V equivalent to $\|\cdot\|$ (cf. (12)) satisfying the Cauchy-Schwarz inequality $a(v, w) \leqslant \|v\|_a \|w\|_a$ for all $v, w \in V$. By the Galerkin orthogonality (G) we thus have

$$\|u - u_h\|^2 = a(u - u_h, u - u_h) \overset{(G)}{=} a(u - u_h, u - v_h) \overset{CS}{\leqslant} \|u - u_h\|_a \|u - v_h\|_a$$

and hence $\|u - u_h\|_a \leqslant \|u - v_h\|_a$ for all $v_h \in V_h$. Hence we can modify the proof of the Lemma:

$$\mu\|u - u_h\|^2 \leqslant a(u - u_h, u - u_h) = \|u - u_h\|_a^2 \leqslant \|u - v_h\|_a^2 \leqslant \beta\|u - v_h\|^2$$

and thus $\|u - u_h\| \leqslant \sqrt{\frac{\beta}{\mu}}\|u - v_h\|$ for all $v_h \in V_h$. $\qquad\square$

Remark 4.1.8 (Discrete solutions approximate solution)

Let $(V_n)_{n\in\mathbb{N}}$ be a Galerkin-scheme. Then we have $\text{dist}(v, V_h) \xrightarrow{h\to 0} 0$ for all $v \in V$. Cea's Lemma implies that $\|u - u_h\| \xrightarrow{h\searrow 0} 0$. But how fast is the convergence?

4.2 The Finite Elements Method

Consider

$$\begin{cases} -u'' = f \text{ on } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{18}$$

Then $V := H_0^1(0,1)$, $a \colon V \times V \to \mathbb{R}$, $(u,v) \mapsto (u,v)_{1,2}$ and $\langle f,v \rangle := \int_0^1 fv\,\mathrm{d}x$ for $v \in V$ is its variational formulation.

We now apply the finite elements method (FEM) to find a Galerkin scheme. For $m \in \mathbb{N}$ we set the step size $h := \frac{1}{m+1}$ and $x_i := i \cdot h$ as the partition of the interval $[0,1]$ for $i \in \{0, \ldots, m+1\}$.

finite elements method

For $i \in \{1, \ldots, m\}$ define

$$\Phi_i(x)^{(h)} := \begin{cases} \frac{1}{h}(x - x_{i-1}), & \text{if } x \in [x_{i-1}, x_i), \\ \frac{1}{h}(x_{i+1} - x), & \text{if } x \in [x_i, x_{i+1}], \\ 0, & \text{else}, \end{cases}$$

which fulfill $\Phi_j(x_i) = \delta_{i,j}$ and

$$\Phi_0^{(h)}(x) := \frac{x_1 - x}{h} \mathbb{1}_{[0,x_1]}(x), \qquad \Phi_{m+1}^{(h)}(x) := \frac{x - x_m}{h} \mathbb{1}_{[x_m,1]}(x).$$

Figure 18: The hat function Φ_i with height 1 in blue. [Source: Wiki/Céa's lemma]

We set $V_h := \text{span}\,((\Phi_k)_{k=1}^m) \subset H_0^1(0,1)$ (we discard $\Phi_0^{(h)}$ and $\Phi_{m+1}^{(h)}$ because of the homogeneous Dirichlet boundary conditions) and show that they form a Galerkin basis.

The discretised problem reads: find a $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle f, v_h \rangle \ \forall v_h \in V_h.$$

As $\{\Phi_1^{(h)}, \ldots, \Phi_m^{(h)}\}$ form a basis of V_h and by linearity of $a(u_n, \cdot)$ and $\langle f, \cdot \rangle$ the discretised problem is equivalent to

$$a(u_h, \Phi_i^{(h)}) = \langle f, \Phi_i^{(h)} \rangle \ \forall i \in \{1, \ldots, m\}.$$

As $\{\Phi_1^{(h)}, \ldots, \Phi_m^{(h)}\}$ form a basis of V_h, we can write the discretised solution $u_h \in V_h$ we are searching for as

$$u_h = \sum_{j=1}^m \bar{u}_i^{(h)} \Phi_i^{(h)}, \qquad \bar{u}_i^{(h)} \in \mathbb{R} \ \forall i \in \{1, \ldots, m\}.$$

In order to find u_h, we only need to find the coefficients $(\bar{u}_i^{(h)})_{i=1}^m$. The problem is thus equivalent to: find $\bar{u}^{(h)} := (\bar{u}_i^{(h)})_{i=1}^m \in \mathbb{R}^m$ such that

$$\sum_{j=1}^m \bar{u}_j^{(h)} a(\Phi_j^{(h)}, \Phi_i^{(h)}) = \langle f, \Phi_i^{(h)} \rangle \qquad \forall i \in \{1, \ldots, m\}$$

by the linearity of $a(\cdot, \Phi_i)$.

With $A_h := (a(\Phi_j^{(h)}, \Phi_i^{(h)}))_{i,j=1}^m \in R^{m \times m}$, $f_h := (\langle f, \Phi_i^{(h)} \rangle)_{i=1}^m \in \mathbb{R}^m$. We therefore have to solve the linear problem
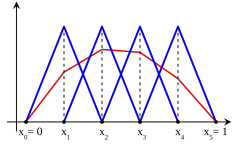
$$A_h \bar{u}^{(h)} = f_h$$

in $\mathbb{R}^m$

The matrix A_h has a particularly nice structure: for $i, j \in \{1, \dots, m\}$ we have

$$(A_h)_{i,j} = a(\Phi_j^{(h)}, \Phi_i^{(h)}) = \int_0^1 \Phi_j^{(h)\prime}(x) \Phi_i^{(h)\prime}(x) \, dx = 0$$

for $i < j - 1$ or $i > j + 1$ since $\text{supp}(\Phi_j^{(h)\prime}) = [x_{j-1}, x_{j+1}]$ and $\text{supp}(\Phi_i^{(h)\prime}) = [x_{i-1}, x_{i+1}]$. Hence A_h is a tridiagonal matrix:

$$A_h = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix},$$

which is even strictly diagonally dominant.

The functions $((\Phi_i^{(h)})_{i=1}^m)_{h>0}$ are not a Galerkin basis, but one can instead consider $(V_{2^m})_{m \in \mathbb{N}}$. How fast does $(u_h)_{h>0}$ converge? By Cea's lemma, we have the following bound on the approximation error

$$|u - u_h|_{1,2} \leqslant \text{dist}(u, V_h),$$

as $\beta = \mu = 1$. We will now bound that approximation error by a interpolation error.

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DEFINITION 4.2.1 (INTERPOLATION OPERATOR)

We call

$$I_h \colon V \to V_h, \qquad u \mapsto \sum_{j=1}^m u(x_i) \Phi_i$$
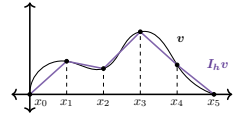
the interpolation operator.

Figure 19: The interpolation operator

Remark 4.2.2 I_h is well-defined since $v \in H_0^1(a,b) \overset{c}{\hookrightarrow} \mathcal{C}([a,b])$ and also linear.

As $I_h u \in V_h$, we have

$$|u - u_h|_{1,2} \leqslant \text{dist}(u, V_h) \leqslant \underbrace{|u - I_h u|_{1,2}}_{\substack{\text{Interpolation} \\ \text{error}}}.$$

THEOREM 4.2.1: INTERPOLATION ERROR: LINEAR FEM IS COMPLETE IN THE LIMIT

The sequence of (linear) FEM spaces $(V_h)_{h \in (0,1)}$ with an equidistant grid is a Galerkin scheme in V, that is, $|u - I_h u|_{1,2} \xrightarrow{h \searrow 0} 0$. For each $m \in \mathbb{N}$, $h := \frac{b-a}{m+1}$ and $v \in H_0^1(a,b) \cap H^2(a,b)$ we have

$$\|v - I_h v\|_{1,2} \leqslant ch\|v\|_{2,2} \qquad \text{(linear convergence rate)}$$
$$\|v - I_h v\|_{0,2} \leqslant ch^2\|v\|_{2,2}. \qquad \text{(quadratic convergence rate)}$$

Remark 4.2.3 The hat functions are not a Galerkin basis since they are not included in each other.

Proof. ① We show that I_h is bounded. We see that for $x \in [x_{i-1}, x_i]$ and $h \in (0,1)$ we have due to the support of the hat functions

$$(I_h v)(x) = v(x_{i-1}) + \frac{x - x_{i-1}}{2} \big(v(x_i) - v_{x_{i-1}}\big)$$

and

$$(I_h v)'(x) = \frac{1}{h}\big(v(x_i) - v_{x_{i-1}}\big)$$

hold. For fixed $h = \frac{b-a}{m+1}$ we have

$$
\begin{aligned}
|I_h(v)|_{1,2}^2 &= \int_a^b |(I_h(v))'(x)|^2 \, \mathrm{d}x \\
&= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} \frac{1}{h^2} \big((v(x_i) - v(x_{i-1}))\big)^2 \, \mathrm{d}x \\
&\overset{\mathrm{FTOC}}{=} \sum_{i=1}^{m+1} \frac{1}{h^2} \int_{x_{i-1}}^{x_i} \left(\int_{x_{i-1}}^{x_i} v'(\xi) \, \mathrm{d}\xi \right)^2 \, \mathrm{d}x \\
&\overset{(\mathrm{H})}{\leqslant} \frac{1}{h^2} \sum_{i=1}^{m+1} h^2 \int_{x_{i-1}}^{x_i} (v'(\xi))^2 \, \mathrm{d}\xi = |v|_{1,2}^2.
\end{aligned}
$$

Hence $\|I_h\|_{L(V,V_h)} \leqslant 1$ holds.

(2) We show the first inequality. Let $v \in H_0^1 \cap H^2$. Then

$$
\begin{aligned}
|v - I_h v|_{1,2}^2 &= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} \left(v'(x) - \frac{1}{h}\big(v(x_{i-1}) - v(x_i)\big) \right)^2 \, \mathrm{d}x \\
&= \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} \left(\frac{1}{h} \int_{x_{i-1}}^{x_i} v'(x) - v'(\xi) \, \mathrm{d}\xi \right)^2 \, \mathrm{d}x \\
&= \frac{1}{h^2} \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} \left(\int_{x_{i-1}}^{x_i} \int_\xi^x v''(\varphi) \, \mathrm{d}\varphi \, \mathrm{d}\xi \right)^2 \, \mathrm{d}x \\
&\overset{(\mathrm{H?})}{\leqslant} \frac{h}{h^2} \sum_{i=1}^{m+1} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} \left(\int_{\xi \wedge x}^{\xi \vee x} v''(\varphi) \, \mathrm{d}\varphi \right)^2 \, \mathrm{d}\xi \, \mathrm{d}x \\
&\leqslant \frac{h^2}{h^2} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} |v''(\varphi)|^2 \, \mathrm{d}\varphi \, \mathrm{d}\xi \, \mathrm{d}x = h^2 |v''|_{0,2}^2
\end{aligned}
$$

holds.

(3) We show that $(V_h)_{h \in (0,1)}$ is a Garlerkin scheme. Let $v \in H_0^1(a,b) = \overline{\mathcal{C}_0^\infty(a,b)}^{|\cdot|_{1,2}}$ and $\varepsilon > 0$. Then there exists a $\psi \in \mathcal{C}_0^\infty(a,b)$ such that

$$|v - \psi|_{1,2} < \frac{\varepsilon}{3}$$

and

$$
\begin{aligned}
|v - I_h v|_{1,2} &\overset{(\mathrm{L})}{\leqslant} |v - \psi|_{1,2} + |\psi - I_h \psi|_{1,2} + |I_h(\psi - v)|_{1,2} \\
&\leqslant \frac{\varepsilon}{3} + h\|\psi\|_{2,2} + |\psi - v|_{1,2} \overset{\mathrm{ex.}}{\leqslant} \frac{2\varepsilon}{3} + h\|\psi\|_{2,2} < \varepsilon.
\end{aligned}
$$

for h sufficiently small ($h \in \left(0, \frac{\varepsilon}{3\|\psi\|_{2,2}}\right)$). Hence for all $\varepsilon > 0$ there exists a $m \in \mathbb{N}$ such that $h = \frac{b-a}{m+1}$ and $\mathrm{dist}(V, V_h) \leqslant |v - I_h v|_{1,2} < \varepsilon$ for all $m \geqslant m_0$.

Since by Céa's lemma we have

$$\underbrace{\|u - u_n\|_V}_{\substack{\text{discretization} \\ \text{error}}} \leqslant \underbrace{\frac{\beta}{\mu} \, \mathrm{dist}(u, V_n)}_{\substack{\text{approximation} \\ \text{error}}} \leqslant \frac{\beta}{\mu} \underbrace{\|v - I_h v\|_V}_{\substack{\text{interpolation} \\ \text{error}}},$$

we get $u_h \xrightarrow{h \to 0} u$ in V. □

Corollary 4.2.4 (todo)

Let $(V_h)_{h \in (0,1)}$ be defined as above and $u \in H_0^1(a,b)$ as the weak solution to (18). Then the sequence $(u_h)_h$ of FEM solutions converges to u with respect to $\|\cdot\|_V$. If furthermore $u \in H_0^1(a,b) \cap H^2(a,b)$, then

$$\|u - u_h\| \leqslant ch\|u\|_{2,2}$$

holds for all $h \in (0,1)$.

Remark 4.2.5 The regularity assumption $v \in H_0^1(a,b) \cap H^2(a,b)$ will be fulfilled in $d = 1$ or suitable assumptions in the domain ($\to$ later). For $f \in L^2(a,b)$ we have

$$\|u\|_{2,2} \leqslant c\|f\|_{0,2}$$

How does the error behave in the L^2-norm? By the Poincaré-Friedrichs-inequality we have

$$\|u - u_h\|_{0,2} \overset{\text{PF}}{\leqslant} c|u - u_h|_{1,2} \leqslant ch\|u\|_{2,2}.$$

Hence the error converges at with the same rates as in H_0^1.

Can we get a better rate?

THEOREM 4.2.2: QUADRATIC CONVERGENCE IN L^2

Under the above assumptions if $u \in H_0^1(a,b) \cap H^2(a,b)$ then

$$\|u - u_h\|_{0,2} \leqslant ch^2\|u\|_{2,2}$$

holds for all $h \in (0,1)$.

Proof. (Aubin-Nitsche Trick) Let $u \in V$, $u_h \in V_h$ as above. Consider the "dual problem"

$$\begin{cases} \text{Find } w \in H_0^1(a,b) \text{ such that} \\ \alpha(u,v) = \langle u - u_h, v \rangle \ \ \forall v \in V \end{cases} \qquad (V')$$

Using the theorem of Lax-Milgram we obtain a unique solution $w \in H_0^1(a,b)$ to (V') and by the estimate from above

$$\|w\|_{2,2} \leqslant C\|u - u_h\|_{0,2}$$

We test with $v = e := u - u_h$ and obtain

$$\|u - u_h\|_{0,2}^2 = \|e\|_{0,2}^2 = (e,e)_{0,2} = \langle e, e \rangle_{V^* \times V} = \alpha(e,w)$$
$$= \alpha(e,w) - \underbrace{\alpha(e,v_n)}_{=0 \ (\star)} = \alpha(e, w - v_h) \leqslant \beta|e|_{1,2}|w - v_h|_{1,2}$$
$$\leqslant \beta ch\|u\|_{1,2}|w - v_h|_{1,2},$$

where $(\star)$ refers to Galerkin orthogonality, i.e. $\alpha(u - u_h, v_h) = 0$ for all $v_h \in V_h$ by choice of u_h.

Taking the infimum over all $v_h \in V_h$ yields

$$\|u - u_h\|_{0,2} \leqslant c\beta h\|u\|_{2,2}\,\text{dist}(w, V_h) \leqslant c\beta h\|u\|_{2,2}\|w - I_h\|_{1,2}$$
$$\leqslant c\beta h\|u\|_{2,2}\|ch\|w\|_{2,2} \leqslant c\beta h^2\|u\|_{2,2}\|u - u_h\|_{0,2}.$$

Dividing by $\|u - u_h\|_{0,2}$ yields the statement. $\qquad\qquad \square$

5 Boundary value problems in multiple space dimensions

5.1 Multidimensional SOBOLEV spaces and weak derivatives

We replace the interval (a, b) with a open (and thus measurable) connected set $\Omega \subset \mathbb{R}^d$. Analogously to the first section we define

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$$L^p(\Omega) := \left\{ u \colon \Omega \to \mathbb{R} : u \text{ measurable } \|u\|_{0,p}^p := \int_\Omega |u(x)|^p \, \mathrm{d}x < \infty \right\},$$

which is a BANACH space for $p \in [1, \infty]$, separable for $p \in [1, \infty)$ and reflexive for $p \in (1, \infty)$. If e.g. Ω is bounded, then $L^q(\Omega) \hookrightarrow L^p(\Omega)$ for $q \geqslant p$ and thus $W^{k,q}(\Omega) \hookrightarrow W^{k,p}(\Omega)$ for $k \in \mathbb{N}$ (cf. Definition later). Analogously we define

$$L_{\mathrm{loc}}^1(\Omega) := \{ u \colon \Omega \to \mathbb{R} \text{ measurable} : u|_K \in L^1(K) \ \forall K \subset \Omega \text{ compact} \}$$

and

$$\mathcal{C}_0^\infty(\Omega) := \{ u \in \mathcal{C}^\infty(\Omega) : \mathrm{supp}(u) \subset \Omega \text{ compact} \}.$$

If Ω is bounded, we can equivalently write $u|_K \in L^1(K)$ for all $K \subset \Omega$ such that $\overline{K} \subset \Omega$.

If Ω is bounded, $\mathrm{supp}(u)$ is always compact.

We also write $K \subset\subset \Omega$ for a compact subset of Ω.

Notation (Multiindices). Let $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $\beta := (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$, where $\mathbb{N}$ includes 0. Then $\alpha + \beta := (\alpha_k + \beta_k)_{k=1}^d$ and $|\alpha| := \sum_{k=1}^d \alpha_k$ and $\alpha! := \prod_{k=1}^d (\alpha_k!)$. For $h = (h_1, \ldots, h_d) \in \mathbb{R}^d$ we set $h^\alpha := \prod_{k=1}^d h_k^{\alpha_k}$.

Furthermore, we define $\alpha \leqslant \beta :\iff \alpha_k \leqslant \beta_k \ \forall k \in \{1, \ldots, d\}$ and $D^\alpha := \partial^\alpha := \prod_{k=1}^d \partial_{x_k}^{\alpha_k}$ and $\partial_{x_k}^{\alpha_k} := \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}} := \frac{\partial^{\alpha_k}}{\partial_k^{\alpha_k}}$.

Example 5.1.1 (Multiindices)
Consider $d = 3$. Then $D^{(1,0,0)} = \partial_{x_1}$, $D^{(1,1,0)} = \partial_{x_1} \partial_{x_2}$ and $D^{(2,0,0)} = \partial_{x_1}^2$.

We use $\mathrm{grad}(u) = \nabla u = (\partial_{x_1} u, \ldots, \partial_{x_d} u)^\mathsf{T}$ for the gradient of u and $\mathrm{div}(u) := \nabla \cdot u = \partial_{x_1} u + \ldots + \partial_{x_d} u$ for the divergence of u. Then $\mathrm{div}(\mathrm{grad}(u)) =: \Delta u = \partial_{x_1}^2 u + \ldots + \partial_{x_d}^2 u$ is the LAPLACE operator applied to u.

DEFINITION 5.1.2 (MULTIDIMENSIONAL WEAK DERIVATIVE)
Let $\alpha \in \mathbb{N}^d$ and $u, v \in L_{\mathrm{loc}}^1(\Omega)$. If

$$\int_\Omega u(x) D^\alpha \varphi(x) \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v(x) \varphi(x) \, \mathrm{d}x$$

holds for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$, then v is called the α-th weak derivative of u and we write $D^\alpha u = v$.

The α-th weak derivative is uniquely defined because the

> ## Theorem 5.1.1: Fundamental Lemma of the Calculus of Variations
>
> Let $v \in L^1_{\mathrm{loc}}(\Omega)$. If
> $$\int_\Omega v(x)\varphi(x)\,\mathrm{d}x = 0$$
> for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$, then $v = 0$ almost everywhere.

Proof. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $K \subset\subset \Omega$. Define $w = \operatorname{sgn}(u)\,\mathbb{1}_K \in L^1_{\mathrm{loc}}(\Omega)$, then we have $\operatorname{supp}(w) \subset K$. We define $w_\varepsilon := \mathcal{J}_\varepsilon * w$. Then, $w_\varepsilon \to w$ almost everywhere on Ω and $\operatorname{supp}(w_\varepsilon) \subset K + B_\varepsilon(0)$, hence $w_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ if ε is small enough by a modification of Theorem 1.2.2.

We test (5) with $\varphi = w_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$, obtaining

$$0 = \int_a^b \underbrace{u(x)w_\varepsilon(x)}_{\overset{\text{a.e.}}{\longrightarrow} u(x)w(x)}\,\mathrm{d}x = \int_{K+B_\varepsilon(0)} u(x)w_\varepsilon(x)\,\mathrm{d}x = \int_a^b u(x)\,\mathbb{1}_{K+B_\varepsilon(0)}(x)w_\varepsilon(x)\,\mathrm{d}x.$$

We have

$$|w_\varepsilon(x)| \le \int_a^b \mathcal{J}_\varepsilon(x-y)\underbrace{|w(y)|}_{\le 1}\,\mathrm{d}y \le 1.$$

For $\varepsilon_0 < \min(c-a, b-d)$ and all $\varepsilon < \varepsilon_0$ we get

$$|u(x)w_\varepsilon(x)| \le |u(x)|\,\mathbb{1}_{K+B_{\varepsilon_0}(0)}(x)$$

This function is integrable on Ω. Lebesgue's Theorem shows

$$0 = \int_a^b u(x)w(x)\,\mathrm{d}x = \int_c^d |u(x)|\,\mathrm{d}x,$$

hence $u \equiv 0$ almost everywhere on K. As $K \subset\subset \Omega$ was chosen arbitrarily, this yields the claim. $\qquad\square$

> **Lemma 5.1.3**
>
> *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ such that*
> $$\int_\Omega u(x)\partial_{x_i}^k \varphi(x)\,\mathrm{d}x = 0$$
> *for some $k \in \mathbb{N}$ and all $i \in \{1,\ldots,n\}$ and all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Then u is equal to a polynomial of degree of at most k almost everywhere.*

Proof. TODO $\qquad\square$

One can show classical and weak derivative coincide for classically differentiable functions and that the other properties hold similarly to the one dimensional case.

> **definition 5.1.4 (Multidimensional Sobolev space)**
>
> For $k \in \mathbb{N}$ and $p \in [1,\infty]$ we define
> $$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}^d\ \text{with}\ |\alpha| \le k\}$$
> with the norm
> $$\|u\|_{k,p} := \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_{0,p}^p\right)^{\frac{1}{p}}.$$

and the seminorm

$$|u|_{k,p} := \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{0,p}^p \right)^{\frac{1}{p}}.$$

for $1 \leqslant p < \infty$ and the obvious modifications for $p = \infty$.

The space $W^{k,p}(\Omega)$ is a Banach space, which is separable for $p \in [1, \infty)$ and reflexive for $p \in (1, \infty)$.

Furthermore, we define $H^k(\Omega) := W^{k,2}(\Omega)$, which is a Hilbert space with the inner product

$$(u, v)_{k,2} := \sum_{|\alpha| \leqslant k} (D^\alpha u, D^\alpha v)_{0,2}.$$

In particular

$$\|u\|_{1,2}^2 = \|u\|_{0,2}^2 + \|\partial_{x_1} u\|_{0,2}^2 + \ldots + \|\partial_{x_d} u\|_{0,2}^2 = \|u\|_{0,2}^2 + \|\nabla u\|_{0,2}^2$$

and $|u|_{1,2} = \|\nabla u\|_{0,2}$. We again define

$$W_0^{k,p}(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{k,p}} \subsetneqq W^{k,p}(\Omega),$$

$H_0^1(\Omega) := W_0^{1,2}(\Omega)$ and

$$H^{-1}(\Omega) := (H_0^1(\Omega))^*.$$

Figure 20: Domain and compact subdomain

Lemma 5.1.5 (local approximation)

Let $u \in W^{k,p}(\Omega)$. Then $u_\varepsilon \xrightarrow[\varepsilon \to 0]{W^{k,p}(\Omega')} u$ holds, where Ω' is an arbitrary compact subdomain of Ω, i.e. $\overline{\Omega'} \subset \Omega$ (sometimes written $\Omega' \subset\subset \Omega$) and $u_\varepsilon := u * J_\varepsilon$, where J_ε is the multidimensional smoothing kernel.

THEOREM 5.1.2: MEYERS-SERRIN: "H = W"

For any $\Omega \subset \mathbb{R}^d$ and $p < \infty$ we have

$$W^{k,p}(\Omega) = \overline{\mathcal{C}^\infty(\Omega) \cap W^{k,p}(\Omega)}^{\|\cdot\|_{k,p}}$$

But this Theorem is not helpful for us if we want to evaluate the solution on the boundary, see Theorem 5.2.1

5.2 Domains

A domain is an open and connected subset.

DEFINITION 5.2.1 (LIPSCHITZ DOMAIN)

A domain $\Omega \subset \mathbb{R}^d$ is a LIPSCHITZ domain and we write $\partial\Omega \in C^{0,1}$, if for every $x_0 \in \partial\Omega$ there exists a $r > 0$ and a LIPSCHITZ continuous function $g\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that (up to a rotation of the coordinate system)

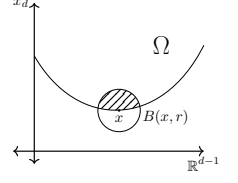$$B(x_0, r) \cap \Omega = \{(x_1, \ldots, x_d) \in B(x_0, r) : x_d > g(x_1, \ldots, x_{d-1})\}$$

Figure 21: Illustration of the definition of a LIPSCHITZ domain.

Remark 5.2.2 Then we also have $B(x_0, r) \cap \partial\Omega = \{x \in B(x_0, r) : x_d = g(x_1, \ldots, x_{d-1})\}$. As Ω is bounded, $\partial\Omega$ is, too and thus is compact, and thus we only need finitely many g to "describe" the boundary.

Remark 5.2.3 We want to use these LIPSCHITZ continuous functions to parametrise the boundary, we know that they are weakly differentiable and their derivative is in L^∞.

Corollary 5.2.4 (To the Theorem of GAUSS, "Partial Integration")

Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$ be a vector field and $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ a scalar valued function and $\Omega \subset \mathbb{R}^d$ a LIPSCHITZ domain. Then

$$\int_\Omega \underbrace{(\nabla \cdot F)(x)}_{div(F)} \varphi(x)\,\mathrm{d}x = -\int_\Omega F(x) \cdot \nabla\varphi(x)\,\mathrm{d}x + \underbrace{\int_{\partial\Omega} \varphi F \cdot \nu\,\mathrm{d}\sigma}_{boundary\ term},$$

where ν is the outer normal and $\cdot$ is the scalar product on $\mathbb{R}^d$.

THEOREM 5.2.1: DENSITY

For a LIPSCHITZ domain Ω, $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $p \in [1, \infty)$.

We define $\mathcal{C}^\infty(\overline{\Omega}) := \{u \in \mathcal{C}^\infty(\Omega) : D^\alpha u \text{ is uniformly continuous } \forall \alpha \in \mathbb{N}^d\}$. Hence $u \in \mathcal{C}^\infty(\Omega)$ can be continuously extended (as its derivatives) to $\overline{\Omega}$.
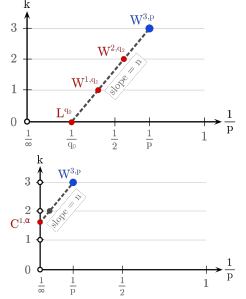
5.3 The Sobolev Embedding Theorem

> **Theorem 5.3.1: Sobolev embedding**
>
> Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $k \in \mathbb{N}$ and $p \in [1, \infty)$. If kp
> 1. $< d$, then (for $m \leqslant k$) $W^{k,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$ if $\frac{1}{q} - \frac{m}{d} \geqslant \frac{1}{p} - \frac{k}{d}$ and thus in particular $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $\frac{1}{q} \geqslant \frac{1}{p} - \frac{k}{d}$ and $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $\frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{d}$.
> 2. $= d$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$.
> 3. $> d$, then $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$, in particular $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}^{\beta,\alpha}(\overline{\Omega})$, where $\beta := k - \left\lfloor \frac{d}{p} \right\rfloor - 1$ and
>
> $$\alpha \in \begin{cases} (0,1), & \frac{d}{p} \in \mathbb{N}, \\ \left(0, \left\lfloor \frac{d}{p} \right\rfloor + 1 - \frac{d}{p}\right), & \text{else.} \end{cases}$$

Figure 22: Here we are "trading" differentiability for integrability: we lose one differentiability but gain $\frac{1}{d}$ integrability. [Source: Wiki]

Remark 5.3.1 Theorem 5.3.1 also holds for fractional Sobolev spaces.

> **Theorem 5.3.2: Rellich-kondrachov**
>
> If $kp < d$, then $W^{k,p}(\Omega) \overset{c}{\hookrightarrow} W^{m,q}(\Omega)$, if $\frac{1}{q} - \frac{m}{d} > \frac{1}{p} - \frac{k}{d}$.

Example 5.3.2 (Embedding of H^1 into L^q with Theorem 5.3.1)
- If $d = 1$, we have $\Omega = (a, b)$. For $k = 1$ and $p = 2$ we have $kp > d$ and thus $W^{k,p}(a,b) = H^1(a,b) \hookrightarrow \mathcal{C}([a,b]) \hookrightarrow L^\infty(a,b)$ by the Theorems we have shown before.

- For $d = 2$, we have $kp = d$ and thus $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1, \infty)$.

- For $d = 3$, we have $kp < d$ and thus $H^1(\Omega) \hookrightarrow L^6(\Omega)$, as $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$.

- For $d = 4$, we still have $kp < d$ and thus $H^1(\Omega) \hookrightarrow L^4(\Omega)$, as $\frac{1}{4} = \frac{1}{2} - \frac{1}{4}$.

Corollary 5.3.3 (Prüfungsprotokoll)
For $p \in [1, \infty)$ and a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ we have

$$W^{1,p}(\Omega) = \{u \in L^1(\Omega) : \partial_j u \in L^p(\Omega) \ \forall j \in \{1, \dots, d\}\}.$$

Proof. "$\subset$": We have $L^p(a,b) \hookrightarrow L^1(a,b)$, so if $u \in L^p(a,b)$, then we have $u \in L^1(a,b)$.

"$\supset$": Let $u \in L^1(a,b)$ with $\partial_j u \in L^p(a,b) \hookrightarrow L^1(a,b)$ for all $j \in \{1, \dots, d\}$. Then $u \in W^{1,1}(\Omega)$.

- If $d = 1$, we have $kp = 1 = d$ and hence $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ by Theorem 5.3.1 ②.

- If $d > 1$, we have $kp = 1 < d$ and hence $W^{1,1}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, q_0]$, where $q_0 := \frac{d}{d-1} \in (1, 2]$.

  – If $q_0 \geqslant p$, we are finished.

  – If $q_0 < p$, we have $u' \in L^p(\Omega) \hookrightarrow L^{q_0}(\Omega)$ and hence $u \in W^{1,q_0}(\Omega)$.

    * If $d = 2$, then $kp = q_0 = 2 = d$ and thus $u \in W^{1,q_0}(\Omega) \hookrightarrow L^p(\Omega)$ by Theorem 5.3.1 ②.

    * If $d \geqslant 3$, we have $q_0 < d$ and thus $W^{1,q_0}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \leqslant \frac{d}{d-2} =: q_1 \in (1, 3]$.

      · If $q_1 \geqslant p$, we are finished.

· If $q_1 < p$, we have $u \in W^{1,q_1}(\Omega)$. If $d = 3$, $kp = q_1 = d$ and hence $W^{1,q_1}(\Omega) \hookrightarrow L^p(\Omega)$ by Theorem 5.3.1 ②.

If $d > 3$, then $q_1 < d$ and thus $W^{1,q_1}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \leqslant \frac{d}{d-3} =: q_2 \in (1, 4]$ by Theorem 5.3.1 ①.

This can be inductively continued until $q_k \geqslant p$. $\qquad\square$

THEOREM 5.3.3: POINCARÉ-FRIEDRICHS INEQUALITY

Let $\Omega \subset \mathbb{R}^d$ be a bounded (LIPSCHITZ) domain. For $u \in W_0^{k,p}(\Omega)$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ we have
$$\|D^\alpha u\|_{k,p} \leqslant C_\Omega |u|_{k,p},$$
so $|\cdot|_{k,p}$ is an equivalent norm on $W_0^{k,p}(\Omega)$.

Remark 5.3.4 (Excursion: Singularities) With polar coordinates one can see that $|\cdot|^{-\gamma} \in L^p(B_1(0))$ if and only if $p < \frac{d}{\gamma}$. Hence the same singularities become less severe in higher dimensions. As an exercise, check for which $p \in [1, \infty]$ the weak derivatives of the above function are in $L^p(B_1(0))$.

5.4 Trace Operators

Motivation. How can we give meaning to "$u|_{\partial\Omega}$" if we only have $u \in W^{k,p}(\Omega)$?

- If $kp > d$, then $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ by Theorem 5.3.1, hence $u|_{\partial\Omega}$ is well defined. In particular, if $k = 1$, then $u|_{\partial\Omega}$ is well defined, if p is large enough, that is, $p > d$. We mostly act in H^1, that is, $p = 2$, so even $d = 2$ is a problem!

- For $kp \leqslant d$ we need trace operators. By Theorem 5.2.1, $\mathcal{C}^{\infty}(\overline{\Omega}) \subset W^{1,p}(\Omega)$ is dense, so for $u \in \mathcal{C}^{\infty}(\overline{\Omega})$, $u|_{\partial\Omega} \in \mathcal{C}(\partial\Omega)$ makes sense. We want to extend this notion from the dense subset to the whole space $W^{1,p}(\Omega)$.

DEFINITION 5.4.1 (TRACE OPERATOR)

Let $\Omega \subset \mathbb{R}^d$ be a LIPSCHITZ domain. Then

$$\operatorname{tr}: \mathcal{C}^{\infty}(\overline{\Omega}) \to L^p(\partial\Omega), \qquad u \mapsto u|_{\partial\Omega}$$

is the trace operator of u.

trace operator

Lemma 5.4.2 (Properties of the trace operator)

The trace operator is linear, bounded and uniquely extendable to an operator $\operatorname{tr}: W^{1,p}(\Omega) \to L^p(\partial\Omega)$.

For $u \in C^{\infty}(\overline{\Omega})$ there exists a $c > 0$ such that

$$\|\operatorname{tr}(u)\|_{L^p(\partial\Omega)} = \|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \leqslant c\|u\|_{W^{1,p}(\Omega)}.$$

Remark 5.4.3 (What is $L^p(\partial\Omega)$?) The boundary $\partial\Omega \subset \mathbb{R}^d$ is a $(d-1)$-dimensional manifold and thus there is an induced $(d-1)$-dimensional (surface) measure on $\partial\Omega$, and hence $L^p(\partial\Omega)$ is well defined.

Remark 5.4.4 For $u \notin \mathcal{C}^{\infty}(\overline{\Omega})$, the quantity $\operatorname{tr}(u)$ cannot be explicitly computed.

THEOREM 5.4.1: CHARACTERISATION OF $W_0^{1,p}(\Omega)$

Under the above conditions we have

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \operatorname{tr}(u) = 0\} = \ker(\operatorname{tr})$$

Thus tr is not injective. Is it surjective? For intuition consider a function being 1 on one part of the boundary and 0 on the other, which is an integrable, but not continuous function. As on the interior of Ω, the function has to be continuous, also satisfying this boundary condition makes the derivative to steep such that the function is not integrable anymore.

We set

$$L^p(\partial\Omega) \supset W^{1-\frac{1}{p},p}(\partial\Omega) := \operatorname{tr}(W^{1,p}(\Omega)),$$

where $W^{1-\frac{1}{p},p}(\partial\Omega)$ is fractional SOBOLEV space. Thus these functions are exactly the functions we can prescribe on the boundary when dealing with non homogeneous DIRICHLET boundary conditions.

Hence $\operatorname{tr}: H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is surjective.

5.5 Variational formulation in multiple dimensions

For a Lipschitz domain Ω, $f, d \colon \Omega \to \mathbb{R}$, $c \colon \Omega \to \mathbb{R}^d$, $A \colon \Omega \to \mathbb{R}^{d \times d}$ consider the second order linear boundary value problem with homogeneous Dirichlet boundary conditions in divergence form

$$\begin{cases} -\nabla \cdot (A(x)\nabla u(x)) + c(x) \cdot \nabla u(x) + d(x)u(x) = f(x), & x \in \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{19}$$

We are searching a solution $u \colon \Omega \to \mathbb{R}$.

For the variational formulation set $V := H_0^1(\Omega)$. For e.g. $f \in L^2(\Omega)$, $\langle f, v \rangle := \int_\Omega f(x)v(x)\,\mathrm{d}x$ defines an element in $H^{-1}(\Omega)$, where $v \in V$. We can find $p \in [1, 2)$ such that $f \in L^p(\Omega)$ induces $f \in H^{-1}(\Omega)$ with Theorem 5.3.1: like in Example 5.3.2 we find that $H_0^1(\Omega) \subset H^1(\Omega) \hookrightarrow L^q$ with $q = \frac{2d}{d-2}$ for $d \geqslant 2$ (for $d = 1$ we have $H^1 \overset{c}{\hookrightarrow} L^2$). Thus

$$|\langle f, v \rangle| \leqslant \|f\|_p \|v\|_{0,q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, that is, $p = \frac{2d}{d+2} \in [1, 2)$.

We multiply with $v \in V$ and integrate over Ω to obtain

$$-\int_\Omega \nabla(A(x)\nabla u(x))v(x)\,\mathrm{d}x + \int_\Omega c(x) \cdot \nabla u(x)v(x) + d(x)u(x)v(x)\,\mathrm{d}x = \langle f, v \rangle.$$

By "partial integration" we obtain

$$-\int_\Omega \nabla(A(x)\nabla u(x))v(x)\,\mathrm{d}x = \int_\Omega A(x)\nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_{\partial\Omega} \underbrace{v(x)}_{\substack{=0 \text{ as } v \in H_0^1(\Omega) \\ \text{hence } \mathrm{tr}(v)=0}} (A(x)\nabla u(x))\nu\,\mathrm{d}\sigma$$

$$= \int_\Omega A(x)\nabla u(x)\nabla v(x)\,\mathrm{d}x,$$

where ν is the outer normal of $\partial\Omega$.

Define the bilinear form

$$a \colon V \times V \to \mathbb{R}, \qquad (u, v) \mapsto \int_\Omega A(x)\nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_\Omega c(x) \cdot \nabla u(x)v(x) + d(x)u(x)v(x)\,\mathrm{d}x$$

Let $d \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)^d$ and $A \in L^\infty(\Omega)^{d \times d}$ all be essentially bounded, then a is well defined and bounded. Define $A \colon V \to V^*$ by $\langle Au, v \rangle := a(u, v)$, which is well defined. The weak formulation of (19) is

$$\left\{ \text{For } f \in H^{-1}(\Omega) \text{ find } u \in H_0^1(\Omega) : Au = f \text{ in } V^* \right.$$

Remark 5.5.1 The Theorem of Lax-Milgram can be applied exactly as in the one-dimensional setting.

Remark 5.5.2 Often, A is symmetric.

Remark 5.5.3 If $A \equiv \mathrm{id}$, then $\nabla \cdot (A(x)\nabla u(x))$ becomes $\nabla \cdot \nabla u(x) = \Delta u(x)$.

Remark 5.5.4 (Different boundary conditions) If we are given inhomogeneous Dirichlet conditions $u|_{\partial\Omega} = g$, where $g \in H^{\frac{1}{2}}(\Omega)$ is a function on $\partial\Omega$, or $\mathrm{tr}(u) = g$, then there can be a $\tilde{u} \in H^1(\Omega)$ with $\mathrm{tr}(\tilde{u}) = g$ and so the condition is well defined. In this case

$\hat{u} := u - \tilde{u} \in H^1_0(\Omega)$ if and only if $\gamma(u) = g$ and the problem $Au = f$ is solved by u if and only if $A(\hat{u} + \tilde{u}) = A\hat{u} + A\tilde{u} = f$ holds, so we can instead solve the problem $A\hat{u} = f - A\tilde{u}$ for $\tilde{u}$.

NEUMANN boundary conditions look like this: $\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n} = g$ on $\partial\Omega$, where $\vec{n}$ is the outer normal.

Mixed boundary conditions can look like this: let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be a partition of $\partial\Omega$, that is, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$. Then the boundary conditions are $u|_{\Gamma_1} = 0$ and $\nabla u \cdot \vec{n} = 0$ on Γ_2. We then consider the closed subspace $V := \{u \in H^1(\Omega) : \text{tr}(u) = 0 \text{ on } \Gamma_1\} \subset H^1(\Omega)$.

Example 5.5.5 (A quasilinear problem)
Consider the quasilinear problem

$$\begin{cases} -\nabla \cdot (a(u)\nabla u) = f, & \text{on } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $a \colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists $m, M > 0$ such that $m \leqslant a(y) \leqslant M$ for all $y \in \mathbb{R}$. Let $\tilde{u} := \int_0^{u(x)} a(s)\,\mathrm{d}s$. Then $\nabla\tilde{u}(x) = \nabla u(x)a(u(x))$. If u is a solution of (5.5.5), then $\tilde{u}$ solves

$$\begin{cases} -\Delta\tilde{u} = f & \text{on } \Omega, \\ \tilde{u}|_{\partial\Omega} = 0, \end{cases}$$

as $u|_{\partial\Omega} = 0$ implies $\tilde{u}|_{\partial\Omega} = 0$. By the Theorem of LAX-MILGRAM, the problem (5.5.5) has a unique solution. Let

$$A \colon \mathbb{R} \to \mathbb{R}, \qquad z \mapsto \int_0^z a(s)\,\mathrm{d}s,$$

which is monotonically increasing. Because of $0 < m \leqslant a(z) \leqslant M$, A is invertible. Thus A^{-1} exists and from $\tilde{u}$, we obtain $u(x) = A^{-1}(\tilde{u}(x))$.

6 Additional Topics

6.1 Inner regularity theory for the LAPLACIAN

Motivation. For $f \in L^2$ consider

$$
\begin{cases}
-u''(x) = f(x), & x \in (a, b), \\
u(a) = u(b) = 0,
\end{cases}
$$

which has a solution $u \in H_0^1(a, b)$, that is, it is one weakly differentiable. One can ask if u is twice weakly differentiable with $u'' = f \in L^2$. If f is even one classically differentiable, then we can differentiate the relation $-u''(x) = f(x)$, to obtain that u'' has to be differentiable as well, that is, u has a third derivative.

In the following Theorem, we see that if the right side f is "better", that is, more regular or integrable, than just defining a functional on H^1 (which guarantees existence), we can expect the solution to be "better" as well. This can only happen in the interior of the domain, as in multiple dimensions, the boundary can be very "bad". But, on a compact subdomain bounded away from the boundary, we can state the following result.

THEOREM 6.1.1: $+2$ INNER REGULARITY ON ANY DOMAIN

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. We consider

$$
\begin{cases}
-\Delta u = f & \text{on } \Omega, \\
u|_{\partial \Omega} = 0
\end{cases}
\tag{20}
$$

If $f \in H^k(\Omega)$ for $k \in \mathbb{N}$, then the unique variational solution $u \in H_0^1(\Omega)$ to (20) satisfies $u \in H^{k+2}(\Omega')$ for any $\Omega' \subset\subset \Omega$ (that is, $u \in H_{\text{loc}}^{k+2}(\Omega)$) and there exists an $c_{\Omega'} > 0$ such that we have

$$
\|u\|_{H^{k+2}(\Omega')} \leqslant c_{\Omega'} \left(\|f\|_{H^k(\Omega)} + \|u\|_{H_0^1(\Omega)} \right).
$$

Remark 6.1.1 This is local and can't be generalised to the whole of Ω without further regularity assumptions on the boundary. If the domain is convex, we are fine.

To show that a function has more regularity, we have to consider its difference quotient.

Figure 23: Left: good, right: bad

Lemma 6.1.2 (Boundedness of the difference quotient)
   Let $p \in (1, \infty)$, $u \in L^p(\Omega)$ and $\Omega \subset \mathbb{R}^d$ a *bounded* domain and $(\tau_h u)(x) := u(x + h)$ be the *shift operator*

   ① Let $u \in W^{1,p}(\Omega)$. For all $\Omega' \subset\subset \Omega$ we then have

   $$
   \|\tau_h u - u\|_{L^p(\Omega')} \leqslant |u|_{W^{1,p}(\Omega)} |h|
   $$

   for all $h \in \mathbb{R}^d$ such that $x + h \in \Omega$ for $x \in \Omega'$, that is $|h| < \text{dist}(\partial\Omega', \partial\Omega)$.

   ② If there exists a $c > 0$ such that for all $\Omega' \subset\subset \Omega$ and for all sufficiently small $|h|$ ($|h| < \text{dist}(\partial\Omega', \partial\Omega)$)

   $$
   \|\tau_h u - u\|_{0,p,\Omega'} \leqslant c|h|
   $$

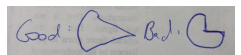   holds, then $u \in W^{1,p}$ and $|u|_{W^{1,p}(\Omega)} \leqslant c$.

Proof. ① Let $\Omega' \subset\subset \Omega$ and $|h|$ be sufficiently small. As $\mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ is dense there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that

$$\|u_n - u\|_{1,p,\Omega} \xrightarrow{n\to\infty} 0.$$

In particular, by the inverse $\triangle \neq$

$$\|\nabla u_n\|_{0,p,\Omega} \xrightarrow{n\to\infty} \|\nabla u\|_{0,p,\Omega}, \qquad \|u_n - u\|_{0,p,\Omega'} \xrightarrow{n\to\infty} 0.$$

and since $\|\cdot\|_{0,p} \leqslant \|\cdot\|_{1,p}$

$$\|\tau_h u_n - \tau_h u\|_{0,p,\Omega'}$$

because of translational symmetry.

Then by the Mean Value Theorem and the Fundamental Theorem of Calculus we obtain

$$\tau_h u_n(x) - u_n(x) = u_n(x+h) - u_n(x) = \int_0^1 \langle h, \nabla u_n(x+\theta h)\rangle \,\mathrm{d}\theta. \qquad (21)$$

Furthermore,

$$
\begin{aligned}
\|\tau_h u_n - u_n\|_{0,p,\Omega'}^p &\overset{\text{Def.}}{=} \int_{\Omega'} |\tau_h u_n(x) - u_n(x)|^p \,\mathrm{d}x \\
&\overset{(21)}{=} \int_{\Omega'} \left|\int_0^1 \langle h, \nabla u_n(x+\theta h)\rangle \,\mathrm{d}\theta\right|^p \,\\
&\overset{\triangle\neq}{\leqslant} |h|^p \int_{\Omega'} \left(\int_0^1 |\nabla u_n(x+\theta h)| \,\mathrm{d}\theta\right)^p \,\mathrm{d}x \\
&\overset{\text{(H)}}{\leqslant} |h|^p \int_{\Omega'} \int_0^1 |\nabla u_n(x+\theta h)|^p \,\mathrm{d}\theta \,\mathrm{d}x \\
&\overset{\text{(F)}}{\leqslant} |h|^p \int_0^1 \int_{\Omega'} |\nabla u_n(x+\theta h)|^p \,\mathrm{d}x \,\mathrm{d}\theta \\
&\leqslant |h|^p \int_0^1 \int_\Omega |\nabla u_n(y)|^p \,\mathrm{d}y \,\mathrm{d}\theta = |h|^p |u_n|_{1,p,\Omega}^p.
\end{aligned}
$$

Hence,

$$\|\tau_h u - u\| \overset{\triangle\neq}{\leqslant} \underbrace{\|\tau_h u - \tau_h u_n\|}_{\xrightarrow{n\to\infty} 0} + \underbrace{\|\tau_h u_n - u_n\|_{0,p,\Omega'}}_{\leqslant |h||u_n|_{1,p,\Omega}} + \underbrace{\|u_n - u\|}_{\xrightarrow{n\to\infty} 0}$$
$$\xrightarrow{n\to\infty} |h||u|_{1,p,\Omega},$$

where $\|\cdot\| = \|\cdot\|_{0,p,\Omega'}$.

② Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\Omega' \subset\subset \Omega$ and $|h|$ small. We assume $\text{supp}(\varphi) \subset \Omega'$.

We have

$$
\begin{aligned}
\int_\Omega [u(x+h) - u(x)]\,\varphi(x)\,\mathrm{d}x &= \int_{\Omega'} [u(x+h) - u(x)]\,\varphi(x)\,\mathrm{d}x \\
&\overset{\text{(H)}}{\leqslant} \|\tau_h u - u\|_{0,q,\Omega'}\|\varphi\|_{0,q,\Omega} \\
&\leqslant c|h|\|\varphi\|_{0,q,\Omega}
\end{aligned}
$$

and with the transformation theorem $(\star)$

$$
\begin{aligned}
\int_\Omega [\tau_h u(x) - u(x)]\,\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} [\tau_h u(x) - u(x)]\,\varphi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \tau_h u(x)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} u(x)\varphi(x)\,\mathrm{d}x
\end{aligned}
$$

$$\stackrel{(\star)}{=} \int_{\mathbb{R}^d} u(y)\tau_{-h}\varphi(y)\,\mathrm{d}y - \int_{\mathbb{R}^d} u(y)\varphi(y)\,\mathrm{d}y$$

$$= \int_{\mathbb{R}^d} \left[\tau_{-h}\varphi - \varphi\right](y)u(y)\,\mathrm{d}y$$

$$= \int_{\Omega} \left[\tau_{-h}\varphi - \varphi\right](y)u(y)\,\mathrm{d}y. \tag{22}$$

Fix one direction $i \in \{1,\ldots,d\}$. We consider $t := he_i$. Then $\|h\| = \|t\|$. For $t > 0$ we have

$$\int_{\Omega} u(y) \underbrace{\frac{\varphi(y - te_i) - \varphi(y)}{t}}_{\xrightarrow{t \searrow 0} -\frac{\partial\varphi(y)}{\partial x_i}}\,\mathrm{d}y \leqslant c\|\varphi\|_{0,q,\Omega}$$

By LEBESGUEs theorem we have

$$-\int_{\Omega} u(y)\frac{\partial\varphi(y)}{\partial x_i}\,\mathrm{d}y \leqslant c\|\varphi\|_{0,q,\Omega} \tag{23}$$

We define

$$g\colon \mathcal{C}_0^\infty(\Omega) \to \mathbb{R},\ \langle g, \varphi\rangle := -\int_{\Omega} u(y)\frac{\partial\varphi(y)}{\partial x_i}\,\mathrm{d}y.$$

As g is linear (in φ) the HAHN-BANACH theorem implies existence of a unique ($\mathcal{C}_0^\infty \subset L^p$ dense) extension to a linear bounded function

$$g\colon L^q(\Omega) \to \mathbb{R} \quad \text{with } \langle g, \varphi\rangle \leqslant c\|\varphi\|_{0,q} \ \forall\varphi \in L^q(\Omega)$$

This shows $g \in (L^q(\Omega))^* \cong L^p$. Hence there exists a $v_i \in L^p(\Omega)$ such that

$$\langle g, \varphi\rangle = \int_{\Omega} v_i\varphi\,\mathrm{d}x \ \forall\varphi \in L^q(\Omega).$$

Hence for $\varphi \in \mathcal{C}_0^\infty(\Omega)$ we have

$$-\int_{\Omega} u(y)\frac{\partial\varphi(y)}{\partial x_i}\,\mathrm{d}y = \langle g, \varphi\rangle = \int_{\Omega} v_i\varphi\,\mathrm{d}x.$$

Hence $\frac{\partial u_i}{\partial x_i} = v_i$, u_i has the weak derivative v_i. $\qquad\qquad\square$

Corollary 6.1.3 (Auxiliary lemma, "partial integration")
  Let $h \in \mathbb{R}^d\backslash\{0\}$, $u,v \in H^1(\mathbb{R}^d)$. We set $D_h u := \frac{\tau_h u - u}{h}$. Then $\nabla D_h u = D_h(\nabla u)$ and

$$\int_{R^d} u(D_{-h}v)\,\mathrm{d}x = \int_{\mathbb{R}^d} D_h v.$$

Proof. The first statement is an exercise and the other one is proven analogously to (22).$\square$

Proof. (of theorem 6.1.1) Let $u \in H_0^1(\Omega)$ be the weak solution of (20) and assume $f \in H^k(\Omega)$.

Let $k = 0$, i.e. $f \in L^2(\Omega)$. Fix $\Omega' \subset\subset \Omega$. Consider $\varphi \in \mathcal{C}_0^\infty(\Omega)$ such that $\varphi|_{\Omega'} \equiv 1$ and $\varphi(x) \in [0,1]$, which can be obtained by a smoothing of $\mathbb{1}_{\Omega'}$. We set $v := u \cdot \varphi \in H^1(\Omega)$ and even $\in H^1(\mathbb{R}^d)$, which is compactly supported in Ω. (Exercise: check this)

With $g := f\varphi - 2\nabla u \cdot \nabla\varphi - u\Delta\varphi \in L^2(\Omega)$ (Exercise: check this), v is the variational solution to

$$\begin{cases} -\Delta v = g, & \text{on } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

For any $w \in H_0^1(\Omega)$ we then have

$$\underbrace{\int \nabla v \cdot \nabla w \, \mathrm{d}x}_{\substack{\text{var. formulation} \\ \text{of the Laplacian}}} = \int (\nabla u)\varphi(\nabla w) \, \mathrm{d}x + \int u(\nabla \varphi)(\nabla w) \, \mathrm{d}x$$

$$= \int \nabla u (\nabla \varphi w - w \nabla \varphi) \, \mathrm{d}x - \int \nabla(u(\nabla \varphi)) w \, \mathrm{d}x$$

$$= - \int \nabla u \nabla \varphi w \, \mathrm{d}x + \int \nabla u \cdot \nabla \varphi w \, \mathrm{d}x$$

$$\quad - \int (\nabla u \cdot \nabla \varphi + u \Delta \varphi) w \, \mathrm{d}x$$

$$= \int (-u \Delta \varphi - 2 \nabla u \nabla \varphi) w \, \mathrm{d}x + \underbrace{\int \nabla u \cdot \nabla(\varphi w) \, \mathrm{d}x}_{= \int g w \, \mathrm{d}x = \int f \varphi w \, \mathrm{d}x},$$

as u is a solution to the RHS. In the second last equality we used that the divergence of the gradient is the Laplacian.

We test $\int_\Omega (\nabla v)(x) \cdot (\nabla w)(x) \, \mathrm{d}x = \int_\Omega g(x) w(x) \, \mathrm{d}x$ with $w = D_{-h} D_h v \in H_0^1(\Omega)$ (or even $H^1(\mathbb{R}^d)$) for h small enough:

$$\int \nabla v \cdot \nabla w \, \mathrm{d}x = \int g w \, \mathrm{d}x \overset{\text{(CS)}}{\leqslant} \|g\|_{0,2,\Omega} \|w\|_{0,2,\Omega}.$$

As, by Lemma 6.1.2 ①

$$\|w\|_{0,2} = \|D_{-h} \underbrace{D_h v}_{\in H^1}\|_{0,2} \leqslant \|\nabla D_h v\|_{0,2} = \|D_h \nabla v\|_{0,2},$$

where the inequality is due to

$$\|\tau_h(D_h(v)) - D_h v\|_{0,2} \leqslant |D_j v|_{1,2} |h|.$$

Furthermore, we have

$$\|g\|_{0,2} = \|f\varphi - 2 \nabla u \cdot \nabla \varphi - u \Delta f\|_{0,2} \leqslant c(\|f\|_{0,2} + \|u\|_{1,2}).$$

On the other hand

$$\int (\nabla v)(x)(\nabla w)(x) \, \mathrm{d}x = \int (\nabla v)(x) \cdot (\nabla(D_{-h} D_h v))(x) \, \mathrm{d}x$$

$$\overset{6.1.3}{=} \int \nabla v(x) (D_{-h}(\nabla D_h v))(x) \, \mathrm{d}x$$

$$\overset{6.1.3}{=} \int (D_h(\nabla v)(x) \cdot (D_h(\nabla v))(x) \, \mathrm{d}x = \int |(D_h(\nabla v))(x)|^2 \, \mathrm{d}x$$

$$= \|D_h \nabla v\|_{0,2}^2.$$

Together we obtain

$$\|D_h(\nabla v)\|_{0,2} \leqslant \|g\|_{0,2},$$

i.e $\|\tau_h(\nabla v) - \nabla v\|_{0,2} \leqslant \|g\| |h|$ for all small $|h|$.

Lemma 6.1.2 ② (applied component-wise) implies that $\nabla v \in H^1(\Omega')^d$, i.e. $v \in H^2(\Omega')$ and

$$\|v\|_{2,2} = \|\nabla v\|_{1,2} + \|v\|_{0,2} \leqslant \|g\|_{0,2} + \|v\|_{0,2}.$$

In particular: $v \in H^2(\Omega')$, but on Ω' we have $v = u$, hence $u \in H^2(\Omega')$. □

6.2 Existence for a nonlinear problem

This subsection follows [Chi12, Chapter 5.1].

Consider the nonlinear problem in divergence form

$$
\begin{cases}
-\nabla \cdot (\alpha(x, u(x)) \nabla u(x)) = f(x), & x \in \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
$$

where Ω is a bounded domain and $\alpha \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Furthermore there exists constants $m, M > 0$ such that $m \leqslant \alpha(x, y) \leqslant M$ holds for almost all $x \in \Omega$ and all $y \in \mathbb{R}$. The variational formulation of (6.2) is

$$
\begin{cases}
\text{For } f \in H^{-1}(\Omega) \text{ find } u \in H_0^1(\Omega) \text{ such that} \\
a(u, v) := \int_\Omega \alpha(x, u(x)) \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x = \langle f, v \rangle \quad \text{for all } v \in H_0^1(\Omega).
\end{cases}
\tag{24}
$$

> **Theorem 6.2.1: Unique solvability of (24) [Chi12, Thm. 5.1]**
>
> The problem (24) has a variational solution.

In the proof we will use a Galerkin scheme and consider the sequence of solutions to the discretised problems and hope that they converge, in a certain sense, to the solution of (24).

Proof. ① As $V := H_0^1(\Omega)$ is separable, by Theorem 4.1.1 there exists a Galerkin scheme $(V_h)_{h>0} \subset V$, that is for all $v \in V$ there exists a sequence $(v_h \in V_h)_{h>0}$ such that $\|v - v_h\|_V \xrightarrow{h \searrow 0} 0$. We consider the discretized problem

$$
\begin{cases}
\text{Find } u_h \in V_h \text{ such that} \\
a(u_h, v_h) = \langle f, v_h \rangle \qquad \text{for all } v_h \in V_h.
\end{cases}
\tag{25}
$$

Recall that V_h is a finite dimensional subspace of $V = H_0^1(\Omega)$ equipped with $|\cdot|_{1,2}$ or $\|\cdot\|_{1,2}$. As $a(u_h, v_h) = \langle f, v_h \rangle$ is a nonlinear equation, we cannot use the Theorem of Lax-Milgram, but instead have to use a fixed-point Theorem.

② For fixed $w_h \in V_h$ we consider the bounded strongly positive bilinear form

$$
a_{w_h} \colon V_h \times V_h \to \mathbb{R}, \qquad (u_h, v_h) \mapsto \int_\Omega \alpha(x, w_h(x)) \nabla u_h(x) \nabla v_h(x) \, \mathrm{d}x.
$$

We have $|a_{w_h}(u_h, v_h)| \leqslant M |u_h|_{1,2} |v_h|_{1,2}$ and $a_{w_h}(u_h, u_h) \geqslant m |u_h|_{1,2}^2$ for all $u_h, v_h \in V_h$. Thus the equation

$$
a_{w_h}(u_h, v_h) = \langle f, v_h \rangle \qquad \forall v_h \in V_h
\tag{26}
$$

has a unique solution $u_h^{(w)} \in V_h$ by the Theorem of Lax-Milgram. We define

$$
T_h \colon V_h \to V_h, \qquad w_h \mapsto u_h^{(w)}
$$

as the operator whose fixed point we want to find.

③ **A-priori estimate.** We test with $v_h = u_h^{(w_h)}$ in (26) and obtain

$$
m |u_h^{(w_h)}|_{1,2}^2 \leqslant a_{w_h}(u_h^{(w_h)}, u_h^{(w_h)}) = \langle f, u_h^{(w_h)} \rangle \leqslant \|f\|_{-1,2} |u_h^{(w_h)}|_{1,2}.
$$

Hence $|T_h(w_h)|_{1,2} = |u_h^{(w_h)}|_{1,2} \leqslant \frac{1}{m} \|f\|_{-1,2}$ for any $w_h \in V_h$ and thus

$$
T_h \colon \overline{B}\left(0, \frac{1}{m} \|f\|_{-1,2}\right) \to \overline{B}\left(0, \frac{1}{m} \|f\|_{-1,2}\right), \qquad w \mapsto u_h^{(w)}
$$

is well defined and maps the nonempty closed convex bounded set $\overline{B}\left(0, \frac{1}{m}\|f\|_{-1,2}\right)$ to itself.

④ The operator T_h is continuous: let $(w_n)_{n\in\mathbb{N}} \subset \overline{B}\left(0, \frac{1}{m}\|f\|_{-1,2}\right)$ be a sequence converging to $w \in \overline{B}\left(0, \frac{1}{m}\|f\|_{-1,2}\right)$ in $H_0^1(\Omega)$. We know that $|T_h(w_n)|_{1,2} \leqslant \frac{1}{m}\|f\|_{-1,2}$ for all $n \in \mathbb{N}$, hence the sequence $(T_h(w_n))_{n\in\mathbb{N}}$ is bounded in V_h. As V_h is finite dimensional, by the Theorem of Bolzano-Weierstrass there exists a subsequence $(w_{n'})_{n'\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ such that $(T_h(w_{n'}))_{n'\in\mathbb{N}}$ converges with respect to $|\cdot|_{1,2}$, that is, there exists a $u_h \in V_h$ such that $|T_h(w_{n'}) - u_h|_{1,2} \xrightarrow{n'\to\infty} 0$. We have to show $u_h = T_h(w)$.

As $w_{n'} \to w$ with respect to $|\cdot|_{1,2}$ and thus in particular with respect to $\|\cdot\|_{1,2}$, there exists a further subsequence $(w_{n''})_{n''\in\mathbb{N}}$ such that $w_{n''}(x) \to w(x)$ almost everywhere in Ω. This implies

$$\alpha(x, w_{n''}(x))\nabla v_h(x) \xrightarrow{n''\to\infty} \alpha(x, w(x))\nabla v_h(x)$$

almost everywhere in Ω. As $|\alpha(x, w_{n''}(x))| \leqslant M$ for almost all $x \in \Omega$ and $v_h \in L^2(\Omega)$ for $v_h \in V_h \subset H_0^1(\Omega)$, the Theorem of Lebesgue shows

$$\int_\Omega |\alpha(x, w_{n''}(x))\nabla v_h(x) - \alpha(x, w(x))\nabla v_h(x)|^2 \, \mathrm{d}x \xrightarrow{n''\to\infty} 0.$$

As $\nabla u_{w_{n''}} = T(w_{n''}) \to u_h$ with respect to $|\cdot|_{1,2}$ thus $\nabla u_{w_{n''}} \to \nabla u_h$ in L^2 we get

$$\int_\Omega \alpha(x, w_{n''}(x))\nabla u_{w_{n''}}(x) \cdot \nabla v_h(x) \, \mathrm{d}x \xrightarrow{n''\to\infty} \int_\Omega \alpha(x, w(x))\nabla u_h(x) \cdot \nabla v_h(x) \, \mathrm{d}x$$

Hence for all $n'' \in \mathbb{N}$ we have

$$\langle f, v_h \rangle = \int_\Omega \alpha(x, w_{n''}(x))\nabla u_{w_{n''}}(x) \cdot \nabla v_h(x) \, \mathrm{d}x \xrightarrow{n''\to\infty} \int_\Omega \alpha(x, w(x))\nabla u_h(x) \cdot \nabla v_h(x) \, \mathrm{d}x$$

Hence $u_h = T(w)$

As $T_h(w)$ does not depend on the subsequence, the subsequence principle shows that $T_h(w_n) \to T_h(w)$ and thus T_h is continuous.

⑤ Brouwer's Fixed Point Theorem shows that T_h has a fixed point $u_h \in \overline{B}\left(0, \frac{1}{m}\|f\|_{-1,2}\right) \subset V_h$, i.e. $\alpha_{u_h}(u_h, v_h) = \alpha(u_h, v_h) = \langle f, v_h \rangle$ for all $v_h \in V_h$ and hence u_h is a solution to (25).

⑥ We consider the sequence $(u_h)_{h>0} \subset V$ of solutions to (25). As $u_h = T_h(u_h)$, we get $|u_h|_{1,2} \leqslant \frac{1}{m}\|f\|_{-1,2}$ irregardless of $h > 0$, hence $(u_h)_{h>0}$ is a bounded sequence in V. We want to show that $(u_h)_{h>0}$ converges to some $u \in H_0^1(\Omega)$ (in some sense) and that u is a solution to (24), i.e.

$$\int \alpha(x, u(x))\nabla u(x)\nabla v(x) \, \mathrm{d}x = \langle f, v \rangle$$

for all $v \in H_0^1(\Omega)$. We know that for each $h > 0$ we have

$$\int \alpha(x, u_h(x))\nabla u_h(x)\nabla v_h(x) \, \mathrm{d}x = \langle f, v_h \rangle$$

for all $v_h \in V_h$.

Fix $v \in H_0^1(\Omega)$ then there exists a sequence $(v_h)_h$ such that $v_h \to v$ in $H_0^1(\Omega)$ by the completeness in the limit of a Galerkin scheme. This implies $\langle f, v_h \rangle \to \langle f, v \rangle$ by the continuity of $\langle f, \cdot \rangle \in H^{-1}(\Omega)$.

We remark that $(u_h)_{h>0}$ is bounded in $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$, hence there is a subsequence $(u_{h'})_{h'}$ of $(u_h)_{h>0}$ which converges to some $u \in L^2$ and thus - up to a subsequence - pointwise almost everywhere, showing

$$\alpha(x, u_{h''}(x)) \to \alpha(x, u(x)) \qquad h'' \searrow 0$$

almost everywhere in Ω by the continuity of α in its second argument. We now have to show that $\nabla u \in L^2$ exists and want to go to the limit in the equation

$$\int_\Omega \alpha(x, u_{h''}) \nabla u_{h''} \nabla v_{h''} \, \mathrm{d}x = \underbrace{\int_\Omega (\alpha(x, u_{h''}) \nabla v_{h''} - \alpha(x, u) \nabla v) \cdot \nabla u_{h''} \, \mathrm{d}x}_{=:(\mathrm{I})}$$
$$+ \underbrace{\int_\Omega \alpha(x, u) \nabla v \cdot \nabla u_{h''} \, \mathrm{d}x}_{=:(\mathrm{II})}.$$

We have

$$(\mathrm{I}) \overset{\mathrm{CS}}{\leqslant} \underbrace{\|\alpha(\cdot, u_{h''}) \nabla v_{h''} - \alpha(\cdot, u) \nabla v\|_{0,2}}_{\to 0} \underbrace{\|\nabla u_{h''}\|_{0,2}}_{\text{bounded}} \to 0.$$

We want to show that

$$(\mathrm{II}) = \int_\Omega \alpha(x, u) \nabla v \cdot \nabla u_{h''} \, \mathrm{d}x \to \int_\Omega \alpha(x, u) \nabla v \cdot \nabla u \, \mathrm{d}x.$$

We can write

$$(\mathrm{II}) =: \langle g, u_{h''} \rangle$$

Then $u_{h''} \mapsto \langle g, u_{h''} \rangle$ is in $H^{-1}(\Omega)$.

We want to find a subsequence such that $\langle g, u_{h''} \rangle$ converges. This is called weak convergence. Indeed, as V is reflexive and $(u_{h''})_{h''} \subset V$ is bounded, there exists a subsequence $(u_{h'''})_{h'''>0} \subset V$ and a $u \in V$ such that

$$\langle g, u_{h''} \rangle \to \langle g, u \rangle \qquad \forall g \in V^*. \qquad \square$$

Example 6.2.1 (Prüfungsprotokoll)
Consider

$$\begin{cases} -\Delta u = f(x, u(x)), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

The Theorem of Zarantonello can't be applied. Wie im obigen Problem die Nichtlinearität entschärfen, in dem man ein w fest wählt und an die Stelle von u in f einsetzen. Nach Zeigen der Lösbarkeit $T \colon w \mapsto u_w$ betrachten und einen Fixpunkt sucht. Ist $f_w \colon x \mapsto f(x, w(x))$ ein Funktion auf $H_0^1(\Omega)$, können wir Lax-Milgram anwenden. Das Korollar aus dem Satz von Lax-Milgram (stetige Abhängigkeit der Lösung von der rechten Seite) zeigt die Stetigkeit von T, also

$$\|u_w - u_{w'}\| \leqslant C\|f_w - f_{w'}\|$$

und wenn f_w stetig in w ist, ist T stetig.

Example 6.2.2
Consider the stationary scalar convection-diffusion equation

$$-\mathrm{div}(A(x)\,\mathrm{grad}\,u(x)) + c(x)\,\mathrm{grad}(u(x)) + d(x)u(x) = f(x), \qquad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a sufficiently smooth, bounded domain. Furthermore assume homogeneous Dirichlet boundary conditions, i.e. $u|_{\partial\Omega} = 0$ and that

(1) The matrix-valued function $A \in L^\infty(\Omega)^{d \times d}$ is symmetric and uniformly positive definite (uniformly elliptic), i.e. there exists a constant $\mu > 0$ such that for all $z \in \mathbb{R}^d$ and almost all $x \in \Omega$ we have $z^\mathsf{T} A(x) z \geqslant \mu \|z\|^2_{\mathbb{R}^d}$.

(2) The vector valued function c is in $L^\infty(\Omega)^d$ and the scalar-valued function d is in $L^\infty(\Omega)$.

Then for all $f \in H^{-1}(\Omega)$ there exists a unique weak solution, if $c \in W^{1,\infty}(\Omega)^d$ and another condition is fulfilled.

TODO

References

[Chi12] M. Chipot, Elements of nonlinear analysis, Birkhäuser Advanced Texts Basler Lehrbücher, Birkhäuser Basel, 2012.

A Appendix

A.1 Elementary Inequalities

Young

Spezialfall $\varphi(x) := \varepsilon p x^{p-1}$ $ab \leqslant \varepsilon a^p + \frac{(p\varepsilon)^{1-q}}{q} b^q$, $\forall \varepsilon > 0$, $a, b \geqslant 0$, $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$

A.2 Additional proofs

Separability of $L^p(I)$ (Lemma 0.0.2)

Proof. (Brezis, 4.13) Let $\mathcal{R} := \{(a_k, b_k) \subset I : a_k, b_k \in \mathbb{Q}\}$ and $\mathcal{E}$ the $\mathbb{Q}$-vector space consisting of all finite linear combinations of the functions $(\mathbb{1}_r)_{r \in \mathcal{R}}$ with rational coefficients. Note that $\mathcal{E}$ is countable.

Given $f \in L^p(I)$ and $\varepsilon > 0$ there exists some $f_1 \in \mathcal{C}_0(I)$ such that $\|f - f_1\|_p < \frac{\varepsilon}{2}$ and a $R := (a_1, b_1) \in \mathcal{R}$ such that $\text{supp}(f_1) \subset R$. Given $\delta > 0$ one can construct a function $f_2 \in \mathcal{E}$ such that $\|f_1 - f_2\|_\infty < \delta$ and $f_2|_{I \setminus R} \equiv 0$: split R into intervals $R_i \subset R$ and define $f_2^{(i)} := C_i \mathbb{1}_{R_i}$, where $C_i \in [0, \delta - (\sup(f|_{R_i} - \inf(f|_{R_i})])$ and define $f_2 := \sum_i f_2^{(i)}$. Therefore, we have

$$\|f_1 - f_2\|_p \leqslant \left(\int_R \|f_1 - f_2\|_\infty \right)^{\frac{1}{p}} = \|f_1 - f_2\|_\infty \cdot |b_1 - a_1|^{\frac{1}{p}} < \delta \cdot |b_1 - a_1|^{\frac{1}{p}}.$$

Therefore, $\|f - f_2\|_p \overset{\triangle \neq}{<} \varepsilon$ provided $\delta > 0$ is chosen so that $\delta |b_1 - a_1|^{\frac{1}{p}} < \frac{\varepsilon}{2}$. $\qquad\square$

Continuity in the L^p-mean

Proof. For $h \in \mathbb{R}$ let $T_h \colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be defined by $T_h u(x) = u(x + h)$. Note that $\|T_h u\|_p = \|u\|_p$ for all $u \in L^p(\mathbb{R})$.

Now, let $u \in L^p(\mathbb{R})$ be fixed and let $\epsilon > 0$ be given. Then we find $\varphi \in C_0^\infty(\mathbb{R})$ such that $\|u - \varphi\|_p < \epsilon/3$. Hence,

$$\|T_h u - u\|_p \leqslant \|T_h(u - \varphi)\|_p + \|T_h \varphi - \varphi\|_p + \|\varphi - u\|_p < \frac{2}{3}\epsilon + \|T_h \varphi - \varphi\|_p.$$

Since $\varphi \in \mathcal{C}^\infty$ we can build upon the similar lemma from DGL I to find $\delta > 0$ such that

$$\|T_h \varphi - \varphi\|_p = \left(\int |\varphi(x + h) - \varphi(x)|^p \, \mathrm{d}x \right)^{1/p} < \frac{\epsilon}{3}$$

for $|h| < \delta$. Thus, for these $|h| < \delta$ we have $\|T_h u - u\|_p < \epsilon$. $\qquad\square$

Simple, step and smooth functions dense in $L^1(\mathbb{R}^d)$

TODO

Standard rules for weak derivatives

Sobolev spaces are Banach spaces

First we show that the Sobolev norm is a norm:

①

②

③

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