Remarques sur les éspaces uniformément convexes (Remarks on uniformly convex spaces) *

Robert Fortet, Bulletin de la S. M. F, 1941, p. 23-46.

Translated and commented by Viktor Stein, March 2022

TODO: substitute normal by orthogonal?

1 Normal projections in uniformly convex spaces

Let B be a real or complex BANACH space and V be a complete subspace of B. Consider the quantity $|X - x|$[1] for fixed $X \in B$ and varying $x \in V$. It has a lower bound $m \geq 0$ ($m = 0$ meaning that $X \in V$).

Definition 1 (Normal projection)**.** If $x \in V$ is such that

$$|X - x| = \inf_{x \in V} |X - x|,$$

then x is a *normal projection of* X *onto* V.

Definition 2 (Strict convexity)**.** The space B is *strictly convex* if $x, y \in B$ fulfilling the condition $|x + y| = |x| + |y|$ implies that $x = \lambda y$ for some $\lambda \geq 0$.

We then have

Lemma 1. *If B is strictly convex, then the normal projection x of X onto V, it it exists, is unique.*

*The properties established in this note are not all new, as will be seen by referring to the various authors we quote in the notes, or to [Pet39, p. 249]. But we thought it would be interesting to show how these properties could be established by extremely elementary procedures.

[1]The symbol $|x|$ denotes either the norm of x or the absolute value of x, depending on whether $x \in B$ or $x \in \{\mathbb{R}, \mathbb{C}\}$.

Proof. Suppose that there are two distinct normal projections x and x' of X onto V. Then we would have

$$|X - x| = |X - x'| = m.$$

This is impossible if $m = 0$ because then necessarily $x = x' = X$. For $m > 0$ we have

$$\left|X - \frac{x + x'}{2}\right| = \left|\frac{X - x}{2} + \frac{X - x'}{2}\right| \leq \left|\frac{X - x}{2}\right| + \left|\frac{X - x'}{2}\right| = m,$$

so $\frac{x+x'}{2}$ is as close to V as x and x', thus if we have

$$\left|X - \frac{x + x'}{2}\right| < m,$$

then m would not be the lower bound of $|X - x|$ for $x \in V$. We must therefore have

$$\left|X - \frac{x + x'}{2}\right| = m \qquad \text{or} \qquad |(X - x) + (X - x')| = |X - x| + |X - x'|.$$

As B is strictly convex, we have

$$X - x' = (X - x)\lambda \qquad \text{with } \lambda = 1$$

or $x = x'$, which establishes the lemma. ☐

Definition 3 (Uniform convexity)**.** The space B is *uniformly convex* [Cla36] if for $x, y \in B$ there exists a function $\delta_k(\varepsilon) > 0$ such that

$$|x| \leq |y| \leq k, \text{ where } k > 0 \qquad \text{and} \qquad |x - y| > \varepsilon$$

imply

$$\left|\frac{x + y}{2}\right| \leq |y| - \delta_k(\varepsilon).$$

We call $\delta_k(\varepsilon)$ the *modulus of uniformity* of B.

Then we have the following theorem:

Theorem 1. *If B is uniformly convex, then the normal projection x of X onto V exists and is unique.*

Proof. If x exists, it is unique be lemma 1 because a uniformly convex space is necessarily strictly convex [Cla36, p. 404].

Let us prove that x exists. There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset V$ such that $\lim_{n\to\infty} |X - x_n| = m$. We can extract a subsequence $(x'_n)_{n\in\mathbb{N}}$ such that

$$|X - x'_{n+1}| \leq |X - x'_n|$$

for all $n \in \mathbb{N}$.

If $(x'_n)_{n\in\mathbb{N}}$ does not converge to a limit x, we can find a $\varepsilon > 0$ such that for values of n as large as we want to and for certain values of p

$$|x'_n - x'_{n+p}| > \varepsilon.$$

But then we would have

$$m \leq \left| \frac{(X - x'_n) + (X - x'_{n+p})}{2} \right| = \left| X - \frac{x'_n + x'_{n+p}}{2} \right|$$

and since B is uniformly convex

$$\frac{1}{2}|(X - x'_n) + (X - x'_{n+p})| \leq |X - x'_n| - \delta_k(\varepsilon)$$

since

$$|(X - x'_n) - (X - x'_{n+p})| = |x'_n - x'_{n+p}| > \varepsilon.$$

Making n so large that $|X - x'_n| < m + \frac{1}{2}\delta_k(\varepsilon)$, this would mean that

$$m \leq m - \frac{1}{2}\delta_k(\varepsilon),$$

a contradiction to $\delta_k(\varepsilon) > 0$.

Hence (x'_n) converges to some limit x such that $|X - x| = m$. $\square$

Let B be a uniformly convex space. Then the normal projection on a linear subspace V is well-defined and unique.

Lemma 2. *In a uniformly convex space B the normal projection onto a linear subspace V is a continuous operator.*

Proof. Let $X, Y \in B$ and x and y be their projections onto V with $\varepsilon := |x - y|$. We have

$$|Y - y| \geq |y - X| - |X - y| \quad \text{and} \quad |Y - x| \leq |Y - x| + |X - x|$$

and by the uniform convexity of B,

$$|X - x| < \left| X - \frac{x + y}{2} \right| \leq |X - y| - \delta_k(\varepsilon)$$

by taking $k \coloneqq |X - y|$. This yields

$$|Y - y| \geq |Y - x| + \delta_k(\varepsilon) - 2|X - Y|$$

Hence as $|Y - x| < |Y - y|$ we have

$$\delta_k(\varepsilon) \leq 2|X - Y|,$$

proving the statement. $\square$

Definition 4 (Normality)**.** We say that y is normal to x if the minimum of $|y - \lambda x|$ is attained at $\lambda = 0$, that is, if the projection of y onto x is zero. Analogously, we define the normality of y to a complete linear subspace V.

Definition 5 (Reciprocal normality)**.** The space B has *reciprocal normality* if for all x, y the fact that y is normal to x entails that x is normal to y.

This is not always the case as we shall see from the example in the ℓ^p spaces.

Definition 6 (Regular norm)**.** The uniformly convex space B has *regular norm* if $y \neq o$ being normal to x implies that

$$|y + \lambda x| = |y| + o(|\lambda|),$$

where $o(|\lambda|)$ is infinitely small with $|\lambda|$ of order greater than 1.

Example 1 (The ℓ^p spaces for $p > 1$)**.** Let ℓ^p for $p > 1$ be the space of points $x = (x_1, \ldots, x_n, \ldots)$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

This space is a Banach space if we set

$$|x| \coloneqq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

It has been shown [Cla36, p. 403] that ℓ^p is uniformly convex. Let $X \coloneqq (x_n)_{n=1}^{\infty}, Y \coloneqq (y_n)_{n=1}^{\infty} \in \ell^p$. We show that if $\lambda = \lambda_1 + i\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$, the function

$$\Phi(\lambda_1, \lambda_2) = |y + \lambda x| = \left(\sum_{n=1}^{\infty} |y_n + \lambda x_n|^p \right)^{\frac{1}{p}}$$

has continuous partial derivatives $\frac{\partial \Phi}{\partial \lambda_1}$ and $\frac{\partial \Phi}{\partial \lambda_2}$ provided $|\lambda| < H$, $|y| > 0$ and $y \neq \lambda_0 x$.

Define

$$A_n \coloneqq |y_n + \lambda x_n|^p \qquad \text{and} \qquad A \coloneqq \sum_{n=1}^{\infty} A_n.$$

We distinguish the A_n into three categories: those for which

- $x_n = y_n = 0$, which play no role in the sum,
- $y_n = 0$ and $|x_n| \neq 0$, which we will call $A_{n'}$,
- $y_n \neq 0$, which we call $A_{n''}$.

Let

$$A' \coloneqq \sum_{n'} A_{n'} \qquad \text{and} \qquad A'' \coloneqq \sum_{n''} A_{n''}.$$

We have

$$\frac{\partial A_{n''}}{\partial \lambda_1} = \frac{p}{2}|y_{n''} + \lambda x_{n''}|^{p-2}(x_{n''}\overline{y_{n''}} + \overline{x_{n''}}y_{n''} + 2\lambda_1|x_{n''}|^2)$$

and an analogous formula for $\frac{\partial A_{n''}}{\partial \lambda_2}$.

This yields

$$\begin{aligned}\frac{\partial A_{n''}}{\partial \lambda_1} - i\frac{\partial A_{n''}}{\partial \lambda_2} &= p x_{n''}|y_{n''} + \lambda x_{n''}|^{p-2}(\overline{y_{n''}} + \overline{\lambda}\overline{x_{n''}}) \\ &= p x_{n''}|y_{n''} + \lambda x_{n''}|^{p-1}\operatorname{sgn}(y_{n''} + \lambda x_{n''}) \eqqcolon B_{n''}.\end{aligned}$$

If $|\lambda| < H$, $B_{n''}$ depends continuously on λ_1 and λ_2 for every n''. As the series

$$\sum_{n''} |B_{n''}| = p\sum_{n''} x_{n''}|y_{n''} + \lambda x_{n''}|^{p-1}\operatorname{sgn}(y_{n''} + \lambda x_{n''}).$$

converges uniformly for $|\lambda| < H$, it follows that for $|\lambda| < H$, A'' has continuous partial derivatives with respect to λ_1 and λ_2.

For A' it is easy to check that

$$\frac{\partial A'}{\partial \lambda_j} = p|\lambda|^{p-1}\sum_{n'} |x_{n'}|^p \frac{\lambda_j}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \quad j \in \{1, 2\},$$

which are well defined and continuous for $|\lambda| < H$.

Hence $\frac{\partial A}{\partial \lambda_1}$ and $\frac{\partial A}{\partial \lambda_2}$ exist and are continuous for $|\lambda| < H$ as A is never zero, we have

$$\frac{\partial \Phi}{\partial \lambda_1} = \frac{1}{p}A^{\frac{1}{p}-1}\frac{\partial A}{\partial \lambda_1},$$

which proves that $\frac{\partial \Phi}{\partial \lambda_1}$ also exists and is continuous and likewise for $\frac{\partial \Phi}{\partial \lambda_2}$.

What is then the necessary and sufficient condition that $Y \neq 0$ is normal to X? It is that, for $\lambda_1 = \lambda_2 = 0$ we have $\frac{\partial \Phi}{\partial \lambda_1} = \frac{\partial \Phi}{\partial \lambda_2} = 0$, which results in

$$\sum_n |y_n|^{p-1} \operatorname{sgn}(y_n) x_n = 0,$$

or

$$\sum_n |y_n|^{p-2} \overline{y_n} x_n = 0.$$

Except in the case of $p = 2$ (then ℓ^p is a Hilbert space) this condition is not symmetric in x and y: thus ℓ^p is not reciprocally normalised.

On the other hand, it is easy to see that ℓ^p has a regular norm; it is sufficient to use the previous results.

We will investigate whether, in a uniformly convex space B, the normal projection is a distributive operator; that is, whether, $X, Y \in B$ and $V \subset B$ being a complete subspace, we have

$$\text{proj norm}_V(X + Y) = \text{proj norm}_V(X) + \text{proj norm}_V(Y)$$

This is true if $X \in V$. In the the general case, let

$$x \coloneqq \text{proj norm}_V X \qquad \text{and} \qquad y \coloneqq \text{proj norm}_V Y.$$

We have $x, y \in V$, so it suffices to establish the property in the particular case where X and Y are normal to V, that is, establishing that if X and Y are normal to V, so is $X + Y$.

Let us first prove the following lemma.

Lemma 3. *In a uniformly convex space B with a regular norm if Z is normal to X and Y, then it is normal to the subspace generated by X and Y.*

Remark 1 (By me). This property holds irrespective of uniform convexity and regularity of the norm [Jam47, p. 274].

It is then easy to establish the following theorem:

Theorem 2. *In a uniformly convex reciprocally normalised space B with regular norm, the normal projection on any complete linear subspace is a continuous linear operator.*

Proof. We have seen in lemma 2 that the normal projection is continuous. By lemma 3 and B being reciprocally normalised, it is linear. $\square$

Remark 2. It follows easily from the preceding theorem that, in a uniformly convex space B with regular norm that has reciprocal normality and is also *separable*, one can find a basis, that is, a sequence of points $(x_k)_{k=1}^{\infty}$ with unit norm that are pairwise orthogonal and such that any point $x \in B$ is uniquely representable by a convergent "series" of the form

$$x = \sum_{k=1}^{\infty} \lambda_k x_k,$$

where the λ_k are real or complex numbers.

Remark 3 (Generalisation). Given a closed and convex subset $E \subset B$, that is $x_1, x_2 \subset E$ implies $\frac{x_1+x_2}{2} \in E$, we can define the projection of any point x onto E. If B is uniformly convex, it is easy to see that this projection exists and is unique.

2 Application to the iteration of linear operators

Let $U \colon B \to B$ be a continuous linear operator. Three subspaces can be associated with U:

- the (not necessarily closed) subspace $V \coloneqq \{U(x) - x : x \in B\}$,
- the subspace V', the closure of V,
- the closed subspace $W \coloneqq \{x \in B : U(x) = x\}$.

Finally, one can consider the set E of points normal to V, if normality can be defined on the space B.

Define

$$U_{n,m} \coloneqq \frac{U^{n+1} + U^{n+2} + \ldots + U^{n+m}}{m}.$$

Note that if $y \in V'$ then

$$\lim_{m \to \infty} U_{n,m}(y) = 0$$

for all n, provided that $\|U^p\|$ is uniformly bounded.

Case 1: B is strictly convex.

Definition 7 (Normality in strictly convex spaces)**.** If B is strictly convex, then x is normal to V if for all $y \in V \setminus \{0\}$ and all $\lambda \neq 0$ we have $|x - \lambda y| > |x|$.

Theorem 3. *If B is strictly convex and $\|U\| = 1$, then $E = W$.*

Proof. $E \subset W$: If $x \in E$, as $U(x) = x + (U(x) - x)$ and $U(x) - x \in V$, if $|U(x) - x|$ were positive, then we would have $|U(x)| > |x|$, contradicting $\|U\| = 1$.

$W \subset E$: If $x \in W$ was not normal to V, then one could find a $y \in V$ such that $|x - y| \leq |x|$, because if we had $|x - y| > |x|$ for all $y \in V$, then x would be normal to V.

So if we have $|x - y| < |x|$ and letting $z \coloneqq x - y$ we have

$$U_{n,m}(x) = x = U_{n,m}(z) + U_{n,m}(y),$$
$$|x| \leq |U_{n,m}(z)| + |U_{n,m}(y)|,$$
$$|x| \leq |z| + |U_{n,m}(y)|$$

and in the limit $|x| \leq |z|$, which is a contradiction.

If we have $x - y = x$, without being able to obtain $|x - y| < |x|$, it is because y is a "projection" of x onto V. This projection, which exist by hypothesis, is moreover unique since B is strictly convex.

Setting $x - y = z$ let $|z - \mu y'|$ for any $y' \in V$,

$$|z - \mu y'| = |x - (y + \mu y')|.$$

If $\mu \neq 0$ and $y' \neq 0$, then the uniqueness of the projection of x implies that

$$|x - (y + \mu y')| > |x| = |z|.$$

It follows that z is normal to V, therefore $z \in E$ and in W and we have

$$U_{n,m}(x) = U_{n,m}(z) + U_{n,m}(y),$$
$$x = z + U_{n,m}(y)$$

and in the limit $x = z$, so x is in E, as is z. □

Case 2: B is uniformly convex. If B is uniformly convex, the normality in V agrees with the normality in V'. Any point x has a projection y onto V' and we can write

$$x = z + y$$

with $z \in W$. So

$$U_{n,m}(x) = z + U_{n,m}(y),$$

thus

$$\lim_{m\to\infty} U_{n,m}(x) = z$$

for all n (there's a typo in the original manuscript here!).

Defining $\overline{U}(x) = z$, the operator thus defined is linear and continuous, and we have (always under the assumption $\|U\| = 1$)

$$\|\overline{U}\| = 1$$

provided $W \neq \emptyset$. Moreover, $\overline{U}$ and $U - \overline{U}$ are orthogonal.

We can then state the following theorem:

Theorem 4. *In a uniformly convex space B any continuous linear operator U of unit norm is the sum of two orthogonal continuous linear operators U and $\overline{U}$ such that*

$$\lim_{m\to\infty} U_{n,m} = \overline{U}$$

for all n.

This theorem was first stated in [Bir39].

3 Linear functionals in uniformly convex spaces

Let Φ be a continuous linear functional on the uniformly convex space B. Consider the complete subspace $V \coloneqq \{x \in B : \Phi(x) = 0\}$, called the characteristic subspace of Φ. If $V = B$, then $\Phi \equiv 0$. Setting aside this case, there is a $x \in B \setminus V$ such that $\Phi(x) = a \neq 0$. Then $B = V \oplus \langle x \rangle$, that is, any $z \in B$ can be uniquely represented as $z = \alpha x + y$, where α is a real or complex number and $y \in V$.

Proof. This is obvious if $z \in V$ (choose $\alpha = 0$). If $z \notin V$, we have

$$\Phi(z) = b \neq 0.$$

Let

$$t \coloneqq x - \frac{a}{b} z$$

Then

$$\Phi(t) = \Phi(x) - \frac{a}{b}\Phi(z) = a - \frac{a}{b} b = 0,$$

so $t \in V$ and we have $z = \frac{b}{a}x - \frac{b}{a}t$, which is of the form $\alpha x + y$. □

It follows that if x is normal to V, any point normal to V is of the form λx. In this case, when you put $z = \alpha x + y$, then y is the projection of z onto V and we have $\Phi(z) = \alpha\Phi(x)$.

In summary, *any linear functional expresses a certain normality.* Moreover,

$$\|\Phi\| = \sup_{\substack{z\in B \\ z\neq 0}} \frac{|\Phi(z)|}{|z|} = \sup_{\substack{z\in B \\ z\neq 0}} \frac{|\alpha||\Phi(x)|}{|z|}$$

and as $|z| \geq |\alpha||x|$, we have

$$\|\Phi\| \leq \frac{|\Phi(x)|}{|x|},$$

so

$$\|\Phi\| = \frac{|\Phi(x)|}{|x|}.$$

You can always choose x such that we have

$$|x| = 1, \qquad \Phi(x) = \|\Phi\|. \tag{1}$$

Definition 8 ((Norm) Vertex of a linear functional)**.** The $x \in B$ determined uniquely by (1) is the *vertex* of Φ. The point $\|\Phi\|x \in B$ is the *norm vertex* of Φ.

The norm vertex of Φ fully characterises Φ.

Reciprocally, let B be a uniformly convex space with regular norm. For $x \in B$, the set of points y to which x is normal forms a closed subspace V. It is easy to show that every point normal to V is for the form λx. If $z \in B$ and y is its projection on V, there is a unique representation

$$z = \alpha x + y,$$

where α is a real or complex number and $y \in V$. If we then set $\alpha = \Phi(z)$, Φ is clearly a continuous linear functional, whose characteristic subspace is V and whose norm is $\frac{1}{|x|}$.

So, in a way, in uniformly convex space with regular norm, any normality relation expressible in norm can be expressed by a linear functional.

In a uniformly convex space B consider two continuous linear functionals Φ_1 and Φ_2 with norm vertices x_1 and x_2 and characteristic subspaces V_1 and V_2. It is easy to see that if $\|\Phi_1 - \Phi_2\|$ tends to zero, without $\|\Phi_1\|$ and $\|\Phi_2\|$ tending to zero, $|x_1 - x_2|$ tends to zero, in a certain sense, uniformly.

To be more precise, let us assume that $\|\Phi_1\| = \|\Phi_2\| = 1$. We have $\|\Phi_1 - \Phi_2\| \geq |\Phi_1(x_1) - \Phi_2(x_1)|$ and if we set $x_1 = \alpha x_2 + y$, where $y \in V_2$, $|\alpha| > 1$, we have $\|\Phi_1 - \Phi_2\| \geq |1 - \alpha|$ or

$$\left|\alpha x_2 + \frac{y}{2}\right| \leq |\alpha x_2 + y| - \delta_1(|y|),$$
$$\delta_1(|y|) \leq 1 - \left|\alpha x_2 + \frac{y}{2}\right| < 1 - |\alpha x_2| = 1 - |\alpha| < |1 - \alpha|.$$

Note that we can without loss of generality assume that $\delta_1(|y|)$ is an increasing function of $|y|$. Denoting by Δ_1 its inverse function, we have

$$|y| \leq \Delta_1(|1 - \alpha|)$$

or

$$|x_1 - x_2| \leq |1 - \alpha| + |y| \leq \|\Phi_1 - \Phi_2\| + \Delta_1(\|\Phi_1 - \Phi_2\|),$$

which establishes the promised result.

Let us consider the converse, that is, ask ourselves if the fact that $|x_1 - x_2|$ tending to 0 implies that $\|\Phi_1 - \Phi_2\|$ tends to zero uniformly, given that $\|\Phi_1\| = \|\Phi_2\| = 1$. We think that is correct only with the additional assumption that B is uniformly regularly normed.

Definition 9. A uniformly convex space B is uniformly regularly normed if there exists a function $o(\varepsilon)$ for $\varepsilon > 0$ infinitely small with ε and of order greater than 1, such that

$$|x + \lambda y| \leq |x| + o(|\lambda|)$$

if $|x| = |y| = 1$ and x is normal to y.

This is a stricter condition that the one for regularly normed spaces, because for the latter we accept that $o(|\lambda|)$ depends on x and y.

Lemma 4. *If B is uniformly convex and uniformly regularly normed, then there exists a function $\varphi(\varepsilon)$ for $\varepsilon > 0$ infinitely small with ε such that, if Φ_1 and Φ_2 are two linear functionals defined on B with unit norm and vertices x_1 and x_2, we have*

$$\|\Phi_1 - \Phi_2\| \leq \varphi(|x_1 - x_2|).$$

Proof. If $z \in B$, define

$$z = \alpha x_2 + y_2$$

with $y_2 \in V_2$ and $\alpha = \Phi_2(z)$ (there's a typo in the original manuscript here!) (as $\Phi_2(x_2) = 1$ and $\Phi_2(y_2) = 0$). Then

$$\Phi_1(z) = \Phi_1(x_2)\Phi_2(z) + \Phi_1(y_2)$$

and thus

$$\begin{aligned}\Phi_1(z) - \Phi_2(z) &= \Phi_1(x_2)\Phi_2(z) + \Phi_1(y_2) - \Phi_2(x_2)\Phi_2(z) \cancel{- \Phi_2(y_2)} \\ &= \Phi_2(z_2)\left[\Phi_1(x_2) - 1\right] + \Phi_1(y_2) = \Phi_2(z_2)\Phi_1(x_2 - x_1) + \Phi_1(y_2)\end{aligned}$$

and hence

$$|\Phi_1(z) - \Phi_2(z)| \leq |z||x_2 - x_1| + |\Phi_1(y_2)|. \tag{2}$$

Defining $t \coloneqq \frac{y_2}{|y_2|}$, we have

$$|\Phi_1(y_2)| = |y_2||\Phi_1(t)| \leq 2|z||\Phi_1(t)|, \tag{3}$$

because y_2, being the projection of z to V_2 is such that $|y_2| \leq 2|z|$. We have

$$|t| = 1 \qquad \text{and} \qquad t = \Phi_1(t)x_1 + y_1 \quad \text{with } y_1 \in V_1.$$

To simplify, we define $\lambda \coloneqq \Phi_1(t)$. Now, x_1 is norm to $t - \lambda x_1$, that is, for all μ we have

$$|x_1 - \mu(t - \lambda x_1)| \geq |x_1| = 1.$$

As

$$x_1 = x_2 + (x_1 - x_2) = x_2 + \varepsilon$$

with $\varepsilon \coloneqq x_1 - x_2$, it follows that

$$\begin{aligned}|x_1 - \mu(t - \lambda x_1)| = |x_2 + \mu\lambda x_2 - \mu t + \varepsilon + \mu\lambda\varepsilon| &\geq 1 \\ \left|x_2 - \frac{\mu}{1+\mu\lambda}t + \varepsilon\right| &\geq \frac{1}{|1+\mu\lambda|}, \\ \left|x_2 - \frac{\mu}{1+\mu\lambda}t\right| + |\varepsilon| &\geq \frac{1}{|1+\mu\lambda|}, \\ 1 + o\left(\left|\frac{\mu}{1+\mu\lambda}\right|\right) + |\varepsilon| &\geq \frac{1}{|1+\mu\lambda|},\end{aligned}$$

since B is uniformly regularly normed. Note that $|\lambda| < 1$. Taking $\mu = -\sqrt{\varepsilon}\frac{|\lambda|}{\lambda}$ yields

$$1 + o\left(\left|\frac{\sqrt{|\varepsilon|}}{1 - |\lambda|\sqrt{|\varepsilon|}}\right|\right) + |\varepsilon| \geq 1 + |\lambda|\sqrt{|\varepsilon|}.$$

Hence, if $|\varepsilon| < 1$, then

$$|\lambda| \leq \sqrt{|\varepsilon|} + \frac{1}{\sqrt{|\varepsilon|}}o(2\sqrt{|\varepsilon|}), \tag{4}$$

by plugging (4) and (3) into (2). $\square$

Remark 4. If we consider two subspaces V_1 and V_2 we can see how they differ by setting
$$\text{angle}(V_1 \text{ with } V_2) = \sup \frac{\text{dist}(y_1, V_2))}{|y_1|},$$
where $y_1 \in V_1$. As $\text{dist}(y_1, V_2) = |\Phi_2(y_1)|$, we see that (4) indicates that: if $|x_1 - x_2|$ tends to 0, the angle of V_1 with V_2 tends uniformly to 0. Note that x_1 defines a "direction" common to all points λx_1. Similarly, for x_2 we can define
$$\text{angle}(\text{direction } x_1 \text{with direction } x_2) = \text{dist}(x_1, \langle x_2 \rangle).$$
This angle is not altered if we replace x_1 with λx_1, where $|\lambda| = 1$.

As before, we show that: if the angle between the direction x_1 and the direction x_2 tends to 0, then the angle of V_1 with V_2 tends uniformly to zero.

We then have

Theorem 5. *The dual space $\overline{B}$ of a uniformly convex space B with uniformly regular norm is uniformly convex.*

Proof. ... □

Example 2 (Application)**.** Consider the space ℓ^p for $p > 1$ and suppose that y is normal to x, where $|y| = |x| = 1$. We study the difference
$$D(\lambda) := |y + \lambda x| - |\lambda|.$$
First, let us study
$$d(\lambda) := \sum_{n=1}^{\infty} |y_n + \lambda x_n|^p - \sum_{n=1}^{\infty} |y_n|^p.$$
It follows from a previous calculation that d is continuously differentiable with respect to $\lambda_1 := \Re(\lambda)$ and $\lambda_2 := \Im(\lambda)$.

For simplicity we assume that $\lambda = \lambda_1$ is real. We first assume that $y_n \neq 0$ for all n. Then we have, by our previous calculations,
$$d'(\lambda) = p \sum_{n=1}^{\infty} x_n |y_n + \lambda x_n|^{p-2} \overline{y_n + \lambda x_n}.$$
For $\theta \in [0, 1]$ it follows that
$$d(\lambda) = \lambda p \sum_{n=1}^{\infty} x_n |y_n + \theta \lambda x_n|^{p-2} \overline{y_n + \theta \lambda x_n}.$$

In the case that $p < 2$, this would not be valid for $y_n + \theta\lambda x_n = 0$, but in this case we would write

$$x_n|y_n + \theta\lambda x_n|^{p-1}\operatorname{sgn}(y_n + \theta\lambda x_n),$$

which is zero, so that what follows remains valid. But we have

$$|y_n + \theta\lambda x_n|^{p-2} = |y_n|^{p-2} + \theta\lambda(p-2)x_n|y_n + \theta'\theta\lambda x_n|^{p-3}\operatorname{sgn}(y_n + \theta'\theta\lambda x_n),$$

where $\theta' \in [0,1]$.

On the other hand, as y is normal to x, we have

$$\sum_{n=1}^{\infty} |y_n|^{p-2}\overline{y_n}x_n = 0,$$

so that d can be written as

$$\begin{aligned} d(\lambda) &= \lambda p \sum_n x_n \left[|y_n|^{p-2}\theta\lambda x_n + \theta\lambda(p-2)x_n(\overline{y_n + \theta\lambda x_n})\operatorname{sgn}(y_n + \theta'\theta\lambda x_n)|y_n + \theta'\theta\lambda x_n|^{p-3}\right] \\ &= \theta\lambda^2 p[A + (p-2)B(\theta'\theta\lambda)], \end{aligned}$$

where

$$A = \sum_{n=1}^{\infty} |y_n|^{p-2}x_n^2$$

and

$$B(\theta'\theta\lambda) = \sum_n |y_n + \theta'\theta\lambda x_n|^{p-3}\overline{y_n + \theta\lambda x_n}\operatorname{sgn}(y_n + \theta'\theta\lambda x_n)x_n^2.$$

Like the series $\sum_n |x_n^2|^{\frac{p}{2}}$ and $\sum_n \left(|y_n|^{p-2}\right)^{\frac{\frac{p}{2}}{\frac{p}{2}-1}}$ converge, it is easy to see that $|A| \le 1$. On the other hand, we have

$$|B| \le \sum_n |y_n + \theta'\theta\lambda x_n|^{p-3}|y_n + \theta\lambda x_n||x_n|^2.$$

Letting

$$z_n = |y_n + \theta\lambda x_n||x_n|^2 \quad \text{and} \quad t_n = |y_n + \theta'\theta\lambda x_n|^{p-3},$$

we have

$$\sum_{n=1}^{\infty} t_n^{\frac{\frac{p}{3}}{\frac{p}{3}-1}} = \sum_{n=1}^{\infty} t_n^{\frac{p}{p-3}} = \sum_{n=1}^{\infty} |y_n + \theta'\theta\lambda x_n|^p < 2^p,$$

$$\sum_{n=1}^{\infty} z_n^{\frac{p}{3}} = \sum_{n=1}^{\infty} |y_n + \theta\lambda x_n|^{\frac{p}{3}}|x_n|^{\frac{2p}{3}}.$$

If

$$u_n := |y_n + \theta\lambda x_n|^{\frac{p}{3}} \quad \text{and} \quad v_n := |x_n|^{\frac{2p}{3}},$$

we have

$$\sum_n u_n^3 = \sum_n |y_n + \theta\lambda x_n|^p < 2^p \quad \text{and} \quad \sum_n v_n^{\frac{3}{2}} = \sum_n |x_n|^p = 1$$

As a result,

$$\sum_{n=1}^{\infty} z_n^{\frac{p}{3}} \leq \left|\sum_{n=1}^{\infty} u_n v_n\right| \leq \left(\sum_n u_n^3\right)^{\frac{1}{3}} \left(\sum_n v_n^{\frac{3}{2}}\right)^{\frac{2}{3}} < 2^{\frac{p}{3}}.$$

Finally,

$$|B| \leq \sum_{n=1}^{\infty} z_n t_n \leq \left(\sum_n t_n^{\frac{p}{p-3}}\right)^{\frac{p-3}{p}} \left(\sum_n z_n^{\frac{p}{3}}\right)^{\frac{3}{p}} < 2^{p-3} \cdot 2 = 2^{p-2}.$$

We conclude

$$|d(\lambda)| \leq |\lambda|^2 p(1 + |p-2|2^{p-2}).$$

In other words, $d(\lambda)$ infinitely small with $|\lambda|$ of order greater than 1, uniformly with respect to x and y. This result remains valid if some $y_n + \theta'\theta\lambda x_n$ cancel each other out, if some y_n cancel and it extends to the case of complex λ.

We can easily deduce the same result for D and it follows that

Theorem 6. *The space ℓ^p for $p > 1$ are uniformly regularly normed.*

We can augment theorem 5 with various statements:

Theorem 7. *If B is uniformly convex and regularly normed, $\overline{B}$ is strictly convex.*

Proof. If $\|\Phi_1 + \Phi_2\| = \|\Phi_1\| + \|\Phi_2\|$ and if z is the vertex of $\Phi_1 + \Phi_2$, we must have

$$|\Phi_1(z)| = \|\Phi_1\| \qquad \text{and} \qquad |\Phi_2(z)| = \|\Phi_2\|.$$

Hence necessarily,

$$z = \lambda_1 x_1 = \lambda_2 x_2 \quad \text{with } |\lambda_1| = |\lambda_2| = 1$$

from which it follows that $x_1 = \frac{\lambda_2}{\lambda_1} x_2$ and therefore that

$$\Phi_1 = \frac{\lambda_2}{\lambda_1} \frac{\|\Phi_1\|}{\|\Phi_2\|} \Phi_2,$$

so

$$\|\Phi_1 + \Phi_2\| = \|\Phi_2\| \left| 1 + \frac{\lambda_2}{\lambda_1} \frac{\|\Phi_1\|}{\|\Phi_2\|} \right| = \|\Phi_2\| + \|\Phi_2\| \left| \frac{\lambda_2}{\lambda_1} \right| \frac{\|\Phi_1\|}{\|\Phi_2\|},$$

from which is it easily follows that $\frac{\lambda_2}{\lambda_1} > 0$, which establishes the theorem. □

Suppose B is uniformly regularly normed and Φ_1 and Φ_2 have unit norm. Consider $\|\Phi_1 + \lambda\Phi_2\|$ and let z be the vertex of $\Phi_1 + \lambda\Phi_2$ with $z = \alpha x_1 + y_1$, where $\alpha = \Phi_1(z)$. Replacing $\Phi_1 + \lambda\Phi_2$ by $(\Phi_1 + \lambda\Phi_2)\,\mathrm{sgn}(\alpha)$, which doesn't change $\|\Phi_1 + \lambda\Phi_2\|$, we can assume $\alpha > 0$, moreover, $\alpha < 1$. We have already seen that $1 - \alpha$ and $|y_1|$ are uniformly infinitely small with $|\lambda|$. We can pose

$$y_1 = [\Phi_1(y_1) + \lambda\Phi_2(y_1)]z + y,$$

where z is normal to y. Moreover, $\Phi_1(y_1) = 0$, so

$$\begin{aligned} y_1 &= \lambda\Phi_2(y_1)z + y, \\ z &= \alpha x_1 + \lambda\Phi_2(y_1)z + y, \\ z - y &= \alpha x_1 + \lambda\Phi_2(y_1)z. \end{aligned}$$

As z is normal to y, we deduce

$$1 \le |\alpha x_1 + \lambda\Phi_2(y_1)z| \le |\alpha| + |\lambda||y_1|,$$

and thus

$$1 - \alpha \le |\lambda||y_1|. \tag{5}$$

Then we have

$$\begin{aligned} \|\Phi_1 + \lambda\Phi_2\| &= \Phi_1(z) + \lambda\Phi_2(z_2) = \alpha + \lambda\alpha\Phi_2(x_1) + \lambda\Phi_2(y_1) &(6)\\ &= 1 - (1 - \alpha) + \lambda\Phi_2(x_1) + \lambda(\alpha - 1)\Phi_2(x_1) + \lambda\Phi_2(y_1). &(7) \end{aligned}$$

As $1 - \alpha$ is infinitely small of order greater than 1 due to (5), the second term in (6) cannot be minimal for $\lambda = 0$ only if $\Phi_2(x_1) = 0$. It is easy to show that this is sufficient, therefore we have

Theorem 8. *Let B be uniformly convex and uniformly regularly normed. The functional Φ_1 is normal to the functional Φ_2 if and only if the vertex x_2 of Φ_2 is normal to the vertex of Φ_1.*

When $\Phi_2(x_1) = 0$, it follows from (6) that

$$\|\Phi_1 + \lambda\Phi_2\| - |\Phi_1| \le o(|\lambda|),$$

where F is uniformly infinitely small of order greater than 1.

As a result, we have

Theorem 9. *If B is uniformly convex and has a uniformly regular norm, then $\overline{B}$ is uniformly convex and has a uniformly regular norm.*

Conclusion. Let B be uniformly convex and have a uniformly regular norm. If to $x \in B$ we associate $\Phi \in \overline{B}$ such that x is the norm vertex of Φ, this correspondence has the following properties:

1. it is invertible, injective and bicontinuous,
2. it preserves the norm,
3. it inverts normality (?)

In general, this correspondence is not linear.

Remark 5. Let $\overline{\overline{B}}$ be the conjugate space of $\overline{B}$. The previous mapping between B and $\overline{B}$ on one hand, and on the other hand between $\overline{B}$ and $\overline{\overline{B}}$ establishes a map between $\overline{B}$ and $\overline{\overline{B}}$ which has the above mentioned properties and we can easily verify that it is linear. B and $\overline{\overline{B}}$ are isomorphic [Pet39].

4 Spaces with reciprocal normality

Let B be a uniformly convex space with reciprocal normality. We have the following result.

In a uniformly convex space B with reciprocal normality, the set of points y to which x is normal forms a complete subspace.

It is easy to check that it is a complete space, it must be shown that it is linear, that is, if x is normal to y and to y_2, it is also normal to $y_1 + y_2$. If this is not the case, there is a λ for which we have

$$|x - \lambda(y_1 + y_2)| < |x|.$$

Consider the three-dimensional subspace $B' := \langle x, y_1, y_2 \rangle$. In this subspace, x is normal to y_1 and to y_2, but not to $y_1 + y_2$ because $x - \lambda(y_1 + y_2)$ is a point of B'. Hence it suffices to show the statement for the three-dimensional uniformly convex space B'.

We can assume $|x| = |y_1| = |y_2| = 1$. Consider the associated three-dimensional space B'', comprised by three orthogonal (?) vectors X, Y_1 and Y_2 such that a point $z = \alpha x + \beta y_1 + \gamma y_2 \in B'$ corresponds to a $Z \in B''$ with coordinates α,

β and γ. The surface locus (??) of Z with $|z| = 1$ is a continuous and convex (even uniformly convex, that is, it contains no line segments) surface Σ.

Let $Z \in \Sigma$. The vectors y to which z is normal have corresponding vectors in $Y \in B''$ parallel to line which touch at Σ in Z without intersecting the interior of Σ, thus which are in the tangent cone of Σ at Z. We claim that this cone is reduced to a plane. If this were not the case, we could find a plain containing the origin and Z and intersection Σ along a certain uniformly convex curve Γ such that Γ has two tangents in Z (one on the right, one on the left), which doe not coincide. Let Z_1 and Z_2 be two vectors both parallel to these tangents such that $|z_1| = |z_2| = 1$. Due to the reciprocal normality, z_1 and z_2 are normal to z and $Z_1, Z_2 \in \Gamma$. Thus, there exist two lines D_1 and D_2 parallel to Z, which touch but not traverse Γ in Z_1 and Z_2. As Γ is uniformly convex, this is only possible if $D_1 = D_2$ and thus also $Z_1 = Z_2$ and thus $z_1 = z_2$, which is a contradiction.

We deduce that theorem 2 is valid even if the space with reciprocal normality is not regular normed.

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