TECHNICAL UNIVERSITY BERLIN

Lecture Notes

Approximation Theory

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Without solutions to the exercises.

Contents

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Introduction

19.10.2021

In approximation theory, our goal is to approximate a function or data points with a simple function, for example a polynomial.

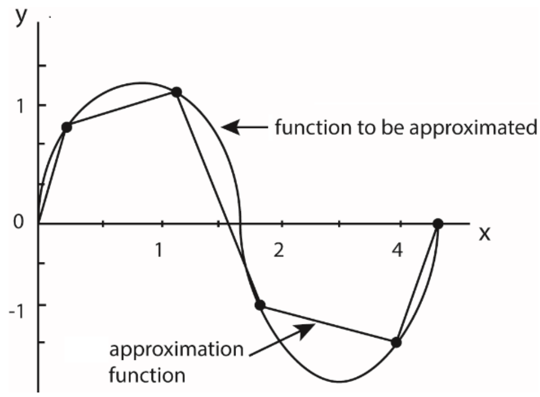


Fig. 1: Approximation of a function by simpler one [LHH15, Fig. 3].

Given this function f and a set $\mathcal{A}$ of nice functions, there are two possible problems:

- for a given $\varepsilon > 0$, find $p_\varepsilon \in \mathcal{A}$ such that we have $\|f - p_\varepsilon\| < \varepsilon$.
- find $p^* \in \mathcal{A}$ such that $\|f - p^*\| = \min_{p \in \mathcal{A}} \|f - p\|$ (best approximation).

The main questions that concern us are the ones of existence, uniqueness, construction (how can we find or construct such a approximation) and measure (e.g. choice of norm).

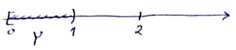
1 Introduction

This chapter roughly follows [Car98, Chp. 1].

best approximation

If not stated otherwise, we consider a normed space $(X, \|\cdot\|)$ and a nonempty subset $Y \subset X$. Can we find a best approximation $y^* \in Y$ such that $\|x - y^*\| = \min_{y \in Y} \|x - y\|$ and is it unique?

Example 1.0.1 (Best approximation: Toy example)
Let $X := \mathbb{R}$, $Y := [0, 1)$ and $x := 2$. Then $\min_{y \in Y} \|x - y\|$ does not exist, but $\inf_{y \in Y} \|x - y\| = 1$. ◇



Hence it seems useful to impose a condition on Y, for example closedness.

Remark 1.0.2 (Necessity of closedness of Y for existence of best approximation)
If $Y \subset X$ is not closed, for $x \in \overline{Y} \setminus Y$, there is no best approximation of x in Y. Indeed, as $x \in \overline{Y}$, there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset Y$ with $y_n \to x$, that is, $\|y_n - x\| \xrightarrow{n \to \infty} 0$. If there were a $y^* \in Y$ such that $\|y^* - x\| = \min_{y \in Y} \|y - x\|$, then $\|y^* - x\| = 0$ and thus $x = y^* \in Y$, contradicting $x \in \overline{Y} \setminus Y$. ○

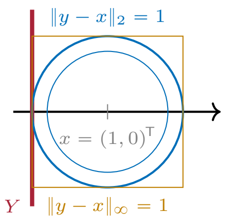


Fig. 2: The point x, the set Y and two 2-norm balls and one ∞-norm ball.

Furthermore, the choice of norm is important.

Example 1.0.3 ((Non-)Uniqueness of the best approximation in $\mathbb{R}^2$)
Let $X := (\mathbb{R}^2, \|\cdot\|_\infty)$, $Y := \operatorname{span}((0, 1)^\top)$ and $x := (1, 0)^\top$. For $y = (0, y_1)^\top \in Y$ we have

$$\|y - x\|_\infty = \|(0, y_1)^\top - (1, 0)^\top\|_\infty = \max(|y_1|, 1) \geqslant 1.$$

If $y_1 \in [-1, 1]$, then $\|x - y\|_\infty = 1 = \min_{y \in Y} \|y - x\|_\infty$, so there are infinitely many best approximations to x in Y. If we instead consider $(\mathbb{R}^2, \|\cdot\|_2)$, then the unique best approximation is $y^* = 0$, as $\|x - y^*\|_2 = \|x\|_2 = 1 < \|x - y\|_2$ for all $y \in Y \setminus \{(0, 0)\}$.

This shows the importance of the chosen norm even in finite-dimensional spaces, where all norms are equivalent. ◇

Definition 1.0.4 (Continuous functions, polynomials)
Let $\mathcal{C}([a, b])$ be equipped with the norm $\|f\|_\infty := \max_{x \in [a,b]} |f(x)|$, $\mathcal{P}_n$ be the set of polynomials of degree at most n and $\mathcal{P} := \bigcup_{n=0}^{\infty} \mathcal{P}_n$ be the space of all polynomials.

Example 1.0.5 (Approximating continuous functions by polynomials)
Let $X := \mathcal{C}([a, b])$ and $Y := \mathcal{P}_n$. Then $\dim(Y) = n + 1$. We will later see that every $f \in X$ has a best approximation in Y.

However for $Y := \mathcal{P}$, we know by Weierstrass's theorem that for all $\varepsilon > 0$ there exists a $p \in Y$ with $\|f - p\|_\infty < \varepsilon$, so there is no best approximation! A classic example is $f(x) = e^x$, whose best approximation in $\mathcal{P}_n$ is its n-th order Taylor expansion. ◇

1.1 Finite dimensional vector spaces

Lemma 1.1.1 (Equivalence of norms)
If V is a finite dimensional vector space, then every two norms $\|\cdot\|$ and $|||\cdot|||$ on V are equivalent, that is, there are constants $A, B > 0$ such that

$$A\|x\| \leqslant |||x||| \leqslant B\|x\| \qquad \forall x \in V.$$

Proof. See [Car98, p. 4]. □

Corollary 1.1.2 (Completeness)
Every finite dimensional normed space is complete.

Corollary 1.1.3 (Compactness)
Every bounded closed subset of a finite dimensional normed vector space is compact.

Corollary 1.1.4 (Closedness)
Every finite dimensional subspace Y of a normed vector space X is closed in X.

Proof idea for the corollaries. There exists a basis $e_1, \dots, e_n$ of V and thus every $x \in X$ can be expanded uniquely as $x = \sum_{i=1}^n a_i e_i$, so there exists a bijection $V \to \mathbb{R}^n$ mapping x to its coefficients $(a_i)_{i=1}^n$. By lemma 1.1.1, any norm on V is equivalent to $\|x\| := \sum_{i=1}^n |a_i|$. For $\mathbb{R}^n$, all the above properties are easy to prove.

Theorem 1.1.1: Existence of best approximation

Let $Y \subset X$ be a finite dimensional subspace. For all $x \in X$ there exists a best approximation $y_x \in Y$ to x from Y, that is,

$$\|x - y_x\| = \min_{y \in Y} \|x - y\|.$$

Proof. Let $x \in X$. Since Y is a vector space, $0 \in Y$ and thus

$$\min_{y \in Y} \|x - y\| \leqslant \|x - 0\| = \|x\|.$$

Hence any best approximation to x from Y must belong to the set

$$K_x := \{y \in Y : \|x - y\| \leqslant \|x\|\} \subset Y,$$

which is bounded and closed and thus compact by corollary 1.1.3. The function

$$f \colon K_x \to \mathbb{R}, \qquad y \mapsto \|x - y\|$$

is continuous on the compact set K_x so its attains a minimum in $y_x \in K$, which is the best approximation to x from Y. □

Remark 1.1.5 This theorem does not grant uniqueness of the best approximation. ○

Lemma 1.1.6 (Continuity of the best approximation map)
Let $Y \subset X$ be a finite dimensional subspace. Assume that for all $x \in X$, there exists a unique best approximation $y_x \in Y$. Then

$$P \colon X \to Y, \qquad x \mapsto y_x \tag{1}$$

is well-defined and continuous.

Proof. Let $x \in X$ and $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \to x$. By the subsequence principle it suffices to show that every subsequence of $(P(x_n))_{n \in \mathbb{N}} \subset Y$ has a subsequence converging to $P(x)$ in order to infer that $P(x_n) \to P(x)$.

Since $x_n \to x$, there exists a $C > 0$ such that $\|x_n\| \leqslant \frac{C}{2}$ for all $n \in \mathbb{N}$. Hence

$$\|P(x_n)\| \leqslant \|P(x_n) - x_n\| + \|x_n\| = \min_{y \in Y} \|y - x_n\| + \|x_n\| \leqslant \|0 - x_n\| + \|x_n\| \leqslant C,$$

so $(P(x_n))_{n\in\mathbb{N}} \subset Y$ is bounded. By corollary 1.1.3, there is a subsequence $(P(x_{n_k}))_{k\in\mathbb{N}}$ and a $y^* \in Y$ such that $P(x_{n_k}) \to y^*$.

We have

$$\begin{aligned}\|y^* - x\| \xleftarrow{k\to\infty} \|P(x_{n_k}) - x_{n_k}\| &= \min_{y\in Y} \|y - x_{n_k}\| \\ &\overset{y=P(x)}{\leqslant} \|P(x) - x_{n_k}\| \xrightarrow{k\to\infty} \|P(x) - x\| = \min_{y\in Y} \|y - x\|.\end{aligned}$$

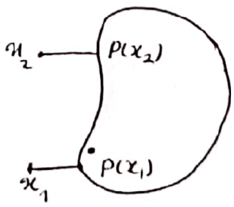


Fig. 3: The projection map P is not linear: $P(x_1) + P(x_2)$ might not belong to Y, so in that case $P(x_1 + x_2) \neq P(x_1) + P(x_2)$.

As the minimiser is unique by assumption, $y^* = P(x)$. □

Remark 1.1.7 (Nonlinearity of the projection map) In general, P is not linear: consider again example 1.0.3 with $\|\cdot\| = \|\cdot\|_2$. Then $P(x) = P(2x)$, but $x \neq 0$. ○

Theorem 1.1.2: Set of best approximations is bounded & convex

Let $Y \subset X$ be subspace and $x \in X$. The set $Y_x \subset Y$ of best approximations of x in Y is bounded and convex.

Proof. If Y_x is empty, the statement holds, so assume $Y_x \neq \varnothing$.

1. **Boundedness.** By definition, Y_x is a subset of the the bounded set

$$S_{d_x} := \{y \in Y : \|x - y\| \leqslant d_x\}, \tag{2}$$

where $d_x := \min_{y\in Y} \|x - y\|$.

2. **Convexity.** Suppose Y_x is not a singleton. For $y_1, y_2 \in Y_x$ and $\lambda \in (0,1)$ define $\tilde{y} := \lambda y_1 + (1-\lambda) y_2$. We have $\tilde{y} \in Y$ as Y is a subspace and thus convex. Then $\|x - y_1\| = \|x - y_2\| = \min_{y\in Y} \|x - y\|$ and

$$\begin{aligned}\|x - \tilde{y}\| &= \|\lambda x + (1-\lambda)x - \lambda y_1 - (1-\lambda) y_2\| \\ &\leqslant \lambda \|x - y_1\| + (1-\lambda)\|x - y_2\| = (\cancel{\lambda} + 1 \cancel{-\lambda}) \min_{y\in Y} \|x - y\| = \min_{y\in Y} \|x - y\|,\end{aligned}$$

so $\|x - \tilde{y}\| = \min_{y\in Y} \|x - y\|$ and hence $\tilde{y} \in Y_x$, so Y_x is convex. □

Remark 1.1.8 (Three cases) The set of best approximations is either empty, a singleton or has infinitely many elements, which is similar to the set of solutions to a linear equation. ○

Lemma 1.1.9
If Y is finite dimensional, then Y_x is closed for every $x \in X$.

In this case, Y_x is convex and compact by Theorem 1.1.2.

Proof. *This is left as an exercise to the reader.* □

1.2 Strictly convex norms

By positive homogeneity of the norm and the triangle inequality, any norm is a convex function and hence its unit ball is a convex set. A special case occurs if the unit ball is a strictly convex set.

Definition 1.2.1 (Strictly convex norm / space)
A norm $\|\cdot\|$ is strictly convex if for all $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = r > 0$ and all $\lambda \in (0,1)$ we have

strictly convex

$$\|\lambda x + (1-\lambda)y\| < r.$$

We say that $(X, \|\cdot\|)$ is a strictly convex space.

Geometrically, this corresponds to the interior of the straight line segment between to two distinct points on the boundary of any $\|\cdot\|$-ball being contained in the interior of that ball.
Remark 1.2.2 (Midpoint-strict convexity) We can without loss of generality choose $r = 1$ in the definition of strict convexity since we can divide $\|\lambda x + (1-\lambda)y\| < r$ by r and $\left\|\frac{x}{r}\right\| = 1$ if $\|x\| = r$.

Further, it suffices to pick the midpoint due to the following: without loss of generality let $\lambda \in (0, \frac{1}{2})$ (otherwise pick $\tilde{\lambda} = 1 - \lambda \in (\frac{1}{2}, 1)$). Then just assuming midpoint-strict convexity, we have for $\|x\| = \|y\| = 1$ that

$$\begin{aligned}\|\lambda x + (1-\lambda)y\| &= \left\|2\lambda \frac{x+y}{2} + (1-2\lambda)y\right\| \\ &\overset{\triangle \neq}{\leqslant} 2\lambda \left\|\frac{x+y}{2}\right\| + (1-2\lambda)\|y\| < 2\lambda + (1-2\lambda) = 1\end{aligned}$$

Hence r-strict convexity and 1-midpoint strict convexity are equivalent. ○

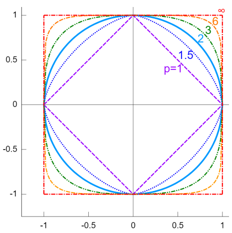


Fig. 4: The unit balls of different p-norms. [Source]

Example 1.2.3 (Strictly convex norm) The p-norm on $\mathbb{R}^n$ is strictly convex for $p \in (1, \infty)$ and not strictly convex for $p \in \{1, \infty\}$. The L^2-norm on $\mathcal{C}([a,b])$ (and also on $L^2([a,b])$) is strictly convex, while the uniform norm is not. ◇

Proof. (HW 1.1) *This is left as an exercise to the reader.* □

Lemma 1.2.4 (Strictly convex norm and triangle inequality)
A space $(X, \|\cdot\|)$ is strictly convex if and only if the triangle inequality is strict for every two nonparallel elements, that is, if $x \neq \lambda y$ and $y \neq \lambda x$ for all $\lambda \in \mathbb{R}$, then

$$\|x + y\| < \|x\| + \|y\|. \tag{3}$$

Proof. " $\Longrightarrow$ ": Assume that $\|\cdot\|$ is strictly convex and let $x, y \in X$ be nonparallel. Then $\frac{x}{\|x\|} \neq \frac{y}{\|y\|}$. As $\|\cdot\|$ is strictly convex and $\frac{\|x\|}{\|x\|+\|y\|} + \frac{\|y\|}{\|x\|+\|y\|} = 1$ and $\left\|\frac{x}{\|x\|}\right\| = 1 = \left\|\frac{y}{\|y\|}\right\|$ hold, we have

$$\left\|\frac{\cancel{\|x\|}}{\|x\|+\|y\|}\frac{x}{\cancel{\|x\|}} + \frac{\cancel{\|y\|}}{\|x\|+\|y\|}\frac{y}{\cancel{\|y\|}}\right\| < 1,$$

implying (3).

" $\Longleftarrow$ ": Assume that the triangle inequality is strict for all nonparallel $x, y \in X$. To show that $\|\cdot\|$ is strictly convex, take $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = r > 0$ (so $x \neq 0 \neq y$) and a $\lambda \in (0,1)$. Then we have

$$\|\lambda x + (1-\lambda)y\| \overset{(3)}{<} \lambda\|x\| + (1-\lambda)\|y\| = (\lambda + 1 - \lambda)r = r,$$

where the strict inequality holds only if $\lambda x \neq \alpha(1-\lambda)y$ for all $\alpha \in \mathbb{R}$ and $(1-\lambda)y \neq \beta\lambda x$ for all $\beta \in \mathbb{R}$.

We show that this is not the case (with one exception, which doesn't cause trouble). Towards contradiction assume there exists an $\alpha \in \mathbb{R}$ with $\lambda x = \alpha(1-\lambda)y$, then

$$\lambda r = \|\lambda x\| = \|\alpha(1-\lambda)y\| = |\alpha|(1-\lambda)r,$$

implying $\alpha = \pm\frac{\lambda}{1-\lambda}$. Plugging this into $\lambda x = \alpha(1-\lambda)y$ yields $x = \pm y$. As $x \neq y$, we only show that if $x = -y$, we also have strict convexity (the case for $(1-\lambda)y = \beta\lambda x$ follows analogously):

$$\|\lambda x + (1-\lambda)(-x)\| = \underbrace{|2\lambda - 1|}_{<1 \text{ as } \lambda\in(0,1)} \|x\| < r.$$

□

Corollary 1.2.5 (Unique best approximation in strictly convex space)
Let Y be a subspace of a strictly convex space $(X, \|\cdot\|)$ and $x \in X$. Then either $Y_x = \varnothing$ or $Y_x = \{y_x\}$.

Proof. By Theorem 1.1.2, it suffices to check the case where the convex set Y_x has infinitely many elements and deduce a contradiction. If $y_1, y_2 \in Y_x$ with $y_1 \neq y_2$, then by convexity of Y_x we have $\lambda y_1 + (1-\lambda)y_2 \in Y_x \subset S_{d_x}$ for all $\lambda \in (0,1)$, but S_{d_x} can not contain the open line segment $\{\lambda y_1 + (1-\lambda)y_2 : \lambda \in (0,1)\}$ because X is strictly convex. □

uniformly convex

Definition 1.2.6 (Uniformly convex space)
A normed vector space $(X, \|\cdot\|)$ is uniformly convex if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x - y\| < \varepsilon$ whenever $\|x\| = \|y\| = 1$ and $\|\frac{1}{2}(x+y)\| > 1 - \delta$.

Theorem 1.2.1: Uniformly convex spaces

1. Uniform convexity implies strict convexity.
2. Every finite dimensional strictly convex space is uniformly convex.
3. Every inner product space is uniformly convex.

Proof. (HW 1.2) *This is left as an exercise to the reader.* □

2 Approximation by Algebraic Polynomials

26.10.2021

2.1 The Weierstrass approximation theorem

Motivation. As $\mathcal{P}_n$ is finite-dimensional, we know by Theorem 1.1.1 that for all $f \in \mathcal{C}([0,1])$ there exists a $p^* \in \mathcal{P}_n$ such that

$$\|f - p^*\|_\infty = \min_{p \in \mathcal{P}_n} \|f - p\|_\infty =: E_n(f), \tag{4}$$

but as $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$ is not strictly convex by example 1.2.3 we can not conclude uniqueness. Since $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, we have $E_n(f) \geqslant E_{n+1}(f)$, so $(E_n(f))_{n\in\mathbb{N}}$ is a decreasing sequence of nonnegative numbers. The following theorem shows that $E_n(f) \searrow 0$ as $n \to \infty$.

Theorem 2.1.1: Weierstrass Approximation Theorem (1885)

For $f \in \mathcal{C}([0,1])$ and $\varepsilon > 0$ there exists a $p \in \mathcal{P}$ with $\|f - p\|_\infty < \varepsilon$.

The results also holds for $\mathcal{C}([a,b])$, since

$$(\mathcal{C}([0,1]), \|\cdot\|_\infty) \to (\mathcal{C}([a,b]), \|\cdot\|_\infty), \qquad f \mapsto f(a + (b-a)\cdot)$$

is an isometry.

There are many proofs of this theorem and we will see a constructive one by Bernstein (1912) first, which explicitly constructs a sequence of approximating polynomials, the Bernstein polynomials.

Definition 2.1.1 (Bernstein (basis) polynomials)

For a *bounded* function f on $[0,1]$, the Bernstein polynomial of degree n is

Bernstein polynomial

$$(B_n f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k},$$

where $(x \mapsto \binom{n}{k} x^k (1-x)^{n-k})_{k=0}^n$ are the Bernstein basis polynomials of degree n.

The Bohman-Korovkin Theorem

We will prove that $B_n f \rightrightarrows f$ ($\rightrightarrows$ means uniform convergence) for $n \to \infty$ on $[0,1]$ for all $f \in \mathcal{C}([0,1])$. This is a consequence of the Bohman-Korovkin Theorem, for which the following notion needs to be introduced.

Definition 2.1.2 (Positive operator)

An operator $T\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ is positive if $f(x) \geqslant 0$ for all $x \in [0,1]$ implies $(Tf)(x) \geqslant 0$ for all $x \in [0,1]$.

Lemma 2.1.3 (Linear, positive $\implies$ bounded)
Every linear positive operator $T\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ is bounded.

Proof. For every $f \in \mathcal{C}([0,1])$ we have $\pm f \leqslant \|f\|_\infty \cdot 1$ pointwise. As T is linear and positive, we have $\pm T(f) \leqslant \|f\|_\infty T(1)$ and thus $|T(f)| \leqslant \|f\|_\infty T(1)$, each pointwise. Hence $\|T\| \leqslant \|T(1)\|_\infty$, so T is bounded. □

Theorem 2.1.2: Bohman-Korovkin

For a sequence of linear positive operators $(T_n \colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1]))_{n\in\mathbb{N}}$, the following are equivalent.

1. $T_n f \rightrightarrows f$ for all $f \in \mathcal{C}([0,1])$.
2. $T_n f_k \rightrightarrows f_k$ for all $k \in \{0,1,2\}$, where $f_k(x) := x^k$.
3. $T_n f_0 \rightrightarrows f_0$ and $\big(t \mapsto (T_n\varphi_t)(t)\big) \rightrightarrows 0$, where $\varphi_t(x) := (x-t)^2$.

Proof. "1 $\implies$ 2": is immediate.

"2 $\implies$ 3": As $\varphi_t(x) = x^2 + t^2 - 2tx$, we have $\varphi_t = f_2 + t^2 f_0 - 2t f_1$ as a function of x, so

$$\begin{aligned}(T_n\varphi_t)(t) &= (T_n f_2)(t) + t^2(T_n f_0)(t) - 2t(T_n f_1)(t) \\ &= (T_n f_2)(t) - t^2 + t^2\big((T_n f_0)(t) - 1\big) - 2t\big((T_n f_1)(t) - t\big).\end{aligned}$$

As each T_n is positive and φ_t is a nonnegative function, we have

$$0 \leqslant (T_n\varphi_t)(t) \leqslant \underbrace{\|T_n(f_2) - f_2\|_\infty}_{\xrightarrow{n\to\infty} 0 \text{ by } 2} + \underbrace{t^2}_{\leqslant 1} \underbrace{\|T_n(f_0) - f_0\|_\infty}_{\xrightarrow{n\to\infty} 0 \text{ by } 2} + \underbrace{2t}_{\leqslant 1} \underbrace{\|T_n(f_1) - f_1\|_\infty}_{\xrightarrow{n\to\infty} 0 \text{ by } 2}$$

for all $t \in [0,1]$. Hence $\big(t \mapsto (T_n\varphi_t)(t)\big) \rightrightarrows 0$ on $[0,1]$.

"3 $\implies$ 1": Let $f \in \mathcal{C}([0,1])$ and without loss of generality $f \not\equiv 0$. Then f is uniformly continuous, that is, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x-t| < \delta$ implies $|f(x) - f(t)| < \varepsilon$ for all $x, t \in [0,1]$. If $|x-t| \geqslant \delta$, then

$$|f(x) - f(t)| \leqslant 2\|f\|_\infty \leqslant 2\frac{(x-t)^2}{\delta^2}\|f\|_\infty = \alpha\varphi_t(x),$$

where $\alpha := \frac{2}{\delta^2}\|f\|_\infty > 0$.

For every $x, t \in [0,1]$ we thus have

$$|f(x) - f(t)| \leqslant \varepsilon + \alpha\varphi_t(x).$$

That means that as a function of x we have

$$-\varepsilon f_0 - \alpha\varphi_t \leqslant f - f(t) f_0 \leqslant \varepsilon f_0 + \alpha\varphi_t.$$

As T_n is linear and positive, these inequalities are preserved when applying T_n:

$$-\varepsilon T_n(f_0) - \alpha T_n(\varphi_t) \leqslant T_n(f) - f(t)T_n(f_0) \leqslant \varepsilon T_n(f_0) + \alpha T_n(\varphi_t).$$

so

$$|(T_n f)(t) - f(t)(T_n f_0)(t)| \leqslant \varepsilon(T_n f_0)(t) + \alpha(T_n\varphi_t)(t).$$

Using the triangle inequality, we have

$$\begin{aligned}|(T_n f)(t) - f(t)| &\leqslant |(T_n f)(t) - f(t)(T_n f_0)(t)| + |f(t)(T_n f_0)(t) - f(t)| \\ &\leqslant \underbrace{\varepsilon}_{\searrow 0} \underbrace{\|T_n f\|_\infty}_{\text{bounded}} + \alpha \underbrace{(T_n\varphi_t)(t)}_{\rightrightarrows 0 \text{ by } 3} + \underbrace{\|f\|_\infty}_{<\infty} \underbrace{\|T_n f_0 - f_0\|_\infty}_{\to 0 \text{ by } 3} \xrightarrow[\varepsilon\searrow 0]{n\to\infty} 0.\end{aligned}$$

□

BERNSTEIN's proof

As B_n is linear and positive, we get the following corollary.

Corollary 2.1.4 (BERNSTEIN theorem)
For all $f \in \mathcal{C}([0,1])$ we have $B_n(f) \rightrightarrows f$.

Proof. By Theorem 2.1.2 it suffices to show that $B_n(f_k) \rightrightarrows f_k$ for $k \in \{0,1,2\}$. We have $B_n(f_0) = 1$ by the binomial theorem and

$$\begin{aligned} B_n(f_1) &= \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} \\ &= x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-k-1} = x. \end{aligned}$$

Lastly, we have

$$\binom{n-1}{k} k = \frac{(n-1)!}{(k-1)!(n-1-k)!} = (n-1)\frac{(n-2)!}{(k-1)!(n-2-(k-1))!} = (n-1)\binom{n-2}{k-1} \quad (5)$$

and hence

$$\begin{aligned} B_n(f_2) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} x^k (1-x)^{n-k} = \frac{1}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} k x^k (1-x)^{n-k} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k} (k+1) x^{k+1} (1-x)^{n-k-1} \\ &\overset{(5)}{=} \frac{n-1}{n} \sum_{k=1}^{n-1} \binom{n-2}{k-1} x^{k+1} (1-x)^{n-1-k} + \frac{1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} (1-x)^{n-1-k} \\ &= \frac{n-1}{n} \sum_{k=0}^{n-2} \binom{n-2}{k} x^{k+2} (1-x)^{n-2-k} + \frac{1}{n} x (x+1-x)^{n-1} \\ &= \frac{n-1}{n} x^2 (x+1-x)^{n-2} + \frac{1}{n} x = \underbrace{\frac{n-1}{n}}_{\to 1} f_2(x) + \underbrace{\frac{1}{n}}_{\to 0} f_1(x) \xrightarrow{n\to\infty} f_2(x). \end{aligned}$$ □

Corollary 2.1.5 (WEIERSTRASS approximation theorem)
Let $f \in \mathcal{C}^1([a,b])$ and $\varepsilon > 0$. Then there exists a $p \in \mathcal{P}$ such that $\|f - p\|_\infty < \varepsilon$ and $\|f' - p'\|_\infty < \varepsilon$.

Proof. (HW 1.3) *This is left as an exercise to the reader.* □

The modulus of continuity

How fast is the convergence of the BERNSTEIN polynomials? This subsection follows [Car98, p. 16 - 18].

DEFINITION 2.1.6 (MODULUS OF CONTINUITY)
The modulus of continuity for a bounded function f on $[a,b]$ is modulus of continuity

$$w_f \colon [0,\infty) \to [0,\infty), \qquad \delta \mapsto \sup\left\{|f(x) - f(y)| : x, y \in [a,b] \text{ with } |x-y| \leqslant \delta\right\}.$$

We have $|f(x) - f(y)| \leqslant w_f(|x-y|)$ and $w_f(0) = 0$.

Remark 2.1.7 If $f \in \mathcal{C}^1([a,b])$, then for all $x, y \in [a,b]$ we have

$$\frac{|f(x)-f(y)|}{|x-y|} \leqslant \|f'\|_\infty$$

and thus $|f(x)-f(y)| \leqslant \|f'\|_\infty |x-y|$. Hence $w_f(\delta) \leqslant \|f'\|_\infty \delta$. This conclusion also holds for LIPSCHITZ functions, where $\|f'\|_\infty$ has to be replaced by the LIPSCHITZ constant of f. ○

Lemma 2.1.8 (Properties of the modulus of continuity)

1. *A function f is uniformly continuous if and only if $\lim_{\delta \searrow 0} w_f(\delta) = 0$.*
2. *The modulus of continuity is monotonically increasing: for $0 < \delta < \delta_1$ we have $w_f(\delta) \leqslant w_f(\delta_1)$.*
3. *The set A of continuous functions that have the same modulus of continuity are uniformly equicontinuous, that is, for all $\varepsilon > 0$ exists a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$ for all $f \in A$.*

Proof. 1. This is immediate as for $\delta > 0$, $w_f(\delta)$ is precisely a number $\varepsilon > 0$ such that $|x-y| \leqslant \delta$ implies $|f(x)-f(y)| \leqslant \varepsilon$.

2. The supremum becomes larger if we take the supremum over more values.

3. Clear. □

DEFINITION 2.1.9 (α-LIPSCHITZ CONTINUITY)
A function $f\colon [a,b] \to \mathbb{R}$ is LIPSCHITZ continuous of order α and we write $f \in \mathrm{Lip}_K^\alpha$ if there exists a positive constant $K > 0$ such that

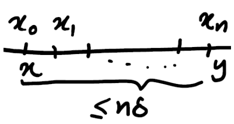
$$|f(x)-f(y)| \leqslant K|x-y|^\alpha.$$

Further, $\mathrm{Lip}^\alpha := \bigcup_{K>0} \mathrm{Lip}_K^\alpha$ is the space of α-LIPSCHITZ functions.

Remark 2.1.10 (Modulus of continuity of LIPSCHITZ functions)
We have $f \in \mathrm{Lip}_K^\alpha$ if and only if $w_f(\delta) \leqslant K\delta^\alpha$. ○

Of course w_f is not linear, but we have the following result.

Lemma 2.1.11 (Bounds on w_f)
For $n \in \mathbb{N}$ we have $w_f(n\delta) \leqslant n w_f(\delta)$ and $w_f(\lambda\delta) \leqslant (1+\lambda) w_f(\delta)$ for all $\lambda, \delta > 0$.



Proof. Take $x, y \in [0,1]$ with $|x-y| < n\delta$ and then consider the following equidistant partition of the interval between them: $n_k := x + \frac{k}{n}(y-x)$ for $k \in \{0, \ldots, n\}$. Then $|n_{k+1} - n_k| = \frac{|y-x|}{n}|k+1-k| < \delta$ and thus

$$|f(x)-f(y)| = \left|\sum_{k=0}^{n-1} f(x_{k+1}) - f(x_k)\right| \overset{\triangle\neq}{\leqslant} \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \leqslant \sum_{k=0}^{n-1} w_f(\delta) = n w_f(\delta).$$

For $\lambda \in \mathbb{R}_+ \setminus \mathbb{N}$ there exists a $n \in \mathbb{N}$ with $\lambda \in (n, n+1)$. As w_f is increasing,

$$w_f(\lambda\delta) \leqslant w_f((n+1)\delta) \leqslant (n+1) w_f(\delta) < (\lambda+1) w_f(\delta)$$

for all $\delta > 0$. □

An error estimate for the BERNSTEIN polynomials

Theorem 2.1.3: Convergence rate Bernstein approximation

For a bounded function f on $[0,1]$, we have

$$\|f - B_n(f)\|_\infty \leqslant \frac{3}{2} w_f\left(\frac{1}{\sqrt{n}}\right).$$

Proof. For $x \in [0,1]$ we have

$$|f(x) - (B_n f)(x)| \overset{\triangle\neq}{\leqslant} \sum_{k=0}^{n} \underbrace{\left|f(x) - f\left(\frac{k}{n}\right)\right|}_{\leqslant w_f\left(\left|x - \frac{k}{n}\right|\right)} \binom{n}{k} x^k (1-x)^{n-k}$$

and

$$w_f\left(\left|x - \frac{k}{n}\right|\right) = w_f\left(\frac{1}{\sqrt{n}}\sqrt{n}\left|x - \frac{k}{n}\right|\right) \overset{2.1.11}{\leqslant} \left(1 + \sqrt{n}\left|x - \frac{k}{n}\right|\right) w_f\left(\frac{1}{\sqrt{n}}\right).$$

Hence

$$\begin{aligned} |f(x) - (B_n f)(x)| &\leqslant w_f\left(\frac{1}{\sqrt{n}}\right) \sum_{k=0}^{n} \left(1 + \sqrt{n}\left|x - \frac{k}{n}\right|\right) \binom{n}{k} x^k (1-x)^k \\ &= w_f\left(\frac{1}{\sqrt{n}}\right) \left(1 + \sqrt{n} \sum_{k=0}^{n} \left|x - \frac{k}{n}\right| \binom{n}{k} x^k (1-x)^k\right) \end{aligned}$$

and by Hölders inequality

$$\begin{aligned} \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right| \binom{n}{k} x^k (1-x)^k\right)^2 &\leqslant \left(\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^2 \binom{n}{k} x^k (1-x)^k\right) \underbrace{\left(\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^k\right)}_{=1} \\ &= \left(\sum_{k=0}^{n} \left(x^2 + \frac{k^2}{n^2} - 2x\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^k\right) \\ &= x^2 + (B_n f_2)(x) - 2x(B_n f_1)(x) \\ &= x^2 + \frac{n-1}{n} x^2 + \frac{1}{n} x - 2x^2 \\ &= \frac{x - x^2}{n} \leqslant \frac{1}{4n} \end{aligned}$$

and thus

$$|f(x) - (B_n f)(x)| \leqslant w_f\left(\frac{1}{\sqrt{n}}\right)\left(1 + \sqrt{n}\frac{1}{2\sqrt{n}}\right) \leqslant \frac{3}{2} w_f\left(\frac{1}{\sqrt{n}}\right).$$

□

Remark 2.1.12 (Bernstein approximation rate for Lipschitz functions)
If $f \in \mathrm{Lip}_K^\alpha$, we have $\|f - B_n(f)\| \leqslant \frac{3K}{2} n^{-\frac{\alpha}{2}}$ by remark 2.1.10. ○

Remark 2.1.13 (Optimality of Theorem 2.1.3)
Unfortunately, we can't hope to qualitatively improve the rate in Theorem 2.1.3: for example if $f(x) := \left|x - \frac{1}{2}\right|$, which is in Lip_1^1, we have

$$\frac{1}{2\sqrt{n}} < \|f - B_n(f)\| \leqslant \frac{3}{2\sqrt{n}}.$$

The first inequality is an exercise or can be looked up in [Riv81, p. 16, Remark 3]. ○

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Remark 2.1.14 ((Dis)Advantages of approximation with Bernstein polynomials) The disadvantages of the Bernstein polynomials are that the convergence $B_n(f) \rightrightarrows f$ is too slow to be useful in applications and that we have $B_n(f) \neq f$ for $f \in \mathcal{P}$, e.g. if $f = f_2$ and $\varepsilon = 10^{-4}$, then we need $n > 2500$ for $\|f_2 - B_n(f_2)\| < \varepsilon$: we have $B_n(f_2) = \frac{n-1}{n} f_2 + \frac{1}{n} f_1$, so

$$\|B_n(f) - f\|_\infty = \frac{1}{n} \max_{x \in [0,1]} |x - x^2| = \frac{1}{4n} < 10^{-4} \iff n > \frac{1}{4} 10^4 = 2500.$$

Also, for $n \geqslant 2$ the best approximation to f_2 from $\mathcal{P}_n$ is f_2 itself.

The advantages of the Bernstein polynomials are that they are linear and positive and that if $f \in \mathcal{C}^m([a,b])$, then $(B_n(f))^{(k)} \rightrightarrows f^{(k)}$ for all $k \in \{0, \dots, m\}$, as is easily deduced from the following theorem. ○

Theorem 2.1.4: Convergence of the derivative $B_n(f)'$

If $f \in \mathcal{C}^1([a,b])$, then $(B_n(f))' \rightrightarrows f'$.

Proof. (HW 2.1) *This is left as an exercise to the reader.* □

2.2 Interpolation

One way to find a polynomial $p(x) := \sum_{k=0}^n c_k x^k$ to approximate a continuous function $f \in \mathcal{C}([a,b])$ is to evaluate f at $(x_k)_{k=0}^n \subset [a,b]$ and require that

$$f(x_k) = p(x_k) \qquad \forall k \in \{0, \dots, n\}.$$

The following theorem shows that we can always find such a polynomial.

Theorem 2.2.1: Interpolation Theorem

If $(x_k)_{k=0}^n$ are distinct points and $(y_k)_{k=0}^n \subset \mathbb{R}$, then there exists a unique polynomial $p \in \mathcal{P}_n$ such that

$$p(x_k) = y_k \qquad \forall k \in \{0, \dots, n\}. \tag{6}$$

Proof. Let $p(x) := \sum_{k=0}^n c_k x^k$. Formulating the interpolation condition (6) as matrix multiplication yields

$$\underbrace{\begin{pmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{pmatrix}}_{=:V} \underbrace{\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}}_{=:c} = \underbrace{\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}}_{=:y}.$$

The matrix V is the Vandermonde matrix with

$$\det(V) = \prod_{1 \leqslant j < i \leqslant n} x_i - x_j.$$

Since the $(x_k)_{k=0}^n$ are distinct, $\det(V) \neq 0$, so V is invertible and there exists a unique solution $c = V^{-1} y$. □

There are two different other ways to find the polynomial $p \in \mathcal{P}_n$ interpolating f at $(x_k)_{k=0}^n$: Lagrange interpolation and the Newton method. By Theorem 2.2.1, both methods yield the same polynomial.

Lagrange interpolation

Given distinct $(x_k)_{k=0}^n$ and values $(y_k)_{k=0}^n \subset \mathbb{R}$, define

$$p(x) := \sum_{k=0}^{n} y_k \ell_k(x) \qquad \text{and} \qquad \ell_k(x) := \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{x - x_j}{x_k - x_j} \tag{7}$$

are the Lagrange basis polynomials, which have degree n and are independent of $(y_k)_{k=0}^n$. Since $\ell_k(x_j) = \delta_{j,k}$, (6) holds. If $f \equiv 1$, then $p(x) \equiv 1$, implying that the $(\ell_k)_{k=0}^n$ are a partition of unity:

$$\sum_{k=0}^{n} \ell_k(x) \equiv 1.$$

Definition 2.2.1 (Lagrange interpolating operator)
The Lagrange interpolating operator is

$$L_n \colon \mathcal{C}([a,b]) \to \mathcal{P}_n, \qquad f \mapsto \sum_{k=0}^{n} f(x_k)\ell_k(x).$$

Lagrange interpolating operator

Does $L_n(f) \rightrightarrows f$? We can not use Theorem 2.1.2, since L_n is linear but not positive, as the ℓ_k are not positive functions.

The operator L_n is self-adjoint since interpolating a polynomial of degree n at $n+1$ points yields a unique polynomial of degree n by Theorem 2.2.1, which thus has to coincide with the original polynomial. We can also calculate

$$\begin{aligned} L_n(L_n(f))(x) &= \sum_{k=0}^{n} (L_n f)(x_k)\ell_k(x) = \sum_{k=0}^{n}\sum_{j=0}^{n} f(x_j)\underbrace{\ell_j(x_k)}_{=\delta_{j,k}}\ell_k(x) \\ &= \sum_{j=0}^{n} f(x_j)\ell_j(x) = (L_n f)(x). \end{aligned}$$

Lemma 2.2.2 (Lebesgue's Theorem)
We have

$$\|L_n\| = \left\| \sum_{k=0}^{n} |\ell_k(\cdot)| \right\|_\infty =: \Lambda_n$$

and

$$\|f - L_n(f)\|_\infty \leqslant (1 + \Lambda_n) E_n(f).$$

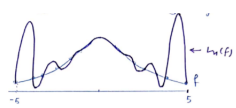
Proof. (HW 3.3) *This is left as an exercise to the reader.* □

The second part of proof works for any projection (when replacing min by inf):

Lemma 2.2.3 (Lebesgue's theorem)
For any linear bounded projection S we have

$$\|f - S\| \leqslant (1 + \|S\|) E_n(f).$$

Example 2.2.4 (Lagrange interpolation gone wrong)
Consider $f(x) := \frac{1}{x^2+1}$ on $[a,b] = [-5,5]$. Then $\|L_n(f) - f\|_\infty \xrightarrow{n\to\infty} \infty$, so increasing the number of sampling points does not help [Epp87]. ◇



If we fix a function f and change the points $(x_k)_{k=0}^n$, we can get $\|L_n(f)-f\|_\infty = \min_{p\in\mathcal{P}_n}\|f-p\|_\infty \xrightarrow{n\to\infty} 0$. But if we fix an array of points

This becomes false if we fix the points.

Theorem 2.2.2: Faber (1914)

Given an array of points

$$x = \begin{cases} x_0^{(0)}, & \text{if } n = 0, \\ \left[x_0^{(1)}, x_1^{(1)}\right], & \text{if } n = 1, \\ \left[x_0^{(1)}, x_1^{(2)}, x_2^{(2)}\right], & \text{if } n = 2, \\ \dots, \end{cases}$$

there exists an $f \in \mathcal{C}([a,b])$ such that $\|L_n(f) - f\|_\infty \to \infty$.

Example 2.2.5 Let $f\colon [-1,1] \to \mathbb{R}$, $x \mapsto |x|$. Then $L_n(f)(x) \to f(x)$ only for $x \in \{0, \pm 1\}$. ◇

Theorem 2.2.2 follows from the next, much stronger theorem.

Theorem 2.2.3: Kharshiladze, Lozinski (1941)

Given a sequence $(T_n\colon \mathcal{C}([a,b]) \to \mathcal{P}_n)_{n\in\mathbb{N}}$ of linear continuous projections, there exists a $f \in \mathcal{C}([a,b])$ such that $\|T_n(f) - f\|_\infty \to \infty$.

Remark 2.2.6 The theorem also holds if $\mathcal{C}([a,b])$ is replaced with the space of 2π-periodic continuous functions and $\mathcal{P}_n$ is replaced by the space of trigonometric polynomials of degree at most n, $\mathcal{T}_n$. ○

Remark 2.2.7 ((Dis)advantages of Lagrange interpolation) Advantages of the Lagrange polynomials are that they are linear and that the basis polynomials only depend on $(x_k)_{k=0}^n$, so interpolating multiple functions at the same points is easy.

The disadvantage of the Lagrange polynomials is that removing / adding one point x_k yields completely different basis functions ℓ_k. ○

Newton's Method

Given distinct $(x_k)_{k=0}^n \subset [a,b]$ and $(y_k)_{k=0}^n \subset \mathbb{R}$, define Newton's polynomial

$$p(x) := \sum_{k=0}^n c_k u_k(x),$$

where

$$u_k(x) := \prod_{j=0}^{k-1}(x - x_j), \qquad \text{and } u_0 \equiv 1.$$

divided differences

The coefficients c_k are defined using divided differences: $c_k := [y_0, \dots, y_k]$, where

$$[y_k] := y_k, \qquad [y_k, y_j] := \frac{y_j - y_k}{x_j - x_k} \text{ for } k \neq j,$$

and

$$[y_{j_0}, \dots, y_{j_m}] := \frac{[y_{j_1}, \dots, y_{j_m}] - [y_{j_0}, \dots, y_{j_{m-1}}]}{x_{j_m} - x_{j_0}} \quad \text{for } j_0 \neq j_m.$$

In the case that $y_k = f(x_k)$, we write $f[x_{j_1}, \dots, x_{j_n}] := [f(x_{j_1}), \dots, f(x_{j_n})]$.

Clearly, p interpolates f, so it agrees with $L_n(f)$. But the advantage over LAGRANGE interpolation is the following result.

THEOREM 2.2.4: ADVANTAGE OF NEWTON INTERPOLATION

Let $f \in \mathcal{C}([a,b])$, $(x_k)_{k=0}^{n+1} \subset [a,b]$ be distinct and p_n the interpolating polynomial of f at $(x_k)_{k=0}^{n}$. Then

$$p_{n+1} := p_n + f[x_0, \dots, x_{n+1}]u_{n+1}$$

interpolates f at $(x_k)_{k=0}^{n+1}$.

Proof. Since $u_{n+1}(x_k) = 0$, for all $k \in \{0, \dots, n\}$, $p_{n+1}(x_k) = p_n(x_k) = f(x_k)$ for all $k \in \{0, \dots, n\}$, so p_{n+1} interpolates f at $(x_k)_{k=0}^{n}$.

Let $q \in \mathcal{P}_{n+1}$ interpolate f at $(x_k)_{k=0}^{n+1}$. By the definition of $f[x_0, \dots, x_n]$, we can see that $q - p_{n+1} \in \mathcal{P}_n$, because the leading coefficient of q must be $f[x_0, \dots, x_{n+1}]$, too. Since $q - p_{n+1}$ has $n+1$ zeros, $q - p_{n+1} \equiv 0$. □

Remark 2.2.8 (Advantages of NEWTON interpolation)
An advantage of the NEWTON basis polynomials is that is if we add or remove a point x_k, the new basis polynomials are easy to calculate from the existing ones. ○

2.3 Approximation via interpolation

CHEBYSHEV polynomials

First we introduce the CHEBYSHEV polynomials.

Lemma 2.3.1 (CHEBYSHEV polynomials of the first and second kind)
For $n \in \mathbb{N}_{\geqslant 1}$, there exist polynomials P_n and Q_{n-1} of degree exactly n and $n-1$ such that

$$\cos(nx) = P_n(\cos(x)) \qquad \textit{and} \qquad \sin(nx) = Q_{n-1}(\cos(x))\sin(x). \tag{8}$$

Proof. (HW 1.4) *This is left as an exercise to the reader.* □

DEFINITION 2.3.2 (CHEBYSHEV POLYNOMIALS OF FIRST / SECOND KIND)
The polynomials defined by (8) are the CHEBYSHEV polynomials of the first and second kind, respectively.

CHEBYSHEV polynomial

Lemma 2.3.3 (Extrema of the CHEBYCHEV polynomials)
The n-th CHEBYCHEV polynomial of first order has $n+1$ extrema at $\left(\cos\left(\frac{k}{n}\pi\right)\right)_{k=0}^{n}$ and n roots at $\left(\cos\left(\frac{2k-1}{2n}\pi\right)\right)_{k=1}^{n}$.

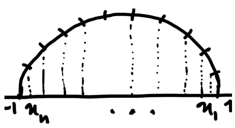


Fig. 5: The zeros of T_n.

Proof. The first order CHEBYCHEV polynomials T_n have $n+1$ extrema: since $T_n(\cos(x)) = \cos(nx)$, we can write $T_n(y) = \cos\big(n\arccos(y)\big)$ for $y \in [-1,1]$ and thus have for $n \geqslant 1$ and for $y \in [-1,1)$

$$T_n'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\cos(n\arccos(y)) = \underbrace{n}_{\neq 0}\sin(n\arccos(y))\underbrace{\frac{1}{\sqrt{1-y^2}}}_{\neq 0} \overset{!}{=} 0 \tag{9}$$

if and only if $\sin(n \arccos(y)) = 0$, that is, $\arccos(y) = \frac{k}{n}\pi$ for $k \in \mathbb{Z}$, that is, $y_k = \cos\left(\frac{k}{n}\pi\right)$. Since arccos maps into $[0, \pi]$, the only $k \in \mathbb{Z}$ that make sense are $k \in \{0, \dots, n\}$. Since $\cos(x) \in [-1, 1]$ for $x \in \mathbb{R}$ and $T_n(y_k) = \cos\left(n\frac{k}{n}\pi\right) = \cos(k\pi) = (-1)^k$, all these points are extrema.

By lemma 2.3.1, T_n has n zeros and we have

$$T_n\left(\cos\left(\frac{2k-1}{2n}\pi\right)\right) = \cos\left(\frac{2k-1}{2}\pi\right) = 0.$$ □

Lemma 2.3.4 (Properties of the Chebychev polynomials)
The Chebyshev polynomials of the first kind T_n and of the second kind U_n have the following properties.

1. $T_n'(x) = nU_{n-1}(x)$.
2. *The polynomial $y = T_n$ is a solution to* $(1 - x^2)y''(x) - xy'(x) + n^2 y(x) = 0$.
3. *We have* $\int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2}\delta_{n,m}$ *for all* $n, m \in \mathbb{N}$ *with* $n > 0$.
4. *The polynomials* $(T_n)_{n\in\mathbb{N}}$ *are linearly independent on every interval* $[a, b]$.

Proof. (HW 2.2) *This is left as an exercise to the reader.* □

If $p \in \mathcal{P}_n$ interpolates $f \in \mathcal{C}([a, b])$ at $(x_k)_{k=0}^n$, we want to measure $\|f - p\|_\infty$.

Definition 2.3.5 (Product function)
The product function for $(x_k)_{k=0}^n$ is $W(x) := \prod_{k=0}^n (x - x_k)$.

Note that W is a polynomial of exactly degree $n + 1$ with leading coefficient one.

Theorem 2.3.1: Upper bound on interpolation error

If $f \in \mathcal{C}^{n+1}([a, b])$ and $p \in \mathcal{P}_n$ interpolates f at $(x_k)_{k=0}^n$, then

$$\|f - p\|_\infty \leqslant \frac{1}{(n+1)!}\|f^{(n+1)}\|_\infty \|W\|_\infty.$$

Proof. We show that for all $x \in [a, b]$ there exists a $\xi_x \in [a, b]$ such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) W(x). \tag{10}$$

- If $x = x_k$ for some $k \in \{0, \dots, n\}$, then the LHS of (10) is zero as p interpolates f but the RHS is also zero since W vanishes on $(x_k)_{k=0}^n$.
- If $x \neq x_k$ for all $k \in \{0, \dots, n\}$, define the scalar

$$\lambda_x := \frac{f(x) - p(x)}{W(x)}$$

and the function $\varphi := f - p - \lambda_x W$. As $f, p, W \in \mathcal{C}^{n+1}([a, b])$, so is φ. We have $\varphi(x_k) = 0$ for all $k \in \{0, \dots, n\}$ and $\varphi(x) = 0$, so φ has at least $n + 2$ zeros. By Rolle's Theorem, φ' has at least $n + 1$ zeros, so, inductively, $\varphi^{(n+1)}$ has at least one zero ξ_x. Hence

$$0 = \varphi^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - 0 - \lambda_x \cdot (n + 1)!$$

and so

$$\lambda_x = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x).$$ □

One way to make this upper bound small is to change the $(x_k)_{k=0}^n$ in such a way that $\|W\|_\infty$ becomes small.

Lemma 2.3.6 (Monic polynomial with minimal $\|\cdot\|$)
Let T_n be the n-th CHEBYCHEV *polynomial for $n \geqslant 1$. The monic polynomial of degree n with minimal supremum norm on $[a,b]$ is*

$$\tilde{T}_n(x) = \frac{1}{2}\left(\frac{b-a}{4}\right)^n T_n\left(\frac{2x-a-b}{b-a}\right).$$

Proof. (HW 1.5) *This is left as an exercise to the reader.* □

Now, lemma 2.3.6 implies the following statement directly.

THEOREM 2.3.2: MINIMIZING THE PRODUCT FUNCTION

The number $\|W\|_\infty$ is minimal when $x_k = \cos\left(\frac{(2k+1)\pi}{2n+1}\right)$ for $k \in \{0,\dots,n\}$.

The x_k above are the roots of the first order CHEBYCHEV polynomial T_{n+1}.

HERMITE interpolation

09.11.2021

We now introduce an interpolation operator family $(T_n)_{n\in\mathbb{N}}$ such that $T_n(f) \rightrightarrows f$, which is a property the LAGRANGE interpolation operator doesn't have.

Let $(x_k)_{k=0}^n$ be distinct and suppose the set $\left(f^{(j)}(x_i)\right)_{j=0}^{J_i}$ is known for all $i \in \{0,\dots,n\}$. We want to find $p \in \mathcal{P}$ interpolating f such that $p^{(k)}$ interpolates $f^{(k)}$:

$$\boxed{p^{(j)}(x_i) = f^{(j)}(x_i) \qquad \forall i \in \{0,\dots,n\},\ j \in \{0,\dots,J_i\}.} \tag{11}$$

We thus (via (11)) have placed $\sum_{i=0}^n (J_i+1)$ conditions on

$$p(x) := \sum_{k=0}^m c_i x^i,$$

where $m+1 := \sum_{i=0}^n (J_i+1)$.

THEOREM 2.3.3: ∃ UNIQUE POLYNOMIAL INTERPOLATING DERIVATIVES

There exists a unique $p \in \mathcal{P}_m$ such that (11) for every data set of distinct $(x_k)_{k=0}^n$ and $\left(f^{(j)}(x_i)\right)_{j=0}^{J_i}$, where $m = \sum_{i=0}^n (J_i+1) - 1$.

Proof. The system has a unique solution if the matrix A of this linear system is invertible. Let $p \in \mathcal{P}_m$ such that

$$p^{(j)}(x_i) = 0 \qquad \forall i \in \{0,\dots,n\},\ j \in \{0,\dots,J_i\},$$

which corresponds to the homogeneous system $Ac = 0$, where $c = (c_k)_{k=0}^m$ are the coefficients of p. Then p has $n+1$ roots $(x_i)_{i=0}^n$, each with multiplicity J_i+1. Hence p has at least $\sum_{i=0}^n (J_i+1) = m+1$ roots, so $p \equiv 0$ and thus $c = 0$. Hence A is injective, so it is invertible. □

HERMITE interpolation method

How can we construct p? We use the HERMITE interpolation method (1878), creating the new set $\{\tilde{x}_0, \dots, \tilde{x}_m\} := \{x_0, \dots, x_0, x_1, \dots, x_1, \dots, x_n, \dots, x_n\}$, where x_i appears $J_i + 1$ times, and then apply the NEWTON method on this set with $f[\tilde{x}_j, \tilde{x}_{j+1}, \dots, \tilde{x}_{j+k+1}] := \frac{f^{(k+1)}(x_i)}{(k+1)!}$ if $\tilde{x}_j = \tilde{x}_{j+1} = \dots = \tilde{x}_{j+k+1}$. In the usual definition for the divided differences, equal nodes $\tilde{x}_i = \tilde{x}_j$ would result in division by 0, which is avoided. This is also called extended NEWTON method.

extended NEWTON method

Example 2.3.7 (HERMITE interpolation for $n = 1 = J_0 = J_1 = 1$)
Consider distinct $x_0, x_1 \in \mathbb{R}$ and the known values $\{f(x_0), f'(x_0), f(x_1), f'(x_1)\}$. Hence $n = J_0 = J_1 = 1$ and thus $m = (1+1) + (1+1) - 1 = 3$. We want to find a $p \in \mathcal{P}_3$ such that $p^{(j)}(x_i) = f^{(j)}(x_i)$ for $i, j \in \{0, 1\}$. We have $f[x_0, x_0] = f'(x_0)$, $f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$ and $f[x_1, x_1] = f'(x_1)$. From this we can calculate the other dividing differences using the following tabular array

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$		
x_0	$f[x_0]$			
		$f[x_0, x_0]$		
x_0	$f[x_0]$		$f[x_0, x_0, x_1]$	
		$f[x_0, x_1]$		$f[x_0, x_0, x_1, x_1]$
x_1	$f[x_1]$		$f[x_0, x_0, x_1]$	
		$f[x_1, x_1]$		
x_1	$f[x_1]$			

to obtain

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 + f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1). \quad \diamond$$

THEOREM 2.3.4: THE EXTENDED NEWTON METHOD

The polynomial constructed by the extended NEWTON method satisfies (11).

Proof. By Theorem 2.3.3, there exists a unique polynomial $p_0 \in \mathcal{P}_m$ with

$$p^{(j)}(x_i) = (p_0)^{(j)}(x_i) = f^{(j)}(x_i) \qquad \forall i \in \{0, \dots, n\},\ j \in \{0, \dots, J_i\}.$$

Hence we can without loss of generality assume that f is a polynomial, i.e. $f = p_0 \in \mathcal{P}_m$.

Let $(x_k)_{k=0}^m$ be the new set of points where repetition is allowed. For any $\delta > 0$, let $(\xi_k)_{k=0}^m$ be distinct such that $|\xi_k - x_k| < \delta$ for all $k \in \{0, \dots, m\}$. As f is a polynomial and hence continuous, the value of $f(\xi_k)$ is close to $f(x_k)$ for all $k \in \{0, \dots, m\}$. If $\frac{f^{(k+1}(x_j)}{(k+1)!}$ occurs in the table of divided differences of p, that means $x_j = x_{j+1} = \dots, x_{j+k+1}$ and hence the corresponding entry in the table of divided differences of $(\xi_k)_{k=0}^m$ is

$$f[\xi_j, \xi_{j+1}, \dots, \xi_{j+k+1}] = \frac{f^{(k+1)}(\xi)}{(k+1)!} \tag{12}$$

for some ξ in the smallest interval including $(\xi_i)_{i=j}^{j+k+1}$. That means $\xi \in (x_j - \delta, x_j + \delta)$. Hence if we make $\delta > 0$ small (12) is very close to $\frac{f^{(k+1)}(x_j)}{(k+1)!}$.

Note that

$$f(x) = \sum_{k=0}^{n} f[\xi_0, \dots, \xi_k] u_k(x).$$

Let $x \in [a, b]$. Then for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - p(x)| < \varepsilon$. Since, by definition, f and p do not depend on δ, we get $p \equiv f$. $\square$

Fejér-Hermite interpolation

The goal of this subsection is to prove the Fejér-Hermite theorem. In 1930, Fejér found out that one can prove the Weierstrass theorem using the Hermite method.

We now change notation; instead of starting indices at 0, we begin at 1.

We now treat the special case of Hermite interpolation, where $J_i = 1$. Hence we have distinct $(x_k)_{k=1}^n$ and given $\{f(x_i), f'(x_i)\}$ for $i \in \{1, \dots, n\}$. We want to find a $p \in \mathcal{P}_{2n-1}$ such that

$$p^{(j)}(x_i) = f^{(j)}(x_i) \qquad \forall i \in \{1, \dots, n\},\ j \in \{0, 1\}.$$

We will use the extension of Newton's method and the Lagrange polynomials.

As an exercise, prove that for $x \neq x_k$

$$\ell_k(x) = \prod_{j \neq k} \frac{x - x_j}{x_k - x_j} = \frac{W(x)}{x - x_k} \frac{1}{W'(x_k)}.$$

This implies

$$\ell_k{}'(x) = \frac{W'(x)(x - x_k)W'(x_k) - W'(x_k)W(x)}{(x - x_k)^2 W'(x_k)^2} = \frac{W'(x)(x - x_k) - W(x)}{(x - x_k)^2 W'(x_k)}$$

for $x \neq x_k$ and thus by L'Hôpital

$$\ell_k{}'(x_k) = \lim_{x \to x_k} \ell_k{}'(x) = \frac{W''(x)\cancel{(x - x_k)} + \cancel{W'(x) - W'(x)}}{2\cancel{(x - x_k)}W'(x_k)} = \frac{W''(x_k)}{2W'(x_k)}. \tag{13}$$

Let

$$p(x) := \sum_{k=1}^n f(x_k)A_k(x) + f_k'(x_k)B_k(x),$$

where

$$A_k(x) := \left(1 - 2(x - x_k)\ell_k{}'(x_k)\right) \ell_k{}^2(x) \quad \text{and} \quad B_k(x) := (x - x_k)\ell_k{}^2(x). \tag{14}$$

Then $A_k, B_k \in \mathcal{P}_{2n-1}$ and

$$A_k(x_j) = B'(x_j) = \delta_{jk} \quad \text{and} \quad A_k'(x_j) = B(x_j) = 0. \tag{15}$$

This implies that $p(x_j) = f(x_j)$ and $p'(x_j) = f'(x_j)$.

Assume that $(x_k)_{k=1}^n$ are distinct. We want to find a $p \in \mathcal{P}_{2n-1}$ such that

$$\boxed{p(x_j) = f(x_j) \quad \text{and} \quad p'(x_j) = 0 \qquad \forall j \in \{1, \dots, n\}.}$$

As we explained above, the following polynomial satisfies those constraints.

Definition 2.3.8 (Fejér-Hermite operator)
The n-th Fejér-Hermite operator is

Fejér-Hermite operator

$$L_n \colon \mathcal{C}([a,b]) \to \mathcal{P}_{2n-1}, \qquad f \mapsto \sum_{i=1}^n f(x_i)A_i.$$

The next theorem was first proven by Fejér.

Theorem 2.3.5: Fejér-Hermite uniform convergence

For $f \in \mathcal{C}([a,b])$ we have $L_n(f) \rightrightarrows f$ if $(x_k)_{k=1}^n$ are the zeros of the first order n-th Chebyshev polynomial.

Proof. (Korovkin, 1959 [Che66, p. 70]) We can without loss of generality assume that $[a,b] = [-1,1]$. We show that for this special choice of $(x_k)_{k=1}^n$, we have

$$L_n(f)(x) = \frac{1}{n^2} T_n(x)^2 \sum_{i=1}^n f(x_i) \frac{1 - x x_i}{(x - x_i)^2}. \tag{16}$$

Then $L_n(f)$ is clearly linear and positive.

Furthermore, $L_n(f_0) = f_0$, since we have $L_n(p) = p$ for any polynomial $p \in \mathcal{P}_{2n-2}$. Indeed, $[L_n(p)]'(x_k) = p'(x_k)$ and $L_n(p)(x_k) - p(x_k) = 0$ for all $k \in \{1, \ldots, n\}$ by (15), so by the mean value theorem there exist $\xi_k \in (x_k, x_{k+1})$ for $k \in \{1, \ldots, n-1\}$ such that $L_n(p)'(\xi_k) - p'(\xi_k) = 0$ for all $k \in \{1, \ldots, n-1\}$. Since $\xi_j \notin (x_k)_{k=1}^n$ for all $j \in \{1, \ldots, n-1\}$, we have found $n + n - 1 = 2n - 1$ zeros of $L_n(p)' - p' \in \mathcal{P}_{2n-2}$. Hence $L_n(p)' = p'$, so $L_n(p) = p + C$ for some constant C. Since $L_n(p)$ and p agree in n points, $C = 0$. (This also shows that $\sum_{k=1}^n A_k \equiv 1$.)

For $\varphi_t(x) := (x - t)^2$ we have

$$L_n(\varphi_t)(t) = \frac{1}{n^2} \underbrace{T_n(t)^2}_{\leqslant 1} \sum_{i=1}^n \cancel{(x_i - t)^2} \overbrace{\frac{1 - t x_i}{\cancel{(t - x_i)^2}}}^{\leqslant 2} \leqslant \frac{1}{n^2} \cdot 1 \sum_{k=1}^n 2 = \frac{2n}{n^2} = \frac{2}{n} \xrightarrow{n \to \infty} 0,$$

so $L_n(\varphi_\cdot)(\cdot) \rightrightarrows 0$. By Theorem 2.1.2, we have $L_n(f) \rightrightarrows f$ for all $f \in \mathcal{C}([-1,1])$.

It remains to show (16). We know that (recall (13) and (14))

$$(L_n f)(x) = \sum_{i=1}^n f(x_i) \left(1 - \frac{W''(x_i)}{W'(x_i)}(x - x_i)\right) \ell_i^2(x). \tag{17}$$

Since $W(x) = \prod_{k=1}^n (x - x_k)$ is monic polynomial and x_k are the roots of T_n, we have $W = \frac{1}{2^{n-1}} T_n$. Hence $\frac{W''}{W'} = \frac{T_n''}{T_n'}$.

By (9) we have

$$T_n'(x) = \frac{n}{\sqrt{1 - x^2}} \sin(n \cos^{-1}(x)).$$

Since $x_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, this implies

$$T_n'(x_k) = \frac{n}{\sqrt{1 - x_k^2}} \sin\left(\left(k - \frac{1}{2}\right)\pi\right) = (-1)^{k+1} \frac{n}{\sqrt{1 - x_k^2}}$$

and thus $T_n'(x_k)^2 = \frac{n^2}{1 - x_k^2}$.

From lemma 2.3.4 we know that the following differential equation is fulfilled:

$$(1 - x^2) T_n''(x) - x T_n'(x) + n^2 T_n(x) = 0,$$

Hence setting $x = x_k$ yields

$$(1 - x_k^2) T_n''(x_k) - x_k T_n'(x_k) + 0 = 0,$$

implying $\frac{T_n''(x_k)}{T_n'(x_k)} = \frac{x_k}{1 - x_k^2}$.

Finally,

$$\ell_k(x) = \frac{W(x)}{(x-x_k)W'(x)} = \frac{\cancel{\frac{1}{2^{n-1}}}T_n(x)}{(x-x_k)\cancel{\frac{1}{2^{n-1}}}T_n'(x)} = \frac{T_n(x)}{(x-x_k)T_n'(x)}$$

and so

$$\ell_k(x)^2 = \frac{T_n(x)^2}{(x-x_k)^2 T_n'(x)^2} = \frac{T_n(x)^2}{(x-x_k)^2 \frac{n^2}{1-x_k^2}} = \frac{1-x_k^2}{n^2}\frac{T_n(x)^2}{(x-x_k)^2}.$$

Plugging all of this in (17) we obtain

$$\begin{aligned}(L_n f)(x) &= \sum_{k=1}^{n} f(x_k)\left(1 - \frac{x_k}{1-x_k^2}(x-x_k)\right)\frac{1-x_k^2}{n^2}\frac{T_n(x)^2}{(x-x_k)^2} \\ &= \frac{1}{n^2}T_n(x)^2 \sum_{k=1}^{n} f(x_k)\frac{1-xx_k}{(x-x_k)^2}.\end{aligned}$$

□

Lemma 2.3.9 (Properties of the Fejér-Hermite operator)

- *The* Fejér-Hermite *operator is positive if and only if the conjugate nodes* $\bar{x}_k := x_k + \frac{1}{2}\frac{1}{\ell_k{}'(x_k)}$ *lie outside of* (a,b).
- *If* $(x_k)_{k=0}^n$ *are the zeros of the n-th first order* Chebyshev *polynomial, then the conjugate nodes are* $\bar{x}_k = \frac{1}{x_k}$.

Proof. (HW 2.4) *This is left as an exercise to the reader.* □

3 Best approximation

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We want to characterise best approximations, that is, find out whether a given function is a best approximation. This chapter roughly follows [Che66, Chp. 4 - 5].

Let $\mathcal{A}$ be a linear subspace of $\mathcal{C}([a,b];\mathbb{R})$. For $f \in \mathcal{C}[a,b]$ consider the min-max-problem

$$\min_{p\in\mathcal{A}} \|f-p\|_\infty = \min_{p\in\mathcal{A}} \max_{x\in[a,b]} |f(x)-p(x)|.$$

Assume that we have a trial $p^* \in \mathcal{A}$. We want to know if p^* is the best approximation for f in $\mathcal{A}$. Would it be possible to find a constant θ and an element $p \in \mathcal{A}$ such that

$$\|f-(p^*+\theta p)\|_\infty < \|f-p^*\|_\infty \ ?$$

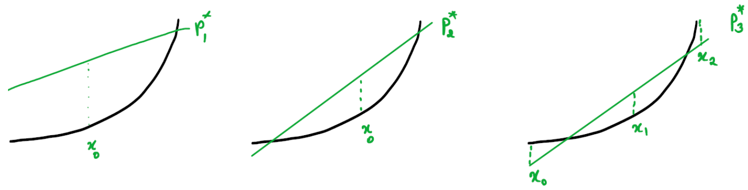

Fig. 6: When trying to find a linear approximation p_i^* to a quadratic polynomial f, it seems to be a good strategy to look at the points where p_i^* and f differ the most and reduce those differences.

3.1 Characterising the best approximation

Let

$$E := E(f-p^*) := \{x \in [a,b] : |f(x)-p^*(x)| = \|f-p^*\|_\infty\}.$$

Theorem 3.1.1: Characterisation of best approximations

Let $f \in \mathcal{C}([a,b])$, $\mathcal{A}$ be a linear subspace of $\mathcal{C}([a,b];\mathbb{R})$ and $p^* \in \mathcal{A}$. Then p^* is a best approximation of f in $\mathcal{A}$ if and only if there is no $p \in \mathcal{A}$ such that

$$\big(f(x)-p^*(x)\big)p(x) > 0 \qquad \forall x \in E. \tag{18}$$

Proof. " $\Longleftarrow$ ": Assume that p^* is not the best approximation. Then there exists a $p \in \mathcal{A}$ such

$$\|f-(p+p^*)\|_\infty < \|f-p^*\|_\infty.$$

Hence

$$|f(x)-p^*(x)-p(x)| < \|f-p^*\|_\infty \qquad \forall x \in [a,b].$$

That means that

$$|f(x)-p^*(x)-p(x)| < |f(x)-p^*(x)| \qquad \forall x \in E. \tag{19}$$

Thus $f(x)-p^*(x)$ and $p(x)$ have the same sign for all $x \in E$, implying (18), because $p(x)=0$ would contradict (19) and $f(x)=p^*(x)$ implies $\|f-p^*\|=0$, contradicting p^* not being the best approximation.

" $\Longrightarrow$ ": Let p^* be the best approximation from $\mathcal{A}$ to f. Assume there exists a $p \in \mathcal{A}$ such that (18) holds. Without loss of generality assume $\|p\|_\infty \leqslant 1$. Let

$$E_0 := \{x \in [a,b] : \big(f(x) - p^*(x)\big)p(x) \leqslant 0\}.$$

Then $E_0 \cap E = \varnothing$ and E_0 is closed and thus compact. Let

$$d := \begin{cases} 0, & \text{if } E_0 = \varnothing, \\ \max_{x \in E_0} |f(x) - p^*(x)|, & \text{else.} \end{cases}$$

Then $d < \|f - p^*\|_\infty$. Let

$$\theta := \frac{1}{2}(\|f - p^*\|_\infty - d) > 0.$$

We show that $p^* + \theta p$ is better approximation of f in $\mathcal{A}$. Let $\xi \in [a,b]$ such that

$$|f(\xi) - (p^* + \theta p)(\xi)| = \|f - (p^* + \theta p)\|_\infty. \tag{20}$$

Case 1: If $\xi \in E_0$, then

$$\begin{aligned} |f(\xi) - p^*(\xi) - \theta p(\xi)| &= |f(\xi) - p^*(\xi)| + \theta|p(\xi)| \leqslant d + \theta \\ &= \|f - p^*\| - 2\theta + \theta = \|f - p^*\| - \theta < \|f - p^*\|, \end{aligned}$$

so by (20) we have $\|f - p^* - \theta p\| < \|f - p^*\|$. The first equality is due to $f(\xi) - p^*(\xi)$ and $p(\xi)$ having opposite signs due to (18).

Case 2: If $\xi \notin E_0$, then

$$\begin{aligned} |f(\xi) - p^*(\xi) - \theta p(\xi)| &< \max(|f(\xi) - p^*(\xi)|, \theta|p(\xi)|) \\ &\leqslant \max\left(\|f - p^*\|, \frac{1}{2}(\|f - p^*\|_\infty - d)\right) \leqslant \|f - p^*\|, \end{aligned}$$

so again by (20) we have $\|f - p^* - \theta p\| < \|f - p^*\|$. □

Example 3.1.1 (Characterisation of best approximations in $\mathcal{P}_n$)
Let $\mathcal{A} = \mathcal{P}_n$. Take $p^* \in \mathcal{A}$ such that $f - p^*$ changes sign $n+1$ times over E. If $p \in \mathcal{P}_n$ is such that (18) holds, then p has $n+1$ zeros and thus is zero, which is a contradiction, so p^* is a best approximation by Theorem 3.1.1. ◇

We want to generalise this to any subspace of $\mathcal{C}[a,b]$ of dimension $n+1$.

3.2 Haar spaces and best approximation

Definition 3.2.1 (Haar space [Haa17])
The functions $(g_k)_{k=0}^n \subset \mathcal{C}([a,b])$ satisfy the Haar condition if every $n+1$ vectors $(g_k(x_j))_{k=0}^n$ for $j \in \{0,\dots,n\}$ are linearly independent, that is, the matrix $(g_k(x_j))_{j,k=0}^n$ is invertible for all sets of distinct points $(x_j)_{j=0}^n \subset [a,b]$. Then $\mathcal{A} := \operatorname{span}(g_0,\dots,g_n)$ is a Haar space and $(g_k)_{k=0}^n$ is a Chebychev system or Haar system. Haar space

Example 3.2.2 (Haar space) For $g_k(x) := x^k$, $\mathcal{A} = \mathcal{P}_n$ is a Haar space and the above matrix is the Vandermonde matrix. Other examples are $g_k(x) := e^{ikx}$ or

$$(1, \sin(x), \cos(x), \dots, \sin(nx), \cos(nx))$$

for $x \in [0, 2\pi)$. ◇

Theorem 3.2.1: Characterisation / Alternation Theorem

Let $\mathcal{A} \subset \mathcal{C}([a,b])$ be a Haar space of dimension $n+1$ and $f \in \mathcal{C}([a,b])$. Then $p^* \in \mathcal{A}$ is a best approximation to f in $\mathcal{A}$ if and only if there exist $n+2$ points $\{\xi_0, \ldots, \xi_{n+1}\}$ such that

1. $a \leqslant \xi_0 < \xi_1 < \ldots < \xi_{n+1} \leqslant b$
2. $|f(\xi_i) - p^*(\xi_i)| = \|f - p^*\|_\infty$ for all $i \in \{0, \ldots, n+1\}$,
3. $f(\xi_{i+1}) - p^*(\xi_{i+1}) = -\big(f(\xi_i) - p^*(\xi_i)\big)$ for all $i \in \{0, \ldots, n\}$.

The set $\{\xi_0, \ldots, \xi_{n+1}\}$ is called an alternating set and it is not unique.

Example 3.2.3 (Applying the Alternation Theorem) We can prove that $p^*(x) = x - \frac{1}{8}$ is *the* best approximation of $f(x) = x^2$ from $\mathcal{P}_1$ on $[0,1]$ with Theorem 3.2.1.

This is left as an exercise to the reader. ◇

Proof. " $\Longleftarrow$ ": Let $p \in \mathcal{A}$. If $\big(f(\xi) - p(\xi)\big)p(\xi) > 0$ for all $\xi \in E$, By the second condition, p has at least $n+2$ zeros. By lemma 3.2.4 1, $p \equiv 0$, contradicting $\big(f(\xi) - p(\xi)\big)p(\xi) > 0$.

" $\Longrightarrow$ ": Find $\xi_0, \ldots, \xi_k$ such that both properties are fulfilled. Then show $k = n$. □

Lemma 3.2.4 (Properties of Haar spaces)
An $(n+1)$-dimensional subspace $\mathcal{A} \subset \mathcal{C}([a,b])$ is a Haar *space if and only if one of the following conditions hold:*

- *If $p \in \mathcal{A}$ has $n+1$ zeros, then $p \equiv 0$.*
- *If $p \in \mathcal{A} \backslash \{0\}$ has $j < n+1$ roots and k of them are in (a,b) and p does not change sign at these roots, then $j + k \leqslant n$.*

Proof. *This is left as an exercise to the reader.* □

Remark 3.2.5 The second condition (and of course the first) holds for $\mathcal{A} = \mathcal{P}_n$: if $p \in \mathcal{A}$ doesn't change sign at those k roots, then they are double roots, so p has $2k + (j-k) = k + j$ roots. The claim follows as p can have at most n roots. ○

Example 3.2.6 ($(x, \ldots, x^n)$ is not a Haar system)
The first condition shows that $(x, \ldots, x^n)$ is not a Haar system: the subspace generated by it is n-dimensional, but it contains nonzero elements with n zeros, such as x^n. ◇

Lemma 3.2.7
If $\mathcal{A}$ is an $(n+1)$-dimensional Haar *space and $(x_j)_{j=0}^k$, where $k \leqslant n$, are distinct, then there exists a $\varphi \in \mathcal{A}$ such that φ changes sign at its roots x_j and has no other zeros.*

Proof. This is left as an exercise to the reader. □

Example 3.2.8 (Rivlin) Let $f\colon [-\pi, \pi] \to \mathbb{R}$, $x \mapsto \sin(4x)$ and p_k^* be the best approximation of f in $\mathcal{P}_k$, where $k \in \mathbb{N}$. Then $p_k^* = 0$ for $k \in \{0, \ldots, 6\}$ and $p_k^* \neq 0$ for $k \geqslant 7$.

Let $\mathcal{A} := \mathcal{P}_k$ for $k \in \{0, \ldots, 6\}$ with $\dim(\mathcal{A}) = k+1$. There exist exactly eight points $\xi_k = \left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \ldots, \frac{3\pi}{4}\right\}$ such that

$$\begin{cases} |f(\xi_i)| = \|f\|_\infty = 1, & i \in \{0, \ldots, 7\}, \\ f(\xi_{i+1}) = -f(\xi_i), & i \in \{0, \ldots, 6\}. \end{cases}$$

This proves the first statement by Theorem 3.2.1. For $\mathcal{A} = \mathcal{P}_k$ with $k \geqslant 7$, we would need $\geqslant 9$ points satisfying the above conditions, which don't exist, so $p_k^* \neq 0$. ◇

23.11.2021

The Alternation Theorem is useful for finding a lower bound for the error.

Theorem 3.2.2: de La Vallée Poussin

Let $\mathcal{A} \subset \mathcal{C}([a,b])$ be an $(n+1)$-dimensional Haar space. Let $f \in \mathcal{C}([a,b])$ and $p^* \in \mathcal{A}$ such that $f - p^*$ alternates in sign at $n+2$ points $a \leqslant \xi_0 < \ldots < \xi_{n+1} \leqslant b$. Then

$$E_n(f) := \min_{p \in \mathcal{A}} \|f - p\|_\infty \geqslant \min_{i \in \{0,\ldots,n+1\}} |f(\xi_i) - p^*(\xi_i)|. \tag{21}$$

The conditions of this theorem are exactly 1 and 3 from Theorem 3.2.1. Together they already imply that p^* is a best approximation by the same argument as in example 3.1.1: suppose there exists a $p \in \mathcal{A}$ such that (18) holds. Then p has $n+1$ zeros, so it is zero by lemma 3.2.4.

Hence we have 1, 3 $\implies$ p^* is best approximation $\iff$ 1 - 3.

Proof. Assume that (21) does not hold:

$$E_n(f) < \min_{i \in \{0,\ldots,n+1\}} |f(\xi_i) - p^*(\xi_i)|.$$

That means there is a $p_0 \in \mathcal{A}$ such that

$$E_n(f) \leqslant \|f - p_0\|_\infty < \min_{i \in \{0,\ldots,n+1\}} |f(\xi_i) - p^*(\xi_i)|, \tag{22}$$

which means that in particular

$$|f(\xi_i) - p_0(\xi_i)| < |f(\xi_i) - p^*(\xi_i)|.$$

Indeed, without loss of generality assume that $f(\xi_0) - p^*(\xi_0) > 0$. Then $f(\xi_0) - p_0(\xi_0) < f(\xi_0) - p^*(\xi_0)$ and $f(\xi_0) - p_0(\xi_0) > f(\xi_1) - p^*(\xi_1)$, and so on.

This shows that $p^* - p_0 = (f - p_0) - (f - p^*)$ alternates in sign at $n+2$ points. As it is continuous, it has at least $n+1$ zeros by the intermediate value theorem. Since $\mathcal{A}$ is a Haar space, $p^* - p_0 \equiv 0$ by lemma 3.2.4, contradicting (22). □

Another corollary of Theorem 3.2.1 is that we can now prove that the best approximation from a Haar space is unique.

3.3 Uniqueness

Note that existence is given by Theorem 1.1.1.

Theorem 3.3.1: Uniqueness

Let $\mathcal{A} \subset \mathcal{C}([a,b])$ be a $(n+1)$-dimensional subspace. Then a function $f \in \mathcal{C}([a,b])$ has a unique best approximation from $\mathcal{A}$ if and only if $\mathcal{A}$ is a Haar space.

Proof. " $\Longleftarrow$ ": Assume that $\mathcal{A}$ is a HAAR space and assume that there is a $f \in \mathcal{C}([a,b])$ and $p^*, q^* \in \mathcal{A}$ such that

$$E_n(f) = \|f - p^*\|_\infty = \|f - q^*\|_\infty.$$

Then

$$\left\|f - \frac{p^* + q^*}{2}\right\|_\infty \leqslant \frac{1}{2}\|f - p^*\|_\infty + \frac{1}{2}\|f - q^*\|_\infty = E_n(f),$$

and thus $E_n(f) = \left\|f - \frac{1}{2}(p^* + q^*)\right\|_\infty$, so $\frac{1}{2}(p^* + q^*)$ is also a best approximation.

By Theorem 3.2.1, there exist $\{\xi_0, \ldots, \xi_n\} \subset [a,b]$ such that

$$f(\xi_{i+1}) - \left(\frac{p^* + q^*}{2}\right)(\xi_{i+1}) = -\left[f(\xi_i) - \left(\frac{p^* + q^*}{2}\right)(\xi_i)\right] = (-1)^i h,$$

for all $i \in \{0, \ldots, n-1\}$, where

$$h := \pm\left\|f - \frac{p^* + q^*}{2}\right\|_\infty.$$

Hence

$$\frac{f(\xi_i) - p^*(\xi_i)}{2} + \frac{f(\xi_i) - q^*(\xi_i)}{2} = (-1)^{i+1} h. \tag{23}$$

Since $|f(\xi_i) - p^*(\xi_i)| \leqslant h$ and $|f(\xi_i) - p^*(\xi_i)| \leqslant h$, (21) implies that

$$f(\xi_i) - p^*(\xi_i) = f(\xi_i) - q^*(\xi_i) = (-1)^{i+1} h.$$

So $q^* - p^* = f - p^* - (f - q^*)$ alternates in sign $n+2$ times. As $\mathcal{A}$ is an $(n+1)$-dimensional HAAR space, $q^* - p^* = 0$ by lemma 3.2.4.

" $\Longrightarrow$ ": See [Che66, p. 81]. □

THEOREM 3.3.2: NONEXISTENCE

Let $\mathcal{A} := \operatorname{span}\{\varphi_0, \varphi_1, \ldots\} \subset \mathcal{C}([a,b])$ be a subspace of $\mathcal{C}([a,b])$ such that for every n, the set $\{\varphi_0, \ldots, \varphi_n\}$ satisfies the HAAR condition. Then no point outside $\mathcal{A}$ has a best approximation from $\mathcal{A}$.

DEFINITION 3.3.1 (MARKOV SYSTEM)

MARKOV system

Such a system $\{\varphi_0, \varphi_1, \ldots\}$ is called a MARKOV system.

Example 3.3.2 The set $(x \mapsto x^k)_{k \in \mathbb{N}_0}$ is a MARKOV system. ◇

Remark 3.3.3 (Corollary: WEIERSTRASS' Theorem)
The WEIERSTRASS theorem follows by taking $\mathcal{A} = \mathcal{P}$. Let $f \in \mathcal{C}([a,b])$, then for every $\varepsilon > 0$, there exists a $p \in \mathcal{A}$ such that $\|f - p\|_\infty < \varepsilon$. ○

We can also state something for 2π-periodic continuous functions, which we will approximation by trigonometric polynomials.

DEFINITION 3.3.4 (PERIODIC FUNCTIONS, TRIGONOMETRIC POLYNOMIALS)
Let $\mathcal{C}_{2\pi}$ be the space of all continuous real-valued 2π-periodic functions, $\mathcal{T}_n$ the set of trigonometric polynomials of degree at most n and $\mathcal{T} := \bigcup_{n \in \mathbb{N}_0} \mathcal{T}_n$.

Corollary 3.3.5 (Weierstrass' Second Theorem (1885))
For every $f \in \mathcal{C}_{2\pi}$ and every $\varepsilon > 0$, there exists a trigonometric polynomial $T \in \mathcal{T}$ such that $\|f - T\|_\infty < \varepsilon$.

Proof. (of corollary 3.3.5 (HW 3.2)) *This is left as an exercise to the reader.* □

Proof. (of Theorem 3.3.2) Let $f \in \mathcal{C}([a,b]) \backslash \mathcal{A}$ and assume that f has a best approximation from $\mathcal{A}$. Hence there is a $g \in \mathcal{A}$ such that

$$E_n(f) = \|f - g\|_\infty =: \varepsilon > 0.$$

That means that the zero function is a best approximation to $f - g \in \mathcal{C}([a,b])$ from $\mathcal{A}_n :=$ span$\{\varphi_0, \dots, \varphi_n\} \subset \mathcal{A}$ for every $n \in \mathbb{N}$.

By the alternation theorem, there exist $n+2$ points such that $f - g - 0$ alternates between ε and $-\varepsilon$. Since that happens for every $n \in \mathbb{N}$ and $f - g$ is continuous on $[a,b]$, we arrive at a contradiction (why?). So f does not have a best approximation from $\mathcal{A}$. □

It is not easy to find the best approximation.

Example 3.3.6 (Best approximation of x^n from $\mathcal{P}_{n-1}$)
We show that the best approximation of $f\colon [-1,1] \to \mathbb{R}$, $x \mapsto x^n$ from $\mathcal{P}_{n-1}([-1,1])$ is

$$p\colon [-1,1] \to \mathbb{R}, \qquad x \mapsto x^n - 2^{-(n-1)} T_n(x),$$

where T_n is the n-th Chebychev polynomial.

We have $p \in \mathcal{P}_{n-1}$, as the leading coefficient of T_n is 2^{n-1}. We look at the error function

$$f(x) - p(x) = x^n - p(x) = 2^{-(n-1)} T_n(x).$$

By lemma 2.3.6, this polynomial has minimum norm among all polynomials of degree n with leading coefficient 1, that is $E_n(f) = \min_{p \in \mathcal{P}_{n-1}} \|f - p\|_\infty = 2^{-(n-1)} \|T_n\|_\infty$.

Alternatively, we can use that T_n has $n+1$ extrema $x_k = \cos(\frac{k\pi}{n})$ with $T_n(x_k) = (-1)^k$ by lemma 2.3.3. Hence

$$|2^{-(n-1)} T_n(x_k)| = \|2^{-(n-1)} T_n\| = 2^{-(n-1)},$$

which is the second condition and $2^{-(n-1)} T_n(x_k) = (-1)^k 2^{-(n-1)}$, which is the third condition. By Theorem 3.2.1, p is the best approximation from $\mathcal{P}_{n-1}$ of $f(x) = x^n$. ◇

4 The inequalities of Markov and Bernstein

4.1 Introduction

30.11.2021

Answering a question raised by Mendeleev, A.A. Markov's inequality (1889) states that if $p \in \mathcal{P}_n$, then

$$\max_{x\in[-1,1]} |p'(x)| \leqslant n^2 \max_{x\in[-1,1]} |p(x)|$$

and we have equality for $p = \alpha T_n$ for any $\alpha \in \mathbb{R}$ (Exercise!). This bounded is optimal. (As $\mathcal{P}_n$ is finite-dimensional, and $p \mapsto p'$ is a linear operation, there exists a constant $M_n \in \mathbb{R}$ such that $\max_{x\in[-1,1]} |p'(x)| \leqslant M_n \max_{x\in[-1,1]} |p(x)|$.)

One can even show that (proven by Markov's brother)

$$\max_{x\in[-1,1]} |p^{(k)}(x)| \leqslant \frac{n^2(n^2-1^2)\dots(n^2-(k-1)^2)}{1\cdot 3\cdot \dots \cdot (2k-1)} \max_{x\in[-1,1]} |p(x)| \qquad \forall k \in \{1,\dots n\},$$

where equality holds for any multiple of T_n.

The analogue of this result for the unit disk in $\mathbb{C}$ is due to Bernstein (1926) and states that if S is a complex algebraic polynomial of degree at most n, then

Bernstein proved the inequality with $2n$ instead, later Feketé showed that the inequality holds with n, attributing his proof to Fejér.

$$\max_{|z|=1} |S'(z)| \leqslant n \max_{|z|=1} |S(z)|$$

and we have equality for $p(z) = \lambda z^n$, where $\lambda \in \mathbb{C}$.

Using the bijection $\theta \leftrightarrow e^{i\theta}$, we can formulate this in terms of trigonometric polynomial: if $S \in \mathcal{T}_n$ then

$$\max_{\theta\in[0,2\pi]} |S'(\theta)| \leqslant n \max_{\theta\in[0,2\pi]} |S(\theta)|.$$

These two inequality are fundamental for inverse theorems.

4.2 Preparatory lemmas

We first need some lemmas.

Lemma 4.2.1 (Lagrange interpolation operator)
Let $x_1,\dots,x_n$ be the zeros of T_n. For $f \in \mathcal{C}([-1,1])$ we have

$$(L_n f)(x) = \frac{1}{n}\left(\sum_{k=1}^{n} f(x_k)(-1)^{k-1}\frac{\sqrt{1-x_k^2}}{x-x_k}\right) T_n(x),$$

where $L_n\colon \mathcal{C}([-1,1]) \to \mathcal{P}_{n-1}$ is the Lagrange interpolation operator.

Proof. In the proof of Theorem 2.3.5 we showed that

$$\ell_k(x) = \frac{\sqrt{1-x_k^2}}{n}(-1)^{k-1}\frac{T_n(x)}{x-x_k},$$

which implies the statement. □

Lemma 4.2.2 (Bounded by $\sqrt{1-x^2}p(x)$)
If $p \in \mathcal{P}_{n-1}$, then

$$\max_{x\in[-1,1]} |p(x)| \leqslant n \max_{x\in[-1,1]} \sqrt{1-x^2}|p(x)|.$$

Proof. Let $x_k := \cos\left(\frac{2k-1}{2n}\pi\right)$ be the zeros of T_n. Then $x_1 = \cos\left(\frac{\pi}{2n}\right) = -\cos\left(\frac{2n-1}{2n}\pi\right) = -x_n$.

We consider two cases.

1. If $x \in [x_n, x_1]$, then $|x| \leqslant x_1$ and thus
$$\sqrt{1-x^2} \geqslant \sqrt{1-x_1^2} = \sin\left(\frac{\pi}{2n}\right) \geqslant \frac{1}{n},$$
where in the last step we use that $x \leqslant \sin(\frac{\pi}{2}x)$ for $x \in [0,1]$. This implies that
$$|p(x)| \leqslant n\sqrt{1-x^2}|p(x)| \leqslant n \max_{x\in[-1,1]} \sqrt{1-x^2}|p(x)| =: M.$$

2. If $x \notin [x_n, x_1]$, then we use lemma 4.2.1. Since $p \in \mathcal{P}_{n-1}$ we have $L_n p = p$. Furthermore, $x - x_j$ and $x - x_k$ have the same sign for all $j, k \in \{1, \dots, n\}$. We have
$$\begin{aligned}|p(x)| = |(L_n p)(x)| &\overset{4.2.1}{\leqslant} \frac{1}{n}\sum_{k=1}^{n} |p(x_k)|\sqrt{1-x_k^2}\frac{|T_n(x)|}{|x-x_k|} \\ &\leqslant \frac{M}{n^2}\sum_{k=1}^{n}\frac{|T_n(x)|}{|x-x_k|} = \frac{M}{n^2}\left|\sum_{k=1}^{n}\frac{T_n(x)}{x-x_k}\right| = \frac{M}{n^2}|T_n'(x)|,\end{aligned}$$
where the last equality is an Exercise. Since $|T_n'(x)| = \frac{n}{\sqrt{1-x^2}}|\sin(n\cos^{-1}(x))|$ and $x = \cos(\cos^{-1}(x))$, we have
$$|T_n'(x)| = n\frac{|\sin(n\cos^{-1}(x))|}{|\sin(\cos^{-1}(x))|} \leqslant n^2,$$
as $|\sin(n\theta)| \leqslant n|\sin(\theta)|$ for all $\theta \in \mathbb{R}$ and all $n \in \mathbb{N}$. (Prove this yourself by induction.) Hence $|p(x)| \leqslant M$ for all $x \notin [x_n, x_1]$. □

The next result result is about odd trigonometric polynomials. Recall that if $S \in \mathcal{T}_n$, then
$$S(\theta) = a_0 + \sum_{k=1}^{n} a_k\cos(k\theta) + b_k\sin(k\theta).$$

By lemma 2.3.1 there exist pairs of polynomials $(P_k, Q_{k-1})_{k=1}^n$, each of degree exactly k and $k-1$ respectively, such that $\cos(k\theta) = P_k(\cos(\theta))$ and $Q_{k-1}(\cos(\theta))\sin(\theta) = \sin(k\theta)$. Hence there are $P \in \mathcal{P}_n$ and $Q \in \mathcal{P}_{n-1}$ such that
$$S(\theta) = P(\cos(\theta)) + Q(\cos(\theta))\sin(\theta).$$

If S is even, then the second terms vanishes, if it is odd, then the first term vanishes due to cos being even and sin being odd.

Lemma 4.2.3 (Odd trigonometric polynomial)
If $S \in \mathcal{T}_n$ is odd, then
$$\max_{\theta\in[0,2\pi]}\left|\frac{S(\theta)}{\sin(\theta)}\right| \leqslant n \max_{\theta\in[0,2\pi]}|S(\theta)|.$$

Proof. Since S is odd, there exists a $Q \in \mathcal{P}_{n-1}$ such that
$$S(\theta) = Q(\cos(\theta))\sin(\theta). \tag{24}$$

Thus by lemma 4.2.2 we have
$$\begin{aligned}\max_{\theta\in[0,2\pi]}\left|\frac{S(\theta)}{\sin(\theta)}\right| &\overset{(24)}{=} \max_{\theta\in[0,2\pi]}|Q(\cos(\theta))| \leqslant n \max_{\theta\in[0,2\pi]}|\sqrt{1-\cos^2(\theta)}Q(\cos(\theta))| \\ &= n\max_{\theta\in[0,2\pi]}|\sin(\theta)Q(\cos(\theta))| \overset{(24)}{=} n\max_{\theta\in[0,2\pi]}|S(\theta)|.\end{aligned}$$
□

4.3 The proofs

Theorem 4.3.1: Bernstein's inequality

Let $n \geqslant 1$ and $S \in \mathcal{T}_n$. Then

$$\boxed{\max_{\theta \in [0,2\pi]} |S'(\theta)| \leqslant n \max_{\theta \in [0,2\pi]} |S(\theta)|.}$$

Proof. Define the function

$$f\colon [0,2\pi] \times [0,2\pi] \to \mathbb{R}, \qquad (\alpha,\theta) \mapsto \frac{1}{2}\big(S(\alpha+\theta) - S(\alpha-\theta)\big).$$

Then $f(\alpha,\cdot)$ is an odd trigonometric polynomial for all $\alpha \in [0,2\pi]$. By lemma 4.2.3 we have

$$\max_{\theta \in [0,2\pi]} \left|\frac{f(\alpha,\theta)}{\sin(\theta)}\right| \leqslant n \max_{\theta \in [0,2\pi]} |f(\alpha,\theta)| \leqslant n \max_{\theta \in [0,2\pi]} |S(\theta)|. \tag{25}$$

Finally,

$$|S'(\alpha)| = \lim_{\theta \to 0} \left|\frac{S(\alpha+\theta) - S(\alpha-\theta)}{2\theta}\right| = \lim_{\theta \to 0} \left|\frac{f(\alpha,\theta)}{\theta}\right| = \lim_{\theta \to 0} \left|\frac{f(\alpha,\theta)}{\sin(\theta)}\right| \overset{(25)}{\leqslant} n \max_{\theta \in [0,2\pi]} |S(\theta)|. \qquad \square$$

Finally, we can prove Markov's inequality.

Theorem 4.3.2: Markov

For $n \geqslant 1$ and $p \in \mathcal{P}_n$ we have

$$\boxed{\max_{x \in [-1,1]} |p'(x)| \leqslant n^2 \max_{x \in [-1,1]} |p(x)|.}$$

Proof. By lemma 4.2.2 we have

$$\max_{x \in [-1,1]} |p'(x)| \leqslant n \max_{x \in [-1,1]} |\sqrt{1-x^2}\,p'(x)| = n \max_{\theta \in [0,2\pi]} |\sin(\theta) p'(\cos(\theta))|. \tag{26}$$

Let $S(\theta) := p(\cos(\theta))$. Then $S'(\theta) = -p'(\cos(\theta))\sin(\theta)$. By Theorem 4.3.1 for $S \in \mathcal{T}_n$ we have

$$\begin{aligned}\max_{x \in [-1,1]} |p'(x)| &\overset{(26)}{\leqslant} n \max_{\theta \in [0,2\pi]} |\sin(\theta) p'(\cos(\theta))| = n \max_{\theta \in [0,2\pi]} |S'(\theta)| \\ &\overset{4.3.1}{\leqslant} n^2 \max_{\theta \in [0,2\pi]} |S(\theta)| = n^2 \max_{\theta \in [0,2\pi]} |p(\cos(\theta))| = n^2 \max_{x \in [-1,1]} |p(x)|. \qquad \square\end{aligned}$$

5 Least squares approximation

5.1 The (continuous) least squares problem

07.12.2021

Let $\mathcal{A} \subset \mathcal{C}([a,b];\mathbb{R})$ be set of approximating functions and $w\colon [a,b] \to (0,\infty)$ a fixed *positive* continuous weight function.

weight function

Definition 5.1.1 (Least squares approximation)
The best weighted least squares approximation from $\mathcal{A}$ to $f \in \mathcal{C}([a,b];\mathbb{R})$ is

best weighted least squares approximation

$$p^* := \operatorname{argmin}_{p\in\mathcal{A}} \int_a^b w(x)\big|f(x) - p(x)\big|^2 \,\mathrm{d}x = \operatorname{argmin}_{p\in\mathcal{A}} \|f - p\|_w,$$

where the weighted scalar product and the induced norm are (for $f, g \in \mathcal{C}([a,b];\mathbb{R})$)

$$(f,g)_w := \int_a^b w(x)f(x)g(x)\,\mathrm{d}x, \qquad \|f\|_w := \sqrt{(f,f)_w}. \tag{27}$$

Without the positivity of w, the scalar product would not be positive definite. We know this construction with $w(x) = \frac{1}{\sqrt{1-x^2}}$ as the weighted scalar product with respect to which the Chebychev polynomials of the first kind are orthogonal.

Hence least squares approximation is best approximation with a weighted norm, so if e.g. $\mathcal{A}$ is a finite dimensional linear subspace, then the best weighted least approximation exists by Theorem 1.1.1.

Lemma 5.1.2 ($\|\cdot\|_w$ strictly convex)
The weighted norm $\|\cdot\|_w$ is strictly convex on $\mathcal{C}([a,b];\mathbb{R})$.

Proof. For $f, g \in \mathcal{C}([a,b];\mathbb{R})$ with $f \neq g$ and $\|f\|_w = \|g\|_w = 1$ we have (we can with loss of generality use midpoint strict convexity)

$$\begin{aligned}\left(\frac{1}{2}\|f+g\|_w\right)^2 &= \frac{1}{4}\int_a^b w(x)|f(x)+g(x)|^2\,\mathrm{d}x < \frac{1}{2}\int_a^b w(x)|f(x)|^2\,\mathrm{d}x + \frac{1}{2}\int_a^b w(x)|g(x)|^2\,\mathrm{d}x \\ &= \frac{1}{2}\|f\|_w^2 + \frac{1}{2}\|g\|_w^2 = 1,\end{aligned}$$

using that $(x+y)^2 < 2(x^2+y^2)$ for $x \neq y$. □

Remark 5.1.3 As $\big(\mathcal{C}([a,b]), (\cdot,\cdot)_w\big)$ is an inner product space, this lemma also follows directly from Theorem 1.2.1 3 and 1, which together say that inner product spaces are strictly convex. ○

Corollary 5.1.4 (Uniqueness of weighted least squares approximation)
If $\mathcal{A} \subset \mathcal{C}([a,b];\mathbb{R})$ is a linear subspace, then either the weighted least squares approximation does not exist or it is unique by corollary 1.2.5.

5.2 The least squares characterisation theorem for inner product spaces

THEOREM 5.2.1: LEAST SQUARES CHARACTERISATION

Let $\mathcal{A} \subset (H, (\cdot,\cdot))$ be linear subspace of an inner product space and $f \in H$. Then $p^* \in \mathcal{A}$ is best approximation from $\mathcal{A}$ to f if and only if the error $e^* := f - p^*$ is orthogonal to $\mathcal{A}$, that is, $(e^*, p) = 0$ for all $p \in \mathcal{A}$.

Proof. " $\Longrightarrow$ ": Suppose there exists a $p \in \mathcal{A}$ such that $(e^*, p) \neq 0$. Then $p^* + \frac{(e^*,p)}{\|p\|^2} p$ is better approximation than p^*:

$$\begin{aligned}\left\| f - p^* - \frac{(e^*,p)}{\|p\|^2} p \right\|^2 &= \|f - p^*\|^2 - 2\frac{(e^*,p)}{\|p\|^2}(e^*,p) + \frac{(e^*,p)^2}{\|p\|^4}\|p\|^2 \\ &= \|f - p^*\|^2 - 2\frac{(e^*,p)^2}{\|p\|^2} + \frac{(e^*,p)^2}{\|p\|^2} \\ &= \|f - p^*\|^2 - \frac{(e^*,p)^2}{\|p\|^2} < \|f - p^*\|^2.\end{aligned}$$

" $\Longleftarrow$ ": Suppose $(e^*, p) = 0$ for all $p \in \mathcal{A}$. For $q^* \in \mathcal{A}$ we have

$$\begin{aligned}\|f - q^*\|^2 - \|f - p^*\|^2 &= \cancel{\|f\|^2} - 2(f, q^*) + \|q^*\|^2 \cancel{- \|f\|^2} + 2(f, p^*) - \|p^*\|^2 \\ &= \|q^*\|^2 - \|p^*\|^2 + 2(f, p^* - q^*) \\ &= \|q^*\|^2 \cancel{- 2(p^*, q^*)} + \|p^*\|^2 + 2(f, p^*) \\ &\quad - 2(f, q^*) - 2(p^*, p^*) + \cancel{2(p^*, q^*)} \\ &= \|q^* - p^*\|^2 + 2(\underbrace{f - p^*}_{=e^*}, \underbrace{p^* - q^*}_{\in \mathcal{A}}) = \|q^* - p^*\|^2.\end{aligned}$$

So p^* is the unique best approximation as

$$\|f - q^*\|^2 = \|f - p^*\|^2 + \|q^* - p^*\|^2 \geqslant \|f - p^*\|^2. \qquad \square$$

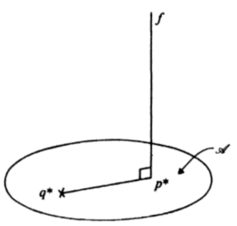

Fig. 7: *The* best approximation $p^* \in \mathcal{A}$ from $\mathcal{A}$ to f and $q^* \in \mathcal{A}$.

Remark 5.2.1 (A geometric viewpoint: orthogonal projection)
We have PYTHAGORAS' Theorem: $\|f - q^*\|^2 = \|f - p^*\|^2 + \|q^* - p^*\|^2$. For $q^* = 0 \in \mathcal{A}$ we get $\|f\|^2 = \|p^*\|^2 + \|f - p^*\|^2$. ○

5.3 Methods of calculation

Choose basis functions $(\varphi_i)_{i=0}^n$ and set $\mathcal{A} := \operatorname{span}((\varphi_i)_{i=0}^n)$. Then it suffices to find coefficients $(c_i^*)_{i=0}^n$ such that $p^* = \sum_{i=0}^n c_i^* \varphi_i$.

THEOREM 5.3.1: EXPLICIT REPRESENTATION

Let $\mathcal{A} \subset H$ be a linear subspace of an inner product space spanned by basis functions $(\varphi_i)_{i=0}^n$ and $f \in H$. If the orthogonality condition

$$(\varphi_i, \varphi_j) = 0 \qquad \forall i \neq j,\ i, j \in \{0, \dots, n\}$$

is satisfied, then the best approximation from $\mathcal{A}$ to f is

$$p^* = \sum_{i=0}^{n} \frac{(\varphi_i, f)}{\|\varphi_i\|^2} \varphi_i.$$

Proof. By orthogonality, the $(\varphi_i)_{i=0}^n$ linearly independent, so the coefficients c_i^* are unique. By Theorem 5.2.1 $p^* = \sum_{j=0}^n c_j^* \varphi_j$ is best approximation if and only if

$$\left(\varphi_i, f - \sum_{j=0}^{n} c_j^* \varphi_j\right) = 0, \qquad \forall i \in \{0, \ldots, n\},$$

as $(\varphi)_{i=0}^n$ is a basis for $\mathcal{A}$ and thus by the linearity of the scalar product in the first argument it suffices to "test" with the basis functions.

This is equivalent to the normal equations

$$(\varphi_i, f) = \sum_{j=0}^{n} c_j^* (\varphi_i, \varphi_j), \qquad \forall i \in \{0, \ldots, n\}.$$

The orthogonality of $(\varphi_i)_{i=0}^n$ implies $(\varphi_i, f) = c_i^* \|\varphi_i\|^2$. □

Remark 5.3.1 (Alternative differential derivation) We can also find coefficients as

$$\begin{aligned}
\operatorname{argmin}_{(c_i)_{i=0}^n} \left\| f - \sum_{i=0}^{n} c_i \varphi_i \right\|^2 &= \operatorname{argmin}_{(c_i)_{i=0}^n} \sum_{i,j=0}^{n} c_i c_j (\varphi_i, \varphi_j) - 2 \sum_{i=0}^{n} c_i (\varphi_i, f) \\
&= \operatorname{argmin}_{(c_i)_{i=0}^n} \underbrace{\sum_{i=0}^{n} c_i^2 \|\varphi_i\|^2 - 2 \sum_{i=0}^{n} c_i (\varphi_i, f)}_{=: g(c)}.
\end{aligned}$$

A necessary condition for the minimum being attained is that for $k \in \{0, \ldots, n\}$

$$\frac{\partial}{\partial c_k} g(c) = 2\|\varphi_k\|^2 c_k - 2(\varphi_k, f) \stackrel{!}{=} 0 \iff c_k = \frac{(\varphi_k, f)}{\|\varphi_k\|^2}.$$ ○

Remark 5.3.2 (Finding an orthogonal basis) Let $\mathcal{A} := \operatorname{span}((\psi_i)_{i=0}^n)$. How can we choose an orthogonal basis of $\mathcal{A}$?

Let $\varphi_0 := \psi_0$, $\mathcal{A}_i := \operatorname{span}(\psi_0, \ldots, \psi_i)$. For $i \geqslant 1$ choose $\widetilde{\psi}_i \in \mathcal{A}_i \setminus \mathcal{A}_{i-1}$, and then let q_i^* be the best approximation from $\mathcal{A}_{i-1}$ to $\widetilde{\psi}_i$. Let

$$\varphi_i := \widetilde{\psi}_i - q_i^*.$$

Because $\varphi_i \perp \mathcal{A}_{i-1}$ by Theorem 5.2.1, we have $\varphi_i \perp \varphi_j$ for all $j < i$. Hence $(\varphi_i)_{i=0}^n$ are orthogonal basis of $\mathcal{A}$. ○

Example 5.3.3 (Application)

Given $f \in H$, an orthogonal basis $(\psi_k)_{k \in \mathbb{N}}$ and an accuracy $\delta > 0$, we want to find $p = \sum_{i=0}^n c_i \psi_i$ such that $\|f - p\| < \delta$. That is, we want to write p as linear combination of *first* $n+1$ elements of basis (this process is sensitive to reordering of basis).

By Theorem 5.3.1, the best approximation to f from $\mathcal{A}_i := \operatorname{span}((\psi_j)_{j=0}^i)$ is

$$p_i^* := \sum_{j=0}^{i} \frac{(\psi_j, f)}{\|\psi_j\|^2} \psi_j,$$

Hence the desired n is least integer such that $\|f - p_n^*\| \leqslant \delta$. By Pythagoras, this is equivalent to $\|f\|^2 - \|p_n^*\|^2 \leqslant \delta^2$, i.e. $\|p_n^*\|^2 \geqslant \|f\|^2 - \delta^2$.

Hence we just have to n so large that

$$\|p_n^*\|^2 = \sum_{j=0}^{n} \frac{(\psi_j, f)^2}{\|\psi_j\|^2}$$

is large enough. ◇

5.4 Orthogonal polynomials with respect to a weight function

Theorem 5.4.1: Recurrence relation for orthogonal monomials

The monic orthogonal polynomials with respect to w are uniquely determined by $Q_0 \equiv 1$, $Q_1(x) := x - a_0$ and for $j \geqslant 1$ by

$$Q_{j+1}(x) := (x - a_j)Q_j(x) - b_j Q_{j-1}(x).$$

where

$$a_j := \frac{(Q_j, xQ_j(\cdot))_w}{(Q_j, Q_j)_w}, \qquad b_j := \frac{(xQ_j, Q_{j-1})_w}{(Q_{j-1}, Q_{j-1})_w} = \frac{\|Q_j\|_w^2}{\|Q_{j-1}\|_w^2}.$$

Proof. 1 **Orthogonality.** We use induction. We have

$$a_0 = \frac{\int_a^b x \cdot 1 \cdot 1 \cdot w(x)\,\mathrm{d}x}{\int_a^b 1 \cdot 1 \cdot w(x)\,\mathrm{d}x} = \frac{\int_a^b x \cdot w(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x}$$

and hence

$$\begin{aligned}(Q_0, Q_1)_w &= \int_a^b (x - a_0)w(x)\,\mathrm{d}x = \int_a^b \left(x - \frac{\int_a^b x \cdot w(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x}\right) w(x)\,\mathrm{d}x \\ &= \int_a^b xw(x)\,\mathrm{d}x - \frac{\int_a^b x \cdot w(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x} \int_a^b w(x)\,\mathrm{d}x = 0.\end{aligned}$$

We assume that $(Q_n, Q_k)_w = 0$ for all $k < n$. We show that $(Q_{n+1}, Q_k)_w = 0$ for all $k < n+1$.

Using the recurrence relation and that $Q_n \perp_w \mathcal{P}_{n-1}$ we have for $k \in \{0, \ldots, n-2, n\}$

$$\begin{aligned}(Q_{n+1}, Q_k)_w &= (xQ_n, Q_k)_w - a_n \cdot (Q_n, Q_k)_w - b_n(Q_{n-1}, Q_k)_w \\ &= (xQ_n, Q_k)_w = (Q_n, xQ_k)_w = 0\end{aligned}$$

and

$$\begin{aligned}(Q_{n+1}, Q_{n-1})_w &= (xQ_n, Q_{n-1})_w - a_n \cdot \underbrace{(Q_n, Q_{n-1})_w}_{=0} - b_n(Q_{n-1}, Q_{n-1})_w \\ &= (xQ_n, xQ_{n-1})_w - (xQ_n, xQ_{n-1})_w = 0.\end{aligned}$$

2 **Uniqueness.** From the recurrence relation we see that $Q_n \in \mathcal{P}_n$ is monic. If $(p_k)_{k=0}^n \subset \mathcal{P}_n$ is a set of monic orthogonal polynomials, then for each $k \in \{1, \ldots, n\}$ we have $p_k - Q_k \in \mathcal{P}_{k-1}$ and $(p_k - Q_k) \perp \mathcal{P}_{k-1}$. Hence $p_k - Q_k = 0$.

These results also imply that

$$(xQ_n, Q_{n-1})_w = (Q_n, xQ_{n-1})_w = (Q_n, \underbrace{xQ_{n-1} - Q_n}_{\in P_{n-1}})_w + (Q_n, Q_n)_w = (Q_n, Q_n)_w,$$

so $b_n = \frac{\|Q_n\|_w^2}{\|Q_{n-1}\|_w^2}$. □

Example 5.4.1 (Chebychev polynomials of first and second kind)
If $w(x) := \frac{1}{\sqrt{1-x^2}}$, then the orthogonal polynomials are $Q_n = 2^{-(n-1)}T_n$, where T_n are the Chebychev polynomials of the first kind.

In this case we have

$$a_0 = \frac{\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x}{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x} = \frac{0}{\pi} = 0, \qquad b_1 = \frac{\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x}{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x} = \frac{\frac{\pi}{2}}{\pi} = \frac{1}{2}$$

and

$$b_n = \frac{\|2^{-(n-1)}T_n\|_w^2}{\|2^{-(n-2)}T_{n-1}\|_w^2} = 2^{2n-4-2n+2}\frac{\|T_n\|_w^2}{\|T_{n-1}\|_w^2} = \frac{1}{4}\frac{\|T_n\|_w^2}{\|T_{n-1}\|_w^2} = \frac{1}{4}\frac{\frac{\pi^2}{4}}{\frac{\pi^2}{4}} = \frac{1}{4}$$

for all $n > 1$.

If $w(x) := \frac{1}{\sqrt{x^2-1}}$, then Q_n are multiples of the Chebychev polynomials of the second kind. ◇

Example 5.4.2 (Jacobi polynomials)
The orthogonal monic polynomials $P_n^{\alpha,\beta}$ corresponding to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ for $\alpha, \beta > -1$ are the Jacobi polynomials. Jacobi polynomial

In particular, $P_n^{-\frac{1}{2},-\frac{1}{2}}$ are the normalised Chebychev polynomials of the first kind, $P_n^{\frac{1}{2},\frac{1}{2}}$ are the normalised Chebychev polynomials of the second kind. ◇

Remark 5.4.3 (Legendre and Hermite polynomials)
If we choose $w \equiv 1$, we get the Legendre polynomials. The polynomials corresponding to $w(x) := \exp(-x^2)$ for $x \in (0, \infty)$ are the Hermite polynomials defined by $H_0 \equiv 1$, $H_1(x) = 2x$ and $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ for $n \geqslant 1$. ○ Legendre polynomials

Lemma 5.4.4 (Minimum weighted norm)
If Q is a monic polynomial of degree exactly n, then there exist $(c_k)_{k=0}^{n-1}$ such that

$$Q = Q_n + \sum_{k=0}^{n-1} c_k Q_k$$

and thus

$$\|Q\|_w^2 = \|Q_n\|_w^2 + \sum_{k=0}^{n-1} |c_k|^2,$$

implying that Q has the minimum weighted norm among monic polynomials of degree n.

Hence $2^{-(n-1)}T_n$ are minimal among the monic polynomials of degree exactly n with respect to $\|\cdot\|_\infty$ *and* $\|\cdot\|_w$.

Lemma 5.4.5 (Q_n has distinct roots)
All roots of Q_n are simple, real and contained in (a, b).

To prove the above statement, we first prove another one.

Lemma 5.4.6
A continuous function $f \in \mathcal{C}([a,b];\mathbb{R})$ *satisfies* $\int_a^b w(x)f(x)p(x)\,\mathrm{d}x = 0$ *for all* $p \in \mathcal{P}_{n-1}$ *if and only if there is a* n*-times differentiable function* u *such that*

$$fw = u^{(n)}, \qquad u^{(k)}(a) = u^{(k)}(b) = 0 \quad \forall k \in \{0,\dots,n-1\}.$$

Proof. (HW 4.2(a)) *This is left as an exercise to the reader.* □

Proof. (of lemma 5.4.5 (HW 4.2(b))) *This is left as an exercise to the reader.* □

Theorem 5.4.2: Erdős-Turán

Let $(Q_n)_{n\in\mathbb{N}}$ be the set of orthogonal polynomials with respect to the weight function $w\colon [a,b] \to (0,\infty)$. Let $L_n f$ be the polynomial of degree n that interpolates f at the zeros of Q_{n+1}, $(x_k)_{k=0}^n$. Then $\|L_n(f) - f\|_w \xrightarrow{n\to\infty} 0$.

Proof. (HW 5.3) 1 We first show that

$$(L_n f)(x) = \sum_{k=0}^n f(x_k)\ell_k(x),$$

where $\ell_k(x) := \frac{Q_{n+1}(x)}{(x-x_k)Q'_{n+1}(x_k)}$. (The ℓ_k are well defined because $(x - x_k)$ is a factor of $Q_{n+1}(x)$, so there is no division by 0.)

This is left as an exercise to the reader.

2 We now show that

$$\int_a^b \sum_{k=0}^n \ell_k^2(x)w(x)\,\mathrm{d}x = \int_a^b \left(\sum_{k=0}^n \ell_k(x)\right)^2 w(x)\,\mathrm{d}x = \int_a^b w(x)\,\mathrm{d}x. \tag{28}$$

This is left as an exercise to the reader.

3 We now show the statement of the theorem.

This is left as an exercise to the reader. □

Lemma 5.4.7 (Upper and lower bounds on $E_n(f)$)
Let f *be formally written as* $f = \sum_{k=0}^\infty c_k Q_k$, *where* $(Q_k)_{k=0}^\infty$ *is the sequence of orthogonal polynomials with respect to a weight function* $w\colon [a,b] \to (0,\infty)$ *and* $\deg(Q_k) = k$. *Then for any* $n \in \mathbb{N}_0$ *we have*

$$\max(\{\alpha_j|c_j| : j \geqslant n+1\}) \leqslant E_n(f) \leqslant \sum_{j=n+1}^\infty \|Q_j\|_\infty |c_j|, \qquad \textit{where } \alpha_k := \frac{(Q_k, Q_k)_w}{(|Q_k|, \mathbb{1})_w}.$$

Proof. (HW 6.3) 1 For $k \geqslant n+1$ we have due to the orthogonality of the $(Q_k)_{k\in\mathbb{N}}$ that

$$(f, Q_k)_w = c_k(Q_k, Q_k)_w = c_k\alpha_k(|Q_k|, \mathbb{1})_w.$$

Let $p \in \mathcal{P}_n$ be the best approximation of f in $\mathcal{P}_n$. Then as $Q_k \perp_w \mathcal{P}_n$ we have

$$\alpha_k|c_k| = \frac{|(f-p, Q_k)_w|}{|(|Q_k|, \mathbb{1})_w|} \leqslant \frac{\|f-p\|_\infty \cancel{|(\mathbb{1}, Q_k)_w|}}{\cancel{(|Q_k|, \mathbb{1})_w}} = E_n(f).$$

2 As the $(Q_k)_{k=0}^{\infty}$ are orthogonal, every $p \in \mathcal{P}_n$ can be written as a linear combination of the form $p = \sum_{k=0}^{n} a_k Q_k$. Hence

$$E_n(f) = \min_{p \in \mathcal{P}_n} \|f - p\|_\infty = \min_{(a_k)_{k=0}^{n} \subset \mathbb{R}} \left\| \sum_{k=0}^{\infty} c_k Q_k - \sum_{k=0}^{n} a_k Q_k \right\|_\infty$$

$$\overset{a_k = c_k}{=} \left\| \sum_{k=n+1}^{\infty} c_k Q_k \right\|_\infty \overset{\triangle \neq}{\leqslant} \sum_{k=n+1}^{\infty} |c_k| \|Q_k\|_\infty. \qquad \square$$

6 FOURIER series

6.1 Fourier series and partial sum

14.12.2021

First, let us fix the notation for certain spaces of 2π-periodic functions.

DEFINITION 6.1.1 (THE FUNCTION SPACES $L^1_{2\pi}$, $L^2_{2\pi}$ AND $\mathcal{C}^k_{2\pi}$)
Let $L^1_{2\pi}$ be the space of 2π-periodic absolutely integrable functions, $L^2_{2\pi}$ be the space of 2π-periodic square integrable functions and $\mathcal{C}^k_{2\pi}$ be the space of k-times continuously differentiable 2π-periodic functions.

DEFINITION 6.1.2 (FOURIER SERIES)

FOURIER series

The FOURIER series of $f \in L^1_{2\pi}$ is

$$x \mapsto \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k[f]\cos(kx) + b_k[f]\sin(kx) = \sum_{k\in\mathbb{Z}} c_k[f]e^{ikx},$$

where

$$a_k[f] := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,\mathrm{d}t, \qquad b_k[f] := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t$$

and $c_k[f] := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,\mathrm{d}t$ are the FOURIER coefficients.

We can pass from the real to the complex form (and the other way around) by

$$2c_0[f] = a_0[f], \quad \text{and} \quad 2c_k[f] = a_k[f] - ib_k[f], \quad 2c_{-k}[f] = a_k[f] + ib_k[f] \text{ for } k \in \mathbb{N}_{>0}\,.$$

Furthermore, $|a_k[f]|, |b_k[f]| \leqslant \frac{1}{\pi}\|f\|_{L^1_{2\pi}}$.

complex harmonics

Remark 6.1.3 (Complex harmonics) The complex harmonics $(x \mapsto e^{ikx})_{k\in\mathbb{Z}}$ are orthonormal with respect to the weighted inner product

$$\langle f, g \rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x, \qquad f, g \in L^2_{2\pi}.$$

For we have $\langle e^{ik\cdot}, e^{ik\cdot} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,\mathrm{d}x = 1$ and for $k \neq j \in \mathbb{Z}$ we have

$$\langle e^{ik\cdot}, e^{ij\cdot} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-j)x}\,\mathrm{d}x = \frac{\sin\big((k-j)\pi\big)}{(k-j)\pi} = 0,$$

as $\sin(\pi\,\mathbb{Z}) = 0$.

Similarly, $\langle \cos(k\cdot), \cos(j\cdot) \rangle = \langle \sin(k\cdot), \sin(j\cdot) \rangle = 0$ for all $j \neq k$ and $\langle \cos(k\cdot), \cos(k\cdot) \rangle = \langle \sin(k\cdot), \sin(k\cdot) \rangle = \frac{1}{2}$ for $k \neq 0$. ◦

DEFINITION 6.1.4 (PARTIAL FOURIER SUM)

partial FOURIER sum

The n-th partial FOURIER sum of $f \in L^1_{2\pi}$ is the trigonometric polynomial of degree n

$$S_n[f](x) := \sum_{|k|\leqslant n} c_k[f]e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{n} a_k[f]\cos(kx) + b_k[f]\sin(kx).$$

Lemma 6.1.5 (Least squares approximation in $\mathcal{T}_n$)
The n-th partial sum $S_n[f]$ is the best least squares approximation from $\mathcal{T}_n$ to f, that is,

$$\|f - S_n[f]\|_2 = \min_{p \in \mathcal{T}_n} \|f - p\|_2$$

for all $f \in L^2_{2\pi}$ and thus in particular for all $f \in \mathcal{C}_{2\pi}$.

Proof. Let $p := \sum_{|k| \leqslant n} d_k e^{ik\cdot} \in \mathcal{T}_n$. Then

$$\begin{aligned}\|f - p\|_2^2 &= \|f\|_2^2 - \langle p, f \rangle - \langle f, p \rangle + \sum_{|k| \leqslant n} |d_k|^2 \\ &= \|f\|_2^2 - \sum_{|k| \leqslant n} |c_k[f]|^2 + \underbrace{\sum_{|k| \leqslant n} |c_k[f] - d_k|^2}_{\geqslant 0} \geqslant \|f\|_2^2 - \sum_{|k| \leqslant n} |c_k[f]|^2\end{aligned}$$

with equality if and only if $c_k[f] = d_k$ for all $|k| \leqslant n$. □

Remark 6.1.6 (Bessel's inequality) Hence $\|f\|_2^2 - \sum_{|k| \leqslant n} |c_k[f]|^2 \geqslant 0$, implying Bessel's inequality

$$\|f\|_2^2 \geqslant \sum_{|k| \leqslant n} |c_k[f]|^2. \tag{29}$$

By the orthonormality of the complex harmonics we have $\|S_n[f]\|_2^2 = \sum_{|k| \leqslant n} |c_k[f]|^2$. Since this holds for every $n \in \mathbb{N}$, we have $\sum_{k \in \mathbb{Z}} |c_k[f]|^2 \leqslant \|f\|_2^2$ and thus $(c_k[f])_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$.

Indeed every $f \in L^2_{2\pi}$ can be written as $\sum_{k \in \mathbb{Z}} c_k[f] e^{ikx}$ and

$$\|f - S_n[f]\|_2^2 = \sum_{|k| > n} |c_k[f]|^2 \xrightarrow{n \to \infty} 0,$$

where the limit is true due to $(c_k[f])_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$. ○

The operator $S_n : L^2_{2\pi} \to \mathcal{T}_n$ is linear and bounded with $\|S_n\|_2 = 1$ by (29) and a projection by lemma 6.1.5. It is even injective.

Lemma 6.1.7 (S_n injective)
The operator $S_n : L^2_{2\pi} \to \mathcal{T}_n$ is injective.

Proof. Since S_n is linear, it suffices to show that only 0 is mapped to 0.

1. First assume that $f \in \mathcal{C}_{2\pi}$ with $c_k[f] = 0$ for all $k \in \mathbb{Z}$. Then for all $p \in \mathcal{T}$ we have $\int_{-\pi}^{\pi} f(x)p(x) \, \mathrm{d}x = 0$.

   By Weierstrass' second theorem, there exists a sequence $(p_n \in \mathcal{T}_n)_{n \in \mathbb{N}}$ such that $p_n \rightrightarrows \overline{f}$. Hence by the uniform convergence we have

   $$\int_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x)p_n(x) \, \mathrm{d}x = 0,$$

   implying $f = 0$ almost everywhere. As f is continuous, $f \equiv 0$.

2. Assume $f \in L^2_{2\pi}$ with $c_k[f] = 0$ for all $k \in \mathbb{Z}$. Define $h : [-\pi, \pi] \to \mathbb{R}$, $x \mapsto \int_{-\pi}^{x} f(y) \, \mathrm{d}y$. Then $h \in \mathcal{C}_{2\pi}$ and $h' = f$ almost everywhere.

   For $k \neq 0$ we have by integration by parts

   $$c_k[h] = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) e^{-ikx} \, \mathrm{d}x$$

$$= \frac{1}{2\pi} \underbrace{\left[h(x) \frac{e^{-ikx}}{-ik} \right]_{x=-\pi}^{\pi}}_{=0 \text{ as } h \text{ is periodic}} - \frac{1}{2\pi} \int_{-\pi}^{\pi} h'(x) \frac{e^{-ikx}}{-ik} \, \mathrm{d}x = \frac{1}{ik} c_k[f] = 0.$$

Thus by linearity of the FOURIER coefficients c_k, the FOURIER coefficients of the continuous 2π-periodic function $h - c_0[h]$ vanish, so by 1, $h \equiv c_0[h]$. We hence have $c_0[h] = h(-\pi) = 0$, so $h = \int_{-\pi}^{\cdot} f(t)\,\mathrm{d}t \equiv 0$, implying $f \equiv 0$. ☐

We have proven the complex harmonics are an orthonormal system in $L^2_{2\pi}$ and that

$$\overline{\operatorname{span}}(\{e^{ik\cdot} : k \in \mathbb{Z}\}) = L^2_{2\pi}.$$

A standard result from Functional Analysis implies that $(e^{ik\cdot})_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2_{2\pi}$.

6.2 Convergence of the partial sums

By the KHARSHILADZE-LOZINSKI theorem, there exists a $f \in \mathcal{C}_{2\pi}$ such that $\|S_n[f] - f\|_\infty \xrightarrow{n\to\infty} \infty$. We show that under certain additional assumptions, we can still guarantee uniform convergence (and more).

THEOREM 6.2.1: UNIFORM & ABSOLUTE CONVERGENCE

If $f \in \mathcal{C}^2_{2\pi}$, then $S_n[f] \rightrightarrows f$ absolutely.

THEOREM 6.2.2: UNIFORM CONVERGENCE

If $f \in \mathcal{C}_{2\pi}$ and $\sum_{k\in\mathbb{Z}} c_k[f] < \infty$, then $S_n[f] \rightrightarrows f$.

Proof. For $x \in [-\pi, \pi]$ we have

$$|S_n[f](x)| \leqslant \sum_{|k|\leqslant n} |c_k[f]e^{ikx}| = \sum_{|k|\leqslant n} |c_k[f]| < \infty,$$

so the result follows by the WEIERSTRASS-M-test. ☐

THEOREM 6.2.3: BERNSTEIN

For $f \in \mathcal{C}^r_{2\pi}$ and $n > 1$ we have

$$\|f - S_n[f]\|_\infty \leqslant c\|f^{(r)}\|_\infty \frac{\ln(n)}{n^r},$$

where c is a constant independent of f and n.

Proof. (For $r = 1$, HW 5.1) The general proof is involved, but for $r = 1$ it is not.

This is left as an exercise to the reader. ☐

It is easier to calculate the partial FOURIER sum using the DIRICHLET kernel.

Definition 6.2.1 (Dirichlet kernel)
For $n \in \mathbb{N}_0$, the n-th Dirichlet kernel is

Dirichlet kernel

$$D_n(x) := \sum_{|k| \leqslant n} e^{ikx} = \begin{cases} \frac{\sin\left((2n+1)\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}, & \text{if } x \in [-\pi, \pi] \backslash \{0\}, \\ 2n+1, & \text{if } x = 0. \end{cases} \tag{30}$$

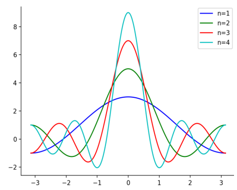

Fig. 8: The Dirichlet kernel D_n for different n.

We have

$$c_j[D_n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{|k| \leqslant n} e^{ikt} e^{-ijt} \, \mathrm{d}t = \sum_{|k| \leqslant n} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-j)t} \, \mathrm{d}t}_{\langle e^{ik\cdot}, e^{ij\cdot} \rangle = \delta_{j,k}} = \begin{cases} 1, & \text{if } |j| \leqslant n, \\ 0, & \text{else.} \end{cases}$$

and

$$\begin{aligned} S_n[f](x) &= \sum_{|k| \leqslant n} c_k[f] e^{ikx} = \sum_{|k| \leqslant n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, \mathrm{d}t \right) e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{|k| \leqslant n} e^{ik(x-t)} \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) \, \mathrm{d}t =: (f * D_n)(x). \end{aligned}$$

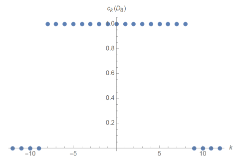

Fig. 9: The Fourier coefficients of the eighth Dirichlet kernel D_8.

Furthermore, D_n is even, $\int_{-\pi}^{\pi} D_n(x) \, \mathrm{d}x = 2\pi$ and $|D_n(x)| \leqslant 2n+1$ with equality only for $x = 0$ (if $n > 0$). Lastly,

$$\boxed{\frac{4}{\pi^2} \log(n) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)| \, \mathrm{d}x \leqslant 3 + \log(n).} \tag{31}$$

Remark 6.2.2 (There exist f such that $S_n[f]$ does not converge to f)
Let $\lambda_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)| \, \mathrm{d}x$. Then $\lambda_n \xrightarrow{n \to \infty} \infty$ by (31). The bound

$$\|S_n[f]\|_\infty = \sup_{x \in [-\pi, \pi]} |S_n[f](x)| \leqslant \sup_{x \in [-\pi, \pi]} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)||D_n(x-t)| \, \mathrm{d}t \leqslant \lambda_n \|f\|_\infty$$

is sharp since we can approximate $\operatorname{sgn}(D_n)$ by a continuous function f with unit norm. Then we get $f(t) D_n(t) \approx \operatorname{sgn}(D_n(t)) D_n(t) = |D_n(t)|$ and thus

$$S_n[f](0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(t) \, \mathrm{d}t \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)| \, \mathrm{d}x = \lambda_n,$$

so S_n is not uniformly bounded, the Banach-Steinhaus theorem implies the existence of a f such that $S_n[f] \not\rightrightarrows f$ for $n \to \infty$. ○

Let $E_n^T(f) := \min_{t_n \in \mathcal{T}_n} \|f - t_n\|_\infty$. By Lebesgue's Theorem and (31) we have

$$E_n^T(f) \leqslant \|f - S_n[f]\|_\infty \overset{(31)}{\leqslant} \underbrace{(4 + \log(n))}_{\leqslant 10, \; n < 400} E_n^T(f). \tag{32}$$

Definition 6.2.3 (Chebychev series)
The Chebychev series of f is

Chebychev series

$$\sum_{k=0}^{\infty} \frac{(f, T_k)_w}{(T_k, T_k)_w} T_k,$$

where $w \colon [-1, 1] \to \mathbb{R}$, $x \mapsto (1 - x^2)^{-\frac{1}{2}}$.

Example 6.2.4 (Chebychev series of arccos (HW 4.1))
This is left as an exercise to the reader. ◇

We can now apply Theorem 6.2.1.

Corollary 6.2.5 (Uniform convergence of Chebychev series)
If $f \in \mathcal{C}^2([-1,1])$, then the Chebychev *series of f converges uniformly to f.*

Proof. Let $\varphi\colon [0,2\pi] \to \mathbb{R}$, $\theta \mapsto f(\cos(\theta))$. Then $S_n[\varphi] \rightrightarrows \varphi$ by Theorem 6.2.1. As cos is even, so is φ. Hence

$$f(\cos(\theta)) = \varphi(\theta) = \sum_{k=0}^{\infty} a_k \cos(k\theta) = \sum_{k=0}^{\infty} a_k T_k(\cos(\theta)),$$

so $f = \sum_{k=0}^{\infty} a_k T_k$, where $a_k \leqslant \frac{2}{k^2}\|\varphi''\|_\infty$: we have using the usual coordinate change $x = \cos(\theta)$ and then preforming integration by parts twice that

$$\begin{aligned} a_k &= \frac{(f,T_k)_w}{(T_k,T_k)_w} = \frac{2}{\pi}\int_{-1}^{1} f(x)T_k(x)\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{2}{\pi}\int_0^\pi \varphi(\theta)\cos(k\theta)\,\mathrm{d}\theta \\ &= \frac{2}{\pi k^2}\int_0^\pi \varphi''(\theta)\cos(k\theta)\,\mathrm{d}\theta \leqslant \frac{2}{\pi k^2}\pi\|\varphi''\|_\infty = \frac{2}{k^2}\|\varphi''\|_\infty \end{aligned}$$ □

Remark 6.2.6 ($E_n^T(f)$ and Chebychev series)
Let $f \in \mathcal{C}([-1,1])$ and $\mathcal{S}_n[f](x) := \sum_{k=0}^{n} a_k T_k(x)$ be the n-th partial Chebychev series of f. Let again $\varphi(\theta) := f(\cos(\theta))$. Then $\mathcal{S}_n[f](\cos(\theta)) = S_n[\varphi](\theta)$. Also,

$$E_n(f) = \min_{p \in \mathcal{P}_n} \|f-p\|_\infty = \min_{T \in \mathcal{T}_n} \|\varphi - T\|_\infty = E_n^T(\varphi). \tag{33}$$

Therefore we have

$$E_n(f) \leqslant \|f - \mathcal{S}_n[f]\|_\infty = \|\varphi - S_n[\varphi]\|_\infty \overset{(32)}{\leqslant} (4+\log(n))E_n^T(\varphi) \overset{(33)}{=} (4+\log(n))E_n(f).$$

The partial Chebychev series is a good approximation for f with respect to $\|\cdot\|_\infty$:

$$E_n(f) \leqslant \|f - \mathcal{S}_n[f]\|_\infty = \left\|\sum_{k=n+1}^{\infty} a_k T_k\right\|_\infty \overset{\triangle\neq}{\leqslant} \sum_{k=n+1}^{\infty} |a_k| \leqslant 2\|\varphi''\|_\infty \sum_{k=n+1}^{\infty} \frac{1}{k^2}.$$ ○

6.3 Uniform convergence

We can get uniform convergence if we replace the Dirichlet kernel by a better kernel.

Definition 6.3.1 (Fejér kernel)
The Fejér kernel is

$$K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k.$$

Fejér kernel

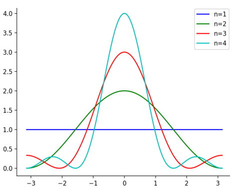

Fig. 10: The Fejér kernel K_n for different n.

Then K_n is even and nonnegative, $\int_{-\pi}^{\pi} K_n(t)\,\mathrm{d}t = 2\pi$.

Lemma 6.3.2 (Sin-representation of the Fejér kernel)
For $t \neq 0$ we have

$$K_n(t) = \frac{\sin^2\left(n\frac{t}{2}\right)}{n\sin^2\left(\frac{t}{2}\right)}.$$

Proof. (Exercise 5.2(a)) We show the equality above. For $t \neq 0$ we have

$$K_n(t) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(t) \overset{(30)}{=} \frac{1}{n}\sum_{k=0}^{n-1} \frac{\sin\left((2k+1)\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}.$$

Hence it remains to show that

$$\sum_{k=0}^{n-1} \frac{\sin\left((2k+1)\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} = \frac{\sin^2\left(n\frac{t}{2}\right)}{\sin^2\left(\frac{t}{2}\right)},$$

that is,

$$\sum_{k=0}^{n-1} \sin\left((2k+1)\frac{t}{2}\right) = \frac{\sin^2\left(n\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}.$$

Multiplying with $\sin\left(\frac{t}{2}\right)$ and using the addition formula yields

$$\begin{aligned}\sin^2\left(n\frac{t}{2}\right) &= \sum_{k=0}^{n-1} \sin\left(\frac{t}{2}\right)\sin\left((2k+1)\frac{t}{2}\right)\\ &= \frac{1}{2}\sum_{k=0}^{n-1} \cos(kt) - \cos((k+1)t) = \frac{1}{2}(1-\cos(nt)),\end{aligned}$$

which is certainly true. □

Remark 6.3.3 (Fourier coefficients of K_n (HW 5.2(b)))
This is left as an exercise to the reader. ○

Let

$$\sigma_n[f] := \frac{1}{n}\sum_{k=0}^{n-1} S_k[f] = K_n * f.$$

This is "more balanced" than $D_n * f = S_n[f]$. Then $\sigma_n \colon \mathcal{C}_{2\pi} \to \mathcal{T}_n$ is linear, bounded, but not a projection.

We have $\|f - \sigma_n[f]\|_2 \xrightarrow{n\to\infty} 0$, as the following theorem shows.

Theorem 6.3.1: Convergence of σ_n (Fejér)

We have $\sigma_n[f] \rightrightarrows f$ for all $f \in \mathcal{C}_{2\pi}$.

Proof. As σ_n is linear and positive, by (a slightly modified version of) the Bohmann-Korovkin theorem it suffices to check the statement for $f_0(x) := 1$, $f_1(x) := \cos(x)$ and $f_2(x) := \sin(x)$. We have, as S_n is a projection,

$$\begin{aligned}\sigma_n(f_0) &= \frac{1}{n}\sum_{k=0}^{n-1} S_k[f] = \frac{1}{n}(f_0 + \ldots + f_0) = f_0,\\ \sigma_n(f_1) &= \frac{1}{n}(0 + f_1 + \ldots + f_1) = \frac{n-1}{n} f_1 \rightrightarrows f_1,\\ \sigma_n(f_2) &= \frac{1}{n}(0 + f_2 + \ldots + f_2) = \frac{n-1}{n} f_2 \rightrightarrows f_2.\end{aligned}$$ □

Corollary 6.3.4
The Weierstrass *approximation theorem.*

approximation identity

Definition 6.3.5 (Approximation identity)
A approximation identity is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{C}_{2\pi}$ of nonnegative functions with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t)\,\mathrm{d}t = 1 \tag{34}$$

and

$$\int_{\delta\leqslant|t|\leqslant\pi} \varphi_n(t)\,\mathrm{d}t \xrightarrow{n\to\infty} 0 \qquad \forall\delta > 0. \tag{35}$$

Example 6.3.6 (Approximation identity, HW 5.2(c))
The Fejér kernel is an approximation identity.

This is left as an exercise to the reader. ◇

Theorem 6.3.2: Approximation identity

If $(\varphi_n)_{n\in\mathbb{N}}$ are an approximation identity and $f \in \mathcal{C}_{2\pi}$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t) f(\cdot + t)\,\mathrm{d}t \rightrightarrows f.$$

Proof. Let $x \in [-\pi, \pi]$. As $[-\pi, \pi]$ is compact and f is continuous, f is uniformly continuous. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(x+t)| < \varepsilon$ for all $|t| < \delta$. Then

$$\begin{aligned}
\left| f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t) f(x+t)\,\mathrm{d}t \right| &\underset{\triangle\neq}{\overset{(34)}{\leqslant}} \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(t)|f(x) - f(x+t)|\,\mathrm{d}t \\
&\leqslant \frac{1}{2\pi}\int_{\delta\leqslant|t|\leqslant\pi} \varphi_n(t)\cdot 2\|f\|_\infty\,\mathrm{d}t + \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_n(t)\cdot\varepsilon\,\mathrm{d}t \\
&\overset{(34)}{=} \frac{\|f\|_\infty}{\pi} \underbrace{\int_{\delta\leqslant|t|\leqslant\pi} \varphi_n(t)\,\mathrm{d}t}_{\xrightarrow{n\to\infty} 0 \text{ by } (35)} + \varepsilon \underset{\varepsilon\searrow 0}{\xrightarrow{n\to\infty}} 0.
\end{aligned}$$

□

Remark 6.3.7 (Other kernels)
There are other kernels, such as the De La Vallée Poussin kernel $V_n(t) := \frac{1}{n}\sum_{k=n}^{2n-1} D_k(t)$, the Landau kernel $L_n(t) := c_n(1-t^2)^n$ or Jackson's kernel $J_n(t) := c_n \frac{\sin^4(nt)}{n^3\sin^4(t)}$, where the constant c_n is chosen such that the kernel integrates to unity. ○

This last part is from [Car98, Ex. 8.11].

Fourier-Legendre series

Definition 6.3.8 (Fourier-Legendre series)
Let $(P_k)_{k=0}^{\infty}$ be the sequence of Legendre polynomials. The Fourier-Legendre series for $f \in \mathcal{C}([-1,1])$ is

$$\sum_{k=0}^{\infty} \langle f, \tilde{P}_k \rangle \tilde{P}_k,$$

where $\tilde{P}_k := \sqrt{\frac{2k+1}{2}} P_k$ and $\langle \cdot, \cdot \rangle$ is the $L^2([-1,1]; R)$ inner product.

Lemma 6.3.9
The partial FOURIER-LEGENDRE *series* $\tilde{S}_n[f] := \sum_{k=0}^{n} \langle f, \tilde{P}_k \rangle \tilde{P}_k$ *converges to* f *in* $L^2([-1,1])$.

Proof. The proof is analogous to the proof for FOURIER series. By comparing the recurrence relation for the orthogonal polynomials and the known recurrence relation $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, we can show

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = \frac{2}{2n+1}\delta_{mn},$$

so $(\tilde{P}_k)_{k=0}^{\infty}$ are orthonormal in $L^2([-1,1])$.

Let $c_k := \langle f, \tilde{P}_k \rangle$. We show an analogue of the BESSEL inequality: for $n \in \mathbb{N}$ and $f \in L^2([-1,1])$ we have due to the orthogonality of the $\tilde{P}_k$ that

$$\begin{aligned} 0 \leqslant \|f - \tilde{S}_n[f]\|_2^2 &= \|f\|_2^2 - 2\langle f, \tilde{S}_n[f] \rangle + \|\tilde{S}_n[f]\|_2^2 \\ &= \|f\|_2^2 - 2\sum_{k=0}^{n} c_k^2 + \sum_{k=0}^{n} c_k^2 = \|f\|_2^2 - \sum_{k=0}^{n} c_k^2. \end{aligned}$$

We obtain $\sum_{k=0}^{n} c_k^2 \leqslant \|f\|_2^2$ and thus in the limit $\sum_{k=0}^{\infty} c_k^2 \leqslant \|f\|_2^2 < \infty$, so $(c_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$. This implies that

$$\left\| f - \sum_{k=0}^{n} c_k \tilde{P}_k \right\|_2^2 \leqslant \sum_{k=n+1}^{\infty} c_k^2 \xrightarrow{n\to\infty} 0. \qquad \square$$

Remark 6.3.10 (Uniform convergence of FOURIER-LEGENDRE series)
Using that $\|P_n\| \leqslant 1$, we get $\|\tilde{S}_n(f)\| \leqslant (n+1)^2\|f\|$, so LEBESGUE's theorem reads $\|f - \tilde{S}_n(f)\| \leqslant Cn^2E_n(f)$. Hence $\tilde{S}_n(f) \rightrightarrows f$ whenever $n^2E_n(f) \to 0$. By a BERNSTEIN-JACKSON theorem this will happen if $f \in \mathcal{C}^2([-1,1])$. ○

7 JACKSON's inequalities

7.1 Jackson's inequality for trigonometric and algebraic polynomials

04.01.2022

THEOREM 7.1.1: JACKSON'S THEOREM I: $\mathcal{C}^1_{2\pi}$

For $f \in \mathcal{C}^1_{2\pi}$ we have

$$E_n^T = \min_{p \in \mathcal{T}_n} \|f - p\|_\infty \leqslant \frac{\pi}{2(n+1)} \|f'\|_\infty.$$

The constant is optimal, but we won't show this.

Proof. 1 Let $\tilde{f}(x) := x$ and $p_n \in \mathcal{T}_n$, which interpolates $\tilde{f}$ at the $2n+1$ points $(x_j := \frac{j\pi}{n+1})_{j=-n}^n$. We show that $\int_{-\pi}^{\pi} |\tilde{f}(x) - p_n(x)| \, \mathrm{d}x = \frac{\pi^2}{n+1}$.

We have $-p_n(-x_j) = -p_n(x_{-j}) = -x_{-j} = x_j$, so $-p_n(-\cdot)$ interpolates $\tilde{f}$ at $(x_j)_{j=-n}^n$. By the interpolation theorem we conclude $p_n = -p_n(-\cdot)$, that is, p_n is odd. Hence there are $(b_k)_{k=1}^n$ such that

$$p_n(x) = \sum_{k=1}^n b_k \sin(kx).$$

Let $e := \tilde{f} - p_n$. We show that e has exactly $2n+1$ simple zeros. As p_n interpolates f, e has at least $2n+1$ zeros.

1 If e has $2n+2$ zeros, then $e' = 1 - p_n'$ has $2n+1$ zeros by ROLLE's theorem. As $e' \in \mathcal{T}_n$, e' can have at most $2n$ zeros, so $e' \equiv 1$, so $p_n' \equiv 1$, which is a contradiction as p_n is a trigonometric polynomial.

2 Suppose there exists a non-simple root x_k with $e(x_k) = e'(x_k) = 0$. By ROLLE's theorem, e' has $2n$ zeros in $\bigcup_{j=-n}^{n-1}(x_j, x_{j+1})$, which is a contradiction.

As the roots are simple, e changes sign at each x_j. Let $s(x) := \operatorname{sgn}(e(x))$. Then $s(x) = (-1)^k$ for $x \in \left(\frac{k\pi}{n+1}, \frac{(k+1)\pi}{n+1}\right)$. Then

$$\int_{-\pi}^{\pi} |x - p_n(x)| \, \mathrm{d}x = \int_{-\pi}^{\pi} (x - p_n(x)) s(x) \, \mathrm{d}x = \int_{-\pi}^{\pi} x s(x) \, \mathrm{d}x - \int_{-\pi}^{\pi} p_n(x) s(x) \, \mathrm{d}x$$

We show that the first term is equal to $\frac{\pi^2}{n+1}$ and the second one vanishes.

We have

$$\begin{aligned}\int_{-\pi}^{\pi} x s(x) \, \mathrm{d}x &= \sum_{k=-(n+1)}^{n} (-1)^k \int_{\frac{k\pi}{n+1}}^{\frac{(k+1)\pi}{n+1}} x \, \mathrm{d}x = \frac{1}{2} \frac{\pi^2}{(n+1)^2} \sum_{k=-(n+1)}^{n} (-1)^k (2k+1) \\ &= \frac{\pi^2}{(n+1)^2} \sum_{k=-(n+1)}^{n} (-1)^k k = (-1)^n \frac{\pi^2}{n+1}.\end{aligned}$$

For the second statement it suffices to show that $A := \int_{-\pi}^{\pi} \sin(kx) s(x) \, \mathrm{d}x = 0$ for all $k \in \{1, \dots, n\}$. We have

$$A = \int_{-\pi}^{\pi} \sin(kx) \underbrace{s(x)}_{=-s\left(x + \frac{\pi}{n+1}\right)} \mathrm{d}x = \int_{-\pi}^{\pi} \sin\left(kx - \frac{k\pi}{n+1}\right) s(x) \, \mathrm{d}x$$

$$= -\int_{-\pi}^{\pi} \sin(kx)\cos\left(\frac{k\pi}{n+1}\right) s(x)\,\mathrm{d}x + \int_{-\pi}^{\pi} \cos(kx)\sin\left(\frac{k\pi}{n+1}\right) s(x)\,\mathrm{d}x.$$

The last summand vanishes as the integrand is odd. Hence we get $A = -\cos\left(\frac{k\pi}{n+1}\right) A$. As $\cos\left(\frac{k\pi}{n+1}\right) \neq -1$, we must have $A = 0$.

2 Using integration by parts, we can relate f and f': for $x \in \mathbb{R}$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} \theta f'(\theta+\pi+x)\,\mathrm{d}\theta &= \theta f(\theta+\pi+x)\Big|_{x=-\pi}^{\pi} - \int_{-\pi}^{\pi} f(\theta+\pi+x)\,\mathrm{d}\theta \\ &= \pi f(2\pi+x) + \pi f(x) - \int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta = 2\pi f(x) - \int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta.\end{aligned}$$

Hence

$$\boxed{f(x) = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta + \int_{-\pi}^{\pi} \theta f'(\theta+\pi+x)\,\mathrm{d}\theta\right).}$$

To approximate f, we replace θ by $p_n(\theta)$ from step 1: for $x \in \mathbb{R}$ let

$$q(x) := \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta + \int_{-\pi}^{\pi} p_n(\theta) f'(\theta+\pi+x)\,\mathrm{d}\theta\right). \tag{36}$$

Then $q \in \mathcal{T}_n$: we have

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(k\theta) f'(\theta+\pi+x)\,\mathrm{d}\theta &= \int_{-\pi}^{\pi} \sin\big(k(\theta-x+\pi)\big) f'(\theta)\,\mathrm{d}\theta \\ &= \cos(kx)\int_{-\pi}^{\pi} \sin\big(k(\theta-\pi)\big) f'(\theta)\,\mathrm{d}\theta \\ &\quad - \sin(kx)\int_{-\pi}^{\pi} \cos\big(k(\theta-\pi)\big) f'(\theta)\,\mathrm{d}\theta.\end{aligned}$$

We have

$$\begin{aligned}\|f-q\|_\infty &= \sup_{x\in[-\pi,\pi]} |f(x)-q(x)| = \sup_{x\in[-\pi,\pi]} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (p_n(\theta)-\theta) f'(\theta+x+\pi)\right| \\ &\overset{\triangle\neq}{\leqslant} \frac{1}{2\pi}\int_{-\pi}^{\pi} |p_n(\theta)-\theta|\|f'\|_\infty\,\mathrm{d}\theta \overset{1}{=} \|f'\|_\infty \frac{\pi^2}{n+1}\frac{1}{2\pi} = \frac{\pi}{2(n+1)}\|f'\|_\infty. \quad \square\end{aligned}$$

Remark 7.1.1 (Mean zero functions) The set of trigonometric polynomials with zero mean

$$\mathcal{T}_n^0 := \left\{p \in \mathcal{T}_n : \int_{-\pi}^{\pi} p(x)\,\mathrm{d}x = 0\right\}$$

is a closed finite-dimensional set.

If $f \in C_{2\pi}^1$ has mean $\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta = 0$, then q from (36) has zero mean, too:

$$\begin{aligned}2\pi\int_{-\pi}^{\pi} q(x)\,\mathrm{d}x &= \int_{-\pi}^{\pi} \underbrace{\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta}_{=0} + \int_{-\pi}^{\pi} p_n(\theta) f'(\theta+\pi+x)\,\mathrm{d}\theta\,\mathrm{d}x \\ &= \int_{-\pi}^{\pi} p_n(\theta) \underbrace{\int_{-\pi}^{\pi} f'(\theta+\pi+x)\,\mathrm{d}x}_{=f(2\pi+x)-f(x)=0}\,\mathrm{d}\theta = 0. \qquad \circ\end{aligned}$$

This yields the following theorem.

THEOREM 7.1.2: JACKSON'S THEOREM FOR MEAN ZERO $\mathcal{C}^1_{2\pi}$ FUNCTIONS

For a mean zero $\mathcal{C}^1_{2\pi}$-function f we have

$$\min_{p\in\mathcal{T}^0_n} \|f-p\|_\infty \leqslant \frac{\pi}{2(n+1)}\|f'\|_\infty. \tag{37}$$

Using the inequality (37), we want to prove JACKSON's theorem for $f \in C^k_{2\pi}$. Let $\tilde{p} \in \mathcal{T}_n$. Then by Theorem 7.1.1 we have

$$\min_{p\in\mathcal{T}_n} \|f-p\|_\infty = \min_{p\in\mathcal{T}_n} \|f-\tilde{p}-p\|_\infty \overset{7.1.1}{\leqslant} \frac{\pi}{2(n+1)}\|f'-\tilde{p}'\|_\infty. \tag{38}$$

Lemma 7.1.2 (Characterising $\mathcal{T}^0_n$)
We have

$$\{\tilde{p}' : \tilde{p} \in \mathcal{T}_n\} = \mathcal{T}^0_n\,. \tag{39}$$

Proof. Let $\tilde{p} \in \mathcal{T}_n$. Then $\int_{-\pi}^{\pi} \tilde{p}'(\theta)\,\mathrm{d}\theta = \tilde{p}(\pi) - \tilde{p}(-\pi) = 0$, so $\tilde{p}' \in \mathcal{T}^0_n$.
If $p \in \mathcal{T}^0_n$, then p is not constant. Hence $\int p(\theta)\,\mathrm{d}\theta \in \mathcal{T}_n$. □

Hence (38) and (39) imply

$$\min_{p\in\mathcal{T}_n} \|f-p\|_\infty \leqslant \frac{\pi}{2(n+1)} \min_{p\in\mathcal{T}^0_n} \|f'-p\|_\infty. \tag{40}$$

If $f \in \mathcal{C}^1_{2\pi}$ has zero mean, then for every nonconstant $\tilde{p} \in \mathcal{T}_n$ we have that $f - \tilde{p} \in \mathcal{C}^1_{2\pi}$ has zero mean and hence

$$\min_{p\in\mathcal{T}^0_n} \|f-p\|_\infty \min_{p\in\mathcal{T}^0_n} \|(f-\tilde{p})-p\|_\infty \overset{(37)}{\leqslant} \frac{\pi}{2(n+1)}\|f'-\tilde{p}'\|_\infty.$$

Hence

$$\min_{p\in\mathcal{T}^0_n} \|f-p\|_\infty \leqslant \frac{\pi}{2(n+1)} \min_{p\in\mathcal{T}_n} \|f'-\tilde{p}'\|_\infty \overset{(39)}{=} \frac{\pi}{2(n+1)} \min_{p\in\mathcal{T}^0_n} \|f'-\tilde{p}\|_\infty. \tag{41}$$

THEOREM 7.1.3: JACKSON'S THEOREM FOR $\mathcal{C}^k_{2\pi}$

If $f \in \mathcal{C}^k_{2\pi}$, then

$$E^T_n(f) \leqslant \left(\frac{\pi}{2(n+1)}\right)^k \|f^{(k)}\|_\infty.$$

Proof. First note that as f is 2π-periodic, so are its derivatives: for $x \in \mathbb{R}$ we have

$$f'(x+2\pi) = \lim_{h\to 0} \frac{f(x+2\pi+h) - f(x+2\pi)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x).$$

As by the Fundamental Theorem of Calculus, the derivative of a peridic function has zero mean, $f', f'', \ldots, f^{(k-1)} \in \mathcal{C}^1_{2\pi}$ have zero mean. Hence

$$\begin{aligned}\min_{p\in\mathcal{T}_n} \|f-p\|_\infty &\overset{(40)}{\leqslant} \frac{\pi}{2(n+1)} \min_{p\in\mathcal{T}^0_n} \|f'-p\|_\infty \overset{(41)}{\leqslant} \left(\frac{\pi}{2(n+1)}\right)^2 \min_{p\in\mathcal{T}^0_n} \|f''-p\|_\infty \\ &\overset{(41)}{\leqslant} \ldots \overset{(41)}{\leqslant} \left(\frac{\pi}{2(n+1)}\right)^{k-1} \min_{p\in\mathcal{T}^0_n} \|f^{(k-1)}-p\|_\infty \overset{(37)}{\leqslant} \left(\frac{\pi}{2(n+1)}\right)^k \|f^{(k)}\|_\infty.\end{aligned}$$ □

Remark 7.1.3 The above upper bound is not sharp: the optimal bound is $\frac{\pi}{2}\frac{1}{(n+1)^k}\|f^{(k)}\|_\infty$. ○

Theorem 7.1.4: Jackson's Theorem for $\mathcal{C}^1[-1,1]$

For $f \in \mathcal{C}^1([-1,1])$ we have $E_n^T(f) \leqslant \frac{\pi}{2(n+1)}\|f'\|_\infty$.

Proof. Let $g(\theta) := f(\cos(\theta))$. Then $g \in \mathcal{C}^1_{2\pi}$ is even and $g'(\theta) = -\sin(\theta)f'(\cos(\theta))$ and hence by Theorem 7.1.1

$$E_n^T(g) \overset{7.1.1}{\leqslant} \frac{\pi}{2(n+1)}\|g'\|_\infty \leqslant \frac{\pi}{2(n+1)}\|f'\|_\infty.$$

We want to show that $E_n^T(g) = E_n(f)$. We have

$$E_n^T(g) = \min_{p\in\mathcal{T}_n} \sup_{\theta\in[-\pi,\pi]} |g(\theta) - p(\theta)|.$$

For $p \in \mathcal{T}_n$ let $\tilde{p} := \frac{1}{2}(p + p(-\cdot)) \in \mathcal{T}_n$. Then $\tilde{p}$ is even. We have, as g is even

$$\|g - \tilde{p}\|_\infty \leqslant \frac{1}{2}\|g-p\|_\infty + \frac{1}{2}\underbrace{\|g - p(-\cdot)\|_\infty}_{\|g-p\|_\infty} = \|g-p\|_\infty.$$

Hence

$$\min_{\substack{p\in\mathcal{T}_n \\ p \text{ even}}} \|g-p\|_\infty \leqslant \min_{p\in\mathcal{T}_n} \|g-p\|_\infty \leqslant \min_{\substack{p\in\mathcal{T}_n \\ p \text{ even}}} \|g-p\|_\infty$$

and thus

$$\min_{p\in\mathcal{T}_n} \|g-p\|_\infty = \min_{\substack{p\in\mathcal{T}_n \\ p \text{ even}}} \|g-p\|_\infty. \tag{42}$$

Thus by the correspondence between algebraic and even trigonometric polynomials we have

$$E_n(f) = \min_{p\in\mathcal{P}_n} \|f-p\|_\infty = \min_{\substack{p\in\mathcal{T}_n \\ p \text{ even}}} \|g-p\|_\infty = \min_{p\in\mathcal{T}_n} \|g-p\|_\infty = E_n^T(g)$$

and thus $E_n(f) = E_n^T(g) \leqslant \frac{\pi}{2(n+1)}\|f'\|_\infty$. □

The following theorem is a generalisation of Theorem 7.1.1, but uses its proof.

Theorem 7.1.5: Jackson's Theorem II: Lip_K^1

If $f \in \mathrm{Lip}_k^1$ is 2π-periodic, then $E_n^T(f) \leqslant \frac{k\pi}{2(n+1)}$.

We have

$$f \in \mathrm{Lip}_K^\alpha \iff |f(x) - f(y)| \leqslant K|x-y|^\alpha \quad \forall x, y \iff w_f(\delta) \leqslant K\delta^\alpha \ \forall \delta \geqslant 0.$$

Hence Theorem 7.1.5 implies that if $w_f(\delta) \leqslant k\delta$, then $E_n^T(f) \leqslant \frac{k\pi}{2(n+1)} \in O\left(\frac{1}{n}\right)$ as $n \to \infty$.

Proof. (of Theorem 7.1.5) For $\delta > 0$ define (or alternatively use (43))

$$\varphi_\delta(x) := \varphi(x) := \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t)\,\mathrm{d}t.$$

Then $\varphi \in \mathcal{C}^1_{2\pi}$ with $\varphi'(x) = \frac{1}{2\delta}\big(f(x+\delta) - f(x-\delta)\big)$. By Theorem 7.1.1 we have

$$E_n^T(\varphi) \leqslant \frac{\pi}{2(n+1)}\|\varphi'\|_\infty.$$

By the Fundamental Theorem of Calculus we have

$$|\varphi'(x)| = \frac{1}{2\delta}|f(x+\delta) - f(x-\delta)| \leqslant \frac{k}{2\delta}|(x+\delta) - (x-\delta)| = k,$$

so $\|\varphi'\|_\infty \leqslant k$.

If $p^* \in \mathcal{T}_n$ is the best approximation of φ, then

$$\begin{aligned}\|f - \varphi\|_\infty &= \sup_x |f(x) - \varphi(x)| \leqslant \frac{1}{2\delta} \sup_x \int_{x-\delta}^{x+\delta} |f(x) - f(t)| \,\mathrm{d}t \\ &\leqslant \frac{k}{2\delta} \sup_x \int_{x-\delta}^{x+\delta} |x - t| \,\mathrm{d}t \leqslant \frac{k}{2\delta} 2\delta^2 = k\delta.\end{aligned}$$

This implies

$$E_n^T(f) \leqslant k\delta + \frac{k\pi}{2(n+1)} \xrightarrow{\delta \searrow 0} \frac{k\pi}{2(n+1)}. \qquad \square$$

THEOREM 7.1.6: JACKSON'S THEOREM III: $C_{2\pi}$

For $f \in \mathcal{C}_{2\pi}$ we have $E_n^T(f) \leqslant w_f\left(\frac{\pi}{n+1}\right)$.

We only prove that $E_n^T(f) \leqslant \frac{3}{2} w_f\left(\frac{\pi}{n+1}\right)$.

Remark 7.1.4 (Non-optimality of BERNSTEIN approximation)
We can get a result for $f \in \mathcal{C}([a,b])$ similarly as we did above. This then is better than the BERNSTEIN approximation, as $\|f - B_n(f)\|_\infty \leqslant \frac{3}{2} w\left(\frac{1}{\sqrt{n}}\right)$ by Theorem 2.1.3. ○

Proof. Defining φ as above, we get

$$\begin{aligned}\|f - \varphi\|_\infty &\overset{\triangle\neq}{\leqslant} \max_x \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} |f(x) - f(t)| \,\mathrm{d}t \leqslant \max_x \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} w_f(|x-t|) \,\mathrm{d}t \\ &\leqslant \frac{1}{\cancel{2\delta}} w_f(\delta) \cdot \cancel{2\delta} = w_f(\delta).\end{aligned}$$

Further,

$$|\varphi'(x)| = \frac{|f(x+\delta) - f(x-\delta)|}{2\delta} \leqslant \frac{w_f\big((x+\delta) - (x-\delta)\big)}{2\delta} = \frac{w_f(2\delta)}{2\delta} \overset{2.1.11}{\leqslant} \frac{w_f(\delta)}{\delta}.$$

Therefore by JACKSON's Theorem I we have

$$E_n^T(\varphi) \leqslant \frac{\pi}{2(n+1)} \|\varphi'\|_\infty \leqslant \frac{\pi}{2(n+1)} \frac{w_f(\delta)}{\delta}.$$

To find the desired upper bound for $E_n^T(f)$ we see that

$$\begin{aligned}E_n^T(f) &= \min_{p \in \mathcal{T}_n} \|f - p\|_\infty \leqslant \min_{p \in \mathcal{T}_n} \|f - \varphi\|_\infty + \|\varphi - p\|_\infty \\ &= \|f - \varphi\|_\infty + E_n^T(\varphi) \leqslant w_f(\delta)\left(1 + \frac{\pi}{2\delta(n+1)}\right) \overset{\delta = \frac{\pi}{n+1}}{=} \frac{3}{2} w_f\left(\frac{\pi}{n+1}\right). \qquad \square\end{aligned}$$

Corollary 7.1.5
The second theorem of WEIERSTRASS.

Corollary 7.1.6 (Dini-Lipschitz)
If $f \in \mathcal{C}_{2\pi}$ and $w_f(\delta)\ln(\frac{1}{\delta}) \xrightarrow{\delta \searrow 0} 0$, then $S_n[f] \rightrightarrows f$.

Proof. We have

$$\|f - S_n[f]\|_\infty \overset{(32)}{\leqslant} (4+\ln(n))E_n^T(f) \overset{7.1.6}{\leqslant} (4+\ln(n))w_f\left(\frac{\pi}{n+1}\right) \xrightarrow{n\to\infty} 0. \qquad \square$$

Theorem 7.1.7: Jackson's Theorem V

For $f\colon [-1,1] \to \mathbb{R}$ we have

$$E_n(f) \leqslant \begin{cases} \frac{\pi M}{2(n+1)}, & \text{if } f \in \mathrm{Lip}_k^1, \\ \left(\frac{\pi}{2}\right)^k \left(\prod_{j=n-k+2}^{n+1} \frac{1}{j}\right) \|f^{(k)}\|_\infty, & \text{if } f \in \mathcal{C}^k([-1,1]), \end{cases}$$

Proof. (1) For $\delta > 0$ define

$$\varphi = \varphi_\delta\colon [-1,1] \to \mathbb{R}, \qquad x \mapsto \frac{1}{2\delta}\int_{\max(x-\delta,1)}^{\min(x+\delta,-1)} f(t)\,\mathrm{d}t. \tag{43}$$

Then $\varphi \in \mathcal{C}^1([-1,1])$ with

$$\varphi'(t) = \frac{1}{2\delta}\begin{cases} f(1) - f(t-\delta), & \text{if } 1-t \leqslant \delta, \\ f(t+\delta) - f(t-\delta), & \text{if } \delta < 1-t. \end{cases}$$

By Jackson's theorem for $\mathcal{C}^1([-1,1])$ we have

$$E_n(\varphi) \leqslant \frac{\pi}{2(n+1)}\|\varphi'\|_\infty.$$

We have $\|\varphi\|_\infty \leqslant K$ as in the lecture. If $p^* \in \mathcal{P}_n$ is the best approximation of φ, then

$$\|f - p^*\|_\infty \leqslant \|f-\varphi\|_\infty + \|\varphi - p^*\|_\infty = \|f-\varphi\|_\infty + E_n(\varphi).$$

We have

$$\begin{aligned}\|f-\varphi\|_\infty &= \sup_x |f(x) - \varphi(x)| \leqslant \frac{1}{2\delta}\sup_x \int_{x-\delta}^{x+\delta} |f(x) - f(t)|\,\mathrm{d}t \\ &\leqslant \frac{K}{2\delta}\sup_x \int_{x-\delta}^{x+\delta} |x-t|\,\mathrm{d}t = \frac{K}{2\delta}\delta^2 = \frac{K}{2}\delta.\end{aligned}$$

This implies

$$E_n(f) \leqslant \frac{K}{2}\delta + \frac{k\pi}{2(n+1)} \xrightarrow{\delta\searrow 0} \frac{k\pi}{2(n+1)}.$$

Alternatively, we can use the corresponding result for trigonometric polynomials: We define $g(\theta) := f(\cos(\theta))$. Then $g \in \mathcal{C}_{2\pi}$ is even. As cos is 1-Lipschitz we have

$$|g(\theta) - g(\tilde{\theta})| \leqslant k|\cos(\theta) - \cos(\tilde{\theta})| \leqslant k|\theta - \tilde{\theta}|.$$

Hence

$$\begin{aligned} E_n(f) &= \min_{p\in\mathcal{P}_n} \max_{x\in[-1,1]} |f(x) - p(x)| = \min_{p\in\mathcal{P}_n} \max_{\theta\in[0,2\pi]} |f(\cos(\theta)) - p(\cos(\theta))| \\ &= \min_{p\in\mathcal{P}_n} \|g - p\circ\cos\|_\infty = \min_{\substack{p\in\mathcal{T}_n \\ S \text{ even}}} \|g - S\|_\infty \overset{(42)}{=} \min_{p\in\mathcal{T}_n} \|g-S\|_\infty = E_n^T(g).\end{aligned}$$

The now result follows by Jackson's theorem II.

② For $f \in \mathcal{C}^k([-1,1])$ and $\tilde{p} \in \mathcal{P}_n$ we have

$$E_n(f) = \min_{p \in \mathcal{P}_n} \|f - p\|_\infty = \min_{p \in \mathcal{P}_n} \|f - \tilde{p} - p\|_\infty \overset{7.1.4}{\leqslant} \frac{\pi}{2(n+1)} \|f' - \tilde{p}'\|_\infty$$

and thus

$$E_n(f) \leqslant \frac{\pi}{2(n+1)} \min_{\tilde{p} \in \mathcal{P}_n} \|f' - \tilde{p}'\|_\infty = \frac{\pi}{2(n+1)} \min_{\tilde{p} \in \mathcal{P}_{n-1}} \|f' - \tilde{p}\|_\infty. \tag{44}$$

Iterating this procedure yields

$$\begin{aligned} E_n(f) &\overset{(44)}{\leqslant} \frac{\pi}{2(n+1)} \min_{\tilde{p} \in \mathcal{P}_{n-1}} \|f' - \tilde{p}\|_\infty \overset{(44)}{\leqslant} \frac{\pi}{2(n+1)} \frac{\pi}{2n} \min_{\tilde{p} \in \mathcal{P}_{n-2}} \|f'' - \tilde{p}\|_\infty \\ &\overset{(44)}{\leqslant} \dots \overset{(44)}{\leqslant} \left(\frac{\pi}{2}\right)^{k-1} \frac{1}{(n+1) \cdot n \cdot \ldots \cdot (n-k+3)} \underbrace{\min_{\tilde{p} \in \mathcal{P}_{n-k+1}} \|f^{(k-1)} - \tilde{p}\|_\infty}_{= E_{n-k+1}(f^{(k-1)})} \\ &\overset{7.1.4}{\leqslant} \left(\frac{\pi}{2}\right)^k \frac{1}{(n+1) \cdot n \cdot \ldots \cdot (n-k+2)} \|f^{(k)}\|_\infty. \end{aligned}$$

□

7.2 Inverse Theorems / Bernstein-Jackson theorems

Bernstein was interested in the converse of Jackson's Theorems: if $E_n(f) \to 0$ with a certain speed, what smoothness does f have? This subsection follows [Che66, Chp. 6.3] and [Car98, p. 76-78].

Theorem 7.2.1: Bernstein I

If $f \in \mathcal{C}_{2\pi}$ and $E_n^T(f) \in O(n^{-\alpha})$ for some $\alpha \in (0,1)$, then $f \in \mathrm{Lip}^\alpha$.

Proof. Let p_n be the best approximation of f from $\mathcal{T}_n$. Let $Q_0 := p_1$ and $Q_n := p_{2^n} - p_{2^{n-1}}$ for $n \geqslant 1$. The sum of these polynomials telescopes and so

$$\sum_{k=0}^n Q_k = p_{2^n} - p_1 + p_1 = p_{2^n}. \tag{45}$$

Further,

$$\|Q_n\|_\infty \overset{\triangle\neq}{\leqslant} \|p_{2^n} - f\|_\infty + \|p_{2^{n-1}} - f\|_\infty \leqslant C(2^{-n\alpha} + 2^{-\alpha(n-1)}) \leqslant C_\alpha 2^{-n\alpha} \xrightarrow{n \to \infty} 0. \tag{46}$$

Hence the sum $\sum_{k=0}^\infty Q_k$ is uniformly convergent by the Weierstrass M-test. Thus

$$\sum_{k=0}^\infty Q_k \overset{(45)}{=} \lim_{n \to \infty} p_{2^n} = f.$$

Let $m \geqslant 1$. By the mean value theorem, there exist $(\xi_n)_{n=0}^{m-1}$, all between x and y, such that

$$\begin{aligned} |f(x) - f(y)| &\overset{\triangle\neq}{\leqslant} \sum_{n=0}^\infty |Q_n(x) - Q_n(y)| \leqslant \sum_{n=0}^{m-1} |Q_n'(\xi_n)||x-y| + 2\sum_{n=m}^\infty \|Q_n\|_\infty \\ &\leqslant \sum_{n=0}^{m-1} \|Q_n'\|_\infty |x-y| + 2\sum_{n=m}^\infty \|Q_n\|_\infty \\ &\overset{4.3.1}{\leqslant} \sum_{n=0}^{m-1} 2^n \|Q_n\|_\infty |x-y| + 2\sum_{n=m}^\infty \|Q_n\|_\infty \end{aligned}$$

$$
\begin{aligned}
&\overset{(46)}{\leqslant} \sum_{n=0}^{m-1} 2^n C_\alpha 2^{-n\alpha}|x-y| + 2\sum_{n=m}^{\infty} C_\alpha 2^{-n\alpha} \\
&= C_\alpha \left(|x-y| \sum_{n=0}^{m-1} 2^{n(1-\alpha)} + 2\sum_{n=m}^{\infty} 2^{-n\alpha} \right) \\
&= C_\alpha \left(|x-y| \frac{2^{m(1-\alpha)}-1}{2^{1-\alpha}-1} + 2^{-m\alpha}\frac{2}{1-2^{-\alpha}} \right).
\end{aligned}
$$

Note that splitting the sums is necessary, since $\sum_{n=0}^{m-1} 2^{n(1-\alpha)} \xrightarrow{m\to\infty} \infty$.

Then if $|x-y| < \delta$, choose m such that

$$\delta 2^{m(1-\alpha)} + 2^{-m\alpha} \leqslant C_3 \delta^\alpha,$$

that is, such that

$$(\delta 2^m)1 - \alpha + (2^m\delta)^{-\alpha} \leqslant C_3.$$

Hence we can choose m such that $1 \leqslant 2^m\delta < 2$, then the term above is $\leqslant C_3|x-y|^\alpha$. □

By corollary 3.3.5, $E_n^T(f) \to 0$, so the case $\alpha = 0$ can not occur. What happens in the case that $\alpha = 1$? We have that $w_f(\delta) \leqslant k\delta$ implies that $E_n^T(f) \in O\left(\frac{1}{n}\right)$ by JACKSON's theorem II, but the converse does not hold, as one can see for the function f considered in example 7.2.3. But instead we get the following result.

Theorem 7.2.2: Bernstein II: $\alpha = 1$

If $f \in \mathcal{C}_{2\pi}$ and $E_n^T(f) \in O(\frac{1}{n})$, then $w(\delta) \leqslant k\delta|\ln(\delta)|$ for small $\delta > 0$.

Proof. Using a similar argument as in the last proof, we get

$$|f(x)-f(y)| \leqslant \sum_{n=0}^{m-1} 2^n \|Q_n\|_\infty |x-y| + 2\sum_{n=m}^{\infty} \|Q_n\|_\infty$$

with $\|Q_n\|_\infty \leqslant \tilde{A}2^{-n\alpha} = \tilde{A}2^{-n}$ because here, $\alpha = 1$. Hence if $|x-y| \leqslant \delta$, then

$$|f(x)-f(y)| \leqslant m\tilde{A}|x-y| + 2^{2-m}\tilde{A} \leqslant 4\tilde{A}(m\delta + 2^{-m}) \overset{!}{\leqslant} k\delta|\log(\delta)|,$$

where the last inequality holds if $\delta 2^m \in [1,2)$ (for details see [Car98, p. 78]). □

In summary,

$$\frac{w_f(\delta)}{\delta} \text{ bounded for all } \delta \implies \left(nE_n^T(f)\right)_{n\in\mathbb{N}} \text{ bounded} \implies \frac{w_f(\delta)}{\delta|\log(\delta)|} \text{ bounded for small } \delta.$$

To get a necessary and sufficient condition for $\alpha = 1$, ZYGMUND defined a new concept of continuity.

Definition 7.2.1 (Zygmund modulus of continuity)
The ZYGMUND modulus of continuity of a bounded function f is

ZYGMUND modulus of continuity

$$w_f^*(\delta) := \sup_x \sup_{|h|<\delta} |f(x+h) - 2f(x) + f(x-h)|, \qquad \delta > 0.$$

The ZYGMUND class is the set of functions $f \in \mathcal{C}_{2\pi}$ such that $\{\frac{w_f^*(\delta)}{\delta} : \delta > 0\}$ is bounded.

Remark 7.2.2 (Relationship between w_f and w_f^*) We have $w_f^*(\delta) \leqslant 2w_f(\delta)$. ○

Theorem 7.2.3: Zygmund

Let $f \in \mathcal{C}_{2\pi}$. Then $E_n^T(f) \in O(\frac{1}{n})$ if and only if $\delta \mapsto \frac{w_f^*(\delta)}{\delta}$ is bounded.

Example 7.2.3 Consider $f(x) := \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}$. As $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, $f \in \mathcal{C}_{2\pi}$ is well-defined. We have $E_n^T(f) \in O(n^{-1})$ for $n \to \infty$. The set $\{\frac{w_f(\delta)}{\delta} : \delta > 0\}$ is unbounded for $\delta \searrow 0$, while the set $\{\frac{1}{\delta} w_f^*(\delta) : \delta > 0\}$ is bounded. ◇

Proof. (HW 7.2) *This is left as an exercise to the reader.* □

Theorem 7.2.4: Bernstein III

Let $f \in \mathcal{C}_{2\pi}$, $E_n^T(f) \in O(n^{-\alpha-p})$, where $p \in \mathbb{N}$ and $\alpha \in (0,1)$. Then $f', \ldots, f^{(p)}$ exist and we have $f^{(p)} \in \mathrm{Lip}^\alpha$.

Proof. As before, let $p_n := \operatorname{argmin}_{p \in \mathcal{T}_n} \|f - p\|_\infty$, $Q_n := p_{2^n} - p_{2^{n-1}}$, $Q_0 = p_1$. Then as we have seen before we have $f = \sum_{n=0}^{\infty} Q_n$ and

$$\|Q_n\|_\infty \leqslant \|p_{2^n} - f\|_\infty + \|f - p_{2^{n-1}}\|_\infty \leqslant A(2^n)^{-p-\alpha} + A(2^{n-1})^{-p-\alpha} = \tilde{A} 2^{-n(p+\alpha)}, \tag{47}$$

where A is a constant and $\tilde{A} := A(1 + 2^{p+\alpha})$ depends on α and p, but not on n. Applying Bernstein's inequality we get

$$\|Q_n^{(p)}\|_\infty \leqslant (2^n)^p \|Q_n\| \overset{(47)}{\leqslant} \tilde{A} 2^{-n\alpha} \xrightarrow{n \to \infty} 0.$$

Thus, $\left(\sum_{k=0}^{n} Q_n^{(p)}\right)_{n \in \mathbb{N}}$ converges uniformly. Since $f = \sum_{n=0}^{\infty} Q_n$, we can prove $f^{(p)}$ exists and $f^{(p)} = \sum_{n=0}^{\infty} Q_n^{(p)}$ as before.

We have

$$E_{2^m}^T(f^{(p)}) \leqslant \left\| f^{(p)} - \sum_{n=0}^{m} Q_n^{(p)} \right\|_\infty \overset{\triangle\neq}{\leqslant} \sum_{n=m+1}^{\infty} \|Q_n^{(p)}\|_\infty \leqslant C 2^{-m\alpha}.$$

For every n choose m such that $2^m \leqslant n \leqslant 2^{m+1}$. Then $2^{-(m+1)} \leqslant n^{-1}$ and thus

$$E_n(f^{(p)}) \leqslant E_{2^m}(f^{(p)}) \leqslant C(2^{-m})^\alpha \leqslant C \left(\frac{n}{2}\right)^\alpha,$$

so $E_n(f^{(p)}) \in O(n^{-\alpha})$. Hence $f^{(p)} \in \mathrm{Lip}^\alpha$ by the first Bernstein theorem. □

Theorem 7.2.5: $f \in \mathcal{C}_{2\pi}^\infty \iff E_n^T(f) \in O(n^{-k})\ \forall k \in \mathbb{N}$

Let $f \in \mathcal{C}_{2\pi}$. Then f is smooth if and only if $n^k E_n^T(f) \to 0$ for all $k \in \mathbb{N}$.

Proof. (HW 7.1) *This is left as an exercise to the reader.* □

Remark 7.2.4 All results of this subsection can also be obtained for algebraic polynomials. ○

8 Approximation by rational functions

This chapter follows [Che66, Chp. 5].

8.1 The continued fraction class

11.01.2022

Let ∂ denote the degree of a polynomial.

Definition 8.1.1 (Rational function, $R^n_m[a,b]$)
A rational function is a quotient of two polynomials:

rational function

$$x \mapsto \frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_m x^m}. \tag{48}$$

The set of bounded rational functions on an interval $[a,b]$ is

$$R^n_m([a,b]) := \{(48) : P \in \mathcal{P}_n,\ Q \in \mathcal{P}_m,\ Q(x) > 0\ \forall x \in [a,b]\}.$$

The condition $Q(x) > 0$ ensures continuity, which we need for approximating functions from $\mathcal{C}([a,b])$.

Theorem 8.1.1: Continued-fraction form

A rational function R can be written in continued-fraction form

$$R = Q_1 + \cfrac{c_2}{Q_2 + \cfrac{c_3}{\ddots + \cfrac{c_k}{Q_k}}}$$

where Q_j is a polynomial and $c_j \in \mathbb{R}$ for all $j \in \{1, \ldots, k\}$.

Example 8.1.2 We have

$$\begin{aligned}\frac{2x^4 - 4x^3 - 2x^2 + 12x - 4}{x^3 - 2x^2 - x + 5} &= 2x + \frac{2x - 4}{x^3 - 2x^2 - x + 5} \\ &= 2x + \frac{2}{\frac{x^3 - 2x^2 - x + 5}{x-2}} = 2x + \frac{2}{x^2 - 1 + \frac{3}{x-2}}.\end{aligned}$$ ◇

Proof. As in the above example, the numerator at some point becomes the denominator. Denote the first numerator and denominator by R_0 and R_1, respectively. Let us assume that $\partial R_0 \geqslant \partial R_1$. For every successive division R_{j-1} by R_j we obtain a quotient Q_j and a remainder R_{j+1} such that

$$R_0 = R_1 Q_1 + R_2, \qquad R_1 = R_2 Q_2 + R_3, \qquad \ldots \qquad R_{k-1} = R_k Q_k. \tag{49}$$

We have $\partial R_{j-1} \leqslant \partial R_j$, so eventually $\partial R_k = 0$. Now we can build R as follows:

$$R = \frac{R_0}{R_1} \overset{(49)}{=} Q_1 + \frac{1}{\frac{R_1}{R_2}} \overset{(49)}{=} Q_1 + \frac{1}{Q_2 + \frac{1}{\frac{R_2}{R_3}}} \overset{(49)}{=} \ldots \overset{(49)}{=} Q_1 + \cfrac{1}{Q_2 + \cfrac{1}{\ddots + \cfrac{1}{Q_k}}}.$$

In the case where $\partial R_0 < \partial R_1$, we write $R = \frac{c_1}{c_1 \frac{R_1}{R_0}}$ with c_1 chosen such that R_1 and R_0 have the same leading coefficient. □

Even though any $R \in R^n_m([a,b])$ will have a "curve-fitting ability" similar to a polynomial $p \in \mathcal{P}_{m+n}$, a lot less operations are needed to evaluate R due to the continued fraction form.

Corollary 8.1.3 (Complexity of evaluating continued fractions)
The rational function $\frac{P}{Q}$ as written in a continued-fraction form with k sub-fractions can be evaluation at any point with at most $\max(\partial P, \partial Q)$ operations.

Proof. 1 To evaluate a continued fraction form, we require $k-1$ multiplications. We can see this with an example: for $k=4$ we have

$$\begin{aligned} P_1 + \frac{c_2}{P_2 + \frac{c_3}{P_3 + \frac{c_4}{P_4}}} &= P_1 + \frac{c_2}{P_2 + \frac{c_3}{\frac{P_3 P_4 + c_4}{P_4}}} \\ &= P_1 + \frac{c_2}{\frac{P_2 P_3 P_4 + c_4 P_2 + c_3 P_4}{P_3 P_4 + c_4}} \\ &= \frac{P_1 P_2 P_3 P_4 + c_4 P_1 P_2 + c_3 P_1 P_4 + c_2 P_3 P_4 + c_4}{c_2 P_3 P_4 + c_2 c_4}. \end{aligned}$$

So we are multiplying every P_j with all $k-1$ divisors.

2 The number of operations thus depends on the number of coefficients of each polynomial P_j, that is, on the its degree. Now assume that $\partial R_0 \geqslant \partial R_1$.

The total number of operations is

$$\sum_{j=1}^{k} \partial P_j = \sum_{j=1}^{k} \partial Q_j \overset{(\star)}{=} \sum_{j=1}^{k} \partial R_{j-1} - \partial R_j = \partial R_0 - \partial R_k = \partial R_0 \leqslant n,$$

where in $(\star)$ we use that $R_{j-1} = Q_j R_j + R_{j+1}$ and thus $Q_j = \frac{R_{j+1} - R_{j-1}}{R_j}$. Hence, as $\partial R_{j-1} \geqslant \partial R_j$, we have $\partial Q_j = \partial \frac{R_{j-1}}{R_j} = \partial R_{j-1} - \partial R_j$.

3 In the case $\partial R_0 < \partial R_1$, we write again $R = \frac{c_1}{c_1 \frac{R_1}{R_0}}$. Then $\frac{c_1 R_1}{R_0}$ can be expressed as a continued fraction, hence it requires at most $\partial R_1 \leqslant m$ operations. □

8.2 Rational best approximation

We can not directly deduce the existence of a best rational approximation to $f \in \mathcal{C}([a,b])$ as the closed and bounded set

$$\{R \in R^n_m([a,b]) : \|R - f\|_\infty \leqslant \|f\|_\infty\}$$

is not compact and $R^n_m([a,b])$ isn't a vector space.

For example the sequence $R_k(x) = \frac{1}{kx+1}$ on the interval $[0,1]$ has property $\|R_k\|_\infty \leqslant 1$ and the limit function is not continuous in 0, so we cannot extract a convergent subsequence.

Theorem 8.2.1: Existence of best rational approximation

For $f \in \mathcal{C}([a,b])$, there exist a best approximation from $R^n_m([a,b])$.

Proof. Let $\delta := \inf_{R \in R_m^n([a,b])} \|f - R\|_\infty$ and $\left(R_k = \frac{P_k}{Q_k}\right)_{k\in\mathbb{N}} \subset R_m^n([a,b])$ be such that $\|f - R_k\|_\infty \to \delta$, where $P_k \in \mathcal{P}_n$ and $Q_k \in \mathcal{P}_m$ such that $\|Q_k\|_\infty = 1$ and $Q_k(x) > 0$ for every $k \in \mathbb{N}$. Without loss of generality we can assume that for every $k \in \mathbb{N}$, we have $\|R_k - f\|_\infty \leqslant \delta + 1$, otherwise pass to an subsequence of $(R_k)_{k\in\mathbb{N}}$ with this property. Thus

$$\|R_k\|_\infty \leqslant \|R_k - f\|_\infty + \|f\|_\infty \leqslant \delta + 1 + \|f\|_\infty =: \varepsilon.$$

Hence $|P_k(x)| = |Q_k(x)||R_k(x)| \leqslant \|Q_k\|_\infty \|R_k\|_\infty \leqslant \varepsilon$, implying that that for every $k \in \mathbb{N}$ (P_k, Q_k) lies in the compact set

$$\left\{(P,Q) \in \mathcal{P}_n \times \mathcal{P}_m : \|P\|_\infty \leqslant \varepsilon,\ \|Q\|_\infty = 1\right\} \qquad \forall k \in \mathbb{N}\,.$$

Due to the compactness we have (up to subsequences) $P_k \to P$ and $Q_k \to Q$ for some $P \in \mathcal{P}_n$ and $Q \in \mathcal{P}_m$. Further, $\|Q\|_\infty = 1$ and there can be at most m points where Q is zero as $\partial Q \leqslant \partial Q_k$. Now, $|P(x)| \leqslant \varepsilon |Q(x)|$ implies that zeros of Q are zeros of P, too and the linear factors corresponding to those zeros can be cancelled.

Consequently, $\frac{P}{Q}$ is well defined and $\frac{P_k}{Q_k} \to \frac{P}{Q}$ and $R \in R_m^n([a,b])$ and since $R_k \to R$, we have $\|R - f\|_\infty = \delta$. □

We want to extend the existence theorem by allowing general fractions $\frac{\sum_{i=0}^n a_i g_i(x)}{\sum_{j=0}^m b_j h_j(x)}$ instead of only rational functions. This is not possible in general and we thus assume that g_i and h_j are analytic on $[a,b]$ for $i \in \{0,\dots,n\}$ and $j \in \{0,\dots,m\}$.

Theorem 8.2.2: Generalised rational best approximation

Each $f \in \mathcal{C}([a,b])$ has a best approximation from

$$\mathbf{R} := \left\{ R \in \mathcal{C}([a,b]) : R(x) \cdot \sum_{j=0}^m b_j h_j(x) = \sum_{i=0}^n a_i g_i(x) \text{ with } \sum_{j=0}^m |b_j| \neq 0 \right\}.$$

Proof. See [Che66, p. 155 - 156]. □

Corollary 8.2.1 (Best approximations from rational trigonometric functions)
Each $f \in \mathcal{C}([a,b])$ has a best approximation from the set of rational trigonometric functions

$$R(\theta) = \frac{\sum_{j=0}^n a_j \cos(j\theta) + b_j \sin(j\theta)}{\sum_{j=0}^m c_j \cos(j\theta) + d_j \sin(j\theta)}.$$

If $[a,b] = [-\pi,\pi]$, then we can always find a best approximation with positive denominator.

8.3 The exchange algorithm

Polynomials are not always suited for approximating functions as no polynomial that is slowly varying when $|x|$ is large can have a sharp peak near the centre of the range of the variable.

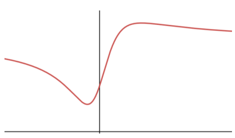

Fig. 11: A rational function which is difficult to approximate with a polynomial.

The exchange algorithm is usually done with polynomials, but a rational function $R \in R_m^n([a,b])$ can be compared to a polynomial with $n+m+1$ degrees of freedom, for example by always mandating that the denominator is one at $x = 0 \in [a,b]$. The reference set has $m+n+2$ elements, so we have a unique solution.

Result: R_N, the best rational approximation of f from $R_m^n([a,b])$.
Set reference $a \leqslant \xi_0 < \ldots < \xi_{m+n+1} \leqslant b$;
for $k \in \{0, 1, \ldots, N\}$ **do**
  find $R_k \in \operatorname{argmin}_{R \in R_m^n([a,b])} \max_{i \in \{0,\ldots,m+n+1\}} |f(\xi_i) - R(\xi_i)|$;
  swap one point in reference by $\eta \in [a,b]$, satisfying $|f(\eta) - R_k(\eta)| = \|f - R_k\|_\infty$.

Algorithm 1: The exchange algorithm.

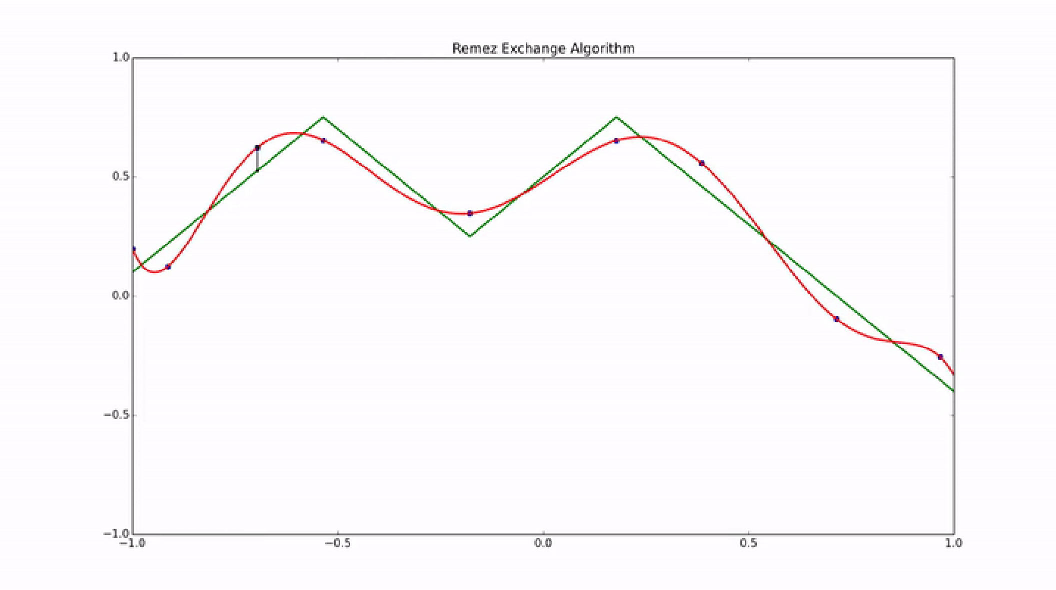

Fig. 12: The exchange algorithm animated [Source: YouTube].

The following theorem gives the equations used to calculate R_k:

Theorem 8.3.1: Finding $\operatorname{argmin}_{R \in R_m^n[a,b]} \max_{i \in \{0,\ldots,m+n+1\}} |f(\xi_i) - R(\xi_i)|$

Let $f \in \mathcal{C}[a,b]$. If $R_k \in R_m^n([a,b])$ and

$$R_k(\xi_i) + (-1)^i h_k = f(\xi_i), \quad i \in \{0, \ldots, n+m+1\} \tag{50}$$

holds for some constant h_k, then

$$R_k \in \operatorname{argmin}_{R \in R_m^n([a,b])} \max_{i \in \{0,\ldots,m+n+1\}} |f(\xi_i) - R(\xi_i)|. \tag{51}$$

Why is R_k the best approximation? We have $R_m^n([a,b]) \subset \mathcal{C}[a,b]$ and

- $a \leqslant \xi_0 < \ldots < \xi_{n+m+1} \leqslant b$,
- $|f(\xi_i) - R_k(\xi_i)| = |h_k|$, $i \in \{0, \ldots, n+m+1\}$,
- $f(\xi_{i+1}) - R_k(\xi_{i+1}) = (-1)^{i+1} h_k = -(-1)^i h_k = -(f(\xi_i) - R_k(\xi_i))$ for all $i \in \{0, \ldots, m+n\}$.

Hence the error $|f - R_k|$ oscillates at $n+m+1$ reference points, so R_k is best approximation of f by Theorem 3.2.1.

Proof. By (51), where the optimal value is $|h_k|$, it suffices to show that if $R \in R_m^n([a,b])$ satisfies

$$\max_{i \in \{0,\ldots,m+n+1\}} |f(\xi_i) - R(\xi_i)| \leqslant |h_k|, \tag{52}$$

then $R = R_k$.

Assume $R \in R_m^n([a,b])$ satisfies (52). Then there exists a $\varepsilon \in [0,1]$ such that

$$\max_{i\in\{0,\dots,m+n+1\}} |f(\xi_i) - R(\xi_i)| = \varepsilon_k |h_k|.$$

Hence for every $i \in \{0,\dots,m+n+1\}$ we have

$$R(\xi_i) - R_k(\xi_i) = (f(\xi_i) - R_k(\xi_i)) - (f(\xi_i) - R(\xi_i)) \overset{(50)}{=} (-1)^i h_k - \varepsilon_k |h_k|.$$

This means that this term is either equal 0 or has the sign of $(-1)^i h_k$. Hence $R - R_k$ has at least $m+n+1$ zeros in $[a,b]$. Since $R, R_k \in R_m^n([a,b])$ are both rational functions, we can write

$$R - R_k = \frac{P}{Q} - \frac{P_k}{Q_k} = \frac{PQ_k - P_kQ}{QQ_k}$$

for polynomials $P, P_k \in \mathcal{P}_m$ and $Q, Q_k \in \mathcal{P}_n$. Hence the numerator $PQ_k - P_kQ$ has at most degree $m+n$, so $R - R_k = 0$. □

Theorem 8.3.2: Characterisation: rational best approximation

A real function $f \in \mathcal{C}([-1,1];\mathbb{R})$ has a unique best approximation r^* from

$$R_{m,n}^{\mathbb{R}} := \left\{ \frac{P}{Q} : Q > 0,\ P, Q \text{ are real},\ P \in \mathcal{P}_m,\ Q \in \mathcal{P}_n \right\}$$

and a function $r = \frac{P}{Q} \in R_{m,n}^{\mathbb{R}}$ equals r^* if f equioscillates (that is, $f(\xi_i) - r(\xi_i) = (-1)^i h$) between $m+n+2-d$ extreme points, where $d = \min\{m - \partial(p), n - \partial(q)\}$.

Then number d is called the defect of the rational function r.

Remark 8.3.1 (Approximation by complex rational functions)
For $f \in \mathcal{C}([-1,1])$, a best approximation from complex rational function exists, but it is not unique [SVN76]. ○

Padé approximation

Padé approximation is the rational function analogue of Taylor polynomial approximation: for $m, n > 0$, $R \in R_m^n$ is the type (m,n) Padé approximation of $f \in \mathcal{C}^{m+n}([-1,1])$ if their Taylor series at $x = 0$ agrees as far as possible. If $P_m^T(x) = \sum_{k=0}^m \frac{1}{k!} f^{(k)}(0) x^k$ is the m-th order Taylor approximation of f, then $(f - P_m)(x) = O(x^{m+1})$ but $(f - R_m^n)(x) = O(x^{\text{maximum}})$, where $R_m^n = \frac{P_n}{Q_m} = \frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^m b_i x^i} \in R_m^n[a,b]$ is the Padé approximation of order (n,m) to f (at the point 0). We can without loss of generality assume that $Q_m(0) = 1$. Then we have Padé approximation

$$f^{(k)}(0) = (R_m^n)^{(k)}(0), \qquad \forall k \in \{0,\dots,m+n\}.$$

To find R_m^n, we can sometimes consider the linearised version of $(f - R_m^n)(x) = O(x^{\text{maximum}})$, which is $f(x)Q_m(x) - P_n(x) = O(x^{\text{maximum}})$. The polynomials P_n and Q_m might have common zeros.

If $R_m^n \in R_m^n([a,b])$ is the Padé approximation of f, then

$$f(x) - R_m^n(x) \in O(x^{n+m+1-d}),$$

where $d := \min(n - \deg(P), m - \deg(Q))$ is the defect of R_m^n.

Remark 8.3.2 (Square block phenomena)
The PADÉ table of a function f is as follows:

$m\backslash n$	0	1	2	3
0	R_0^0	R_0^1	R_0^2	$\dots$
1	R_1^0	R_1^1	$\ddots$	$\ddots$
2	$\ddots$	$\vdots$	$\ddots$	$\ddots$

Table 1: The PADÉ table of a function f.

Usually, we find blocks consisting of the same function, e.g. $R_0^0 = R_0^1 = R_1^0 = R_1^1$ and $R_2^0 = R_3^0 = R_2^1 = R_3^1$ for $f = \cos$ or $R_m^0 = 0$ for all $m \in \mathbb{N}$ and $R_1^0 = R_2^0 = R_1^1 = R_2^1$ for $f = \sin$. ○

Example 8.3.3 (PADÉ approximation of order $(1,2)$ of $\tan(x)$ (HW 7.3))
This is left as an exercise to the reader. ◇

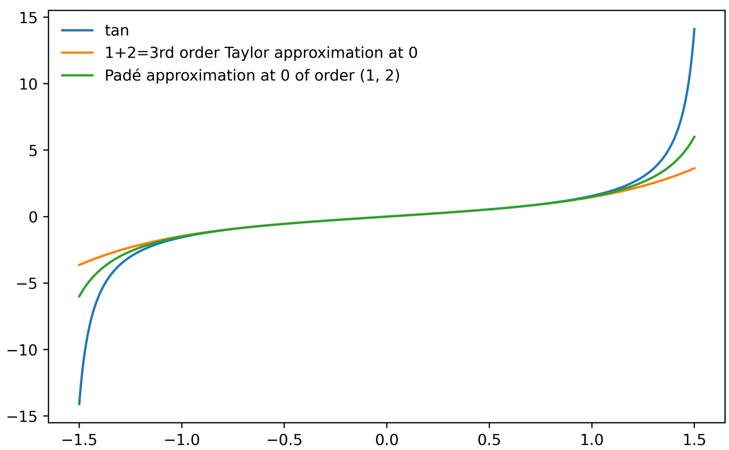

Fig. 13: One notices that for larger x, the PADÉ approximation of f of order (n, m) approximates f better than the TAYLOR SERIES of degree $m + n$.

In most cases PADÉ approximation approximates better than the TAYLOR polynomial of degree $m + n$: sometimes, the TAYLOR polynomials grow fast outside of an interval, while the PADÉ approximation follows the function on a larger interval.

9 The Stone approximation theorem and multivariate interpolation

25.01.2022

There have been many attempts to generalize Weierstrass' approximation theorem, which is one of the most important theorems in approximation theory and states that $\overline{\mathcal{P}} = \mathcal{C}([a,b])$. We want to replace $[a,b]$ by a compact metric space.

This chapter follows [Che66, Chp. 6.1].

9.1 The Stone approximation theorem

Let X be a compact metric space and $\mathcal{C}(X)$ be the algebra (together with the pointwise multiplication) of all continuous functions of X. The vector space $\mathcal{C}(X)$ is an algebra because we have $(f \cdot g)(x) = f(x) \cdot g(x)$, $(f+g)h = fh + gh$ and $f(g+h) = fg + fh$ and $\alpha(fg) = (\alpha f)g = (\alpha g)g = f(\alpha g)$, where α is a scalar and $(f \cdot g)h = f \cdot (g \cdot h)$. algebra

We want to the characterise all subalgebras (subset of an algebra, closed under the induced operations), which are dense in $C(X)$. For example, $\mathcal{C}([a,b])$ is an algebra and $\mathcal{P}$ is a dense subalgebra.

Theorem 9.1.1: Stone-Weierstrass

If X is a compact metric space and $\mathcal{A} \subset \mathcal{C}(X)$ is a subalgebra such that

1. $1 \in \mathcal{A}$,
2. $\mathcal{A}$ separates points of X, that is for $x \neq y \in X$, then there exists a $f \in \mathcal{A}$ with $f(x) \neq f(y)$,

then $\overline{\mathcal{A}} = \mathcal{C}(X)$.

Proof. 1 First we prove that if $f \in \overline{\mathcal{A}}$, then $|f| \in \overline{\mathcal{A}}$. Let $f \in \overline{\mathcal{A}}$ and $\varphi(x) := |x|$. Then $\varphi \in \mathcal{C}([-\|f\|_\infty, \|f\|_\infty])$. There is a set of polynomials $(p_n) \subset \mathcal{P}$ such that $p_n \rightrightarrows \varphi$. Then $p_n(f(x)) \rightrightarrows \varphi(f(x))$, implying that $p_n \circ f \rightrightarrows |f|$. If $\mathcal{A}$ is an algebra, so is $\overline{\mathcal{A}}$ (Exercise). Hence $p_n f \in \overline{\mathcal{A}}$. Hence $|f| \in \overline{\mathcal{A}}$.

2 We now prove that $f, g \in \overline{\mathcal{A}}$, then $\min(f,g), \max(f,g) \in \overline{\mathcal{A}}$.

We write $\min(f,g)(x) := \min(f(x), g(x)) = \frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|)$. Analogously, $\max(f,g)(x) = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|)$. By the previous point, $\min(f,g), \max(f,g) \in \overline{\mathcal{A}}$.

3 We now show that if $p, q \in X$ with $p \neq q$ and λ, μ are constant, then there exists a $\varphi \in \mathcal{A}$ such that $\varphi(p) = \lambda$ and $\varphi(q) = \mu$.

By condition 2, there exists a $F \in \mathcal{A}$ such that $F(p) \neq F(q)$. By 1, $1 \in \mathcal{A}$. Define $\varphi(x) := \frac{\lambda - \mu}{F(p) - F(q)} F + \frac{\mu F(p) - \lambda F(q)}{F(p) - F(q)} 1$.

4 We now show that $\overline{\mathcal{A}} = \mathcal{C}(X)$.

Towards contradiction assume there exists a $f \in \mathcal{C}(X)$ and a $\varepsilon > 0$ such that $\|f - g\| \geqslant \varepsilon$ for all $g \in \mathcal{A}$. Let $p, q \in X$ and take $\varphi_{p,q} \in \mathcal{A}$ such that $\varphi_{p,q}(p) = f(p)$ and $\varphi_{p,q}(q) = f(q)$. Now define the open set

$$V_{p,q} := \{x \in X : \varphi_{p,q}(x) < f(x) + \varepsilon\}.$$

Then $p \in V_{p,q}$ because $\varphi_{p,q}(p) = f(p) < f(p) + \varepsilon$. Hence $X = \bigcup_{p \in X} V_{p,q}$. As X is compact, for any fixed $q \neq p$ we get that $X = \bigcup_{j=1}^{n} V_{p_j,q}$. (Note that the p_j depends on q.)

Let $\varphi_q = \min(\varphi_{p_1,q}, \dots, \varphi_{p_n,q})$ and $V_q := \{x \in X : \varphi_q(x) > f(x) - \varepsilon\}$ is an open set. Further, $q \in V_q$, because $\varphi_q(q) > \varphi_{p_i,q}(q) = f(q) > f(q) - \varepsilon$. Hence $X = \bigcup_{q \in X} V_q$. As X is compact, we have $X = \bigcup_{k=1}^{m} V_{q_k}$. Let $g(x) := \max(\varphi_{q_1}, \dots \varphi_{q_m})$. Then you can see (Exercise!) that $f(x) - \varepsilon < g(x) < f(x) + \varepsilon$ for all $x \in X$, so $\|g - f\|_\infty < \varepsilon$, which contradicts the assumption. □

Corollary 9.1.1

The first and second WEIERSTRASS *theorems follow by taking* $X = [a,b]$ *or* $[0, 2\pi]/_{0 \sim 2\pi}$.

Corollary 9.1.2 (Multivariate polynomials)

If $X \subset \mathbb{R}^d$ *is compact, then the polynomials in* d *variables on* X *are dense in* $\mathcal{C}(X)$.

9.2 Multivariate polynomials

Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. How can we define a multivariate polynomial $\sum_\alpha c_\alpha x^\alpha$?

Let $\mathbb{N}_0^d := \{\alpha = (\alpha_1, \dots, \alpha_d) : \alpha_i \in \mathbb{N}_0\}$ and $|\alpha| := \sum_{i=1}^d \alpha_i$ for $\alpha \in \mathbb{N}_0^d$. Further, let $x^\alpha = \prod_{k=1}^d x_k^{\alpha_k}$ for $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$.

DEFINITION 9.2.1 (MULTIVARIATE MONOMIAL, POLYNOMIAL)
The function $x \mapsto x^\alpha$ is a monomial. A polynomial is $p(x) = \sum_{\alpha \in I} c_\alpha x^\alpha$, where $I \subset \mathbb{N}_0^d$ is finite. The degree of p is $\max(\{|\alpha| : \alpha \in I, c_\alpha \neq 0\})$.

The degree of the zero polynomial is $-\infty$.

Example 9.2.2 If $d = 3$ then x_1, x_2, x_3, x_1x_2, $x_1^2x_2$ are all monomials of degree 1, 1, 1, 2 and 3, respectively. ◇

DEFINITION 9.2.3 (MULTIVARIATE FACTORIAL, BINOMIAL COEFFICIENT)
If $\alpha, \beta \in \mathbb{N}_0^d$ fulfil $\beta \leqslant \alpha$, that is, $\beta_i \leqslant \alpha_i$ for all $i \in \{1, \dots, d\}$, then $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$ and 0 else, where $\alpha! := \prod_{k=1}^d \alpha_k!$.

THEOREM 9.2.1: BINOMIAL THEOREM

For $x, y \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$ we have

$$(x+y)^\alpha = \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} x^\beta y^{\alpha - \beta}.$$

Proof. Induction on d. □

THEOREM 9.2.2: MULTINOMIAL THEOREM

For $x, y \in \mathbb{R}^d$ and $k \in \mathbb{N}$ we have

$$(\langle x, y \rangle)^k = \sum_{|\alpha| = k} \frac{k!}{\alpha!} x^\alpha y^\alpha.$$

Proof. Preform induction on d and in the induction step use that for $x, y \in \mathbb{R}^{d+1}$ and $\alpha = (\alpha_1, \ldots, \alpha_{d+1})$ we have

$$\{\alpha \in \mathbb{N}_0^{d+1} : |\alpha| = k\} = \bigcup_{j=0}^{k} \{\alpha \in \mathbb{N}_0^{d+1} : \alpha_{d+1} = j,\ \alpha_1 + \ldots + \alpha_d = k - j\}. \qquad \square$$

Definition 9.2.4 (Multidimensional polynomials $\mathcal{P}_n(\mathbb{R}^d)$)
The linear space of all polynomials of degree at most n in $\mathbb{R}^d$ is denoted by $\mathcal{P}_n(\mathbb{R}^d)$.

If $p \in \mathcal{P}_n(\mathbb{R}^d)$, then we can represent it as $p(x) = \sum_{|\alpha| \leqslant n} c_\alpha x^\alpha$.

Theorem 9.2.3: Monomials are basis of $\mathcal{P}_n(\mathbb{R}^d)$

The monomials $x \mapsto x^\alpha$ with $|\alpha| \leqslant n$ form a basis for $\mathcal{P}_n(\mathbb{R}^d)$.

Proof. We perform induction on d.

We know this theorem is true in $d = 1$. For the induction hypothesis assume the statement is true for d.

Assume that $\sum_{\alpha \in I} c_\alpha x^\alpha = 0$, where $I \subset \mathbb{N}_0^{d+1}$ is finite. Then $I = \bigcup_{j=0}^{n} I_j$, where $I_j := \{\alpha \in I : \alpha_{d+1} = j\}$.

We can write

$$\sum_{\alpha \in I} c_\alpha x^\alpha = \sum_{j=0}^{n} \left(\sum_{\alpha \in I_j} c_\alpha x_1^{\alpha_1} \ldots x_d^{\alpha_d} \right) x_{d+1}^{j} = 0$$

if and only if $\sum_{\alpha \in I_j} c_\alpha x_1^{\alpha_1} \ldots x_d^{\alpha_d} = 0$, which can be written as $\sum_{\alpha \in I_j} \tilde{x}^{\tilde{\alpha}} = 0$, where $\tilde{x} = (x_1, \ldots, x_d)$ and $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_d)$. Now use the induction hypothesis to obtain that $c_\alpha = 0$ for $\alpha \in I_j$, where $j \in \{0, \ldots, n\}$ and hence $c_\alpha = 0$ for all $\alpha \in I$.

Hence the monomials $x \mapsto x^\alpha$ are linearly independent. $\square$

Theorem 9.2.4: Dimension of $\mathcal{P}_n(\mathbb{R}^d)$

The dimension of $\mathcal{P}_n(\mathbb{R}^d)$ is equal to $\binom{n+d}{d}$.

Proof. Preform induction on d and use that $\sum_{k=0}^{n} \binom{n-k+d}{d} = \sum_{k=0}^{n} \binom{k+d}{d} = \binom{n+d+1}{d+1}$. $\square$

Theorem 9.2.5: No Haar spaces for $d \geqslant 2$

There are no n-dimensional Haar spaces of continuous functions on $\mathbb{R}^d$ for $n, d \geqslant 2$.

Proof. Assume that we have a Haar space with dimension $n \geqslant 2$. Then we have a Haar system $(u_k \colon \mathbb{R}^d \to \mathbb{R})_{k=1}^{n}$. That means if $(x_k)_{k=1}^{n} \subset \mathbb{R}^d$ are distinct, then $A := [u_i(x_j)]_{i,j=1}^{n}$ is invertible, that is, $\det(A) \neq 0$.

Select a closed path in $\mathbb{R}^d$ that contains x_1 and x_2 but no other points $x_3, \ldots, x_n$. Moving x_1 and x_2 continuously towards each other along this path, we can make x_1 and x_2 exchange their places. This corresponds to exchanging the first and second column in A. Hence the

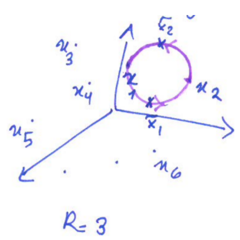

Fig. 14: A closed path in $\mathbb{R}^d$ containing only x_1 and x_2.

continuous function $A \mapsto \det(A)$ changes sign. Hence det has to be zero somewhere on the path, which contradicts that $\{u_1, \ldots, u_n\}$ is a HAAR system. $\square$

We conclude that the set $\{x^\alpha : |\alpha| \leqslant n\}$ can not be a HAAR system of $\mathcal{P}_n(\mathbb{R}^d)$ if $n, d \geqslant 2$. That means if $m = \dim(\mathcal{P}_n(\mathbb{R}^d))$, then interpolation for sets of m distinct points is not possible!

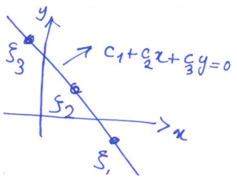

Fig. 15: We only get a solution if the ξ_i are colinear.

Example 9.2.5 Let us consider $\mathcal{P}_1(\mathbb{R}^2)$, which is a 3-dimensional linear space by Theorem 9.2.4 with basis $\{1, x_1, x_2\}$ by Theorem 9.2.3. Let $\xi_i = (x^{(i)}, y^{(i)}) \in \mathbb{R}^2$ for $i \in \{1,2,3\}$. We want to find $p \in \mathcal{P}_1(\mathbb{R}^2)$ such that $p(\xi_i) = 0$ for $i \in \{1,2,3\}$. Let $p(x_1, x_2) = c_1 + c_2 x_1 + c_3 x_2$. Formulating the interpolation problem in matrix-vector form yields

$$\underbrace{\begin{pmatrix} 1 & x^{(1)} & y^{(1)} \\ 1 & x^{(2)} & y^{(2)} \\ 1 & x^{(3)} & y^{(3)} \end{pmatrix}}_{=:A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus if the nodes ξ_1, ξ_2 and ξ_3 are colinear, then the matrix A is singular and interpolation of the data is not possible. $\diamond$

But we get the following positive result.

THEOREM 9.2.6: $\mathcal{P}_n(\mathbb{R}^d)$ CAN INTERPOLATE $n+1$ DISTINCT POINTS

For distinct $(x_i)_{i=0}^n \subset \mathbb{R}^d$ and arbitrary $(y_i)_{i=0}^n$, there exists a $p \in \mathcal{P}_n(\mathbb{R}^d)$ such that $p(x_i) = y_i$.

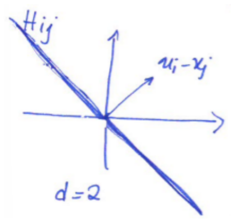

Fig. 16: The subspace $H_{i,j}$.

Proof. Let $x_0, \ldots, x_n$ be distinct points in $\mathbb{R}^d$. Then we can find $u \in \mathbb{R}^d$ such that $\langle u, x_i \rangle \neq \langle u, x_j \rangle$ for any $i \neq j$ because defining $H_{i,j} := \{y \in \mathbb{R}^d : \langle y, x_i - x_j \rangle = 0\}$ for $1 \leqslant i < j \leqslant n$ we get $\bigcup_{1 \leqslant i < j \leqslant n} H_{i,j} \neq \mathbb{R}^d$. Hence there exists a $u \in \mathbb{R}^d$ such that $\langle u, x_i - x_j \rangle \neq 0$ for all $i \neq j$, that is, $(\langle u, x_i \rangle)_{i=0}^n \subset \mathbb{R}$ are distinct scalars.

By the interpolation theorem there is a $q \in \mathcal{P}_n(\mathbb{R})$ such that $q(\langle u, x_i \rangle) = y_i$. Let $p(x) := q(\langle u, x \rangle) = \sum_{k=0}^n c_k \langle u, x \rangle^k$. Applying the multinomial theorem yields that $p \in \mathcal{P}_n(\mathbb{R}^d)$ with $p(x_i) = q(\langle u, x_i \rangle) = y_i$, so p interpolates the data (x_i, y_i). $\square$

10 Positive definite functions

01.02.2022

10.1 Positive definite functions

This section follows [CL09, Chp. 12].

In the following let X be a vector space.

Definition 10.1.1 ((Strictly) positive definite function)
A function $\varphi\colon X \to \mathbb{C}$ is positive definite if for all $n \in \mathbb{N}$ and every n distinct points $x_1, \dots, x_n \in X$ and $\alpha \in \mathbb{C}^n$ we have

positive definite

$$\alpha^* A^{(\varphi)} \alpha = \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \varphi(x_j - x_k) \geqslant 0, \tag{53}$$

where $A^{(\varphi)} := (\varphi(x_i - x_j))_{i,j=1}^{n}$, and strictly positive definite if (53) is strict for all $\alpha \in \mathbb{C}^n \setminus \{0\}$.

strictly positive definite

Example 10.1.2 (Complex exponential map in real inner product space)
If $(X, \langle \cdot, \cdot \rangle)$ is a real inner product space, then $g_y\colon X \to \mathbb{C}$, $x \mapsto \exp(i\langle y, x \rangle)$ is positive definite for every $y \in X$: for all $u \in \mathbb{C}^n$ and all $x_1, \dots, x_n \in X$ we have

$$\sum_{j,k=1}^{n} \overline{u_k} u_j e^{i\langle y, x_k - x_j \rangle} = \sum_{k=1}^{n} \overline{u_k} e^{i\langle y, x_k \rangle} \sum_{j=1}^{n} u_j e^{-i\langle y, x_j \rangle} = \left| \sum_{j=1}^{n} \overline{u_j} e^{i\langle y, x_j \rangle} \right|^2 \geqslant 0. \qquad \diamond$$

Remark 10.1.3 Any non-negative linear combination of positive definite functions is positive definite. ○

Example 10.1.4 The cosine is positive definite on $\mathbb{R}$ by remark 10.1.3 and example 10.1.2, as $\cos = \frac{1}{2}(g_1 + g_{-1})$. ◇

Remark 10.1.5 (Creating positive definite functions $f\colon X \to \mathbb{C}$)
If $f\colon \mathbb{R} \to \mathbb{C}$ is positive definite, then we can define a positive definite function $f^*\colon X \to \mathbb{C}$ for any vector space X: choose a linear function $\varphi\colon X \to \mathbb{R}$ and define $f^* := f \circ \varphi$. Then

$$u^* A^{(f^*)} u = \sum_{j,k=1}^{n} \overline{u_k} u_j f^*(x_k - x_j) = \sum_{j,k=1}^{n} \overline{u_k} u_j f(\varphi(x_k) - \varphi(x_j)) = u^* \tilde{A}^{(f)} u \geqslant 0,$$

where $\tilde{A}^{(f)} = \Big(f\big(\varphi(x_i) - \varphi(x_j)\big) = f\big(\tilde{x}_i - \tilde{x}_j\big) \Big)_{i,j}$ where $\tilde{x}_k := \varphi(x_k)$ are distinct. ○

Elementary properties of positive semidefinite matrices

Lemma 10.1.6
If $A \in \mathbb{C}^{n,n}$ has $v^* A v = 0$ for all $v \in \mathbb{C}^n$, then $A = 0$.

Proof. For $u, v \in \mathbb{C}^n$ we have

$$0 = (u+v)^* A (u+v) = \underbrace{u^* A u}_{=0} + v^* A u + u^* A v + \underbrace{v^* A v}_{=0} = v^* A u + u^* A v.$$

Replacing u by iu yields

$$0 = i v^* A u - i u^* A v$$

Combining $v^*Au + u^*Av = 0$ and $v^*Au - u^*Av = 0$ yields $v^*Au = 0$ for all $u, v \in \mathbb{C}^n$. Replacing v by Au yields $0 = v^*Au = \|Au\|^2$, so $Au = 0$ for all $u \in \mathbb{C}^n$, implying $A = 0$. Alternatively, $A_{i,j} = e_i^* A e_j = 0$ for all $i, j \in \{1, \dots, n\}$, so $A = 0$. □

Theorem 10.1.1: Characterising Hermitian matrices

A matrix $A \in \mathbb{C}^{n,n}$ is Hermitian if and only if $v^*Av \in \mathbb{R}$ for all $v \in \mathbb{C}^n$.

Proof. " $\Longrightarrow$ ": For $u \in \mathbb{C}^n$ we have $\overline{u^*Au} = u^*A^*u = u^*Au$, so $u^*Au \in \mathbb{R}$.

" $\Longleftarrow$ ": Let $B := \frac{1}{2}(A + A^*)$ and $C := \frac{1}{2i}(A - A^*)$, which are both Hermitian with $A = B + iC$. Then for $u \in \mathbb{C}^n$

$$\underbrace{u^*Au}_{\in\mathbb{R}} = \underbrace{u^*Bu}_{\in\mathbb{R}} + i\underbrace{u^*Cu}_{\in\mathbb{R}}$$

can only hold in $u^*Cu = 0$. By lemma 10.1.6, $C = 0$, so $A = A^*$. □

Corollary 10.1.7
A positive semidefinite matrix is Hermitian.

Lemma 10.1.8
If $A \in \mathbb{C}^{n,n}$ is positive (semi)definite, then all its eigenvalues and its determinant are positive (nonnegative).

Remark 10.1.9 The converse is only true if A is Hermitian: consider $A = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} 1 & -1 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) = -2 < 0$. ○

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvector of A and $v \in \mathbb{C}^n \setminus \{0\}$ be the corresponding eigenvector with $Av = \lambda v$. Then, as A is positive definite, we have

$$0 < v^*Av = v^*\lambda v = \lambda\|v\|^2,$$

so $\lambda > 0$. The second statement follows from the fact that the determinant is the product of the eigenvalues. □

The Schur product

Definition 10.1.10 (Schur product)
The Schur product of $A \in \mathbb{C}^{n,n}$ and $B \in \mathbb{C}^{n,n}$ is $A \cdot_S B := (A_{i,j}B_{i,j})_{i,j=1}^n \in \mathbb{C}^{n,n}$.

Example 10.1.11 We have

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot_S \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}.$$ ◇

The next result is also called Schur's lemma.

Lemma 10.1.12 (Positive semidefinite functions closed under $\cdot_S$)
The Schur *product of positive semidefinite matrices is positive semidefinite.*

Lemma 10.1.13 (Characterising positive semidefinite matrices)
For $A \in \mathbb{C}^{n,n}$ is positive semidefinite if and only if there is a $B \in \mathbb{C}^{m,n}$ such that $A = B^*B$.

The proof of lemma 10.1.13 (in [CL09, Lemma 12.4, p. 80]) shows that we can take B to be square.

Proof. (of lemma 10.1.12) By lemma 10.1.13, for $A, B \in \mathbb{C}^{n,n}$ there exists $E, F \in \mathbb{C}^{n,n}$ such that $A = E^*E$ and $B = F^*F$. Let $G \in \mathbb{C}^{n^2,n}$ be described by $G_{(\mu,\nu),j} := F_{\mu,j}E_{\nu,j}$ for $\mu, \nu, j \in \{1, \dots, n\}$. Then for $i, j \in \{1, \dots, n\}$ we have

$$\begin{aligned}(G^*G)_{i,j} &= \sum_{\mu,\nu=1}^{n} \overline{F_{\mu,i}E_{\nu,i}}F_{\mu,j}E_{\nu,j} = \sum_{\mu=1}^{n} \overline{F_{\mu,i}}F_{\mu,j} \sum_{\nu=1}^{n} \overline{E_{\nu,i}}E_{\nu,j} \\ &= (F^*F)_{i,j}(E^*E)_{i,j} = B_{i,j}A_{i,j},\end{aligned}$$

so $C = G^*G$ is positive semidefinite. □

Corollary 10.1.14 (Postcomposition with special analytic function)
Let $f\colon X \to \mathbb{C}$ be a positive definite function and $\varphi\colon \mathbb{C} \to \mathbb{C}$ an analytic function having nonnegative TAYLOR *coefficients on the disk $\{z \in \mathbb{C} : |z| \leqslant f(0)\} =: D$. Then $\varphi \circ f$ is positive definite.*

Proof. As f is positive definite, for all $x \in X$ we have $|f(x)| \leqslant f(0)$: as the matrix $A^{(f)} = \begin{pmatrix} f(0-0) & f(x-0) \\ f(x-0) & f(0-0) \end{pmatrix}$ must have nonnegative determinant $\det(A^{(f)}) = f(0)^2 - |f(x)|^2 \overset{!}{\geqslant} 0$ and $f(0) = \det(f(x_1 - x_1)) \geqslant 0$ is real. Hence $f(x) \in D$. Let $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k$. Then

$$(\varphi \circ f)(x) = \sum_{j=0}^{\infty} a_j f(x)^j$$

is positive definite as nonnegative linear combination of products of positive definite functions by remark 10.1.3 and lemma 10.1.12. □

Lemma 10.1.15 (Uniform continuity)
Let $(X, \|\cdot\|)$ be a normed vector space. If $f\colon X \to \mathbb{C}$ is positive definite and continuous in 0, then

$$|f(x) - f(y)|^2 \leqslant 2f(0)\Re(f(0) - f(y-x)),$$

so f is uniformly continuous.

Proof. 1 Let $x, y \in X$ and $x_1 := x$, $x_2 := 0$ and $x_3 := x - y$. As f is positive definite, the matrix

$$\begin{pmatrix} f(x_1 - x_1) & f(x_1 - x_2) & f(x_1 - x_3) \\ f(x_2 - x_1) & f(x_2 - x_2) & f(x_2 - x_3) \\ f(x_3 - x_1) & f(x_3 - x_2) & f(x_3 - x_3) \end{pmatrix} = \begin{pmatrix} f(0) & f(x) & f(y) \\ f(-x) & f(0) & f(y-x) \\ f(-y) & f(x-y) & f(0) \end{pmatrix}$$

is positive semidefinite and hence HERMITIAN by corollary 10.1.7, so

$$\begin{pmatrix} f(0) & f(x) & f(y) \\ f(-x) & f(0) & f(y-x) \\ f(-y) & f(x-y) & f(0) \end{pmatrix} = \begin{pmatrix} f(0) & f(x) & f(y) \\ \overline{f(x)} & f(0) & f(y-x) \\ \overline{f(y)} & \overline{f(y-x)} & f(0) \end{pmatrix} =: A^{(f)}.$$

Multiplying out $u^*A^{(f)}u \geqslant 0$ gives

$$\|u\|^2 f(0) + 2\Re\big(\overline{u_1}u_2 f(x) + \overline{u_1}u_3 f(y) + \overline{u_2}u_3 f(y-x)\big) \geqslant 0.$$

Choosing $u = \left(t, \frac{|f(x)-f(y)|}{f(x)-f(y)}, -\frac{|f(x)-f(y)|}{f(x)-f(y)}\right)$, where $t \in \mathbb{R}$, yields

$$\begin{aligned}0 &\leqslant (t^2+2)f(0) + 2\Re\left(t\frac{|f(x)-f(y)|}{f(x)-f(y)}f(x) - t\frac{|f(x)-f(y)|}{f(x)-f(y)}f(y) - f(y-x)\right)\\ &= (t^2+2)f(0) + 2\Re\left(t|f(x)-f(y)| - f(y-x)\right)\\ &= f(0)t^2 + 2|f(x)-f(y)|t + 2\Re\big(f(0)-f(y-x)\big).\end{aligned}$$

Hence this quadratic polynomial is nonnegative, so it has nonpositive discriminant, that is,

$$4|f(x)-f(y)|^2 - 8f(0)\Re\big(f(0)-f(y-x)\big) \leqslant 0,$$

which is the statement.

2 As $|f(x)| \leqslant f(0)$, we have $f(0) = 0$ only if $f \equiv 0$. Thus let $\varepsilon > 0$ and $f(0) > 0$. Since f is continuous in 0, there is a $\delta > 0$ such that for all $z \in X$ we have $\|z\| < \delta$ we have $|f(0) - f(z)| < \frac{\varepsilon^2}{2f(0)}$.

For $x, y \in X$ with $\|x-y\| < \delta$ we have

$$\begin{aligned}|f(x)-f(y)| &\leqslant \sqrt{2f(0)\Re\big(f(0)-f(y-x)\big)} \leqslant \sqrt{2f(0)|f(0)-f(y-x)|}\\ &< \sqrt{2f(0)\frac{\varepsilon^2}{2f(0)}} = \varepsilon.\end{aligned}$$

□

10.2 Interpolation by translates of a single function

Strictly positive definite functions are related to polynomial interpolation in higher dimensions.

Assume $f\colon X \to \mathbb{R}$ is a function and $x_1, \dots, x_n \in X$ are distinct. We want to find an interpolating function $g\colon X \to \mathbb{R}$ of the form $g = \sum_{j=1}^m a_j \varphi(\cdot - \nu_j)$ such that

$$g(x_k) = f(x_k) \qquad \forall k \in \{1, \dots, n\}. \tag{54}$$

Here we suppose that the distinct translation parameters $\nu_1, \dots, \nu_m \in X$ are known and the function $\varphi\colon X \to \mathbb{R}$ is fixed, while the coefficients $a_1, \dots, a_m \in \mathbb{R}$ are unknown. Then (54) is a system of n equations with m unknowns.

Example 10.2.1 Consider $X := \mathbb{R}$ and $\varphi(x) := e^{\lambda x}$ for some $\lambda \in \mathbb{R}$. Then $\varphi(\cdot - \nu_j) = e^{-\lambda\nu_j}\varphi$, so shifts of φ only generate a one-dimensional subspace, which is undesirable. ◇

Remark 10.2.2 We look at the easiest case, where $X = \mathbb{R}^d$, $m = n$ and $\nu_j = x_j$. In this case, the interpolation problem can be written as

$$\underbrace{\begin{pmatrix}\varphi(x_1-x_1) & \dots & \varphi(x_1-x_n)\\ \vdots & \ddots & \vdots\\ \varphi(x_n-x_1) & \dots & \varphi(x_n-x_n)\end{pmatrix}}_{=A^{(\varphi)}}\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix} = \begin{pmatrix}f(x_1)\\ \vdots\\ f(x_n)\end{pmatrix}.$$

We want $A^{(\varphi)}$ to be nonsingular for all choices of $x_1, \dots, x_n$. This is the case for example if the function φ is strictly positive definite, because then $A^{(\varphi)}$ is positive definite for all sets of distinct points $\{x_1, \dots, x_n\}$. ○

Example 10.2.3 Any n distinct translates of $\varphi(x) := x^{-1}$ span an n-dimensional Haar space on any interval $I \subset \mathbb{R}$ that does not include the translation points $\nu_1, \ldots, \nu_n$. ◇

Proof. For any set of distinct points $x_1, \ldots, x_n \in I$, the Cauchy matrix

Cauchy matrix

$$\begin{pmatrix} \varphi(x_1 - \nu_1) & \ldots & \varphi(x_1 - \nu_n) \\ \vdots & \ddots & \vdots \\ \varphi(x_n - \nu_1) & \ldots & \varphi(x_n - \nu_n) \end{pmatrix} = \begin{pmatrix} \frac{1}{x_1 - \nu_1} & \ldots & \frac{1}{x_1 - \nu_n} \\ \vdots & \ddots & \vdots \\ \frac{1}{x_n - \nu_1} & \ldots & \frac{1}{x_n - \nu_n} \end{pmatrix} =: A$$

is invertible as by [Che66, p. 195],

$$\det(A) = \frac{\prod_{1 \leqslant j < i \leqslant n} (x_i - x_j)(\nu_j - \nu_i)}{\prod_{1 \leqslant i,j \leqslant n} (x_i - \nu_j)} \neq 0,$$

as $\nu_j \notin I$ for all $j \in \{1, \ldots, n\}$ and the $\{x_1, \ldots, x_n\}$ and $\{\nu_1, \ldots, \nu_n\}$ are each distinct. Hence $\left(\varphi(\cdot - \nu_j)\right)_{j=1}^n$ is a Haar system, which spans an n-dimensional Haar space. □

Example 10.2.4 Let $X := \mathbb{R}$ and $\varphi(x) := x^n$. Then any $n+1$ translates of φ span $\mathcal{P}_n$. ◇

Proof. Let $p \in \mathcal{P}_n$ with $p(x) = \sum_{k=0}^n a_k x^{n-k}$. We want to find $b_0, \ldots, b_n \in \mathbb{R}$ such that $p(x) = \sum_{j=0}^n b_j (x - \nu_j)^n$. By the binomial theorem we have

$$\sum_{j=0}^n b_j (x - \nu_j)^n = \sum_{j=0}^n b_j \sum_{k=0}^n \binom{n}{k} x^{n-k} (-\nu_j)^k = \sum_{k=0}^n \left(\binom{n}{k} \sum_{j=0}^n b_j (-\nu_j)^k \right) x^{n-k}.$$

Hence we need to solve

$$\sum_{j=0}^n b_j (-\nu_j)^k = \frac{k!(n-k)!}{n!} a_k, \qquad \forall k \in \{0, \ldots, n\}$$

for $b_0, \ldots, b_n$. The system matrix is a Vandermonde matrix, so a unique solution exists. □

08.02.2022

Theorem 10.2.1: Fourier transform is strictly positive definite

If $f \in L^1(\mathbb{R}^d) \backslash \{0\}$ is continuous and nonnegative, then

$$\hat{f} \colon \mathbb{R}^d \to \mathbb{R}, \qquad \omega \mapsto \int_{\mathbb{R}^d} f(x) e^{-i\omega^\top x} \, \mathrm{d}x$$

is strictly positive definite.

Theorem 10.2.2: Strictly positive definite, Fourier transform

If $f \in L^1(\mathbb{R}^d)$ is continuous, then f is strictly positive definite if and only if f is bounded, $\hat{f} \geqslant 0$ and $\hat{f} \not\equiv 0$.

Corollary 10.2.5 (Gaussians are strictly positive definite)
The Gaussian $\Phi \colon \mathbb{R}^d \to \mathbb{R}$, $x \mapsto e^{-\alpha \|x\|_2^2}$, where $\alpha > 0$, is strictly positive definite by Theorem 10.2.2 as

$$\hat{\Phi}(\omega) = \left(\frac{\pi}{\alpha}\right)^{\frac{d}{2}} e^{-\frac{1}{2} \left\| \frac{\omega}{\sqrt{2\alpha}} \right\|_2^2}.$$

Example 10.2.6 (Interpolation with translates of a Gaussian)
Hence by remark 10.2.2 for data $\left((x_k, f_k)\right)_{k=1}^n \subset \mathbb{R}^d \times \mathbb{R}$ there exist unique $(c_k)_{k=1}^n \subset \mathbb{R}$ such that

$$\sum_{j=1}^n c_j e^{-\alpha\|x_i - x_j\|_2^2} = f_k \qquad \forall k \in \{1, \dots, n\}.$$ ◇

Remark 10.2.7 (Tradeoff: Condition vs. representation)
For large α, the Gaussian Φ is "pointy", so we don't get a nice representation of f but $(\Phi(x_i - x_j))_{i,j=1}^n$ is very close to a diagonal matrix and well-conditioned.

For small $\alpha > 0$, we get a better representation, but a worse interpolation matrix condition. ○

Example 10.2.8 (Using $\sum_{j=1}^n c_j e^{x x_j}$) Any data $\left((x_k, f_k)\right)_{k=1}^n \subset \mathbb{R}^d \times \mathbb{R}$ can be interpolated by

$$g(x) := \sum_{j=1}^n c_j e^{x x_j},$$

since $A := [e^{x_i x_j}]_{i,j=1}^n$ is positive definite: by corollary 10.2.5,

$$0 < [e^{-\|x_i - x_j\|_2^2}]_{i,j=1}^n = [e^{-\|x_i\|_2^2} e^{2 x_i x_j} e^{-\|x_j\|_2^2}]_{i,j=1}^n = DAD,$$ ◇

If $A > 0$ and $\det(B) \neq 0$, then $B^* A B > 0$.

where $D := \operatorname{diag}\left(e^{-\|x_1\|_2^2}, \dots, e^{-\|x_n\|_2^2}\right)$, which is invertible.

10.3 Radial and completely monotone functions

radial

Definition 10.3.1 (Radial function)
A function $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ is radial if

$$\Phi(x) = \Phi(y) \quad \text{for all } x, y \in \mathbb{R}^d \text{ with } \|x\|_2 = \|y\|_2.$$

Hence Φ is radial if there exists a function $\varphi\colon [0, \infty) \to \mathbb{R}$ such that $\Phi = \varphi \circ \|\cdot\|_2$.
Remark 10.3.2 For $d = 1$, all even functions are radial. ○

Example 10.3.3 Gaussians, such as $e^{-\alpha\|x\|_2^2}$ for $\alpha \geqslant 0$ are radial. ◇

We want to interpolate any data in $\mathbb{R}^d$ with linear combinations of translates of a radial function Φ. By remark 10.2.2 it is sufficient if $\Phi = \varphi \circ \|\cdot\|_2$ is strictly positive definite.

As we will see soon, the following condition can be placed on φ such that Φ is (strictly) positive definite.

complete monotone

Definition 10.3.4 (Completely monotone function)
A function $\varphi\colon [0, \infty) \to \mathbb{R}$ is complete monotone if $\varphi \in \mathcal{C}([0, \infty); \mathbb{R}) \cap \mathcal{C}^\infty((0, \infty); \mathbb{R})$ and $(-1)^k \varphi^{(k)}$ is nonnegative for all $k \in \mathbb{N}_0$.

Example 10.3.5 (Completely monotone function) The functions $x \mapsto e^{-\alpha x}$ and $x \mapsto \alpha$ for $\alpha \geqslant 0$, $\ln\left(\frac{\cdot + 2}{\cdot + 1}\right)$ as well as $(\cdot + \beta)^{-\alpha}$ for $\beta > 0$ and $\alpha \geqslant 0$ are completely monotone. ◇

Lemma 10.3.6 (Completely monotone functions closed under $+$, $\cdot$)
The class of completely monotone functions is closed under addition, multiplication and scalar multiplication (like the positive definite functions).

Proof. For two completely monotone functions f and g we have

$$(-1)^n (f \cdot g)^{(n)} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k} \underbrace{(-1)^k f^{(k)}}_{\text{nonnegative}} \underbrace{(-1)^{n-k} g^{(n-k)}}_{\text{nonnegative}}.$$

All other properties are clear or an exercise. □

Theorem 10.3.1: Schoenberg, 1938

A function $\varphi\colon [0,\infty) \to \mathbb{R}$ is completely monotone if and only if

$$\Phi\colon \mathbb{R}^d \to \mathbb{R}, \qquad x \mapsto \varphi(\|x\|_2^2) \tag{55}$$

is positive definite for all $d \in \mathbb{N}_{>0}$.

Proof. See [Wen04, Thm. 7.13, p. 93-94]. □

But only positive definiteness is not enough for interpolation, according to remark 10.2.2 we need *strict* positive definiteness.

Theorem 10.3.2: Strictly positive def. & completely monotone

Let $\varphi\colon [0,\infty) \to \mathbb{R}$. Then $\Phi\colon \mathbb{R}^d \to \mathbb{R}$, $x \mapsto \varphi(\|x\|_2)$ is strictly positive definite for all $d \in \mathbb{N}_{>0}$ if and only if $\varphi \circ \sqrt{\cdot}$ is completely monotone and non-constant.

Proof. See [Wen04, Thm. 7.14, p. 95]. □

Example 10.3.7 Data $\big((x_k, f_k)\big)_{k=1}^n \subset \mathbb{R}^d \times \mathbb{R}$ can be interpolated by functions of the form

$$\sum_{j=1}^{n} \frac{c_j}{\sqrt{1+\|\cdot - x_j\|_2^2}}, \qquad \sum_{j=1}^{n} \frac{c_j}{1+\|\cdot - x_j\|_2^2}, \qquad \sum_{j=1}^{n} c_j e^{-\|\cdot - x_j\|_2^2}, \qquad \sum_{j=1}^{n} c_j e^{\cdot x_j}.$$

In these functions it is easy to add/subtract data points x_j. ◇

Proof.

- We set $\Phi(\cdot - x_j) := \frac{1}{\sqrt{1+\|\cdot - x_j\|_2^2}} \overset{!}{=} \varphi(\|\cdot - x_j\|_2)$, so $\varphi(x) = \frac{1}{\sqrt{1+x^2}}$ and $\varphi(\sqrt{x}) = (1+x)^{-\frac{1}{2}}$ is completely monotone and nonconstant by example 10.3.5. Hence Φ is strictly positive definite by Theorem 10.3.2.
- We set $\Phi(\cdot - x_j) := \frac{1}{1+\|\cdot - x_j\|_2^2} \overset{!}{=} \varphi(\|\cdot - x_j\|_2)$, so $\varphi(x) = \frac{1}{1+x^2}$ and $\varphi(\sqrt{x}) = \frac{1}{1+x}$ is completely monotone and nonconstant by example 10.3.5. Hence Φ is strictly positive definite by Theorem 10.3.2.
- The last two are examples 10.2.6 and 10.2.8. □

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