Technische Universität Berlin

Lecture Notes

Topology

read by Michael Joswig in the winter semester 2021/2022
With 124 figures and commutative diagrams!

Contents

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These notes are neither endorsed by the lecturer nor the university and make no claim to accuracy or correctness.

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0.1 A primer on graph theory (Tutorial 0)

Definition 0.1.1 (Graph, cycle, forest, (spanning) tree)
A undirected finite graph (without multi-edges) is a pair (V, E) of vertices V with $|V| < \infty$ and edges $E \subset \{M \subset \mathcal{P}(V) : |M| = 2\}$. graph

A cycle is a minimal set of edges $e_1, \dots, e_n$ such that $e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n$ is still connected for all $i \in \{1, \dots, n\}$.

A forest is an undirected graph in which any two vertices are connected by at most one path.

A tree is a connected and cycle-free graph. A spanning tree of G is a subgraph of G containing all vertices (but not necessarily all edges). The number of vertices of G is $v(G)$ and the number of edges is $e(G)$.

Lemma 0.1.2
We have $e(T) \geqslant v(T) - 1$ for any connected graph T and equality if and only if T is a tree.

Proof. (1) cf. CoMa II.

(2) " $\implies$ ": Assume $e(T) = v(T) - 1$, then T can't have any closed cycle as a closed cycle has as many edges as it has vertices. Assume there is no path between two vertices, then $e(T) < v(T) - 1$, so T is a connected without cycles, so it is a tree.

" $\impliedby$ ": Let $T = (V, E)$ be a tree. As T doesn't have closed cycles, $e(T) \leqslant v(T) - 1$. If $e(T) < v(T) - 1$, T would not be connected.

Solution from the Tutorials: Perform induction over the number of edges. The base case consists of two nodes being connected by one edge; the property is fulfilled. Induction step: if $e(T) = n$, then there must be at least one leaf. Remove this leaf. □

Lemma 0.1.3
Any connected undirected graph G contains a spanning tree.

Proof. Just delete edge which doesn't make graph disconnected (for every cycle, delete one edge from it), as $e \leqslant 2^v$, this process terminates.

Solution from the Tutorials: Let $e_1, \dots, e_n$ be the edges of G. Kruskal's algorithm finds a minimal spanning tree. Sort edges such that $e_1 < \dots < e_n$ (in terms of cost). Start with $T_0 = (V, E_0)$, where V are the vertices of G and $E_0 = \emptyset$. Then perform the iteration $E_{i+1} = E_i \cup \{e_i\}$ if $E_i \cup \{e_i\}$ is cycle-free, else $Ei + 1 = E_i$. Then $T = (V, E_m)$ is a spanning tree and the algorithm has run-time $O(m + n)$, where $n := |V|$. □

Exercise: Give a set-theoretic (inner and outer) description of the cube and the octahedron from the lecture. Solution: One can write the octahedron, which has six vertices, as $\operatorname{conv}(\pm e_1, \pm e_2, \pm e_3)$. The cube can be written as $[-1, 1]^3$. The octahedron and cube are dual to each other.

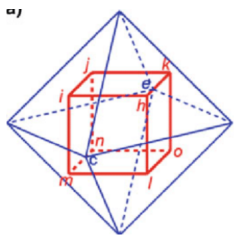

Fig. 1: The dual graph of the octahedron (seen as a cell complex) is a cube.

The dual graph of some polyhedron or cell complex can obtained by the following:

- nodes $\leftrightarrow$ maximal cells of the polyhedron
- edges $\leftrightarrow$ two adjacent maximal cells, that is, two maximal cells sharing one codim?

1 Introduction

1.1 Euler's Theorem

This course will follow [1].

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The EUCLIDEAN space $\mathbb{E}^n$ is $\mathbb{R}^n$ equipped with the topology induced by $\|x\|^2 := \sum_{k=1}^n x_k^2$. Our first notion of a *shape* is a polyhedron, which is pieced together by objects that are easy to understand.

DEFINITION 1.1.1 (POLYHEDRON, FACE, EDGE, (DEGREE OF) VERTEX)
A polyhedron is a collection of finitely many convex planar polygons, so called faces, in $\mathbb{E}^3$ such that

- each edge is contained in *exactly* two faces,
- at each vertex, we can label the incident faces $Q_1, \ldots, Q_n$ such that $Q_i \cap Q_{i+1}$ and $Q_n \cap Q_1$ are edges for $i \in \{1, \ldots, n-1\}$. The number n is the degree of the vertex.

polyhedron

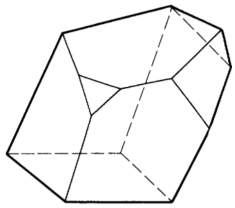

Fig. 2: A nonconvex polyhedron with convex faces.

The surface in figure 2 is a polyhedron.

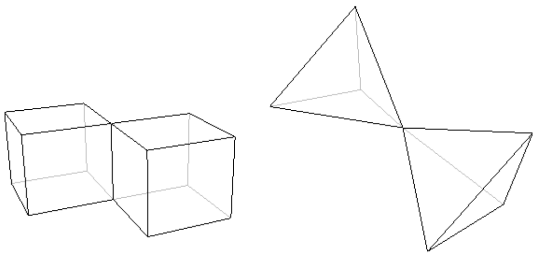

Fig. 3: These objects are not polyhedra because an edge is contained in more than two faces (on the left) or incident faces to the central vertex can't be labelled as in Definition 1.1.1 (on the right). [Source]

DEFINITION 1.1.2 (EULER CHARACTERISTIC)
Let v, f and e be number of vertices, faces and edges of a polyhedron P respectively. The EULER characteristic of P is

$$\chi(P) := v - e + f.$$

EULER characteristic

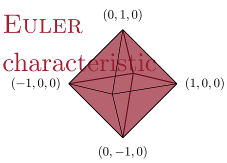

Fig. 4: An octahedron.

Example 1.1.3 (Octahedron) Consider the octahedron which has 6 vertices, 12 edges and 8 faces, so its EULER characteristic is $\chi(P) = 6 - 12 + 8 = 2$. ◇

The following is the first theorem in topology, period.

THEOREM 1.1.1: EULER'S POLYHEDRON THEOREM (1750)

Let P be a polyhedron such that

1. any two vertices of P can be joined by a chain of edges (connectedness),
2. any (not self-intersecting) loop on P, which consists of straight line segments (not necessarily edges) separates P into two pieces (genus zero).

The EULER characteristic of P is equal to 2.

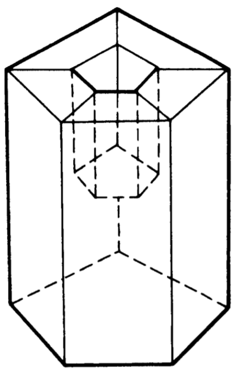

Fig. 5: The prism over a pentagon with a smaller pentagon pushed through has 20 vertices, 35 edges and 17 faces and thus

The JORDAN curve theorem (see later) can and should be used to prove Theorem 1.1.1.

Counterexample 1.1.4 (Polyhedra with EULER characteristic not equal to two) Consider (the surface of) a cube with a smaller cube removed from its interior, then condition (1) is violated. This object has 16 vertices, 24 edges and 12 faces, so its EULER characteristic is $\chi(P) = 16-24+12 = 4$. The object in figure 7 has EULER characteristic $20-40+20 = 0$ and violates condition (2). ◇

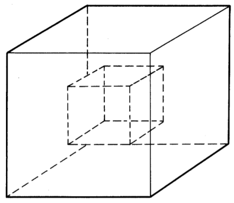

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Fig. 6: A cube with a smaller cube removed from its interior.

We will later see a proof with all the details. For now, to get some intuition, we will only sketch it. However, every step in this intuitive proof can be made rigorous with means learned later.

Proof. (Sketch) The vertices and the edges of the polyhedron P form a graph G. By condition (1), G is a connected graph. Hence G has a spanning tree T by lemma 0.1.3.

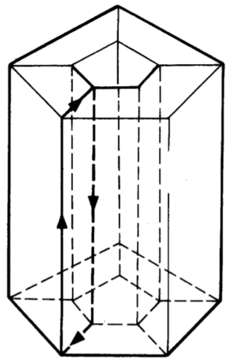


Fig. 7: A prism with a hole straight through the centre.

Define (another) abstract graph Γ, where

- a face A of P yields a node $\hat{A}$ of Γ
- $\hat{A}$ and $\hat{B}$ span an edge of Γ if the corresponding edge A and B share an edge of P, which is not contained in the spanning tree T.

Now one can show that Γ is connected (using that T is a tree). The edges of Γ correspond to the "non-edges" of T and the tree is "too slim" to disconnected Γ. Moreover, condition (2) forces that Γ does not contain any topological loops, so Γ is a tree.

Hence the polyhedron P "decomposes" into two trees, T and Γ. As T is a spanning tree, $v = v(T)$. Every edge of the polyhedron either contributes to T or it is responsible for an edge of Γ. The faces of P correspond to the the nodes of Γ. In summary,

$$v - e + f = v(T) - \big(e(T) + e(\Gamma)\big) + v(\Gamma) = v(T) - e(T) + v(\Gamma) - e(\Gamma) = 1 + 1 = 2,$$

as $v(G) - e(G) = 1$ for all trees G. □

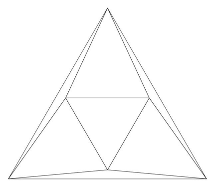


Fig. 8: The SCHLEGEL diagram of an octahedron.

To provide a planar picture (SCHLEGEL diagram) of the octahedron, the "outside" plane also represents a face.

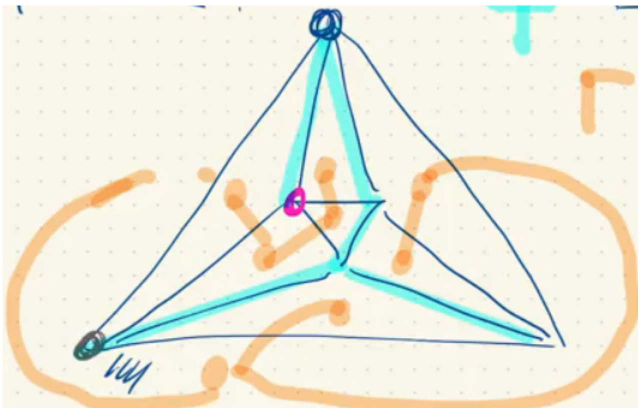


Fig. 9: SCHLEGEL diagram for the octahedron together with the trees T and Γ from the of Theorem 1.1.1.

1.2 Topological equivalence

We will have different notions making the word *shape* precise and the polyhedron is the first one. We will also have different notions of topological equivalence and the following is a first one.

Definition 1.2.1 (Global continuity)
Let $X \subset \mathbb{E}^M$ and $Y \subset \mathbb{E}^N$. A function $f\colon X \to Y$ is continuous if for all $x \in X$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x' \in X$ we have that $\|x' - x\| < \delta$ implies $\|f(x') - f(x)\| < \varepsilon$.

Definition 1.2.2 (Homeomorphism)
A function $f\colon X \to Y$ is a homeomorphism if f is bijective and f as well as f^{-1} are continuous. We then write $X \approx Y$ and say that X and Y are homeomorphic.

homeomorphism

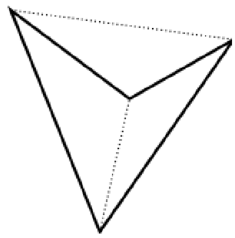


Fig. 10: Dividing a nonconvex quadrilateral into two convex triangles.

Being homeomorphic is a symmetric property.

Remark 1.2.3 Notice that both a convex and a nonconvex planar polygon are homeomorphic to the planar unit disk $\mathbb{B}^2 := \{x \in \mathbb{E}^2 : \|x\| \leqslant 1\}$. But we can divide the nonconvex planar quadrilateral into two convex triangles. The first shape has four vertices, four edges and one face, so its Euler characteristic is $4 - 4 + 1 = 1$, while the second shape has four vertices, five edges and two faces, so its Euler characteristic is also 1. This is not a coincidence.

Therefore it "suffices" to consider only convex polygons in the definition of a polyhedron. ○

Example 1.2.4 (Tetrahedron and unit sphere are homeomorphic)
Consider the surface/boundary of a tetrahedron Θ (a platonic solid, the convex hull of four points which are affinely independent in $\mathbb{E}^3$) and the unit sphere $\mathbb{S}^2 := \{x \in \mathbb{E}^3 : \|x\| = 1\}$.

The radial projection $f\colon \Theta \to \mathbb{S}^2$, $t \mapsto \frac{t}{\|t\|}$ (connect a point $t \in \Theta$ to the origin and extend that ray until it intersects $\mathbb{S}^2$) is a homeomorphism. ◇

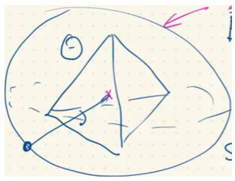


Fig. 11: Radial projection from a tetrahedron onto the unit sphere, which is a homeomorphism.

This also works for any convex polytope whose interior contains the origin, so up to translation, for every convex polytope, and also in higher dimensions.

Let us connect the homeomorphism with the construction in the proof of Euler's theorem. We can decompose the tetrahedron Θ into T and Γ as follows:

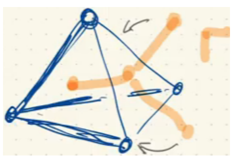


Fig. 12: The grey arrows show to which faces the orange nodes correspond.

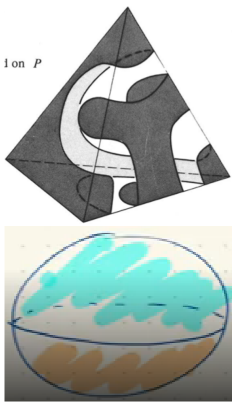


Fig. 13: The thickening of the graphs and their homeomorphic counterparts on $\mathbb{S}^2$.

Consider "thickening up" T and Γ, creating two, in some way two-dimensional, closed disks homeomorphic to $\mathbb{B}^2$ (which have nontrivial intersection), giving a cell decomposition of the sphere $\mathbb{S}^2$. Thickening a tree always gives a disc, though thickening a graph with loops will give a space with holes in it.

This should be compared with the following: The unit sphere $\mathbb{S}^2 \subset \mathbb{E}^3$ has an equator, which divides it into two hemispheres, which each are homeomorphic to $\mathbb{B}^2$.

Example 1.2.5 Consider the following four shapes.

1. An open cylinder in $\mathbb{E}^3$ (can be viewed as product of open interval and a circle) without its boundary.

2. A one-sheeted hyperboloid, all points $(x, y, z) \in \mathbb{E}^3$ with $x^2 + y^2 - z^2 = 1$, whose horizontal cross-sections are circles and whose vertical cross-sections are hyperbolas.
3. An open annulus consists of all points $(x, y) \in \mathbb{E}^2$ with $1 < x^2 + y^2 < 3$.
4. A sphere $\mathbb{S}^2$ without the north and south pole, $(0, 0, \pm 1)$.

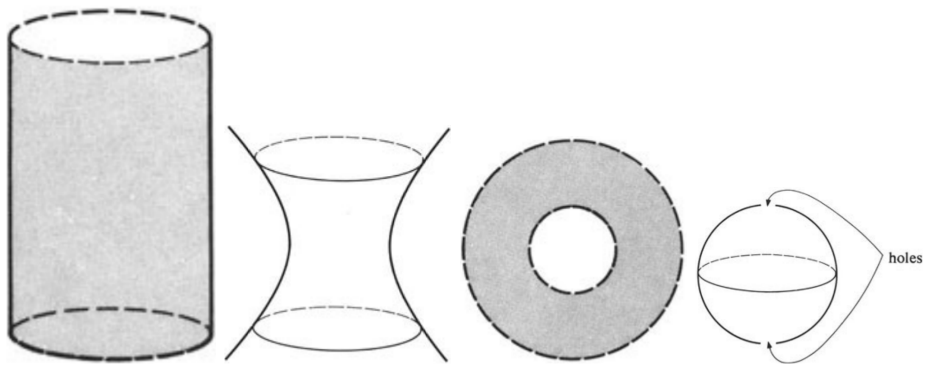


Fig. 14: An open cylinder, one-sheeted hyperboloid, an open annulus and the doubly punctured 2-sphere in $\mathbb{E}^3$.

Here, the notion of *shape* is just "open subset of Euclidean space", there is not polyhedral structure. Any two of the above shapes are homeomorphic.

We only show that 2 $\approx$ 3. We write the points of 2 in cylindrical polar coordinates (r, ϑ, z), where z specifies the height, r the radius of the circular cross section at that height and $\vartheta \in [0, 2\pi)$ the angle of the point. There is a equation relating r, ϑ and z (Homework).

We write the points on 3 in polar coordinates (r', ϑ'). Define

$$f\colon (-\infty, \infty) \to (1, 3), \qquad x \mapsto \frac{x}{1 + |x|} + 2.$$

and

$$\eta\colon 2 \to 3, \qquad (r, \vartheta, z) \mapsto (f(z), \vartheta).$$

One can check that this is a homeomorphism. The function f is continuous as the composition of continuous maps and a bijection. ◇

1.3 Surfaces

The point of this section is to provide sufficiently interesting examples of topological spaces and topological methods.

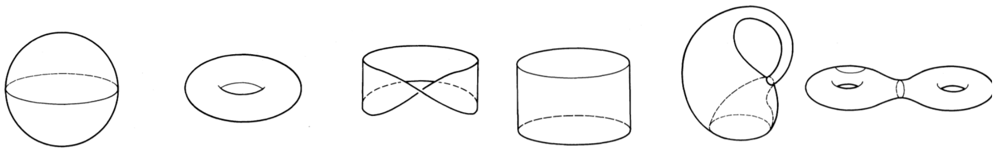
The following definition still depends on the embedding into $\mathbb{E}^n$.

Definition 1.3.1 (Surface (Version 1))
A surface is a subset $S \subset \mathbb{E}^n$ such that each point has a neighbourhood (the intersection of an open ball in $\mathbb{E}^n$ intersected with S) which is homeomorphic to the closed disk $\mathbb{B}^2$. surface

Beware that these surfaces may have a boundary.

Example 1.3.2 (Surfaces)
Surfaces include the sphere $\mathbb{S}^n$, the torus $\mathbb{S}^1 \times \mathbb{S}^1$, the Möbius strip, the cylinder $\mathbb{S}^1 \times [0, 1]$, the Klein bottle, which can be embedded in $\mathbb{E}^4$ without self-intersections, but not into

$\mathbb{E}^3$ without self-intersections (this is a topological result), the punctured double torus and polyhedra.



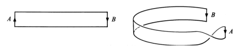
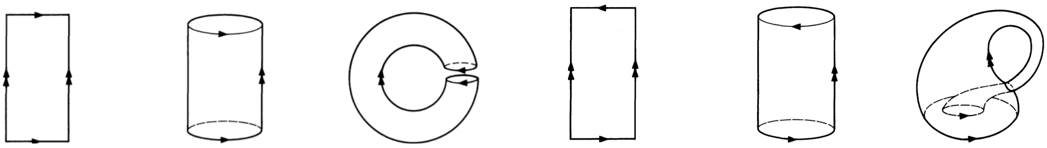


Fig. 15: A Möbius band can be obtained by identifying opposite edges with opposite orientations.

We can construct surfaces by taking a planar polygon and identifying its edges in a suitable way:



Note that the notion of a homeomorphism does not depend on the embedding of the surface. We can also twist the ends of the Möbius strip three times but then there does not exist a homeomorphism of $\mathbb{E}^3$.

The torus is homeomorphic to the pentagon-pushthrough.)

1.4 Abstract spaces

Definition 1.4.1 (Neighbourhood in $\mathbb{E}^n$)
Let $p \in \mathbb{E}^n$. A set $N \subset \mathbb{E}^n$ is a neighbourhood of p in $\mathbb{E}^n$ if there exists a $r > 0$ such that the closed n-ball/disk $B^n(p, r) := \{x \in \mathbb{E}^n : \|x - p\| \leqslant r\}$ is contained in N.

Lemma 1.4.2 (Continuity in Euclidean spaces)
A function $f\colon \mathbb{E}^m \to \mathbb{E}^n$ is continuous if and only if for all $x \in \mathbb{E}^m$ and for all neighbourhoods N of $f(x) \in \mathbb{E}^n$, the full preimage $f^{-1}(N)$ is a neighbourhood of $x \in \mathbb{E}^m$.

Definition 1.4.3 (Topological space, Neighbourhood, Topology)
A topological space is a set X equipped with a nonempty collection $\mathcal{N}_x$ of subsets of X (called neighbourhoods) for each $x \in X$ such that

topological
neighbourhood

1. $x \in N$ for all $N \in \mathcal{N}_x$,
2. $N \cap N' \in \mathcal{N}_x$ for all $N, N' \in \mathcal{N}_x$,
3. for all $N \in \mathcal{N}_x$ and all $U \subset X$, $N \subset U$ implies $U \in \mathcal{N}_x$ ("neighbourhoods are large enough"),
4. for all $N \in \mathcal{N}_x$, their interior $\mathring{N} := N^\circ := \{z \in N : N \in \mathcal{N}_z\}$ is in $\mathcal{N}_x$.

The collection $(\mathcal{N}_x)_{x \in X}$ is a topology on X.

topology

We now take lemma 1.4.2 as definition for continuity of maps between topological spaces.

Definition 1.4.4 (Continuity in topological spaces)
A map $f\colon X \to Y$ between topological spaces is continuous if for all $x \in X$ and for all neighbourhoods $N \in \mathcal{N}_{f(x)}$, we have $f^{-1}(N) \in \mathcal{N}_x$.

Example 1.4.5 (Euclidean topology)
The neighbourhoods from Definition 1.4.1 define a topology on $\mathbb{E}^n$ (for the last property, halve the radius). ◇

Example 1.4.6 (Subspace topology)
Let X be a topological space and $Y \subset X$ a subset. For $y \in Y$ and $N \in \mathcal{N}_y \subset X$ we declare $N \cap Y$ to be a neighbourhood of y with respect to Y. This is the subspace topology induced by Y on X. subspace topology

(Exercise: check axioms) ◇

Surfaces inherit the subspace topology from $\mathbb{E}^n$.

Example 1.4.7 (Metric topology)
Let (X, d) be a metric space. For $\varepsilon > 0$ and $x \in X$, let $B(x, \varepsilon) := \{y \in X : d(x, y) \leqslant \varepsilon\}$ Then $N \subset X$ is a neighbourhood of x in X if there exists a $\varepsilon > 0$ such that $B(x, \varepsilon) \subset N$. This is the metric topology on X (e.g. $\mathbb{E}^n$). ◇

Example 1.4.8 (Cofinite topology)
One set can carry different topologies. Consider $X := \mathbb{R}$ where N is a neighbourhood of $x \in \mathbb{R}$ if $x \in N$ and $\mathbb{R} \setminus N$ is finite. This is the cofinite topology (can be defined on any set), which is distinct from the Euclidean topology, that is, there exists no homeomorphism between $\mathbb{R}$ equipped with the cofinite topology and $\mathbb{R}$ equipped with the Euclidean topology.

TODO: proof ◇

Definition 1.4.9 (Surface w/o boundary (coordinate free version))
A surface without boundary is a topological space such that

- each point has a neighbourhood which is homeomorphic to (an open disc in) $\mathbb{E}^2$,
- any two points have disjoint neighbourhoods (Hausdorff property)

A closed surface - a 2-manifold without boundary - is either orientable (e.g. torus) or non-orientable (e.g. Klein bottle).

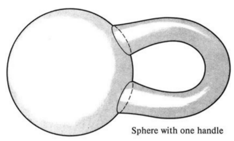


Fig. 16: The sphere with a handle is obtained by cutting two holes into the sphere and attaching a cylinder and has genus 1 (is hence homeomorphic to the torus).

1.5 Classification of surfaces

Construction: start with some surface, e.g. $\mathbb{S}^2$. We can now produce new surfaces by local modifications.

Definition 1.5.1 (Attaching a handle / cross-cap)
For a surface S,

- attaching a handle means removing two disjoint disks and gluing in a cylinder C such that the two bounding circles of C are glued to the boundary of circles of the two disjoint circles.
- attaching a cross-cap meaning removing a disk and gluing in a Möbius strip (notice that the boundary of a Möbius strip is a circle).

The sphere $\mathbb{S}^2$ with a glued in Möbius strip is homeomorphic to $\mathbb{RP}^2$ and $\mathbb{S}^2$ with two glued in Möbius strips is homeomorphic to the Klein bottle.

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Definition 1.5.2 (Closed surface)
A surface is closed if it is connected and compact (and without boundary).

Example 1.5.3 (Closed surface) Any polyhedral surface is closed. ◇

The following theorem comes in many forms in the literature and we will prove it later.

Theorem 1.5.1: Surface Classification Theorem

Any closed surface is homeomorphic to either $\mathbb{S}^2$ or $\mathbb{S}^2$ with finitely many handles attached or $\mathbb{S}^2$ with finitely many cross-caps attached.
No two of these surfaces are homeomorphic.

The surfaces with handles attached and those with cross-caps are distinguished by orientability (which is related to the embeddability into $\mathbb{E}^3$).

(A sphere with n handles is orientable and has genus n, so its Euler characteristic is $2-2n$.)

2 Continuity

Our goal for the next few sections is now to prove Theorem 1.5.1. We consider the following more abstract approach to topology.

2.1 Open and closed sets

Let X be a topological space (we henceforth identify the set with the topological space). The neighbourhood axioms give us a notion of open sets.

Definition 2.1.1 (Open set)
A subset $O \subset X$ is open if $O \in \mathcal{N}_x$, that is, O is a neighbourhood of x for all $x \in O$. open

Why did we even define neighbourhood systems? Continuity with respect to the neighbourhoods is a local property, similar to the continuity definition in $\mathbb{E}^n$. It is a special property that a map is continuous if and only if it is continuous at every point (this is for example not true for boundedness of functions).

Definition 2.1.2 (Hausdorff property)
A topological space X is Hausdorff if for any two distinct points $x, y \in X$ there exists disjoint open neighbourhoods $U_x \in \mathcal{N}_x$ and $U_y \in \mathcal{N}_y$ of x and y, respectively. Hausdorff

Lemma 2.1.3 (Properties of open sets)
Let I be an index set such that $O_i \subset X$ is an open set for all $i \in I$. Then

1. $\bigcup_{i\in I} O_i$ *is open,*
2. $\bigcap_{i\in I} O_i$ *is open, provided I is finite,*
3. *the sets $\varnothing$ and X are open.*

Proof. 1 Let $x \in \bigcup_{i\in I} O_i$. Then there exists a $i \in I$ such that $x \in O_i$. As O_i is open, it is a neighbourhood of x. By 3, $\bigcup_{i\in I} O_i \supset O_i$ is a neighbourhood of x as well.

2 Let $x \in \bigcap_{i\in I} O_i$. Then $x \in O_i$ for all $i \in I$. As O_i is open, it is a neighbourhood of x. By 2, $\bigcap_{i\in I} O_i$ is a neighbourhood of x.

3 The empty set is trivially open. The whole space X is open, as each point has a neighbourhood. □

Remark 2.1.4 If N is a neighbourhood of $x \in X$, then $\mathring{N}$ is open by construction. ○

Example 2.1.5 (Open sets in $\mathbb{E}^n$) In $\mathbb{E}^3$, a set is open if and only if it contains a (small) open ball around every point. Examples include

- linear halfspaces defined by a vector $z \in \mathbb{E}^3 \setminus \{0\}$: $\{x \in \mathbb{E}^3 : \langle x, z \rangle > 0\}$,
- open balls $\{(x, y, z) \in \mathbb{E}^3 : x^2 + y^2 + z^2 < 1\}$. ◇

The property concerning intersections assumes the family of sets to be intersected to be finite and this is necessary as the following example shows.

Example 2.1.6 The intersection of infinitely many open sets does not need to be open: consider $O_\varepsilon := (0, 1+\varepsilon)$ for $\varepsilon > 0$, which are open in $\mathbb{R}$. Then

$$\bigcap_{\varepsilon>0} O_\varepsilon = (0,1]$$

is not open as a subset of $\mathbb{R}$ because $1 \in (0,1]$ does not have neighbourhood which is contained in $(0,1]$ (as this would require an open interval with 1 in the interior to be a subset of $(0,1]$). (Equipping $(0,1]$ with the subspace topology from $\mathbb{R}$, the entire set $(0,1]$ is an open subset of itself.) ◇

The conditions of lemma 2.1.3 can be turned into a definition of a topological space: we can ask of a family of subsets to satisfy these axioms. From this, we can cook up the definition of a neighbourhood as a set containing the point and an open set.

Lemma 2.1.7 (Definition of a topology with open sets)
Let T be a family of subsets of a set X. We call the elements of T open and we require the properties from lemma 2.1.3. A set $N \subset X$ is a neighbourhood of $x \in X$ if there exists an open set $O \in T$ such that $x \in O \subset N$. Then the collection of neighbourhoods T is a topology on X.

Proof. Exercise. □

Definition 2.1.8 (Subspace topology (open sets definition))
Let X be a topological space and $Y \subset X$. The open sets of Y with respect to the subspace topology on X are precisely the sets $O \cap Y$, where O is open in X.

Example 2.1.9 (Subspace topology on $[0,1)$ and $\mathbb{S}^1$) The open sets in $[0,1)$ with the subspace topology induced by $\mathbb{R}$ are arbitrary unions of the sets (a,b) with $0 \leqslant a < b \leqslant 1$ and $[0,b)$ with $b \in [0,1]$ and the open sets in $\mathbb{S}^1$ equipped with the subspace topology induced by $\mathbb{R}^2$ are arbitrary unions of open circle segments. ◇

Example 2.1.10 (Continuous bijection without noncontinuous inverse)
Let $C := \{z \in \mathbb{C} : |z| = 1\} \approx \mathbb{S}^1$. Then $f\colon [0,1) \to C$, $x \mapsto e^{2\pi i x}$ is bijective and continuous as the composition of continuous maps, but f^{-1} is not continuous: the set $[0, \frac{1}{2})$ is open in $[0,1)$, but $(\varphi^{-1})^{-1}([0,1)) = \varphi([0,1))$ is an half-open circle segment, which is not open in $\mathbb{S}^1$, as there is no neighbourhood of $\varphi(0)$ fully contained in $\varphi([0,1))$. ◇

This doesn't mean that $[0,1)$ and $\mathbb{S}^1$ are not homeomorphic. But as compactness (like being Hausdorff or (path)connected) is a topological invariant (that is, preserved under homeomorphism), these two spaces can't be homeomorphic as $[0,1)$ is not compact, but $\mathbb{S}^1$ is.

Similarly, $[0,1]$ is not homeomorphic to $\mathbb{S}^1$.

Proof. Suppose there was a continuous bijection $f\colon [0,1] \to \mathbb{S}^1$ with continuous inverse. Pick $x_0 \in (0,1)$. Then $f|_{[0,1]\setminus\{x_0\}} \to \mathbb{S}^1\setminus\{f(x_0)\}$ is still a homeomorphism, as the restriction of continuous/bijective functions remains continuous/bijective. The set $[0,1]\setminus\{x_0\}$ is not connected, while the set $\mathbb{S}^1\setminus\{f(x_0)\}$ is even path connected. As the continuous image of a connected set is connected, this is a contradiction to $(f|_{[0,1]\setminus\{x_0\}})^{-1}$ being continuous. □

Remark 2.1.11 (Topological invariants) Properties like compactness or connectedness are binary; a space is either connected or not. For more subtle properties, Topology gives an answer in terms of (homotopy and homology) groups. ○

Example 2.1.12 (Metric topology) Let (X, d) be a metric space with the metric topology $T := \{O \subset X : \forall x \in O \exists \varepsilon > 0 : x \in B_\varepsilon(x) \subset O\}$. Then the subspace topology $T|_U$ for some subset $U \subset X$ is metrisable, as $d|_{U \times U}$ is a metric on U. ◇

metric topology

We can define a topology on any set, so being a topological space is a void statement.

Definition 2.1.13 (Discrete topology)
Let X be a set and define *every* $O \subset X$ to be open. This is the discrete topology on X.

discrete topology

Closed sets

Definition 2.1.14 (Closed)
A subset $A \subset X$ is closed if its complement $X \backslash A$ is open.

closed

Example 2.1.15 (Closed sets) Let $X := \mathbb{E}^2$. The sets $\mathbb{S}^1$, $\mathbb{B}^2$ and $\{(x, y) \in \mathbb{E}^2 : x \geqslant y^2\}$ are closed in $\mathbb{E}^2$, while the set $\{(x, y) \in \mathbb{E}^2 : x > y^2\}$ is open. ◇

Example 2.1.16 (Non-example) Let $X := \mathbb{E}^1$. Then the set $[0, 1) := \{x \in \mathbb{R} : 0 \leqslant x < 1\}$ is neither open nor closed in X. Hence the EUCLIDean topology on $\mathbb{E}^1$ is distinct from the discrete topology on $\mathbb{E}^1$. ◇

Hence we can create another set of axioms by passing to complements:

Lemma 2.1.17 (Properties of closed sets)
Let I be an index set such that A_i is an closed set for all $i \in I$. Then

1. *$\bigcap_{i \in I} O_i$ is closed,*
2. *$\bigcup_{i \in I} O_i$ is closed, provided I is finite,*
3. *the sets $\varnothing$ and X are closed.*

Proof. Follows from lemma 2.1.3 by using DEMORGAN's rules for exchanging complements with intersections and unions. □

2.2 Limit points

We know limit points from the beginning of our Analysis I lecture, but there is a more general viewpoint expressed in abstract topology.

Let X be a topological space.

Definition 2.2.1 (Limit point)
Let $A \subset X$ be a subset. A point $x \in X$ is a limit point (or accumulation point) of A if $(A \backslash \{x\}) \cap N \neq \varnothing$ for all neighbourhoods $N \in \mathcal{N}_x$ of x.

limit point

Remark 2.2.2 (Intuition for limit points) Let X be a metric space, $A \subset X$ and x a limit point of A. Consider the balls $B_{\frac{1}{n}}(x)$ with centre $x \in X$ and radii $\frac{1}{n}$ for $n \in \mathbb{N}_{>0}$. As $A \backslash \{x\} \cap B_{\frac{1}{n}}(x) \neq \varnothing$, there exists a sequence $\left(x_n \in A \backslash \{x\} \cap B_{\frac{1}{n}}(x)\right)_{n \in \mathbb{N}} \subset A$ that converges to x. Hence limit points are limits of sequences in A. ○

Example 2.2.3 (Limit points in $\mathbb{E}$) Let $X := \mathbb{E}^1$ and $A := \{\frac{1}{n} : n \in \mathbb{N}_{>0}\}$. Then $0 \notin A$ is the only limit point of A (in Calculus we wrote $\lim_{n\to\infty} \frac{1}{n} = 0$).

Now let $A := [0, 1)$. Each point of A is a limit point of A and additionally, $1 \notin A$ also is. ◇

Example 2.2.4 (Limit points in $\mathbb{E}^n$) Let $X := \mathbb{E}^n$. Let $A \subset X$ be the set of points with rational coordinates. Then $\mathbb{E}^n$ is the set of limit points of A.

If A is the set of points with integer coordinates, which has the same cardinality as the previous set A, then there are no limit points of A. ◇

Definition 2.2.5 (Convergence)
A sequence $(a_n)_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ if for all $O \in \mathcal{N}_x$ there exists a $N_O \in \mathbb{N}$ such that $a_n \in O$ for all $n \geqslant N_O$.

Theorem 2.2.1: Closed sets and limit points

A set is closed if and only if it contains all its limit points.

Proof. " $\implies$ ": Let A be closed. Then $X\backslash A$ is open, so $X\backslash A \in \mathcal{N}_x$ for all $x \in X\backslash A$. Hence no point in $X\backslash A$ can be a limit point of A: suppose $x \in X\backslash A$ were a limit point of A, then $A\backslash\{x\} \cap N = A \cap N \neq \varnothing$ for all $N \in \mathcal{N}_x$, but this is not true for $N = X\backslash A \in \mathcal{N}_x$. Thus A contains all its limit points.

" $\impliedby$ ": Suppose A contains all its limit points. Let $x \in X\backslash A$. Then x is not a limit point. Hence there exists a neighbourhood $N \in \mathcal{N}_x$ such that $A \cap N = \varnothing$, so $N \subset X\backslash A$. Hence $X\backslash A$ is a neighbourhood of each of its points, so it is open, hence A is closed. □

Example 2.2.6 (Limit points in the cofinite topology) Let $T \subset P(X)$ be comprised of complements of finite subsets and the emptyset. We first show that T is a topology.

1. We have $\varnothing \in T$ by definition and also $X \in T$, as the complement of X is empty and thus finite.
2. Let $O_i = X\backslash F_i$ be in T, where $F_i \subset X$ is finite for all $i \in I$. Then $O_i \cap O_j = X\backslash(F_i \cup F_j) \in T$ as $F_i \cup F_j$ is finite for $i, j \in I$.
3. Furthermore, $\bigcup_{i\in I} O_i = X\backslash \bigcap_{i\in I} F_i \in T$, as $\bigcap_{i\in I} F_i \subset F_j$ is finite.

Let $A \subset X$ and $x \in X$ be a limit point of A. Then every open set U with $x \in U$ intersects $A\backslash\{x\}$. Hence $A\backslash\{x\} \not\subset F$ for any finite set $F \subset X$. Hence if A is infinite, all points in X are limit points of A. We have $A\backslash\{x\} \subset A$, so if A is finite, then it has no limit point. ◇

2.3 Closures and bases

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Definition 2.3.1
The closure of $A \subset X$, $\overline{A}$, is the union of all its limit points.

The set $\overline{A}$ is the smallest closed set containing A.

Proof. (1) We first show that $\overline{A}$ is closed. Let $x \in X \backslash \overline{A}$. Then x is not a limit point of A, so there exists an open neighbourhood $U \in \mathcal{N}_x$ such that $(A \backslash \{x\}) \cap U = A \cap U = \varnothing$. As U is open, U is a neighbourhood for any of its points. Suppose that $U \cap \overline{A} = U \cap (\overline{A} \backslash A) \neq \varnothing$, i.e. there exists a $y \in U$, which is a limit point of A. Then $(A \backslash \{y\}) \cap U = A \cap U \neq \varnothing$, as $U \in \mathcal{N}_y$, which contradicts $U \cap A = \varnothing$. Hence $U \cap \overline{A} = \varnothing$, so $U \subset X \backslash \overline{A}$. Hence $X \backslash \overline{A}$ is open, so $\overline{A}$ is closed.

(2) Let B be closed with $A \subset B$. As $\overline{A}$ is closed by (1), every limit point of A lies in B, so $\overline{A} \subset B$. $\square$

Corollary 2.3.2 (Characterisation of closed sets)
A set $A \subset X$ is closed if and only if $A = \overline{A}$.

DEFINITION 2.3.3 (DENSE, INTERIOR, FRONTIER)
Let $A \subset X$. Then A is dense if $\overline{A} = X$. The interior of A, $\mathring{A}$, is the union of all open sets contained in A. The frontier of A is $\overline{A} \cap \overline{X \backslash A}$.

interior
frontier

The frontier of the whole space X is empty.

Example 2.3.4 The sets $\mathbb{Q} \subset \mathbb{R}$ and $(\mathbb{B}^2)^\circ \subset \mathbb{B}^2$ are dense. As

$$\mathbb{S}^1 = \mathbb{B}^2 \cap \overline{\mathbb{E}^2 \backslash \mathbb{B}^2} = \mathbb{S}^1 \cap \overline{\mathbb{E}^2 \backslash \mathbb{S}^1},$$

$\mathbb{S}^1$ is the frontier of both $\mathbb{S}^1$ and $\mathbb{B}^2$. $\diamond$

Lemma 2.3.5 (Second countable $\implies$ separable)
If X has a countable basis, then it is separable - it contains a countable dense subset.

Proof. Let $B := (B_n)_{n \in \mathbb{N}} \subset P(X)$ be a basis of X. For each $B_n \in B$ choose any $x_n \in B_n$ and set $D := (x_n)_{n \in \mathbb{N}}$, which is countable. To show that D is dense in X, we show that it intersects every open subset of X. Let $O \subset X$ be open. Then we can write $O = \bigcup_{j \in J} B_j$ for some subset $J \subset \mathbb{N}$. Hence $O \cap D = \{x_n : n \in J\} \neq \varnothing$. $\square$

The product space $\{f : X \to R\} =: \mathbb{R}^X$ equipped with the product topology is not separable.

Only using that $\overline{A} = \bigcap_{\substack{A \subset B \subset X \\ B \text{ closed}}} B$, we can show the following lemma.

Lemma 2.3.6 (Characterisation of the closure)
We have $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.

Corollary 2.3.7 (Characterisation of the frontier)
A point $x \in X$ lies in the frontier of A if and only if every neighbourhood of x intersects A and $X \backslash A$.

Proof. (of lemma 2.3.6) "$\implies$": Let $x \in \overline{A}$. Then $x \in B$ for all closed B with $A \subset B \subset X$. Let $U \subset X$ be a neighbourhood of x, then there exists an open set O with $x \in O \subset U$. Suppose $U \cap A = \varnothing$. Then $O \cap A = \varnothing$. Hence $X \backslash O$ is a closed set containing A, but $x \notin X \backslash O$, which is a contradiction.

"$\impliedby$": Let $x \in X$ such that $N \cap A = \varnothing$ for all $N \in \mathcal{N}_x$. We have to show that for all closed sets B with $A \subset B$ we have $x \in B$. Let B with $A \subset B$ be a closed set. Suppose $x \in X \backslash B$. As B is closed, $X \backslash B$ is open, so $X \backslash B \in \mathcal{N}_x$. Hence $(X \backslash B) \cap A \neq \varnothing$, which contradicts $A \subset B$. $\square$

Lemma 2.3.8 (Closure and the subspace topology)
Let $A \subset Y \subset X$ and let $\overline{A}^Y$ be the closure of A in Y and $\overline{A}^X$ be the closure of A in X. Then $\overline{A}^Y = \overline{A}^X \cap Y$.

Proof. (My alternative) Let $\tilde{B} \subset Y$ be closed. Then there exists a closed set $B \subset X$ such that $\tilde{B} = B \cap Y$. Hence

$$\overline{A}^Y = \bigcap_{\substack{A \subset \tilde{B} \subset Y \\ \tilde{B} \text{ closed}}} \tilde{B} = \bigcap_{\substack{A \subset B \subset X \\ B \text{ closed}}} B \cap Y = \overline{A}^X \cap Y.$$

□

Proof. "$\supset$": Let $x \in \overline{A}^X \cap Y$. We want to show that $x \in \overline{A}^Y$. Let U be a neighbourhood of x in Y, that is, $x \in U$ and U is open in Y. We want to show that $U \cap A \neq \varnothing$. As U is open in Y, there exists an open set $O \subset X$ such that $U = O \cap Y$. As $x \in \overline{A}^X \cap Y$,

$$\varnothing \neq A \cap O = A \cap (Y \cap O) = A \cap U.$$

"$\subset$": Let $x \in \overline{A}^Y$. As $A \subset Y$, we have $\overline{A}^Y \subset Y$. It remains to show that $x \in \overline{A}^X$. Let O be a neighbourhood of x in X. We want to show that $O \cap A \neq \varnothing$. We have $(O \cap Y) \cap A \neq \varnothing$ by assumption. □

There is a, in a sense, dual statement for the relative interior.

Lemma 2.3.9
Let $A \subset Z$ and $\operatorname{int}_Z(A)$ be the interior of A in Z. Then we have $\operatorname{int}_X(A) \subset \operatorname{int}_Z(A)$.

Proof. Let $x \in \operatorname{int}(A)$. Then there exists an open (in X) set $O \subset A$ with $x \in O$. We must show that there exists an open (in Z) set $O' \subset A$ with $x \in O'$. As $O \subset A \subset Z$, one possibility is

$$O' := O = O \cap Z,$$

which is open in Z. □

Counterexample 2.3.10 ($\operatorname{int}_X(A) \subsetneq \operatorname{int}_Z(A)$)
Take $X := \mathbb{R}$ with the standard topology, $Z := [0, 2)$ and $A := [0, 1)$. Then $\operatorname{int}(A) = (0, 1) \subsetneq [0, 1) = \operatorname{int}_Z(A)$. ◇

Lemma 2.3.11
Metric spaces are Hausdorff.

Proof. Let $x, y \in (X, d)$ be distinct and $r := \frac{1}{2} d(x, y)$. Then $B(x, r)$ and $B(y, r)$ are disjoint open neighbourhoods of x and y, respectively. □

Lemma 2.3.12 (Characterisation of Hausdorff spaces)
A topological space X is Hausdorff *if and only if for every $x \in X$ we have $\{x\} = \bigcap_{U \in \mathcal{N}_x} \overline{U}$.*

Proof. "$\implies$": "$\subset$": is clear since every neighbourhood of x and thus its closure must contain x.

"$\supset$": Let $x \in X$ and suppose that $y \in \overline{U}$ for all $U \in \mathcal{N}_x$. Towards contradiction suppose that $x \neq y$. As X is Hausdorff, there exist disjoint open sets $U_x, U_y \subset X$ with $x \in U_x$

and $y \in U_y$. Then $y \in \overline{U_x}$ by assumption, but as $U_x \cap U_y = \varnothing$, we have $y \in \overline{U_x} \backslash U_x$, so y is a limit point of U_x. Hence

$$U_x \backslash \{y\} \cap N = U_x \cap N \neq \varnothing \qquad \forall N \in \mathcal{N}_y .$$

Choosing $N = U_y$ yields that $U_x \cap U_y \neq \varnothing$, which is a contradiction to $U_x \cap U_y = \varnothing$.

" $\Longleftarrow$ ": Let $x, y \in X$ be distinct. As $\{x\} = \bigcap_{U \in \mathcal{N}_x} \overline{U}$, x is the *only* point contained in all $\overline{U}$, where $U \in \mathcal{N}_x$. Hence there exists a $U_x \in \mathcal{N}_x$ such that $y \notin \overline{U_x}$. As $U_x \in \mathcal{N}_x$, there exists an open set $O_x \subset U_x$ with $x \in O_x$. As $y \notin \overline{U_x}$, we also have $y \notin O_x$. Now, $O_y := X \backslash \overline{U_x}$ is an open neighbourhood of y not containing x. We have found two open disjoint neighbourhoods O_x, O_y of x and y, respectively: as $O_x \subset \overline{U_x}$, we have

$$O_x \cap (X \backslash \overline{U}_x) = \varnothing.$$

□

Corollary 2.3.13
In HAUSDORFF *spaces, the singletons are closed.*

Counterexample 2.3.14 (Non-HAUSDORFF space with closed singletons)
Consider an *infinite* set X with the cofinite topology $T = \{A \subset X : X \backslash A \text{ is finite or } A = \varnothing\}$. Then $X \backslash \{x\} \in T$ for all $x \in X$, as $\{x\}$ is finite. Hence $\{x\}$ is closed, that is, $\{x\} = \overline{\{x\}}$. But (X, T) is not HAUSDORFF as any two non-empty open subsets are not disjoint: for $U, V \in T \backslash \{\varnothing\}$, the intersection $U \cap V = X \backslash \big((X \backslash U) \cup (X \backslash V)\big)$ is the complement of a finite set in X. As X is infinite, $U \cap V$ must be infinite and in particular not empty. ◇

Bases and subbases

DEFINITION 2.3.15 (BASIS)
Let β be a collection of open subsets of X. If each open subset of X is the union of sets in β, then β is a basis of the topology on X. basis

THEOREM 2.3.2: CONSTRUCTION OF TOPOLOGIES WITH BASES

Let $\beta \subset 2^X$ be a non empty collection of subsets of X. If all finite intersections of subsets are contained in β and $\bigcup\{Y : Y \in \beta\} = X$, then β is a basis for a topology on X.

Put differently: let $B \subset P(X)$ set of subsets with $\varnothing \in B$, $B_1 \cap B_2 \in B$ if $B_1, B_2 \in B$, and $X = \bigcup_{B_i \in B} B_i$. Then $T := \{\bigcup_{A \in B'} A : B' \subset B\}$ is a topology on X.

This follows from the axioms of the topology. It is more efficient to give a basis to describe the topology instead giving all open sets, all neighbourhoods, or all closed sets.

Some properties already hold if they are true on the basis, like continuity (see next subsection) and compactness (see next section).

Example 2.3.16 (Basis) On any ordered field such as $\mathbb{E}^1$, we can defined intervals and the open intervals always form a basis, turning the ordered field into an ordered topological field.

As $\mathbb{Q} \subset \mathbb{E}^1$ is dense, the open intervals with rational endpoints for a countable basis of $\mathbb{E}^1$. ◇

Definition 2.3.17 (Subbasis)
A subbasis $\mathcal{F}$ of a topology on X (in terms of open sets) induced by any family of subsets containing $\varnothing$ and X, is the topology whose basis consis of all finite intersections of sets in $\mathcal{F}$.

2.4 Continuous functions

There are at least three ways to speak abstractly about mathematics; set theory, logic and category theory. Topological spaces form a category. The morphisms in the category of topological spaces are precisely the continuous functions.

Let X, Y and Z be topological spaces.

Theorem 2.4.1: Characterisation of continuity

A function $f\colon X \to Y$ is continuous if and only if for all open sets $O \subset Y$, the full preimage $f^{-1}(O) \subset X$ is open.

Proof. " $\implies$ ": Let f be continuous and $O \subset Y$ be open. Then O is a neighbourhood of any of its points, so $f^{-1}(O)$ is a neighbourhood of any of its points (by definition). Hence $f^{-1}(O)$ is open.

" $\impliedby$ ": Let $x \in X$ and $U \subset X$ a neighbourhood of $f(x)$. Then there exists a open set $O \subset U$ with $f(x) \in O \subset U$. Then $x \in f^{-1}(O) \subset f^{-1}(U)$. As $f^{-1}(O)$ is open, $f^{-1}(U)$ is a neighbourhood of x. □

Definition 2.4.1 (Map)
A map is a continuous function.

map

Theorem 2.4.2: Composition of map

The composition of maps is a map.

Proof. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. Let $O \subset Z$ be open, then $g^{-1}(Z)$ is open (as g is a map), so $(g \circ f)^{-1}(Z) = f^{-1}(g^{-1}(Z))$ is open, as f is a map. □

Theorem 2.4.3: Restriction of maps is a map

Let $A \subset X$ be equipped with the subspace topology. If $f\colon X \to Y$ is a map, then the restriction $f|_A\colon A \to Y$ is a map, too.

Proof. Let $O \subset X$ be open. We have $(f|_A)^{-1}(O) = f^{-1}(O) \cap A$, which is open in the subspace topology as f is a map. □

Definition 2.4.2 (Identity, inclusion map)
The map $\mathrm{id}_X\colon X \to X$, $x \mapsto x$ is the identity map. For any $A \subset X$, $\mathrm{id}_X|_A\colon A \to X$ is the inclusion map.

The identity map is indeed continuous, as the preimage of any set is itself.

Definition 2.4.3 (Coarser, finer)
Let X be a set and $T_1, T_2 \subset 2^X$ be topologies on X. If $T_1 \subset T_2$, then T_1 is coarser than T_2 and T_2 is finer than T_1.

The coarsest topology on any set is the trivial topology $\{\varnothing, X\}$ and the finest is the discrete toplogy 2^X.

The subspace topology is the coarsest topology on A such that the inclusion is continuous.

Theorem 2.4.4: Characterisations of continuity

The following are equivalent.
1. $f \colon X \to Y$ is a map.
2. If β is a basis for the topology on Y, then for every open set $O \in \beta$, $f^{-1}(O) \subset X$ is open.
3. $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.
4. $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all $B \subset Y$.
5. $f^{-1}(B)$ is closed for all closed sets $B \subset Y$.

Proof. "1 $\implies$ 2": This is trivial.

"2 $\implies$ 3": Let β be a basis for the topology on Y and $A \subset X$. Clearly, $f(A) \subset \overline{f(A)}$. Let $x \in X$ be a limit point of A, which is not in A, i. e. $f(x) \notin f(A)$.

We show that $f(x)$ is a limit point of $f(A)$. Consider a neighbourhood $N \in \mathcal{N}_{f(x)}$ of $f(x) \in Y$. As β is a base, there exists a $B \in \beta$ such that $f(x) \in B \subset N$. By 2, $f^{-1}(B)$ is open in X, so $f^{-1}(B)$ is a neighbourhood of x. As x is a limit point of A, $f^{-1}(B) \cap A \neq \varnothing$, so $B \cap f(A) \neq \varnothing$. As $B \subset N$, this implies $N \cap f(A) \neq \varnothing$, so $f(x)$ is a limit point of $f(A)$.

"3 $\implies$ 4": For any $A \subset X$, let $B := f(A) \subset Y$. Note that it is sufficient to consider such B, as for arbitrary $B' \subseteq Y$ it holds $f^{-1}(B') = f^{-1}(B' \cap f(X))$. Then

$$f(\overline{f^{-1}(B)}) = f(\overline{A}) \subset \overline{f(A)} = \overline{B}$$

and thus $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

"4 $\implies$ 5": Let $B \subset Y$ be closed by corollary 2.3.2. Then $\overline{B} = B$. By 4,

$$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) = f^{-1}(B),$$

so $f^{-1}(B)$ is closed ($\overline{A} \subset A$ implies $A = \overline{A}$).

"5 $\implies$ 1": Let $O \subset Y$ be open. Then

$$f^{-1}(O) = f^{-1}(Y \backslash (Y \backslash O)) = f^{-1}(Y) \backslash f^{-1}(Y \backslash O) = X \backslash f^{-1}(Y \backslash O).$$

As $Y \backslash O$ is closed, so is $f^{-1}(Y \backslash O)$, so $f^{-1}(O)$ is open. □

Example 2.4.4 Consider again the map from example 2.1.10. A base for the topology on C is given by the open circular intervals, e.g. $I_1 := \{f(x) : x \in (0, \frac{1}{4})\}$ or $I_2 := \{f(x) : |x| \in (0, \frac{1}{4})\}$. Then $f^{-1}(I_1) = (0, \frac{1}{4})$ is open in X and so is $f^{-1}(I_2) = (0, \frac{1}{4}) \cup (1 - \frac{1}{4}, 1 - 0)$.

Now consider $I_3 := \{f(x) : x \in [0, \frac{1}{4})\}$. Then $f^{-1}(I_3) = [0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$ is still open in $[0, 1)$. The function f is continuous, but f^{-1} is not continuous, as $f(f^{-1}(I_3))$ is not open. ◇

Definition 2.4.5 (Homeomorphism)
A function $f\colon X \to Y$ is a homeomorphism if f is bijective and f and f^{-1} are maps.

Distance functions and Tietze's extension theorem

Definition 2.4.6 (Distance to a set)
For $A \subset (X,d)$, the distance from $x \in X$ to A is $d(x,A) := \inf_{a\in A} d(x,a)$. The distance from A to $B \subset (X,d)$ is $d(A,B) := \inf_{x\in B} d(x,A)$.

Lemma 2.4.7 (Distance function and closure)
Let (X,d) be a metric space. We have $x \in \overline{A}$ if and only if $d(x,A) = 0$. The map $d(\cdot,A)\colon X \to \mathbb{R}$ is continuous.

Proof. TODO ☐

Example 2.4.8 For closed disjoint subsets $A, B \subset X$, there exists a surjective map $f\colon X \to [-1,1]$ with $f(A) = \{1\}$ and $f(B) = \{-1\}$. This map is

$$x \mapsto \frac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)}.$$

This map is well defined as $d(x,A) \geqslant 0$ and one of the summands $d(x,A)$ and $d(x,B)$ hence is always positive, and thus so is their sum. The first two conditions are fulfilled and the last one, too, as for $x \in X\backslash(A \cup B)$,

$$\begin{aligned}\left|\frac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)}\right| < 1 &\iff -d(x,A) - d(x,B) < d(x,A) - d(x,B) < d(x,A) + d(x,B)\\ &\iff 0 < d(x,A) < d(x,A) + d(x,B)\end{aligned}$$

is always true. ◇

Lemma 2.4.9
Let $A, B \subset (X,d)$ be closed and disjoint. Then there exist disjoint open subsets $U, V \subset (X,d)$ with $A \subset U$ and $B \subset V$.

Proof. Let $f := \frac{d(\cdot,B)-d(\cdot,A)}{d(\cdot,B)+d(\cdot,A)}$. Then $U := f^{-1}([-1,0))$ and $V := f^{-1}((0,1])$ are open and disjoint. ☐

Example 2.4.10
The sets $A := \{(x,y) \in \mathbb{R}^2 : y \leqslant 0\}$ and $B := \{(x,y) \in \mathbb{R}^2 : y = e^x\}$ fulfil $d(A,B) = 0$ even though they are closed and disjoint.

But neither A nor B can be compact: suppose A were compact, then $d(\cdot,B)|_A\colon A \to \mathbb{R}$ has a compact and therefore closed image, as it is a subspace of $\mathbb{R}$. Suppose $d(A,B) = 0$, then 0 is in aforementioned image and thus $A \cap B \neq \varnothing$. ◇

Theorem 2.4.5: Tietze extension theorem

Let X be a metric Hausdorff space, $A \subset X$ closed and $f\colon A \to \mathbb{R}$ a map. Then there exists a map $g\colon X \to \mathbb{R}$ with $g|_A = f$.

Counterexample 2.4.11 (Tietze extension theorem)
In the non-Hausdorff space from counterexample 2.3.14 with $X := \mathbb{N}$, the restriction $f := \mathrm{id}|_A$, where $A := \{0,1\}$ is a closed set, is a map. Let $g\colon X \to \mathbb{R}$ be a continuous extension of f. Now let $O \subset \mathbb{R}$ be open such that $g^{-1}(P)$ is finite, e.g. $O = B(0,\varepsilon)$, where $\varepsilon > 0$ is sufficiently small. Then $g^{-1}(O)$ is not open, so g is not continuous. ◇

$$\begin{tikzcd} \mathbb{R} \arrow[r, "h", "\approx"'] & (0,1) \\ A \arrow[u, "f"] \arrow[r, hook] & X \arrow[u, "g"'] \arrow[ul, "h^{-1}\circ g"'] \end{tikzcd}$$

Fig. 17: A commutative diagram for the first step of the proof of the Tietze extension theorem.

Proof. ① Suppose we have proven the theorem for bounded functions. Let $f\colon A \to \mathbb{R}$ be unbounded. We have a homeomorphism $h\colon \mathbb{R} \to (0,1)$ (e.g. $\frac{2}{\pi}\arctan$), an inclusion map $A \to X$ and an extension $g\colon X \to (0,1)$ of the bounded function $h \circ f\colon A \to (0,1)$. Then $h^{-1} \circ g$ extends f continuously.

② Suppose that $|f(x)| \leqslant M$ for all $x \in C$. The idea is that we represent $f = \sum_{k=1}^{\infty} g_k$ as a series. Let

$$A_1 := f^{-1}\left(\left[\frac{M}{3},\infty\right)\right) \quad \text{and} \quad B_1 := f^{-1}\left(\left(-\infty,-\frac{M}{3}\right]\right),$$

which are closed in A and, as A is closed, also closed in X.

Similarly to example 2.4.8, there exists a map

$$g_1\colon X \to \left[-\frac{M}{3},\frac{M}{3}\right]$$

with $g_1(A_1) = \left\{\frac{M}{3}\right\}$, $g_1(B_1) = \left\{-\frac{M}{3}\right\}$ and $g_1\big(X\backslash(A_1 \cup B_1)\big) \subset \left(-\frac{M}{3},\frac{M}{3}\right)$.

Hence $|f(x) - g_1(x)| \leqslant \frac{2}{3}M$ for all $x \in A$. Let

$$A_2 := (f-g_1)^{-1}\left(\left[\frac{2}{3}M,\infty\right)\right) \quad \text{and} \quad B_2 := (f-g_1)^{-1}\left(\left(-\infty,-\frac{2}{3}M\right]\right).$$

Then again there exists a $g_2\colon X \to \left[-\frac{2}{3}M,\frac{2}{3}M\right]$ such that $g_2(A_2) = \ldots$, $g_2(B_2) = \ldots$ and …. Hence $\left|\big(f(x) - g_1(x)\big) - g_2(x)\right| < \frac{4}{9}M$ for all $x \in A$.

In the m-th iteration we obtain a function

$$g_m\colon X \to \left[-\frac{2^{m-1}}{3^m}M,\frac{2^{m-1}}{3^m}M\right]$$

such that

$$\left|f(x) - \sum_{k=1}^{m} g_k(x)\right| \leqslant \frac{2^m}{3^m}M$$

for all $x \in A$ and $|g_m(x)| \leqslant \frac{2^{m-1}}{3^m}M$ for all $x \in X\backslash A$. By the Weierstrass-M-test, we have $f(x) = \sum_{k=1}^{\infty} g_k(x)$ and as $\sum_{k=1}^{\infty} g_k$ converges uniformly, it is a continuous function on the whole of X. □

Disks

08.11.2021

Like we "glued together" polyhedra from convex polygons, we want to produce topological spaces by glueing together disks.

Definition 2.4.12 (Disk)
A disk is a topological space homeomorphic to $\mathbb{B}^2$.

disk

Lemma 2.4.13 (Homeomorphism preserving frontier)
If A is a disk and $h\colon A \to \mathbb{B}^2$ is a homeomorphism, then $h^{-1}(\mathbb{S}^1) = \partial A$.

In particular $h^{-1}(\mathbb{S}^1)$ does not depend on h.

Note that in general, homeomorphisms do not map the boundary to the boundary.

Proof. Exercises. □

We have $\partial B := \{x \in B : \forall U \in \mathcal{N}_x \subset 2^B : U \not\approx \mathring{B}^2\}$. Let $h\colon B \to \mathbb{B}^2$ be a homeomorphism, where B is a disk. Then $h(\partial B) = \mathbb{S}^1$.

Lemma 2.4.14 (Homeomorphism extension)
Any homeomorphism from the frontier of a disk to itself can be extended to a homeomorphism of the entire disk.

Proof. Let A be a disk. Then there exists a homeomorphism $h\colon A \to \mathbb{B}^2$. Let $g\colon \partial A \to \partial A$ be a homeomorphism. Then $(h|_{\partial A}) \circ g \circ (h^{-1}|_{\mathbb{S}^1})\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism as the concatenation of homeomorphisms. Define

$$f\colon \mathbb{B}^2 \to \mathbb{B}^2, \qquad x \mapsto \begin{cases} 0, & \text{if } x = 0, \\ \|x\| \cdot (h \circ g \circ h^{-1})\left(\frac{x}{\|x\|}\right), & \text{else.} \end{cases}$$

On $\mathbb{S}^1$ we have $f \equiv h \circ g \circ h^{-1}$. Hence $h^{-1} \circ f \circ h\colon A \to A$ extends g.

Exercise: show that f is a homeomorphism. □

Lemma 2.4.15 (Union of two frontier-intersecting disks is a disk)
Let $A, B \subset X$ with $A, B \approx \mathbb{B}^d$ and intersect only along their frontiers ($\approx \mathbb{S}^{d-1}$) in a homeomorphic copy of $\mathbb{B}^{d-1}$. Then $A \cup B \approx \mathbb{B}^d$.

Consider two disks intersecting in just two points. Then their union is not a disk.

Proof. We only consider $d = 2$, where A and B intersect in an interval homeomorphic to $[0, 1]$. Let γ be the arc $A \cap B$. We can find complementary arcs α, β with $\alpha \cup \gamma = \partial A$ and $\beta \cup \gamma = \partial B$ such that $\alpha \cap \gamma$ and $\beta \cap \gamma$ are each sets with exactly two points.

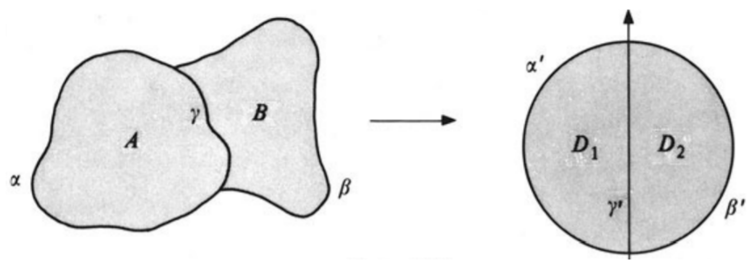

Fig. 18: We instead call D_1 and D_2 A' and B', respectively.

The y-axis divides $\mathbb{B}^2$ into two halves. Let

$$A' := \{(x, y) \in \mathbb{B}^2 : x \geqslant 0\} \qquad \text{and} \qquad B' := \{(x, y) \in \mathbb{B}^2 : x \leqslant 0\}.$$

Then $\mathbb{B}^2 = A' \cup B'$. Let $\alpha' := A \cap \mathbb{S}^1$ and $\beta' := B \cap \mathbb{S}^1$. We have $\alpha \approx \alpha'$ by lemma 2.4.13. This homeomorphism can be extended to $\partial A = \alpha \cup \gamma \approx \alpha' \cup \gamma' \approx \mathbb{S}^1$. By lemma 2.4.14, $A \approx A'$. Similarly, $B \approx B'$. Hence $A \cup B \approx A' \cup B' \approx \mathbb{B}^2$. □

2.5 Space-filling curves

Let $\Delta := \mathrm{conv}(\{x, y, z\})$ be a triangle in $\mathbb{E}^2$ (where x, y and z don't lie on the same line) such that $x + y + z = 0$ (its barycentre is the origin).

Define the curve in Δ (a map from an interval into Δ)

$$f\colon [0,1] \to \Delta, \qquad t \mapsto \begin{cases} (1-2t)x, & \text{if } t \in [0, \frac{1}{2}], \\ (2t-1)y, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

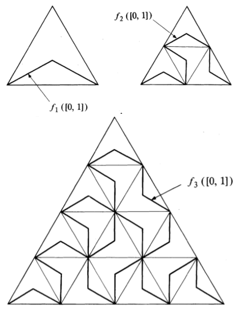

Fig. 19: The iterative construction of the paths f_k.

We have $f(\frac{1}{2}) = 0$ and $(1-2t)x \in \Delta$ for all $t \in [0, \frac{1}{2}]$, so the map is well defined. We iteratively define $f_2, f_3, \ldots$ by subdividing Δ into 4 smaller copies and using a rescaled f_k to define f_{k+1}:

$$f_{k+1}\colon [0,1] \to \Delta, \qquad t \mapsto \begin{cases} \frac{1}{2^{k-1}}\left(f_k(2^k t) + x\right), & \text{if } t \in [0, \frac{1}{2^k}], \\ \ldots, & \text{if } t \in [\frac{1}{2^m}, \frac{1}{2^{k-1}}], \\ \ldots \end{cases}$$

Assume that Δ is a regular triangle with edge length $\frac{1}{2}$. Suppose $n \geqslant m$. For $t \in [0,1]$, there exists a small triangle (in generation m) with edge length $\frac{1}{2^m}$ which contains $f_n(t)$ and $f_m(t)$. Hence

$$\|f_m(t) - f_n(t)\| \leqslant \frac{1}{2^m} \qquad \forall t \in [0,1],$$

so $(f_m)_{m\in\mathbb{N}}$ is uniformly convergent. Let $f\colon [0,1] \to \Delta$ denote the limit map. We want to show that f is space-filling in Δ, i.e. that it is surjective.

Proof. We have

$$\forall n \in \mathbb{N}\ \forall p \in \Delta\ \exists t \in [0,1] : \|f_n(t) - p\| \leqslant \frac{1}{2^n}. \tag{1}$$

This already implies that $\overline{f([0,1])} = \Delta$. Let $p \in \Delta$ and $U \in \mathcal{N}_p \subset \mathcal{P}(\mathbb{E}^2)$. Choose $N \in \mathbb{N}$ such that $B(p, 2^{1-N}) \subset U$. Further, by (1) there is a $t_0 \in [0,1]$ such that

$$\|p - f_n(t_0)\| \leqslant \frac{1}{2^N}.$$

By the triangle inequality,

$$\|p - f(t_0)\| \leqslant \|p - f_N(t_0)\| + \|f_N(t_0) - f(t_0)\| \leqslant \frac{1}{2^N} + \frac{1}{2^N} = \frac{1}{2^{N-1}},$$

so $f(t_0) \in U$. Thus $f([0,1]) = \Delta$, as $f([0,1])$ is closed (by a compactness result from later). □

Instead of Δ we could also have chosen $\mathbb{B}^2$ or a tetrahedron.

3 Compactness and connectedness

10.11.2021

3.1 Closed and bounded subsets of $\mathbb{E}^n$

Let X be a topological space.

Definition 3.1.1 (Open cover, subcover)
A family $\mathcal{F}$ of open subsets of X is an open cover of X if $\bigcup_{F \in \mathcal{F}} F = X$. A family $\mathcal{F}'$ is a subcover of $\mathcal{F}$ if $\mathcal{F}'$ is a open cover and $\mathcal{F}' \subset \mathcal{F}$.

open cover
subcover

Example 3.1.2 Let $X := \mathbb{E}^n$. Then $\mathcal{F} := \{\mathring{B}^n(x,1) : x \in \mathbb{Z}^n\}$ is an infinite open cover without a proper subcover: as $\mathring{B}^n(x,1)$ only contains $x \in \mathbb{Z}^n$ and no other points from $\mathbb{Z}^n$, taking away one set from $\mathcal{F}$ will lead to it not being a cover. ◇

Example 3.1.3 Consider $X = [0,1]$ with the subspace topology of $\mathbb{R}$. Let

$$\mathcal{F} := \left\{\left[0, \frac{1}{10}\right), \left(\frac{1}{3}, 1\right]\right\} \cup \left\{\left(\frac{1}{n+2}, \frac{1}{n}\right) : n \geqslant 2\right\},$$

which is an infinite open cover. But there is a finite subcover:

$$\mathcal{F}' := \left\{\left[0, \frac{1}{10}\right), \left(\frac{1}{3}, 1\right]\right\} \cup \left\{\left(\frac{1}{n+2}, \frac{1}{n}\right) : n \in \{2, \dots, 9\}\right\}$$

We will later show that *every* open cover of $[0,1]$ has a finite subcover. ◇

Definition 3.1.4 (Compact)
The space X is compact if every open cover of X has a finite subcover.

compact

We will later prove that a subset $X \subset \mathbb{E}^n$ is compact if and only if it is closed and bounded. In particular, compact subsets of $\mathbb{E}^n$ have to be closed.

3.2 The Heine-Borel Theorem

Theorem 3.2.1: Heine-Borel

A closed interval of $\mathbb{E}^1$ is compact.

Proof. ("Creeping-along") Let $[a,b] \subset \mathbb{E}^1$ be a closed interval be equipped with the subspace topology. Suppose $\mathcal{F}$ is an open cover of $[a,b]$. Define

$$X := \{x \in [a,b] : [a,x] \text{ is contained in union of some finite subfamily of } \mathcal{F}\}.$$

Then $X \neq \varnothing$ since $a \in X$. Furthermore, $X \subset [a,b]$ is bounded. Then $s := \sup(X)$ exists since $\mathbb{R}$ is complete.

We show that $s \in X$ and $s = b$.

1. Pick $O \in \mathcal{F}$ with $s \in O$. As O is open, there exists a $\varepsilon > 0$ such that $(s - \varepsilon, s] \subset O$. Further, if $s < b$, then $(s - \varepsilon, s + \varepsilon) \subset O$ $(\star)$. If $x \in X$, then for all $y \in [a,x]$ we have $y \in X$ $(\star\star)$. Since s is the least upper bounded of X, we have $s - \frac{\varepsilon}{2} \in X$. By $(\star\star)$,

$[a, s - \frac{\varepsilon}{2}] \subset X$. By the definition of X, $[a, s - \frac{\varepsilon}{2}]$ is contained in the union of some finite subfamily $\mathcal{F}' \subset \mathcal{F}$, so

$$[a, s] \subset \bigcup_{F' \in \mathcal{F}'} F' \cup \{O\},$$

so $s \in X$.

2 Assume $s < b$. By ($\star$),

$$\left[a, s + \frac{\varepsilon}{2}\right] \subset \bigcup_{F' \in \mathcal{F}'} F' \cup \{O\},$$

Hence $s + \frac{\varepsilon}{2} \in X$, which is a contradiction to $s = \sup(X)$. □

Corollary 3.2.1 (Continuous image of compact interval is bounded)
Let $f\colon [a, b] \to \mathbb{R}$ be a map. Then the image $f([a, b])$ is bounded.

Proof. Let $x \in [a, b]$. Then there exists a open neighbourhood $O_x \in \mathcal{N}_x$ in $[a, b]$ such that $|f(x') - f(x)| < 1$ for all $x' \in O_x$. The set $\mathcal{F} := \{O_x : x \in [a, b]\}$ is an open cover of $[a, b]$. By Theorem 3.2.1, there exist finitely many $x_1, \ldots, x_k \in [a, b]$ such that $\bigcup_{j=1}^{k} O_{x_j} = [a, b]$. If $x \in O_{x_1}$, then $|f(x)| \leqslant |f(x_1)| + 1$. Hence

$$\max_{x \in [a,b]} |f(x)| \leqslant 1 + \max\big\{|f(x_j)| : j \in \{1, \ldots, k\}\big\}.$$

□

3.3 Properties of compact spaces

We can generalise the previous corollary.

Theorem 3.3.1: Continuous image of a compact space

The continuous image of a compact space is compact.

Proof. Let $f\colon X \to Y$ be a surjective map and X be compact. We show that Y is compact. Let $\mathcal{F}$ be any open cover of Y. For $O \in \mathcal{F}$, the full preimage $f^{-1}(O)$ is open in X as f is a map. Since X is compact, there exists a finite subcover $O_1, \ldots, O_n \in \mathcal{F}$ such that $X = \bigcup_{k=1}^{n} f^{-1}(O_k)$. As f is surjective, $f(f^{-1}(O_k)) = O_k$ for all $k \in \{1, \ldots, n\}$. Hence $Y = \bigcup_{k=1}^{n} O_k$, so Y is compact. □

Definition 3.3.1 (Compact subset)
Let $C \subset X$. If C is compact with respect to the subspace topology induced by X, then C is a compact subset of X.

Theorem 3.3.2: Closed subset of compact space

A closed subset of compact space is compact.

Proof. Let $C \subset X$ be closed and X be compact. Let $\mathcal{F}$ be an open cover of C, that is, $C = \bigcup_{F \in \mathcal{F}} F$. As C is closed, $X \backslash C$ is open. Hence $X = C \cup (X \backslash C) = (\bigcup_{F \in \mathcal{F}} F) \cup \{X \backslash C\}$, so $\tilde{\mathcal{F}} := \mathcal{F} \cup \{X \backslash C\}$ is an open cover of X. As X is compact, there exists an finite subcover $\tilde{F}'$ of X. As $\mathcal{F}$ only covers C, $X \backslash C$ has to be part of $\tilde{F}'$, that is

$$X = \left(\bigcup_{k=1}^{n} F_k \right) \cup (X \backslash C),$$

where $F_k \in \mathcal{F}$. Then $C = \bigcup_{k=1}^{n} F_k$, so there is a finite subcover of $\mathcal{F}$, so C is compact. □

Compactness and Hausdorffness

Lemma 3.3.2 (Properties of compact Hausdorff spaces)
Let X be Hausdorff and let $K \subset X$ be compact. Then

$$K = \bigcap \{\overline{O} : O \supset K, O \text{ open}\}.$$

Let $K' \subset X$ another compact set, disjoint from K. Then there are disjoint open sets O and O' of X with $K \subset O$ and $K' \subset O'$.

Proof. TODO, HW 4.1 □

Definition 3.3.3 (Neighbourhood of a set)
Let $M \subset X$. Then U is a neighbourhood of M if $U \in \mathcal{N}_x$ for all $x \in M$ (or equivalently: if there exist an open set O with $M \subset O \subset U$).

Theorem 3.3.3: Hausdorff space

Let X be a Hausdorff space and $A \subset X$ be a compact subset. Then for each point $x \in X \backslash A$, there exist disjoint neighbourhoods of x and A. In particular A is closed.

Proof. Let $x \in X \backslash A$. As X is Hausdorff, for any $a \in A$, there exists open neighbourhoods $U_a, V_a \subset X$ such that $x \in U_a$, $a \in V_a$ and $U_a \cap V_a = \varnothing$. Then $(V_a)_{a \in A}$ is an open cover of A. As A is compact, there exists a finite subcover such that $A \subset \bigcup_{k=1}^{n} V_{a_k} =: V$. Furthermore, $U := \bigcap_{k=1}^{n} U_{a_k}$ is an open neighbourhood. Then U and V are the disjoint neighbourhoods of x and A. □

Theorem 3.3.4: Map from compact to HD space

Let X be compact and Y a Hausdorff space. If $f : X \to Y$ is a bijective map, then f is a homeomorphism.

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Proof. Let $C \subset X$ be closed. Then C is compact by Theorem 3.3.2. As f is continuous, by Theorem 3.3.1 $f(C)$ is compact. By Theorem 3.3.3, $f(C)$ is closed in Y, so f^{-1} is continuous, so f is a homeomorphism. □

Theorem 3.3.5: Bolzano-Weierstrass

An infinite subset of a compact space has a limit point.

Proof. Suppose every $x \in X$ is not a limit point of A. Then for all $x \in X$ there exists an open $N_x \in \mathcal{N}_x$ such that $(A\backslash\{x\}) \cap N_x = \varnothing$. That is, for $x \in X$ we have

$$N_x \cap A = \begin{cases} \varnothing, & \text{if } x \notin A, \\ \{x\}, & \text{if } x \in A. \end{cases}$$

Then $(N_x)_{x \in X}$ is an open cover of X. As X is compact, we have $X = \bigcup_{k=1}^{n} N_{x_k}$. Hence

$$A = A \cap X = \bigcup_{k=1}^{n} (N_{x_k} \cap A) = \{x_1, \ldots, x_n\} \cap A,$$

so $A \subset \{x_1, \ldots, x_n\}$ is finite. □

Theorem 3.3.6: Heine-Borel

Let $C \subset \mathbb{E}^n$ be compact. Then C is closed and bounded.

Proof. As $\mathbb{E}^n$ is Hausdorff and $C \subset \mathbb{E}^n$ is compact, C is closed by Theorem 3.3.3.

Let $\mathcal{F}$ be an open cover of $\mathbb{E}^n$ by balls centred at integer coordinates with fixed radius $r > 1$. Then $\{F \cap C : F \in \mathcal{F}\}$ is an open cover of C. As C is compact, $C = \bigcup_{k=1}^{n} F_k \cap C$ for $(F_k)_{k=1}^{n} \subset \mathcal{F}$, so $C \subset \bigcup_{k=1}^{n} F_k$ is bounded. □

The following theorem is useful in (linear, discrete and other types of) optimisation.

Theorem 3.3.7: Bounded map on compact domain

Let X be compact. A map $f \colon X \to \mathbb{R}$ is bounded and attains its bounds.

Proof. As X is compact and f is a map, $f(X) \subset \mathbb{R}$ is compact by Theorem 3.3.1 and thus closed and bounded by Theorem 3.3.6. Hence f is bounded. As $f(X)$ is closed, $\inf(f(X)), \sup(f(X)) \in f(X)$, so the bounds are attained. □

Definition 3.3.4 (Diameter)

The diameter of a metric space X is diameter

$$\operatorname{diam}(X) := \sup_{x,y \in X} d(x, y) \in [0, \infty].$$

Because the product of compact spaces is compact (proven later) and the distance to a compact set is a continuous function, the diameter of a compact set is finite.

Theorem 3.3.8: Lebesgue Lemma

Let (X, d) be a compact metric space and $\mathcal{F}$ be an open cover of X. Then there exists a $\delta > 0$ (the Lebesgue number of $\mathcal{F}$) such that any subset of diameter less than δ is contained in some $F \in \mathcal{F}$.

Proof. Exercise. □

Local compactness and one-point compactifications

Definition 3.3.5 (Local compactness)
A topological space X is locally compact it every point in X has a compact neighbourhood. locally compact

Lemma 3.3.6 (Properties of locally compact spaces)
Any compact or discrete space is locally compact. A closed subset of a locally compact space is locally compact. Local compactness is preserved by homeomorphism

Proof. (1) Let X be compact, $x \in X$ and $N \in \mathcal{N}_x$. Then $\overline{N} \subset X$ is a closed and thus by Theorem 3.3.2 compact neighbourhood of x.

Let X be discrete. For $x \in X$, the singleton $\{x\}$ is open and thus a neighbourhood of x. Furthermore, it is compact.

(2) Let X be locally compact, $A \subset X$ be closed and $x \in A$. As X is locally compact, there exists a compact neighbourhood $N \subset X$ of x. Then $N \cap A \subset N$ is compact as a closed subset of a compact set (as A is closed).

(3) Let X be locally compact and $f\colon X \to Y$ a homeomorphism. We show that Y is locally compact.

Let $y \in Y$. As X is locally compact, there exists a compact neighbourhood $N \subset X$ of $f^{-1}(y) \in X$ and thus an open set $O \subset N$ with $y \in f(O) \subset f(N)$. As N is compact and f is a map, $f(N)$ is a compact neighbourhood of y. As f^{-1} is a map, $f(O)$ is open. □

Counterexample 3.3.7 (Local compactness) $\mathbb{Q} \subset \mathbb{R}$ is not locally compact.

Let $x \in \mathbb{Q}$ and $N \subset \mathbb{Q}$ be a neighbourhood of x. We show that N is not compact. Since N is a neighbourhood of x, there exists an open set $O := (\varepsilon_1, \varepsilon_2) \cap \mathbb{Q}$ with $x \in O$ such that $\varepsilon_1, \varepsilon_2 \in \mathbb{R}\backslash\mathbb{Q}$. Now $\mathcal{F} := \{(0, 1 - 2^{-n}) \cap \mathbb{Q} : n \in \mathcal{N}_{\geqslant 1}\}$ is an open cover of $(0, 1) \cap \mathbb{Q}$, which doesn't have an finite subcover: enumerate the intervals in $\mathcal{F}$ by F_n, then for every finite subset $A \subset \mathcal{F}$, there is a F_{n_0} with the greatest n_0 among the $F_n \in A$. But then any $x \in (1 - 2^{-n_0}, 1) \cap \mathbb{Q}$ is not covered by A. Via scaling and translation, $\mathcal{F}$ can be transformed into an open cover $\widetilde{\mathcal{F}}$ of O. Lastly, $\widetilde{F}_1 := (-\infty, \varepsilon_1] \cap N \subset \mathbb{Q}$ and $\widetilde{F}_2 := [\varepsilon_2, \infty) \cap N \subset \mathbb{Q}$ are open in N and disjoint from O because $\varepsilon_1, \varepsilon_2 \notin \mathbb{Q}$. Hence $\widetilde{\mathcal{F}} \cup \{\widetilde{F}_1, \widetilde{F}_2\}$ is an open cover of N without finite subcover.

Holger's solution. Let $i\colon \mathbb{Q}^d \to \mathbb{R}^d$ be the continuous inclusion map. For $x \in \mathbb{Q}^d$ we have $x \in B_\varepsilon^{\mathbb{Q}}(x) := B_\varepsilon(x) \cap \mathbb{Q}^d$. Suppose there exists a $\varepsilon > 0$ such that $B_\varepsilon^{\mathbb{Q}}(x) \subset N \subset \mathbb{R}$, where N is compact. As $\mathbb{R}^d$ is Hausdorff and N is compact, $\overline{i(N)} = \overline{N} = N$ and thus

$$\overline{B}_\varepsilon(x) = \overline{i(B_\varepsilon^{\mathbb{Q}}(x))} \subset \overline{i(N)} = N \subset \mathbb{Q}^d,$$

which is a contradiction. ◇

Definition 3.3.8 (One-point compactification)
Let (X, τ_X) be a non-compact topological space. The one-point compactification of X the set $X^+ := X \cup \{\infty\}$ with the topology one-point compactification

$$\tau^+ := \tau_X \cup \{(X\backslash C) \cup \{\infty\} : C \subset X \text{ closed and compact}\}.$$

Lemma 3.3.9 (τ^+ is a topology)
The topology on X^+ is a topology.

Proof. As $\varnothing$ is closed and compact, $X^+ = (X\backslash\varnothing) \cup \{\infty\} \in \tau^+$.

Let $A, B \in \tau^+$. If $A, B \in \tau_X$, then $A \cap B \in \tau_x \subset \tau^+$. If $A \in \tau_X$ and $B \in \tau^+\backslash\tau_X$, then $\infty \notin A$, so there exists a closed and compact set $C \subset X$ such that

$$A \cap B = A \cap ((X\backslash C) \cup \{\infty\}) = A \cap (X\backslash C) \in \tau_X,$$

as $C \cup (X\backslash A)$ is closed as the union of closed sets.

If $A, B \in \tau^+\backslash\tau_X$, then there exist closed and compact sets $C_1, C_2 \subset X$ such that

$$\begin{aligned} A \cap B &= \big((X\backslash C_1) \cup \{\infty\}\big) \cap \big((X\backslash C_2) \cup \{\infty\}\big) \\ &= \big((X\backslash C_1) \cap (X\backslash C_2)\big) \cup \{\infty\} = \big(X\backslash(C_1 \cup C_2)\big) \cup \{\infty\} \in \tau^+, \end{aligned}$$

as the union of closed/compact sets is closed/compact.

Let $(A_i)_{i\in I} \subset \tau^+$, $J_1 \subset I$ be the set of indices of sets in τ_X and $J_2 \subset I$ the indices of sets in $\tau^+\backslash\tau_X$. Then there exist closed and compact sets $C_i \subset X$ such that $A_i = (X\backslash C_i) \cup \{\infty\}$ for $i \in J_2$. Hence

$$\begin{aligned} \bigcup_{i\in I} A_i &= \left(\bigcup_{i\in J_1} A_i\right) \cup \left(\bigcup_{j\in J_2} (X\backslash C_j) \cup \{\infty\}\right) = \left(\bigcup_{i\in J_1} X\backslash(X\backslash A_i)\right) \cup \left(\bigcup_{j\in J_2} (X\backslash C_j)\right) \cup \{\infty\} \\ &= \left(\bigcup_{i\in J_1,\ j\in J_2} X\backslash\big((X\backslash A_i) \cap C_j\big)\right) \cup \{\infty\} = X\backslash\left(\bigcap_{i\in J_1,\ j\in J_2} ((X\backslash A_i) \cap C_j)\right) \cup \{\infty\}. \end{aligned}$$

As $A_i \in \tau_X$ is open, $X\backslash A_i$ is closed and thus $(X\backslash A_i) \cap C_j$ is closed for all $i \in J_1$ and $j \in J_2$. Furthermore, $(X\backslash A_i) \cap C_j \subset C_j$ is compact as the closed subset of a compact set. Hence their intersection is closed and thus compact as a closed subset of a compact set. Thus $\bigcup_{i\in I} A_i \in \tau^+$. □

Lemma 3.3.10 (Properties of the one-point compactification)
The one-point compactification (X^+, τ^+) is compact and $X \subset X^+$ is dense.

Proof. Let $\mathcal{F}$ be an open cover X^+. Then there exists a $F_0 = (X\backslash C) \cup \{\infty\} \in \mathcal{F}$ for some compact closed set $C \subset X$. Without loss of generality we can assume that $\mathcal{F}$ contains no other sets of this form (from $\tau^+\backslash\tau_X$). Now, $\mathcal{F}\backslash\{F_0\}$ is an open cover of C. As C is compact, there exists a finite subcover $\widetilde{\mathcal{F}} \subset \mathcal{F}\backslash\{F_0\}$ of C. Hence $\widetilde{\mathcal{F}} \cup \{F_0\}$ is an finite subcover of X^+, so X^+ is compact.

To show that $X \subset X \cup \{\infty\}$ is dense, we show that ∞ is a limit point of X, that is, $X \cap N \neq \varnothing$ for all $N \in \mathcal{N}_\infty$. For every $N \in \mathcal{N}_\infty$ there exists a $O := (X\backslash C) \cup \{\infty\}$ with $\infty \in O \subset N$. We have $O \cap X = X\backslash C \neq \varnothing$, because otherwise $X = C$ would be compact, a contradiction. □

Lemma 3.3.11
The one-point compactification X^+ is Hausdorff *if and only if X is a locally compact* Hausdorff *space.*

Example 3.3.12 The sphere $\mathbb{S}^n$ is realized as the one-point compactification of $\mathbb{R}^n$ via the stereographic projection $\mathbb{S}^n\backslash\{e_n\} \approx \mathbb{R}^n$.

TODO ◇

Lemma 3.3.13 (Maps between one-point compactifications)
Let X, Y be Hausdorff and let $f\colon X \to Y$ be a map. Then the following statements are equivalent:

1. *The preimage $f^{-1}(C)$ of every compact set $C \subset Y$ is compact.*
2. *The extension $f^+\colon X^+ \to Y^+$ of f with $f^+(\infty) = \infty$ is continuous.*

Remark 3.3.14 (Why we need X to be non-compact)
If X were already compact, then $(X\backslash X) \cup \{\infty\} = \{\infty\}$ is open in X^+. Hence X is not dense in X^+ anymore and X^+ is not connected, as X and $\{\infty\}$ are two open nonempty disjoint subset spanning the whole of X^+. ○

Lemma 3.3.15
Let (X_1, T_1) and (X_2, T_2) be a compact Hausdorff *spaces. Let $X \subset X_1, X_2$ be a non-compact subspace of both X_1 and X_2 such that $X_1\backslash X = \{\infty_1\}$ and $X_2\backslash X = \{\infty_2\}$ holds. Then $(X_1, T_1) \approx (X_2, T_2)$.*

Application of this lemma. The one point compactification works for non-compact, locally compact Hausdorff spaces. For example, let $\mathbb{S}^n\backslash\{e_n\} \approx \mathbb{R}^n$ be X and $X_1 = \mathbb{S}^n \supset X$, then $X_1 \approx X_2$, where X_2 is the one point compactification of $\mathbb{R}^n$, so $\mathbb{S}^n$ can be realised as the one point compactification of $\mathbb{R}^n$.

We can also infer that the one point compactification of the two non-homeomorphic spaces (because of connectedness) $X\colon [0,1) \not\approx Y := [0,1]\backslash\{\frac{1}{2}\}$ is $[0,1]$.

Proof. Define

$$f\colon X_1 \to X_2, \qquad x \mapsto \begin{cases} x, & \text{if } x \in X, \\ \infty_2, & \text{if } x = \infty_1. \end{cases}$$

As X_1 is compact and X_2 is Hausdorff, f is a homeomorphism if f is continuous.

Let $O \subset X_2$ be open with $\infty_2 \in O$. Then there exists a open set $\tilde{O} \subset X$ such that $O = \tilde{O} \cup \{\infty_2\}$. Then $f^{-1}(O) = \tilde{O} \cup \{\infty_1\} \subset X_1$ is open, as its complement is $X\backslash\tilde{O}$, which is closed. □

Lemma 3.3.16
Let X be a Hausdorff *space. If $A \subset X$ is closed, show that X/A is homeomorphic to the one-point compactification of $X\backslash A$.*

In particular, CX is homeomorphic to the one-point compactification of $X \times [0,1)$.

Proof. (HW 7.1) We define the function

$$f\colon X/A \to (X\backslash A)^+, \qquad [x] \mapsto \begin{cases} \infty, & \text{if } x \in A, \\ x, & \text{else.} \end{cases}$$

Then f is bijective. Furthermore, X/A is compact as the quotient of a compact space and $(X\backslash A)^+$ is Hausdorff by Homework 4.3(c) because $X\backslash A$ is Hausdorff (as a subset of the Hausdorff space X) and locally compact: let $x \in X\backslash A$. Then $\{x\}$ and A are closed and thus (as X is compact) compact disjoint subsets of X. By homework exercise 4.1(b), there exists open disjoint sets $U, V \subset X$ with $x \in U$, $A \subset V$ and $U \cap V \neq \varnothing$. Then $X\backslash V$ is a closed and thus compact neighbourhood of x.

To show that f is the desired homeomorphism, it hence suffices to show its continuity. Let $a \in A$ and $q\colon X \to X/A$ be the quotient map. Let $O \subset (X\backslash A)^+$ with $\infty \in O$ be open. Then there exists an open set $\hat{O} \subset X\backslash A$ such that $O = \hat{O} \cup \{\infty\}$. Then

$$f^{-1}(O) = f^{-1}(\hat{O}) \cup f^{-1}(\{\infty\}) = \bigcup_{x\in\hat{O}} [x] \cup [a] \subset X/A$$

and thus

$$q^{-1}(f^{-1}(O)) = \bigcup_{x\in\hat{O}} q^{-1}([x]) \cup q^{-1}([a]) = \hat{O} \sqcup A \subset X.$$

As $X\backslash(\hat{O} \sqcup A) = (X\backslash A)\backslash\hat{O}$ is closed, $q^{-1}(f^{-1}(O))$ is open, so $f^{-1}(O)$ is open. $\square$

3.4 Product spaces

Via products, we can construct new spaces from known ones.

Example 3.4.1 (Topology on the cylinder)
Consider the cylinder $C := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0,1]\}$, which is, as a set, equal to $\mathbb{S}^1 \times [0,1]$. Given topologies of $\mathbb{S}^1$ and I, respectively, we want to define a topology on C which agrees with the subspace topology from $\mathbb{E}^3$. (This is true as $\mathbb{S}^1$ and I both inherit the subspace topology from $\mathbb{E}^2$ and $\mathbb{E}^1$, respectively and $\mathbb{E}^3 \approx \mathbb{E}^2 \times \mathbb{E}^1$ (cf. later).) $\diamond$

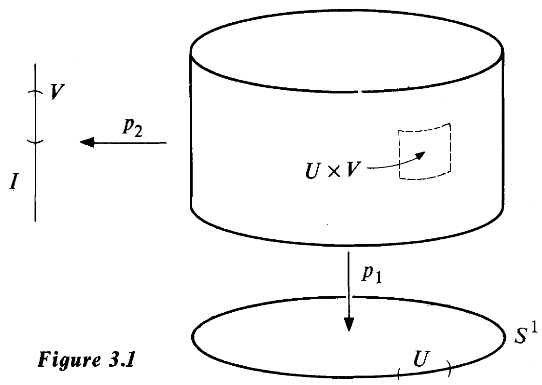

Figure 3.1

Fig. 20: Topology on the cylinder. An open square on the cylinder is mapped by the projections p_1 and p_2 onto open sets in $\mathbb{S}^1$ and I, respectively.

Definition 3.4.2 (Product topology)
For topological spaces X and Y, $\mathbb{B} := \{U \times V : U \subset X, V \subset Y \text{ open}\}$ is a basis for the product topology on $X \times Y$. The space $X \times Y$ equipped with the product topology is a product space. product topology

The set $\mathbb{B}$ is a basis, since it contains $X \times Y$, so $\bigcup_{B\in\mathbb{B}} B = X \times Y$ and for all $U \times V, U' \times V' \in \mathbb{B}$ we have

$$(U \times V) \cap (U' \times V') = \underbrace{(U \cap U')}_{\substack{\text{open}\\ \text{in } X}} \times \underbrace{(V \cap V')}_{\substack{\text{open}\\ \text{in } Y}} \in \mathbb{B},$$

as X and Y are topological spaces.

Remark 3.4.3 This construction generalises for finitely many factors immediately. We have $\mathbb{E}^n \approx \mathbb{E}^1 \times \ldots \times \mathbb{E}^1$ (n factors). $\circ$

Theorem 3.4.1: Product topology

With respect to the product topology on X and Y, the projections $p_X: X \times Y \to X$, $(x, y) \mapsto x$ and p_Y (defined analogously) are continuous and open (they map open sets to open sets). The product topology is the coarsest topology on $X \times Y$ for which p_X and p_Y are maps.

Remark 3.4.4 For product spaces we always assume $X \neq \varnothing \neq Y$. ○

Theorem 3.4.2: Universal property of the product

Let Z be a topological space. A function $f: Z \to X \times Y$ is a map if and only if $p_X \circ f: Z \to X$ and $p_Y \circ f$ are maps.

$$\begin{tikzcd} X & Z \arrow[l, "p_X \circ f"'] \arrow[dl, "f"] \arrow[d, "p_Y \circ f"] \\ X \times Y \arrow[u, two heads, "p_X"] \arrow[r, two heads, "p_Y"'] & Y \end{tikzcd}$$

Fig. 21: The universal property of the product.

Proof. " $\Longrightarrow$ ": The compositions of maps are maps.

" $\Longleftarrow$ ": It suffices to check continuity on the basis. Let $U \subset X$ and $V \subset Y$ be open, then

$$f^{-1}(U \times V) = (p_X \circ f)^{-1}(U) \cap (p_Y \circ f)^{-1}(V)$$

is open, where the equality is due to e.g. $(p_X \circ f)^{-1}(U) = f^{-1}(p_X^{-1}(U)) = f^{-1}(U \times Y)$. □

Theorem 3.4.3: Product of Hausdorff spaces

The product of two Hausdorff spaces is a Hausdorff space.

Proof. Let X and Y be Hausdorff and $(x_1, y_1), (x_2, y_2) \in X \times Y$ be distinct. Without loss of generality we assume that $x_1 \neq x_2$. As X is Hausdorff, there exists two open disjoint neighbourhoods $U_1, U_2 \subset X$ of x_1 and x_2, respectively. Let $V_1, V_2 \subset Y$ be any open neighbourhoods of y_1 and y_2, respectively. Then $U_1 \times V_1$ and $U_2 \times V_2$ are open neighbourhoods of (x_1, y_1) and (x_2, y_2), respectively. They are disjoint:

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \varnothing \times (V_1 \cap V_2) = \varnothing.$$ □

Theorem 3.4.4: Product of compact spaces

The product space $X \times Y$ is compact if and only if X and Y are compact.

One ingredient in the proof of " $\Longleftarrow$ " is the following lemma.

Lemma 3.4.5 (Compactness and bases)
Let $\mathbb{B}$ be a basis of X. Then X is compact if and only if every open cover of X by sets in $\mathbb{B}$ has a finite subcover.

Proof. (of lemma 3.4.5) " $\Longrightarrow$ ": Is immediate.

" $\Longleftarrow$ ": Let $(F_i)_{i \in I}$ be an open cover of X. Then for all $i \in I$ there exist $(O_j^{(i)})_{j \in J_i}$ such that $F_i = \bigcup_{j \in J_i} O_j^{(i)}$. Then $\{O_j^{(i)} : j \in J_i, i \in I\}$ is an open cover of X with elements from $\mathbb{B}$. By assumption, there exists a finite subcover: $X = \bigcup_{\ell=1}^n O_{j_\ell}^{(i_\ell)}$. But then adding sets yields $X = \bigcup_{\ell=1}^n F_{i_\ell}$, so there is a finite subcover. □

Proof. (of Theorem 3.4.4) " $\Longrightarrow$ ": Let $X \times Y$ be compact. As the projections p_X and p_Y are maps by Theorem 3.4.1, their images $X = p_X(X \times Y)$ and $Y = p_Y(X \times Y)$ are compact by Theorem 3.3.1.

" $\Longleftarrow$ ": By lemma 3.4.5, it suffices to show that any cover of $X \times Y$ by elements from $\mathbb{B}$ has a finite subcover. Let thus $(B_i := U_i \times V_i)_{i \in I} \subset \mathbb{B}$ be an open cover of $X \times Y$.

Let $x \in X$. We can consider only the elements $B_i^{(x)} = U_i^{(x)} \times V_i^{(x)} \in \mathbb{B}$ such that $x \in U_i^{(x)}$ and get a cover $\left(B_i^{(x)}\right)_i$ of $\{x\} \times Y$. As x is in all sets $B_i^{(x)}$ and Y is compact, we can choose a finite subcover $\{x\} \times Y \subseteq \bigcup_{i=1}^{n_x} \left(U_i^{(x)} \times V_i^{(x)}\right)$. Because all the $U_i^{(x)}$ are open sets, so is their finite intersection $U_x := \bigcap_{i=1}^{n_x} U_i^{(x)}$. (This intersection is not empty because x is in it.) Hence,

$$U_x \times Y \subseteq \bigcup_{i=1}^{n_x} \left(U_i^{(x)} \times V_i^{(x)}\right).$$

This way we can find a cover of X via the $U_x, x \in X$. Because X is compact, there are finitely many $x_1, \ldots, x_m \in X$ such that $X = \bigcup_{j=1}^{m} U_{x_j}$. Then $X \times Y$ is covered by:

$$X \times Y = \bigcup_{j=1}^{m} U_{x_j} \times Y \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_{x_j}} U_i^{(x_j)} \times V_i^{(x_j)}$$

and this is a finite subcover of $(B_i)_{i \in I}$ of order $\sum_{j=1}^{m} n_{x_j} < \infty$. □

Theorem 3.4.5: Heine-Borel

A subset $C \subset \mathbb{E}^n$ is compact if and only if it is closed and bounded.

Proof. " $\Longrightarrow$ ": This is Theorem 3.3.6.

" $\Longleftarrow$ ": Let C be closed and bounded. As C is bounded, there exists a $r > 0$ such that $C \subset [-r, r]^n \subset (\mathbb{E}^1)^n \approx \mathbb{E}^n$ and $[-r, r]^n$ is compact by Theorem 3.2.1 and Theorem 3.4.4. As C is a closed subset of a compact set, it is compact by Theorem 3.3.2. □

Remark 3.4.6 It is possible to define the product topology for products with arbitrarily many (e.g. unaccountably many) factors. Tychonoff's theorem generalises Theorem 3.4.4 to that case (but it is equivalent to the axiom of choice). ○

3.5 Connectedness

Definition 3.5.1 (Connected)
A space X is connected if for all nonempty subsets $A, B \subsetneq X$ with $X = A \cup B$ we have $\overline{A} \cap B \neq \varnothing$ or $A \cap \overline{B} \neq \varnothing$. connected

An equivalent definition is the following.

Definition 3.5.2 (Connected)
A space X is connected if it is not the union of two nonempty disjoint proper open subsets.

Proof. (Of the equivalence) TODO □

Example 3.5.3 (Connected) The sets $\mathbb{E}^n$ and $[0, 1]^n$ are connected for all $n \in \mathbb{N}$. ◇

Definition 3.5.4 (Connected components)
The connected components of a topological space X are its maximally connected subsets.

connected components

Connected components are disjoint, as the following lemma shows.

Lemma 3.5.5 (Tut 2)
Let (X, τ) be a topological space and $T_1, T_2 \subset X$ be connected with $T_1 \cap T_2 \neq \varnothing$ and $X = T_1 \cup T_2$. Then X is connected.

Proof. Assume that $X = O_1 \sqcup O_2$, where $O_1, O_2 \in \tau$. Taking the intersection of this equality with T_1 and T_2 yields

$$T_1 = (O_1 \cap T_1) \sqcup (O_2 \cap T_1) \qquad \text{and} \qquad T_2 = (O_1 \cap T_2) \sqcup (O_2 \cap T_2)$$

As T_1 is connected, one of the sets has to be empty. Without loss of generality let $O_2 \cap T_1 = \varnothing$ and hence $O_1 \cap T_1 = T_1$, so $T_1 \subset O_1$.

Suppose that $O_2 \cap T_2 \neq \varnothing$. Then $O_2 \cap T_2 = T_2$ by connectedness of T_2, so $T_2 \subset O_2$. We have $T_1 \cap T_2 \neq \varnothing$ but $O_1 \cap O_2$, which is a contradiction to $T_1 \subset O_1$ and $T_2 \subset O_2$. Hence $O_2 \cap T_2 = \varnothing$ and thus $O_1 \cap T_2 = T_2$.

Hence
$$X = T_1 \cup T_2 = (O_1 \cap T_1) \cup (O_2 \cap T_2) = O_1 \cap (T_1 \cup T_2) = O_1 \cap X$$
and thus $O_1 = X$, so $O_2 = \varnothing$ and hence X is connected. □

Hence there is a underlying equivalence relation induced on X, where $x \sim y$ if and only if x and y are in the same connected component of X, that is, there is a connected subspace $(U, T|_U)$ of (X, T) containing both x and y.

Lemma 3.5.6
The above relation is an equivalence relation.

Thus a space is connected precisely if it has a single connected component. By lemma 3.5.6 every connected subspace $U \subset X$ lies in a unique connected component C_U.

Lemma 3.5.7
The set C_U is the inclusion-wise maximal connected subspace of X containing U.

Lemma 3.5.8
Connected components are closed.

Proof. Let C be a connected component of X. Then C is connected. We later show that $\overline{C}$ is also connected. As C is maximal and $C \subset \overline{C}$, we have $C = \overline{C}$, so C is closed by corollary 2.3.2.

Suppose there exist nonempty proper open disjoint subsets $O_1, O_2 \subset \overline{C}$ such that $\overline{C} = O_1 \sqcup O_2$. Then $C = (O_1 \cap C) \sqcup (O_2 \cap C)$ is a disjoint union of open proper subsets, so without loss of generality $O_1 \cap C = \varnothing$ and $O_2 \cap C = C$, implying $C \subset O_2$. Suppose there exists a $x \in O_1$. Then $x \in \overline{C} \backslash C$, so x is a limit point of C, so $C \cap N \neq \varnothing$ for all $N \in \mathcal{N}_x$. Choosing $N = O_1 \in \mathcal{N}_x$ yields $C \cap O_1 \neq \varnothing$, which is a contradiction. Hence $O_1 = \varnothing$, so $\overline{C}$ is connected. □

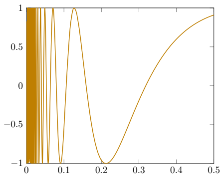

Fig. 22: The topologist's sine curve.

Example 3.5.9 (Topologist's sine curve) The topologist's sine curve

$$X := \{0\} \times [-1,1] \cup \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x > 0 \right\} \subset \mathbb{R}^2$$

has only one connected component: the set $\{0\}\times[-1,1]$ is connected and so is $\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0\right\}$ as the image of a connected set under a continuous function. But they also belong to the same connected component: take p in $\{0\} \times [-1,1]$, for example $p = 0$, then there are points of $\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0\right\}$ in every neighbourhood of p. Thus $\{0\} \times [-1,1]$ is not open in X and hence $\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0\right\}$ is not closed in X and cannot be a connected component on its own. ◇

3.6 Joining points by paths

Definition 3.6.1 (Path-connected)
A space X is path-connected if any two points $x, y \in X$ can be connected by a path, that is, there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

path-connected

Theorem 3.6.1: Path-connected $\implies$ connected

Path-connected spaces are connected.

Proof. Let $A \subsetneq X$ be a nonempty open subset. Suppose A is also closed. Pick $x \in A$ and $y \in X \backslash A$. As X is path-connected, there exists a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Because $0 \in \gamma^{-1}(A)$, $\gamma^{-1}(A)$ is a nonempty subset of $[0,1]$, which is both open and closed as γ is a map by Theorem 2.4.4. This contradicts $[0,1]$ being connected. □

Definition 3.6.2 (Concatenation of paths)
Let $x, y, z \in X$. The concatenation of a path α from x and y and a path β from y to z is

$$\alpha\beta\colon [0,1] \to X, \qquad t \mapsto \begin{cases} \alpha(2t), & \text{if } t \in [0, \frac{1}{2}], \\ \beta(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

The concatenation is a map as composition of maps and well-defined as $\alpha(2\frac{1}{2}) = \alpha(1) = y = \beta(0) = \beta(2\frac{1}{2} - 1)$.

Theorem 3.6.2: Open connected sets

Let $X \subset \mathbb{E}^n$ be open and connected. Then X is path-connected.

Proof. For $x \in X$ let

$$U(x) := \{y \in X : \exists \text{ map } \gamma\colon [0,1] \to X \text{ with } \gamma(0) = x,\ \gamma(1) = y\}$$

This should also hold if $\mathbb{E}^n$ is replaced by a locally convex TVS, but the proof has to be modified.

be the path-component of x. As $x \in U(x)$, $U(x) \neq \varnothing$. Furthermore, $U(x)$ is path-connected: for $y, z \in U(x)$ there exists paths from y to x and from x to z, whose concatenation is a path from y to z.

Further, $U(x)$ is open: let $y \in U(x)$. Since $X \subset \mathbb{E}^n$ is open, there exists a $\varepsilon > 0$ such that $B := B^n(y, \varepsilon) \subset X$. Let $z \in B$. As B is convex, z is connected to y by a straight line path in $B \subset X$. Via concatenation, z is joined to x by a path and thus $z \in U(x)$.

Assume $U(x) \subsetneq X$. Then $X \backslash U(x) = \bigcup_{y \in X \backslash U(x)} U(y)$ is open. Hence $U(x)$ is a nontrivial open and closed subset of X, contradicting that X is connected. $\square$

4 Identification spaces

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We follow the general strategy of defining various notions of shapes, the purpose of topology being to make these notions precise. Everything discussed so far was a mild generalisation of material known from Calculus. The following is different: we will talk about genuine topological constructions you will have probably not encountered before.

4.1 Initial and final topology

Let X and Y be sets and $f\colon X \to Y$ be a surjective function.

Suppose Y is a topological space. We can then use the topology on Y and the function f to define a topology on X such that f is a map. An obvious choice is the discrete topology on X. The following is more efficient:

Definition 4.1.1 (Initial topology)
The initial topology on X is $\tau_X = \{f^{-1}(A) : A \in \tau_Y\}$, the coarsest topology such that f is a map.

Now suppose that instead X is a topological space. We can then use the topology on X and the function f to define a topology on Y such that f is a map. Again an obvious choice is the trivial topology $\{\varnothing, Y\}$ on Y.

Definition 4.1.2 (Final topology)
The final topology on Y is $\{O \subset Y : f^{-1}(O) \subset X \text{ open}\}$, which is the finest topology such that f is a map.

This can be generalised as follows.

Definition 4.1.3 (Initial topology)
Let X be a set and J a index set. For every $j \in J$, let Y_j be a topological space and $f_j\colon X \to Y_j$ a function. The initial topology with respect to the family $(f_j)_{j\in J}$ is a topology on X such that

initial topology

A All functions $f_j\colon X \to Y_j$ are maps.

B For every topological space Z, every function $g\colon Z \to X$ is a map if and only if every composition $f_j \circ g$, where $j \in J$, is a map.

$$\begin{tikzcd} X \arrow[r, "f_j"] & Y_j \\ Z \arrow[u, "g"] \arrow[ur, "f_j \circ g"'] & \end{tikzcd}$$

Fig. 23: Commutative diagram for the characteristic property **B** of the initial topology on X.

Lemma 4.1.4
An initial topology on X exists and is unique.

Proof. Existence. The initial topology has the subbasis $\{f_j^{-1}(U_j) : U_j \subset Y_j \text{ open}, j \in J\}$. Then **A** is clearly fulfilled. Let $g\colon Z \to X$ be a function and $f_j \circ g$ be a map for every $j \in J$. Let $O \subset X$ be open. Then O is the union and finite intersection of sets of the form $f_j^{-1}(U_j)$, where $U_j \subset Y_j$ is open. Hence g^{-1} is the union and finite intersection of sets of the form $g^{-1}(f_j^{-1}(U_j)) = (f_j \circ g)^{-1}(U_j)$, which is open as $f_j \circ g$ is a map.

Uniqueness. Analogous to the proof of lemma 4.1.10. □

Example 4.1.5 (Product topology is initial topology wrt to projections)
For a finite index set J, the cartesian product $X := \prod_{j\in J} Y_j$, the initial topology on X with respect to the projection maps $p_j : X \to Y_j$ coincides with the product topology. ◇

Proof. We show that the initial topology on $X \times Y$ with respect to the projections $p_X : X \to X \times Y$ and $p_Y : Y \to X \times Y$ is the product topology on $X \times Y$.

The subbasis in this case is

$$\begin{aligned} B &:= \{p_X^{-1}(U) : U \subset X \text{ open}\} \cup \{p_Y^{-1}(V) : V \subset Y \text{ open}\} \\ &= \{U \times Y : U \subset X \text{ open}\} \cup \{X \times V : V \subset Y \text{ open}\}. \end{aligned}$$

Taking finite intersections of elements in B precisely yields $\{U \times V : U \subset X \text{ open}, V \subset Y \text{ open}\}$, which is exactly the basis of the product topology. □

Now, reconsider an arbitrary index set J and again the cartesian product

$$\prod_{j\in J} Y_j = \left\{ \text{functions } x : J \to \bigcup_{j\in J} Y_j \text{ with } x(j) \in Y_j \quad \forall j \in J \right\}$$

and define the product topology as the initial topology with respect to the projection maps $p_j : \prod_{j\in J} Y_j \to Y_j$, $x \mapsto x(j)$ for $j \in J$. TODO: Determine a basis for the product topology in this general case.

Remark 4.1.6 (Motivation: Embedding in linear algebra)
If V and W are $\mathbb{K}$ vector spaces and $f : V \to W$ is $\mathbb{K}$-linear and injective, then $f(V) \subset W$ is a subspace of W that can be identified with V. ○

Note that the subspace topology on $A \subset X$ is the initial topology with respect to the inclusion map $A \hookrightarrow X$. This can be generalised.

Definition 4.1.7 (Embedding)
A map $\iota : A \to X$ is called embedding if ι is injective and A carries the initial topology with respect to the funciton ι. embedding

Lemma 4.1.8 (Characterisation of embeddings)
A function $\iota : A \to X$ is an embedding if and only if ι is a homeomorphism onto its image $\iota(A) \subset X$ equipped with the subspace topology.

Proof. The initial topology on A with respect to ι is $\tau_A := \{\iota^{-1}(O) : O \subset X \text{ open}\}$.

Hence a subset $U \subset A$ is open if and only if there exists an open set $O \subset X$ such that $\iota^{-1}(O) = U$, which is the case if and only if there exists an open set $O \subset X$ such that $\iota^{-1}(O \cap \iota(A)) = U$. This is the case precisely if there exists an open set $O \subset X$ such that $\iota(\iota^{-1}(O \cap \iota(A))) = \iota(U)$. Due to injectivity of ι, this is equivalent to $O \cap \iota(A) = \iota(u)$, which is equivalent to $\iota(U) \subset \iota(A)$ being open. □

We can generalise the final topology similarly.

Definition 4.1.9 (Final topology)
Let X be a set and J a index set. For every $j \in J$, let Y_j be a topological space and $f_j : Y_j \to X$ be a function. The final topology with respect to the family $(f_j)_{j\in J}$ is topology on X such that final topology

A' All functions $f_j\colon Y_j \to X$ are maps.

B' For every topological space Z every function $g\colon X \to Z$ g is a map if and only if the composition $g \circ f_j$, where $j \in J$, is a map.

$$\begin{tikzcd} (X, T') \arrow[d, "g"'] & Y_j \arrow[l, "f_j"'] \arrow[dl, "g \circ f_j"] \\ Z & \end{tikzcd}$$

Fig. 24: Commutative diagram for the characteristic property **B'** of the final topology T' on X.

Lemma 4.1.10

A final topology on X exists and is unique.

Proof. Existence. The final topology on X is $\{O \subset X : f_j^{-1}(O) \subset Y_j \text{ open } \forall j \in J\}$.

Uniqueness. Consider the commutative diagram

$$\begin{tikzcd} Y_j \arrow[r, "f_j"] \arrow[dr, "x \mapsto f_j(x)"'] & (X, \tau_{\boldsymbol{B'}}) \arrow[d, "i"] \\ & (X, \tau_{\boldsymbol{A'}}) \end{tikzcd}$$

Fig. 25: Commutative diagram for the proof of uniqueness of the final topology.

where $\tau_{\boldsymbol{B'}}$ is topology fulfilling **B'** and analogously for $\tau_{\boldsymbol{A'}}$.

We want to show that $\tau_{\boldsymbol{A'}} \subset \tau_{\boldsymbol{B'}}$, which is equivalent to i being a map (the continuity of i is $U \in \tau_{\boldsymbol{A'}} \implies U \in \tau_{\boldsymbol{B'}}$). Applying **B'** to i yields that i is a map if and only if $i \circ f_j$ is a map for all $j \in J$, which is exactly **A'** applied to $i \circ f_j$.

Now suppose T_1 and T_2 both fulfil **A'** and **B'**. As T_1 fulfils **A'** and T_2 fulfils **B'**, we have $T_1 \subset T_2$. Analogously, we have $T_2 \subset T_1$, so $T_1 = T_2$. □

We will later see that the identification topology on any identification space $X/\sim$ is the final topology with respect to the projection map $q\colon X \to X/\sim$.

The disjoint union of topological spaces (also called coproduct) is dual to the product of topological spaces in the following sense.

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, dotted, "f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd} \qquad \begin{tikzcd} & Y & \\ X_1 \arrow[ur, "f_1"] \arrow[r, "\varphi_1"'] & X_1 \coprod X_2 \arrow[u, dotted, "f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, "\varphi_2"] \end{tikzcd}$$

Fig. 26: Reversing the arrows in the commutative diagram for the product yields the commutative diagram for the disjoint union.

Definition 4.1.11 (Disjoint union)

The disjoint union $X := \coprod_{i \in I} X_i := \bigcup_{i \in I} X_i \times \{i\}$ carries the final topology with respect to the canonical injections

disjoint union

$$\varphi_i\colon X_i \to X, \qquad x \mapsto (x, i).$$

The universal property of the disjoint union states that if all f_j are maps, then there exists exactly one map f as in figure 26. This implies that f is a map if and only if $f_j = f \circ \varphi_j$ are all maps.

4.2 The identification topology

Example 4.2.1 (Construction of a Möbius strip)
We can construct a Möbius strip by gluing a piece of paper. We want to formalise this process.

Let $R \coloneqq \{(x, y) \in \mathbb{E}^2 : 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 1\}$ be a rectangle (a product space). We want to identify points on the boundary of R. We partition the points (equivalently: define an equivalence relation $\sim$) into

1. $\{(0, y), (3, 1-y)\}$ for every $y \in [0, 1]$,
2. $\{(x, y)\}$ for all $x \in (0, 3)$ and $y \in [0, 1]$. ◇

We say that all the type-2 points are only equivalent to themselves and $(0, y)$ is equivalent to $(3, 1-y)$ for every $y \in [0, 1]$. Let $M \coloneqq R/\sim$ be the Möbius strip as a set.

We define the canonical projection $\pi\colon R \to M$, $(x, y) \mapsto [(x, y)]_\sim$. The identification topology (or quotient topology) on M is the finest topology on M such that π is continuous. Hence $O \subset M$ is open if and only if $\pi^{-1}(O) \subset R$ is open.

As M is the range of the function π (for product spaces we took the coarsest topology making the projection continuous, but there the domain of the projection was the product space), we pick the finest topology such that π is continuous.

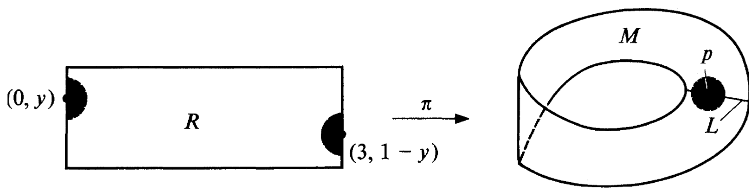

Fig. 27: Open sets in the identification topology of the Möbius strip.

We now generalise this example to arbitrary topological spaces and equivalence relations.

Let X be a topological space with an equivalence relation $\sim$. We define $Y \coloneqq X/\sim$ and the projection $\pi\colon X \to Y$, $x \mapsto [x]_\sim$.

Definition 4.2.2 (Identification topology)
The identification (or quotient) topology on Y is the finest topology such that π is a map.

Compare the following theorem to Theorem 3.4.2 and notice that figure 28 is figure 21 with the arrows reversed.

Theorem 4.2.1: Universal property of the quotient

Let Z be a topological space. A function $f\colon Y \to Z$ is a map if and only if $f \circ \pi\colon X \to Z$ is a map.

$$\begin{tikzcd} X \arrow[r, "f\circ\pi"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, "f"'] & \end{tikzcd}$$

Fig. 28: The universal property of the quotient space states that this diagram commutes.

Proof. " $\implies$ ": The composition of maps is a map.

" $\impliedby$ ": Let $U \subset Z$ be open. Then $f^{-1}(U)$ is open in Y if and only if $\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U)$ is open in X. □

Definition 4.2.3 (Identification map)
A surjective map $f\colon X \to Y$ is an identification map (or quotient map) if the topology on Y is the finest topology such that f is a map.

identification map

In other words: a surjective map $f\colon X \to Y$ is a quotient map if it is surjective and $T_Y := \{O \subset Y : f^{-1}(O) \subset X \text{ open}\}$.

Example 4.2.4 (Surjective map $f\colon X \to Y$ such that $X/\sim_f \not\approx Y$)
Consider $i\colon X \to (X, \tau)$, where τ is any topology coarser than the topology on X. Then $(X/\sim_i) = X$.

More concrete: let $w\colon [0,1] \twoheadrightarrow \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$. Then $(\mathbb{S}^1, \tau) \approx [0,1]/\sim_w$, where τ is the Euclidean topology. So $\mathbb{S}^1$ carries the finest topology such that w is continuous. Choosing $\tau_{\mathbb{S}^1} = \{\varnothing, \mathbb{S}^1\}$ breaks this, because it is not the finest topology. ◇

Definition 4.2.3 says that f is compatible with the quotient construction. One can see this the other way around: for a given topological space X and a surjective map $f\colon X \to Z$, f defines an equivalence relation on X (and hence a topology on Z) via $x \sim_f x'$ if and only if $f(x) = f(x')$, which has the property that the projection $\pi\colon X \to X/\sim_f$ satisfies "$\pi = f$".

$$\begin{tikzcd} X \arrow[r, "g\circ f"] \arrow[d, "\pi"'] \arrow[dr, two heads, "f"] & A \\ X/\sim_f \arrow[r, leftrightarrow, "\approx"'] & Z \arrow[u, "g"'] \end{tikzcd}$$

Fig. 29: Mapping properties of the identification map.

Theorem 4.2.2: Mapping properties of the quotient

If $f\colon X \to Z$ is an identification map, then
1. $Z \approx X/\sim_f$.
2. $g\colon Z \to A$ is a map if and only if $g \circ f$ is a map.

Proof. 2 is just the proof of Theorem 4.2.1 because by definition, Z carries the finest topology making f continuous: let $O \subset A$ be open. Then $g^{-1}(O)$ is open if $f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O) \subset X$ is open, which is fulfilled as $g \circ f$ is a map.

For 1, let $\pi\colon X \to X/\sim_f$ be the quotient map and $h\colon X/\sim_f \to Z$ be defined by $[x]_{\sim_f} \mapsto f(x)$, which is well defined because if $y \in [x]_{\sim_f}$, then $f(y) = f(x)$ by the definition of $\sim_f$. We now show that h is bijective. First, h is injective by construction: take $[x_1]_{\sim_f}, [x_2]_{\sim_f} \in X/\sim_f$ with $h([x_1]_{\sim_f}) = h([x_2]_{\sim_f})$. By the definition of $\sim_f$, this means $f(x_1) = f(x_2)$, which is the case if and only if $x_1 \in [x_2]_{\sim_f}$, that is, $[x_1]_{\sim_f} = [x_2]_{\sim_f}$. Secondly, h is surjective because f is: for $y \in Y$ there exists a $x \in X$ with $f(x) = y$. Then $h([x]_{\sim_f}) = f(x) = y$. We have $h \circ \pi = f$.

Also, h is a map: let $O \subset Y$ be open. Then $h^{-1}(O) \subset X/\sim_f$ is open if and only if $\pi^{-1}(h^{-1}(O)) = (h \circ \pi)^{-1}(O) = f^{-1}(O)$ is open, which is true as f is a map. Lastly, h^{-1} is a map by 2 (choosing $A = X/\sim_f$) as $\pi = h^{-1} \circ f$ is a map. □

The following theorem yields a more easy to check criterion for identification maps.

Theorem 4.2.3: Open/closed, surjective map $\Longrightarrow$ quotient map

If a surjective map $f\colon X \to Y$ is open or closed, then it is an identification map.

Proof. Suppose f is open. Let $U \subset Y$ be such that $f^{-1}(U)$ is open in X. As f is onto, $f(f^{-1}(U)) = U$, so the only way to make $f^{-1}(U)$ open is to make U is open. As Y is open in Y, the given topology on Y is the finest topology such that f is a map. Hence f is an identification map. □

Corollary 4.2.5 (Surjective map from compact to Hausdorff space)
Let $f\colon X \to Y$ be a surjective map. If X is compact and Y is Hausdorff*, then f is an identification map.*

Proof. Let $C \subset X$ be closed. As X is compact, C is compact by Theorem 3.3.2. Hence $f(C) \subset Y$ is compact by Theorem 3.3.1. As Y is Hausdorff, $f(C)$ is closed by Theorem 3.3.3. Hence f is closed. By Theorem 4.2.3, f is an identification map. $\square$

The notion of CW-complexes (patching together closed disks of various dimension under certain conditions) only works due to corollary 4.2.5.

Lemma 4.2.6
Let X, Y_1 and Y_2 be topological spaces and let $q_1\colon X \to Y_1$ and $q_2\colon X \to Y_2$ be identification maps such that for all $x, x' \in X$ we have $q_1(x) = q_1(x')$ if and only if $q_2(x) = q_2(x')$. Then there is a unique map $h\colon Y_1 \to Y_2$ with $h \circ q_1 = q_2$. In particular, h is a homeomorphism.

$$\begin{tikzcd} X \arrow[r, two heads, "q_1"] \arrow[dr, "q_2"'] & Y_1 \arrow[d, dotted, "\exists h"] \\ & Y_2 \end{tikzcd}$$

Fig. 30

Proof. We define $h\colon Y_1 \to Y_2$ via $y \mapsto q_2(x)$ for some $x \in q_1^{-1}(y)$, which is the only choice set-theoretically. Then h is well-defined as for all $x, x' \in X$ we have $q_1(x) = q_1(x')$ if and only if $q_2(x) = q_2(x')$. As $h \circ q_1 = q_2$ is a map, h is a map by the universal property.

Analogously $g\colon Y_2 \to Y_1$ is a map as $g \circ q_2 = q_1$. We have $h \circ g = \mathrm{id}_{Y_2}$ and $g \circ h = \mathrm{id}_{Y_1}$ and thus $h^{-1} = g$, so h is a homeomorphism. $\square$

Lemma 4.2.7
We have $[0,1]/\{0,1\} \approx \mathbb{S}^1$.

$$\begin{tikzcd} I \arrow[r, two heads, "g"] \arrow[dr, two heads, "\pi"'] & \mathbb{S}^1 \arrow[d, dotted, leftrightarrow, "\exists \approx"] \\ & {[0,1]/_{0\sim 1}} \end{tikzcd}$$

Fig. 31: Commutative diagram for $\mathbb{S}^1 \approx [0,1]/\{0,1\}$.

Proof. Let $g\colon [0,1] \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$ Then g is a surjective map. As $[0,1]$ is compact and $\mathbb{S}^1$ is Hausdorff, g is an identification map by corollary 4.2.5. There exists a quotient map $[0,1] \to [0,1]/\sim_g = [0,1]/\{0,1\}$. Hence by lemma 4.2.6 there exists a homeomorphism $h\colon [0,1]/\{0,1\} \to \mathbb{S}^1$ defined by $h^{-1}([t]) = e^{2\pi i t}$. $\square$

Example 4.2.8 (Torus) Let $X := [0,1] \times [0,1] \subset \mathbb{E}^2$. Partition X into

1. $\{(0,0),(0,1),(1,0),(1,1)\}$ (the set of vertices of the square),
2. $\{(x,0),(x,1)\}$ for every $x \in (0,1)$,
3. $\{(0,y),(1,y)\}$ for every $y \in (0,1)$,
4. $\{(x,y)\}$ for $x, y \in (0,1)$.

The torus is $T := X/\sim$.

For $\mathbb{S}^1 \subset \mathbb{C}$ consider

$$f\colon X \to \mathbb{S}^1 \times \mathbb{S}^1, \qquad (x,y) \mapsto (e^{2\pi i x}, e^{2\pi i y}).$$

Hence $\sim = \sim_f$ and thus $T = X/\sim \approx \mathbb{S}^1 \times \mathbb{S}^1$ by Theorem 4.2.2. $\diamond$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, two heads, "\pi"'] & \mathbb{S}^1 \times \mathbb{S}^1 \arrow[d, dotted, leftrightarrow, "\exists \approx"] \\ & T \end{tikzcd}$$

Fig. 32: The torus is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.

Definition 4.2.9 (Cone)
Let X be a topological space. On $X \times I$ consider the equivalence relation with partition

1. $X \times \{1\}$,
2. $\{(x,t)\}$ for every $(x,t) \in X \times [0,1)$.

Then $CX := (X \times I)/\sim$ is the cone on X.

cone

Example 4.2.10 (Geometric cone) Let $X \subset \mathbb{E}^n$ be compact (e.g. a polyhedron) and the apex $a \in \mathbb{E}^{n+1} \setminus (\mathbb{E}^n \times \{0\})$, e.g. $a = (0, \dots, 0, 1)$ with n zero coordinates. Then the subspace

apex

$$G := \{ta + (1-t)x : x \in X \times \{0\}, t \in [0,1]\} \subset \mathbb{E}^{n+1}$$

is homeomorphic to the cone on X: define

$$f\colon X \times I \to G, \qquad (x,t) \mapsto ta + (1-t)x,$$

which is a surjective map. As $\sim = \sim_f$ (Exercise), the statement follows by lemma 4.2.6. ◇

$$\begin{tikzcd} X \times I \arrow[r, "f"] \arrow[dr, two heads, "\pi"'] & G \arrow[d, dotted, leftrightarrow, "\exists"', "\approx"] \\ & CX \end{tikzcd}$$

Fig. 33

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Example 4.2.11 Let X be a topological space and $A \subset X$. Consider the equivalence relation $\sim$ on X with partition (1) A and (2) $\{x\}$ for every $x \in X \setminus A$. Special cases include

- the cone over X, $CX = (X \times I)/\sim \approx (X \times I)/(X \times \{1\})$,
- $\mathbb{B}^n/\mathbb{S}^{n-1} \approx \mathbb{S}^n$. ◇

Proof. (For the second special case: $\mathbb{B}^n/\mathbb{S}^{n-1} \approx \mathbb{S}^n$) Consider

$$f\colon \mathbb{B}^n \to \mathbb{S}^n, \qquad x \mapsto \begin{cases} (h_2 \circ h_1)(x), & \text{if } x \in \mathbb{B}^n \setminus \mathbb{S}^{n-1}, \\ p, & \text{if } x \in \mathbb{S}^{n-1}, \end{cases}$$

where $p \in \mathbb{S}^n$ is arbitrary but fixed and $h_1\colon \mathbb{B}^n \setminus \mathbb{S}^{n-1} \to \mathbb{E}^n$ is any homeomorphism (e.g. radially applying arctan) and $h_2\colon \mathbb{E}^n \to \mathbb{S}^n \setminus \{p\}$ is the (inverse of the) stereographic projection, which is a homeomorphism. Then f is a surjective map. As $\mathbb{B}^n$ is compact and $\mathbb{S}^n$ is Hausdorff, f is an identification map by corollary 4.2.5. □

$$\begin{tikzcd} \mathbb{B}^n/\mathbb{S}^{n-1} & & \\ \mathbb{B}^n \arrow[u, two heads, "\pi"] \arrow[rr, "\exists f"] & & \mathbb{S}^n \arrow[ull, dotted, "\approx"'] \\ \mathbb{B}^n \setminus \mathbb{S}^{n-1} \arrow[u, hook] \arrow[dr, "h_1", "\approx"'] & & \mathbb{S}^n \setminus \{p\} \arrow[u, hook] \\ & \mathbb{E}^n \arrow[ur, "h_2", "\approx"'] & \end{tikzcd}$$

Fig. 34: Visualising the construction in the proof of $\mathbb{B}^n \setminus \mathbb{S}^{n-1} \approx \mathbb{S}^n$.

Here's a different, more complicated proof.

Proof. The idea is to find a surjective map $w\colon \mathbb{D}^{n+1} \to \mathbb{S}^{n+1}$ such that $w(\mathbb{S}^n) = \{N\}$, where $N \in \mathbb{S}^{n+1}$ is some point and $f|_{\mathbb{D}^{n+1} \setminus \mathbb{S}^{n+1}}$ is injective.

First, let $f\colon \mathbb{S}^n \times \mathbb{D}^1 \to \mathbb{D}^{n+1}$, $(x,s) \mapsto \frac{s+1}{2}x$, which is continuous and surjective. (Note that $f(\mathbb{S}^n \times \{r\}) = \tilde{r}\mathbb{S}^n$.) We have $x \sim_f x'$ if and only if $x, x' \in \mathbb{S}^n \times \{-1\}$. As $\mathbb{S}^n \times \mathbb{D}^1$ is compact and $\mathbb{D}^{n+1}$ is Hausdorff, f is an identification map.

Now, let $g\colon \mathbb{S}^n \times \mathbb{D}^1 \to \mathbb{S}^{n+1}$, $(x,s) \mapsto (\sqrt{1-s^2}x_1, \dots, \sqrt{1-s^2}x_{n+1}, s)$, which maps $\mathbb{S}^1$ to circles of radius $\sqrt{1-s^2}$ on $\mathbb{S}^2$. As g is continuous and surjective, it is an identification map. We have $x \sim_g x'$ if and only if $x = x'$ or $x, x' \in \mathbb{S}^n \times \{1\}$ or $x, x' \in \mathbb{S}^n \times \{-1\}$.

We hence have $\mathbb{D}^{n+1} \approx (\mathbb{S}^n \times \mathbb{D}^1)/(\mathbb{S}^n \times \{-1\})$ and $\mathbb{S}^{n+1} \approx (\mathbb{S}^n \times \mathbb{D}^1)/(\mathbb{S}^n \times \{1,-1\})$. Hence $x \sim_f x'$ implies $x \sim_g x'$ and so $\sim_f$ is stronger than $\sim_g$. There exists a map $w\colon \mathbb{D}^{n+1} \to \mathbb{S}^{n+1}$ with $w \circ f = g$. As g is surjective, w is, too. Further, w is continuous by the universal property, as $\mathbb{D}^{n+1}$ has the identification topology with respect to f. As $\mathbb{D}^{n+1}$ is compact and $\mathbb{S}^{n+1}$ is Hausdorff, w is a quotient map.

$$\begin{tikzcd} & \mathbb{D}^{n+1} \approx (\mathbb{S}^n \times \mathbb{D}^1)/(\mathbb{S}^1 \times \{1\}) \arrow[d, dotted, two heads, "w"] \\ \mathbb{S}^n \times \mathbb{D}^1 \arrow[ur, two heads, "f"] \arrow[r, two heads, "g"'] & \mathbb{S}^{n+1} \approx (\mathbb{S}^n \times \mathbb{D}^1)/(\mathbb{S}^n \times \{\pm 1\}) \end{tikzcd}$$

Fig. 35: Visualising the construction in the second proof for $\mathbb{B}^n \setminus \mathbb{S}^{n-1} \approx \mathbb{S}^n$.

We have to show that $x \sim_w x'$ if and only if $x, x' \in \mathbb{S}^n$. We have $g(y) = g(y')$ if and only if $y, y' \in \mathbb{S}^n \times \{1\}$ or $y, y' \in \mathbb{S}^n \times \{-1\}$. Hence $w(x) = w(x')$ if and only if $x, x' \in f(\mathbb{S}^n \times \{1\})$ or

$x, x' \in f(\mathbb{S}^n \times \{-1\})$ (the latter condition is equivalent to $x = x' = 0$). Hence $w(x) = w(x')$ if and only if $x, x' \in \mathbb{S}^n$. $\square$

Theorem 4.2.4: Properties carried over to quotient spaces

Let X be a topological space with some equivalence relation $\sim$ and let $Y := X/\sim$. If X is compact/(path-)connected, then so is Y.

Proof. The first statement is Theorem 3.3.1 as the projection is a map.

Suppose Y were not connected, then there exists a nonempty proper open and closed subset $U \subset Y$. But then $\pi^1{-}1(U) \subset X$ would be a proper (because π is surjective) nonempty open and closed subset, contradicting the connectedness of X.

Pick $y_0, y_1 \in Y$ and a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) \in \pi^{-1}(\{y_0\})$ and $\gamma(1) \in \pi^{-1}(\{y_1\})$. Then $\pi \circ \gamma$ is a path joining y_0 and y_1. $\square$

Example 4.2.12 (Quotients of HD spaces need not be HD)
The space $X := \mathbb{E}^1$ is Hausdorff, but $X/\sim$ where $r \sim s$ if $r - s \in \mathbb{Q}$ is not Hausdorff. ◇

Glueing and attaching maps

Definition 4.2.13 (Glueing map (with universe))
Let X and Y be subsets of some topological space ("universe") U. The topologies on X, Y and $X \cup Y$ are the subspace topologies induced by U. Let Z be another space and let $f\colon X \to Z$ and $g\colon Y \to Z$ be functions with $f(p) = g(p)$ for all $p \in X \cap Y$. Then the function obtained from f and g by glueing is

$$f \cup g\colon X \cup Y \to Z, \qquad q \mapsto \begin{cases} f(q), & \text{if } q \in X, \\ g(q), & \text{if } q \in Y \end{cases}$$

$$\begin{tikzcd} X \arrow[d, hook] \arrow[dr, "f"] & \\ X \cup Y \arrow[r, dashed, "f \cup g"] & Z \\ Y \arrow[u, hook] \arrow[ur, "g"'] & \end{tikzcd}$$

Fig. 36: The glueing of the maps f and g.

Lemma 4.2.14 (Glueing lemma)
If X and Y are closed in $X \cup Y$ and f and g are maps, then $f \cup g$ is a map.

Proof. Let Z be closed. As f is a map, $f^{-1}(Z) \subset X$ is closed. As X is closed in $X \cup Y$, $f^{-1}(Z) \subset X \cup Y$ is closed. Similarly, $g^{-1}(Z) \subset X \cup Y$ is closed, so $(f \cup g)^{-1}(Z) = f^{-1}(Z) \cup g^{-1}(Z)$ is closed, making $f \cup g$ a map. $\square$

Remark 4.2.15 The proof shows that the statement remains valid if we instead require X and Y to be open in $X \cup Y$. ○

Let $Y' := \{y' : y \in Y\}$, where $'\colon Y \to Y'$ is bijective. Then $X \cap Y' = \varnothing$. Exercise: find $X \times Y \to X + Y := X \cup Y'$.)

Let $X + Y$ be the disjoint union of X and Y. Consider the map (!)

$$j\colon X + Y := X \cup Y' \to X \cup Y \subset U, \qquad p \mapsto p,$$

whose restriction to X resp. Y is the inclusion into $X \cup Y$. As a base for the topology on $X + Y$ we take the union of the topologies on X and Y. **Should be the final topology with respect to the canonical inclusions ι_1, ι_2??**

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[drr, hook, "\iota_1"] & & \\
Z & X \cup Y \arrow[l, "f \cup g"'] & & X + Y = X \cup Y' \arrow[ll, "j"] \\
& Y \arrow[ul, "g"] \arrow[urr, hook, "\iota_2"] \arrow[rr, "\prime"] & & Y' \arrow[u, hook]
\end{tikzcd}$$

Fig. 37: The glueing of f and g in the disjoint union of X and Y.

Then

$$(f \cup g) \circ j \colon X + Y \to Z$$

is a map if and only if f and g are maps as $\big((f \cup g) \circ j\big)^{-1}(U) = f^{-1}(U) + g^{-1}(U)$ for $U \subset Z$. Combining this with the second statement from Theorem 4.2.2 proves the following theorem.

Theorem 4.2.5: Different version of glueing lemma

If j is an identification map and f and g are maps, then $f \cup g$ is a map.

Remark 4.2.16 The glueing lemma is a special case this theorem. ○

Remark 4.2.17 If j is an identification map, then $O \subset X \cup Y$ is open if and only if $O \cap X \subset X$ and $O \cap Y \subset Y$ are open. We say that $X \cup Y$ has the identification topology. ○

Remark 4.2.18 This generalises to the case of an arbitrary union of sets (for details see [1, p. 70]). ○

Real projective spaces

We describe $\mathbb{R}\mathrm{P}^n$ as incidence geometry. Consider the vector space $\mathbb{R}^{n+1}$. We declare the points/lines/k-planes/$(n-1)$-planes to be the one/two/$(k+1)$/n-dimensional subspaces of $\mathbb{R}^{n+1}$. With respect to the inclusion $(\mathbb{R}^{n+1}, \{k\text{-planes} : k \in \{0, \dots, n\}\})$ forms a partially ordered set and the group $\mathrm{GL}_{n+1}(\mathbb{R})$ operates on it (i.e. maps k-planes to k-planes such that the partial order stays intact).

As an example, consider $n = 2$. In the real projective space $\mathbb{R}\mathrm{P}^2$, any two distinct lines meet in a point (there are no parallel lines) and are contained in a unique line.

We now equip the set of points of $\mathbb{R}\mathrm{P}^n$ with a topology in several ways.

1. Consider the equivalence relation $\sim$ on $\mathbb{E}^{n+1} \backslash \{0\}$ given by $p \sim q$ if there exists a $\lambda \in \mathbb{R}$ such that $\lambda p = q$. The equivalence classes are the one-dimensional subspaces, so exactly the points in $\mathbb{R}\mathrm{P}^n$. Hence $(\mathbb{E}^{n+1} \backslash \{0\}) / \sim$ defines a topology on $\mathbb{R}\mathrm{P}^n$.

2. Consider $\mathbb{S}^n \subset \mathbb{E}^{n+1}$ with the equivalence relation $p \sim -p$. Then $\mathbb{R}\mathrm{P}^n \approx \mathbb{S}^n / \sim$, since every one dimensional subspace intersects $\mathbb{S}^n$ in exactly two points, which are antipodal.

3. Consider $\mathbb{B}^n \subset \mathbb{E}^n$ with an equivalence relation with the partition a) $\{p, -p\}$ for all $p \in \mathbb{S}^{n-1} = \partial \mathbb{B}^n$ and b) $\{q\}$ for all $q \in \mathbb{B}^n \backslash \mathbb{S}^{n-1}$. Then $\mathbb{B}^n / \sim \approx \mathbb{R}\mathrm{P}^n$.

Remark 4.2.19 We have $\mathbb{R}\mathrm{P}^1 \approx \mathbb{S}^1$: consider the unit circle as a piece of string and then lay one half onto the other such that antipodal points are on top of each other. We can project this onto the plane and the image will be exactly $\mathbb{S}^1$. Every point on this projected $\mathbb{S}^1$ will have exactly two preimages in the later called covering space. ($\mathbb{S}^1$ is its own covering space via $z \mapsto z^2$). ○

Lemma 4.2.20 (Properties of projective space)
Consider the first formulation of the projective space $\mathbb{RP}^n$. The quotient map $q\colon X := \mathbb{R}^{n+1}\setminus\{0\} \to X/\sim = \mathbb{RP}^n$ is open. Furthermore, $\mathbb{RP}^n$ is compact, Hausdorff and path-connected.

Proof. 1 Let $U \subset X$ be open. For $\lambda \in \mathbb{R}\setminus\{0\}$, let $\lambda U := \{\lambda u : u \in U\} \subset X$, which is open: let $x = \lambda u \in \lambda U$. As U is open, there exists a $\varepsilon > 0$ such that $B(u, \varepsilon) \subset U$. Then $B(\lambda u, |\lambda|\varepsilon) = B(x, |\lambda|\varepsilon) \in \lambda U$. Then

$$\pi\left(\bigcup_{\lambda\in\mathbb{R}\setminus\{0\}} \lambda U\right) = \bigcup_{\lambda\in\mathbb{R}\setminus\{0\}} \pi(\lambda U) = \bigcup_{\lambda\in\mathbb{R}\setminus\{0\}} \pi(U) = \pi(U), \tag{2}$$

so by the surjectivity of the quotient map π (S) we obtain that

$$\pi^{-1}(\pi(U)) \overset{(2)}{=} \pi^{-1}\left(\pi\left(\bigcup_{\lambda\in\mathbb{R}\setminus\{0\}} \lambda U\right)\right) \overset{(S)}{=} \bigcup_{\lambda\in\mathbb{R}\setminus\{0\}} \lambda U$$

is open as the union of open sets. By the definition of the quotient topology, this means that $\pi(U)$ is open.

2 The restriction $\pi|_{\mathbb{S}^n}$ remains surjective, since for $x \in X$ we can choose the representative $\frac{x}{\|x\|} \in \mathbb{S}^{n-1}$ of $[x]_\sim$. As π is continuous by definition and $\mathbb{S}^n$ is compact, $X/\sim$ is compact as the continuous image of a compact set by Theorem 3.3.1.

3 Consider $x \sim_a y$ if $x \sim \pm y$ and

$$\begin{tikzcd}
X := \mathbb{R}^{n+1}\setminus\{0\} \arrow[r, "g"] \arrow[d, "x\mapsto\frac{x}{\|x\|}"', "f"] & (\mathbb{R}^{n+1}\setminus\{0\})/\sim = \mathbb{R}P^n \arrow[d, "{[x]_\sim\mapsto[\frac{x}{\|x\|}]_{\sim_a}}"', "h"] \\
\mathbb{S}^n \arrow[r, "q_a"] & \mathbb{S}^n/\sim_a = \mathbb{R}P^n
\end{tikzcd}$$

Then f is open: let $U \subset X$, then $f(U) = \bigcup_{\lambda>0} \lambda U \cap \mathbb{S}^n \subset \mathbb{S}^n$ is open, as $x \mapsto \lambda x$ is a homeomorphism, so λU is open. Hence f is a quotient map.

Furthermore, q_a is open: let $O \subset \mathbb{S}^n$ be open. Then $q_a^{-1}(O)$ is open as $q_a^{-1}(q_a(O)) = O \cup (-O)$ is open as the union of open sets (again $x \mapsto -x$ is a homeomorphism, so $-O$ is open). Hence q_a is a quotient map, too.

Hence g is open, as h is a homeomorphism and f and q_a are open.

To show that $\mathbb{S}^n\setminus \sim_a$ is Hausdorff, let $\tilde{x}, \tilde{y} \in \mathbb{S}^n/\sim_a$ be distinct. Take $x, y \in \mathbb{S}^n$ with $q_a(x) = \tilde{x}$ and $q_a(y) = \tilde{y}$. As $\mathbb{S}^n$ is Hausdorff, there exists open disjoint open neighbourhoods U_0, U_0' with $x \in U_0$ and $y \in U_0'$ and disjoint open neighbourhoods U_1, U_1' such that $x \in U_1$ and $-y \in U_1'$. Let

$$U := q_a(U_0 \cap U_1) \qquad \text{and} \qquad q := q_a(U_0' \cap (-U_1'))$$

Then $\tilde{x} \in U$ as $x \in U_0 \cap U_1$ and $\tilde{y} \in V$ as $y \in U_0' \cap (-U_1')$. Then by the distributive law for sets we have

$$\begin{aligned}
q^{-1}(U \cap V) &= q^{-1}(U) \cap q^{-1}(V) \\
&= [(U_0 \cap U_1) \cup (-1)\cdot(U_0 \cap U_1)] \cap \left[(U_0' \cap (-U_1')) \cup (-1)\cdot(U_0' \cap (-U_1'))\right] \\
&= (U_0 \cap U_1) \cap (U_0' \cap (-U_1')) \cup \ldots \cup \ldots \cup \ldots = \varnothing \cup \varnothing \cup \varnothing \cup \varnothing = \varnothing.
\end{aligned}$$

Hence $U \cap V = \varnothing$.

Lastly, as $\mathbb{S}^n$ is is compact, so is $\mathbb{S}^n/\sim_a$ by Theorem 4.2.4 and as $(\mathbb{R}^{n+1}\backslash\{0\})/\sim = h^{-1}(\mathbb{S}^n/\sim_a)$ is the continuous image of a compact set, it is compact as well.

4. Clearly, X is path-connected and thus connected and thus so is $X/\sim$ by Theorem 4.2.4.
□

Lemma 4.2.21
The projective space $\mathbb{R}\mathrm{P}^n = \mathbb{S}^n/\sim_a$, where $\sim_a$ is the antipodal identifying relation is homeomorphic to $\mathbb{D}^n/\sim$, where $\sim$ is the equivalence relation on $\mathbb{D}^n$ identifying antipodal points on the boundary.

Proof. Let

$$f\colon \mathbb{D}^n \to \mathbb{S}^n, \qquad (x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_n, \sqrt{1 - \|x\|^2}\right). \tag{3}$$

Then $\sim_{q_a \circ f} = \sim$. Furthermore, $\mathbb{D}^n$ is compact and $\mathbb{R}\mathrm{P}^n$ is Hausdorff, so $q_a \circ f$ is an identification map. □

$$\begin{tikzcd} \mathbb{D}^n \arrow[r, "q_a \circ f"] \arrow[d, "f"'] & \mathbb{S}^n/\sim_a \\ \mathbb{S}^n \arrow[ur, "q_a"'] & \end{tikzcd}$$

Fig. 38: Commutative diagram for the equivalence of two description of the projective space.

TODO: this was also Exercise 6.1. Compare to our Ansatz.

Attaching maps

Definition 4.2.22 (Attaching maps)
Let X and Y be topological spaces and $A \subset X$ and $f\colon A \to Y$ a map. We define an equivalence relation $\sim$ on $X + Y$ with partition

- $\{a, f(a)\} \in X + Y$ for all $a \in A$,
- $\{y\}$ for all $y \in Y\backslash f(A)$
- $\{x\}$ for all $x \in XA$.

We set $X \cup_f Y := (X + Y)/\sim$ and say that f is an attaching map and X is *glued* to Y along the map f.

In the relevant examples, X will be complicated, while Y will be simple, e.g. $\mathbb{S}^n$ or $\mathbb{B}^n$.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, hook] \\ Y \arrow[r, hook] & X + Y \arrow[d, two heads, "q"] \\ & X \cup_f Y \end{tikzcd}$$

attaching map

Fig. 39: The glueing of X to Y along f.

To be more precise on the nature of the equivalence relation: let $\iota_X\colon X \to X + Y$ and $\iota_Y\colon Y \to X + Y$ be the canonical inclusions. Further, we define $\sim_f$ to be the equivalence relation on $X + Y$ generated by all relations of the form $\iota_X(a) \sim_f \iota_Y(f(a))$ for $a \in A$. This determines $\sim_f$ on $X + Y$ as for all $x, x' \in X$ and $y, y' \in Y$ we have

1. $\iota_Y(y) \sim_f \iota_Y(y')$ if and only if $y = y'$,
2. $\iota_X(x) \sim_f \iota_Y(y)$ if and only if $x \in A$ and $f(x) = y$,
3. $\iota_Y(y) \sim_f \iota_X(x)$ if and only if $x \in A$ and $f(x) = y$, and
4. $x \sim_f x'$ if and only if $x = x'$ or $x, x' \in A$ and $f(x) = f(x')$.

We get the composition maps

$$\chi\colon X \xrightarrow{\iota_X} X + Y \xrightarrow{q} X \cup_f Y,$$
$$\iota\colon Y \xrightarrow{\iota_Y} X + Y \xrightarrow{q} X \cup_f Y,$$

where $q\colon X + Y \to X \cup_f Y$ denotes the quotient map. Then $X + Y$ and $X \cup_f Y$ are understood to have the final topology for the pair (ι_X, ι_Y) resp. the quotient map q: a set $O \subset X \cup_f Y$ is open if and only if $q^{-1}(O) \subset X + Y$ is open precisely if $\iota_X^{-1}(q^{-1}(O)) \subset X$ and $\iota_Y^{-1}(q^{-1}(O)) \subset Y$ are open.

Example 4.2.23 Draw a picture for $X \cup_f Y$. What is the connection to polyhedral surfaces?

TODO ◇

Lemma 4.2.24
The map $\iota\colon Y \to X \cup_f Y$ is an embedding, but $\chi\colon X \to X \cup_f Y$ is not.

Proof. First, ι is injective: let $x, y \in Y$ with $\iota(x) = \iota(y)$. Then $q(i_Y(x)) = q(i_Y(y))$, that is $i_Y(x) \sim i_Y(y)$, which is precisely the case if $x = y$ by 1.

Let $V \subset Y$ be open. Then $\chi^{-1}(\iota(V)) = f^{-1}(V) \subset A$ is open, so there exists an open set $U' \subset X$ such that $U' \cap A = f^{-1}(V)$.

Define $U := \iota(V) \cup \chi(U')$. Then $U \subset X \cup_f Y$ is open as ι is injective and thus

$$\iota^{-1}(U) = \underbrace{\iota^{-1}(\iota(V))}_{=V} \cup \underbrace{\iota^{-1}(\chi(U'))}_{=f(U' \cap A) \subset V} = V \subset Y$$

is open and

$$\begin{aligned}\chi^{-1}(U) = \underbrace{\chi^{-1}(\iota(V))}_{=f^{-1}(V)} \cup \underbrace{\chi^{-1}(\chi(U'))}_{\overset{?????}{=} U' \cup f^{-1}(f(U' \cap A))} &= f^{-1}(V) \cup \big(U' \cup \underbrace{f^{-1}(f(f^{-1}(V)))}_{=f^{-1}(V) \subset U'}\big)\\ &= f^{-1}(V) \cup \big(U' \cup f^{-1}(V)\big) = U' \subset X\end{aligned}$$

is open, too. □

Theorem 4.2.6: Characteristic mapping property of attaching maps

For any topological space Z and any pair of maps $a\colon Y \to Z$ and $b\colon X \to Z$ with $a \circ f = b|_A$ there exists a unique map $c\colon X \cup_f Y \to Z$ such that $c \circ \iota = a$ and $c \circ \chi = b$.

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "\chi"] \arrow[ddr, "b"] & \\
Y \arrow[r, "\iota"] \arrow[drr, "a"'] & X \cup_f Y \arrow[dr, dotted, "\exists! c"] & \\
 & & Z.
\end{tikzcd}$$

Fig. 40: The characteristic mapping property for the space $X \cup_f Y$ and the maps χ, ι and f states that the following diagram commutes.

Proof. By the universal property of the quotient $(X+Y) \to (X+Y)/\sim_f$ there is a bijection between the following two sets:

$$\{g\colon (X+Y)/\sim_f \to Z : g \text{ is a map}\} \qquad \text{and} \qquad \{\varphi\colon X+Y \to Z : \varphi \text{ is a map}, \sim_\varphi = \sim_f\}.$$

By the universal property of the disjoint union there exists a bijection from the latter set to the set

$$\{(a\colon X \to Z, b\colon Y \to Z) : a, b \text{ are maps such that } f(x) = y \implies b(y) = a(x)\}.$$

The condition $f(x) = y \implies b(y) = a(x)$ precisely states that $b \circ f = a$, i.e. that the diagram figure 41 commutes. □

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "a"] \\
Y \arrow[r, "b"] & Z
\end{tikzcd}$$

Fig. 41

Example 4.2.25 Consider the MÖBIUS strip M and a closed disk $D \approx \mathbb{B}^2$. As $\partial M \approx \mathbb{S}^1$, there exists a homeomorphism $h\colon \partial D \to \partial M$ (via $h\colon \mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^2$). Then $M \cup_h D \approx \mathbb{RP}^2$.

Similarly, $\mathbb{RP}^2$ is obtained by glueing a 2-disk $\mathbb{D}^2$ to the boundary of a MÖBIUS strip $\mathbb{D}^2 \times \mathbb{D}^2 / \sim$ which is again a sphere $\mathbb{S}^1$. Here, the equivalence relation $\sim$ identifies points $(-1, -x)$ with $(1, x)$ for $x \in [0, 1]$. ◇

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, hook] \arrow[d, two heads, "h"] & \mathbb{D}^2 \arrow[d, two heads, "\chi"] \\ \mathbb{S}^1 \arrow[r, "\iota"] & \mathbb{D}^2 \cup_h \mathbb{S}^1 \end{tikzcd}$$

Fig. 42

Proof. (For the first statement) Consider the commutative diagram figure 42. In this case, χ is surjective, as $Y := \mathbb{S}^1 \subset \mathbb{D}^2 =: X$. Furthermore, χ is a quotient map: let $\chi^{-1}(O) \subset \mathbb{D}^2$ be open. We want to show that $O \subset \mathbb{D}^2 \cup_h \mathbb{S}^1$ is open. As $\chi^{-1}(O) \subset \mathbb{D}^2$ is open, $h^{-1}(\iota^{-1}(O)) = \chi^{-1}(O) \cap \mathbb{S}^1 \subset \mathbb{S}^1$ is open. As h is surjective, we have $h(h^{-1}(\iota^{-1}(O))) = \iota^{-1}(O)$. As h doubles angles, it is open: the basis of the topology on $\mathbb{S}^1$ are open connected arcs, which are mapped to open connected arcs. Hence $\sim_\chi$ is the equivalence relation only identifying antipodal points on the boundary of $\mathbb{D}^2$. □

Proof. (For the second statement) Consider again the map $f\colon \mathbb{D}^2 \to \mathbb{S}^2$ from (3). By dividing $\mathbb{S}^1$ into four quarter arcs, we find a homeomorphism h onto the rectangle $\mathbb{D}^1 \times \mathbb{D}^1$. One possibility is (I think...)

$$h(e^{i\theta}) = \begin{cases} 1 + \left(\frac{4}{\pi}\theta - 1\right) i, & \text{if } \theta \in [0, \frac{\pi}{2}], \\ \left(3 - \frac{4}{\pi}\theta\right) + i, & \text{if } \theta \in [\frac{\pi}{2}, \pi], \\ -1 + \left(5 - \frac{4}{\pi}\theta\right) i, & \text{if } \theta \in [\pi, \frac{3\pi}{2}], \\ \left(\frac{4}{\pi}\theta - 7\right) - i, & \text{if } \theta \in [\frac{3\pi}{2}, 2\pi]. \end{cases} \quad \text{or} \quad h(e^{2\pi i\theta}) = \begin{cases} 1 + (8\theta - 1)\, i, & \text{if } \theta \in [0, \frac{1}{2}], \\ (3 - 8\theta) + i, & \text{if } \theta \in [\frac{1}{2}, 1], \\ -1 + (5 - 8\theta)\, i, & \text{if } \theta \in [1, \frac{3}{4}], \\ (8\theta - 7) - i, & \text{if } \theta \in [\frac{3}{4}, 1]. \end{cases}.$$

The MÖBIUS strip $M := (\mathbb{D}^1 \times \mathbb{D}^1)/\sim$ is compact by Theorem 4.2.4 and $\mathbb{RP}^n$ is HAUSDORFF.

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, hook] \arrow[d, "h"] & \mathbb{D}^2 \arrow[d, two heads, "q_a \circ f"'] \arrow[ddr, bend left, "a", "\approx"'] & \\ \mathbb{D}^1 \times \mathbb{D}^1 \arrow[d, "q_M"'] & \mathbb{RP}^2 \arrow[dr, dotted, "\exists! c"] & \\ M \arrow[ur, two heads, "\exists g"] \arrow[rr, two heads, "b", "\text{quotient map}"'] & & \mathbb{D}^2 \cup_{q_M \circ h} M \arrow[ul, dotted, bend left, "\exists! d"] \end{tikzcd}$$

Fig. 43

By the universal property of attaching maps we have $c \circ d = \mathrm{id}_{\mathbb{D}^2 \cup_{q_M \circ h} M}$ and $d \circ c = \mathrm{id}_{\mathbb{RP}^2}$. □

4.3 | Topological groups

24.11.2021

It turns out that topology is particularly rich if we mix it with other structures. All kinds of algebraic structures can be put together with topology in more than one way. The first scenario we will look at of this kind is where we have a group and equip it with a topology.

Definition 4.3.1 (Topological group)
Let $(G, \cdot)$ be a group equipped with a Hausdorff topology. If the functions

$$m \colon G \times G \to G, \qquad (g, h) \mapsto g \cdot h \qquad \text{(Multiplication)}$$
$$i \colon G \to G, \qquad g \mapsto g^{-1} \qquad \text{(Inversion)}$$

are maps, then G is a topological group.

topological group

Here, $G \times G$ is equipped with the product topology.

Example 4.3.2 (Topological groups in $\mathbb{E}^n$)
Clearly, $(\mathbb{E}^n, +)$ is a topological group.

Consider the subgroup $\mathbb{S}^1 \subset (\mathbb{C} \setminus \{0\}, \cdot) =: \mathbb{C}^\times$. The multiplication is $m(e^{i\theta}, e^{i\varphi}) = e^{i(\theta+\varphi)}$ and the inverse is $e^{i\theta} \mapsto e^{-\theta}$. Both functions are maps. (Furthermore, $\mathbb{C}^\times$ can be equipped with a topology making it into a topological group.) ◇

Remark 4.3.3 Let $(G, \cdot)$ be a arbitrary group with the discrete topology. This topology is Hausdorff and hence G is a topological group. Hence just assuming that we have a topological group doesn't say anything about the group because any group is possible. ○

Example 4.3.4 (Product of topological groups)
If G and H are topological groups, then $G \times H$ (equipped with the product topology) becomes a topological group. A topological group lives in the category of topologies **Top** and the category of groups **Grp**. In both categories, there is a notion of product: there is a group structure and a topological structure on the product. For example, the topological n-torus $\mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is a topological group. ◇

Example 4.3.5 (Matrix topological groups)
The general linear group of the reals of order n, $\mathrm{GL}(n) := \mathrm{GL}_n(\mathbb{R}) := \{M \in \mathbb{R}^{n\times n} : \det(M) \neq 0\}$ equipped with matrix multiplication is a non-compact topological group.

The orthogonal group $\mathrm{O}(n) := \mathrm{O}_n(\mathbb{R}) := \{M \in \mathrm{GL}(n) : M^{-1} = M^\intercal\}$ and the special orthogonal group $\mathrm{SO}(n) := \mathrm{SO}_n(\mathbb{R}) := \{M \in O(n) : \det(M) = 1\}$ are compact topological groups.

The topologies of those matrix topological groups are the trace topologies induced by any of the equivalent norms on $\mathbb{R}^{n\times n}$. ◇

Let G be a topological group and let $H < G$ be a subgroup.

Lemma 4.3.6
The factor map $q \colon G \to G/H := \{gH : g \in G\}$ is open. The closure $\overline{H}$ of H in G is also a subgroup.

Proof.

Lemma 4.3.7
The space G/H is discrete is and only if H is an open subgroup.

Proof.

Lemma 4.3.8
Let H be a normal subgroup, i.e. G/H with the induced group operation is a group (i.e. $g_1 \sim g_2$ if $g_1^{-1}g_2 \in H$) or, equivalently, $gHg^{-1} \subset H$ holds for every $g \in G$. Then $\overline{H}$ is a normal subgroup of G as well.

Proof.

Normal subgroups are kernels of group homomorphisms.

Lemma 4.3.9
Let U be a neighbourhood of the neutral element $e \in G$. Then there is a neighbourhood V of e such that $VV^{-1} \subset U$ holds.

Proof.

TODO: Can one replace e with any $g \in G$?

Lemma 4.3.10
Let G be compact and H be a closed subgroup. Then H is open if and only if H has finite index in G.

The index of H in G is the number of cosets, i.e. $\#G/H$. We have $G = \bigsqcup_{[g]\in G/H} gH$

Proof. " $\implies$ ": Let H be open, then gH is open. Hence $(gH)_{[g]\in G/H}$ is an open cover G. As G is compact, there exists a finite subcover, so $\#G/H < \infty$.

" $\impliedby$ ": We can write

$$H = G \setminus \bigsqcup_{g \notin H} gH.$$

As H has finite index in G, we are only taking away finitely many closed sets, so H is open. □

TODO: Describe the smallest normal subgroup of G containing a fixed set $S \subset G$.

Definition 4.3.11 (Linear Lie group)
A subgroup $G \leqslant \mathrm{GL}(n)$ is a linear Lie group if G is closed.

Example 4.3.12 (Linear Lie group)
The $\mathrm{GL}(n)$, $O(n)$ and $SO(n)$ are linear Lie groups and there are many more. ◇

Lemma 4.3.13 (From HU Topology Exam)
Let e be the identity of the topological group G. Then $\pi_1(G, e)$ (cf. later) is ABELIAN.

Proof. TODO. □

Remark 4.3.14 There is a sophisticated structure theory for Lie groups which is associated with all kinds of questions of symmetry. It came about historically when solving systems of differential equations subject to symmetry. It occurs in crystallography, reflection groups, non-associative algebra, Combinatorics, Mathematical Physics and many other fields. The notion of a linear Lie group is a key mathematical notion of the 20th century.

Special properties of $\mathrm{GL}(n)$ makes it possible to define linear Lie groups so easily in topological terms despite the fact that the main properties of $\mathrm{GL}(n)$ can be best expressed in terms of differential equations, differentiable maps and their derivations. But just because $\mathrm{GL}(n)$ is such a strong structure, it suffices to define linear Lie groups as relevant substructures in terms of topological notions, which in general are much weaker than differential geometry notions. ○

Epilogue. This ends this first large chunk of topology in this course, where we started out with some basic objects - polyhedra - to play with and wrote down the classification theorem for surfaces, which we intend to prove. Then we walked our way through the basic definitions: open and closed sets, closure, neighbourhoods. We studied continuous functions and ways to invent topologies via the initial and final topology constructions. We talked about attaching maps and now we had examples for topological groups.

5 Fundamental group

Now we are equipped to tackle the first really interesting topic of this course - we switch to topology for grown-ups.

We first give a very short introduction to category theory.

A good introduction is [9].

Definition 5.0.1 (Category)
A category C consists of

category

- a class of objects, $\mathrm{Ob}(C)$,
- for all $X, Y \in \mathrm{Ob}(C)$ a (hom)set $\mathrm{Mor}_C(X, Y)$ of morphisms $f : X \to Y$ such that $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms f, g and h that and for all $X \in \mathrm{Ob}(C)$ there exists an identity map $\mathrm{id}_X \in \mathrm{Mor}_C(X, X)$ such that $\mathrm{id}_X \circ f = f$ for all $f \in \mathrm{Mor}(Z, X)$ and any $Z \in \mathrm{Ob}(C)$ and $g \circ \mathrm{id}_X = g$ for all $g \in \mathrm{Mor}(X, Z)$ and any $Z \in \mathrm{Ob}(C)$, where
$$\circ : \mathrm{Mor}_C(Y, Z) \to \mathrm{Mor}_C(X, Y) \to \mathrm{Mor}_C(X, Z), \qquad (f, g) \mapsto f \circ g$$
is the composition map.

A class is "large", while a set is "small". A set is a class. The actual definition of a class is to complicated to mention here.

Example 5.0.2 (Category) The class of topological spaces together with the maps as morphisms is a category, as is the class of pointed topological spaces (X, p) with $p \in X$ together with the basepoint preserving maps $f : (X, p) \to (Y, f(p))$.

Let R be a ring. The class of R-modules together with the R-homomorphisms as morphisms is a category. Examples are the category **Grp** of groups together with group homomorphisms and the category of $\mathbb{K}$ vector spaces with $\mathbb{K}$-linear vector space homomorphisms. ◇

Definition 5.0.3 (Functor)
A functor $F : C \to D$ is a mapping between categories C and D such that $F : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$, $A \mapsto F(A)$ and $F : \mathrm{Mor}_C(X, Y) \to \mathrm{Mor}_D(F(X), F(Y))$ for all $X, Y \in C$, $f \mapsto F(f)$ fulfil

functor

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$ for all $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, G)$.

Definition 5.0.4 (Isomorphism)
A morphism $f \in \mathrm{Mor}_C(X, Y)$ is a isomorphism in C if there exists a $g \in \mathrm{Mor}(Y, X)$ such that $f \circ g = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$.

Lemma 5.0.5
Functors map isomorphisms to isomorphisms.

Proof. Let $f \in \mathrm{Mor}_C(X, Y)$ be a isomorphism and g be as in Definition 5.0.4. Then $F(f) \circ F(g) = F(f \circ g) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and analogously $F(g) \circ F(f) = \mathrm{id}_{F(Y)}$, so $F(f)$ is an isomorphism. □

5.1 Homotopic maps

We want to associate a group with any topological space called the fundamental group. To find out how this group will be defined we have to talk about homotopic maps.

Definition 5.1.1 (Loop)
Let $\alpha\colon I \to X$ be a path in X. Then α is a loop based at $p \in X$ if $\alpha(0) = p = \alpha(1)$.

loop

Any constant path is a loop.

Lemma 5.1.2
A loop in X is the same as a map $\mathbb{S}^1 \to X$.

$$\begin{tikzcd} I \arrow[r, two heads, "q"] \arrow[d, "\alpha"'] & \mathbb{S}^1, \arrow[dl, dotted, "\exists\tilde{\alpha}"] \\ X & \end{tikzcd}$$

Fig. 44

Proof. We have quotient map $q\colon [0,1] \to [0,1]/\{0,1\} \approx \mathbb{S}^1$ and a path $\alpha\colon [0,1] \to X$. Now α respects the equivalence relation $0 \sim 1$ as $\alpha(0) = \alpha(1)$. By the mapping property of the final topology, there must exist a map $\tilde{\alpha}\colon \mathbb{S}^1 \to X$ such that figure 44 commutes.

Given a map $f\colon \mathbb{S}^1 \to X$, let $\alpha := f \circ q$. □

Definition 5.1.3 (Product of loops)
If $\alpha, \beta\colon I \to X$ are both loops based at $p \in X$, then their product is the following loop based at p:

$$\alpha \bullet \beta\colon I \to X, \qquad s \mapsto \begin{cases} \alpha(2s), & s \in [0, \frac{1}{2}], \\ \beta(2s-1), & s \in [\frac{1}{2}, 1]. \end{cases}$$

This is precisely the concatenation of paths discussed before.

Consider another topological space Y.

Definition 5.1.4 (Homotopy (relative to a set))
Let $f, g\colon X \to Y$ be two maps. Then f is homotopic to g if there exists a map $F\colon X \times I \to Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. Then F is a homotopy and we write $f \simeq_F g$.

homotopy

If additionally $A \subset X$ and $F(a, \cdot) = f(a)$ for all $a \in A$, then F is a homotopy relative to A and we write $f \simeq_F g$ rel A.

The homotopy interpolates between f and g. If $f \simeq_F g$ rel A, then $f(a) = F(a, 1) = g(a)$ for all $a \in A$, that is, $f|_A \equiv g|_A$.
Remark 5.1.5 Let $\alpha, \beta\colon I \to X$ be loops based at $p \in X$. Then a homotopy F from α to β relative to the endpoints $\{0, 1\}$ is a map $F\colon I \times I \to X$ with $F(\cdot, 0) = \alpha$, $F(\cdot, 1) = \beta$ and $F(0, \cdot) = p = F(1, \cdot)$.

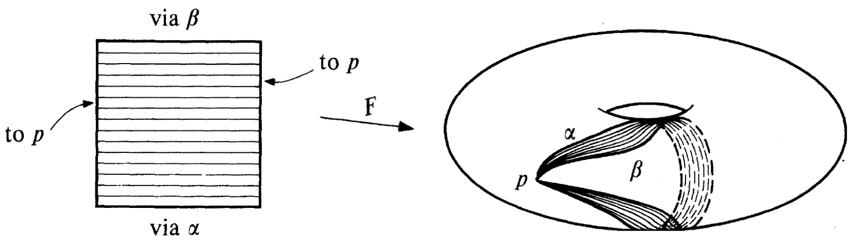

Fig. 45: A homotopy F relative to $\{0, 1\}$ of two loops α and β based at p.

Example 5.1.6 (Straight line homotopy) Let $C \subset \mathbb{E}^n$ be convex, X be a topological space and $f, g \colon X \to C$ be maps. Then

$$F \colon X \times I \to C, \qquad (x,t) \mapsto (1-t)f(x) + tg(x)$$

is a homotopy from f to g, well-defined as C is convex.

If, moreover, there exists a subspace $A \subset X$ such that $f|_A = g|_A$, then F is a homotopy relative to A. The map F is a straight line homotopy.

Hence any two maps into a convex set are homotopic. Hence convex sets are boring from a topological point of view and it was not an accident that polyhedra are patched together from boring shapes. $\diamond$

Example 5.1.7 Consider maps $f, g \colon X \to \mathbb{S}^n$ such that $f(x) \neq -g(x)$ for all $x \in X$ $(\star)$. Seeing $\mathbb{S}^n$ as a subset of $\mathbb{E}^{n+1}$, this allows us to define a straight line homotopy in $\mathbb{E}^{n+1}$ as $\mathbb{E}^{n+1}$ is convex. But this would leave the target space $\mathbb{S}^n$. By $(\star)$, we can project from 0 to $\mathbb{S}^n$ as follows: the map

$$F \colon X \times I \to \mathbb{S}^n, \qquad (x,t) \mapsto \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is a homotopy from f to g. It is well defined because $(1-t)f(x) + tg(x)$ can only vanish for $t = \frac{1}{2}$ (as f and g map to $\mathbb{S}^n$), but $(\star)$ ensures this does not happen.

So even though $\mathbb{S}^n$ is not convex, but only the boundary of a convex set, the assumption $(\star)$ guarantees the existence of a homotopy. $\diamond$

Example 5.1.8 (Homotopic loops on $\mathbb{S}^1$) **29.11.2021**
Consider the following loops in $\mathbb{S}^1 = e^{2\pi i[0,1]}$ based at $e^0 = 1 + 0i$:

$$\alpha(s) := \begin{cases} e^{4\pi i s}, & \text{if } s \in [0, \frac{1}{2}], \\ e^{4\pi i(2s-1)}, & \text{if } s \in [\frac{1}{2}, \frac{3}{4}], \\ e^{8\pi i(1-s)}, & \text{if } s \in [\frac{3}{4}, 1]. \end{cases}, \qquad \beta(s) := e^{2\pi i s}.$$

The loop α traverses $\mathbb{S}^1$ twice counterclockwise and then once clockwise, while β only traverses $\mathbb{S}^1$ once counterclockwise. Consider the function

$$F(s,t) := \begin{cases} e^{4\pi i \frac{s}{t+1}}, & \text{if } s \in [0, \frac{t+1}{2}], \\ e^{4\pi i(2s-t-1)}, & \text{if } s \in [\frac{t+1}{2}, \frac{t+3}{4}], \\ e^{8\pi i(1-s)}, & \text{if } s \in [\frac{t+3}{4}, 1], \end{cases}$$

which is a map by the glueing lemma. Hence $\alpha \simeq_F \beta$ rel $\{0,1\}$.

Hence, up to homotopy, traversing $\mathbb{S}^1$ in one direction and then in the opposite direction is like not moving at all. Thus for loops on $\mathbb{S}^1$ it is only important how many times (in a signed matter) they go around the circle, which will be made precise later by $\pi_1(\mathbb{S}^1) = \mathbb{Z}$. $\diamond$

Lemma 5.1.9 ($\simeq$ is equivalence relation)
The relation $\simeq$ is an equivalence relation on $\{f \colon X \to Y : f \text{ is a map}\}$.

Proof. (1) For $F(\cdot, t) = f$ we have $f \simeq_F f$.

(2) If $f \simeq_F g$, let $G(\cdot, t) = F(\cdot, 1-t)$. Then $g \simeq_G f$.

3 If $f \simeq_F g$ and $g \simeq_G h$, then $f \simeq_H h$, where

$$H(\cdot, t) := \begin{cases} F(\cdot, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ G(\cdot, 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \tag{4}$$

□

Lemma 5.1.10 ($\simeq$ rel A is equivalence relation)
For any subset $A \subset X$, $\simeq$ rel A is an equivalence relation on the set of maps $f\colon X \to Y$ which agree on A.

$$\begin{tikzcd} X \arrow[r, "f", "g"'] \arrow[dr, "u\circ f", "v\circ g"'] & Y \arrow[d, "u", "v"'] \\ & Z \end{tikzcd}$$

Fig. 46

Remark 5.1.11 ($f \simeq g$, $u \simeq v \implies u \circ f \simeq v \circ g$)
Let Z be a topological space. If $f, g\colon X \to Y$ are maps homotopic relative to $A \subset X$ via F and $u\colon Y \to Z$ is a map, then $u \circ f \simeq_{u \circ F} u \circ g$ rel A. If $u \simeq_G v$ rel $f(A)$, then $u \circ f \simeq_H v \circ g$ rel A where

$$H(x, t) := \begin{cases} G(f(x), 2t), & \text{if } t \in [0, \frac{1}{2}], \\ G(g(x), 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

○

5.2 Construction of the fundamental group

For a fixed $p \in X$ consider the set with an operation

$$(\{\alpha : \alpha \text{ is a loop based at } p\}, \bullet).$$

Then concatenating a loop with the constant path does not change its image. We can also revert the sense of direction by considering the inverse path $\alpha^{-1}(t) := \alpha(1-t)$. We have associativity in the same sense as above.

To get a group, we take the homotopy classes with respect to p.

Definition 5.2.1 (Homotopy class)
Let X be a topological space with a base point $p \in X$. For two loops $\alpha, \beta\colon I \to X$ based at p, the homotopy class of α,

$$\langle \alpha \rangle := \{\alpha' : \alpha' \text{ is a loop based at } p \text{ with } \alpha \simeq \alpha' \text{ rel } \{0,1\}\},$$

is the equivalence class of α with respect to the equivalence relation $\simeq$ rel $\{0,1\}$.

We define the operation

$$\langle \alpha \rangle \cdot \langle \beta \rangle = \langle \alpha \bullet \beta \rangle,$$

where $\bullet$ denotes concatenation.

This is independent of the choice of representative: if $\alpha' \in \langle \alpha \rangle$ and $\beta' \in \langle \beta \rangle$, then $\alpha' \simeq_F \alpha$ rel $\{0,1\}$ and $\beta' \simeq_G \beta$ rel $\{0,1\}$. Then $\alpha' \bullet \beta' \simeq_H \alpha \bullet \beta$ rel $\{0,1\}$, where

$$H(x, t) := \begin{cases} F(2x, t), & \text{if } x \in [0, \frac{1}{2}], \\ G(2x-1, t), & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Theorem 5.2.1: Fundamental group

For $p \in X$ the set

$$\pi_1(X,p) := \{\langle\, \alpha \,\rangle : \alpha \text{ is a loop in } X \text{ based at } p\}$$

equipped with the above multiplication is a group, the fundamental group of X with base point p.

fundamental group

Proof. Let α, β and γ be loops in X based at p.

- **Associativity.** We have, as $(\alpha \bullet \beta) \bullet \gamma \simeq \alpha \bullet (\beta \bullet \gamma)$,

$$\begin{aligned}(\langle\, \alpha \,\rangle \cdot \langle\, \beta \,\rangle) \cdot \langle\, \gamma \,\rangle &= \langle\, \alpha \bullet \beta \,\rangle \cdot \langle\, \gamma \,\rangle = \langle (\alpha \bullet \beta) \bullet \gamma \,\rangle \\ &= \langle\, \alpha \bullet (\beta \bullet \gamma) \,\rangle = \langle\, \alpha \,\rangle \cdot (\langle\, \beta \bullet \gamma \,\rangle) = \langle\, \alpha \,\rangle \cdot (\langle\, \beta \,\rangle \cdot \langle\, \gamma \,\rangle).\end{aligned}$$

- **Neutral element.** Define the constant path $e \colon I \to X$, $t \mapsto p$. We need to show that $\langle\, e \,\rangle \langle\, \alpha \,\rangle = \langle\, \alpha \,\rangle \langle\, e \,\rangle = \langle\, \alpha \,\rangle$. Define $f \colon I \to I$, $s \mapsto 0$ if $s \in [0, \frac{1}{2}]$ and $s \mapsto 2s - 1$ if $s \in [\frac{1}{2}, 1]$, which has $f(0) = 0$ and $f(1) = 1$. Then $f \simeq \mathrm{id}_I$ rel $\{0,1\}$ as I is convex. Hence

$$e \bullet \alpha = \alpha \circ f \simeq \alpha \circ \mathrm{id}_I \ \text{ rel } \{0,1\} = \alpha.$$

  The other equality instead uses $f(x) = 2x$ for $x \in [0, \frac{1}{2}]$ and 1 for $x \in [\frac{1}{2}, 1]$.

- **Inverses.** For a loop α in X based at $p \in X$, let $\alpha^{-1} \colon I \to X$, $t \mapsto \alpha(1-t)$ be the inverse loop of α. We need to show that $\langle\, \alpha^{-1} \,\rangle \langle\, \alpha \,\rangle = \langle\, \alpha \,\rangle \langle\, \alpha^{-1} \,\rangle = \langle\, e \,\rangle$. Define $f \colon I \to I$, $s \mapsto 2s$ if $s \in [0, \frac{1}{2}]$ and $s \mapsto 2(1-s)$ if $s \in [\frac{1}{2}, 1]$, which has $f(0) = f(1) = 0$ and $g \colon I \to I$, $s \mapsto 0$. As I is convex, $f \simeq g$ rel $\{0,1\}$. Hence

$$\alpha \bullet \alpha^{-1} = \alpha \circ f \simeq \alpha \circ g \text{ rel } \{0,1\} = e.$$

  Again the other equality is basically the same. □

Theorem 5.2.2: π_1 of path-connected space independent of p

If X is path-connected and $p, q \in X$, then there is a group isomorphism $\pi_1(X,p) \to \pi_1(X,q)$.

Proof. As X is path-connected, there exists a path γ from p to q. Then γ^{-1} is a path from q to p. For any loop $\alpha \colon I \to X$ based at p, the path

$$\tilde{\gamma}_*(\alpha) := \gamma^{-1} \bullet \alpha \bullet \gamma$$

is a loop based at q. If $\alpha \simeq_F \alpha'$, then $\tilde{\gamma}_*(\alpha) \simeq_G \tilde{\gamma}_*(\alpha')$ via $G(\cdot, t) := \gamma \bullet F(\cdot, t) \bullet \gamma^{-1}$. Hence $\tilde{\gamma}_*$ is constant on equivalence classes $\langle\, \alpha \,\rangle$, so it induces a map

$$\gamma_* \colon \pi_1(X,p) \to \pi_1(X,q), \qquad \langle\, \alpha \,\rangle \mapsto \langle\, \tilde{\gamma}_*(\alpha) \,\rangle, \tag{5}$$

which is a well-defined group homomorphism: for loops α and β at p we have

$$\begin{aligned}\gamma_*(\langle\, \alpha \,\rangle \cdot \langle\, \beta \,\rangle) = \gamma_*(\langle\, \alpha \bullet \beta \,\rangle) &= \langle\, \gamma \bullet (\alpha \bullet \beta) \bullet \gamma^{-1} \,\rangle = \langle\, \gamma \bullet \alpha \bullet \gamma^{-1} \bullet \gamma \bullet \beta \bullet \gamma^{-1} \,\rangle \\ &= \langle\, \gamma \bullet \alpha \bullet \gamma^{-1} \,\rangle \cdot \rangle\, \gamma \bullet \beta \bullet \gamma^{-1} \,\rangle = \gamma_*(\langle\, \alpha \,\rangle) \cdot \gamma_*(\langle\, \beta \,\rangle).\end{aligned}$$

Further, is bijective with $(\gamma_*)^{-1} = (\gamma^{-1})_*$. □

Remark 5.2.2 (Notation)
If X is path-connected, then it is common to write $\pi_1(X)$ instead of $\pi_1(X,p)$, but this doesn't describe $\pi_1(X,p)$ as a set but only up to homomorphism type. ○

Example 5.2.3 (Induced homomorphism on fundamental group)
Let $f\colon X \to Y$ be a map between topological spaces. Pick a base point $p \in X$ and consider $f(p) \in Y$ as a base point for Y. For any $\alpha \in \langle\alpha\rangle \in \pi_1(X,p)$ we have $f \circ \alpha \in \langle f\circ\alpha\rangle \in \pi_1(Y, f(p))$. For $\alpha' \in \langle\alpha\rangle$ we have $f\circ\alpha' \in \langle f\circ\alpha\rangle$ by remark 5.1.11. Hence we obtain a function

$$f_*\colon \pi_1(X,p) \to \pi_1(Y,f(p)), \qquad \langle\alpha\rangle \mapsto \langle f\circ\alpha\rangle. \tag{6}$$

$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[dr, "f\circ\alpha"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Fig. 47

We have $f\circ(\alpha\bullet\beta) = (f\circ\alpha)\bullet(f\circ\beta)$ for loops α,β in X at p, so f_* is a group homomorphism:

$$\begin{aligned} f_*(\langle\alpha\rangle\cdot\langle\beta\rangle) &= f_*(\langle\alpha\bullet\beta\rangle) = \langle f\circ(\alpha\bullet\beta)\rangle \\ &= \langle(f\circ\alpha)\bullet(f\circ\beta)\rangle = \langle f\circ\alpha\rangle\cdot\langle f\circ\beta\rangle = f_*(\langle\alpha\rangle)\bullet f_*(\langle\beta\rangle). \end{aligned}$$ ◇

Theorem 5.2.3: $_*$ is covariant functor

Let X, Y and Z be topological spaces and $f\colon X\to Y$ and $g\colon Y\to Z$ be maps. For $p\in X$ let $q := f(p)$ and $r := g(q)$. Then

$$(g\circ f)_* = g_*\circ f_*\colon \pi_1(X,p)\to\pi_1(Z,r).$$

This theorem shows that the fundamental group can be applied to commutative diagrams:

$$\begin{tikzcd} (X,p) \arrow[r, "f"] \arrow[dr, "g\circ f"'] & (Y,q) \arrow[d, "g"] \\ & (Z,r) \end{tikzcd} \qquad \begin{tikzcd} \pi_1(X,p) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*=g_*\circ f_*"'] & \pi_1(Y,q) \arrow[d, "g_*"] \\ & \pi_1(Z,r) \end{tikzcd}$$

Fig. 48: The fundamental group is a covariant functor and hence can be applied to commutative diagrams.

In terms of category theory, the fundamental group

$$\pi_1\colon \mathbf{Top}_\bullet \to \mathbf{Grp}, \qquad \begin{cases} (X,p)\mapsto \pi_1(X,p), \\ \big(f\colon (X,p)\to(Y,f(p))\big)\mapsto\big(f_*\colon \pi_1(X,p)\to\pi_1(Y,f(p))\big) \end{cases}$$

is a functor, where $\mathbf{Top}_\bullet$ is the category of pointed spaces with the base point preserving maps as morphisms.

Corollary 5.2.4 (Fundamental groups of homeomorphic spaces are isomorphic)
Let $h\colon X\to Y$ be a homeomorphism. Then h_* and h_*^{-1} are group isomorphisms.

Proof. By Theorem 5.2.3,

$$h_*^{-1}\circ h_*\colon \pi_1(X,p)\to\pi_1(X,p)$$

is equal to $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,p)}$ and

$$h_*\circ h_*^{-1}\colon \pi_1(Y,h(p))\to\pi_1(Y,h(p))$$

is equal to $(\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y,h(p))}$. □

Hence the fundamental group is a topological invariant of topological spaces: if $X \approx Y$, then $\pi_1(X,p) \cong \pi_1(Y,q)$.

Lemma 5.2.5 (Homotopic maps induce same map on π_1)
If two maps $f, g\colon (X,p) \to (Y,q)$ (that means: $f(p) = g(p) = q$) are homotopic relative to $\{p\}$, then they induce the same maps on fundamental groups, i.e. $f_* = g_*$.

Proof. As $f \simeq g$ rel $\{p\}$, there exists a homotopy $H\colon X \times [0,1] \to Y$ such that $H(\cdot,0) = f$, $H(\cdot,1) = g$ and $H(p,\cdot) = f(p) = g(p)$. Now let $\alpha\colon I \to X$ be a loop at p and

$$H_\alpha\colon [0,1]\times[0,1] \to Y, \qquad (x,t) \mapsto H(\alpha(x),t).$$

$$\begin{tikzcd}
X \times I \arrow[dr, "H"'] & (X,p) \arrow[l, hook'] \arrow[d, "g"', "f"] & I \arrow[l, "\alpha"'] \arrow[d, hook] \\
 & (Y,q) & I \times I \arrow[l, "H_\alpha"]
\end{tikzcd}$$

Fig. 49

Then H_α is a map as a composition of maps and $H_\alpha(\cdot,0) = H(\alpha,0) = f\circ\alpha$ and analogously $H_\alpha(\cdot,1) = g\circ\alpha$. Furthermore, $H_\alpha(a,\cdot) = H(\alpha(a),\cdot) = H(p,\cdot) = q$ for $a \in \{0,1\}$. Hence $f\circ\alpha \simeq_{H_\alpha} g\circ\alpha$ rel $\{0,1\}$, so

$$f_*(\langle\alpha\rangle) = \langle f\circ\alpha\rangle = \langle g\circ\alpha\rangle = g_*(\langle\alpha\rangle).$$ □

5.3 Calculations

How can we compute the fundamental group? Let us start with a trivial example which is nonetheless worth keeping in mind. 01.12.2021

Example 5.3.1 (Fundamental group of convex set in $\mathbb{E}^n$)
Let $C \subset \mathbb{E}^n$ be a convex subset. Then $\pi_1(C)$ is trivial and independent of the base point as C is path-connected. ◇

Definition 5.3.2 (Simply connected space)
A path-connected space with trivial fundamental group is simply connected. simply connected

Remark 5.3.3 We thus have

$$\text{simply connected} \implies \text{path-connected} \implies \text{connected}.$$ ○

The fundamental group of $\mathbb{S}^1$

We identify $\mathbb{S}^1$ with complex numbers of the form $e^{2\pi i t}$ for $t \in [0,1]$, and we pick $1 = e^0$ as the base point for $\mathbb{S}^1$. We define $\pi\colon [0,1] \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$. For $n \in \mathbb{Z}$ consider the path

$$\gamma_n\colon [0,1] \to [0,n], \qquad t \mapsto nt.$$

$$\begin{tikzcd}
I \arrow[r, "\pi"] \arrow[dr, "\gamma_n"'] & \mathbb{S}^1 \\
 & {[0,n]} \arrow[u, "\pi_n"']
\end{tikzcd}$$

Fig. 50

from 0 to n in the real line $\mathbb{E}^1$. Lastly, consider the loop based at 1

$$\pi_n := \pi\circ\gamma_n\colon [0,1] \to \mathbb{S}^1, \qquad t \mapsto e^{2\pi i \gamma_n(t)}.$$

Hence $\langle\pi_n\rangle \in \pi_1(\mathbb{S}^1, 1)$.

Theorem 5.3.1: $\pi_1(\mathbb{S}^1, 1) \cong (\mathbb{Z}, +)$

The function

$$\Phi\colon (\mathbb{Z}, +) \to (\pi_1(\mathbb{S}^1, 1), \cdot), \qquad n \mapsto \langle\, \pi_n \,\rangle \tag{7}$$

is a group isomorphism.

The proof will be based on the subsequent steps.

Lemma 5.3.4
The function (7) is a homomorphism.

Proof. Let $m, n \in \mathbb{Z}$ and define the map $\sigma\colon [0,1] \to \mathbb{R}$, $s \mapsto \gamma_n(s) + m = ns + m$, which is a path from m to $m+n$.

Then $\pi \circ \sigma = \pi \circ \gamma_n$ and the concatenated path $\gamma_m \bullet \sigma$ joins 0 to $m+n$. As $\mathbb{E}^1$ is convex, $\gamma_m \bullet \sigma \simeq \gamma_{m+n}$ rel $\{0, 1\}$.

Hence
$$\begin{aligned}\Phi(m+n) &= \langle\, \pi \circ \gamma_{m+n} \,\rangle = \langle\, \pi \circ (\gamma_m \bullet \sigma) \,\rangle = \langle (\pi \circ \gamma_m) \bullet (\pi \circ \sigma) \rangle \\ &= \langle\, \pi \circ \gamma_m \,\rangle \cdot \langle\, \pi \circ \sigma \,\rangle = \langle\, \pi \circ \gamma_m \,\rangle \cdot \langle\, \pi \circ \gamma_n \,\rangle = \Phi(m) \cdot \Phi(n).\end{aligned}$$
□

Consider
$$U := \bigcup_{n \in \mathbb{Z}} \left(n - \frac{1}{2}, n + \frac{1}{2}\right) = \mathbb{E}^1 \setminus (\mathbb{Z} + \frac{1}{2}) \subset \mathbb{E}^1 .$$
Then $\pi\colon \mathbb{E}^1 \to \mathbb{S}^1$ maps U to $\mathbb{S}^1 \setminus \{-1\}$. Similarly, for
$$V := \bigcup_{n \in \mathbb{Z}} (n, n+1) = \mathbb{E}^1 \setminus \mathbb{Z} \subset \mathbb{E}^1$$
we have $\pi(V) = \mathbb{S}^1 \setminus \{1\}$.

Then $U \cup V$ is an open cover of $\mathbb{E}^1$ and $\pi(U) \cup \pi(V)$ is an open cover of $\mathbb{S}^1$.

The following is an important lemma of which we will have many generalisations.

Lemma 5.3.5 (Path-lifting)
Let σ be a path in $\mathbb{S}^1$ beginning at 1. Then there is a unique path $\tilde{\sigma}$ in $\mathbb{E}^1$ beginning at 0 with $\pi \circ \tilde{\sigma} = \sigma$.

$$\begin{tikzcd} & \mathbb{S}^1 \\ I \arrow[ur, "\sigma", "0\mapsto 1"'] \arrow[dr, "\tilde{\sigma}"', "0\mapsto 0"] & \\ & \mathbb{E}^1 \arrow[uu, "\pi", "0\mapsto 1"'] \end{tikzcd}$$

lift

Fig. 51: The commutative diagram for lemma 5.3.5.

We will call $\tilde{\sigma}$ the lift of the path σ with respect to the starting points 0 and 1 (!) and π is a projection.

Proof. The open sets $\sigma^{-1}(\pi(U))$ and $\sigma^{-1}(\pi(V))$ form an open cover of $[0,1]$. By the Lebesgue lemma, there exists a subdivision $0 = t_0 < t_1 < \ldots < t_m = 1$ such that for all $i \in \{0, \ldots, m-1\}$ we either have $[t_i, t_{i+1}] \subset \sigma^{-1}(\pi(U))$ or $[t_i, t_{i+1}] \subset \sigma^{-1}(\pi(V))$.

Since $1 \in \pi(U) \setminus \pi(V)$, we have $\sigma([0, t_1]) \subset \pi(U)$. The map π restricted to $(-\frac{1}{2}, \frac{1}{2})$ is a homeomorphism onto $\mathbb{S}^1 \setminus \{-1\}$. Let $f := (\pi|_{(-\frac{1}{2}, \frac{1}{2})})^{-1}\colon \mathbb{S}^1 \setminus \{-1\} \to (-\frac{1}{2}, \frac{1}{2})$ be its inverse. For $s \in [0, t_1]$ let $\tilde{\sigma}(s) := (f \circ \sigma)(s)$.

This is the beginning of an inductive construction. Suppose that $\tilde{\sigma}|_{[0, t_k]}$ is defined.

1. If $[t_k, t_{k+1}] \subset \sigma^{-1}(\pi(U))$ and, say, $\tilde{\sigma}(t_k) \in (n - \frac{1}{2}, n + \frac{1}{2})$, then let g be the inverse of the homeomorphism $\pi|_{(n-\frac{1}{2}, n+\frac{1}{2})}$. For $s \in [t_k, t_{k+1}]$, let $\tilde{\sigma}(s) := (g \circ \sigma)(s)$.
2. If $[t_k, t_{k+1}] \subset \sigma^{-1}(\pi(V))$ and say $\tilde{\sigma}(t_k) \in (n, n+1)$, then let h be the inverse of the homeomorphism $\pi|_{(n,n+1)}$. For $s \in [t_k, t_{k+1}]$ set $\tilde{\sigma}(s) := (h \circ \sigma)(s)$.

The function $\tilde{\sigma}$ is unique as we didn't have any choice in the construction. □

Corollary 5.3.6 (Surjectivity)
The homomorphism (7) *is onto.*

Proof. Let α be a loop in $\mathbb{S}^1$ based at 1. Then its lift is $\tilde{\alpha}$. Then $e^{2\pi i\tilde{\alpha}(1)} = (\pi \circ \tilde{\alpha})(1) = \alpha(1) = 1$, so $\tilde{\alpha}(1) \in \mathbb{Z}$. Then

$$\Phi(\tilde{\alpha}(1)) = \langle \pi \circ \gamma_{\tilde{\alpha}(1)} \rangle = \langle \pi \circ \tilde{\alpha} \rangle = \langle \alpha \rangle$$

because $\gamma_{\tilde{\alpha}(1)} \simeq \tilde{\alpha}$ rel $\{0,1\}$ as $\tilde{\alpha}(0) = 0 = \gamma_{\tilde{\alpha}(1)}(0)$ and $\tilde{\alpha}(1) = \gamma_{\tilde{\alpha}(1)}(1)$ and $\mathbb{E}^1$ is convex. □

Definition 5.3.7 (Degree)
The number $\tilde{\alpha}(1) \in \mathbb{Z}$ is the degree of α.

degree

Lemma 5.3.8 (Homotopy lifting)
If $F\colon I \times I \to \mathbb{S}^1$ is a map such that $F(0,\cdot) = F(1,\cdot) = 1$, then there exists a unique map $\tilde{F}\colon I \times I \to \mathbb{R}$ such that $\pi \circ \tilde{F} = F$ and $\tilde{F}(0,\cdot) = 0$.

$$\begin{tikzcd}
I \times I \arrow[r, "F", "(0,\cdot)\mapsto 1"'] \arrow[dr, "\tilde{F}", "(0,\cdot)\mapsto 0"'] & \mathbb{S}^1 \\
& \mathbb{R} \arrow[u, "\pi", "0\mapsto 1"']
\end{tikzcd}$$

Fig. 52: Homotopy lifting.

Again, we call $\tilde{F}$ to be the lift of F with respect to the starting points 0 and 1.

Proof. Similar to lemma 5.3.5. □

We used the simply connected space $\mathbb{R}$ to analyse the fundamental group of $\mathbb{S}^1$, which is a paradigm we will analyse later.

Proof. (of Theorem 5.3.1) It remains to show that Ψ is injective. Since Φ is a homomorphism, it suffices to show that $\Phi^{-1}(\langle e \rangle) = 0$, where e is the constant path based at 1.

As Φ is surjective, there exists a $n \in \mathbb{Z}$ such that $\Phi(n) = \langle e \rangle$. Suppose γ is a path in $\mathbb{R}$ from 0 to n such that $\pi \circ \gamma \simeq_F e$ rel $\{0,1\}$ for a suitable homotopy F.

By lemma 5.3.8, there exists a unique lift $\tilde{F}\colon I \times I \to \mathbb{R}$ such that $\pi \circ \tilde{F} = F$ and $\tilde{F}(0,\cdot) = 0$.

Let $P := (\{0,1\} \times [0,1]) \cup ([0,1] \times \{0\}) \subset I \times I$ describe three of the edges of the unit square. As F is a homotopy relative to $\{0,1\}$, we have $F(0,\cdot) = F(1,\cdot) = 1 \in \mathbb{S}^1$. As $\pi \circ \gamma \simeq_F e$, we have $F(\cdot,0) = 1$. Hence $F|_P \equiv 1$, so by $\pi \circ \tilde{F} = F$ there exists a $m \in \mathbb{Z}$ such that $\tilde{F}|_P \equiv m$. As $(0,0) \in P$, we have $\tilde{F}(0,0) = 0$, so $m = 0$.

Then $\tilde{\gamma} := \tilde{F}(\cdot,1)$ is a path in $\mathbb{R}$ and by construction a lift of $\pi \circ \gamma$ which begins at 0. By uniqueness of lifts with fixed starting points we have $\tilde{\gamma} = \gamma$. Then

$$n = \gamma(1) = \tilde{\gamma}(1) = \tilde{F}(1,1) = m = 0$$

as $(1,1) \in P$. □

We will later say that $\mathbb{R}$ is a universal cover of $\mathbb{S}^1$.

Definition 5.3.9 (Base point preserving variant)
Let $\mathbb{S}^1 \subset \mathbb{C}$ have basepoint $1 \in \mathbb{S}^1$ and $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map. The base point preserving variant $\overline{f}$ of f is

$$\overline{f}\colon (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1), \qquad z \mapsto \frac{f(z)}{f(1)}.$$

We have $\overline{f}(1) = 1$.

Definition 5.3.10 (Mapping degree)
Let $\alpha\colon \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ be a group isomorphism, which exists by Theorem 5.3.1. The degree $\deg(f)$ of a map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the group homomorphism making the following diagram commutative:

$$\begin{tikzcd} \pi_1(\mathbb{S}^1, 1) \arrow[r, "(\overline{f})_*"] \arrow[d, "\alpha"] & \pi_1(\mathbb{S}^1, 1) \arrow[d, "\alpha"] \\ \mathbb{Z} \arrow[r, "n \mapsto n\cdot\deg(f)"] & \mathbb{Z} \end{tikzcd}$$

Lemma 5.3.11 (Homotopic maps have equal degree)
Let $f, g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be homotopic maps. Then $\deg(f) = \deg(g)$.

Proof. By the commutative diagram, it suffices to show that $(\overline{f})_* = (\overline{g})_*$. This follows from lemma 5.2.5 by showing that $\overline{f} \simeq \overline{g}$ rel $\{1\}$. As $f \simeq_F g$, we have $\overline{f} \simeq_G \overline{g}$ relative to $\{1\}$, where $G(z,t) := \frac{F(z,t)}{F(1,t)}$, as $G(\cdot, 0) = \frac{F(\cdot,0)}{F(1,0)} = \frac{f}{f(1)} = \overline{f}$ and analogously for g. Lastly, $G(1, \cdot) = 1$ by construction. □

For showing that $\pi_1(\mathbb{S}^n) = 1$ for all $n \geqslant 2$, we first prove the following theorem, which is a special case of the Seifert-van-Kampen theorem we will see later.

Theorem 5.3.2: Simply connectedness

Let X be a topological space with open subspaces $U, V \subset X$ such that $X = U \cup V$. If U and V are both simply connected and $U \cap V$ is path connected, then X is simply connected.

Proof. As U, V and $U \cap V$ are path-connected, so is X.

Pick a base point $p \in U \cap V$. Consider a loop $\alpha\colon I \to X$ in X based at p. By the Lebesgue lemma, there exists a subdivision

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

such that $\alpha([t_k, t_{k+1}]) \subset U$ or $\alpha([t_k, t_{k+1}]) \subset V$.

Without loss of generality we can assume that $\alpha(t_i) \in U \cap V$ for all $i \in \{1, \ldots, n-1\}$ because we can just "delete" t_i, that are in not in $U \cap V$ without loosing the above property.

As $U \cap V$ is path-connected, there exist paths γ_i in $U \cap V$ from p to $\alpha(t_i)$ for $i \in \{1, \ldots, n-1\}$. If $\alpha([0,1]) \subset U$, then the concatenation $\alpha|_{[0,t_1]} \circ \gamma_1^{-1}$ is a loop based at p. As U is simply connected, we have

$$\alpha|_{[0,t]} \circ \gamma_1^{-1} \simeq e \quad \text{rel } \{0, 1\}.$$

Since the assumptions are symmetric in U and V, we have

$$\begin{aligned} \alpha = (\alpha|_{[0,t]}) \circ \ldots (\alpha|_{[t_{n-1},t_n]}) \\ &\simeq \underbrace{(\alpha|_{[0,t]})\gamma_1^{-1}}_{\simeq e \text{ rel } \{0,1\}} \gamma_1 \circ \ldots (\alpha|_{[t_{n-1},t_n]}) \quad \text{rel } \{0,1\} \\ &\simeq \gamma_1 \circ \ldots (\alpha|_{[t_{n-1},t_n]}) \quad \text{rel } \{0,1\}. \end{aligned}$$

This works similarly, if $\alpha([0, t_1]) \subset V$. Now proceed inductively to show that $\alpha \simeq e$ relative to $\{0, 1\}$. $\square$

Corollary 5.3.12 (Fundamental group of the spheres)
We have $\pi_1(\mathbb{S}^n, p) = \{e\}$ for all $p \in \mathbb{S}^n$ and all $n \geqslant 2$.

Proof. Let $q \in \mathbb{S}^n \backslash \{p\}$. As $\mathbb{S}^n \backslash \{p\} \approx \mathbb{E}^n$, $\mathbb{S}^n \backslash \{p\}$ is simply connected. We can decompose $\mathbb{S}^n$ as a union of open simply connected subsets:

$$\mathbb{S}^n = (\mathbb{S}^n \backslash \{p\}) \cup (\mathbb{S}^n \backslash \{q\}).$$

As $n \geqslant 2$, their intersection

$$(\mathbb{S}^n \backslash \{p\}) \cap (\mathbb{S}^n \backslash \{q\}) = \mathbb{S}^n \backslash \{p, q\}$$

is path-connected. By Theorem 5.3.2, $\pi_1(\mathbb{S}^n, p)$ is trivial. $\square$

Product spaces

Theorem 5.3.3: Fundamental group of a product

Let X and Y be topological spaces, $p \in X$ and $q \in Y$. Then

$$\pi_1(X \times Y, (p, q)) \cong \pi_1(X, p) \times \pi_1(Y, q).$$

Proof. By example 5.2.3, the projection maps $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ induce group homomorphisms

$$(p_1)_* \colon \pi_1(X \times Y, (p, q)) \to \pi_1(X, p), \qquad (p_2)_* \colon \pi_1(X \times Y, (p, q)) \to \pi_1(Y, q).$$

They yield a group homomorphism

$$\begin{gathered} (p_1)_* \times (p_2)_* \colon \pi_1(X \times Y, (p, q)) \to \pi_1(X, p) \times \pi_2(Y, q), \\ \langle \alpha \rangle \mapsto \big((p_1)_*(\langle \alpha \rangle), (p_2)_*(\langle \alpha \rangle)\big) = \big(\langle p_1 \circ \alpha \rangle, \langle p_2 \circ \alpha \rangle\big). \end{gathered}$$

Let α be a loop in $X \times Y$ with the base point $(p, q) \in X \times Y$ such that $\pi_1 \circ \alpha \simeq_F e_p$ relative to $\{0, 1\}$ and $\pi_2 \circ \alpha \simeq_G e_q$ relative to $\{0, 1\}$. For $H = (F, G)$ we have $\alpha \simeq_H e_{(p,q)}$, where e_x is the constant path at x. Hence $(p_1)_* \times (p_2)_*$ is injective.

If β is a loop in X based at p and γ is a loop in Y based at q, then

$$\alpha \colon I \to X \times Y, \qquad t \mapsto (\beta(t), \gamma(t))$$

is a loop in $X \times Y$ based at (p, q) with $((p_1)_* \times (p_2)_*)(\alpha) = (\beta, \gamma)$. Hence $(p_1)_* \times (p_2)_*$ is onto and hence a isomorphism. $\square$

Example 5.3.13 We have $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(\mathbb{S}^n \times \mathbb{S}^m) = \{e\}$ for all $m, n \geqslant 2$. $\diamond$

5.4 Homotopy type

What kinds of differences does π_1 measure?

Definition 5.4.1 (Homotopy equivalence)
Two topological spaces are homotopy equivalent and we write $X \simeq Y$ if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X \qquad f \circ g \simeq \mathrm{id}_Y.$$

Then g is the homotopy inverse of f and vice versa.

homotopy equivalent

This property relaxes the property of begin homeomorphic; homeomorphic spaces are homotopy equivalent.

Remark 5.4.2 (Equivalence relation) Homotopy equivalence is an equivalence relation of any set of topological spaces.

The set of topological spaces does not exist.

Example 5.4.3 Each convex subset of $\mathbb{E}^n$ is homotopy equivalent to a point. ◇

Example 5.4.4 ($\mathbb{E}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$)
We have $\mathbb{E}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$ (and not $\mathbb{E}^n \setminus \{0\} \approx \mathbb{S}^{n-1}$!) via

$$f\colon \mathbb{E}^n \setminus \{0\} \to \mathbb{S}^{n-1}, \qquad x \mapsto \frac{x}{\|x\|} \qquad \text{and} \qquad g\colon \mathbb{S}^{n-1} \to \mathbb{E}^n \setminus \{0\}, \qquad y \mapsto y.$$

We have $g \circ f = \mathrm{id}_{\mathbb{S}^{n-1}}$. With

$$F\colon (\mathbb{E}^n \setminus \{0\}) \times I \to \mathbb{E}^n \setminus \{0\}, \qquad (x,t) \mapsto (1-t)x + tf(x) \tag{8}$$

we have $f \circ g \simeq_F \mathrm{id}_{\mathbb{E}^n \setminus \{0\}}$, because $F(x,t) = 0$ if and only if $t = \frac{1}{1-\|x\|}$, but $t \in [0,1]$ and $\frac{1}{1-\|x\|} \in \mathbb{R} \setminus [0,1]$ as $\|x\| > 0$. ◇

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Definition 5.4.5 (Deformation retraction, deformation retract)
A homotopy $G\colon X \times I \to X$ relative to $A \subset X$ is a deformation retraction if

- $G(a, \cdot) = a$ for all $a \in A$,
- $G(\cdot, 0) = \mathrm{id}_X$,
- $\{G(x,1) : x \in X\} \subset A$.

For $f\colon A \hookrightarrow X$, $a \mapsto a$ and $g\colon X \to A$, $x \mapsto G(x,1)$ we have $f \circ g \simeq_G \mathrm{id}_X$ and $g \circ f \simeq_{G|_{A \times I}} \mathrm{id}_A$. Hence $X \simeq A$, so A is a deformation retract of X and we say that X retracts to A.

deformation retraction

Example 5.4.6 A deformation retraction from $X := \mathbb{E}^n \setminus \{0\}$ to $A := \mathbb{S}^{n-1} \subset X$ is (8). ◇

Example 5.4.7 A deformation retraction from the cylinder $\mathbb{S}^1 \times I$ to the circle $\mathbb{S}^1$ (identified with $A := \mathbb{S}^1 \times \{0\}$) is

$$G\colon (\mathbb{S}^1 \times I) \times I \to \mathbb{S}^1 \times I, \qquad (z,s,t) \mapsto (z, s(1-t)).$$

Then $G(z,0,t) = (z,0)$, $G(z,s,0) = (z,s)$ and $G(z,s,1) = (z,0) \in A$. ◇

This construction can make the computation of the fundamental group easier. By corollary 5.2.4, the fundamental group is a homotopy invariant. Hence in order to compute the fundamental group of X it suffices to find the fundamental group of any deformation retract of X.

The following theorem is statement about the functor π_1 on the level of morphisms.

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "f_*"] \arrow[d, "g_*"'] & \pi_1(Y, f(p)) \arrow[dl, "\gamma_*"] \\ \pi_1(Y, g(p)) & \end{tikzcd}$$

Fig. 53

Theorem 5.4.1: Functorial properties of π_1

Let $f, g\colon X \to Y$ be maps with $f \simeq_F g$. Then, for a base point $p \in X$,

$$g_*\colon \pi_1(X, p) \to \pi_1(Y, g(p))$$

equals the composition $\gamma_* \circ f_*$, where

$$f_*\colon \pi_1(X, p) \to \pi_1(Y, f(p))$$

and the path joining $f(p)$ and $g(p)$,

$$\gamma\colon I \to Y, \qquad s \mapsto F(p, s),$$

induces a map (via (5))

$$\gamma_*\colon \pi_1(Y, f(p)) \to \pi_1(Y, g(p)), \qquad \langle\, \alpha \,\rangle \mapsto \langle\, \gamma^{-1} \bullet \alpha \bullet \gamma \,\rangle.$$

Proof. Let α be a loop in X based at p. Then $g_*(\langle\, \alpha \,\rangle) = \langle\, g \circ \alpha \,\rangle$ and

$$(\gamma_* \circ f_*)(\langle\, \alpha \,\rangle) = \langle\, \gamma^{-1} \bullet (f \circ \alpha) \bullet \gamma \,\rangle.$$

Hence we have to show that $g \circ \alpha \simeq \gamma^{-1} \bullet (f \circ \alpha) \bullet \gamma$ relative to $\{0, 1\}$.

Let

$$G\colon I \times I \to Y, \qquad (s, t) \mapsto F(\alpha(s), t)$$

and

$$H\colon I \times I \to Y, \qquad (s, t) \mapsto \begin{cases} \gamma(1 - 4s), & \text{if } 0 \leqslant s \leqslant \frac{1-t}{4}, \\ G\left(\frac{4s+t-1}{3t+1}, t\right), & \text{if } \frac{1-t}{4} \leqslant s \leqslant \frac{1+t}{2}, \\ \gamma(2s - 1), & \text{if } \frac{1+t}{2} \leqslant s \leqslant 1. \end{cases}$$

By the glueing lemma, G and H are continuous.

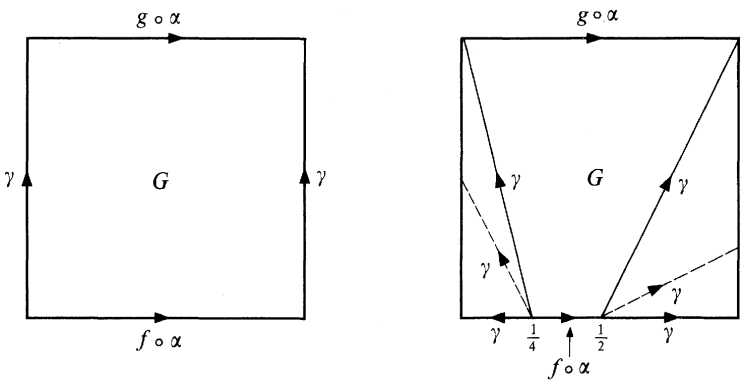

Fig. 54

Hence $g \circ \alpha \simeq_H \gamma^{-1} \bullet (f \circ \alpha) \bullet \gamma$ relative to $\{0, 1\}$. $\square$

This is statement about the functor π_1 on the level of objects.

Theorem 5.4.2: $X \simeq Y \implies \pi_1(X) \cong \pi_1(Y)$

Let X and Y be path-connected. If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

This theorem together with the previous theorem says that π_1 even is a functor on the category of topological spaces modulo homotopy equivalence, often simply called the homotopy category.

Proof. Let $f\colon X \to Y$ and $g\colon Y \to X$ be maps with $\mathrm{id}_X \simeq_F g \circ f$ and $\mathrm{id}_Y \simeq_G f \circ g$. Let $q \in Y$ and $p := g(q)$ be base points. Define $\gamma\colon I \to X$, $s \mapsto F(p, s)$. Then $\gamma(0) = F(p, 0) = \mathrm{id}_X(p) = p$ and $\gamma(1) = F(p, 1) = g(f(p))$, so γ is a path in X joining p with $g(f(p))$.

Applying Theorem 5.4.1 to $f = \mathrm{id}_X$ and $g = g \circ f$ yields that

$$(g \circ f)_* = g_* \circ f_* = \gamma_*\colon \pi_1(X, p) \to \pi_1(X, g(f(p)))$$

is an isomorphism. Hence f_* is injective (its left inverse up to conjugation is g_*). A similar argument using G instead of F shows that f_* is surjective, so f_* is bijective. □

Example 5.4.8 We have $\mathbb{S}^1 \times I \simeq \mathbb{S}^1$ (this is in fact a deformation retract). Hence $\pi_1(\mathbb{S}^1 \times I) \cong \pi(\mathbb{S}^1) \cong \mathbb{Z}$ by Theorem 5.4.2. Let M be the Möbius strip. We have $M \not\approx \mathbb{S}^1 \times I$ but $M \simeq \mathbb{S}^1$, so $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ by Theorem 5.4.2. ◇

Remark 5.4.9 In Theorem 5.4.2 it would suffice that either X or Y are path-connected: as path-connectedness is a homotopy invariant: let $X \simeq Y$ and X be path-connected, then Y is path connected.

Indeed, let $f\colon X \to Y$ and $g\colon Y \to X$ be such that $f \circ g \simeq_F \mathrm{id}_Y$. Further, let $y_1, y_2 \in Y$. Then there exists a path $\gamma\colon [0, 1] \to X$ with $\gamma(0) = g(y_1)$ and $\gamma(1) = \gamma(y_2)$. Hence the path $f \circ \gamma\colon [0, 1] \to Y$ has $(f \circ \gamma)(0) = (f \circ g)(y_1)$ and $(f \circ \gamma)(1) = (f \circ g)(y_2)$. Hence $F(y_1, \cdot) \bullet (f \circ \gamma) \bullet F(y_2, 1 - \cdot)$ is a path from y_1 to $(f \circ g)(y_1)$ to $(f \circ g)(y_2)$ to y_2. ○

Contractible spaces

Definition 5.4.10 (Contractible)
A space X is contractible if $\mathrm{id}_X \simeq e_p$ for some $p \in X$. contractible

Here $e_p\colon X \to X$, $x \mapsto p$ denotes the constant map at $p \in X$ on X.

Remark 5.4.11 X is contractible if and only if X has the homotopy type of a point. ○

Remark 5.4.12 (Contractible spaces are path connected)
Let X be contractible. Then there exists a $p \in X$ and a homotopy $F\colon X \times I \to X$ with $F(\cdot, 0) = \mathrm{id}_X$ and $F(\cdot, 1) = e_p$. Let $x \in X$. Then $F(x, \cdot)$ is a path from x to p. Hence X is path connected. ○

Theorem 5.4.3: Contractible spaces

Let X be contractible.

- Then X is simply connected.
- Let Y be a space with maps $f, g\colon Y \to X$. Then $f \simeq g$.
- Then id_X is homotopic to e_x for all $x \in X$.

The second property, stating that maps into contractible spaces are homotopic to each other, generalises the fact that the fundamental group of convex spaces is trivial.

Proof.
- As X is path-connected, $\pi_1(X) = \pi_1(\{p\})$ is trivial by Theorem 5.4.2: we have $X \simeq \{p\}$ via $f\colon X \to \{p\}$, $x \mapsto p$ and the inclusion $g\colon \{p\} \to X$, as $(f \circ g)(p) = p$, so $f \circ g = \mathrm{id}_{\{p\}}$ and $g \circ f = e_p \simeq \mathrm{id}_X$.
- We have
$$f = \mathrm{id}_X \circ f \simeq e_p \circ f = e_p \circ g \simeq \mathrm{id}_X \circ g = g.$$
- By the previous point the maps $\mathrm{id}_X\colon X \to X$ and $e_x\colon X \to X$ are homotopic for every $x \in X$. □

In a way, contractible spaces are boring because their fundamental group is trivial. There may be something interesting left, which we can see now.

Example 5.4.13 (Dunce hat)
Consider a triangle with all three edges identified with each other in the following fashion. We can define it as $T/\sim$, where T is the convex hull of the three unit vectors in $\mathbb{R}^3$ and the equivalence relation $\sim$ with the partition, $\{x\}$ if $x = (a, b, c)$, where $a + b + c = 1$ and $0 < a, b, c < 1$ and the three vertices $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $\{(1 - s, 0, s), (0, 1 - s, s), (s, 1 - s, 0)\}$ for any $s \in (0, 1)$.

This dunce hat is contractible. ◇

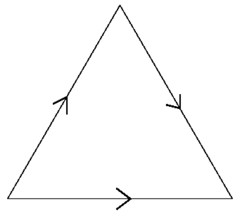

Fig. 55: The dunce hat.

5.5 Covering maps

Definition 5.5.1 (Cover(ing map))
Let X and Y be topological spaces. A map $p\colon Y \to X$ is a covering map or cover (of X) if every point $x \in X$ has an open neighbourhood V for which $p^{-1}(V)$ decomposes into a disjoint union of open sets $U_i \subset Y$ such that $p|_{U_i}\colon U_i \to V$ is a homeomorphism.

covering map

In particular, a covering map is always surjective.

A cover $p\colon Y \to X$ is a space over X.

Covers of X form a category $\mathrm{Cov}(X)$, where the objects are the covers $p\colon Y \to X$ of X and the morphisms are the homeomorphisms $f\colon Y \to Z$, where $p_1\colon Y \to X$ and $p_2\colon Z \to X$ are covers of X, with $p_2 \circ f = p_1$.

$$\begin{tikzcd} Y \arrow[rr, "f", "\approx"'] \arrow[dr, "p_1"'] & & Z \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Fig. 56: The category of covers consists of covers and homeomorphisms between their codomains.

Example 5.5.2 Let $I \neq \varnothing$ be a discrete space. The projection $p_X\colon X \times I \to X$ onto the first coordinate is the trivial cover of X. ◇

trivial cover

Lemma 5.5.3
A map $p\colon Y \to X$ is a cover of X precisely if every point $x \in X$ has an open neighbourhood such that $p|_{p^{-1}(V)}$ is homeomorphic (as a space over V) to some trivial cover $V \times I_V$ over V.

Proof. (HW 7.5) TODO □

Lemma 5.5.4
If X is connected, then the discrete spaces I_V from above are all homeomorphic to the same space I.

In this case the cardinality of I is the number of sheets of the cover p.

Proof. (HW 7.5) TODO □

Definition 5.5.5 (Universal cover)
A cover $p\colon Y \to X$ is universal if Y is simply connected.

Example 5.5.6 The map $\mathbb{C}^* \to \mathbb{C}^* := \mathbb{C}\setminus\{0\}$, $z \mapsto z^n$ is a n-sheeted cover. As $\pi_1(\mathbb{C}^*) = \pi_1(\mathbb{S}^1) = \mathbb{Z} \neq \{1\}$ (due to $\mathbb{C}^* \simeq \mathbb{S}^1$), this is not a universal cover.

The quotient map $\mathbb{S}^n \to \mathbb{R}\mathrm{P}^n$ for $n \geqslant 2$ is a n-sheeted universal cover, as $\pi_1(\mathbb{S}^n)$ is trivial. ◇

Example 5.5.7 The map $[0,1] \to \mathbb{S}^1$, $t \mapsto e^{2\pi i n t}$ is not a cover of because the preimages of points from $\mathbb{S}^1$ have different cardinalities (all sheets must have the same cardinality).

The map $\mathbb{D}^2 \to \mathbb{D}^2 / \sim_a \approx \mathbb{R}\mathrm{P}^2$ is not a cover for the same reason. ◇

Definition 5.5.8 (Deck transformation)
Let X be connected and locally simply connected and $p\colon \tilde{X} \to X$ be a universal cover. The group of deck transformations is

$$\operatorname{Deck}(\tilde{X} \to X) := \{f\colon \tilde{X} \to \tilde{X} : f \text{ homeomorphism},\ p \circ f = p\}.$$

$$\begin{tikzcd} \tilde{X} \arrow[r, "p"] \arrow[d, "f"'] & X \\ \tilde{X} \arrow[ur, "p"] & \end{tikzcd}$$

Fig. 57: Commutative diagram for a deck transformation f.

Theorem 5.5.1: Deck transformations and π_1

We have
$$\operatorname{Deck}(\tilde{X} \to X) \cong \pi_1(X)$$

Example 5.5.9 We have $\operatorname{Deck}(\mathbb{S}^2 \to \mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2\mathbb{Z}$, because there are only two homeomorphisms $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that $q \circ f = q$, where $q\colon \mathbb{S}^2 \to \mathbb{R}\mathrm{P}^2$ is the quotient map realising antipodal identification: multiplication by 1 and multiplication by -1. ◇

Example 5.5.10 The map $\mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$ is a universal cover with countably many sheets. (Bild: Spirale über $\mathbb{S}^1$) We have

$$\operatorname{Deck}(\mathbb{R} \to \mathbb{S}^1) = \{f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x + m : m \in \mathbb{Z}\} \cong \mathbb{Z} = \pi_1(\mathbb{S}^1).$$ ◇

Remark 5.5.11 (Why does this work so nicely?) For $x \in X$ the functor

$$\operatorname{Fib}_x\colon \{\text{covers of } X\} \to \{\text{sets}\}, \qquad (Y \xrightarrow{p} X) \mapsto p^{-1}(x)$$

is representable, that is,

$$\operatorname{Fib}_x(Y \to X) = \operatorname{Hom}_X(\tilde{X}_x, Y)$$

for all covers $Y \to X$. Here, $\tilde{X}_x = \{\text{homotopy classes of } w\colon [0,1] \to X \text{ with } w(0) = x\}$ is the universal cover of X at x (doesn't really depend on x) and $\operatorname{Hom}_X(\tilde{X}_x, Y)$ is the set of homeomorphisms $f\colon \tilde{X} \to Y$ of covers $\tilde{X} \to X$ and $Y \to X$ of X. ○

5.6 Brouwer's Fixed Point Theorem

13.12.2021

Theorem 5.6.1: Brouwer's Fixed Point Theorem

Every map $f\colon \mathbb{B}^n \to \mathbb{B}^n$ has a fixed point for $n \geqslant 1$.

Proof. (For $n = 2$.) Suppose $f(x) \neq x$ for all $x \in \mathbb{B}^2$. For $x \in \mathbb{B}^2$ let $g(x)$ be the unique point of intersection of $\mathbb{S}^1$ and the ray from $f(x)$ to x in the direction of x.

Then $g\colon \mathbb{B}^2 \to \mathbb{S}^1$ is a map as a composition of maps. For $x \in \mathbb{S}^1$ we have $g(x) = x$. Hence g is a retraction from $\mathbb{B}^2$ to $\mathbb{S}^1$. Then $\mathbb{B}^2 \simeq g(\mathbb{B}^2) = \mathbb{S}^1$ by Theorem 5.4.2, which is a contradiction as $\pi_1(\mathbb{B}^2) \neq \pi_1(\mathbb{S}^1)$. □

Remark 5.6.1 (Application of Brouwer's Fixed Point Theorem for $d = 2$)
There is no draw in the two player game HEX [8]. ○

Proof. (For $n = 1$) Replace π_1 by π_0 in the above proof, where π_0 is the set of path-connected components (which is not a group). □

We can't do the proof for $n > 2$ yet, one can replace π_1 by π_{n-1} (homotopy group) or by H_{n-1} (homology group), which we will introduce later.

In the proof we get the following commutative diagram, to which we then apply π_1:

$$
\begin{tikzcd}
 & \mathbb{D}^2 \arrow[dd, "r"', "\text{retract}"] & & & \{1\} \cong \pi_1(\mathbb{D}^2) \arrow[dd, "\pi_1(r)=r_*"] \\
\mathbb{S}^1 \arrow[ur, "\iota"] \arrow[dr, "\mathrm{id}"'] & & \overset{\pi_1}{\longmapsto} & \mathbb{Z} \cong \pi_1(\mathbb{S}^1) \arrow[ur, "\pi_1(\iota)"] \arrow[dr, "\pi_1(\mathrm{id}_{\mathbb{S}^1})=\mathrm{id}_{\pi_1(\mathbb{S}^1)}"'] & \\
 & \mathbb{S}^1 & & & \mathbb{Z} \cong \pi_1(\mathbb{S}^1)
\end{tikzcd}
$$

Fig. 58: "Proof of Brouwer's fixed point theorem by diagram".

But the second diagram can't commute because 2 is mapped by ι_* to 1 (the only possibility) and as r_* is a group homomorphism, we have $r_*(1) = 1$, but 2 is mapped by $\mathrm{id}_{\pi_1(\mathbb{S}^1)}$ to $1 \neq 2$.

5.7 Separation in the plane

Theorem 5.7.1: Jordan's Curve Theorem

Let $J \subset \mathbb{E}^2$ be a Jordan curve, that is $J \approx \mathbb{S}^1$. Then $\mathbb{E}^2 \backslash J$ has exactly two (path)components.

By Theorem 3.6.2, path-connectedness and connectedness are equivalent for open subset of $\mathbb{E}^n$ such as $\mathbb{E}^2 \backslash J$.

Proof. Identify $\mathbb{E}^2$ with $\mathbb{E}^2 \times \{0\} \subset \mathbb{E}^3$ and let $h\colon \mathbb{E}^2 \to \mathbb{S}^2 \backslash \{(0,0,1)\}$ be a homeomorphism, e.g. the inverse of the stereographic projection.

Pick $p \in h(J)$. Let $k\colon \mathbb{E}^2 \to \mathbb{S}^2 \backslash \{p\}$ be a homeomorphism and consider the following subspace of $\mathbb{E}^2$:

$$L := k^{-1}(h(J)\backslash\{p\}) \approx \mathbb{E}^1 .$$

The spaces $\mathbb{E}^2\backslash J$, $\mathbb{S}^2\backslash h(J)$ and $\mathbb{E}^2\backslash L$ all have the same number of components: we have

$$\begin{aligned}\mathbb{E}^2\backslash L &= k^{-1}(\mathbb{S}^2\backslash\{p\})\backslash k^{-1}(h(J)\backslash\{p\})\\ &= k^{-1}\big((\mathbb{S}^2\backslash\{p\})\backslash(h(J)\backslash\{p\})\big) = k^{-1}(\mathbb{S}^2\backslash h(J)) \approx \mathbb{S}^2\backslash h(J).\end{aligned}$$

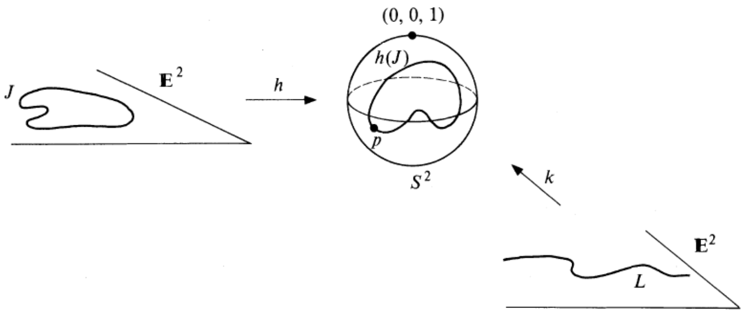

Fig. 59

Assume $\mathbb{E}^2\backslash L$ is connected. As L is closed, $\mathbb{E}^2\backslash L$ is path-connected by Theorem 3.6.2. Let

$$H^+ := \{(x,y,z) \in \mathbb{E}^3 : z > 0\}, \qquad \text{and} \qquad H^- := \{(x,y,z) \in \mathbb{E}^3 : z < 0\}$$

as well as

$$U := H^+ \cup \{(x,y,z) \in \mathbb{E}^3 : (x,y) \in \mathbb{E}^2\backslash L,\ z \in (-1,0]\}$$

and

$$V := H^- \cup \{(x,y,z) \in \mathbb{E}^3 : (x,y) \in \mathbb{E}^2\backslash L,\ z \in [0,1)\},$$

which are all open sets.

Then $U \cup V = \mathbb{E}^3\backslash L$ and $U \cap V \approx (\mathbb{E}^3\backslash L) \times (-1,1)$ is path-connected as the product of path-connected sets. By Theorem 5.3.2 $\mathbb{E}^3\backslash L$ is path-connected.

There is a homeomorphism $\eta\colon \mathbb{E}^3 \to \mathbb{E}^3$ such that $\eta(L)$ is the z-axis Z (will be proven shortly.) Then $\mathbb{E}^3\backslash L \cong \mathbb{E}^3\backslash Z \simeq \mathbb{E}^2\backslash\{0\} \approx \mathbb{S}^1$. By Theorem 5.4.2 we have $\pi_1(\mathbb{E}^3\backslash L) \cong (\mathbb{Z},+)$, which is a contradiction to $\mathbb{E}^3\backslash L$ being path-connected.

For the existence of η choose a homeomorphism $f\colon L \to \mathbb{E}^1$ and consider

$$L_1 := \{(x,y,f(x,y)) : (x,y) \in L\} \subset \mathbb{E}^3\,.$$

Then $L_1 \approx L \approx \mathbb{E}^1$.

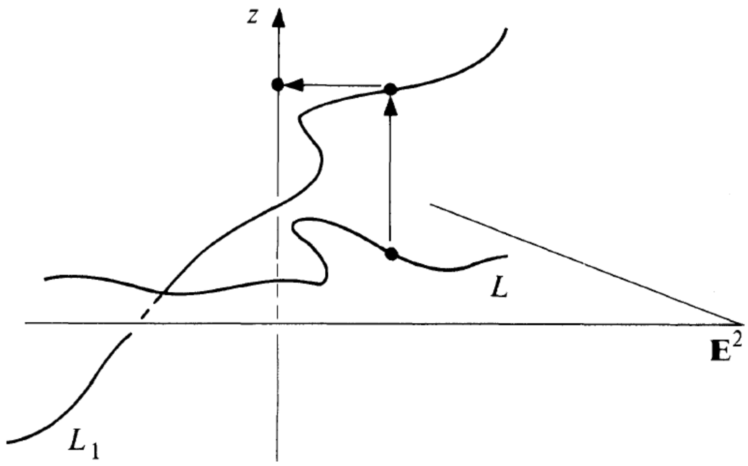

Fig. 60

Each plane parallel to $\{(x, y, z) \in \mathbb{E}^3 : z = 0\}$ intersects L_1 is a unique point, as f is a bijection. As L is closed, we can extend f to a map $g\colon \mathbb{E}^2 \to \mathbb{E}^1$ by TIETZE's extension theorem. Define

$$\eta_1\colon \mathbb{E}^3 \to \mathbb{E}^3, \qquad (x, y, z) \mapsto (x, y, z + g(x, y)).$$

Then $\eta_1(L) = L_1$. Define

$$\eta_2\colon \mathbb{E}^2 \to \mathbb{E}^3, \qquad (x, y, z) \mapsto \left(x - \left(f^{-1}(x)\right)_1, y - \left(f^{-1}(z)\right)_y, z\right),$$

which is a homeomorphism with $\eta_2(L_1) = Z$. Then $\eta := \eta_1 \circ \eta_2$ is the desired homeomorphism. □

Theorem 5.7.2: ?

Let $A \subset \mathbb{E}^2$ be a curve which doesn't closed up, that is, $A \approx [0, 1]$. Then $\mathbb{E}^2 \setminus A$ is path-connected.

5.8 The boundary of a surface

Definition 5.8.1 (Surface, topological k-manifold)
A (topological) k-manifold is a HAUSDORFF space such that each point has a neighbourhood either homeomorphic to $\mathbb{E}^k$ or $\mathbb{E}^k_+ := \{(x_1, \ldots, x_k) \in \mathbb{E}^k : x_k \geqslant 0\}$.

A surface is a 2-manifold. surface

Example 5.8.2 Polyhedra are surfaces. ◇

Definition 5.8.3 (Interior, boundary)
Let S be a surface. The interior of S is the set of all $x \in \mathbb{S}$ such that there exists a neighbourhood $N \in \mathcal{N}$ with $N_x \approx \mathbb{E}^2$. interior

The boundary of S is the set of all $x \in \mathbb{S}$ such that there exists a homeomorphism $f\colon \mathbb{E}^2_+ \to N$, where $N \in \mathcal{N}_x$ and $f(0) = x$. boundary

Definition 5.8.4 (Closed)
A surface is closed if it is compact and has empty boundary. closed

We show that the concepts of interior and boundary are well defined.

Theorem 5.8.1: Welldefinedness of boundary/interior

The interior and boundary of a surface are disjoint.

Proof. Assume $x \in S$ is in the interior and the boundary of S. Then there exist $U, V \in \mathcal{N}_x$ and homeomorphisms $f\colon \mathbb{E}^2 \to U$ and $g\colon \mathbb{E}^2_+ \to V$ with (without loss of generality $f(0) = g(0) = x$).

Choose a closed half disk $D_1 \subset \mathbb{E}^2_+$ centered at zero such that $f(D_1) \subset V$. Let $\varphi := g^{-1} \circ f|_{D_1}\colon D_1 \to \mathbb{E}^2_+$, which is a homeomorphism onto its image with $\varphi(0) = 0$. Thus

$\varphi(D_1)$ is a neighbourhood of 0 in $\mathbb{E}^2_+$. Now choose a closed disk $D_2 \subset \mathbb{E}^2$ centered at 0 such that $D_2 \subset \varphi(D_1)$. Let $r\colon \mathbb{E}^2 \setminus \{0\} \to \partial D_2$ be the radial projection onto ∂D_2. Then $r|_{\varphi(D_1)\setminus\{0\}}\colon \varphi(D_1)\setminus\{0\} \to \partial D_2$ retracts $\varphi(D_1)\setminus\{0\}$ onto ∂D_2. Hence

$$r_*\colon \pi_1(\varphi(D_1)\setminus\{0\}) \to \pi_1(\partial D_2)$$

is a surjective group homomorphism, which is a contradiction to $\varphi(D_1)\setminus\{0\}$ being contractible. □

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Theorem 5.8.2: Homeomorphisms preserve boundary

Let S_1 and S_2 be surfaces which are homeomorphic via $h\colon S_1 \to S_2$. Then h maps the interior of S_1 homeomorphically onto the interior of S_2 (and likewise for the boundaries).

When discussing fundamental groups, we encountered pointed spaces. The pair (surface, its boundary) is similar. The above theorem states that a homeomorphism of surfaces induces (by restriction) a homeomorphism of such pairs.

Proof. Let x be in the interior and boundary of S_1. Then there exists a neighbourhood U of x and a homeomorphism $f\colon \mathbb{E}^2 \to U$. As h is a homeomorphism, too, $h(U)$ is a neighbourhood of $h(x) \in S_2$. Hence $h(x)$ is in the interior of S_2.

The converse follows from looking at h^{-1}.

By Theorem 5.8.1, the boundary is the complement of the interior, proving the second claim. □

Corollary 5.8.5
Homeomorphic surfaces have homeomorphic boundaries.

Example 5.8.6 The cylinder and the Möbius strip are not homeomorphic, as the boundary of the cylinder $\mathbb{S}^1 \times I$ is $\mathbb{S}^1 \sqcup \mathbb{S}^1$ and the boundary of the Möbius strip is (homeomorphic to) $\mathbb{S}^1$ (the latter is path-connected, while the former is not even connected, so they can not be homeomorphic).

But the Möbius strip and the cylinder are homotopy equivalent since they both retract to $\mathbb{S}^1$. ◇

6 Triangulations

6.1 Triangulations

We lack ways to construct interesting topological spaces.

Let $V := \{v_0, \ldots, v_k\} \subset \mathbb{E}^n$. (We could also instead pick a discrete, but not necessarily finite set V, but would have to modify a few things).

Definition 6.1.1 (Affine hull)
The affine hull of V is $\operatorname{aff}(V) := \left\{ \sum_{j=0}^{k} \lambda_j v_j : \sum_{j=0}^{k} \lambda_j = 1 \right\}$.

The affine hull of two distinct points is the unique line going through both of them.
Remark 6.1.2 The affine hull of V is an affine subspace (of dimension at most k), that is, the solution set of a system of k not necessarily homogeneous linear equations. ○

Definition 6.1.3 (General position)
The set V is in general position if $\dim(V) = k$.

We say that the points in V are affinely independent. The set V is general position if and only if $(v_j - v_0)_{j=1}^{k}$ are linearly independent.
Remark 6.1.4 (The affine category) Affine maps $f \colon V \to W$ between affine spaces V and W are characterised by $f\left(\sum_{k=1}^{n} \lambda_k x_k\right) = \sum_{k=1}^{n} \lambda_k f(x_k)$ if $\sum_{k=1}^{n} \lambda_k = 1$ (the restriction $\sum_{k=1}^{n} \lambda_k = 1$ is not imposed for linear maps, so affine maps are more general than linear maps). Affine spaces together with affine maps form the category **Aff**. Like linear maps are determined by their values on a basis, an affine map is determined by its values on an affinely independent set. ○

Lemma 6.1.5 (Affine and linear independence)
The set $\{x_0, \ldots, x_m\} \subset \mathbb{R}^n$ is affinely dependent if and only if the set $\{(1, x_0), \ldots, (1, x_m)\} \subset \mathbb{R}^{n+1}$ is linearly dependent.

Hence affine independence is "linear independence in one dimension higher".

Proof. The set $\{(1, x_0), \ldots, (1, x_m)\} \subset \mathbb{R}^{n+1}$ is linearly dependent if and only if there exists coefficients $(\lambda_k)_{k=1}^{m}$ (not all zero) such that

$$\begin{pmatrix} 1 \\ x_0 \end{pmatrix} = \sum_{k=1}^{m} \lambda_k \begin{pmatrix} 1 \\ x_k \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{m} \lambda_k \\ \sum_{k=1}^{m} \lambda_k x_k \end{pmatrix}.$$

This is the case if and only if $x_0 = \sum_{k=1}^{m} \lambda_k x_k$ with $\sum_{k=1}^{m} \lambda_k = 1$, that is, $(x_k)_{k=0}^{m}$ is affinely independent, as $(x_k - x_0)_{k=1}^{m}$ are linearly dependent:

$$\sum_{k=1}^{m} \lambda_k (x_k - x_0) = \sum_{k=1}^{m} \lambda_k x_k - \sum_{k=1}^{m} \lambda_k x_0 = \sum_{k=1}^{m} \lambda_k x_k - x_0 = 0.$$ □

This is useful for programming, because we can encode polytopes as cuts at height 1 of cones.

Definition 6.1.6 (Convex hull)
The convex hull of V is

$$\operatorname{conv}(V) := \left\{ \sum_{j=0}^{k} \lambda_j v_j : \sum_{j=0}^{k} \lambda_j = 1, \lambda_i \geqslant 0 \ \forall i \in \{0, \dots, k\} \right\}.$$

The convex hull of V is a convex polytope.

Definition 6.1.7 (k-simplex)
The convex hull $\operatorname{conv}(V)$ is a k-simplex if V is in general position. k-simplex

In the case that V is in general position we call the points v_j, $j \in \{0, \dots, k\}$ vertices.

Definition 6.1.8 (Relative interior of a simplex)
The relative interior of the simplex $\sigma = \operatorname{conv}(x_1, \dots, x_m)$ is $\operatorname{relint}(\sigma) = \{\sum_{k=1}^{m} \lambda_k x_k : 0 < \lambda_k < 1, \sum_{k=1}^{m} \lambda_k = 1\}$.

Definition 6.1.9 ((proper) face)
Let $\sigma = \operatorname{conv}(V)$ be a k-simplex. If $W \subset V$, then $\tau := \operatorname{conv}(W)$ is a simplex, too, called a face of σ. We write $\tau \leqslant \sigma$. Further, τ proper face of σ if $W \notin \{\varnothing, V\}$.

The number of k-dimensional faces of a n-simplex is $\binom{n}{k+1}$.

Definition 6.1.10 (Geometric simplicial complex)
A finite collection of simplices in $\mathbb{E}^n$ is a geometric simplicial complex if any two simplices from the collection meet in a common face (which may be empty).

A subcomplex is a subset of a simplicial complex, which itself is a simplicial complex. subcomplex

Definition 6.1.11 (Dimension)
The dimension $\dim(K)$ of a geometric simplicial complex K is the maximal dimension of a simplex in K.

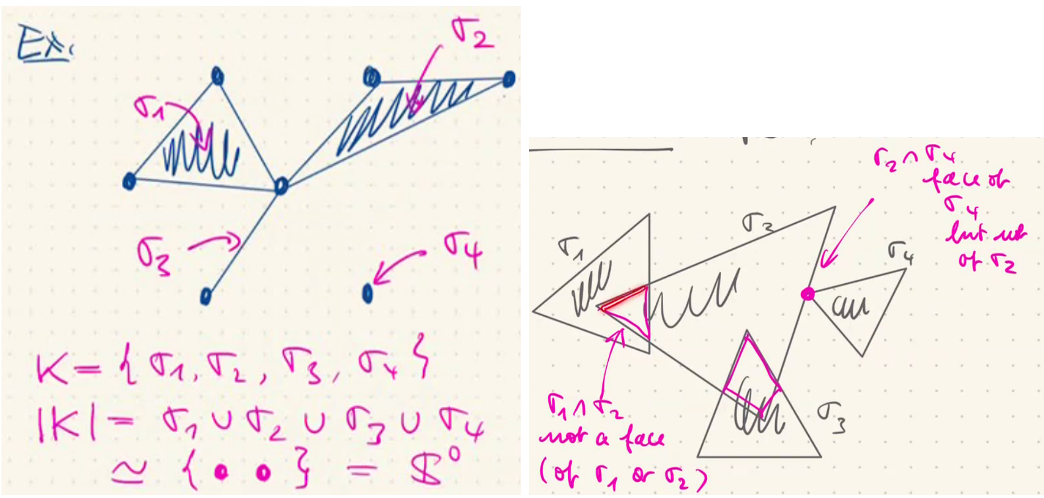

Fig. 61: Example and non-example for a simplicial complex.

The following object is occasionally called a polyhedron.

Definition 6.1.12 (Realisation)
The realisation of a simplicial complex $K := (\sigma_j)_{j=1}^k$ in $\mathbb{E}^n$ is the subspace

$$|K| := \bigcup_{j=1}^{k} \sigma_j \subset \mathbb{E}^n .$$

While a geometric simplicial complex K is a combinatorial object that can be described by finite data, its realisation $|K|$ is a topological space.

Which topological spaces can be described via simplices? A space is triangulable if it is homeomorphic to the realisation of a geometric simplicial complex.

Definition 6.1.13 (Triangulation)
Let X be a topological space. A pair (K, X) is a triangulation of X if K is a simplicial complex and $h\colon |K| \to X$ is a homeomorphism. triangulation

The space X is triangulable if it has a triangulation.

Remark 6.1.14 (Triangulability of surfaces) Every closed surface is triangulable (almost follows from what we did in section 1.1).

Moivre proved in 1952 (proof is highly nontrivial) that every closed 3-manifold is triangulable. Whitney proved in 1957 that every closed differentiable submanifold of $\mathbb{E}^n$ is triangulable. In 2016 is was proven that there exist closed non-triangulable 5-manifolds (and thus also 6, 7, ...-manifolds) [10]. ○

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Example 6.1.15 Let K be a geometric simplicial complex in $\mathbb{E}^n \cong \mathbb{E}^n \times \{0\} \subset \mathbb{E}^{n+1}$. Pick $v \in \mathbb{E}^{n+1} \setminus (\mathbb{E}^n \times \{0\})$. Then for a k-simplex $\sigma \in K$, the vertices of $\sigma \cup \{v\}$ are in general position by construction. Then the join of σ to v,

$$\sigma * v := \operatorname{conv}(\sigma \cup \{v\}), \tag{9}$$

is a $(k+1)$-simplex in $\mathbb{E}^{n+1}$. ◇

Definition 6.1.16 (Cone of simplicial complex)
Let K be a geometric simplicial complex in $\mathbb{E}^n \cong \mathbb{E}^n \times \{0\} \subset \mathbb{E}^{n+1}$. The cone of K with apex $v \in \mathbb{E}^{n+1} \setminus (\mathbb{E}^n \times \{0\})$ is

$$CK := K \cup \big\{\sigma * v : \sigma \in (K \cup \{\varnothing\})\big\}.$$

The CK is a geometric simplicial complex.

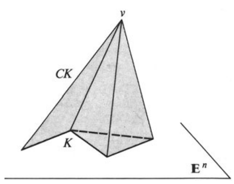

Fig. 62: The cone of a simplicial complex.

The cone of abstract simplicial complex K is the smallest abstract simplicial complex containing $\big\{\sigma \cup \{v\} : \sigma \in K \cup \{\varnothing\}\big\}$.

Lemma 6.1.17
We have $|CK| \approx C|K|$.

Remark 6.1.18 The former is the realisation of a combinatorial objection and the latter is the known Euclidean cone of the realisation of the geometric simplicial complex K. ○

Proof. (HW, TODO) Exercise. □

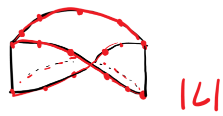

Fig. 63: A simplicial complex L supporting the boundary of a MÖBIUS strip.

Example 6.1.19 (Realising $\mathbb{RP}^2$ in $\mathbb{E}^4$) Let K be a geometric simplicial complex triangulating the MÖBIUS strip M, that is, $|K| \approx M$. The edges of K, which lie in the boundary, form a subcomplex L of K. The union $K \cup CL$ is a geometric simplicial complex in $\mathbb{E}^4$ with $|K \cup CL| \approx \mathbb{RP}^2$.

We can not realise $\mathbb{RP}^2$ in $\mathbb{E}^3$, so this is, in some sense, optimal. (Every non-orientable surface can be embedded in $\mathbb{R}^4$.) ◇

Simplicial maps

We now know the objects in the category of geometric simplicial complexes, but we still need to define the relevant morphisms.

Notation. For a geometric simplicial complex K, we write $\mathrm{Vert}(K)$ for the vertices of K.

DEFINITION 6.1.20 (SIMPLICIAL MAP)
Let K and L be geometric simplicial complexes. A function $\varphi\colon \mathrm{Vert}(K) \to \mathrm{Vert}(L)$ is a simplicial map if for all $\sigma \in K$ we have $\varphi(\sigma) \in L$.

simplicial map

Remark 6.1.21 A simplicial map induces a continuous function $|\varphi|\colon |K| \to |L|$ as each point $p \in |K|$ is contained in some simplex $\sigma \in K$ by definition. Then $\{\sigma \in K : p \in \sigma\}$ is nonempty. Hence

$$\tau := \bigcap_{\substack{\sigma \in K \\ p \in \sigma}} \sigma$$

is the smallest simplex in K containing p.

Let $\tau := \mathrm{conv}(v_0, \ldots, v_k)$ for some $k+1$ vertices $v_0, \ldots, v_k \in \mathrm{Vert}(K)$. The point p lies in the relative interior of τ because if it wouldn't, it would lie in a distinct face of τ, contradicting that τ is the smallest such face. Hence there exist unique positive coefficients $(\lambda_j)_{j=0}^k$ such that

$$v = \sum_{j=0}^{k} \lambda_j v_j.$$

Every point $p \in K$ is either a vertex, lies in a face or in an open edge segment. This is a disjoint decomposition of K into relatively (that is, in the affine hull) open subsets.

Then $|\varphi|(p) := \sum_{j=0}^k \lambda_j \varphi(v_j)$ is well-defined by the uniqueness of the $(\lambda_j)_{j=0}^k$. We have affinely extended φ via barycentric coordinates. ○

Now, $\varphi(\sigma) \in L$ for $\sigma \in K$ makes sense.

DEFINITION 6.1.22 (SIMPLICIAL ISOMORPHISM)
A bijective simplicial map whose inverse is a simplicial map is a simplicial isomorphism.

simplicial isomorphism

If K and L are simplicially isomorphic, then K and L have the same number of simplices of each dimension and these simplices exhibit the same pattern of intersections.

Remark 6.1.23 If $\varphi\colon K \to L$ is a simplicial isomorphism, then $|\varphi|\colon |K| \to |L|$ is a homeomorphism. ○

Remark 6.1.24 The cones of a simplex K with different apices $v \neq w$ are simplicially isomorphic (the isomorphism is the identity on the vertices of K and sends v to w), so the choice of apex doesn't really matter. ○

Counterexample 6.1.25 (Bijective simplicial map φ but φ^{-1} not simplicial map)
One can construct a bijective simplicial map whose inverse is not a simplicial map by taking as L a two-dimensional geometric simplicial complex and letting K be the one-skeleton of L, that is, L without its faces. The simplicial map φ (defined on the vertices!) will be the identity, but the faces from L are mapped nowhere by φ^{-1}. ◇

The next theorem states that $n+1$ terms in a convex combination suffice to describe $\mathbb{R}^n$.

Theorem 6.1.1: Carathédory

For $x \in S \subset \mathbb{R}^n$ there exists $(\lambda_k)_{k=1}^{n+1} \subset \mathbb{R}_{\geqslant 0}$ and $(x_k)_{k=1}^{n+1} \subset$ such that $x = \sum_{k=1}^{n+1} \lambda_k x_k$.

Corollary 6.1.26 (conv(S) compact for S compact)
If $S \subset \mathbb{R}^d$ is compact, then so is $\operatorname{conv}(S)$.

Proof. Exercise. □

Lemma 6.1.27 (Easy to prove facts)
Let K be a geometric simplicial complex in $\mathbb{E}^n$.

- *The set $|K| \subset \mathbb{E}^n$ is compact.*
- *Each point in $|K|$ is contained in the relative interior of a unique simplex, called that points carrier.*
- *Taking the simplices in K separately and giving the identification topology to the union gives exactly $|K|$.*
- *We have the following equivalences: $|K|$ is connected $\iff$ $|K|$ is path-connected $\iff$ $K^{\leqslant 1}$ is a connected graph, where $K^{\leqslant 1}$ is the one-skeleton of K, the graph consisting of the vertices and edges of K.*

We can also define geometric simplicial complexes for discrete (instead of finite) sets V, e.g. a tiling of $\mathbb{E}^2$ with regular triangles. In that case, $|K|$ will only be locally compact.

Proof.
- By corollary 6.1.26, a simplex is a compact set. Hence $|K|$ is closed and bounded and hence compact by Theorem 3.2.1.
- Let $x \in K$ with $x \in \operatorname{relint}(\sigma) \cap \operatorname{relint}(\tau)$. Then $\sigma \cap \tau$ is a face of σ and of τ. Furthermore, $\sigma \cap \tau$ cannot be a proper face of σ, so $\sigma \cap \tau = \sigma$. Hence $\sigma \subset \tau$. Analogously, $\tau \subset \sigma$, so $\tau = \sigma$.
- The simplices of K are closed subsets of $|K|$, since they are closed in $\mathbb{E}^n$. Hence if $C \subset |K|$ and $C \cap A \subset A$ is closed for each simplex $A \in K$, then $C \cap A \subset |K|$ must be closed. Hence the *finite* union $C = \bigcup_{A \in K} C \cap A \subset |K|$ is closed . Hence the closed subsets of $|K|$ are precisely those which intersect each simplex of K in a closed set. In other words, $|K|$ has the identification topology. (???TODO)
- TODO □

The following, combinatorial definition is crucial.

Definition 6.1.28 (Abstract simplicial complex)
Let V be a finite set. A subset $K \subset 2^V$ is an abstract simplicial complex with vertex set V if for all $\sigma \in K$ and all $\tau \subset \sigma$ we have $\tau \in K$.

abstract simplicial complex

Remark 6.1.29 Assume that $\#V = n$ and without loss of generality that $V = [n] := \{1, \dots, n\}$. Let K be an abstract simplicial complex with vertex set V ("on V"). Then

$$L := \{\operatorname{conv}(\{e_i : i \in \sigma\}) : \sigma \in K\} \subset \mathbb{E}^n = \operatorname{span}(e_1, \dots, e_n)$$

is a geometric simplicial complex (as K is an abstract simplicial complex, each face of L is contained in L and the faces can't intersect in a bad way because each subset of $(e_i)_{i=1}^n$ is an ONB). ○

There's another way to realise an abstract simplicial complex over $[n]$.

Remark 6.1.30 Let S be an abstract simplicial complex over $[n]$. It can be realised in two ways by glueing simplices from S, which is the correct way for category theory people, because it is a colimit construction.

For an finite set M define

$$\Delta^M := \left\{ x \in \mathbb{R}^M : \sum_{i \in M} x_i = 1, x_i \geqslant 0 \right\}.$$

Then $\Delta^{\{0,\dots,n\}}$ is the standard n-simplex.

We document the relations $\tau \subset \sigma$ of S in the following way (this is a vast generalisation of HA5.6). We build a directed system via inclusions

$$i_\tau^\sigma \colon \Delta^\tau \to \Delta^\sigma, \qquad x \mapsto (x_k \mathbb{1}_{k \in \tau})_{k \in \sigma}.$$

Then define

$$|S| := \left(\bigsqcup_{\sigma \in S} \Delta^\sigma \right) /\!\sim$$

where $(\Delta^\sigma \ni x) \sim (x' \in \Delta^{\sigma'})$ if there exists a $y \in \Delta^{\sigma \cap \sigma'}$ such that $i^\sigma_{\sigma \cap \sigma'}(y) = x$ and $i^{\sigma'}_{\sigma \cap \sigma'}(y) = x'$. Now set

$$\chi_\sigma^S \colon \Delta^\sigma \overset{i_\sigma}{\hookrightarrow} \bigsqcup_{\sigma \in S} \Delta^\sigma \xrightarrow{q} |S|$$

where i_σ is the canonical inclusion into the coproduct and q is the quotient map. The coproduct $\bigsqcup_{\sigma \in S} \Delta^\sigma$ has the final topology with respect to the $(i_\sigma)_\sigma$ and $|S|$ carries the final topology with respect to the $(\chi_\sigma^S)_{\sigma \in S}$.

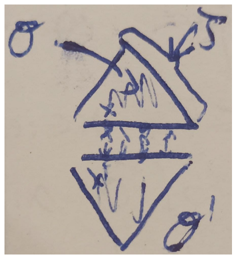

Fig. 64: The glueing of two edges of two simplices together.

A directed system is good because every map from it to an arbitrary topological space X factors through the realisation $|S|$. ○

Lemma 6.1.31
Let S and T be abstract simplicial complexes with $T \subset S$. We say that T is a subcomplex of S. Then there exists a closed map j as in figure 65.

$$\begin{tikzcd}
 & {|T|} \arrow[dd, dotted, "\exists j"] \\
\Delta^\sigma \arrow[ur, "\chi_\sigma^T"] \arrow[dr, "\chi_\sigma^S"'] & \\
 & {|S|}
\end{tikzcd}$$

Fig. 65

Proof. As χ_σ^S respects $\sim_T$, j exists. We have $\chi_\sigma^T(x) = \chi_\sigma^T(x')$ if and only if there exists a $y \in \Delta^{\sigma \cap \sigma'}$ such that $i^\sigma_{\sigma \cap \sigma'}(y) = x$ and $i^{\sigma'}_{\sigma \cap \sigma'}(y) = x'$ if and only if $\chi_\sigma^S(x) = \chi_\sigma^S(x')$. (this is not precise / correct?)

Hence j is a map, which is induced by the quotient and injective. We have to show that for a closed set $A \subset |T|$, the set $j(A) \subset |S|$ is closed, that is, that $(\chi_\sigma^S)^{-1}(j(A)) \subset \Delta^\sigma$ is closed for all $\sigma \in S$ (final topology).

We have ("intersect A with all $\tau \subset T$")

$$(\chi_\sigma^S)^{-1}(j(A)) = \bigcup_{\substack{\tau \in \sigma \\ \tau \in T}} i_\tau^\sigma\left((\chi_\tau^T)^{-1}(A)\right).$$

As A is closed and χ^T_τ is a map, $(\chi^T_\tau)^{-1}(A)$ is closed. Furthermore, it is bounded, so it is compact. As i^σ_τ is continuous, $i^\sigma_\tau\left((\chi^T_\tau)^{-1}(A)\right)$ is compact by Theorem 3.3.1. Finally, $(\chi^S_\sigma)^{-1}(j(A))$ is compact as the finite union of compact sets. By Theorem 3.3.6, it is thus closed. $\square$

Still todo: What is a path between two vertices in an abstract simplicial complex S? Show that this induces an equivalence relation $\sim$. Let C be an equivalence class and $\sigma \in S$ any simplex. Show that $\sigma \cap C = \varnothing$ or $\sigma \subset C$. Describe the connected components of $|S|$.

Forgetting the coordinates of the vertices turns a geometric simplicial complex into an abstract simplicial complex (this will be made precise later).

Example 6.1.32 Consider the following abstract drawing (which is not a drawing in $\mathbb{E}^2$!).

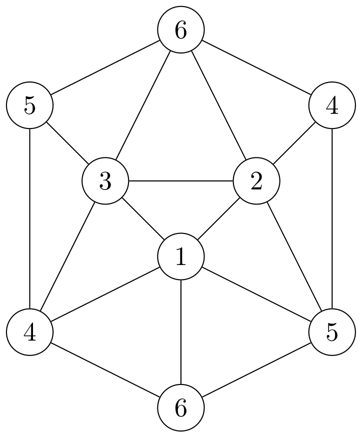

Fig. 66: A six vertex triangulation of $\mathbb{R}\mathrm{P}^2$.

Let $V := [6]$ and

$$K := \overline{\{356, 236, 246, 345, 134, 123, 156, 125, 245, 146\}},$$

where $\overline{\cdot}$ denotes "take all subsets". (These vertex combinations each form a face of the above two-dimensional geometric simplicial complex.)

We have $|K| \approx \mathbb{R}\mathrm{P}^2 \approx \mathbb{D}^2 / \sim_a$, because the hexagonal polygon 645645 is homeomorphic to a disk and the labelling of the vertices suggests antipodal identification on the boundary of that disk, which precisely is $\sim_a$.

A platonic solid called the icosahedron has 20 faces and 12 vertices and is symmetric. The above graph shows half of an icosahedron (10 faces, 6 edges) with points identified in such a way that ... $\diamond$

6.2 Barycentric subdivision

05.01.2022

We introduce a specific construction to create one simplicial complex from another one.

Definition 6.2.1 (Barycentric coordinates, barycentre)
Let $\tau := \operatorname{conv}(v_0, \dots, v_k) \subset \mathbb{R}^n$ be a k-simplex. Then each point $x \in \tau$ can be *uniquely* written as $x = \sum_{j=0}^k \lambda_j v_j$, where $\lambda_j \geqslant 0$ for all $j \in \{0, \dots, k\}$ and $\sum_{j=0}^k \lambda_j = 1$. The coefficients $(\lambda_j)_{j=0}^k$ are the barycentric coordinates of x.

The barycentre of τ is the point barycentre

$$\beta_\tau := \frac{1}{k+1}\sum_{j=0}^{k} v_j \in \tau.$$

Remark 6.2.2 (Concerning the uniqueness)
Suppose $x \in \operatorname{conv}(v_0,\dots,v_k)$ can be written as $x = \sum_{j=0}^k \lambda_j v_j = \sum_{j=0}^k \mu_j v_j$ with $\lambda_j, \mu_j \geqslant 0$ and $\sum_{j=0} k\lambda_j = \sum_{j=0}^k \mu_j = 1$. Then

$$0 = \sum_{j=0}^k (\lambda_j - \mu_j)v_j = \sum_{j=1}^k (\lambda_j - \mu_j)v_j + (\lambda_0 - \mu_0)v_0 = \sum_{j=1}^k (\lambda_j - \mu_j)(v_j - v_0),$$

so $\lambda_j = \mu_j$, as $(v_j - v_0)_{j=1}^k$ are linearly independent. ○

Definition 6.2.3 (First barycentric subdivision)
Let K be a geometric simplicial complex. The (first) barycentric subdivision K^1 of K is the geometric simplicial complex with $\operatorname{Vert}(K^1) = \{\beta_\sigma : \sigma \in K\backslash\{\varnothing\}\}$ such that $\operatorname{conv}(\beta_{\sigma_0},\dots,\beta_{\sigma_k})$ is a face if and only if there exists a $\varphi \in \operatorname{Sym}(\{0,1,\dots,k\})$ such that barycentric subdivision

$$\sigma_{\varphi(0)} < \sigma_{\varphi(1)} < \ldots < \sigma_{\varphi(k)}.$$

As the barycentre of a zero-dimensional face (a vertex) of K is the vertix itself, we have $\operatorname{Vert}(K) \subset \operatorname{Vert}(K^1)$.

In barycentric subdivision, each triangle gets divided into 6 smaller triangles and every edge gets divided into two equally long edges of half the length of the original edge. Generally, each k-face gets divided into $(k+1)!$ smaller k-faces.

Lemma 6.2.4
The barycentric subdivision of a geometric simplicial complex is a geometric simplicial complex.

Proof. We prove this by induction of the number of faces of K.

If the number of faces is equal to one, then K consists a point and thus so does K^1.

Let the number of faces of K be equal to n and $A \in K$ a cell of maximal dimension. (Consider a triangle with an "extra arm". Then there a two maximal cells (the triangle and the extra arm) of different dimension.) Let $L := K\backslash\{A\}$. Then $|L| = |K|\backslash \operatorname{relint}(A)$. By the induction hypothesis, L^1 is a geometric simplicial complex.

We now have to check that the new cells add "nicely" to L^1, that is,

1. Any simplex $\sigma \in K^1\backslash L^1$ intersect every simplex $\gamma \in L^1$ nicely, that is, such that $\sigma \cap \gamma$ is a face of both σ and γ.
2. For $\sigma, \sigma' \in K^1\backslash L^1$ we have that $\sigma \cap \sigma'$ is a face of both σ and σ'.

Let $\sigma = \operatorname{conv}(\beta_{\sigma_0},\dots,\beta_{\sigma_{k-1}},\beta_A) \in K^1\backslash L^1$ and $\tau := \operatorname{conv}(\beta_{\sigma_0},\dots,\beta_{\sigma_{k-1}}) \in L^1 \cap \sigma$.

1. For $\gamma \in L^1$ we have
$$\sigma \cap \gamma = \sigma \cap |L^1| \cap \gamma = \tau \cap \gamma.$$
Hence $\sigma \cap \gamma$ is a face of τ and hence also of σ. As L^1 is a geometric simplicial complex, $\sigma \cap \gamma$ is also a face of γ.

(2) Similar: if $\sigma' \in K^1 \backslash L^1$, then we can construct $\tau' = \sigma' \cap |L^1| \in |L^1$ such that $\tau \cap \tau'$ and $\sigma \cap \sigma'$ are nice. TODO!! □

Definition 6.2.5 (Barycentric subdivision of abstract SC)
The (first) barycentric subdivision of an abstract simplicial complex K has as vertices the faces of K, that is, $K \backslash \{\varnothing\}$, and as faces the flags of K.

Example 6.2.6 Consider the abstract simplicial complex

$$K := \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}.$$

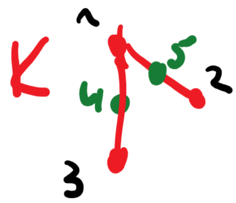

Fig. 67: The geometric realisation of the abstract simplicial complex K and its first barycentric subdivision.

Forming the first barycentric subdivision of the realisation of K as a geometric simplicial complex would add new vertices 4 and 5 and replace the edges $\{1,2\}$ and $\{1,3\}$ with $\{1,4\}$ and $\{4,2\}$ and $\{1,5\}$ and $\{5,3\}$, respectively.

The first barycentric subdivision of K is

$$K^{(1)} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{\{3\}, \{1,3\}\}, \{\{1\}, \{1,3\}\}, \{\{1\}, \{1,2\}\}, \{\{2\}, \{1,2\}\}\}$$

with the correspondence $5 \rightsquigarrow \{1,2\}$ and $4 \rightsquigarrow \{1,3\}$ and the added flags represent the new edges. ◇

Definition 6.2.7 (Mesh)
The mesh (size) of a complex K is

$$\mu(K) := \max(\{\operatorname{diam}(\sigma) : \sigma \in K\}),$$

where $\operatorname{diam}(\sigma) := \max(\{\|x - y\| : x, y \in \sigma\})$.

Lemma 6.2.8

(1) *Each simplex of K^1 is contained in a simplex of K.*

(2) *We have $|K^1| = |K|$.*

(3) *If $\dim(K) = n$, then $\mu(K^1) \leqslant \frac{n}{n+1}\mu(K)$.*

Proof. (1)

(2)

(3) *Sketch.* Let us consider $d = 2$. As the notions are invariant under affine transformations, we can consider a regular triangle with diameter (=length of the sides) 1. Intersecting a 3-simplex with a plane yields again a triangle, so the higher-dimensional case can be deduced from the $d = 2$. □

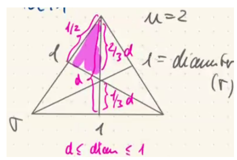

Fig. 68

Definition 6.2.9 (m-th barycentric subdivision)
The m-th barycentric subdivision is

barycentric subdivision

$$K^m := (K^{m-1})^1.$$

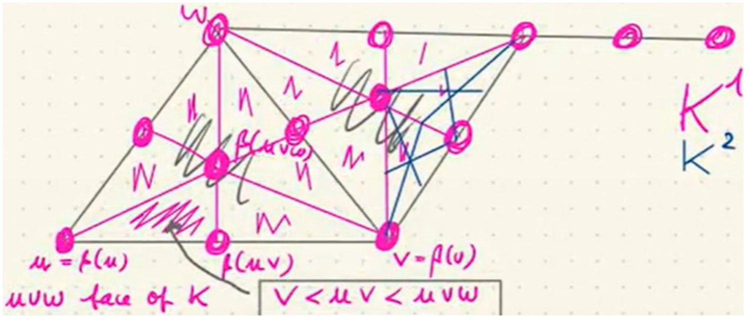

Fig. 69: The first two barycentric subdivisions of a two-dimensional geometric simplicial complex.

6.3 Simplicial approximation

We have geometric simplicial complexes as models of topological spaces and simplicial maps as combinatorial models of continuous maps. How well do they describe continuous functions?

Let K and L be geometric simplicial complexes and $f\colon |K| \to |L|$ be a map. How does the map f relate to the combinatorial structure given by the complexes?

Definition 6.3.1 (Simplicial approximation)
A simplicial map $s\colon K \to L$ is a simplicial approximation of the continuous function f if $s(x)$ lies in the carrier of $f(x)$ for each $x \in |K|$.

simplicial approximation

Being a simplicial approximation means that the combinatorial structure imposed by the geometric simplicial complex on both K and L is fine enough (i.e. the mesh is small enough): no matter how one maps the point x to $f(x)$, there is a simplicial map s, such that $s(x)$ is close to $f(x)$.

If $x_1, \ldots, x_m$ are in general position, then every point $y \in \operatorname{conv}(x_1, \ldots, x_m)$ (and analogously every $y \in \operatorname{aff}(x_1, \ldots, x_m)$), has a unique representation as $y = \sum_{k=1}^{m} \lambda_k x_k$. This implies that simplicial maps only need to be defined on the vertices of the geometric simplicial complexes. A simplicial map $f\colon S \to T$ induces a unique piecewise affine map : $|S| \to |T|$ with "regions of linearity" being the maximal simplices.

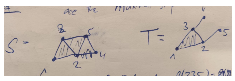

Fig. 70: The two simplicial complexes S and T.

A simplicial map would be $f(i) = 1$ for all $i \in [5]$ or $f(1) = f(5) = 1$, $f(2) = 2$, $f(3) = f(4) = 3$. The map $f(i) = i$ for all $i \in [5]$ is *not* a simplicial map because $f(235)$ is not a face of T.

Lemma 6.3.2
If s is a simplicial approximation of $f\colon |K| \to |L| \subset \mathbb{E}^n$, then $|s| \simeq f$.

Proof. Define

$$F\colon |K| \times I \to \mathbb{E}^n, \qquad (x,t) \mapsto (1-t)|s|(x) + tf(x).$$

For $x \in |K|$, there exists a face $\sigma \in L$ such that $|s|(x), f(x) \in \sigma$. Since σ is convex, the straight line homotopy F stays inside σ.

Hence the image of the F is contained in $|L|$ and $|s| \simeq_F f$. □

–

Let $f\colon |K| \to |L|$ be any map, which we want to approximate by a simplicial map $s\colon \operatorname{Vert}(K) \to \operatorname{Vert}(L)$. The unique piecewise affinely linear continuation $\tilde{s}$ is a simplicial approximation of f if for all $x \in |K|$ we have that $s(x)$ lies in the carrier of $f(x)$, which is the unique simplex σ of L with $f(x) \in \operatorname{relint}(\sigma)$.

Beware: this doesn't mean that for every $x \in |K|$ the points $s(x)$ and $f(x)$ have the same carrier! TODO

Example 6.3.3 Consider $f\colon |K| \to |L|$, $x \mapsto x^2$, where K is consists of the two edges connecting the vertices 0 and $\frac{1}{3}$ and 1 in that order and $|L|$ is the same for 0, $\frac{2}{3}$ and 1. This map has no simplicial approximation $s\colon \operatorname{Vert}(K) \to \operatorname{Vert}(L)$: we need to have $s(0) = 0$ and $s(1) = 1$. If $s(\frac{1}{3}) = 0$, then $\tilde{s} = \operatorname{id}$ and s is not a simplicial map. We must have $s(\frac{1}{3}) = \frac{2}{3}$. Then $s\colon \{0, \frac{1}{3}, 1\} \to \{0, \frac{2}{3}, 1\}$ is not a simplicial approximation f as $f(\frac{1}{2}) = \frac{1}{4} \in [0, \frac{2}{3}]$ but $\frac{1}{2} \in [\frac{1}{2}, 1]$. ◇

–

The following theorem states that we can replace continuous maps between realisations of geometric simplicial complexes by simplicial maps.

Theorem 6.3.1: Simplicial approximation

Let $f\colon |K| \to |L|$ be a map. Then there exists a $m \in \mathbb{N}$ such that there exists a simplicial approximation $s\colon K^m \to L$ to f.

The proof requires a definition and a lemma.

Definition 6.3.4 (Open star)
Let K be a complex and $v \in \operatorname{Vert}(K)$. The open star of v in K, $\operatorname{star}(v, K)$ is the union of interiors of those simplices of K which have v as a vertex.

The open star is an open subset of $|K|$.

Lemma 6.3.5
Vertices $v_0, \ldots, v_k \in \operatorname{Vert}(K)$ of a simplicial complex K are the vertices of a simplex A of K if and only if the intersection of their open stars is nonempty.

Proof. If $v_0, \ldots, v_k$ are the vertices of a simplex A of K, then $\operatorname{int}(A) \subset \operatorname{star}(v_i, K)$ for all $i \in \{0, \ldots, k\}$.

If $x \in \bigcup_{i=0}^{k} \operatorname{star}(v_i, K)$ and A is the carrier of x, then each v_i must be a vertex of A, so $v_0, \ldots, v_k$ are vertices of a face of A. □

Proof. (of Theorem 6.3.1) TODO!!!! □

6.4 The edge group of a simplicial complex

The edge group will give us a way to compute the fundamental groups of topological spaces, and in particular, simplicial complexes.

Let K be a geometric simplicial complex and $V := \operatorname{Vert}(K)$.

Definition 6.4.1 (Edge path, edge loop)
An edge path in K is a sequence $(v_0, \ldots, v_k)$ in V such that the edge (or point: we allow $v_{i-1} = v_i$) $v_{i-1}v_i := \operatorname{conv}(\{v_{i-1}, v_i\})$ lies in K for all $i \in [k]$.

edge path

If $v_0 = v_k$, then this sequence is an edge loop based at v_0.

The edge paths in a geometric simplicial complex are paths in the 1-skeleton, where it is allowed to repeat vertices. Further, edge loops are closed paths in the 1-skeleton.

Definition 6.4.2 (Equivalent edge paths)
Two edge paths are equivalent if they can be transformed into another by finitely many operations of the following kind:

- $(u, v, w) \leftrightarrow (u, w)$ if $\operatorname{conv}(u, v, w) \in K$ (taking a shortcut).
- $(u, u) \leftrightarrow u$.

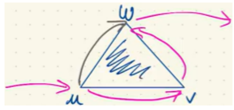

Fig. 71: Taking a shortcut.

Definition 6.4.3 (Edge group)
The edge group $E(K, v)$ of K based at $v \in V$ consists of equivalence classes of edge loops at v with respect to the multiplication

edge group

$$[v_0, \ldots, v_k][v_k, v_{k+1}, \ldots, v_m] := [v_0, \ldots, v_m],$$

where $v_0 = v_k = v_m = v$.

Theorem 6.4.1: $E(K, v) \cong \pi_1(|K|, v)$

The edge group $E(K, v)$ is isomorphic to $\pi_1(|K|, v)$.

Note that $E(K, v)$ has a *finite* description and thus fundamental groups obtained this way will always be finitely generated.

Proof. Each edge loop in K gives rise to a loop in $|K|$. Equivalent edge loops yield homotopic loops in $|K|$. This way we obtain a homomorphism of groups

$$\Phi\colon E(K, v) \to \pi_1(|K|, v).$$

First we want to show that Φ is onto. Let $\alpha\colon I \to |K|$ be a loop based at v. Consider the realisation $I = |L|$ for the simplicial complex $L := [0, 1] \subset \mathbb{E}^n$ (with the faces being the emptyset and the vertices). Now apply simplicial approximation to get a simplicial map $s\colon L^m \to K$ such that $|s|\colon |L^m| \to |K|$ is homotopic to α. Then $\operatorname{Vert}(L^m) = \{\frac{i}{2^m} : i \in \{0, \ldots, 2^m\}\}$ and $v_i := s\left(\frac{i}{2^m}\right)$. Then $v_0 = v_{2^m} = v$. Hence $\varphi([v_0, v_1, \ldots, v_{2^m}]) = \langle\, |s| \,\rangle = \langle\, \alpha \,\rangle$.

We now show that Φ is injective, that is, that the fibre of the neutral element of $\pi_1(|K|, v)$ only consists of the neutral element of $E(K, v)$. This means that every contractible (or: 0-homotopic) loop at v can be contracted (by applying the two operations introduced above) in $K^{\leq 2}$.

As α is null-homotopic to the constant path at v, there exists a homotopy $F\colon I \times I \to |K|$ which satisfies $F(\cdot, 0) = \alpha$ and $F(0, \cdot) = F(1, \cdot) = F(\cdot, 1) = v$. Define $a_k \in I \times \{0\}$ such that $F(a_j) = v_j$ for $j \in \{0, \ldots, k-1\}$. Let $c_0 = a_0$, $c_1 = (0, 1)$, $c_2 = (1, 1)$ and $c_3 = (1, 0)$ be the corners of $I \times I$. Let S denote the geometric simplicial complex in figure 72 with all faces filled in.

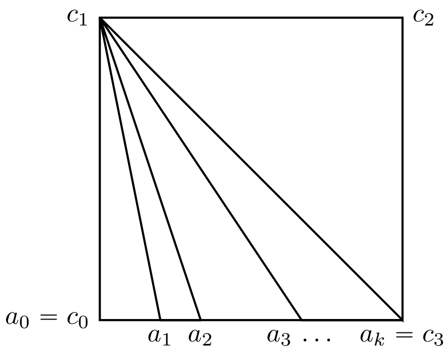

Fig. 72: A simplicial complex in $I \times I$.

Then the edge paths $a_0 a_1 \dots a_{k-1} c_3$ and $c_0 c_1 c_2 c_3$ are equivalent: let $\sim$ denote equivalence. Then because $\operatorname{conv}(a_j, a_{j+1}, c_1)$ is a face, we have

$$a_0 a_1 a_2 \dots a_k \sim a_0 c_1 a_2 \dots a_k \sim \dots \sim a_0 c_1 a_k \sim a_0 c_1 c_2 c_3 = c_0 c_1 c_2 c_3.$$

For any $m \in \mathcal{N}$, consider the barycentric subdivision $S^{(m)}$ of S. The two equivalent paths on the edges of $I \times I$ from above thus have $2^m - 1$ additional vertices inserted between each pair of consecutive vertices. Lets call those paths p_1 and p_2. Then p_1 and p_2 are equivalent, too.

By the simplicial approximation theorem, there exists a barycentric subdivision $S^{(m)}$ and a simplicial approximation $s\colon |S^{(m)}| \to |K|$ of the map F. Applying the simplicial map to two equivalent edge paths preserves their equivalence. Thus s applied to p_2 gives the vertex v repeated, which is equivalent to the constant edge loop at p. Lastly, applying s to p_1 gives an edge loop equivalent to $v_0 \dots v_k$. □

Partially ordered sets

Definition 6.4.4 (Partially ordered set)
A partially ordered set (poset) $(P, \leqslant)$ is a set P with a reflexive, antisymmteric and transitive relation.

partially ordered set

There is a bijection between posets and acyclic digraphs via the HASSE-diagram, a non-redundant representation of posets, where transitivity is implicit. The nodes of a HASSE-diagram are the elements of P and the directed arcs represent the covering relations in the sense that a directed edge from $x \in P$ to $y \in P$ means that $x \leqslant y$ and for all $z \in P$ with $x \leqslant z \leqslant y$ we have $z \in \{x, y\}$.

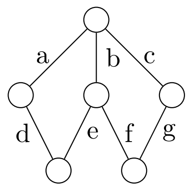

Fig. 73: A HASSE diagram of a poset $\{1, \dots, 6\}$.

From the HASSE diagram we can form an abstract simplicial complex $O(P)$ - order complex of P - the whose vertices are the edges of $O(P)$ and whose faces are the flags in P. For the above HASSE diagram, we would have $K = \overline{\{ad, be, bg, eg\}}$.

Conversely, if K is an abstract simplicial complex with vertex set V, then the face poset of K is $P(K) :=$ (faces of $K, \subset$). The order complex of the HASSE diagram in figure 74 has vertices $\varnothing$, 1, 2, 3, 12, 23, 13 and 123 and its faces are the flags of faces in K, e.g. $\{\varnothing, 1, 13, 123\}$ or $\{2, 12, 123\}$.

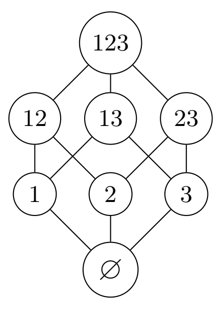

Fig. 74: The HASSE di-

Hence from K we can form $P(K)$ and from $P(K)$ we can form $O(P(K))$. Is $K = O(P(K))$? No, but $K^1 \cong O(P(K))$, where $\cong$ denotes a simplicial isomorphism. Here, barycentric subdivision arises naturally.

10.01.2022

Finitely presented groups

Finitely presented group can be written as

$$\langle g_1, \ldots, g_k \mid r_1 = \ldots = r_\ell = 1 \rangle := \mathrm{Free}(g_1, \ldots, g_k)/(\text{normal subgrp generated by } (r_k)_{k=1}^{\ell}),$$

where g_i are called generators and r_m are relators, which are reduced words in g_i and g_j^{-1}. We only quotient out a *normal* subgroup because arbitrary quotients of groups need not be groups.

Example 6.4.5 (Free groups)
For example,

$$\mathrm{Free}(a, b, c) = \{e, a, b, c, a^{-1}, b^{-1}, c^{-1}, a^2, ab, ac, ab^{-1}, \ldots\}$$

is an infinite non-abelian group.

We have $\langle g \mid g^n = 1 \rangle \cong \mathbb{Z}/n\mathbb{Z}$, which is the cyclic group of order n and is abelian. ◇

Example 6.4.6 (Free group representation of Sym(3))
Consider

$$G := \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^2 = 1 \rangle = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\},$$

where the equality is due to that in the quotient we have $s_1 s_2 s_1 = s_2 s_1 s_2$: we have

$$s_1 s_2 s_1 = s_2 s_1 s_2 \iff s_2^{-1} s_1 s_2 s_1 s_2^{-1} s_1^{-1} = e$$

and

$$s_2^{-1} s_1 s_2 s_1 s_2^{-1} s_1^{-1} \overset{(s_1 s_2)^3 = 1}{=} s_2^{-1} s_1 s_2 \cancel{s_1 s_1} s_2 s_1 s_2 = s_2^{-1} s_1 \cancel{s_2 s_2} s_1 s_2 = s_2^{-1} \cancel{s_1 s_1} s_2 = s_2^{-1} s_2 = e.$$

Thus $(s_1 s_2)^2 = s_2 s_1$ and $(s_2 s_1)^2 = s_1 s_2$. Hence $|G| = 6$. We want to show that $G \cong \mathrm{Sym}(3)$. Let $s_1 \mapsto (1\ 2)$, $s_2 \mapsto (2\ 3)$, then $s_1 s_2 \mapsto (1\ 3\ 2)$ and $(s_1 s_2)^2, (s_1 s_2)^3 \mapsto (1\ 2\ 3)$. In particular, we have $s_2 s_1 s_2$, $s_1 s_2 s_1 \mapsto (1\ 3)$. ◇

We now know how to find the fundamental group of any triangulable space. But to analyse $E(K, v)$ we still have to go through all possible edge paths in $K^{\leqslant 1}$. This can be improved in the following way.

Example 6.4.7 ($G(K, L)$)
Assume that $|K|$ is connected (which is equivalent to $K^{\leqslant 1}$ being a connected graph) and thus path-connected by lemma 6.1.27. This suffices because if $|K|$ is not connected, we can apply the following procedure to every connected component.

Let L be any subcomplex of K containing all vertices of K, that is, $\mathrm{Vert}(L) = \mathrm{Vert}(K) = \{v_1, \ldots, v_n\}$. Further assume that L is simply connected, that is, all cycles can be contracted.

For example L could be a spanning tree of $K^{\leqslant 1}$, which is acyclic and thus simply connected. We define a new group in terms of generators and relations: we have the generators g_{ij}

for whenever the edge $\mathrm{conv}(v_i, v_j)$ between v_i and v_j lies in K and the relations $g_{i,j} = 1$ if $\mathrm{conv}(v_i, v_j) \in L$ and $g_{i,j}g_{j,k} = g_{i,k}$ if $\mathrm{conv}(v_i, v_j, v_k) \in K$ (recall Definition 6.4.2). Hence

$$G(K, L) := \left\langle (g_{ij})_{i,j} \mid (g_{\tilde{i},\tilde{j}})_{\tilde{i},\tilde{j}} \right\rangle$$

is a finitely presented group, as it can be described by a finite list of generators and relations. ◇

Theorem 6.4.2

We have $G(K, L) \cong E(K, v)$ for $v \in \mathrm{Vert}(K)$.

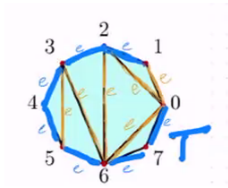

Fig. 75: TODO: the fundamental group of the octagon is trivial.

Proof. (Sketch) Let E_i be a path from v to the vertex v_i of K. Then $\Phi(g_{ij}) := E_i v_i v_j E_j^{-1}$ is a map $G(K, L) \to E(K, v)$. Further, $\Theta(v v_k v_\ell v_m \ldots v_n v) := g_{0k} g_{k\ell} g_{\ell m} \ldots g_{n0}$ is a map $E(K, v) \to G(K, L)$. □

Example 6.4.8 (Calculating $G(K, L)$)
In figure 76 is not a triangulation of the Möbius strip, as $q(\sigma_1) \cap q(\sigma_3) = q(\{1, 3\})$, which is no edge of $q(\sigma)$, where q is the quotient map. In figure 77 we can see a triangulation of Möbius strip. Hence we get $ab = bc = e$, which yields $b = a^{-1} = c^{-1}$ and thus $a = b^{-1} = c$. Hence $\pi_1(M) = \langle a, b, c \mid ab = bc = e \rangle = \langle a \rangle \cong (\mathbb{Z}, +)$. ◇

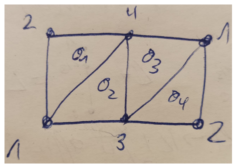

Fig. 76: Not a triangulation of the Möbius strip.

The fundamental group of the torus is

$$\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z} \neq \pi_1(\mathbb{S}^2) = \pi_1(\partial\Delta_3) = \pi_1(\Delta_3) = \{e\}$$

as Δ_3 is convex and the fundamental group is only interested in the 2-skeleton of the complex. In this case $\partial\Delta_3$ is the 2-skeleton of the 3-simplex Δ_3. We have $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle = \langle a, b \mid aba^{-1}b^{-1} = e \rangle = \mathrm{Ab}(\langle a, b \rangle)$, where Ab denotes the abelianisation of a group.

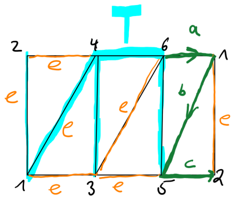

Fig. 77: A triangulation of the Möbius strip.

The construction of $G(K, L)$ as been described as a quotient: first we constructed the (infinite) free group with the generators $g_{i,j}$ and then "quotienting out" the spanning tree by introducing the relations, which we can also write as relators $g_{i,j} g_{j,k} g_{i,k}^{-1} = 1$. So $G(K, L)$ is the (free group on generators) / (normal subgroup generated by relators).

This gives a description of $\pi_1(|K|, v)$ as finitely presented group, But it is undecidable to check if an arbitrary finitely presented group is trivial. Hence it is undecidable to check if a simplicial complex is simply connected.

There is a semi-decision procedure, the Todd-Coxeter algorithm.

Example 6.4.9 Consider the one-dimensional geometric simplicial complex K in figure 78. Then $G(K, L)$ is the (non-abelian!) free group generated by g_{23}, g_{45} and $g_{6,7}$. We write $G(K, L) = \langle g_{23}, g_{45}, g_{6,7} \rangle$. ◇

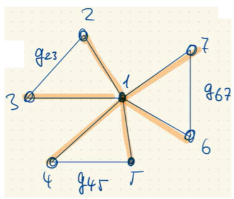

Fig. 78: A one-dimensional geometric simplicial complex K.

A free group is abelian if and only if it has exactly one generator and then it is isomorphic to $(\mathbb{Z}, +)$.

Example 6.4.10 ($\pi_1(\mathbb{R}\mathrm{P}^2)$ via edge group)
We know a triangulation of the projection plane $\mathbb{R}\mathrm{P}^2$. Using the ordering $a < b < c < d < e < f$ on the vertices and picking a spanning tree as in figure 79, we get that

$$\pi_1(\mathbb{R}\mathrm{P}^2) = \langle g \mid g^2 \rangle = \mathrm{Free}(g)/\langle g^2 = e \rangle \cong \mathbb{Z}/2\,\mathbb{Z}\,.$$

◇

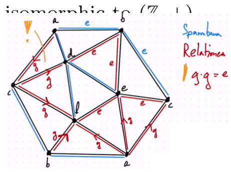

Fig. 79: Spanning tree L of a triangulation K of the projective plane $\mathbb{R}\mathrm{P}^2$ and the generators of $G(K, L)$.

Remark 6.4.11 (Important) Every finitely presented group occurs as the fundamental group of a 2-dimensional simplicial complex. ○

Hence whenever we have a finitely presented group, we can construct a 2-dimensional simplicial complex such that the given group occurs as the fundamental group of that complex. This means that the topology that one obtains by passing to simplicial complexes "knows everything about the group". In a way, all kinds of intricacies that exist for finitely presented groups are inherited by the topology of simplicial complexes.

Remark 6.4.12 (Undecidability result) This is a reason why these things are - in a certain algorithmic way - rather complicated: a basic question is if a (path connected) topological space is simply connected. But it is undecidable if finite simplicial complexes are simply connected. ○

7 Classification of surfaces

We intend to prove Theorem 1.5.1, which, among other things asserts, that $\mathbb{S}^2$ and $\mathbb{S}^2$ with finitely many handles attached comprise all orientable surfaces, while $\mathbb{S}^2$ with finitely many cross-caps comprise all non-orientable surfaces.

We will see that orientable surfaces (defined by the above theorem as $\mathbb{S}^2$ with m handles attached, where $m \in \mathbb{N}_{\geqslant 0}$) can be embedded in $\mathbb{E}^3$, while *closed* surfaces containing a MÖBIUS strip cannot be embedded in $\mathbb{E}^3$.

The MÖBIUS strip has a boundary, so it is not closed.

If we mix cross-caps and handles, we always end up with a non-orientable surface.

THEOREM 7.0.1: MIXING CROSS-CAPS AND HANDLES

Modifying the sphere $\mathbb{S}^2$ by adding m handles and $n > 0$ disjoint cross-caps is homeomorphic to $\mathbb{S}^2$ with $2m + n$ disjoint cross-caps.

Proof. Later. □

The proof of the following pretty recent result is highly non-trivial.

THEOREM 7.0.2: PERELMAN, 2002 (POINCARE CONJECTURE)

Every simply connected, closed 3-manifold is homeomorphic to $\mathbb{S}^3$.

From remark 6.1.14 we know that every 3-manifold is triangulable. The above decidable-result doesn't help, but RUBINSTEIN and THOMPSON found an algorithm for recognising $\mathbb{S}^3$ [11] [12]. Hence not every finitely presented group can occur as the fundamental group of a closed connected 3-manifold. This is in contrast to 2-manifolds (see theorem before). But every finitely presented group occurs as the fundamental group of a 5-manifold.

7.1 Triangulation and orientation

THEOREM 7.1.1: RADO, 1923

Every closed surface is triangulable.

This is significant because when talking about closed surfaces, we can use everything we learned about simplicial complexes.

Lemma 7.1.1 (Properties of the triangulation of a closed surface)
Let $S \approx |K|$ be a connected closed surface with triangulation K. Then

- $\dim(K) = 2$,
- *K is pure, that is, each facet (maximal face with respect to inclusion) is 2-dimensional,*
- *each edge of K is contained in exactly two triangles,*
- *any two vertices of K can be joined by an edge-path,*
- *each vertex v is contained in at least three triangles, which together form a cone with apex v.*

A complex with these five properties will be called a combinatorial surface.

Example 7.1.2 Consider the simplicial complex for $\mathbb{RP}^2$ with $v = 3$. Then the circle 12465 is a homeomorphic description of a cone together with apex 3. $\diamond$

Definition 7.1.3 (Orientation of a simplex)
An orientation of a simplex $\sigma := \operatorname{conv}(v_0, \dots, v_k)$ is an ordering $v_0, v_1, \dots, v_k$ of the vertices up to an even permutation (a product of an even number of transpositions).

orientation

Remark 7.1.4 (There are exactly two orientations)
If the dimension of the simplex is at least 1 (that is, there are at least two vertices), then there are exactly two orientations (as the symmetric group of at least two elements has the alternating group as a proper subgroup of index two). We use a $+$ and $-$ to indicate this. $\circ$

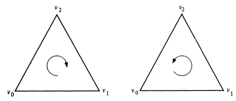

Fig. 80: The two orientations of a triangle.

Definition 7.1.5 (Induced orientation)
The orientation of a face $\tau = \operatorname{conv}(v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k) < \sigma$ induced by an orientation $v_0, \dots, v_k$ of σ is

$$\begin{cases} v_0 \dots v_{i-1} v_{i+1} \dots v_k, & \text{if } i \text{ is even,} \\ -v_0 \dots v_{i-1} v_{i+1} \dots v_k, & \text{if } i \text{ is odd.,} \end{cases}$$

where the minus denotes to opposite orientation.

As a demonstration, consider the left triangle in figure 80. The orientation of the simplex $\sigma := \operatorname{conv}(v_0, v_1, v_2)$ is $v_0 v_1 v_2$ (equivalent: $v_2 v_0 v_1$ or $v_1 v_2 v_0$). The induced orientation of the faces $\operatorname{conv}(v_0, v_1)$ and $\operatorname{conv}(v_1, v_2)$ are $v_0 v_1$ and $v_1 v_2$, respectively, as 2 and 0 are even. However, the orientation of $\operatorname{conv}(v_0, v_2)$ is $-v_0 v_2$, that is, the edge goes from v_2 to v_0.

Definition 7.1.6 (Orientable triangulation)
A triangulation K is orientable if there exists orientations of all triangles such that each edge receives opposite orientations from its two triangles.

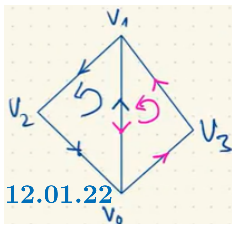

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Fig. 81: Edges receive opposite orientations.

Lemma 7.1.7
Let the surface $S := |K|$ be the polyhedron defined by K. If S is orientable, then K is orientable.

Proof. Assume that S is orientable. Pick any triangle and orient it. In the dual graph, take a spanning tree rooted at the triangle chosen, along which we can uniquely spread the orientation as in figure 81. This will always work - the only thing that can go wrong is that two nodes in this tree have incompatible orientations. We show that this can't happen.

If this were the case, then there is a sequence of triangles $\sigma_1, \dots, \sigma_k$ which are adjacent such that σ_i and σ_{i+1} are adjacent (they share an edge) and have compatible orientations for $i \in \{1, \dots, k-1\}$, but σ_1 and σ_k are adjacent but have no compatible orientations. We join the barycentres β_{σ_i} to the barycentres of the edges $\beta_{\sigma_i \cap \sigma_{(i+1) \bmod k}}$ and $\beta_{\sigma_{i-1} \cap \sigma_i}$, obtaining a simple closed polygonal path in the polyhedron $|K^1|$. "Thickening" that path yields a Möbius strip, contradicting that S is orientable.

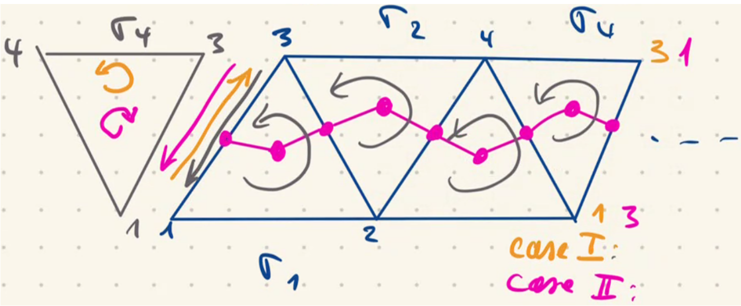

Fig. 82: In case 1, the surface is orientable, in case 2, it is not. This is a bit of a simplified picture because there could be unwanted identifications of vertices.

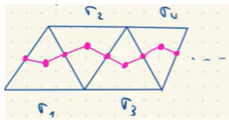

Fig. 83: A path contained in K^1.

DEFINITION 7.1.8 (THICKENING)
Let L be a one-dimensional subcomplex in K^1. The thickening of L is the subcomplex of K^2 of the triangles (and their faces) which meet L.

thickening

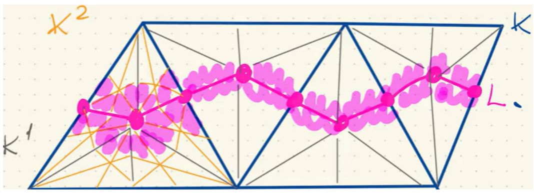

Fig. 84: The thickening of the above path.

Remark 7.1.9 The thickening of L is a closed neighbourhood of $|L|$ in $|K|$ whose polyhedron is homotopy equivalent to $|L|$. ○

Lemma 7.1.10
The thickening of a tree is homeomorphic to a disk. The thickening of a simple closed polygonal curve is either a cylinder or a MÖBIUS *strip.*

Proof. Exercise. □

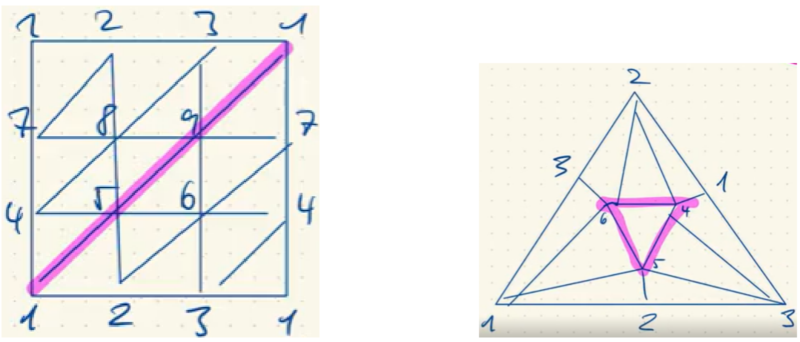

Fig. 85: **Left:** A triangulation of $T \approx \mathbb{S}^1 \times \mathbb{S}^1$. The closed loop 1591 gives a cylinder.
Right: A triangulation $|R| \approx \mathbb{RP}^2$. The closed loop 4564 is a MÖBIUS strip.
The torus is orientable, while the projective plane is not.

Consider a square, where the horizontal edges are identified and have the same direction, while the vertical edges are identified but oriented differently. This is a KLEIN bottle. The thickening of a vertical path will give a cylinder, while the thickening of a horizontal path will be homeomorphic to a Möbius strip.

Hence **orientability is a topological invariant**: if two surfaces are homeomorphic, either both are orientable or both are not. The distinction is: we can go through all possible polygonal loops in some suitable triangulation (fine enough - maybe we need to pass through the barycentric subdivision and then in the second barycentric subdivision) we can construct the thickening as a two-dimensional subcomplex. Then this is either a Möbius strip or a cylinder. Whenever we find at least one Möbius strip, the surface is not orientable, if we find none, it is orientable.

Lemma 7.1.11 (HW 10.1)
For a triangulation L of a closed surface S we have

$$V \geqslant \frac{7+\sqrt{49-24\chi(S)}}{2}.$$

Proof. As S is a closed surface, we have $V \geqslant 4$, as a triangulation with 3 vertices is a triangle, which can not be a closed surface. Furthermore, $\chi(S) \leqslant 2$, such that the term in the square root is positive. Thus $2V-7 \geqslant 0$ and we square both sides of the inequality and simplify to obtain

$$V^2 - 7V \geqslant -6\chi(S).$$

As S is a closed surface, each edge is contained in exactly two faces, so we have $3F = 2E$. Plugging this into $\chi = V - E + F$ yields

$$-6\chi(S) = -6(V-E+F) = -6V + 6E - 4E = -6V + 2E,$$

which plugged into the above inequality yields the equivalent inequality

$$V^2 - V \geqslant 2E.$$

We now show that this inequality is true. Let $\deg(v)$ be the *degree* of a vertex v, that is, the number of edges it is contained in. As a vertex can maximally be connected to every other vertex in the triangulation, we have $\deg(v) \leqslant V-1$ for all vertices v. Lastly, we have seen in CoMa that $\sum_{v \text{ vertex}} \deg(v) = 2E$ holds for any finite simple graph. Hence

$$2E = \sum_{v \text{ vertex}} \deg(v) \leqslant \sum_{v \text{ vertex}} (V-1) = V(V-1) = V^2 - V.$$

□

7.2 Euler characteristics

Definition 7.2.1 (f-vector, Euler characteristic)
Let L be a d-dimensional simplicial complex. Then $f_k(L)$ is the number of k-dimensional simplices of L, $(f_j)_{j=0}^d$ is the f-vector (or face vector) of L and the Euler characteristic of L is f-vector

$$\chi(L) := \sum_{k=0}^{d} (-1)^k f_k(L).$$

We will later see that the Euler characteristic is a topological invariant, while e.g. the f-vector is not invariant under barycentric subdivision.

Example 7.2.2 The Euler characteristic of the Möbius strip and the cylinder are both 0, as one can see from the following triangulations:

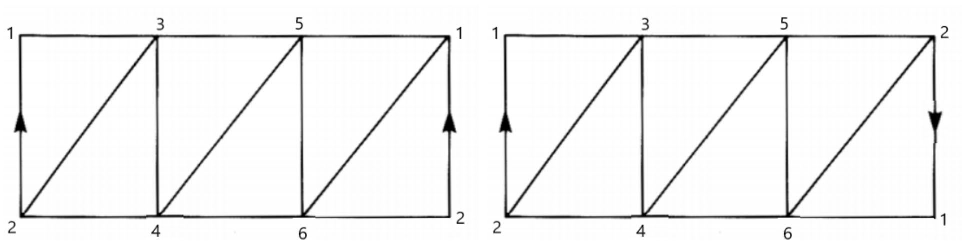

Fig. 86: Triangulations of the cylinder and the Möbius strip with same number of vertices, edges and faces.

Lemma 7.2.3 (Euler characteristic of connected graphs)
Let T be a connected graph (= 1-dimensional simplicial complex). Then $\chi(T) \leqslant 1$ and $\chi(T) = 1$ if and only if T is a tree.

Proof. From CoMa I we know that T is a tree if and only if $f_0(T) = f_1(T) + 1$ (f_0 is the number of vertices and f_1 the number of edges). Hence $\chi(T) = f_0(T) - f_1(T) = 1$.

As trees are minimal connected graphs, a graph that is not a tree has more vertices, so $\chi(T) \leqslant 1$. □

Example 7.2.4 Let K be a triangulation of a surface with spanning tree T in $K^{\leqslant 1}$. We construct a dual graph Γ, whose vertices are the (barycentres of) triangles (can be identified with the maximal simplices of K) and whose edges correspond to edges in K which are not edges in T (i.e. adjacency pairs of maximal simplices, which are not documented in T). Referring to the barycentres means that we can realise this a geometric simplicial complex $\Gamma^1 \leqslant K^1$.

T gives rise to the thickening of T^1 in K^1, the two-dimensional subcomplex $N(T) \leqslant K^2$. Similarly, from Γ we obtain the thickening $N(\Gamma) \leqslant K^2$. By lemma 7.1.10, $N(T)$ is homeomorphic to a disk. Since we don't know if Γ is a tree, the topological structure of $N(\Gamma)$ is unclear.

We have $N(T) \cup N(\Gamma) = |K^2|$ and $N(T) \cap N(\Gamma) = \partial N(T) \approx \mathbb{S}^1$. Lastly, T is connected, as T is acyclic. ◇

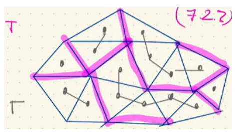

Fig. 87: The fragment of a triangulation of a surface and (parts of) the graphs T and Γ (small error in lower left corner)

17.01.2022

Lemma 7.2.5
We have $\chi(K) \leqslant 2$ for a combinatorial closed surface K.

Proof. Choose a spanning tree T in K and construct the complementary graph Γ as above. Then $\chi(K) = \chi(T) + \chi(\Gamma)$, because each face of the triangulation either contributes to T (if it is a vertex or an edge of T) or to Γ (if it is a triangle or if it bijectively corresponds an edge which is not in T) in such a way that the signs match. As T is a tree, $\chi(T) \leqslant 1$ and as Γ is a connected simple graph, $\chi(\Gamma) \leqslant 1$ by lemma 7.2.3. □

The following theorem is similar to Theorem 1.1.1.

Theorem 7.2.1: Characterisation I

The following statements are equivalent:

1. Every polygonal curve in $|K|$ made up of edges in K^1 separates $|K|$.
2. $\chi(K) = 2$.
3. $|K| \approx \mathbb{S}^2$.

Proof. "1 $\implies$ 2". As in the proof of Euler's theorem, it follows that Γ is a tree. Hence $\chi(K) = \chi(T) + \chi(\Gamma) = 1 + 1 = 2$.

"2 $\implies$ 3". If $\chi(K) = 2$, then $\chi(\Gamma) = 1$ by the above formula and hence Γ is a tree. Thus $|K| = |N(T)| \cup |N(\Gamma)|$ is a union of two disks glued at their boundaries, so $|K| \approx \mathbb{S}^2$.

"3 $\implies$ 1". This follows from the proof of the Jordan curve theorem. □

Remark 7.2.6 (Easy exercises) Let L and M be simplicial complexes (in fact, they can be any cell complexes) which intersect in a common subcomplex $L \cap M$. (There is a subcomplex of L which triangulates $L \cap M$, which is also a subcomplex of M). Then

$$\chi(L \cup M) = \chi(L) + \chi(M) - \chi(L \cap M)$$

by the inclusion-exclusion principle. ○

Furthermore, $\chi(L^1) = \chi(L)$

Proof. (For a triangulation) We have $\chi(L) = V_0 - E_0 + F_0$ and

$$\begin{aligned}\chi(L^1) = V_1 - E_1 + F_1 &= (V_0 + E_0 + F_0) - (2E_0 + 6F_0) + 6F_0 \\ &= V_0 + E_0 + F_0 - 2E_0 - 6F_0 + 6F_0 = V_0 - E_0 + F_0.\end{aligned}$$

□

7.3 Surgery

Definition 7.3.1 (Combinatorial surface)
A combinatorial surface K is a two-dimensional manifold S with a fixed triangulation K.

Let K be a combinatorial surface. Assume there exist a simple closed polygonal curve L (is a subcomplex) which does not separate $|K|$. By Theorem 7.2.1, $|K| \not\approx \mathbb{S}^2$. Let N be the thickening of L in K^2. By lemma 7.1.10, $|N|$ is either homeomorphic to a cylinder or a Möbius strip.

Let M be the subcomplex complementary to N in K^2 (cf. thickening of dual graph Γ).

Case 1. $|N| \approx$ cylinder. Then $|\partial N| = |\partial M|$ are two disjoint circles. Let $L_1, L_2 \leqslant K^2$ be one-dimensional subcomplexes supporting those circles. Then

$$K_* := M \cup CL_1 \cup CL_2$$

is a combinatorial surface because L_1 and L_2 are circles, so their cones CL_1 and CL_2 are disks. This surface is simpler than the original surface K.

Case 1. $|N| \approx$ Möbius strip. Then $\partial|N| = \partial|M| \approx \mathbb{S}^1$ is supported by a one-dimensional subcomplex $L \leqslant K^2$. Then $K_* := M \cup CL$ is a combinatorial surface, which is again simpler than K.

In both cases we say that K_* is obtained from K by doing surgery along L resp. $L_1 \cup L_2$.

Remark 7.3.2 This can be seen as a symmetric process: we can go from K to K_*, which is topologically simpler, or we can start with K_* to K by glueing in a cylinder or a Möbius strip.

The following operation decreases the Euler characteristic $\chi(K) = 2$ of $|K| \approx \mathbb{S}^2$ (for example realised as a tetrahedron) and is inverse to the procedure outlined above.

The following is a non-orientable operation: remove a 2-face S of K to obtain K'. Then $\chi(K') = 4-6+3 = 1$, so the Euler characteristic is reduced by 1. Glue in a Möbius strip M via $\partial M \approx \mathbb{S}^1$ to ∂S to obtain K''. Then

$$\chi(K'') = \chi(K') + \chi(M) - \chi(C) = 1 + 0 - 0 = 1,$$

where C is the common subcomplex of K' and M, which is a triangle without the face, having $\chi(C) = 3 - 3 + 0 = 0$.

An orientable operation is: take away two disjoint disks, that is, 2-faces, from K to obtain a new complex K'. Then

$$\chi(K') = 4 - 6 + 2 = 0.$$

Now glue in a cylinder Z to obtain K''. The common subcomplex of Z and K' is $\mathbb{S}^1 \sqcup \mathbb{S}^1$, which has $\chi(\mathbb{S}^1 \sqcup \mathbb{S}^1) = 2\chi(\mathbb{S}^1) = 0$. Hence

$$\chi(K'') = \chi(K') + \chi(Z) - \chi(\mathbb{S}^1 \sqcup \mathbb{S}^1) = 0 + 0 - 0 = 0.$$

○

Example 7.3.3 (Applying surgery to the torus (torus - handle = sphere))
Let $|K|$ be the torus and choose a closed non-separating edge-path $|L| \approx \mathbb{S}^1$ in $|K|$. Let N be its thickening. Then $M \subset \mathbb{R}^3$ is a cylinder and $L_1 \approx L_2 \approx \mathbb{S}^1$ be its boundary. Choosing apexes $A_1, A_2 \in \mathbb{R}^4$ for cones on L_1 and L_2 we obtain a (homeomorphic description of the) sphere.

◇

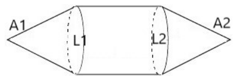

Fig. 88: A cylinder with a cone on each of its boundary circles is homeomorphic to a sphere.

Lemma 7.3.4
We have $\chi(N) = 0$.

Proof. We have

$$N = \bigcup_{v \in \mathrm{Vert}(L^1)} \mathrm{star}(v, K^2),$$

where denotes the closed star, the subcomplex generated by the faces containing v. For $v, w \in \mathrm{Vert}(L^1)$, which are neighbours, the intersection

$$\mathrm{star}(v, K^2) \cap \mathrm{star}(w, K^2)$$

consists of three vertices and two edges. We have $\chi(\mathrm{star}(v, K^2)) = 1$ because $\mathrm{star}(v, K^2)$ is a disk.

Hence by remark 7.2.6

$$\begin{aligned}\chi(\mathrm{star}(v, K^2) \cup \mathrm{star}(w, K^2)) &= \chi(\mathrm{star}(v, K^2)) + \chi(\mathrm{star}(w, K^2)) \\ &\quad - \chi(\mathrm{star}(v, K^2) \cap \mathrm{star}(w, K^2)) \\ &= 1 + 1 - (3 - 2) = 1.\end{aligned}$$

We apply induction until we find the final closing vertex. Hence

$$\chi(N) = \chi(\text{everything before closing}) + 1 - (6 - 4) = 2 - 6 + 4 = 0.$$

□

Theorem 7.3.1: Surgery increases Euler characteristic

We have $\chi(K_*) > \chi(K)$.

Proof. Case 1: $|N|$ is a cylinder. Then, as L_1 and L_2 are disjoint,

$$\begin{aligned}\chi(K_*) &= \chi(M) + \chi(CL_1 \cup CL_2) - \chi(M \cap (CL_1 \cup CL_2)) \\ &= \chi(M) + \chi(CL_1) + \chi(CL_2) - \chi(L_1) - \chi(L_2) \\ &= \chi(M) + 1 + 1 - 0 - 0 = \chi(M) + 2.\end{aligned}$$

Case 2: $|N|$ is a Möbius strip. Then

$$\chi(K_*) = \chi(M) + \chi(CL) - \chi(L) = \chi(M) + 1.$$

Lastly, in both cases we have

$$\chi(K) = \chi(K^2) = \chi(M) + \chi(N) - \chi(M \cap N) = \chi(M) + 0 - 0 = \chi(M),$$

as $M \cap N$ is a circle. □

Definition 7.3.5 (Combinatorial sphere)
A combinatorial surface K is a combinatorial sphere if $|K| \approx \mathbb{S}^2$.

Corollary 7.3.6
Any combinatorial surface can be modified into a combinatorial sphere by a finite number of surgeries.

Proof. (Of Theorem 1.5.1) Triangulate the surface S and the preform surgery on the combinatorial surface until it becomes a combinatorial sphere. (Make sure that the disks constructed on the way are pairwise disjoint.) Reverse the surgeries: if the surface is (non)orientable, only do (non)orientable operations. □

Remark 7.3.7 The simple closed polygonal loops used for the surgeries give rise to a system of generators of $\pi_1(S)$ after choosing a base point. This allows to conclude that any surfaces constructed have distinct fundamental groups. ○

7.4 Surface symbols

19.01.2022

We started out with $\mathbb{S}^2$ and added p handles, which we will denote by $H(p)$. This is an orientable surface with genus p. Further, let $M(q)$ be $\mathbb{S}^2$ with q Möbius strips glued in, which is a non-orientable surface.

These closed loops (todo!) form a system of generators for the fundamental group and the genus is the number of generators. For example, $H(1) \approx \mathbb{S}^1 \times \mathbb{S}^1$ and $M(1) \approx \mathbb{RP}^2$ (because $\mathbb{S}^2$ with a circle removed is homeomorphic to $\mathbb{D}^2$ and $\mathbb{D}^2 \cup_{z \mapsto z^2} \mathbb{S}^1 \approx \mathbb{RP}^2$).

Example 7.4.1 We can use the following as a model for constricting $H(p)$ for any p. We construct $H(2)$ as the quotient of an octagon.

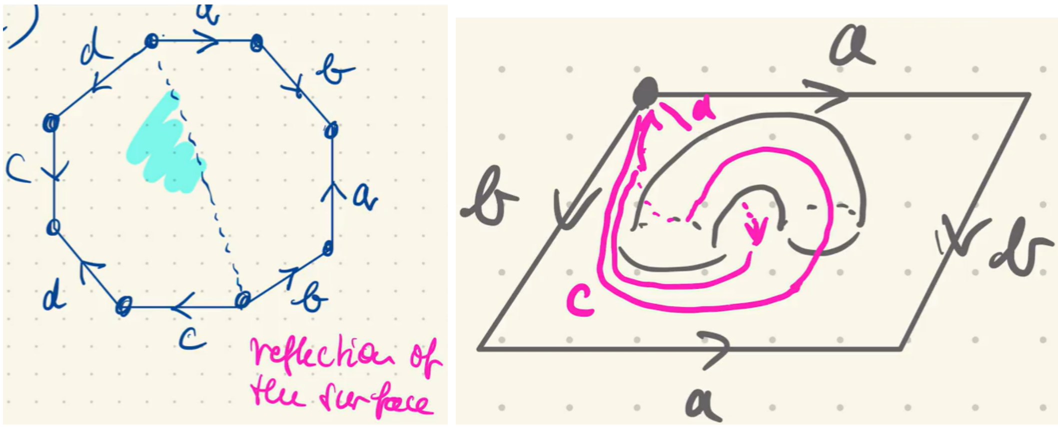

Fig. 89: Two representations of the surface $H(2)$.

In general, $H(p)$ will be the quotient of a $4p$-gon.

We have

$$\pi_1(H(2)) = \langle\, a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \,\rangle,$$

as $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ represents the path taken along the edge path of the surface, which can be contracted, as the octagon is convex. The word $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ is a surface symbol for $H(2)$. (A symbol where the letters do not occur in pairs cannot produce a surface.) surface symbol

It is nontrivial, that is the only the only relation necessary. This arises from the analysis of the fundamental group of the torus $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \langle\, a, b \mid aba^{-1}b^{-1} \,\rangle$, which arises from glueing one handle into $\mathbb{S}^2$. Iterating this shows the desired result. ◇

Remark 7.4.2
One might have noticed that surface symbol for the Klein bottle looks different to the standard one. They can be transformed into each other as follows:

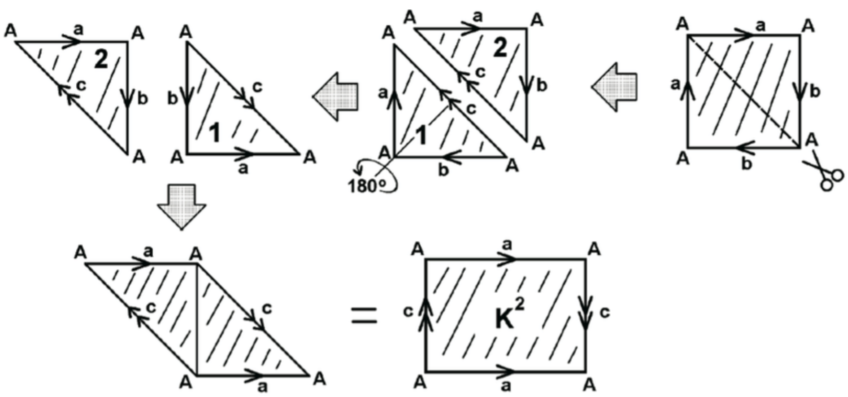

Fig. 90: Transitioning between two representations of the Klein bottle; each as quotient of a square [4].

Lemma 7.4.3 ($\pi_1(H(p))$ and $\pi_1(M(q))$)
For $p \geqslant 0$ we have

$$\pi_1(H(p)) \cong \left\langle a_1, b_1, \ldots, a_p, b_p \,\middle|\, \prod_{k=1}^{p} a_k b_k a_k^{-1} b_k^{-1} \right\rangle$$

and for $q \geqslant 1$

$$\pi_1(M(q)) \cong \left\langle a_1, \dots, a_q \;\middle|\; \prod_{k=1}^{p} a_k^2 \right\rangle.$$

Proof. (Sketch) A surface with genus g can be obtained by quotienting an $4g$-gon (see figure 91 for the case $g = 2$) with $2g$ edge labels $a_1, b_1, \dots, a_g, b_g$ and the relation $\prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1}$ e. Using van-Kampen yields

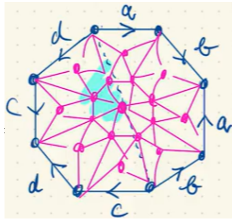

Fig. 91: A triangulation of the quotient of an octagon.

$$\begin{tikzcd}
\pi_1(U_0 \cap U_1) = \mathbb{Z} \arrow[rr, "i_1"] \arrow[d, "i_0"] & & \pi_1(U_0) = \{e\} \arrow[d] \\
\pi_1(U_1) \arrow[r] & \pi_1(8) = (\mathbb{Z} * \dots * \mathbb{Z}) / \left\langle \prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1} = e \right\rangle &
\end{tikzcd}$$

because the map i_0 sends the generator of $\mathbb{Z}$, x to $\prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1}$ and i_1 sends it to e.

The non-orientable case is similar and homework. □

Remark 7.4.4 (Euler characteristic of $H(p)$ and $M(q)$)
We have

$$\chi(H(p)) = 2 - 2p \qquad \text{and} \qquad \chi(M(q)) = 2 - q.$$

This can be explained as follows: we know that $\chi(K) \leqslant 2$ and that reducing by one handle means going from K to K_*, that is, from $H(p)$ to $H(p-1)$, reducing the EULER characteristic by 2. In the non-orientable case going from $M(q)$ to $M(q-1)$ reduces the EULER characteristic only by 1. ○

The classification theorem can now be summarised as follows.

THEOREM 7.4.1: CLASSIFICATION

The closed surfaces are characterised by orientability and genus (or equivalently: orientability and EULER characteristic).

Example 7.4.5 ($\chi(\mathbb{R}P^2)$ via a triangulation)
We get from the triangulation in figure 79, that

$$\chi(\mathbb{R}P^2) = 6 - 15 + 10 = 1,$$

which is odd, so $\mathbb{R}P^2$ is non-orientable by remark 7.4.4. ◇

Example 7.4.6 (Fundamental group of the figure eight and its generalisations)
Triangulating the figure eight as two not filled in triangles sharing one vertex we obtain that $\pi_1(8) = \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{Z} * \mathbb{Z}$ and furthermore $\pi_1(\mathbb{S}^1 \vee \dots \vee \mathbb{S}^1) = \mathbb{Z} * \dots * \mathbb{Z}$ ◇

Remark 7.4.7 (Classification theorem from the view point of connected sums)
The connected sum of two surfaces S_1 and S_2, $S_1 \# S_2$ is obtained by removing a disk from each surface and connecting those holes via a cylinder. We get

$$\begin{aligned}
\chi(S_1 \# S_2) &= \chi(S_1) + \chi(S_2) - 2\chi(\text{disk}) + \chi(\mathbb{S}^1 \times I) = \chi(S_1) + \chi(S_2) - 2 \cdot 1 + 0 \\
&= \chi(S_1) + \chi(S_2) - 2.
\end{aligned}$$

We have

$$H(p) \approx \mathbb{S}^2 \# \underbrace{(\mathbb{S}^1 \times \mathbb{S}^1) \# \dots \# (\mathbb{S}^1 \times \mathbb{S}^1)}_{p \text{ times}} \qquad \text{and} \qquad M(q) \approx \mathbb{S}^2 \# \underbrace{(\mathbb{R}P^1) \# \dots \# (\mathbb{R}P^1)}_{q \text{ times}},$$

as by the previous computation

$$\chi\big(\mathbb{S}^2\#\underbrace{(\mathbb{S}^1\times\mathbb{S}^1)\#\ldots\#(\mathbb{S}^1\times\mathbb{S}^1)}_{p\text{ times}}\big) = 2+p\cdot 0-2p = 2-2p = \chi(H(p)),$$

$$\chi\big(\mathbb{S}^2\#\underbrace{(\mathbb{RP}^1)\#\ldots\#(\mathbb{RP}^1)}_{q\text{ times}}\big) = 2+q\cdot 1-2q = 2-q = \chi(M(q)).$$

○

The SEIFERT-VAN-KAMPEN theorem

The goal of the SEIFERT-VAN-KAMPEN theorem is to separate the space X, of which we wish to find the fundamental group, into simpler spaces U_0 and U_1 such that we know the fundamental groups of U_0, U_1 and $U_0\cap U_1\neq\varnothing$. We can then find $\pi_1(X)=\pi_1(U_0\cup U_1)=\pi_1(U_0)*\pi_1(U_1)/N$, where N is natural.

For this to work we obviously require that U_0, U_1 and $U_0\cap U_1$ are path-connected and the basepoint lies in $U_0\cap U_1$. The SEIFERT-VAN-KAMPEN theorem states that the following is a pushout of groups:

$$\begin{tikzcd}
\pi_1(U_0\cap U_1) \arrow[rr, "i_1"] \arrow[dd, "i_0"] & & \pi_1(U_1) \arrow[dl, "j_0"] \arrow[dd] \\
& \pi_1(U_0)*\pi_1(U_1) \arrow[dr] & \\
\pi_1(U_1) \arrow[ur, "j_1"] \arrow[rr] & & \pi_1(X)=(\pi_1(U_0)*\pi_1(U_1))/N
\end{tikzcd}$$

Fig. 92: Commutative diagram for the SEIFERT-VAN-KAMPEN theorem.

Here, N is chosen such that the diagram commutes, that is

$$N=\Big\langle (j_1\circ i_1)(a)\cdot(j_0\circ i_0)(a)^{-1} : a\in\pi_1(U_0\cap U_1)\Big\rangle.$$

Example 7.4.8 ($\pi_1(8)$ with VAN-KAMPEN)
Applying VAN-KAMPEN to the figure eight and choosing U_0 and U_1 to be the one of the two triangles, respectively, we get the following pushout:

$$\begin{tikzcd}
\pi_1(U_0\cap U_1)=\{e\} \arrow[r] \arrow[d] & \pi_1(U_0)=\mathbb{Z} \arrow[d] \\
\pi_1(U_1)=\mathbb{Z} \arrow[r] & \pi_1(8)=\mathbb{Z}*\mathbb{Z}/\{e\}=\mathbb{Z}*\mathbb{Z}\,.
\end{tikzcd}$$

Fig. 93: A VAN-KAMPEN diagram for determining the fundamental group of the figure eight.

Example 7.4.9 (Fundamental group of the dunce hat (HW 10.39))
The dunce hat D can be realised as the adjunction space $\mathbb{D}^2\cup_\beta\mathbb{S}^1$, where $\beta=\mathrm{id}_{\mathbb{S}^1}$. Hence VAN KAMPEN's Theorem implies that the following is a pushout of groups:

$$\begin{tikzcd}
\pi_1(\mathbb{S}^1)=\mathbb{Z} \arrow[r, hook, "g\mapsto e"] \arrow[d, hook, "\mathrm{id}"'] & \pi_1(\mathbb{D}^2)=\{e\} \arrow[d, hook] \\
\pi_1(\mathbb{S}^1)=\mathbb{Z} \arrow[r, hook] & \pi_1(D)=\langle g\mid g=e\rangle=\{e\}.
\end{tikzcd}$$

Lets try to motivate this a bit more rigorously. Let U_0 be a solid triangle from the interior of the dunce hat. Then $U_0 \approx \mathbb{D}^2$. Let U_1 be $D \setminus \operatorname{int}(U_0)$. Then the triangle $U_0 \cap U_1 \approx \mathbb{S}^1$ has fundamental group $\mathbb{Z}$. Lastly, U_1 retracts onto its boundary. We show below in a picture why it is homeomorphic to $\mathbb{S}^1$, so that the upper commutative diagram is justified.

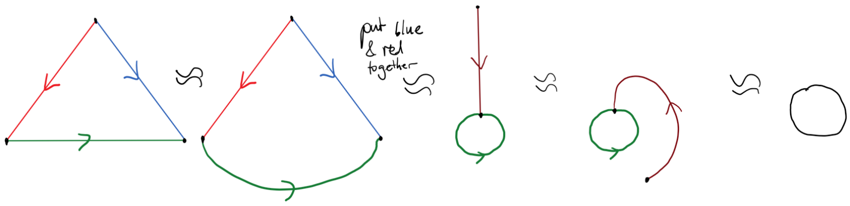

Fig. 94: A drawing depicting that $U_1 \approx \mathbb{S}^1$.

8 Simplicial homology

So far we dealt with the fundamental group, which is an algebraic object that we associate with a topological space, in particular if it is given by a finite triangulation, in which case it is a finitely represented group. This is a topological invariant: if we pass through a different triangulation of the same surface and compute the fundamental group via edge groups, then we will arrive at an isomorphic group. This can be used to distinguish spaces, in particular surfaces.

But dealing with these finitely presented groups can be difficult because of the ensuing non-decidability result: if we have a finitely presented group, then we can not say if it is finite or not by an algorithmic procedure. Hence this is a bit unwieldy sometimes and thus it is useful to look into other algebraic invariants, one of them being simplicial homology.

Let us first recap some Algebra.

Let G be a group and define the commutator $[a, b] := aba^{-1}b^{-1}$ for $a, b \in G$. Then G is ABELIAN if and only if the commutator is trivial for all $a, b \in G$. We have $[a, b]ba = ab$, so the commutator is the "price to pay" to exchange the order of multiplication of a and b, it collects everything that measures the deviation from being ABELIAN. The commutator subgroup is $G' := \langle \{[a, b] : a, b \in G\} \rangle$, the subgroup generated by all commutators. The commutator subgroup is a normal subgroup, so it occurs as a kernel of a group homomorphism. commutator subgroup

DEFINITION 8.0.1 (ABELIANISATION)
The abelianised quotient of G (or: abelisation of G) is $G^{\mathrm{Ab}} := G/G'$, the largest ABELIAN factor of G. abelisation

For example, is $\mathbb{A}_5^{\mathrm{Ab}}$ is trivial.

Algebra reminder: Modules

Let R be a commutative ring with multiplicative unit 1.

DEFINITION 8.0.2 (R-MODULE)
An R-module is an ABELIAN group $(G, *)$ together with a ring R and a ring homomorphism $\varphi\colon R \to \mathrm{End}(G)$, which is the (not unique) R-module structure on G. R-module

Here, $\mathrm{End}(G)$ denotes the group endomorphisms of G with respect to $*$.

Hence for all $a, b \in G$ and $s, r \in R$ we have by choosing "$r = \varphi(r)$" and using the additivity and multiplicativity of φ that

$$r(a + b) = ra + rb, \qquad (r + s)a = ra + sa, \qquad r(sa) = (rs)a, \qquad 1a = a.$$

Example 8.0.3 We could pick $(G, *) = (\mathbb{R}^k, +)$ and $R = \mathbb{R}$. ◇

Example 8.0.4 (R is a field) Let K be a field. Every K-vector space is a K-module, that is, a K-module *is* a K-vector space. ◇

Example 8.0.5 ($R = \mathbb{Z}$) There exists exactly one $\mathbb{Z}$-module structure, since every group homomorphism $\mathbb{Z} \to \mathrm{End}(G)$ is uniquely determined by $\varphi(1)$, as $\mathbb{Z} = \langle 1 \rangle$. Hence a $\mathbb{Z}$-module is an ABELIAN group. This is the most important case for applications. The $\mathbb{Z}$-homology is also called integral homology. ◇

Definition 8.0.6 (Free module)
Let X be a set. Then the free $R-$module with basis X is the set of unique formal linear combinations

free $R-$module

$$\bigoplus_{x \in X} R := \left\{ \sum_{i=1}^{n} r_i x_i : r_i \in R, x_i \in X, n \in \mathbb{N} \right\}$$

where $\sum_{i=1}^{n} r_i x_i + \sum_{i=1}^{n} s_i x_i := \sum_{i=1}^{n} (r_i + s_i) x_i$.

A R-submodule of a free R-module is a free R-module e.g. if R is a field or $R = \mathbb{Z}$.

Example 8.0.7 (Non-free modules) The $\mathbb{Z}$-module $\mathbb{Z}_2 \times \mathbb{Z}$ is not free, as $0 \cdot (1, 0) = (0, 0) = 2 \cdot (1, 0)$, so the formal linear combinations are not unique. In this case, $\mathbb{Z}_2$ is the torsion part and $\mathbb{Z}$ is the free part, cf. later. ◇

Remark 8.0.8 (Some information about finitely generated Abelian groups)
Each finitely generated Abelian group G has two uniquely determined subgroups:

- F, which is a free subgroup,
- T, which is a finite subgroup called the *torsion subgroup*,

such that $G = F \times T$ (inner direct product). In particular, finitely generated $\mathbb{Z}$-modules have the form

$$\underbrace{\mathbb{Z}^r}_{\substack{\text{free}\\\text{part}}} \oplus \underbrace{\mathbb{Z}_{a_1} \oplus \ldots \oplus \mathbb{Z}_{a_k}}_{\text{torsion part}},$$

where $a_1 | a_2 | \ldots | a_k$ and $|$ denotes "divides". (This is the fundamental theorem of finitely generated abelian groups.) ○

Definition 8.0.9 (Torsion element)
A torsion element is a $a \in G \backslash \{e\}$ such that there exists a $r_a \in R \backslash \{0\}$ with $ra = 0$.

8.1 Homology modules

Let K be a simplicial complex, whose vertices $V := \mathrm{Vert}(K) := \{v_1, \ldots, v_n\}$ are totally ordered. Fix a commutative ring R with multiplicative unit 1. Homology modules will be modules over R.

Definition 8.1.1 (Chain module of K, $C_q(K; R)$)
The q-th simplicial chain module of K is $C_q(K; R)$, the set of all formal linear combinations of q-dimensional faces of K with coefficients in R.

chain module

The chain module of K is a free R-module, where the addition and multiplication are inherited coefficient-wise from the addition and multiplication in R, respectively.

The q-faces form (by construction) an R-basis of $C_q(K; R)$. Hence

Bases over modules work a little different than bases over vector spaces. In a free module, the thing is quite similar. For instance, if we have a $\mathbb{Z}$-module, then by tensoring with a quotient field, like $\mathbb{Q}$, we can pass to a rational vector space. So it is a different thing, but it is always nearby.

$$\boxed{\partial_q \underbrace{(w_0 w_1 \ldots w_q)}_{\text{oriented } q\text{-face}} := \sum_{j=0}^{q} (-1)^j \underbrace{w_0 w_1 \ldots \widehat{w_j} \ldots w_q}_{\text{oriented } (q-1)\text{-face}} \in C_{q-1}(K; R)}$$

defines an R-linear map by extension.

Definition 8.1.2 (q-th simplicial boundary operator)
For $q \geqslant 1$, $\partial_q \colon C_q(K;R) \to C_{q-1}(K;R)$ is the q-th simplicial boundary operator of K.

boundary operator

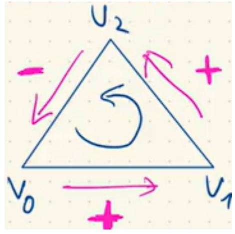

Fig. 95: An edge $v_i v_j$ receives a plus if $v_i < v_j$ and a minus else.

Example 8.1.3 Let $\sigma := v_0 v_1 v_2$ be an oriented triangle with $v_0 < v_1 < v_2$. Then

$$\partial_2(\sigma) = v_1 v_2 - v_0 v_2 + v_0 v_1 \in C_1(\sigma, \mathbb{Z}).$$

Note that every cyclic permutation $\tilde{\sigma}$ of σ has the same value $\partial_2(\tilde{\sigma}) = \partial_2(\sigma)$. Further,

$$\begin{aligned}\partial_1(\partial_2(\sigma)) &= \partial_1(v_1 v_2 - v_0 v_2 + v_0 v_1) = \partial_1(v_1 v_2) - \partial_1(v_0 v_2) + \partial_1(v_0 v_1)\\ &= (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0.\end{aligned}$$

In lemma 8.1.5 we will see how $\partial_1(\partial_2(\sigma)) = 0$ generalises. ◇

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Definition 8.1.4 (∂_0)
Let $\partial_0 := 0$ or define $C_{-1}(K;R) = R \cdot \varnothing$ (all R-multiples of the empty set) and $\partial_0(v) = \varnothing$ for $v \in V$.

The first definition is the usual definition, whilst the second one is reduced homology. Using reduced or standard homology only affects H_0. As we will see, for connected complexes K we have $H_0(K;R) \cong R$ in standard homology but $H_0(K;R) = \{0\}$ because $H_0 = \ker(\partial_0) = \{0\}$ *in reduced homology.*

Lemma 8.1.5 (Most important lemma: $\partial^2 = 0$)
We have $\partial_q \partial_{q+1} = 0$ for all $q \geqslant 0$.

Previously we used subscripts for the boundary maps. Because we can kind of see the collection of all the boundaries as being parts working on a certain graded piece of the chain module, if the context is clear, we omit the index.

Proof. Consider $q+2$ vertices $w_0, \dots, w_{q+1}$ of a $(q+1)$-dimensional simplex. Then

$$\begin{aligned}\partial^2(w_0 \dots, w_{q+1}) &= \partial\left(\sum_{k=0}^{q+1} (-1)^k w_0 \dots \widehat{w_k} \dots w_{q+1}\right) \overset{\text{(L)}}{=} \sum_{k=0}^{q+1} (-1)^k \partial\left(w_0 \dots \widehat{w_k} \dots w_{q+1}\right)\\ &= \sum_{k=0}^{q+1} (-1)^k \left(\sum_{j=k+1}^{q+1} (-1)^{j-1} w_0 \dots \widehat{w_k} \dots \widehat{w_j} \dots\dots w_{q+1}\right.\\ &\qquad \left. + \sum_{k=0}^{q+1} (-1)^k \sum_{j=0}^{k-1} (-1)^j w_0 \dots \widehat{w_j} \dots \widehat{w_k} \dots\dots w_{q+1}\right) = 0,\end{aligned}$$

where we use that ∂ is linear in (L) and the hat indicates that this vertex is omitted. Each ordered q-simplex occurs twice, but with opposite sign, hence the term is zero. □

Definition 8.1.6 (Boundary module, module of cycles)
The q-th (simplicial) boundary module is $B_q(K;R) := \operatorname{im}(\partial_{q+1}) \subset C_q(K;R)$.
The q-th (simplicial) cycle module is $Z_q(K;R) := \ker(\partial_q) \subset C_q(K;R)$.

boundary module
cycle module

Since ∂_{q+1} and ∂_q are R-linear, $B_q(K;R)$ and $Z_q(K;R)$ are free R-submodules of the R-module $C_q(K;R)$. By lemma 8.1.5,

$$B_q(K;R) \leqslant Z_q(K;R),$$

so we can take the quotient.

Definition 8.1.7 (Simplicial homology module)
The q-th simplicial homology module is

homology module

$$\boxed{H_q(K;R) := Z_q(K;R)/B_q(K;R).}$$

Definition 8.1.8 (Betty number)
The q-th Betty number β_q of K is the free rank of $H_q(K;\mathbb{Z})$.

Betty number

The Betty numbers are topological invariants.

Remark 8.1.9 Tensoring with $\mathbb{R}$ or $\mathbb{Q}$ "kills" torsion, e.g.

The universal coefficient theorem states that approximately, $H_i(X;\mathbb{Z}) \times \mathbb{R} \cong H_i(X;\mathbb{R})$.

$$(\mathbb{Z}_2 \oplus \mathbb{Z}) \otimes \mathbb{R} = \cancel{(\mathbb{R} \otimes \mathbb{Z}_2)} \oplus (\mathbb{R} \otimes \mathbb{Z}) = \mathbb{R}\,.$$

Note that while, in general, $H_k(K;\mathbb{Z}) = \mathbb{Z}^f \oplus \mathbb{Z}_{a_1} \oplus \ldots \oplus \mathbb{Z}_{a_n}$, we have $H_k(K;\mathbb{Q}) = \mathbb{Q}^f$, so only the torsion part is kept and clearly $\dim(H_k(K;\mathbb{Q})) = f$. Hence can tensor $H_q(K;\mathbb{Z})$ with $(\mathbb{Q},+)$, to obtain a $\mathbb{Q}$-vector space which has dimension β_q. ○

Note that e.g. $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 = \mathbb{Z}_{\mathrm{ggT}(2,2)} = \mathbb{Z}_2$.

Theorem 8.1.1: Euler characteristic and Betty numbers

For an n-dimensional simplicial complex K we have

$$\boxed{\chi(K) = \sum_{k=0}^{n} (-1)^k f_i(K) = \sum_{k=0}^{n} (-1)^k \dim(H_k(K;\mathbb{F})),}$$

where $\mathbb{F}$ is any field of characteristic zero, such as $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Proof. As $H_k(K;\mathbb{F}) = Z_k(K;\mathbb{F})/B_k(K;\mathbb{F})$, we have $\dim(H_k(K;\mathbb{F})) = \dim(Z_k(K;\mathbb{F})) - \dim(B_k(K;\mathbb{F}))$ as well as $\dim(\ker(\partial_q)) + \dim(\operatorname{im}(\partial_q)) = \dim(C_q(K;\mathbb{F})) = f_q(K)$ and thus

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \beta_k &= \sum_{k=0}^{n} (-1)^k \big(\dim(\ker(\partial_k)) - \dim(\operatorname{im}(\partial_{k+1}))\big) \\
&= \sum_{k=0}^{n} (-1)^k \dim(\ker(\partial_k)) + (-1)^{k+1} \dim(\operatorname{im}(\partial_{k+1})) \\
&= \dim(\ker(\partial_0)) + \sum_{k=1}^{n} (-1)^k \big(\dim(\ker(\partial_k)) + \dim(\operatorname{im}(\partial_k))\big) \\
&= f_0(K) + \sum_{k=1}^{n} (-1)^k f_k(K) = \chi(K).
\end{aligned}$$

□

Definition 8.1.10 (Homology class)
For $c \in Z_q(K;R)$ the homology class of c is

homology class

$$[c] := c + B_q(K;R).$$

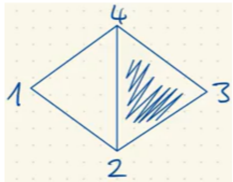

Fig. 96: A simplicial complex with f-vector $(4, 5, 1) = (\dim(C_0), \dim(C_1), \dim(C_2))$.

Example 8.1.11 Let K be the simplicial complex from figure 96, $R = \mathbb{Z}$. We abbreviate $C_q := C_q(K; R)$.

Then C_0 is a free ABELIAN module over $\mathbb{Z}$ generated by the vertices 1, 2, 3, 4, so $C_0 \cong \mathbb{Z}^4$. If we adopt the convention $\partial_0 = 0$, then $Z_0 = C_0$.

Further, C_1 is generated by the five edges 12, 14, 23, 24 and 34, so we get a free ABELIAN group of rank 5, $\mathbb{Z}^5$. By definition,

$$B_0 = \partial(C_1) = \langle 2-1, 4-1, 3-2, 4-2, 4-3 \rangle = \langle 2-1, 3-2, 4-3 \rangle \cong \mathbb{Z}^3,$$

as $(4-1)-(2-1) = 4-2$ and $(4-3)+(3-2)+(2-1) = 4-1$. Note that $2-1 \neq 1 \in \mathbb{N}$, but instead it is the integer linear combination of the vertices 2 and 1 with the coefficients 1 and $-1 \in \mathbb{Z}$.

Then $i + B_0 = j + B_0$ for all vertices $1 \leqslant i, j \leqslant 4$. Hence $H_0 = Z_0/B_0 \cong \mathbb{Z}^4/\mathbb{Z}^3 \cong \mathbb{Z}$ is generated by $[1] = [2] = [3] = [4]$. This is since K is connected. We have $\beta_0 = 1$, because $\mathbb{Z}$ is the free group of rank 1.

Z_1 is generated by $1\,2 + 2\,4 - 1\,4$ (the triangle 124) and $2\,3 + 3\,4 - 2\,4$ - they form a cycle basis. For example,

$$1\,2 + 2\,3 + 3\,4 - 1\,4 = (1\,2 + 2\,4 - 1\,4) - (2\,3 + 3\,4 - 2\,4).$$

Hence Z_1 is a free ABELIAN group of rank 2, i.e. $Z_1 \cong \mathbb{Z}^2$. Alternative: we have

$$\partial_1 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \in \mathbb{Q}^{4\times 5}$$

and hence $\dim(\ker(\partial_1)) = 2$ and so $Z_1 = \mathbb{Z}^2$.

C_2 is generated by the only face 234, so $C_2 = \langle 234 \rangle \cong \mathbb{Z}$. We have $\partial(234) = 3\,4 - 2\,4 + 2\,3$ and thus $B_1 = \langle 2\,3 + 3\,4 - 2\,4 \rangle$. Hence $H_1 = [1\,2 + 2\,4 - 1\,4] + B_1 \cong \mathbb{Z}$, because one of the generators of Z_1, namely $2\,3 + 3\,4 - 2\,4$ gets "killed" by B_1. We thus have $\beta_1 = 1$. Note that $\pi_1(K) \cong H_1(K; \mathbb{Z})$.

We have $\partial_2([234]) = [34] - [24] + [23]$, so $\partial_2 = (0, 0, 1, -1, 1)^\mathsf{T}$. Hence $\dim(B_1) = 1$ and $\dim(H_1) = 2 - 1 = 1$. Tensoring $\partial_2 \colon \mathbb{Z} \to \mathbb{Z}^5$, $x \mapsto (0, 0, -1, 1, 1)^\mathsf{T} x$ with $\mathbb{R}$ we get $\partial_2 \otimes \mathbb{R} \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto (0, 0, 1, -1, 1)^\mathsf{T} x$. Thus $\ker(\partial_2)$ and $\operatorname{ran}(\partial_3)$ are trivial, so $H_2 = \{0\}/\{0\} = \{0\}$.

Further, $Z_2 = 0$, so $H_2 = 0$ and thus $\beta_2 = 0$. As there are no faces of dimension 3 and higher, $C_k = 0$ for all $k \geqslant 3$ ◇

GAUSS elimination for computing homology over fields is called SMITH normal form for EUCLIDEAN domains (e.g. $\mathbb{Z}$).

Calculate yourself: $H_k(\mathbb{R}\mathrm{P}^2; \mathbb{Q})$ for $k \in \{0, 1, 2\}$.

Example 8.1.12 Now consider the the cone CK of the simplicial complex with vertices 0, 1, 2, 3 and new apex 4 in the previous example. The facets of CK are 1234, 014 and 034. We have

$$\begin{array}{ccccccccc} C_4 & & C_3 & & C_2 & & C_1 & & C_0 \\ \{0\} & \xrightarrow{\partial_4} & \mathbb{Z} & \xrightarrow{\partial_3} & \mathbb{Z}^5 & \xrightarrow{\partial_2} & \mathbb{Z}^9 & \xrightarrow{\partial_1} & \\ & & 1234 & & 123, 134, 124, 234, 014, 034 & & & & \end{array}$$

and $\partial_3 = [-1, -2, 1, 1, 0, 0]^\mathsf{T}$. We can show that $H_0(CK;\mathbb{Z}) = \mathbb{Z}$ and $H_r(CK;\mathbb{Z}) = \{0\}$ for all $r > 0$ with the SMITH normal form (SNF). ◇

Example 8.1.13 Consider a triangulation of the figure eight with two not filled in triangles and 4 vertices. We have

$$\begin{array}{ccc} C_1 & & C_0 \\ \mathbb{Z}^6 & \xrightarrow{\partial_1} & \mathbb{Z}^5 \\ 01, 02, 12, 23, 24, 34 & & 0, 1, 2, 3, 4 \end{array}$$

and

$$\partial_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{\text{SNF}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$H_0(CK) = \bigoplus_{i=1}^{r} \mathbb{Z}_{a_i} \oplus \mathbb{Z}^{k-r-s},$$

where k is the rank of C_0, r is the number of non-zero diagonal entries a_i in the SNF and $s := \operatorname{rank}(\partial_0)$.

In our case $a_i = 1$, $r = 4$, $k = 5$ and $s = 0$ and thus $H_0(CK) \cong \mathbb{Z}$. ◇

THEOREM 8.1.2: CHARACTERISING $H_0(K;R)$

We have $H_0(K;R) \cong R^C$, where C is the number of connected components of K.

Proof. If v and w are vertices of K in the same connected component, then the equivalence classes in the 0-th homology agree ("they are 0-homologous"): $[v] = [w]$. Indeed, we can join v and w by an edge path $vv_1 \dots v_k w$ in which no consecutive vertices are equal. Then $\partial((vv_1) + (v_1v_2) + \dots + (v_kw)) = w - v$, so $w - v \in \operatorname{im}(\partial_1) = B_0(K;R)$. Then $w = v + w - v \in v + B_0(K;R) = [v]$.

Furthermore, vertices which lie in different components of $|K|$ are not 0-homologous and R-multiples of a single vertex can never be a boundary. □

For $p \in \mathbb{N}$ and topological spaces $(X_k)_k$ we have

$$\boxed{H_p\left(\bigsqcup_k X_k\right) = \bigoplus H_p(X_k).} \qquad \text{(EILENBERG-STEENROD axiom \#4)}$$

Example 8.1.14 Consider again the object C depicted in figure 6. Then $C \approx \mathbb{S}^2 \sqcup \mathbb{S}^2$. We have $H_0(C;R) \cong \mathbb{Z}^2$, $H_1(C;\mathbb{Z}) = H_1(\mathbb{S}^2;\mathbb{Z}) \oplus H_1(\mathbb{S}^2;\mathbb{Z}) \cong \{0\}$ and $H_2(C;\mathbb{Z}) = H_2(\mathbb{S}^2;\mathbb{Z}) \oplus H_2(\mathbb{S}^2;\mathbb{Z}) \cong \mathbb{Z}^2$. ◇

Example 8.1.15 (Homology of torus, the KLEIN bottle and $\mathbb{R}\mathbf{P}^2$)
We have $H_2(\mathbb{S}^1 \times \mathbb{S}^1;R) \cong R$ and this does not depend on the triangulation. If K is the KLEIN bottle, then $H_2(K;R) \cong \{0\}$.

We have $H_0(\mathbb{RP}^2;\mathbb{Z}) = H_2(\mathbb{RP}^2;\mathbb{Z}) = \{0\}$ and $H_1(\mathbb{RP}^2;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Any vertex is a generator of $H_0(\mathbb{RP}^2;\mathbb{Z})$. A generator of $H_1(\mathbb{RP}^2;\mathbb{Z})$ is the loop $w := abca \in \ker(\partial_1) = Z_1(\mathbb{RP}^2;\mathbb{Z}) \subset C_1(\mathbb{RP}^2)$ in figure 79, which is not the boundary of a triangle. Hence $[w] \in H_1(\mathbb{RP}^2; Z)$ is not the trivial class. But traversing w twice, i.e. $abcabca$ bounds the whole complex, so its class is trivial. ◇

Example 8.1.16 (Homology of a cone)
Let L be an simplicial complex and $K := CL$. Then $H_0(K;R) \cong R$ as K is connected and $H_q(K;R) = 0$ for $q \geqslant 1$. ◇

Example 8.1.17 Let Δ^{n+1} be the $(n+1)$-dimensional simplex and $\Sigma^n := \{\tau \in \Delta^{n+1} : \dim(\tau) \leqslant n\}$ to be the set of proper subcomplexes of dimension at most n. Then $|\Sigma^n| \approx \mathbb{S}^n$. We have $H_0(\Sigma^n;R) \cong H_n(\Sigma^n;R) \cong R$ and $H_q(\Sigma^n;R) = 0$ if $q \in \{1,\dots,n-1\}$. ◇

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Let K be a connected abstract simplicial complex with totally ordered vertex set and with a vertex v. Each edge loop $\alpha = vv_1v_2\dots v_kv$ based at v gives rise to a simplicial 1-chain with integer coefficients

$$z(\alpha) := (vv_1) + (v_1v_2) + \dots + (v_kv) \in C_1(K;\mathbb{Z})$$

provided that subsequent vertices are distinct, i.e. $v_i \neq v_{i+1}$. The order matters: $(v_i, v_{i+1}) = -(v_{i+1}v_i)$. We have $\partial(z(\alpha)) = 0$ because α is closed and thus each vertex appears exactly twice in the linear combination. Hence $z(\alpha) \in Z_1(K;\mathbb{Z})$.

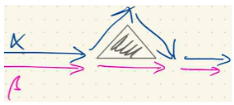

Fig. 97: Parts of the equivalent edge loops α and β (which are directed paths in $K^{\leqslant 1}$) and a triangle ∂ in K. The difference of α and β will be exactly the boundary of ∂ up to sign, so this is a 1-boundary in K.

For another chain β, which is equivalent to α in the edge path group of K, we get $z(\beta) - z(\alpha) \in B_1(K;\mathbb{Z})$, i.e. $z(\beta) - z(\alpha)$ is a 1-boundary of K. This yields a homomorphism of groups

$$\varphi\colon \pi_1(|K|,v) \to H_1(K;\mathbb{Z}), \qquad [\alpha] \mapsto [z(\alpha)], \tag{10}$$

where $[\alpha]$ is the homotopy class in the edge group and $[z(\alpha)]$ is the homology class.

It is unreasonable to presume that this is an isomorphism, as π_1 may be non-abelian, while H_1 is abelian. While they are not isomorphic they are something else, which is as closed as possible.

Theorem 8.1.3: Hurewicz: $H_1(K;\mathbb{Z}) \cong \pi_1(|K|,v)^{\mathrm{Ab}}$

The homomorphism (10) is onto and its kernel is the commutator subgroup of $\pi_1(|K|,v)$.

Proof. Skipped, see [1, p. 182]. □

$$\begin{tikzcd} E(K,v) \arrow[r, two heads, "\varphi", "\gamma\mapsto\gamma"'] \arrow[d] & H_1(K;\mathbb{Z}) \\ E(K,v)/\sim \arrow[ur, "\cong"'] & \end{tikzcd}$$

Fig. 98: Hurewicz' theorem gives this commutative diagram, where $\sim$ denotes the commutator subgroup.
When φ is a group homomorphism into an Abelian group, then the commutator subgroup is always a subgroup of $\ker(\varphi)$.

Hence (if K is connected)

$$\boxed{H_1(K;\mathbb{Z}) \cong \pi_1(|K|,v)^{\mathrm{Ab}}.}$$

Example 8.1.18 (Homology of a graph)
Assume that the graph $G = (V,E)$ with $\#V = n$ and $\#E = m$ is connected. Then $H_0(G;R) = R$ by Theorem 8.1.2. Furthermore, $H_k(G;R)$ is trivial for $k \geqslant 2$, as G has no faces of dimension $k \geqslant 2$. We have $H_1(G;R) = Z_1(G;R)/B_1(G;R)$. As $C_2(G;R)$ is trivial, B_1 is, too.

We pick a spanning tree (a maximal collapsible (and thus contractible) subcomplex) T in G, where the edges not contained in the tree are the generators of $\pi_1(G)$. There are no relations, because they are determined by the two-dimensional faces of G, which don't exist. Then T has $n-1$ edges, so $\pi_1(G)$ is the free group on $m-(n-1) = m-n+1$ generators. By Theorem 8.1.3, $H_1(G;R) = \pi_1(G)^{\mathrm{Ab}} = R^{m-n+1}$. ◇

Remark 8.1.19 (Other R) As other field $R = \mathbb{K}$ we only need the prime fields $\mathbb{Z}_p$ or $\mathbb{Q}$, as only 1, -1 and 0 appear as coefficients in the definition of ∂, and the field generated by the coefficients are precisely the prime fields. ○

Example 8.1.20 (The first three $\mathbb{Z}$-homology groups of surfaces with genus g)
Let K be a combinatorial surface of genus g. Then K is connected, so $H_0(K;\mathbb{Z}) = \mathbb{Z}$ and we have the following table:

	K is orientable	K is non-orientable
$H_0(K;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$
$H_1(K;\mathbb{Z})$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{g-1}\times\mathbb{Z}_2$
$H_2(K;\mathbb{Z})$	$\mathbb{Z}$	$\{0\}$

Table 1: The first three $\mathbb{Z}$-homology groups of surfaces with genus g. Note that a non-orientable surface of genus 0 does not exist, so everything is well-defined.

The lowest left entry can be explained as follows: as $C_3(K) = \{0\}$, we have $H_2(K;R) \cong \ker(\partial_2)$. As intersecting triangles have opposite orientations, they cancel out when applying ∂_2 to the whole of K, so K is a generator of $H_2(K;R)$.

For instance if K is orientable and $g=0$, then $|K| \approx \mathbb{S}^2$ and $H_2(K;\mathbb{Z}) = \mathbb{Z}$ and $H_1(K;\mathbb{Z}) = \pi_1(|K|)^{\mathrm{Ab}} = \{0\}$, as we computed before. For the torus we have $H_2(T;\mathbb{Z}) \cong \mathbb{Z}$ and $H_1(T;\mathbb{Z}) \cong \mathbb{Z}^2$, as we computed before (both 1 and -1 can each be generators of $\mathbb{Z}$, each representing one choice of orientation).

Further, we have $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = H_1(\mathbb{R}\mathrm{P}^2) = H_1(M(1))$ and $\mathbb{Z}_2\times\mathbb{Z} = H_1(M(2))$, where $M(2)$ is one representation of the Klein bottle K. (Recall that $\pi_1(K) = \langle\, a,b \mid a^2b^2 = e\,\rangle$ so indeed $H_1(K) = \pi_1(K)^{\mathrm{Ab}}$.) ◇

Remark 8.1.21 (Can't detect orientability with H_1 for $R = \mathbb{Z}_2$)
For $R = \mathbb{Z}_2$ we have $H_1(\mathbb{S}^1\times\mathbb{S}^1;\mathbb{Z}_2) = \mathbb{Z}_2^2 = H_1(\text{Klein bottle};\mathbb{Z}_2)$, so we can differentiate between an oriented and a non-oriented surface by looking at their first homology. ○

Remark 8.1.22 (Homology with $\mathbb{Z}_2$ coefficients)
We have $H_0(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) = H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) = \{0\}$ and $H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) = \mathbb{Z}_2$ (and $H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Q}) = \{0\}$).

Any cycles modulo $\mathbb{Z}$ are also cycles over any finite field, as the is a ring homomorphism $\mathbb{Z}\to\mathbb{Z}_p$ sending 0 to 0. But when considering $\mathbb{Z}_2$ coefficients, boundaries may change and there might be more cycles.

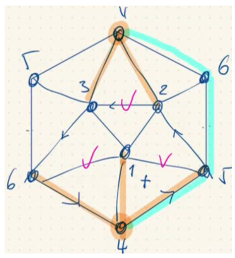

Fig. 99: A triangulation of $\mathbb{R}\mathrm{P}^2$ with a spanning tree in yellow.

We have (cf. figure 99)

$$\partial_2(145-125+123+136-146+256-246-234+356+345) = 2(45+45-46),$$

which is not equal to 0 if $R=\mathbb{Z}$, but it is equal to 0 if $R = \mathbb{Z}_2$, as there $2=0$. Lastly, $145-125+123+136-146+256-246-234+356+345$ is a generator for $H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) = \mathbb{Z}_2$.

For example, $45-25+24$ and $45+56-46$ are equal in $H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z})$ as

$$45-25+24-(45+56-46) = -25+34-56+46 = \partial_2(246-256).$$ ○

How can we prove the result above? One of the key things we did not yet discuss is: "to what extend does the homology depend on the triangulation?" We already stated that it does not, but we haven't proved it. If we accept that, one way to prove it is as follows: we talked about the surface symbols and then we can compute it from there directly (Exercise!).

8.2 | Topological invariance

So what about this invariance - what can we say about the homology being independent of the triangulation?

Here are some easy to prove facts, the arguments being like in the case of fundamental groups.

Lemma 8.2.1

- *Simplicial maps induce homomorphisms in homology.*
- *Subdivision preserves homology.*

The following theorem is also similar to the situation of fundamental groups.

Theorem 8.2.1

Let K and L be simplicial complexes. The map $f\colon |K| \to |L|$ induces a homomorphism of R-modules
$$f_*\colon H_q(K;R) \to H_q(L;R)$$
in each dimension $q \geqslant 0$.

Proof. Based on simplicial approximation: using that subdivision preserves homology and that $|K|$ and $|L|$ are compact, we can pass to a sufficiently fine barycentric subdivision by reducing the mesh, such that this sufficiently fine subdivisions of K and L are good enough to approximate f well enough. □

We can look at the special case where f is the identity map.

Theorem 8.2.2: $\mathrm{id}_* = \mathrm{id}$

If $f\colon |K| \to |K|$ is the identity map, then for all $q \geqslant 0$, $f_*\colon H_q(K;R) \to H_q(K;R)$ is the identity, too.

Theorem 8.2.3: $(g \circ f)_* = g_* \circ f_*$

If M is another simplicial complex and $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$ are maps, then
$$(g \circ f)_* = g_* \circ f_*\colon H_q(K;R) \to H_q(M;R).$$

The two previous theorems are relevant for notions of category theory: they state that
$$H_q(\cdot;R)\colon \mathbf{Top} \to R\text{-}\mathbf{Mod}, \qquad \begin{cases} X \mapsto H_q(X;R), \\ (f\colon X \to Y) \mapsto \big(f_*\colon H_q(X;R) \to H_q(Y;R)\big) \end{cases}$$

is a covariant functor from the category of finite simplicial complexes to the category of finitely generated ABELIAN groups or from the category of topological spaces to the category of R-modules.

There is the following special case: a simplicial map $f\colon S \to T$ induces a map

$$c_k(f)\colon C_k(S) \to C_k(T),\ [v_0,\dots,v_k] \mapsto \begin{cases} 1\cdot[f(v_0),\dots,f(v_k)], & \#\{f(v_0),\dots,f(v_k)\} = m+1,\\ 0, & \text{else.}\end{cases}$$

on chain modules C_k, which are free ABELIAN R-modules over k-simplices, that is, $C_k(S) = \bigoplus_{\sigma\in S:\#\sigma=k+1} R$. This in turn yields a map

$$f_l\colon H_k(S) \to H_k(T), \qquad [c] \mapsto \big[(c_k(f)(c)\big],$$

which is well defined as $(C_k(f))_k$ is a chain map, that is, it maps cycles to cycles and boundaries to boundaries, i.e. the following diagram commutes:

$$\begin{tikzcd} C_{k+1}(S) \arrow[r, "\partial_{k+1}"] \arrow[d, "c_{k+1}(f)"'] & C_k(S) \arrow[r, "\partial_k"] \arrow[d, dashed, "c_k(f)"'] & C_{k-1}(S) \arrow[d, "c_{k-1}(f)"] \\ C_{k+1}(T) \arrow[r, "\partial_{k+1}"'] & C_k(T) \arrow[r, "\partial_k"'] & C_{k-1}(T) \end{tikzcd}$$

Fig. 100: TODO: verify with a calculation on the basis elements that this diagram commutes.

For example, if $x \in C_k(S)$ is mapped to 0 by ∂_k, then $(c_k(f))(x)$ is mapped to 0 by ∂_k, too, as $c_{k-1}(f)$ is a group homomorphism mapping 0 to 0 and the right square of the above diagram commutes.

Similarly, $y \in C_k(S)$ is a boundary in S if there exists a $z \in C_{k+1}(S)$ such that $\partial_{k+1}(y) = z$. But then also $c_k(f)(z)$ is a boundary in T because $\partial_{k+1}(c_k(f)(y)) = c_k(f)(z)$ and $c_k(f)(y) \in C_{k+1}(T)$ because the left square of the above diagram commutes.

THEOREM 8.2.4: $f \simeq g \implies f_* = g_*$

If $f, g\colon |K| \to |L|$ are homotopic, then the induced maps f_* and g_* in homology are equal.

Corollary 8.2.2 (Simplicial homology is a homotopy invariant)
If $|K| \simeq |L|$, then $H_q(K;R) \cong H_q(L;R)$ for all $q \geqslant 0$.

Let us finish this subsection with a general application of homology.

THEOREM 8.2.5: $\mathbb{S}^m$ AND $\mathbb{S}^n$ ARE NOT HOMOTOPY EQUIVALENT

If $m \neq n$ are distinct integers, then $\mathbb{S}^m$ and $\mathbb{S}^n$ are not homotopy equivalent.

Proof. We have $H_m(\mathbb{S}^m) = \mathbb{Z} \neq \{e\} = H_m(\mathbb{S}^n)$. To compute this we can pick any model of the sphere, e.g. the boundary of any simplex. Alternatively we could use the cone construction or a MAYER-VIETORIS exact sequence (a type of VAN-KAMPEN theorem for homology). Also, one could use CW-complexes and obtain $\mathbb{S}^n$ by glueing the boundary of $\mathbb{D}^n$,

$\mathbb{S}^{n-1}$ to a point (recall that $\mathbb{S}^n \approx \mathbb{D}^n \setminus \mathbb{S}^{n-1}$), and thus attaching a n-cell to a 0-dimensional cell. We get the following chain complex in CW-homology:

$$\begin{array}{ccccccccccccc} n+1 & & n & & n-1 & & \dots & & 1 & & 0 & & -1 \\ 0 & \xrightarrow{b} & R & \xrightarrow{a} & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

Fig. 101

Then $H_n(\mathbb{S}^n) = \ker(a)/\mathrm{im}(b) = R$. $\square$

Corollary 8.2.3 ($\mathbb{E}^m \approx \mathbb{E}^n \iff m = n$)
We have $\mathbb{E}^m \approx \mathbb{E}^n$ if and only if $m = n$.

Proof. Let $h\colon \mathbb{E}^m \to \mathbb{E}^n$ be a homeomorphism preserving the origin. Then $\mathbb{S}^{m-1} \simeq \mathbb{E}^m \setminus \{0\} \overset{h}{\approx} \mathbb{E}^n \setminus \{0\} \simeq \mathbb{S}^{n-1}$ can only be true if $n = m$ by Theorem 8.2.5. $\square$

Recall Brouwer's fixed point theorem: a continuous map from a ball $\approx \mathbb{B}^n$ to itself has a fixed point. We proved this for the disk $\mathbb{B}^2$ by employing the topological properties of $\partial\,\mathbb{B}^2 \approx \mathbb{S}^1$, whose fundamental group is $\mathbb{Z}$. The same proof goes through for arbitrary dimensions, we just have to make a few amendments: we have to use H_{n-1} instead of π_1: assuming $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has no fixed point, we can construct a retraction $r\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ are before. As r is a retraction, we get the commutative diagram on the left, to which we apply $H_{n-1}(\cdot; R)$:

$$\begin{tikzcd} & \mathbb{D}^n \arrow[dd, "r"] \\ \mathbb{S}^{n-1} \arrow[ur, hook, "i"] \arrow[dr, "\mathrm{id}"'] & \\ & \mathbb{S}^{n-1} \end{tikzcd} \qquad \begin{tikzcd} & \{0\} \arrow[dd, "r_*"] \\ R \arrow[ur, "i_*"] \arrow[dr, "\mathrm{id}"'] & \\ & R \end{tikzcd}$$

Fig. 102

we use the homology of the sphere obtained above. Furthermore, $\mathbb{D}^n$ is convex and thus contractible, so it has trivial homology. The right diagram can not commute because the one-element $1 \in R$ is sent to 0 by i_*, which is mapped to 0 by the homomorphism r_*, where 1 is mapped to 1 by id.

9 Persistent homology

31.01.2022

We completed the classification of surfaces and the discussion of simplicial homology. We could continue to discuss the classification of higher-dimensional manifolds but before we introduce the topic of topological data analysis, in one concrete instance: persistent homology. It became a fairly involved and comprehensive topic in the last 20 years and we will only give the beginnings of that theory.

topological data analysis

We will follow [5].

Persistent homology can be used to measure the scale or resolution of a topological feature, it starts with a filtration and then produces the persistent homology modules by following the embeddings induced in homology. This is most often done for the Vietoris-Rips complex (cf. later).

9.1 Čech complexes

The main purpose of this subsection is to introduce a certain class of simplicial complexes, which can serve as examples a bit later, because some aspects of persistent homology are quite abstract and we want to have some specific examples in mind.

Definition 9.1.1 (Nerve complex)
Let F be a finite collection of closed convex sets in $\mathbb{E}^d$. Then the nerve complex is an abstract simplicial complex on F as vertex set:

nerve complex

$$\operatorname{Nrv}(F) := \left\{X \subset F : \bigcap X \neq \varnothing\right\}.$$

If $Y \subset X$ and $\bigcap X \neq \varnothing$, then $\bigcap Y \neq \varnothing$, so the nerve complex is an abstract simplicial complex.

Theorem 9.1.1: Nerve theorem

We have
$$\operatorname{Nrv}(F) \simeq \bigcup\{x : x \in F\} \subset \mathbb{E}^d.$$

Definition 9.1.2 (Čech complex)
Let $S \subset \mathbb{R}^d$ be a finite set of points and $r \geqslant 0$. Then

$$\check{\mathrm{C}}\mathrm{ECH}(r) := \left\{\sigma \subset S : \bigcap_{x\in\sigma} B_x(r) \neq \varnothing\right\}$$

is the Čech complex of S with respect to the radius r. Here, $B_x(r)$ is the closed ball of all points with distance at most r from x.

Čech complex

Remark 9.1.3 The Čech complex is an abstract simplicial complex (as the nerve complex is) on the vertex set S. Theorem 9.1.1 implies that $\check{\mathrm{C}}\mathrm{ECH}(r) \simeq \bigcup_{x\in S} B_x(r)$. ○

Example 9.1.4 Consider the set of five points in figure 103, which are midpoints of balls of equal radius. If k balls all intersect each other, then their k midpoints form a k-simplex – if two circles intersect, draw an edge between their midpoints, if three circles intersect mutually, then their centres form a triangle, a two-dimensional face of the complex. For this

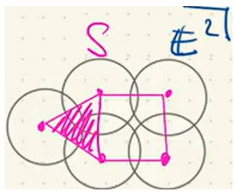

Fig. 103: A Čech complex.

particular radius, this Čech complex C is a two-dimensional, non-pure (there are maximal faces of dimension 1 and dimension 2) simplicial complex.

We have $C \approx \mathbb{S}^1$, because there is a hole between the four right-most circles and the only face can be contracted. ◇

Remark 9.1.5 (Properties of Čech complexes) ① We have $\text{Čech}(0) = S$ as a 0-dimensional complex.

② The final complex $\text{Čech}(\infty)$ is a $(|S| - 1)$-dimensional simplex on the vertex set S, where ∞ denotes a sufficiently large radius $r > \operatorname{diam}(S)$.

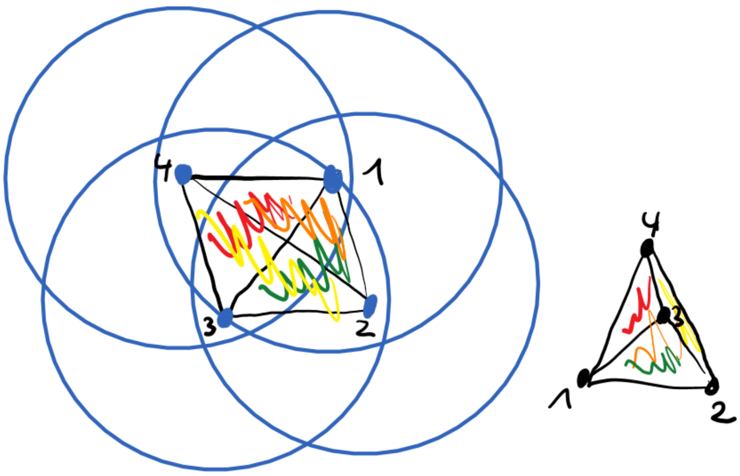

Fig. 104: $\text{Čech}(\infty)$, where $\#S = 4$, is homeomorphic to a 3-simplex, e.g. boundary the tetrahedron.

③ If $r \leqslant r'$, then $\text{Čech}(r) \leqslant \text{Čech}(r')$, where $\leqslant$ denotes "is a subcomplex of".

Hence we get a filtration - a sequence of subcomplexes that is contained in each other - of $\text{Čech}(\infty)$ (which is a simplex). In between $r = 0$ and "$r = \infty$" we get some things which depend on the geometry of S, while $\text{Čech}(0)$ and $\text{Čech}(\infty)$ only depend on $|S|$. ○

9.2 Persistent homology modules

Let $K_0 \leqslant K_1 \leqslant \ldots \leqslant K_n = K$ be some filtration $K_\bullet$ of a complex K. Then (by definition) for $0 \leqslant i \leqslant j \leqslant n$, we have $K_i \leqslant K_j$ and the induced (by the inclusion of subcomplexes $K_i \hookrightarrow K_j$) homomorphisms $f_p^{i,j} \colon H_p(K_i; R) \to H_p(K_j; R)$, where R is a commutative ring with 1.

Definition 9.2.1 (Persistent homology module)
The p-th persistent homology module of the filtration $K_\bullet$ with respect to R is

persistent homology module

$$\boxed{H_p^{i,j} := H_p^{i,j}(K_\bullet; R) := \operatorname{im}(f_p^{i,j}).}$$

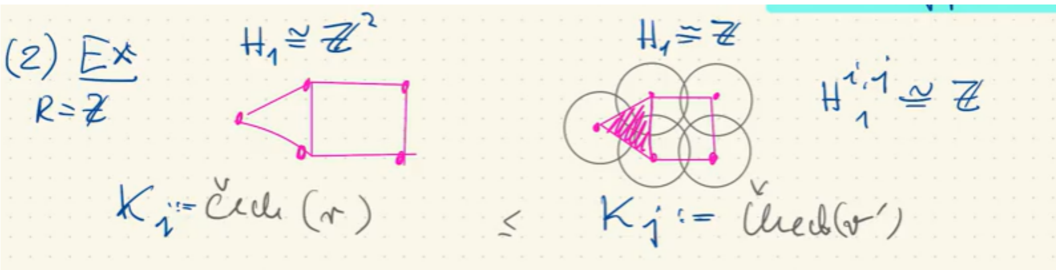

Fig. 105: The smaller radius r is chosen such that the three left circles don't all intersect in one common intersection but only pairwisely. Let $K_i := \text{Čech}(r)$. Then $H_1(K_i;\mathbb{Z}) \cong \mathbb{Z}^2$ and $H_1(K_j;\mathbb{Z}) \cong \mathbb{Z}$. The inclusion $K_i \hookrightarrow K_j$ kills one generator (the left not filled out triangle) - maps it to zero - and maps the other one (not filled out square) isomorphically. Hence $H_1^{i,j}(K_\bullet;\mathbb{Z}) \cong \mathbb{Z}$.

An explicit description of the persistent homology modules is

$$\boxed{H_p^{i,j}(K_\bullet;R) = Z_p(K_i;R)/[B_p(K_j;R) \cap Z_p(K_i;R)],}$$

where use that $Z_p(K_i;R) \leqslant Z_p(K_j;R)$. In particular, $H_p^{i,i} = H_p(K_i;R)$.

Definition 9.2.2 (Homology class is born)
A homology class $\gamma \in H_p(K_i)$ is born at K_i (or: at the i-th step) if $\gamma \notin H_p^{i-1,i}$.

Hence a homology class is born if it didn't occur in the image of the map induced by the embedding before. We see the filtration as a process of building up the complex K in several time steps.

There is a dual notion.

Definition 9.2.3 (Homology class dies)
If γ is born at K_i, then it dies entering K_j if $f_p^{i,j-1}(\gamma) \notin H_p^{i-1,j-1}$ but $f_p^{i,j}(\gamma) \in H_p^{i-1,j}$.

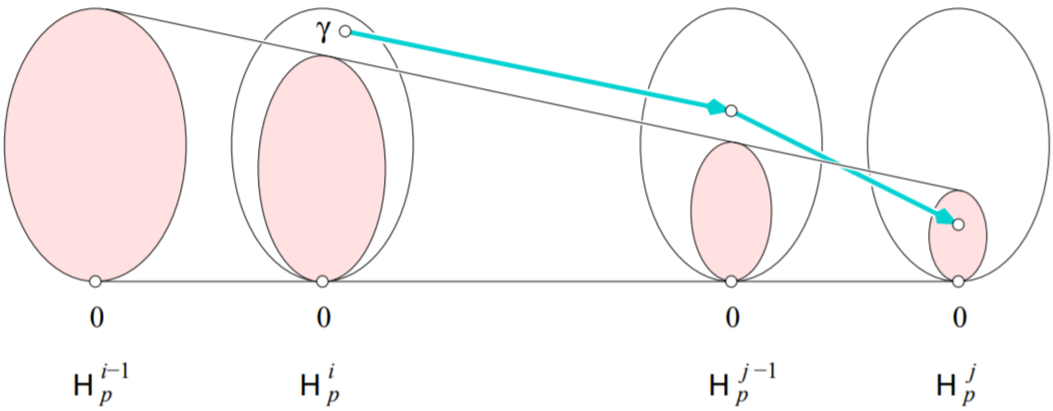

Fig. 106: The ovals represent homology modules. Time goes by from left to right and $j > i$. The origins of every homology module are denoted by 0. Here, γ is born at K_i, because we don't have $\gamma \in H_p^{i-1,i}$ and it dies entering K_j.

What's the idea of all of this? This is applied in topological data analysis, where we have a set of points and define a filtration which grows over time for larger radii and we keep track of what the filtration does with respect to additional information concerning the homology.

9.3 Persistence diagrams

Let K be a complex with fixed filtration $K_0 \leqslant K_1 \leqslant \ldots \leqslant K_n$ and let the coefficient ring R be a field.

One can also do this in principal ideal domains, where $\dim_R(H_p^{i,j})$ is still well defined.

Definition 9.3.1 (Persistent Betti number)
(For $p \in \{0, \ldots, n\}$) let $\beta_p^{i,j} := \dim_R(H_p^{i,j})$ (dimension as an R-vector space) be the p-th persistent Betti number of $K_\bullet$ with respect to R. If $i < j$, let

$$\mu_p^{i,j} := \left(\beta_p^{i,j-1} - \beta_p^{i,j}\right) - \left(\beta_p^{i-1,j-1} - \beta_p^{i-1,j}\right).$$

Hence $\mu_p^{i,j}$ is the number of independent p-dimensional homology classes born in K_i which die entering K_j. Indeed, the first difference on the right hand side counts the classes that are born at or before K_i and die entering K_j, while the second difference counts the classes that are born at or before K_{i-1} and die entering K_j.

A convention is $\mu_p^{i,i} := \infty$.

Here we see that for arbitrary rings the situation is more interesting and the only invariant that is relevant for vector spaces if the field is fixed, is the dimension. Hence we can reduce the structural information to counting - μ is a certain difference of vector space dimensions, so we can think of independent cycles from picking a basis of a homology as a vector space over R.

Lemma 9.3.2 (Fundamental Lemma of Persistent Homology)
For all $0 \leqslant k \leqslant \ell \leqslant n$ we have

$$\beta_p^{k,\ell} = \sum_{i=0}^{k} \left(\sum_{j=\ell+1}^{n} \mu_p^{i,j} + \mu_p^{i,\infty} \right).$$

The persistent Betty number $\beta_p^{k,\ell}$ counts the number of independent p-dimensional homology classes in the persistent homology vector space with parameters k and ℓ. They are split into various pieces, depending on when they are born and when they die. In order to contribute to $\beta_p^{k,\ell}$, they have to be born before k and die after ℓ (die entering K_ℓ).

For this to make sense we make an convention: $\mu_p^{i,\infty}$ shall denote the number of independent p-dimensional homology classes born in K_i, which still "live" in $K_n = K$. (We have $\mu_p^{i,\infty} = 0$ for the Vietoris-Rips complex (cf. later).)

This is an important property. It says the diagram encodes the entire information about persistent homology groups.

Definition 9.3.3 (Persistence diagram)
The p-th persistence diagram of the filtration with respect to R is the point configuration

persistence diagram

$$\{(i,j) : \mu_p^{i,j} \geqslant 1\} \in \overline{\mathbb{R}}^2 := \mathbb{R} \times (\mathbb{R} \cup \{\infty\})$$

with multiplicities $\mu_p^{i,j}$.

Example 9.3.4 Let the complex K be generated by a triangle and all its faces. The filtration consists of $K_0 = \{\varnothing\}$ and for $i \in \{1, \dots, 7\}$, K_i is the subcomplex generated by the first i faces, where we choose an enumeration of the faces such that faces of lower dimension have lower numbers, like in figure 107. This is not a ČECH filtration. Further, let $R = \mathbb{Z}_2$ (but the diagram below would not change for other fields).

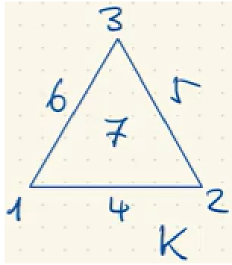

Fig. 107: An enumeration of the faces of a complex K generated by a triangle.

Everything we did for homology we can also do for reduced homology, to make it a bit more interesting.

In these persistence diagrams, everything happens above the diagonal.

The first vertex, 1, creates a connected complex and the 0-th reduced homology is trivial. Hence the homology dies in step 1. Then, 2 and 3 create two independent components. If we continue and pass to 4, which connects 1 and 2, one connected component gets killed - at step 4 we only have two connected components. After introducing the edge 5, the whole complex becomes connected, so the homology generator born in time step 3 dies in time step 5. Adding the edge 6 generates a 1-cycle. which dies at time step 7, which is represented by the dot in the diagram for $p = 1$.

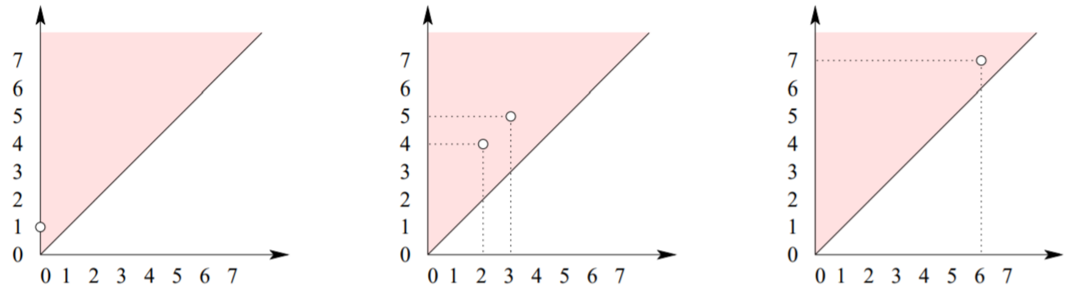

Fig. 108: The $p = -1$-th, 0-th and first persistence diagram. Here, birth is noted in the horizontal axis, whereas death is recorded on the vertical axis. The indices on the axis denote the filtration of K. The pink homology class in the diagram for $p = -1$ is generated by $\varnothing$ and is born at step 0 and dies in step 1.

So this is a persistence diagram of a filtration of a two-dimensional simplicial complex generated by a triangle and its faces and it arises just from labelling those faces. ◇

There is a connection to, among other things, Morse theory.

Example 9.3.5 (Toy example with only one edge)
Consider K_0 to be two vertices and K_1 be K_0 and the line connecting the two vertices. Then $H_0(K_0; \mathbb{Z}) \cong \mathbb{Z}^2$ by Theorem 8.1.2 with the generators $(1,0)$ and $(0,1)$ corresponding to the vertices. We have $H_0(K_1; \mathbb{Z}) \cong \mathbb{Z}$ by Theorem 8.1.2 and thus $\beta_0^{0,1} = 1$. Further

$$f_0^{0,1} \colon H_0(K_0; \mathbb{Z}) \cong \mathbb{Z}^2 \to \mathbb{Z} \cong H_0(K_1; \mathbb{Z}), \qquad (1,0), (0,1) \mapsto 1.$$

Thus, for example $f_0^{0,1}(1, -1) = 1 - 1 = 0$. ◇

Example 9.3.6 (Toy example with a triangle)
Consider K_0 to consist of three vertices, K_1 begin the hollow triangle with vertex set K_0 and finally, K_2 be the filled out version of K_1. Then $H_1(K_0)$ and $H_1(K_2)$ are trivial, while $H_1(K; R) \cong R$ is generated by the loop going around the triangle once. This homology class is born in K_1 and dies entering K_2. ◇

Example 9.3.7 (More elaborate example)
Consider the following (part of a) filtration.

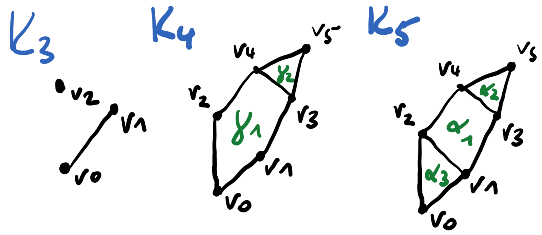

Fig. 109

We have $H_1(K_4;\mathbb{Z}) \cong \mathbb{Z}^2$ and $H_1(K_5;\mathbb{Z}) \cong \mathbb{Z}^3$. The generators of the former are

$$\gamma_1 := [(v_3,v_1)+(v_1,v_0)+(v_0,v_2)+(v_2,v_4)+(v_4,v_3)],\ \gamma_2 := [(v_3,v_5)+(v_5,v_4)+(v_4,v_3)].$$

Further,

$$f_1^{4,5}\colon H_1(K_4;\mathbb{Z}) \to H_1(K_5;\mathbb{Z}), \qquad (0,1) \mapsto (0,1,0),\ (1,0) \mapsto (1,0,1), \qquad \diamond$$

where $(1,0)$ represents γ_1 and $(0,1)$ represents γ_2 and the generators of $H_1(K_5;\mathbb{Z})$ are (in that order!) α_1, α_2 and α_3.

Hence α_1 and α_3 are born in K_5, as $(1,0,0),(0,0,1) \notin \operatorname{im}(f_1^{4,5})$.

If we add $K_0 := \{v_0\}$, $K_1 := \{v_0,v_1\}$ and $K_2 := \{v_0,v_1,v_2\}$, we get the following persistence diagrams, where green represents H_0 and blue represents H_1.

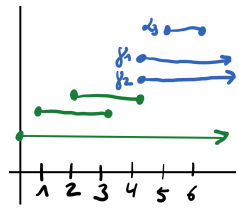

Fig. 110

9.4 Vietoris-Rips complexes

02.02.2022

Let us talk about different filtrations. We saw two ones already - the Čech filtration and the filtration we exemplified with the triangle. The following is the most important one for applications.

First, let $G=(V,E)$ be a finite simple graph with vertex set V and edge set E.

Definition 9.4.1 (Clique complex)
The clique complex of G is the following simplicial complex on V:

$$C(G) := \{\sigma \subset V : \forall u,v \in \sigma \text{ with } u \neq v : \{u,v\} \in E\}.$$

clique complex

Remark 9.4.2 Clique complexes are sometimes called flag simplicial complexes. A simplicial complex has the flag property if it arises as the clique complex of some graph. This is an important subclass of simplicial complexes ○

In graph theory, a clique is a subset $\sigma \subset V$ such that any two points in σ are connected by an edge.

Example 9.4.3 The clique complex of (V, E) contains both V and E as 0 and 1-dimensional faces. Its two-dimensional faces are the triangles as in a triangle each vertex is connected to each other vertex, whereas for a quadrangle this is not the case. ◇

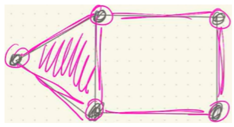

Fig. 111: A clique complex of a graph on five nodes.

We start out with a finite point set $S \subset \mathbb{R}^d$ and a radius $r \geqslant 0$. Consider the 1-dimensional simplicial complex $G(r) := \check{\text{C}}\text{ECH}(r)^{\leqslant 1}$ as a graph.

DEFINITION 9.4.4 (VIETORIS-RIPS COMPLEX)
The VIETORIS-RIPS complex is $\text{VR}(r) := C(G(r))$

VIETORIS-RIPS complex

This yields a filtration by choosing radii r.

The VIETORIS-RIPS complex is more commonly used compared to the ČECH complex because it is easy to compute as we only need the graph and filling in the triangles and simplices of higher dimensions is fairly easy. On the other hand, the ČECH complex was motivated via the nerve theorem: the ČECH complex is topologically well-explained because it encodes the topology of a union of disks of a common radius.

How is the VIETORIS-RIPS complex related to this particular situation? By construction, $\check{\text{C}}\text{ECH}(r) \leqslant \text{VR}(r)$ - the clique complex is the largest simplicial complex with a given 1-skeleton.

We wish to have a reverse relation $\check{\text{C}}\text{ECH}(r) \leqslant \text{VR}(r) \leqslant ?$ to sandwich $\text{VR}(r)$.

Lemma 9.4.5 (VIETORIS-RIPS lemma)
We have $\text{VR}(r) \leqslant \check{\text{C}}\text{ECH}(\sqrt{2}r)$.

Note that the factor $\sqrt{2}$ is independent of the ambient dimension.

Proof. We use barycentric coordinates. This means that it suffices to study a regular simplex and then the rest goes over via an affine map. This is pretty much the same as when we talked about barycentric subdivision and the mesh.

Remember that for the topological invariance of the homology groups, for instance, we invoked simplicial approximation, which rests on being able to change the combinatorics of a geometric simplicial complex by making the simplices arbitrarily small and not changing the topology. One way to achieve this is by barycentric subdivision.

This is a simplex in dimension d. It suffices to look at a regular simplex. This has vertices $100\ldots$, $010\ldots$ and $001\ldots$. The midpoint is $\frac{1}{d}11\ldots1$ and the side length is $\sqrt{2}$, because the edge lie in a two-dimensional plane.

The VIETORIS-RIPS complex and the ČECH complex have a common 1-skeleton. We are interested in the situation where we have circles that intersect tangentially, which can be achieved by considering circles centred at the vertices with radius $\frac{\sqrt{2}}{2}$. The distance of a vertex to the midpoint of the triangle is $\frac{1}{d}\sqrt{(d-1)d} < 1$.

This means that if we multiply all edge lengths by $\sqrt{2}$, then the distance of a vertex to the midpoint of an edge will be $\frac{\sqrt{2}}{2} \cdot \sqrt{2} = 1$, which is larger than the distance of a vertex to the midpoint of the triangle.

Hence if we make the circles larger by a factor of $\sqrt{2}$, they contain the midpoint. So then we see everything in ČECH complex with radius scaled by $\sqrt{2}$, which contains $\text{VR}(r)$. □

In summary, the VIETORIS-RIPS complex is almost the same as the ČECH filtration and thus is well-defined topologically, but it has advantages when it comes to computations.

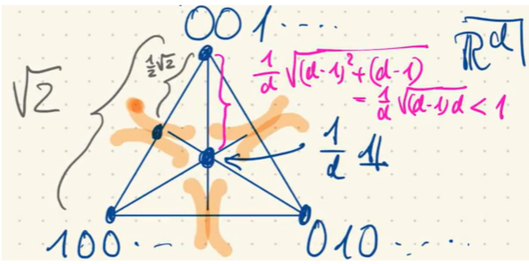
001 . . . .
$\mathbb{R}^d$
$\sqrt{2}$
$\frac{1}{2}\sqrt{2}$
$\frac{1}{d}\sqrt{(d-1)^2+(d-1)}$
$= \frac{1}{d}\sqrt{(d-1)d} < 1$
$\frac{1}{d}\mathbb{1}$
100 . . .
010 . . . .

Fig. 112

10 Combinatorial topology

10.1 Collapsability

Let K be a finite abstract simplicial complex K and $\tau, \sigma \in K$ be are faces such that $\tau \lessdot \sigma$, that is, σ is a facet of τ, it has exactly one dimension less: $\dim(\tau) = \dim(\sigma) - 1$.

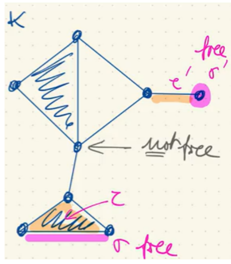

Fig. 113: An abstract simplicial complex K with two regular pairs and one non-free face indicated (there are more).

Definition 10.1.1 (Free face, regular pair)
A face $\sigma \in K$ is free if there is a unique $\tau \in K$ such that $\sigma \lessdot \tau$. In that case, (σ, τ) is a regular pair of K.

If (τ, σ) is a regular pair, then $K \backslash \{\tau, \sigma\}$ is again a simplicial complex due to the uniqueness of the larger face τ ("nobody else is missing σ other than τ").

Definition 10.1.2 (Elementary collapse)
The complex $K \backslash \{\tau, \sigma\}$ is the complex obtained from K by an elementary collapse.

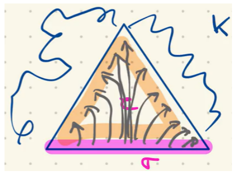

Fig. 114: The triangle shown is one small part of a simplicial complex K. The pink and beige faces are a regular pair and the grey arrows indicate the collapse, which is even a deformation retract.

Lemma 10.1.3 ($K \backslash \{\sigma, \tau\} \simeq K$)
If (σ, τ) is a regular pair of K, then complex obtained from K by an elementary collapse $K \backslash \{\sigma, \tau\}$ is homotopy equivalent to K.

Proof. As shown in figure 114. □

This easy combinatorial operation, if it is possible, simplifies the complex because it reduces the number of faces but retains the topological information because it keeps the homotopy type. For example, fundamental and homology groups are not changed by this operation.

We want to define a sequence of simplicial complexes beginning with $K_0 := K$. Let us assume that K has a regular pair (σ_1, τ_1). Let $K_1 := K_0 \backslash \{\sigma_1, \tau_1\}$. Assume K_1 has a regular pair (σ_2, τ_2). Then define $K_2 := K_1 \backslash \{\sigma_2, \tau_2\}$. We continue this procedure until we end up with a single point K_ℓ. By lemma 10.1.3, $K = K_0 \simeq K_1 \simeq K_2 \simeq \ldots \simeq K_\ell$, so K is homotopy equivalent to a point, that is, contractible.

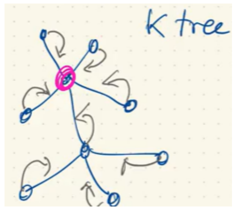

Fig. 115: We can first "collapse" the leaves, because a leaf and its edge are a regular pair. Removing those pairs retains the tree property. Hence we can continue this process until the tree is collapse into the pink vertex.

Definition 10.1.4 (Collapsible)
Contractible spaces which have such a sequence of regular pairs are collapsible.

Hence a collapsible complex is contractible.

Example 10.1.5 Every tree K like in figure 115 is a collapsible 1-dimensional complex. ◇

The one-dimensional simplicial complexes are graphs. As soon as we have a cycle we have non-trivial homology, so the trees are exactly the collapsible 1-dimensional complexes.

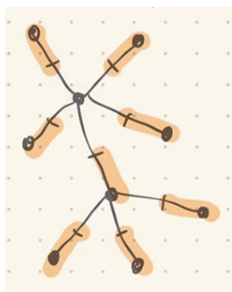

Fig. 116: The regular

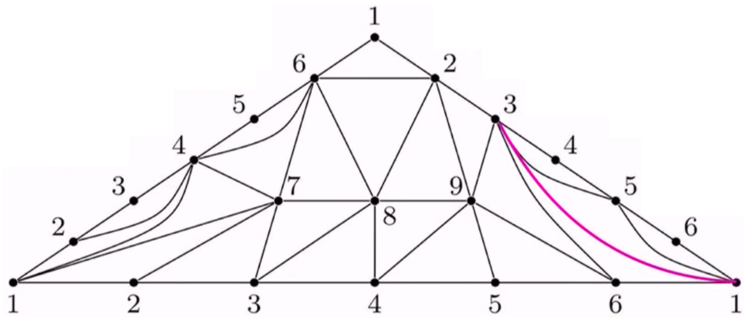

Fig. 117: This is a two-dimensional simplicial complex on nine vertices, which is a triangulation of the dunce hat. It is contractible, but not collapsible: it it were, we would need a sequence of regular pairs. But there is no free face. One can even show that no triangulation of the dunce hat is collapsible.

The above figure also reveals $\chi(\text{Dunce hat}) = 9 - 30 + 22 = 1$.

Example 10.1.6 The MÖBIUS strip is not collapsible, because it retracts to $\mathbb{S}^1$, which is not contractible and thus not collapsible. The spheres $\mathbb{S}^n$, $n \geqslant 1$ are not contractible, as are the orientable surfaces of genus $g \geqslant 1$. ◇

10.2 CW complexes

07.02.2022

We had simplicial complexes as a combinatorial model to describe certain topological spaces. We saw that this is sometimes very tedious because we need very many cells if the space gets complicated. In a way, these CW complexes are a more efficient generalisation, because we can describe more complicated spaces with fewer cells.

DEFINITION 10.2.1 (FINITE CW COMPLEX)
Let $K^{(0)}$ be a finite nonempty set of points (discrete topology). Inductively assume that $K^{(n-1)}$ is given. Pick n-dimensional balls ("cells") $\mathbb{B}_1^n, \dots, \mathbb{B}_\ell^n$ (disjoint copies) and ("attaching") maps $f_k \colon \partial \mathbb{B}_k^n \approx \mathbb{S}^{n-1} \to K^{(n-1)}$ for $k \in \{1, \dots, \ell\}$. Define $K^{(n)} := K^{(n-1)} \cup_f Y$, where $Y := \bigsqcup_{k=1}^{\ell} \mathbb{B}_k^n$. This is a finite CW complex of dimension n if $\ell \geqslant 1$. CW complex

CW means "closure-weak", as the attaching construction is called "weak topology" on the disjoint set of balls of various dimension.

By construction the CW complexes shown above have finitely many cells, but there is a version for infinitely many cells, but we don't need that.

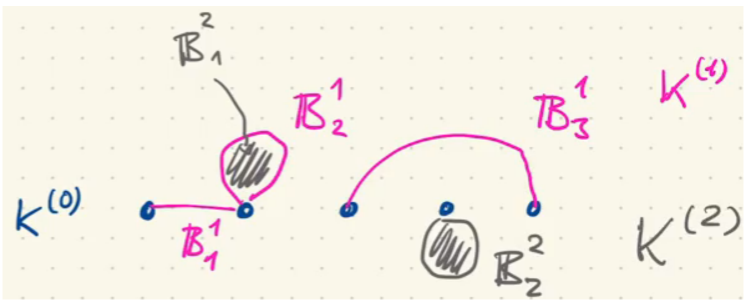

Fig. 118: A CW complex of dimension two. $\mathbb{B}_2^2$ being attached to a 0-cell means that this point sits on $\partial \mathbb{B}_2^2 \approx \mathbb{S}^2$.

It is important to be able to skip dimensions and we will see that in an examples.

Example 10.2.2 Each finite simplicial complex may be viewed as a CW complex: the vertices are $K^{(0)}$, then glue in the edges and so on. The attaching maps are just the usual boundary maps you get from simplices, which are trivial in this case.

In this sense, CW complexes generalise simplicial complexes. ◇

In order to see why this is useful, we can construct spheres as a CW complex.

Example 10.2.3 (CW complex for $\mathbb{S}^2$) The n-sphere $\mathbb{S}^n$ can be constructed as a single point glued with the constant map $f\colon \partial\mathbb{B}^n \approx \mathbb{S}^{n-1} \to \{\bullet\}$ to a ball $\mathbb{B}^n$: $\mathbb{S}^n \approx \{\bullet\} \cup_f \mathbb{B}^n$. This is essentially the one-point compactification of $\mathring{\mathbb{B}}^n$.

This only requires one 0-dimensional cell and one n-cell, while the simplicial complex description of $\mathbb{S}^n$ as the boundary of a $(n+1)$-dimensional simplex has $n+2$ vertices and all together 2^{n+1} many faces of all dimensions, so this is way more than the above description.

In this sense, CW complexes are much more parsimonious and simple, the price to pay being that the attaching maps are more complicated. ◇

Example 10.2.4 (Less parsimonious CW complex for $\mathbb{S}^d$)
We can glue $\mathbb{S}^{d-1}$ to a point and then glue the two hemispheres $\mathbb{B}^n$ to this sphere to obtain $\mathbb{S}^d$ with the f-vector $(1, 0, \ldots, 0, 1, 2)$. ◇

Example 10.2.5 (CW complex for $\mathbb{R}\mathrm{P}^2$)
We can realise $\mathbb{R}\mathrm{P}^2$ as $\mathbb{R}^3 \setminus \{0\}$ modulo (non-zero) scalar multiples, that is,

$$\mathbb{R}\mathrm{P}^2 = \{[x, y, z]_\sim : (x, y, z) \in \mathbb{R}^3 \setminus \{0\},$$

where $(x, y, z) \sim (x', y', z')$ if there exists a $\lambda \in \mathbb{R}\setminus\{0\}$ such that $\lambda(x, y, z) = (x', y', z')$. For any representative, we write homogeneous coordinates $(x : y : z) \in [x, y, z]_\sim$.

Consider the two subspaces of $\mathbb{R}\mathrm{P}^2$, $p := (1 : 0 : 0)$ and a real projective line $L := \{(x : y : 0) : (x, y) \in \mathbb{R}^2 \setminus \{0\}\} \approx \mathbb{R}\mathrm{P}^1 \approx \mathbb{S}^1$.

An encoding of $\mathbb{R}\mathrm{P}^2$ as a CW complex is

$$\mathbb{R}\mathrm{P}^2 = \{p\} \sqcup L\setminus\{p\} \sqcup (\mathbb{R}\mathrm{P}^2 \setminus L)$$

as a disjoint union of a 0-cell, a circle without a point (1-cell) and the interior of a 2-dimensional ball.

As we didn't skip any dimensions (this is a regular CW complex), we can collect the number of k-cells in an alternative f-vector $f = (1, 1, 1)$, whose alternating sum equals the Euler characteristic: $1 - 1 + 1 = 1 = \chi(\mathbb{R}\mathrm{P}^2)$. ◇

Example 10.2.6 (CW complex for $\mathbb{S}^1 \times \mathbb{S}^1$)
The oriented surface of genus 1 is $\mathbb{S}^1 \times \mathbb{S}^1$ (up to homeomorphy). It can be realised as a CW complex with one 0-cell, two 1-cells and one 2-cell, which one can "read off" from the representation as the quotient of a square. Again, we get $\chi(\mathbb{S}^1 \times \mathbb{S}^1) = 0 = 1 - 2 + 1$.

[todo: picture] ◇

Remark 10.2.7 (CW homology)
For CW complexes one can also define homology groups. The only difference is the precise definition of the boundary maps. As before, C_k is the free module of k-cells and $H_k =$ im/ker. ○

10.3 Discrete Morse Theory

This is a fairly recent idea by Robin Forman (1998). The purpose of this section is to relate the topics of CW complexes and simplicial complexes in a systematic way and to relate it to the computation of homology. In short, Morse matchings are a clever way to calculate homology.

Let K be a finite simplicial complex.

Definition 10.3.1 (Hasse diagram)
The Hasse diagram $H(K)$ is a digraph with nodes corresponding to the faces of K receives directed edges (or: arcs) from the covering relations of inclusion: if σ and τ are faces, then (σ, τ) is an arc if $\sigma \lessdot \tau$.

We could also orient all edges the other way around, this is just the usual convention.

Morse matching

Definition 10.3.2 (Morse matching)
A matching μ in $H(K)$ is a Morse matching if reversing the arcs in μ does not produce a directed cycle.

If $(\sigma, \tau) \in \mu$, it is a regular pair. If $\sigma \in K$ is a face which does not occur in any arc of μ, then it is critical.

Example 10.3.3 (Empty matching is a Morse matching)
Let K be an arbitrary simplicial complex and $\mu = \varnothing$. The Hasse diagram $H(K)$ does not contain directed cycles, so the empty matching μ is a Morse matching. All faces are critical. ◇

A matching in a graph $G = (V, E)$ is a set of pairwise non-adjacent edges, none of which are loops; that is, no two edges share common vertices.

This example shows that having a Morse matching (or: discrete Morse function) on a simplicial complex itself does not provide any extra information. An interesting Morse matching should have as few critical cells as possible.

Example 10.3.4
Let K be collapsible. Then there exists a collapsing sequence to a point (by regular pairs), e.g. starting with the free face 234 and choosing the regular face $(23, 234)$ and in the next steps choosing $(12, 124)$, $(1, 14)$, $(2, 24)$ and $(3, 34)$.

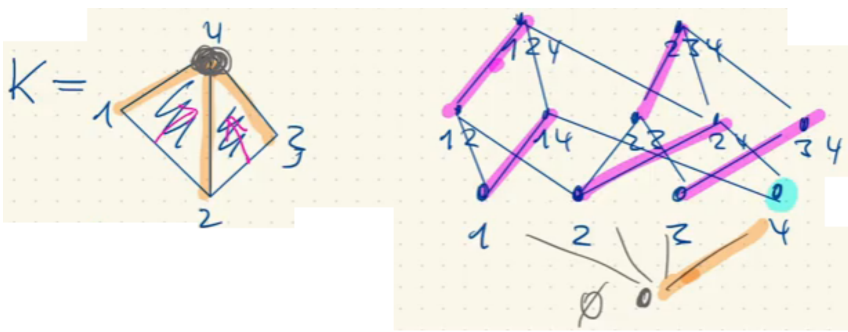

Fig. 119: A two-dimensional simplicial complex K (left) and its Hasse diagram $H(K)$ (right). The pink arrows show the first two collapsing steps and the yellow spanning tree is the resulting reduced complex. The pink edges in the Hasse diagram represent the regular pairs and the only point not contained in any is the one corresponding to the vertex 4, to which the complex is being collapsed to. It is critical. Including the empty set in the Hasse diagram, this collapsing strategy gives rise to a perfect matching.

We end with a few results concerning Morse matchings by Robin Forman which relate classical notions of graph theory - matchings in finite graphs - with classical questions in topology [7].

A perfect matching occurs if every vertex of the graph is incident to an edge of the matching.

Let K be a simplicial complex with a Morse matching μ. Define c_k to be the number of critical k-cells.

Theorem 10.3.1: [7, Thm. 1.5]

K is homotopy equivalent to a CW complex with c_k k-cells for all k.

Proof. Contract the regular pairs. What remains is a CW complex (you can't do this in the realm of simplicial complexes) with exactly c_k k-faces. □

Theorem 10.3.2: Strong Morse inequality [7, Thm. 1.8]

For any $k \geqslant 0$, we have

$$c_k - c_{k-1} + c_{k-2} + \ldots \pm c_0 \geqslant \beta_k - \beta_{k-1} + \beta_{k-2} + \ldots \pm \beta_0,$$

where β_k is the k-Betti number with respect to any field.

This is e.g. a relationship to rational homology.

Note that collapsibility is characterised by $\sum_{k=0}^{d} c_k = 1$. One sphere theorem is the following.

Theorem 10.3.3: Sphere theorem

If $\sum_{k=0}^{d} c_k = 2$ (the Morse matching has exactly two critical cells), then $K \simeq \mathbb{S}^n$, where n is the component which is non-negative (except $c_0 \geqslant 1$).

Proof. See [7, Thm. 8.2, p. 26]. □

10.4 Combinatorial manifolds

08.02.2022

We have been discussing abstract simplicial complexes and CW complexes. The examples mostly dealt with surfaces and rather simple complexes. In this section we talk about higher dimensional spaces, which are interesting.

Let us look at a specific type of simplicial sphere (triangulation of the sphere).

Definition 10.4.1 (PL-d-sphere)
Let S be a triangulation of $\mathbb{S}^d$. Then S is a piecewise-linear d-sphere (PL d-sphere) if S is piecewise linearly homeomorphic to $\partial\Delta^{d+1}$.

A simplicial sphere S is a PL-d-sphere if and only if there exist subdivisions of S and $\partial\Delta^{d+1}$, which are simplicially isomorphic.

Remark 10.4.2 This is a rather strong notion. It is not straight forward to give an example that shows why this is a necessary assumption - it is not easy to write down a non-PL triangulation of the sphere. ○

Example 10.4.3 (PL d-sphere)
Let P be an abstract simplicial $(d+1)$-polytope (the convex hull of finitely many points in $\mathbb{R}^{d+1}$ in affinely independent position). Then the boundary complex ∂P is a PL d-sphere. ◇

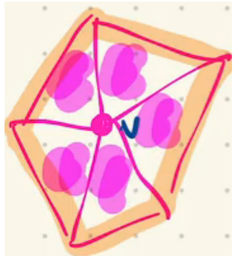

Fig. 120: If σ is a vertex in a surface, then its star are the pink triangles and their non-yellow edges and its link are the yellow edges.

Definition 10.4.4 (Star, link of a face)
Let K be a simplicial complex and $\sigma \in K$ a face. The star of σ with respect to K is

$$\operatorname{star}_K(\sigma) := \langle \tau \in K : \sigma \subset \tau \rangle \leqslant K,$$

where $\langle X \rangle$ denotes the smallest subcomplex containing X.

The link of σ with respect to K is

$$\operatorname{link}_K(\sigma) := \langle \tau \backslash \sigma : \sigma \subset \tau \rangle \leqslant \operatorname{star}_K(\sigma).$$

Definition 10.4.5 (Combinatorial manifold)
A simplicial complex K is a combinatorial d-manifold if link $_K(\sigma)$ is a PL $(d-k-1)$-sphere for each k-face $\sigma \in K$.

combinatorial d-manifold

Example 10.4.6 (Combinatorial manifold) The six vertex triangulation of $\mathbb{RP}^1$ is a combinatorial manifold: the edges are each contained in two triangles, so their links are 0-spheres and the links of vertices have distinct vertices, so they are circles.

In general, any triangulated surface is a combinatorial manifold. ◇

Counterexample 10.4.7 (Combinatorial manifold)
The dunce hat D is not a combinatorial manifold because $\operatorname{link}_D(12) = \{4, 6, 7\} \not\approx \mathbb{S}^{2-1-1}$. ◇

Theorem 10.4.1

If K is a combinatorial manifold, then $|K|$ is a manifold.

Proof. (Sketch for surfaces) Each edge of a surface is contained in exactly two triangles of the triangulation. This means that the link of an edge σ contains two elements and thus $\operatorname{link}_K(\sigma) \approx \mathbb{S}^0 = \mathbb{S}^{2-1-1}$.

We can order the triangles around any vertex v of the surface such that the triangles each share an edge. This closes up a cycle, so $\operatorname{link}_K(v) \approx \mathbb{S}^1 = \mathbb{S}^{2-0-1}$.

We need to prove that each point has a neighbourhood homeomorphic to a d-dimensional ball. The stars of the vertices form neighbourhoods of the vertices. The boundary of the link is homeomorphic to an embedded sphere, so the neighbourhood is a ball.

There are more points in the polyhedron $|K|$ induced by K than vertices in K but we can proceed through the 1-skeleton of K. If a point lies in the relative interior of an edge, its link is a PL sphere so we can in this way construct appropriate neighbourhoods of that point. □

The following result will relate the two new notions - the PL sphere and combinatorial manifold - with the Morse matchings.

Theorem 10.4.2: Whitehead 1939 [13], Forman 1998 [6]

A combinatorial d-manifold is a PL d-sphere if and only if there exists a subdivision with a Morse matching with exactly two critical cells, one 0-cell and one d-cell.

This is a characterisation of PL spheres within the class of combinatorial manifolds.

Remark 10.4.8 (Proof idea) Recall Theorem 10.3.3, which is, in a way, more general. Here, it is just known that often under mild assumptions homotopy among manifolds translates into a homeomorphism, because there are not so many way to properly deform a manifold without violating the manifold property. Hence this theorem is a consequence of Theorem 10.3.3. ○

We are still missing explicit constructions of higher dimensional manifolds.

Example 10.4.9 (Poincaré homology 3-sphere) Consider a 3-polytope - the regular dodecahedron with its vertices labelled. There are six pairs of vertices at the barycentres of the pentagonal faces that also occur on the opposite facet such that the vertices making up the pentagon are labelled cyclically shifted. This should be compared to the surface symbols, where we started with a 2-polytope and identified facets (= edges) on the boundary. This is generalised here.

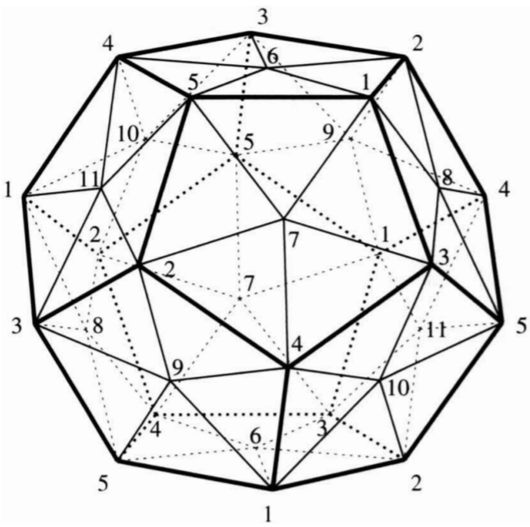

Fig. 121: An $\mathbb{A}_5$-invariant triangulation of the Poincaré homology 3-sphere [2, Fig. 2]. One needs vertices in the interior of the the homology 3-sphere to subdivide between identified vertices in the barycentres of the pentagonal faces, so one ends up with $11 + 12 + 1 = 24$ vertices (the 12 vertices come from the icosahedron placed inside).

This is (easy to check) a triangulation of a 3-manifold, called the Poincaré homology 3-sphere.

This is a combinatorial 3-manifold P. We have

$$H_i(P;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i \in \{0,3\}, \\ \{0\}, & \text{otherwise} \end{cases},$$

The dodecahedron has an automorphism group which is isomorphic to a double cover of the alternating group of degree 5, $\mathbb{A}_5$, which still operates on this triangulation.

so the homology groups of P and $\mathbb{S}^3$ are identical. Why is this not a sphere? Because $\pi_1(P) = \mathbb{A}_5$, so P is not simply connected, while $\mathbb{S}^3$ is.

This does not contradict $H_1(P;\mathbb{Z}) \cong \pi_1(P)^{\mathrm{Ab}}$, as $\mathbb{A}_5' = \mathbb{A}_5$, so $\mathbb{A}_5^{\mathrm{Ab}}$ is trivial. ◇

The suspension is a generalisation of the cone construction.

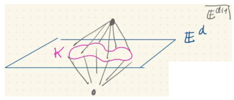

Fig. 122: The suspension of a space.

suspension

Example 10.4.10 (Suspension) Consider an simplicial complex K, whose realisation lies in $\mathbb{E}^d$. Picking two points in $\mathbb{E}^{d+1} \setminus \mathbb{E}^d$ on opposite sides of $\mathbb{E}^d$ and forming the cone of $|K|$ with both of those apices yields the suspension (dt. *Einhängung*) of K, ΣK as the union of two cones. ◇

(More generally, ΣX is the quotient space obtained from $X \times [-1, 1]$ by identifying $X \times \{1\}$ and $X \times \{-1\}$), each.)

Theorem 10.4.3: Canon, 1979 [3]

Let H be a homology d sphere. Then $\Sigma^2 H \simeq \mathbb{S}^{d+2}$, where Σ^2 is the double suspension, the suspension of the suspension.

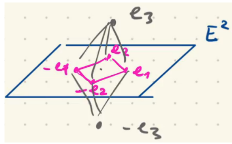

Fig. 123: The suspension of a square.

Example 10.4.11 (Suspension) Consider a quadrangle in the plane containing the origin, i.e. $Q = \text{conv}(\pm e_1, \pm e_2)$. Then $\text{conv}(\Sigma Q)$ is an octahedron, whose boundary is homeomorphic to $\mathbb{S}^2$.

In general, if $K \approx \mathbb{S}^d$, then $\Sigma K \approx \mathbb{S}^{d+1}$. ◇

Instead letting H only be a homology sphere is a much weaker assumption.

Remark 10.4.12 By Theorem 10.4.3 we have $\Sigma^2 P \approx \mathbb{S}^5$, but $\Sigma^2 P$ is not a PL sphere. ○

Index

References

[1] M.A. Armstrong. *Basic Topology.* Springer Science & Business Media, Berlin Heidelberg, 2013.

[2] Anders Björner and Frank H Lutz. Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the poincaré homology 3-sphere. *Experimental Mathematics*, 9(2):275–289, 2000.

[3] James W Cannon. Shrinking cell-like decompositions of manifolds. codimension three. *Annals of Mathematics*, pages 83–112, 1979.

[4] Paulo Henrique Costa, Renato Vianna, and Scott Northrup. Self intersections in immersions of the projective plane. *Proc. of the International Research Experience for Students in Mathematics (IRES)*, 2007.

[5] Herbert Edelsbrunner and John Harer. *Computational topology: an introduction.* American Mathematical Soc., 2010.

[6] Robin Forman. Morse theory for cell complexes. *Advances in mathematics*, 134(1):90–145, 1998.

[7] Robin Forman. A user's guide to discrete morse theory. *Séminaire Lotharingien de Combinatoire [electronic only]*, 48:B48c–35, 2002.

[8] David Gale. The game of hex and the brouwer fixed-point theorem. *The American Mathematical Monthly*, 86(10):818–827, 1979.

[9] S.M. Lane. *Categories for the Working Mathematician.* Graduate Texts in Mathematics. Springer New York, 2013.

[10] Ciprian Manolescu. Pin(2)-equivariant seiberg-witten floer homology and the triangulation conjecture. *Journal of the American Mathematical Society*, 29(1):147–176, 2016.

[11] Joachim H Rubinstein. An algorithm to recognize the 3-sphere. In *Proceedings of the International Congress of Mathematicians*, pages 601–611. Springer, 1995.

[12] Abigail Thompson. Thin position and the recognition problem for S^3. *Mathematical Research Letters*, 1(5):613–630, 1994.

[13] John Henry Constantine Whitehead. Simplicial spaces, nuclei and m-groups. *Proceedings of the London mathematical society*, 2(1):243–327, 1939.

Topological Space	compact	Hausdorff	path-connected	orientable	χ	Fundamental group	Homology
connected simple graph	y	/	y		0		$H_1(\cdot; R) = R^{\lvert E\rvert - \lvert V\rvert + 1}$
convex set	/	y	y			0	$H_k(\cdot; R) = 0,\ k \geqslant 1.$
$(\mathbb{S}^1)^n$	y	y	y	y	0	$\mathbb{Z}^n$	
$\mathbb{S}^n,\ n \geqslant 2$	y	y	y	y	$2 \cdot \mathbb{1}_{n \text{ even}}$	0	$H_k(\mathbb{S}^n; R) = R$ for $k \in \{0, m\}$, $\{0\}$ e
Möbius strip	y	y	y	n	0	$\mathbb{Z}$	$H_0(M; R) = H_1(M; R) \cong R$, else
Dunce hat	y		y	/	1	0	0
CX			y	/		0	$H_q(CX; R) = 0\ q \geqslant 1.$
$\mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1$	y	y	y	y?	$1 - n$	$\mathbb{Z} * \ldots * \mathbb{Z}$	$H_1(\cdot; \mathbb{Z}) = \mathbb{Z}^n$
$\mathbb{R}\mathrm{P}^n,\ n \geqslant 3$ odd	y	y	y	y		$\mathbb{Z}_2$	
$\mathbb{R}\mathrm{P}^n,\ n \geqslant 3$ even	y	y	y	n		$\mathbb{Z}_2$	
$M(p),\ p \geqslant 0$	y	y	y	y	$2 - 2g$	$\left\langle a_1, b_1, \ldots, a_p, b_p : \prod_{k=1}^{p} a_k b_k a_k^{-1} b_k^{-1} = e \right\rangle$	$H_1(M(p); \mathbb{Z}) = \mathbb{Z}^{2g},\ H_2(M(p); \mathbb{Z}) =$
$H(q),\ q \geqslant 1$	y	y	y	n	$2 - g$	$\left\langle a_1, \ldots, a_q : \prod_{k=1}^{q} a_k^2 = e \right\rangle$	$H_1(H(q); \mathbb{Z}) = \mathbb{Z}^{g-1} \times \mathbb{Z}_2,\ H_2(H(q); \mathbb{Z}$

Fig. 124: Various topological spaces together with characteristic properties and topological invariants from the lecture.

Ideologically philosophical trends The numbers give the percentages of modern mathematicians belonging to those trends.

4% **Formalism** (Hilbert, Russell, Whitehead, who wrote "Principia Mathematica", which breaks by Gödel's incompleteness thoerems.)

1% **Intutionism / Constructivism / Finitism.** No AOC, no proofs by contradiction. Prominent example: Brouwer.

95% **Platonism.** Platonisches Höhlengleichnis, Platonische Ideenlehre.