Numeral systems

A numeral system is used to represent numbers of a given set (e.g., integers or rational numbers) using a set of digits or symbols. The decimal numeral system is perhaps the most familiar example of such a system. It is a base-10 positional numeral system with the decimal digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The value of an integral decimal number $c_{n-1}c_{n-2}\cdots c_0$ with n digits may be expressed in the expanded form

$$c_{n-1}\cdot 10^{n-1} + c_{n-2}\cdot 10^{n-2} + \cdots + c_0 \cdot 10^0,$$

i.e., the position of each digit determines its value. For example, the decimal number 492 is equal to

$$4\cdot 10^2 + 9\cdot 10^1 + 2\cdot 10^0.$$

Non-integral values can be expressed by introducing a *decimal separator* "." (aka *decimal point*), and negative values are represented by adding a leading minus sign. For example, a decimal number with n digits before and l digits after the decimal separator may be expressed as

$$s\,(c_{n-1}c_{n-2}\cdots c_0.d_1d_2\cdots d_l),$$

where $s \in \{-1,+1\}$ represents the sign. The corresponding value is

$$s\left(\sum_{j=0}^{n-1} c_j\cdot 10^j + \sum_{j=1}^{l} d_j \cdot 10^{-j}\right).$$

Digits to the left of the decimal separator are multiplied by a nonnegative exponentiation of 10, according to the position of the digit, whereas as digits to the right of the decimal separator are multiplied by a negative exponentiation of 10. Thus, the decimal separator separates the integral part and the fractional part of the number. For integers, it is customary to omit the decimal separator (and the digits to the right of it), and for nonnegative numbers, it is customary to omit the plus sign.

A decimal number with a finite number of digits after the decimal separator is called a *terminating decimal*. Such numbers can be expressed as a decimal fraction $c/10^l$ for some $c \in \mathbb{Z}$ and $l \in \mathbb{N}_0$. In contrast, an *infinite decimal* (aka *non-terminating decimal*) has infinitely many digits after the decimal separator. The decimal representation of a rational number is either a terminating decimal (e.g., $1/4 = 0.25$) or a so-called *recurring decimal* (aka *repeating decimal*). A recurring decimal is an infinite decimal with a group of digits (not all equal to zero) that repeats indefinitely. For example, we have that $1/11 = 0.\overline{09}$ where the "bar" is used to indicate the recurring group of digits. Irrational numbers, on the other hand, are always infinite decimals with a non-repeating sequence of digits. The decimal expansions of a number x is unique if x is not a decimal fraction, and otherwise x has two decimal expansions. For example, the rational number $1/4$ has the two decimal expansions 0.25 and $0.24\overline{9}$.

The decimal numeral system is a special case of a more general family of positional numeral systems in which the *positional notation* for a number x is of the form

$$x = s\,(c_{n-1}c_{n-2}\cdots c_0.d_1d_2\cdots)_b.$$

Here $s \in \{-1,+1\}$ represents the sign, the *radix* (aka *base*) b is an integer greater than or equal to two, and each digit is one of b unique symbols (e.g., the standard set of digits $\{0,\ldots,b-1\}$). For example, the base-2 numeral system (aka the *binary* numeral system) uses the digits 0 and 1, and the base-16 numeral system (aka the *hexadecimal* numeral system) uses the sixteen symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, and f. Note that the base is sometimes omitted if it is obvious from the context. The separator "." is called the *radix point*, and it separates positions with a nonnegative exponent from those with negative exponent, i.e., the expanded form of x is given by

$$x = s\left(\underbrace{\sum_{j=0}^{n-1} c_j \cdot b^j}_{\text{integral part}} + \underbrace{\sum_{j=1}^{\infty} d_j \cdot b^{-j}}_{\text{fractional part}}\right).$$

The representation of a number x in base b is terminating if and only if x can be expressed as c/b^k for some $c \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. The following table shows the decimal, binary, and hexadecimal representations of different rational numbers.

Number	Decimal ($b=10$)	Binary ($b=2$)	Hexadecimal ($b=16$)
18	18 or $17.\overline{9}$	10010 or $10001.\overline{1}$	12 or $11.\overline{\text{f}}$
11	11 or $10.\overline{9}$	1011 or $1010.\overline{1}$	b or $\text{a}.\overline{\text{f}}$
4	4 or $3.\overline{9}$	100 or $11.\overline{1}$	4 or $3.\overline{9}$
1	1 or $0.\overline{9}$	1 or $0.\overline{1}$	1 or $0.\overline{9}$
$1/3$	$0.\overline{3}$	$0.\overline{01}$	$0.\overline{5}$
$1/4$	0.25 or $0.24\overline{9}$	0.01 or $0.00\overline{1}$	0.4 or $0.3\overline{\text{f}}$
$1/10$	0.1 or $0.0\overline{9}$	$0.0\overline{0011}$	$0.1\overline{9}$
$-1/2$	-0.5 or $-0.4\overline{9}$	-0.1 or $-0.0\overline{1}$	-0.8 or $-0.7\overline{\text{f}}$