Module 11 solutions

Exercises

1. Let f_n denote the number of function calls required to compute F_n recursively, i.e.,

$$f_n = f_{n-1} + f_{n-2} + 1, \quad n \geq 2$$

with $f_0 = f_1 = 1$. The number of function calls required to compute the 5th Fibonacci number F_5 is shown in the following table:

n	F_n (nth Fibonacci number)	f_n (function calls)	2^n	$2^{n/2}$
0	0	1	1	1
1	1	1	2	1.414
2	1	3	4	2
3	2	5	8	2.828
4	3	9	16	4
5	5	15	32	5.657
⋮	⋮	⋮	⋮	
k	$F_{k-1} + F_{k-2}$	$f_{k-1} + f_{k-2} + 1$	2^k	$2^{k/2}$

The number of function calls can be verified with the following program:

```c
#include <stdlib.h>
#include <stdio.h>

int ncalls = 0;

unsigned long fibonacci(unsigned long n) {
  ncalls+=1;
  unsigned long Fn;
  if ( n == 0 )
    Fn = 0;
  else if ( n == 1 )
    Fn = 1;
  else
    Fn = fibonacci(n-1) + fibonacci(n-2);
  return Fn;
}

int main(int argc, const char * argv[]) {

  unsigned long n=0;
  if (argc != 2) {
    fprintf(stdout,"Usage: %s n\n",argv[0]);
```

```
    exit(EXIT_SUCCESS);
  }
  n = atoi(argv[1]);
  fprintf(stdout,"fibonacci(%lu) = %lu\n",n,fibonacci(n));
  fprintf(stdout,"ncalls = %d\n",ncalls);

  return EXIT_SUCCESS;
}
```

2. What is the time complexity of this recursive implementation?

   The time complexity can be expressed in terms of function evaluations since the time required for each function evaluation (excluding the time spent on other function calls) can be upper bounded by a constant. It follows from the table and the recursive expression $f_n = f_{n-1} + f_{n-2} + 1$ that $f_n > f_{n-1}$ for $n \geq 2$, and hence

   $$f_n = f_{n-1} + f_{n-2} + 1 \leq 2\, f_{n-1}, \quad n \geq 3.$$

   This implies that the number of function calls is upper bounded by 2^n (since f_n is at most twice as large as f_{n-1}), or equivalently, $f_n = O(2^n)$ which implies that **the cost is** (at most) **exponential**.

   Remark: It is also possible to obtain an exponential lower bound on the number of function evaluations. Indeed, the inequality $f_n > f_{n-1}$ also implies that

   $$f_n = f_{n-1} + f_{n-2} + 1 \geq 2 f_{n-2}, \quad n \geq 3,$$

   which means that the number of function calls *at least* doubles when increasing n by two. Thus, as can be verified from the table, the exponential sequence $2^{n/2}$ is a lower-bound for the sequence f_n. The so-called "Big Theta" notation may be used instead of the "Big O" notation to indicate that our function is not only bounded above asymptotically, it is also bounded below: we have $f_n = \Theta(2^n)$ which means that the asymptotic growth rate is exponential.

3. What is the space complexity of the recursive implementation of the Fibonacci function?

   Although the time complexity is exponential, the space complexity is only *linear*. This is because the *depth* of the recursion is linear in n. This can be verified with the following program:

```
#include <stdlib.h>
#include <stdio.h>

int depth = 0;
int max_depth = 0;

unsigned long fibonacci(unsigned long n) {
    unsigned long Fn;
    depth += 1;
    max_depth = depth > max_depth ? depth : max_depth;
    if ( n == 0 ) Fn = 0;
    else if ( n == 1 ) Fn = 1;
```

```
    else Fn = fibonacci(n-1) + fibonacci(n-2);
    depth -= 1;
    return Fn;
}

int main(int argc, char * argv[]) {

  unsigned long n;
  if ( argc != 2 ) {
    fprintf(stderr, "Usage: %s n\n", argv[0]);
    exit(EXIT_FAILURE);
  }
  n = atoi(argv[1]);
  printf("fibonacci(%lu) = %lu\n", n, fibonacci(n));
  printf("Recursion depth(%lu): %d\n", n, max_depth);

  return 0;
}
```

Compile and run the program for different values of n. It is easy to verify that the maximum number of simultaneous calls on the stack is equal to n.

4. Rewrite the Fibonacci function so that you avoid recursive function calls. The return type should be an unsigned long. What is the time complexity of your implementation? What is the space complexity?

   The recursive function calls can be avoided by computing the Fibonacci numbers sequentially:

```
unsigned long fib_sequential(unsigned long n) {
    unsigned long f, fm = 1, fmm = 0;
    if ( n == 0 ) return 0;
    else if ( n == 1 || n == 2 ) return 1;
    for (int i=2;i<=n;i++) {
        f = fm + fmm;
        fmm = fm;
        fm = f;
    }
    return f;
}
```

   This implementation has time complexity $O(n)$ (i.e., linear time complexity; this is because of the loop) and the space complexity is $O(1)$ (it does not depend on n).

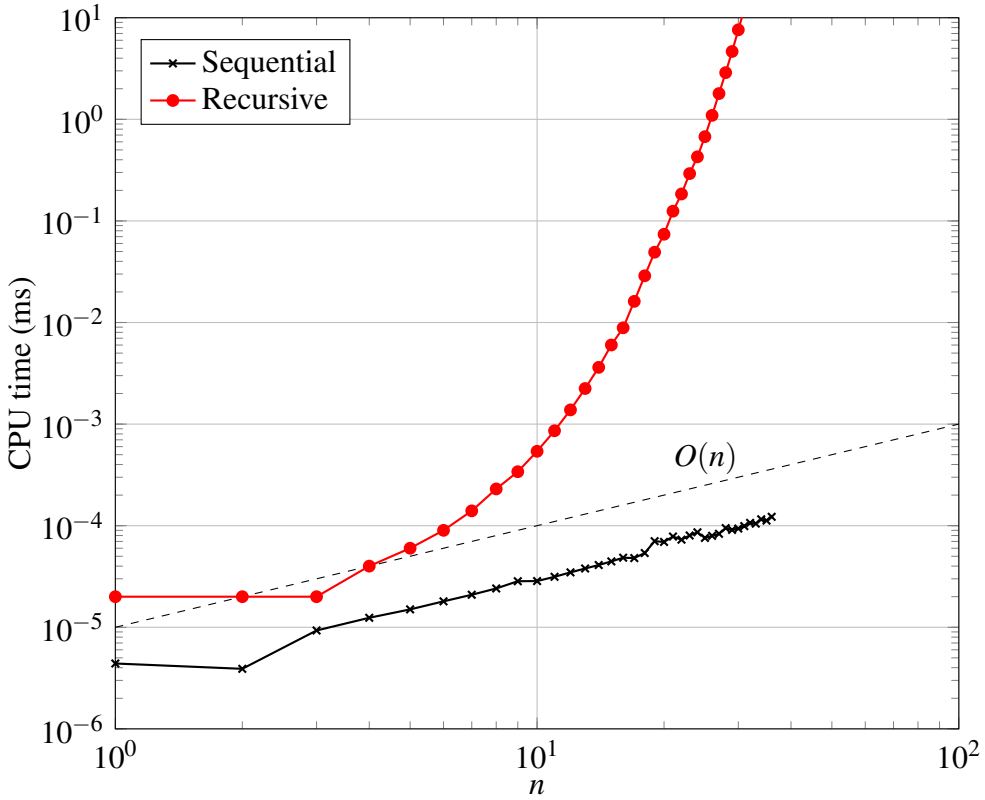
   Of course, it is also possible to use the definition

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

   to compute the nth Fibonacci number without the recursion. The complexity of this approach depends on how the term φ^n is computed.

5. Write a program that measures the CPU time for the recursive variant of the Fibonacci function and your modified version for different values of n.

```c
#include <stdio.h>
#include <time.h>

unsigned long fib_sequential(unsigned long n);
unsigned long fib_recursive(unsigned long n);

#define NREPEAT_S 10000
#define NREPEAT_R 100
#define mytimer clock

/* Time difference in milliseconds */
#define delta_t(a,b) (1.0e3*((b)-(a))/CLOCKS_PER_SEC)

int main(void) {

  clock_t t1,t2;
  double Ts,Tr;
  unsigned long fn;

  for (int n=1;n<=36; n+=1) {
    // Measure time for sequential version
    t1 = mytimer();
    for (int k=0;k<NREPEAT_S;k++)
        fn = fib_sequential(n);
    t2 = mytimer();
    Ts = delta_t(t1,t2)/NREPEAT_S;

    // Measure time for recursive version
    t1 = mytimer();
    for (int k=0;k<NREPEAT_R;k++)
        fn = fib_recursive(n);
    t2 = mytimer();
    Tr = delta_t(t1,t2)/NREPEAT_R;

    // Print results (time in ms)
    printf("%2d %.3e %.3e\n",n,Ts,Tr);
  }

  return 0;
}
```



6. Find the largest Fibonacci number that can be represented as an unsigned long?

   The largest unsigned b-bit integer is $2^b - 1$. If we approximate the nth Fibonacci number by

   $$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \approx \frac{\varphi^n}{\sqrt{5}},$$

   we obtain the following (approximate) equation

   $$\frac{\varphi^n}{\sqrt{5}} \approx 2^b.$$

   Solving for n and rounding down yields

   $$n \approx \left\lfloor \frac{b\log(2) + \log(5)/2}{\log(\varphi)} \right\rfloor.$$

   Thus, on systems where an `unsigned long` consists of 4 bytes (32 bits), we obtain $n \approx 47$, and on systems where an `unsigned long` is 8 bytes, we get $n \approx 93$. Indeed, the 93rd Fibonacci number is 12,200,160,415,121,876,738 whereas the 94th Fibonacci number is 19,740,274,219,868,223,167 which is larger than $2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$. This can be verified with the non-recursive implementation from one of the previous exercises.