22.1 Theory and examples

After a very elementary chapter about extremal properties of graphs, it is time to see how the study of their cycles can give valuable information in combinatorial problems. We will assume in this chapter some familiarity with basic concepts of graph theory that can be found in practically any book of combinatorics. We prefer to do so, because recalling all definitions would require a large digression and would largely diminish the quantity of examples presented. And since the topic is very subtle and the problems are in general difficult, we think it is better to present many examples. We would like to thank Adrian Zahariuc for the large quantity of interesting results and solutions that he communicated to us.

We start with a simple, but important result. It was extended by Erdős in a much more difficult to prove statement: if the number of edges of a graph on n vertices is at least $\frac{(n-1)k}{2}$ then there exists a cycle of length at least $k+1$ (if $k>1$). Let us remain modest and prove the following much easier result :

Example 1. In a graph G with n vertices, every vertex has degree at least k. Prove that G has a cycle of length at least $k+1$.

Solution. The shortest solution uses the extremal principle. Indeed, consider the longest chain $x_0, x_1, ..., x_r$ in G and observe that this maximality property ensures that all vertices adjacent to x_0 are in this longest chain. Or, the degree of x_0 being at least k, we deduce that there exists a vertex x_i adjacent to x_0 such that $k \le i \le r$. Therefore $x_0, x_1, ..., x_i, x_0$ is a cycle of length at least $k+1$.

Any graph with n vertices and at least n edges must have a cycle. The following problem is an easy application of this fact:

Example 2. Suppose $2n$ points of an $n \times n$ grid are marked. Prove that there exists a $k > 1$ and $2k$ distinct marked points $a_1, a_2, ..., a_{2k}$ such that for all i, a_{2i-1} and a_{2i} are in the same row, while a_{2i} and a_{2i+1} are in the same column.

IMC 1999

Solution. Here it is not difficult to discover the graph to work on. It is enough to look at the n lines and n columns as the two classes of a bipartite graph. We connect two vertices if the intersection of the corresponding row and column is marked. Clearly, this graph has $2n$ vertices and $2n$ edges, so there must exist a cycle. But the existence of a cycle is equivalent (by the definition of the graph) to the conclusion of the problem.

The following example is an extremal problem in graph theory, of the same kind as Turan's theorem. This type of problem can go from easy or even trivial to extremely complex and complicated results. Of course, we will discuss just the first type of problem.

Example 3. Prove that every graph on $n \geq 4$ vertices and $m > \frac{n+n\sqrt{4n-3}}{4}$ edges has at least one 4-cycle.

Solution. Let us count, in two different ways, the number of triples (c, a, b) where a, b, c are vertices such that c is connected to both a and b. For a fixed vertex c, there are $d(c)^2 - d(c)$ possibilities for the pair (a, b), where $d(c)$ denotes the valence of c. It follows that there are at least $\sum\limits_c (d(c)^2 - d(c))$ triples. By the Cauchy-Schwarz inequality, if m represents the number of edges of the graph, then

$$\sum_c d(c)^2 - d(c) \geq \frac{4m^2}{n} - 2m \tag{22.1}$$

Now, if there are no 4-cycles, then for fixed a and b there is at most one vertex c that appears in a triple (a, b, c). Hence we obtain at most $n(n-1)$ triples. It follows that $\frac{4m^2}{n} - 2m \leq n^2 - n$, which implies that $m \leq \frac{n+n\sqrt{4n-3}}{4}$, a contradiction.

Recall that a graph in which every vertex has degree 2 is a disjoint union of cycles. It turns out that this very innocent observation is more than helpful in some quite challenging problems. Here are some examples, taken from different contests:

Example 4. A company wants to build a 2001×2001 building with doors connecting pairs of adjacent rooms (which are 1×1 squares, two rooms being adjacent if they have a common edge). Is it possible for every room to have exactly 2 doors?

[Gabriel Caroll]

Solution. Let us analyze the situation in terms of graphs: suppose such a situation is possible, and consider the graph G with vertices representing the rooms and connecting two rooms if there exists a door between them. Then the hypothesis says that the degree of any vertex is 2. Thus G is a union of disjoint cycles $C_1, C_2, \dots, C_p$. However, observe that any cycle has even length, because the number of vertical steps is the same in both directions and the same holds for horizontal steps. Therefore the number of vertices of G, which is the sum of lengths of these cycles, is an even number, a contradiction.

Reading the solution to the following problem, one might say that it is extremely easy: there is no tricky idea behind it. But there there are many possible approaches that can fail, and this probably explains its presence on the list of problems proposed for the IMO 1990.

Example 5. Let E be a set of $2n-1$ points on a circle, with $n > 2$. Suppose that precisely k points of E are colored black. We say that this coloring is admissible if there is at least one pair of black points such that the interior of one of the arcs they determine contains exactly n points of E. What is the smallest k such that any coloring of k points of E is admissible?

IMO 1990

Solution. Consider G the graph having vertices the black points of E and join two points x, y by an edge if there are n points of E on one of the two open arcs determined by x and y. Thus the problem becomes: what is the least k such that among any k vertices of this graph at least two are adjacent? The problem becomes much easier with this statement, because of the fact that the degree of any vertex in G is clearly 2, thus G is a union of disjoint cycles. It is clear that for a single cycle of length r, the least value of k is $1 + \left\lfloor \frac{r}{2} \right\rfloor$. Now, observe that if $2n-1$ is not a multiple of 3 then G is actually a cycle (because $(\gcd(n+1, 2n-1) = 1)$, while in the other case G is the union of three disjoint cycles of length $\frac{2n-1}{3}$. Therefore the least k is $n = \left\lfloor \frac{2n-1}{2} \right\rfloor + 1$ if $2n-1$ is not a multiple of 3 and $n-1 = 3\left\lfloor \frac{2n-1}{6} \right\rfloor + 1$ otherwise.

Finally, a more involved example using the same idea, but with some complication which are far from obvious.

Example 6. Consider in the plane the rectangle with vertices $(0,0), (m,0)$ $(0,n), (m,n)$, where m and n are odd positive integers. Partition it rectangle into triangles satisfying the following conditions: 1) Each triangle has at least one side (called the good side; the sides that are not good will be called bad) on a line $x = j$ or $y = k$ for some nonnegative integers j, k, such that the height corresponding to that side has length 1; 2) Each bad side is common for two triangles of the partition. Prove that there are at least two triangles having two good sides each.

IMO 1990 Shortlist

Solution. Let us define a graph G having as vertices the midpoints of the bad sides and as edges the segments connecting the midpoints of two bad sides in a triangle of the partition. Thus, any edge is parallel to one of the sides of the rectangle, being at distance $k + \frac{1}{2}$ from the sides of the rectangle, for a suitable integer k. Also, it is clear that any vertex has degree at most 2, so we have three cases. The easiest is when there exists an isolated vertex. Then the triangles that have the side containing that vertex as common side have two good sides. Another easy case is when there exists a vertex x having degree 1. Then x is the end of a polygonal line formed by edges of the graph, and having the other end a point y, which is the midpoint of a side in a triangle having two good sides. The conclusion follows in this case, too. Thus, it remains to cover the "difficult" case when all vertices have degree 2. Actually, we will show that this case is impossible. Observe that until now we haven't used the hypothesis that m, n are odd. This suggests looking at the cycles of G. Indeed, we know that G is a union of disjoint cycles. If we manage to prove that the number of squares traversed by any cycle is even, it would follow that the table has an even number of unit squares, which is impossible, because mn is odd. Divide first the rectangle by its lattice points into mn unit squares. So, fix a cycle and observe that from the hypothesis it follows that the center of any square is contained in only one cycle. Now, by alternatively coloring the cells of the rectangle with white and black, we obtain a chessboard in which every cycle passes alternatively on white and black squares, so it passes through an even number of squares. This proves the claim and shows that G cannot have all vertices of degree 2.

The next problem is already unobvious, and the solution is not immediate, because it requires two arguments which are completely different: a construction and a proof of optimality. Starting with some special cases is often the best way to proceed, and this is indeed the key here.

Example 7. Let n be a positive integer. Suppose that n airline companies offer trips to citizens of N cities such that for any two cities there exists a direct flight in both directions. Find the least N such that we can always find a company which can offer a trip in a cycle with an odd number of landing points.

Adapted after IMO 1983 Shortlist

Solution. By starting with small values of n, we can guess the answer: $N = 2^n + 1$. But it is not obvious how to prove both that for 2^n the assertion in the problem is not always true and the fact that for $2^n + 1$ cities the conclusion always holds. Let us start with the first claim: the result is not always true if we allow only 2^n cities. Indeed, let the cities be $C_0, C_1, ..., C_{2^n-1}$. Write every number smaller than 2^n in base 2 with n digits (we allow zeros in the first positions), and let us join two cities C_i and C_j by a flight offered by an airline company A_1 if the first digit of i and j is different, by a flight offered by A_2 if the first digits are identical, but the second digit differs in the two numbers and so on. Because the i-th digit is alternating in the vertices of a cycle for company A_i, it follows that all cycles realized by A_i are even. Therefore $N \geq 2^n + 1$. Now, we prove by induction that the assertion holds for $N \geq 2^n+1$. For $n = 1$ everything is clear, so assume the result for $n - 1$. Suppose that all cycles in the graph of flights offered by company A_n are even (otherwise we have found our odd cycle). Therefore the graph of flights offered by A_n is bipartite, that is there exists a partition $B_1, B_2, ..., B_m$, $D_1, D_2, ..., D_p$ of the cities such that any flight offered by A_n connects one of the cities B_j with one of the cities D_k. Because $m+p = 2^n+1$, we may assume that $m \geq 2^{n-1}+1$. But then the cities $B_1, B_2, ..., B_m$ are connected only by flights offered by $A_1, A_2, ..., A_{n-1}$, so by the induction hypothesis one of these companies can offer an odd cycle. This finishes the induction step and shows that $N = 2^n + 1$ is the desired number. Here comes a very challenging problem with a very beautiful idea:

Example 8. On an infinite checkerboard are placed 111 non-overlapping corners, L-shaped figures made of 3 unit squares. Suppose that for any corner, the 2×2 square containing it is entirely covered by the corners. Prove that one can remove each number between 1 and 110 of the corners so that the property will be preserved.

St. Petersburg 2000

Solution. We will argue by contradiction. Assuming that by removing any 109 corners the property is no longer preserved, it would follow that no 2×3 rectangle is covered by 2 corners. Now, define the following directed graph with vertices on the corners: for a fixed corner C, draw an edge from it to the

corner that helps covering the 2×2 square containing C. It is clear that if in a certain corner there is no entering edge, we may safely remove that corner, contradiction. Therefore, in every corner there exists an entering edge and so the graph constructed has the property that every edge belongs to some cycle. We will prove that the graph cannot be a cycle of 111 vertices. Define the "center" of a corner as the center of the 2×2 square containing it. The first observation, that no two corners can cover a 2×3 rectangle, shows that in a cycle the x coordinate of the centers of the vertices are alternatively even and odd. Thus the cycle must have an even length, which shows that the graph itself cannot be a cycle. Therefore, it has at least two cycles. But then we may safely remove all the corners except those in a cycle of smallest length and the property will be preserved, thus again a contradiction.
The following result is particularly nice:

Example 9. There are n competitors in a table-tennis contest. Any 2 of them play exactly once against each other and no draws are possible. We know that no matter how we divide them into 2 groups A and B, there is some player from A who defeated some player from B. Prove that at the end of the competition, we can sit all the players at a round table such that everyone defeated his or her right neighbor.

Solution. Clearly, the problem refers to a tournament graph, that is, a directed graph in which any two vertices are connected in exactly one direction. We have to prove that this graph contains a Hamiltonian circuit. Take the longest elementary cycle, $v_1, v_2, \ldots, v_m$ with pairwise distinct vertices, and take some other vertex v. Unless all edges come either out of v or into v, there is some i such that $v_i v$ and $v v_{i+1}$ are edges. Then, $v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_n$ is a longer elementary cycle, contradiction. Therefore, there are only two kinds of vertices $v \in V - \{v_i\}$: (type A) those for which all $v v_i$ are edges; and (type B) those for which all $v_i v$ are edges. If there is some edge ba with a of type A and b of type B, then we can construct once again a longer circuit: $b, a, v_1, \ldots, v_n$. Therefore, for any $a \in A$ and $b \in B$, ab is an edge. Consider the partition $V = B \cup (A \cup \{v_i\})$. Due to the hypothesis, since all edges between the two classes point towards B, we must have $B = \emptyset$. But, once again, $V = A \cup \{v_i\}$

is a forbidden partition, so $A = \emptyset$. Therefore, the circuit is Hamiltonian.

Before discussing the next problem, we need to present a very useful result, which is particularly easy to prove, but has interesting applications. This is why it will be discussed as a separate problem and not as a lemma:

Example 10. Prove that a graph is bipartite if and only if all of its cycles have even length.

Solution. One part of the result is immediate: if the graph is bipartite then obviously it cannot have odd cycles, because there is no internal edge in one of the two classes of the partition. The converse is a little bit trickier. Suppose that a graph G has no odd cycles and start your "journey" with an arbitrary vertex v and color this vertex white. Continue your trip through the vertices of the graph, by coloring all neighbors of the initial vertex in black. Continue in this manner, by considering this time every neighbor of v as an initial point of a new trip and color new vertices by the described rule, avoiding vertices that are already assigned a color. We must prove that you can do your trip with no problem. But the only problem that may occur is to have two paths to a certain vertex (called a problem vertex), each leading to a different color. But this is impossible, since all cycles are even. Indeed, any two paths from v to this problem vertex must have the same parity. Therefore we have a valid coloring of the vertices of the graph, and by construction this proves that G is bipartite. And here is an application:

Example 11. A group consists of n tourists. Among any 3 of them there are 2 who are not familiar. For every partition of the tourists in 2 buses, we can always find 2 tourists that are in the same bus who are familiar with each other. Prove that there is a tourist who is familiar with at most $\frac{2n}{5}$ tourists.

Bulgaria 2004

Solution. Construct a graph G on n vertices corresponding to the n tourists. We construct the edge ab if and only if the tourists a and b are familiar with each other. By the hypothesis, G is not bipartite, so it must have an odd cycle. Let $a_1, a_2, ..., a_l$ be the smallest odd cycle. Since l is odd and $l > 3$, we must have $l \geq 5$. It is clear that there are no other edges among the a_i except $a_i a_{i+1}$. If some vertex v is connected to a_i and a_j, it is easy to show that the "distance" between i and j is 2, that is $|i-j|$ equals 2 or $l-2$, since otherwise we would have a smaller odd cycle. Therefore, every vertex which does not belong to the cycle is adjacent to at most 2 a_i's. Even more, every vertex of the cycle is connected to exactly 2 a_i's. Therefore, if $c(v)$ is the number of edges between v and the vertices of the cycle, $c(v) \leq 2$, so

$$\sum_{i=1}^{l} d(a_i) = \sum_{v \in V} c(v) \leq 2n \Rightarrow d(a_k) \leq \frac{2n}{l} \leq \frac{2n}{5} \tag{22.2}$$

for some k. The solution ends here.

At first glance, the following has nothing to do with graphs and cycles. Well, it does! Here is a beautiful solution by Adrian Zahariuc:

Example 12. In each square of a chessboard is written a positive real number such that the sum of the numbers in each row is exactly 1. It is known that for any 8 squares, no two in the same row or column, the product of the numbers written in these squares does not exceed the product of the numbers on the main diagonal. Prove that the sum of the numbers on the main diagonal is at least 1.

St. Petersburg 2000

Solution. First, let us label the rows and the columns $1, 2, ..., 8$, consecutively, in increasing order. Suppose by way of contradiction that the sum of the numbers on the main diagonal is less than 1. Then on row k there is some cell (k, j) such that the number written in it is greater than the number written in cell (j, j), that is, the one in the same column, on the main diagonal. Color (k, j) red and draw an arrow from row k to row j. Some of these arrows must form a loop. From each row belonging to the loop we choose the red cell, and from all other rows we choose the cell on the main diagonal. All these 8 cells lie in different rows and different columns and their product exceeds the product of the numbers on the main diagonal, a contradiction. Therefore our assumption is false, and the sum of the numbers on the main diagonal is at least 1.

And for the die-hards, here are two very difficult problems communicated to us by Adrian Zahariuc:

Example 13. There are two airline companies in Wonderland. Any pair of cities is connected by a one-way flight offered by one of the companies. Prove that there is a city in Wonderland from which any other city can be reached via airplane without changing the company.

Iranian TST 2006

Solution. We would rather reformulate the problem in terms of graph theory: given a bichromatic (say, red and blue) tournament $G(V, E)$ (i.e. a directed graph in which there is precisely one edge between any pair of vertices). We have to prove that there is a vertex v such that, for any other vertex u, there is a monochromatic directed path from v to u. Such a point will be called "strong". Let $|V| = n$. We will prove the claim by induction on n.
The base case is trivial. Suppose it is true for $n - 1$; we will prove it for n. Now suppose by way of contradiction that the claim fails for some G. By the inductive hypothesis we know that for each $v \in V$ there is some $s(v) \in V - \{v\}$ which is a strong point in $G - \{v\}$. Clearly, $s(v) \neq s(v')$ for all $v \neq v'$, since

otherwise $s(v)$ would be strong in G. Let $f = s^{-1}$, i.e. $s(f(v)) = v$ for all v. It is clear that from v we can reach all points through a monochromatic path except $f(v)$. For each v, draw an arrow from v to $f(v)$. These arrows must form a loop. If this loop does not contain all n vertices of the graph, by the inductive hypothesis we must have a strong vertex in this graph, which contradicts the fact that we can't reach $f(v)$ from v. Hence, this loop is a Hamiltonian circuit $v_1, v_2, ..., v_n$. Let $v_{n+1} = v_1$. From v_i, we can reach all vertices except v_{i+1} because $v_{i+1} = f(v_i)$. We can't reach v from u through paths of both colors since, in that case, from u we could reach all the points we could reach from v, including $f(u)$, which is false.

For $v \neq f(u)$, let $c(uv)$ be the color of all paths from u to v. It is clear that $c(uv) \neq c(vf(u))$. We have $c(uv) \neq c(vf(u)) \neq c(f(u)f(v))$, so $c(uv) = c(f(u)f(v))$ for $u \neq v \neq f(u)$. In other words, $c(v_k v_{k+m}) = c(v_{k+1} v_{k+m+1})$. From here, it is easy to fill in the details. Basically, we just have to take $m > 1$ coprime with n to get that we can travel between any two points through paths of color $c(v_0 v_m)$ and we are done.

Example 14. Does there exist a 3-regular graph (that is, every vertex has degree 3) such that any cycle has length at least 30?

St. Petersburg 2000

Solution. Even though the construction will not be easy, the answer is: yes, there does. We construct a 3-regular graph G_n by induction on n such that any cycle has length at least n. Take $G_3 = K_4$, the complete graph on 4 vertices. Now, suppose we have constructed $G_n(V, E)$ and label its edges $1, 2, ..., m$. Take an integer $N > n2^m$ and let $V' = V \times \mathbb{Z}_N$. If the edge numbered k in G_n is ab, we draw an edge in $G_{n+1}(V', E')$ between (a, x) and $(b, x + 2^k)$ for all $x \in \mathbb{Z}_N$. It is clear that G_{n+1} is 3-regular. We show that G_{n+1} has the desired property, i.e. it contains no cycle of length less than $n + 1$. Suppose $(a_1, x_1), ..., (a_t, x_t)$ is a cycle with $t \leq n$. Clearly, $a_1, a_2, ..., a_t$ is a cycle of G_n.

Therefore $t = n$, and all a_i are distinct. We have

$$\boxed{0 = (x_1 - x_2) + (x_2 - x_3) + \cdots + (x_n - x_1) \equiv \sum_{j=1}^{n} \pm 2^{k_j} (\operatorname{mod} N). \quad (22.3)}$$

This sum is nonzero since all k_j are distinct, and also it is at most $n2^m < N$ in absolute value, a contradiction. Therefore this graph has all the desired properties, and the inductive construction is complete.