

Non-Abelian Anyons:  
Statistical Repulsion and  
Topological Quantum Computation

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## Chapter 1

# Introduction

## Chapter 2

# How anyons arise

### 2.1 Particle statistics

### 2.2 The braid group

The braid group  $B_n$  (representing braids on  $n$  strands) is the group with generators  $\sigma_1, \dots, \sigma_{n-1}$  and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2, \quad (2.1a)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \quad (2.1b)$$

**Lemma 2.2.1.** *All generators of the braid group are conjugate.*

*Proof.* First,  $\sigma_i$  is conjugate to  $\sigma_{i+1}$  for all  $i$ , as seen by

$$\begin{aligned} (\sigma_i \sigma_{i+1}) \sigma_i (\sigma_i \sigma_{i+1})^{-1} &= (\sigma_i \sigma_{i+1} \sigma_i) \sigma_{i+1}^{-1} \sigma_i^{-1} \\ [\text{eq. (2.1b)}] &= (\sigma_{i+1} \sigma_i \sigma_{i+1}) \sigma_{i+1}^{-1} \sigma_i^{-1} \\ &= \sigma_{i+1} \sigma_i \sigma_i^{-1} \\ &= \sigma_{i+1}. \end{aligned}$$

Finally, conjugation is transitive, thus all braid generators are conjugate.  $\square$

**Definition 2.2.1.** TODO

$$U_p = \sigma_1 \sigma_2 \cdots \sigma_p \sigma_{p+1} \sigma_p \cdots \sigma_2 \sigma_1,$$

### 2.3 Representation theory

### 2.4 Physical realizability

### 2.5 Anyons with or without quantum kinetic energy

Interferometry Bonderson.

Wavefunction density Lundholm.

## Chapter 3

# Lower bound for the kinetic energy in a gas of anyons

This chapter gives an outline of a method to estimate lower bounds for the kinetic energy of a gas of anyons, more generally known as Lieb-Thirring inequalities. This chapter is based on [6].

### 3.1 Preliminaries

Consider  $N$  anyons in  $\mathbb{R}^2$ , with coordinate  $x_1, \dots, x_N$ . This collection of anyons are described by an  $N$ -particle wave function  $\psi \in L^2(\mathbb{R}^{2N})$ , i.e. a square-integrable complex-valued function. The kinetic energy  $T$  of the system of  $N$  anyons is by definition

$$T := \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |\Delta_j \psi|^2 dx$$

where  $\Delta_j$  acts on  $x_j$ . In order to give bounds for the kinetic energy we shall determine the kinetic energy for pairs of anyons, by factoring out the dynamics of two anyons from the system of  $N$  anyons.

Among the  $N$  anyons, let all except two of them be fixed. Let  $x_j, x_k \in \mathbb{R}^2$  denote the coordinates for the two free anyons and define

$$x_{\text{cm}} = \frac{1}{2}(x_j + x_k), \quad x_{\text{rel}} = \frac{1}{2}(x_j - x_k).$$

Consider a frame of reference where the center of mass is at the origin, i.e.  $x_{\text{cm}} = 0$ . Let  $(r, \varphi)$  be the polar coordinates for the relative coordinates  $x_{\text{rel}}$ . The  $N$ -particle wave function can thus be parametrized as

$$\psi = \psi(x', r, \varphi),$$

where  $x'$  is the coordinates for the anyons that we considered to be fixed. If we also freeze out the radial dependence by fixing  $r$ , we factor out the angle-dependence from the wave function and write

$$\psi(\varphi) := \psi(x', r, \varphi).$$

We shall consider the angular dynamics of the two free anyons. As the free anyons circle each other, i.e. as  $\varphi$  increases from 0 to  $\pi$ , a number of fixed anyons may be enclosed. As we shall see, the number of encircled anyons will play a crucial role, therefore we separate the state space  $\mathbb{R}^2$  as follows.

Separate the state space into open annuli (regions between two concentric circles) such that none of the fixed  $N-2$  anyons are at the interior of an annulus, cf. fig. 3.1. This gives a separation of  $\mathbb{R}^2$  into regions with increasing number  $p$  of the  $N-2$  fixed anyons. More explicitly, each annulus is centered in  $x_{\text{cm}}$ . Furthermore, let  $r_1 \leq r_2 \leq \dots \leq r_{N-2}$  be the radii of the  $N-2$  fixed anyons. The innermost annulus is an open disc (degenerate annulus) with radius  $r_1$ , the second innermost annulus is the region between circles of radius  $r_1$  and  $r_2$ , etc. Note that the circles with radii  $r_1, r_2, \dots, r_{N-2}$  are not contained in any annuli, since the anyons cannot pass through each other, and we are considering angular motion. If the radii of two fixed anyons coincide, then there is clearly no annulus separating them.

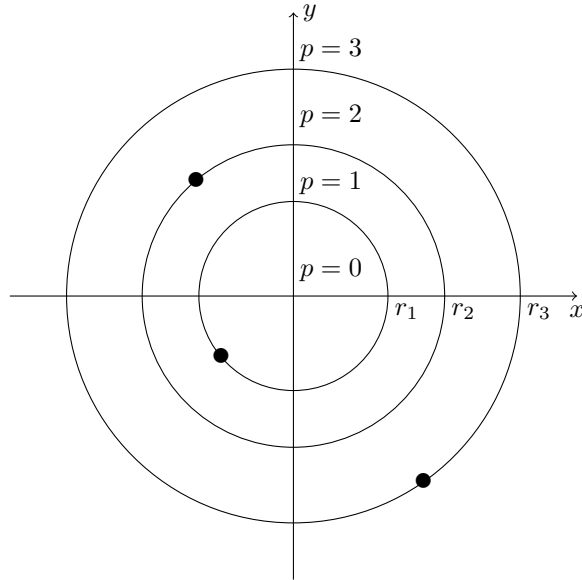


Figure 3.1: Illustration of annuli separating  $\mathbb{R}^2$  into regions with increasing number  $p$  of contained anyons. A blob ( $\bullet$ ) denotes a fixed anyon. In each annulus, exchange of the two free anyons, i.e. as  $\varphi$  increases from 0 to  $\pi$ , a given number  $p$  of fixed anyons will be enclosed.

In the general case, exchange of a pair of anyons introduces an anyonic phase  $U_p \in U(n)$  as discussed in chapter 2. The exchange operator  $U_p$  may depend on the number  $p$  of anyons that get encircled in the exchange loop. With the wave function  $\psi = \psi(\varphi)$  parametrized by the relative angular coordinate, we thus get the boundary condition

$$\psi(\pi) = U_p \psi(0). \quad (3.1)$$

This essentially alters the geometry of the space, splitting it in half. It suffices to consider the region  $0 \leq \varphi \leq \pi$ , i.e. it suffices to consider half annuli.

We have set out to estimate the energy of the system, this amounts to solving

the Schrödinger equation, in particular solving for the lowest energy  $\lambda$ ,

$$H\psi = \lambda\psi$$

where  $H = -\nabla^2 = -\frac{\partial^2}{\partial\varphi^2}$  is the Hamiltonian of the system. So far we have just one boundary condition, eq. (3.1), while the Schrödinger equation is a second order differential equation, requiring two boundary conditions to give a unique solution. However, since we are primarily interested the energies of the system, it suffices to compute the spectrum of  $H$ . With this in mind, we write the Hamiltonian  $H = -\nabla^2$  as a square  $H = D^2$ , where

$$D = -i\partial_\varphi, \quad 0 \leq \varphi \leq \pi, \text{ subject to boundary conditions given by } U_p,$$

and use the spectral theorem to compute the spectrum of  $H$  as the squared spectrum of  $D$ , i.e.

$$\sigma(H) = \{\lambda^2 : \lambda \in \sigma(D)\}.$$

Similarly, we have that the ground state energy is the infimum of  $\sigma(H)$ , i.e.

$$\lambda_0 = \inf \sigma(H) = \inf \{\lambda^2 : \lambda \in \sigma(D)\}.$$

The spectrum of  $D$  is straight forward to compute and we present the result as a lemma.

**Lemma 3.1.1.** *The eigenfunctions  $u$  and eigenvalues  $\lambda$  of  $D$  are such that*

$$Du = \lambda u \quad \Longleftrightarrow \quad -iu'(\varphi) = \lambda u(\varphi),$$

*having general solution*

$$u(\varphi) = Ce^{i\lambda\varphi}$$

*for some constant  $C \in \mathbb{C}^n$ .*

We seek an expression for the spectrum of  $D$ , the above lemma and the boundary condition eq. (3.1) gives

$$u(\pi) = U_p u(0) \implies Ce^{i\lambda\pi} = U_p C,$$

that is,  $e^{i\lambda\pi}$  is an eigenvalues of  $U_p$ . We have the following result.

**Lemma 3.1.2.** *The spectrum of  $D$  (and by extension the spectrum of  $H$ ) is given by*

$$\sigma(D) = \{\lambda : e^{i\lambda\pi} \text{ is an eigenvalue of } U_p\}.$$

*Remark 3.1.1.* Recall that  $U_p$  is unitary, so all eigenvalues of  $U_p$  are on the complex unit circle. Hence, each eigenvalue  $e^{i\lambda\pi}$  of  $U_p$  with  $0 \leq \lambda < 2$ , gives a family of possible energy levels  $\{(\lambda + 2n)^2 \mid n \in \mathbb{Z}\}$  to the system.

### 3.2 Abelian anyons

Although non-Abelian anyons are the main focus of this thesis, we give a quick overview of the the case of Abelian anyons. For more details, see [6]

Abelian anyons are characterized by the fact that the exchange operator is one dimensional, i.e. an element of  $U(1)$ , hence it is of the form  $e^{i\alpha\pi}$ . Take this to be  $U_0$ , i.e. simple exchange of a pair of Abelian anyons, with the exchange loop enclosing no other anyons. It is then easy to realize that

$$U_p = e^{i\alpha(1+2p)\pi} \quad (3.2)$$

because the left anyon performs an exchange with the  $p$  inner anyons, likewise for the right anyon, finally the two anyons undergoing exchange are exchanged with each other, resulting in  $1 + 2p$  exchanges of  $\alpha$ -type anyons, as illustrated in fig. 3.2.

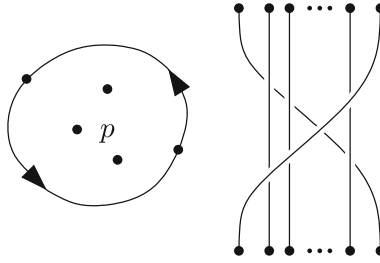


Figure 3.2: Exchange of a pair of Abelian  $\alpha$  anyons around  $p$  fixed  $\alpha$  anyons. Figure taken from [6].

**Proposition 3.2.1** (Ground state energy for a pair of abelian anyons). *Consider a pair of Abelian anyons with anyonic phase  $\alpha$  enclosing  $p$  fixed anyons (also with anyonic phase  $\alpha$ ). The ground state energy is bound from below by*

$$\begin{aligned} & \inf_{p,q \in \mathbb{Z}} (\alpha(1+2p) + 2q)^2 = \\ & = \begin{cases} \frac{1}{\nu^2}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ with } \mu \in \mathbb{Z}, \nu \in \mathbb{N}_+ \text{ relatively prime and } \mu \text{ odd,} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

*Proof.* By eq. (3.2) we have the anyonic exchange operator as

$$U_p = e^{i\alpha(1+2p)\pi}.$$

By lemma 3.1.2 we have that the boundary condition gives

$$\begin{aligned} u(\pi) &= e^{i\alpha(1+2p)\pi} u(0) \\ \iff e^{i\lambda\pi} &= e^{i\alpha(1+2p)\pi} \\ \iff \lambda &= \alpha(1+2p) + 2q, \quad q \in \mathbb{Z}. \end{aligned}$$

Thus we see that the spectrum for  $D$  and  $H = D^2$ , respectively, is

$$\begin{aligned} \sigma(D) &= \{\alpha(1+2p) + 2q : q \in \mathbb{Z}\}, \\ \sigma(H) &= \{(\alpha(1+2p) + 2q)^2 : q \in \mathbb{Z}\}. \end{aligned}$$



Hence, the energy  $\lambda^2$  is minimized for

$$E_0 = \inf_{p,q \in \mathbb{Z}} (\alpha(1+2p) + 2q)^2.$$

A number-theoretic result, found as proposition 5 in [6], shows that this can be written as

$$E_0 = \begin{cases} \frac{1}{\nu^2}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ with } \mu \in \mathbb{Z}, \nu \in \mathbb{N}_+ \text{ relatively prime and } \mu \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

□

From this we have an immediate corollary for bosons ( $\alpha = 0$ ) and fermions ( $\alpha = 1$ ).

**Corollary 3.2.2.** *With  $\alpha = 0$ , the anyonic phase reads*

$$U_p = e^{i\alpha(1+2p)\pi} = 1$$

for all  $p$ , i.e. bosons do not “see” each other, and they have a zero ground state energy.

With  $\alpha = 1$ , the anyonic phase reads

$$U_p = e^{i\alpha(1+2p)\pi} = e^{i(1+2p)\pi},$$

giving us the boundary condition

$$e^{i\lambda\pi} = e^{i(1+2p)\pi} \iff \lambda = 1 + 2p + 2q, \quad q \in \mathbb{Z}$$

Since  $p$  is an integer, this shows that also fermions do not “see” the enclosed  $p$  anyons. However, the anyons in the pair do “see” each other, in the sense that their wave-function changes sign after they have been exchanged, as expected. Finally, this also shows that fermions have a non-zero ground state energy  $E_0 = 1$ .

### 3.3 Non-Abelian anyons

Consider the same setting as above, with the distinction that the anyons are now non-Abelian. That is, consider a pair of non-Abelian anyons, with wave-function  $\psi \in L^2(\mathbb{R}; \mathbb{C}^n)$  parametrized by the relative angle  $\varphi$ , such that the pair of anyons encloses  $p$  fixed anyons as  $\varphi$  increases from 0 to  $\pi$ , giving rise to an anyonic phase  $U_p \in U(n)$ .

A standard characterization of unitary matrices is as follows.

**Lemma 3.3.1.** *Let  $U \in U(n)$ , the  $n$  eigenvalues (counted with multiplicity) of  $U$  are on the form  $e^{i\beta_j\pi}$  for  $j = 1, 2, \dots, n$ . Furthermore,  $e^{i\alpha\pi}$  where  $\alpha = \frac{1}{n} \sum_j \beta_j$  is referred to as the Abelian phase of  $U_p$ . It is the Abelian part in the factorization of  $U(n) = U(1) \times SU(n)$ . Note that if  $\det U = e^{i\theta\pi}$  we have  $\alpha = \theta/n$ .*

*Proof.* We only show the  $U(n) = U(1) \times SU(n)$  decomposition. Rewrite  $U$  as  $U = e^{i\alpha} (e^{-i\alpha} U)$  where  $e^{i\alpha}$  is the Abelian part and  $e^{-i\alpha} U$  is the  $SU(n)$  part. It suffices to show that  $\det(e^{-i\alpha\pi} U) = 1$ ,

$$\begin{aligned} \det(e^{-i\alpha\pi} U) &= (e^{-i\alpha\pi})^n e^{i(\sum_j \beta_j)\pi} \\ &= \left(e^{-i(\sum_j \beta_j)\pi/n}\right)^n e^{i(\sum_j \beta_j)\pi} = 1. \end{aligned}$$

On the other hand,  $1 = \det(e^{-i\alpha\pi} U) = e^{-i\alpha\pi n} e^{i\theta\pi} = e^{i(-\alpha n + \theta)\pi}$ , thus  $\alpha = \theta/n$  (modulo 2).  $\square$

Next, we show that the Abelian phase shifts the eigenvalues uniformly along the complex unit circle.

**Proposition 3.3.2.** *Consider non-Abelian anyons with arbitrary (unitary)  $U_p \in U(n)$  exchange operator. The Abelian part of the non-Abelian anyonic phase  $U_p$  can always be chosen so that the ground state has zero energy.*

*Proof.* By changing the Abelian phase, each eigenvalue is shifted. This is obvious if we diagonalize  $U_p$ . In particular, adding  $-\beta_j$  to the Abelian phase, the  $j$ :th eigenvalue is shifted to  $e^{i(\beta_j - \beta_j)} = 1$ , corresponding to zero energy.  $\square$

As we shall see in the following chapter, the exchange operator  $U_p$  has a rather involved dependence on the particular type of non-Abelian anyons that are considered. In order to characterize the energy bound, via the eigenvalues of  $U_p$ , we must first dive into the framework of abstract anyon models in order to characterize  $U_p$ .

## Chapter 4

# Abstract anyon models

In this chapter we shall present the framework of abstract anyon models, culminating in a characterization of anyon braiding. That is, we will show how the braid group representation is computed for a given anyon model.

Anyon models can be modeled by unitary braided tensor categories, see [10]. However, setting up this framework in full generality is redundant for our purposes and we take a more straight forward approach, which is also most common in the literature. The benefit of the categorical approach is that consistency of braiding and fusion is most naturally shown this way. We shall see the ties between the straight forward approach and the categorical model in when discussing the consistency conditions in section 4.6.

This chapter is in part based on [8, 10, 26]. We shall be using the notation that is standard in the literature, and when possible make use of and extend the fusion diagram notation, which greatly clarifies braiding of fusion states.

Starting with section 4.7 we show how a given abstract anyon model gives rise to representations of the braid group.

### 4.1 Preliminaries

An abstract anyon model consists of a set of labels representing different types of anyons. These labels are known as anyonic charge, topological charge or superselection labels. Anyons can be combined, or fused, to give an anyon of some charge, possibly in different ways. This is modeled by a fusion algebra

$$a \times b = \sum_c N_{ab}^c c$$

representing the possible outcomes from fusion of anyons of type  $a$  and  $b$ . The fusion multiplicities  $N_{ab}^c$  are non-negative integers denoting the number of distinct ways  $a$  and  $b$  can fuse to  $c$ . In each anyon model there is the trivial label 1, representing the vacuum, with the property  $N_{a1}^b = N_{1a}^b = \delta_{ab}$ , i.e. 1 fuses trivially with every other charge. Furthermore, to each charge  $a$  there is a corresponding conjugate charge  $\bar{a}$  representing the antiparticle of  $a$ , with the property  $N_{ab}^1 = \delta_{b\bar{a}}$ , i.e.  $a$  fuses to the vacuum only with its antiparticle.

The  $N_{ab}^c$  distinguishable ways in which  $a$  and  $b$  can fuse to  $c$  can be regarded as an orthonormal basis of a Hilbert space  $V_{ab}^c$ . This is the state space, or fusion

space, of anyons of type  $c$  resulting from fusion of  $a$  and  $b$ . The states of  $V_{ab}^c$  are called fusion states and we denote the basis for  $V_{ab}^c$  by

$$\{|ab; c, \mu\rangle, \quad \mu = 1, 2, \dots, N_{ab}^c\}$$

where  $|ab; c, \mu\rangle$  represents the  $\mu$ :th way in which  $a$  and  $b$  can fuse to  $c$ .

The splitting space  $V_c^{ab}$  is the dual space of the fusion space  $V_{ab}^c$ , it is the state space of particles with anyonic charge  $a$  and  $b$  that can be split from an anyon of charge  $c$ .

More generally, the fusion space of anyons of type  $c$  resulting from fusion of anyons of type  $a_1, \dots, a_n$  is denoted by  $V_{a_1 a_2 \dots a_n}^c$ . This space has a canonical decomposition

$$V_{a_1 a_2 \dots a_n}^c \cong \bigoplus_{b_1, b_2, \dots, b_{n-2}} V_{a_1 a_2}^{b_1} \otimes V_{b_1 a_3}^{b_2} \otimes V_{b_2 a_4}^{b_3} \dots \otimes V_{b_{n-2} a_n}^c$$

with an associated canonical orthonormal basis with elements being the fusion states

$$|a_1 a_2; b_1, \mu_1\rangle \otimes |b_1 a_3; b_2, \mu_2\rangle \otimes \dots \otimes |b_{n-3} a_{n-1}; b_{n-2}, \mu_{n-2}\rangle \otimes |b_{n-2} a_n; c, \mu_{n-1}\rangle$$

for all possible  $b_j$  and  $\mu_j$ . For convenience we write  $b_0 = a_1$  and  $b_{-1} = 1$ .

Many anyon models of interest have the property that  $N_{ab}^c \leq 1$  for all particle types  $a, b$  and  $c$ . When this is the case, such as for the Fibonacci anyons that we shall consider in chapter 5, the multiplicity label  $\mu$  can be ignored. This makes the model easier to handle, and we introduce the fusion diagram notation.

## 4.2 Fusion diagrams

Consider an abstract anyon model with  $N_{ab}^c \leq 1$ . That is, if fusion of  $a$  and  $b$  to  $c$  is possible, this happens in exactly one way. Then, the multiplicity label  $\mu$  in  $|ab; c, \mu\rangle$  can be ignored, because the only possibility is  $\mu = 1$ , if  $\mu = 0$  the state is not valid. In this case we introduce the fusion diagram notation for fusion states and write

$$|ab; c; \mu\rangle =: |ab; c\rangle =: \frac{b}{a \quad c}.$$

The diagram should be read left/top to right/bottom, i.e.  $a$  fuses with  $b$  resulting in  $c$ . The diagram notation is primarily useful when considering fusion of several anyons and extends in a natural way;

$$\frac{b \quad c}{a \quad e \quad d}$$

denotes fusion of  $a, b, c$  to  $d$  such that  $a$  fuses with  $b$  resulting in the intermediate charge  $e$ , finally  $e$  fuses with  $c$  to give the resulting charge  $d$ .

The canonical basis for the fusion space  $V_{a_1 a_2 \dots a_n}^c$  with canonical decomposition

$$V_{a_1 \dots a_n}^c \cong \bigoplus_{b_1, b_2, \dots, b_{n-2}} V_{a_1 a_2}^{b_1} \otimes V_{b_1 a_3}^{b_2} \otimes V_{b_2 a_4}^{b_3} \dots \otimes V_{b_{n-2} a_n}^c$$

can thus be written in terms of fusion diagrams, assuming trivial fusion multiplicities, as

$$\left\{ \begin{array}{c} \begin{array}{ccccccc} & a_2 & & a_3 & & & \\ & | & & | & & & \\ a_1 & b_1 & b_2 & & b_{n-3} & b_{n-2} & c \end{array} \cdots \begin{array}{ccc} & a_{n-1} & a_n \\ & | & | \\ b_{n-3} & b_{n-2} & c \end{array} \end{array} \middle| \begin{array}{l} \text{for all possible intermediate} \\ \text{charges } b_1, b_2, \dots, b_{n-2} \end{array} \right\}.$$

We can thus think of the basis as consisting of a tuple of possible intermediate charges

$$(b_1, b_2, \dots, b_{n-2}).$$

TODO: Mention explicitly that we are representing all this with  $\mathbb{C}^n$  matrices and how to map back and forth, and then ignore it.

The real advantage of writing fusions states with fusion diagrams is that braiding is much easier to represent. This will be extremely useful now as we proceed in developing the abstract model for anyons, and characterize braiding.

### 4.3 The $R$ -matrix: Commutativity of fusion

The result of fusing  $a$  with  $b$  must be the same as fusing  $b$  with  $a$ . That is, fusion is commutative,

$$a \times b = b \times a.$$

This gives rise to a natural isomorphism

$$R : V_{ba}^c \rightarrow V_{ab}^c$$

between the corresponding fusion spaces which can be represented by a unitary matrix  $R_{ab}^c$  in the canonical basis

$$|ba, c, \mu\rangle = \sum_{\mu'} (R_{ab}^c)_{\mu}^{\mu'} |ab, c, \mu'\rangle.$$

As we have seen, if the fusion multiplicities  $N_{ab}^c \leq 1$  for all labels  $a, b, c$ , we can disregard explicit mention of the fusion multiplicities. From now on we shall do so, and note that it is straight forward to add these indices back if needed. In the diagrammatic notation we then have

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \bigcirc \\ \diagup \quad \diagdown \\ c \end{array} = R_{ab}^c \begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ \bigcirc \\ \diagdown \quad \diagup \\ c \end{array}. \quad (4.1)$$

In this case it is clear that  $R$ -matrix  $R_{ab}$  is diagonal in the canonical basis and we have

$$(R_{ab})_{ij} = \delta_{ij} R_{ab}^i.$$

## 4.4 The $F$ -matrix: Associativity of fusion

The result of fusing multiple anyons must be independent of which anyons are fused first. That is, fusion is associative,

$$(a \times b) \times c = a \times (b \times c)$$

This gives rise to a natural isomorphism between the two decompositions of the fusion space

$$V_{abc}^d \cong \bigoplus_f V_{ab}^f \otimes V_{fc}^d \cong \bigoplus_e V_{bc}^e \otimes V_{ae}^d.$$

The first decomposition should be understood as first fusing  $a$  with  $b$  in all possible ways giving an intermediate charge  $f$ , followed by fusing  $c$  with  $f$  to give the final charge  $d$ . The second decomposition should be understood as first fusing  $b$  with  $c$  in all possible ways giving an intermediate charge  $e$ , followed by fusing  $a$  with  $e$  to give the final charge  $d$ .

The first of these two decompositions is referred to as the standard decomposition and the second is the fusion decomposition. We denote this isomorphism by

$$F : \bigoplus_f V_{ab}^f \otimes V_{fc}^d \rightarrow \bigoplus_e V_{bc}^e \otimes V_{ae}^d$$

and it can be represented by the matrix  $F_{abc}^c$ , satisfying the equation

$$\sum_f (F_{abc}^d)_{ef} \frac{b \ c}{a \ f \ d} = \frac{b \ c}{a \ e \ d}. \quad (4.2)$$

That is, the  $F$ -matrix is the change of basis matrix from the standard basis to the fusion basis of the fusion space  $V_{abc}^d$ .

The following lemma will be useful when computing the  $F$ -matrix.

**Lemma 4.4.1.** *Consider the fusion space  $V_{abc}^d$ , when one of the particle types is trivial, i.e.  $a, b, c$  or  $d$  equals 1, then  $\dim V_{abc}^d = 1$  and the corresponding  $F$ -matrix  $F_{abc}^d$  is trivial. More explicitly we have*

$$\begin{aligned} F_{1bc}^d &= (F_{1bc}^d)_{db} = 1, \\ F_{a1c}^d &= (F_{a1c}^d)_{ca} = 1, \\ F_{ab1}^d &= (F_{ab1}^d)_{bd} = 1, \\ F_{abc}^1 &= (F_{abc}^1)_{\bar{a}\bar{c}} = 1. \end{aligned}$$

*Proof.* With  $a = 1$  we have, by definition of the  $F$ -matrix,

$$\frac{b \ c}{1 \ e \ d} = \sum_f (F_{1bc}^d)_{ef} \frac{b \ c}{1 \ f \ d}.$$

From the fusion diagram on the right hand side we read out  $1 \times b = f$  from the first fusion, this is valid only for  $b = f$ . Similarly, on the left hand side the final

fusion reads  $1 \times e = d$ , implying  $e = d$ . Since the indices  $e$  and  $f$  are forced, this implicitly shows that the corresponding fusion spaces is one-dimensional. The other results follow analogously,

$$\begin{aligned} \frac{1 \ c}{a \ d} &= \sum_f (F_{a1c}^d)_{ef} \frac{1 \ c}{a \ f \ d} \implies e = c, f = a, \\ \frac{b \ 1}{a \ d} &= \sum_f (F_{ab1}^d)_{ef} \frac{b \ 1}{a \ f \ d} \implies e = b, f = d, \\ \frac{b \ c}{a \ 1} &= \sum_f (F_{abc}^1)_{ef} \frac{b \ c}{a \ f \ 1} \implies e = \bar{a}, f = \bar{c}. \end{aligned}$$

The result can also be realized by noting that three anyons of type  $a, b$  and  $c$ , where one of them is the trivial type 1, is uniquely determined by their total charge. Indeed, these three anyons are really just two, since the trivial particle  $q$  fuses trivially, it can be added or removed in the representation without changing anything. Thus, the corresponding  $F$  matrix must be trivial in this case.  $\square$

## 4.5 The $B$ -matrix: Braiding of standard fusion states

We shall now consider braiding on the standard fusion states. This can be realized by applying the  $F$ -matrix to put the state in a basis where the  $R$  matrix can be applied immediately, followed by reverting back to the standard basis via  $F^{-1}$ . That is, using the  $F$  and  $R$ -matrix we obtain the relation

$$\begin{aligned} \frac{b \ c}{a \ e \ d} &= \sum_f \left( (F^{-1})_{acb}^d \right)_{ef} \frac{b \ c}{a \ f \ d} \\ &= \sum_f \left( (F^{-1})_{acb}^d \right)_{ef} R_{bc}^f \frac{b \ c}{a \ f \ d} \\ &= \sum_g \sum_f \left( (F^{-1})_{acb}^d \right)_{ef} R_{bc}^f (F_{abc}^d)_{fg} \frac{b \ c}{a \ g \ d}. \end{aligned}$$

From this we define the  $B$ -matrix as

$$(B_{abc}^d)_{eg} = \sum_f \left( (F^{-1})_{acb}^d \right)_{ef} R_{bc}^f (F_{abc}^d)_{fg}.$$

Symbolically we write this as

$$B = F^{-1} R F.$$

To sum up, the  $B$ -matrix braids the standard fusion states according to

$$\frac{b \ c}{a \ e \ d} = \sum_g (B_{abc}^d)_{eg} \frac{b \ c}{a \ g \ d}.$$

As a consequence of the definition of the  $B$ -matrix and lemma 4.4.1 we have the following lemma, which will be useful when computing the  $B$ -matrix.

**Lemma 4.5.1.** *Consider the fusion space  $V_{abc}^d$ , when one of the particle types is trivial, i.e.  $a, b, c$  or  $d$  equals 1, then  $\dim V_{abc}^d = 1$  and the corresponding  $B$ -matrix  $B_{abc}^d$  is one-dimensional,*

$$\begin{aligned} B_{1bc}^d &= (B_{1bc}^d)_{cb} = R_{bc}^d, \\ B_{a1c}^d &= (B_{a1c}^d)_{da} = R_{1c}^c = 1, \\ B_{ab1}^d &= (B_{ab1}^d)_{ad} = R_{b1}^b = 1, \\ B_{abc}^1 &= (B_{abc}^1)_{\bar{b}\bar{c}} = R_{bc}^{\bar{a}}. \end{aligned}$$

## 4.6 The pentagon and hexagon equations

When considering an anyon model, it is ultimately the  $B$ -matrix that determines the properties of interest. The  $B$ -matrix gives the phase change introduced when braiding anyons. In the previous section we saw how the  $B$ -matrix determine the braid group representation. This is the relevant property both for the study the dynamics of anyons, but also for developing methods for quantum computation with anyons, known as topological quantum computation.

We have seen that the  $B$ -matrix is computed from the  $F$  and  $R$ -matrices. These matrices are in turn determined by what is known as the pentagon equation

$$(F_{12c}^5)_a^d (F_{a34}^5)_b^c = (F_{234}^d)_c^c (F_{1e4}^5)_b^d (F_{123}^b)_a^e \quad (4.3)$$

and hexagon equation

$$R_{ac}^g (F_{bac}^d)_e^g R_{ab}^e = \sum_f (F_{bca}^d)_f^g R_{af}^d (F_{abc}^d)_e^f. \quad (4.4)$$

In these equations, all indices are taken as arbitrary particle labels.

These equations are known as coherence conditions for fusion and braiding. The diagrammatic version of these equations, found as commutative diagrams in figs. 4.1 and 4.4 and figs. 4.2 and 4.4 make the point clearer, and shows that these equations are commutativity constraints for fusion and braiding. Indeed, the pentagon equation is the formal constraint for associativity of fusion,

$$(a \times b) \times c = a \times (b \times c).$$

As previously hinted, anyon models can be described by braided tensor categories. In this setting, the pentagon and hexagon equations are precisely Mac Lane's coherence theorem [27], showing that no further conditions are required for consistent fusion and braiding. Further details can be found in [10, 8].

Solving the pentagon and hexagon equations is in general highly non-trivial. The equations are multivariate polynomial equations and require elaborate techniques to be solved. First one must fix the gauge freedom that comes from the choice of basis for the fusion space, next an appropriate Gröbner basis can be used to solve the system. See [26] for more details.



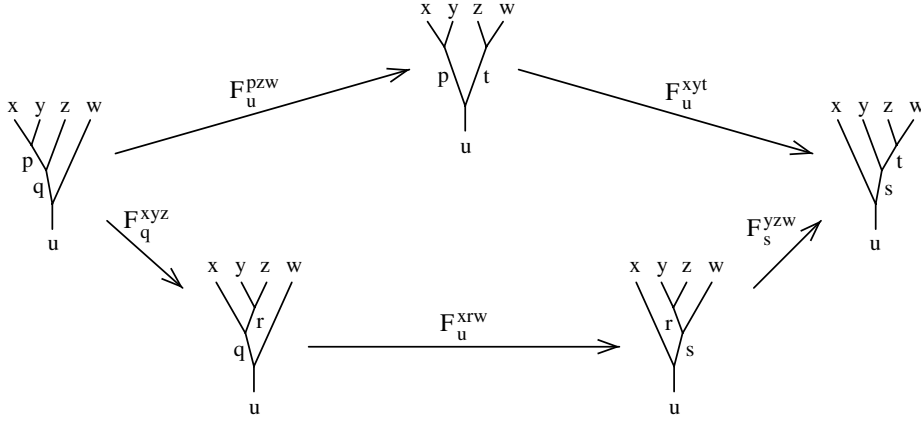


Figure 4.1: The pentagon equation in terms of fusion diagrams. Figure taken from [10]. Note that the  $F$ -matrix have super- and sub-scripts reversed.

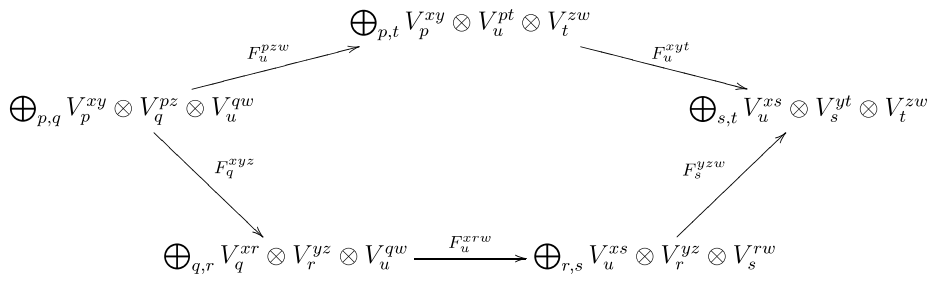


Figure 4.2: The pentagon equation in terms of fusion spaces. Figure taken from [10]. Note that the fusion spaces and the  $F$ -matrix have super- and sub-scripts reversed.

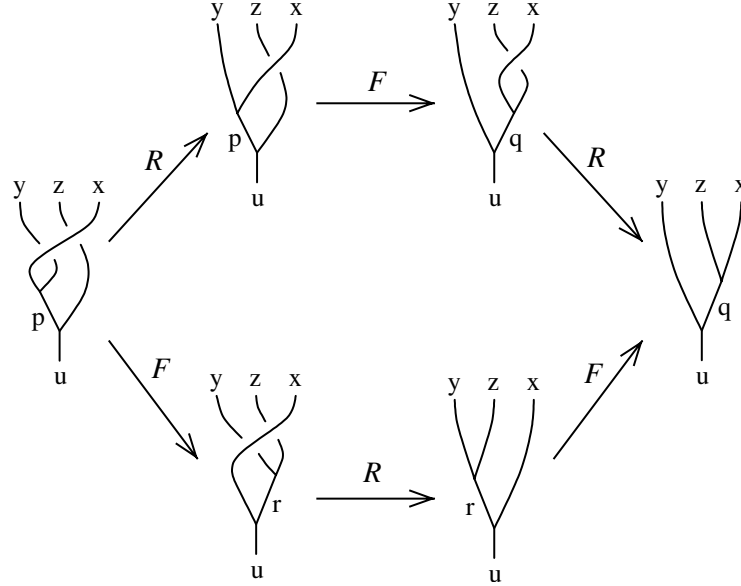


Figure 4.3: The hexagon equation in terms of fusion diagrams. Figure take from [10].

$$\begin{array}{ccc}
 \bigoplus_p V_p^{yx} \otimes V_u^{pz} & \xrightarrow{F_u^{yxz}} & \bigoplus_q V_u^{yq} \otimes V_q^{xz} \\
 \uparrow \bigoplus_p R_p^{xy} \otimes \text{id}_{V_u^{pz}} & & \downarrow \bigoplus_q \text{id}_{V_u^{yq}} \otimes R_q^{xz} \\
 \bigoplus_p V_p^{xy} \otimes V_u^{pz} & & \bigoplus_q V_u^{yq} \otimes V_q^{zx} \\
 \downarrow F_u^{xyx} & & \uparrow F_u^{yzx} \\
 \bigoplus_r V_u^{xr} \otimes V_r^{yz} & \xrightarrow{\bigoplus_r (\text{id}_{V_r^{yz}} \otimes R_u^{xr}) \cdot \text{swap}} & \bigoplus_r V_r^{yz} \otimes V_u^{rx}
 \end{array}$$

Figure 4.4: The hexagon equation in terms of fusion spaces. Figure take from [10]. Note that the fusion spaces,  $F$ -matrix and  $R$ -matrix have super- and subscripts reversed.

## 4.7 Representations of the braid group

TODO: Representations of the braid group in abstract anyon models.

## 4.8 Braid group representations $\rho(\sigma_j)$ : Braiding of general fusion states

Consider the fusion space  $V_{a_1 \dots a_n}^c$  in a given anyon model, the representation  $\rho(\sigma_j)$  gives the anyonic phase introduced to the total wave function when particles  $a_j$  and  $a_j + 1$  are exchanged (counter)clockwise. Recall from chapter 2 that the braid group is generated by  $\sigma_1, \dots, \sigma_n$ , thus it suffices to give expressions for  $\rho(\sigma_j)$  in order to be able to compute any braid in a given anyon model.

Before showing how  $\rho(\sigma_j)$  is computed, we begin by making our notation slightly more flexible. Consider the fusion space  $V_{a_1 \dots a_n}^c$ . As we have seen, the fusion states are on the form

$$\begin{array}{ccccccc} & a_2 & a_3 & & a_{n-1} & a_n & \\ & | & | & & | & | & \\ a_1 & b_1 & b_2 & \cdots & b_{n-3} & b_{n-2} & c \end{array}.$$

We shall sometimes write such states on the form of the standard basis states of  $V_{1a_1 \dots a_n}^c$ , i.e. as

$$\begin{array}{ccccccc} & a_1 & a_2 & & a_{n-1} & a_n & \\ & | & | & & | & | & \\ 1 & a_1 & b_1 & \cdots & b_{n-3} & b_{n-2} & c \end{array}.$$

The reason being that it is then simpler to represent braiding of  $a_1$  with  $a_2$ . This observation allows us to extend the fusion diagrams with trivial charges when convenient.

**Lemma 4.8.1.** *In a general anyon model, in the standard basis of the fusion space  $V_{a_1 \dots a_n}^c$  we have*

$$\rho(\sigma_1)_{ij} = \delta_{ij} R_{a_1 a_2}^j \iff \rho(\sigma_1) = R_{a_1 a_2}.$$

*Proof.* In the general case, the representation  $\rho(\sigma_1)$  for the first generator  $\sigma_1$  that braids  $a_1$  with  $a_2$ , is given by

$$\rho(\sigma_1) \left( \begin{array}{ccccccc} & a_1 & a_2 & & & & \\ & | & | & & & & \\ 1 & a_1 & b_1 & \cdots & & & \end{array} \right) = \left( \begin{array}{ccccccc} & a_1 & a_2 & & & & \\ & | & | & & & & \\ 1 & a_2 & b_1 & \cdots & & & \end{array} \right) = \sum_g \left[ \left( B_{1a_1 a_2}^{b_1} \right)_{a_2 g} \left( \begin{array}{ccccccc} & a_1 & a_2 & & & & \\ & | & | & & & & \\ 1 & g & b_1 & \cdots & & & \end{array} \right) \right]$$

Since 1 fuses trivially we must have  $g = a_1$  and thus  $\rho(\sigma_1)$  is one-dimensional with

$$\rho(\sigma_1) = \left( B_{1a_1 a_2}^{b_1} \right)_{a_2, a_1} = \sum_f \left( (F^{-1})_{1a_2 a_1}^{b_1} \right)_{a_2 f} R_{a_1 a_2}^f \left( F_{1a_1 a_2}^{b_1} \right)_{f a_1} = R_{a_1 a_2}^{b_1},$$

where the last equality follows from lemma 4.4.1.  $\square$

*Remark 4.8.1.* Above,  $\rho(\sigma_1)$  is expressed in the basis for the reduced space  $V_{a_1 a_2} = \bigoplus_{b_1} V_{a_1 a_2}^{b_1}$ , having basis determined by the possible anyonic labels  $b_1$ .

The full space  $V_{a_1 \dots a_n}^c$  has basis states  $(b_1, \dots, b_{n-2})$  and we implicitly used the operator on the space reduced to fusion states  $(b_1)$ .

Extending  $\rho(\sigma_1)$  to the full space is straight forward: In order to determine the action of  $\rho(\sigma_1)$  on a given fusion state in the basis of the full fusion space it suffices to consider the action of  $\rho(\sigma_1)$  on the labels  $a_1$  and  $a_2$  in the given fusion state. This information is precisely captured in the representation of  $\rho(\sigma_1)$  in the reduced basis. This results in repeating blocks in the matrix representing  $\rho(\sigma_1)$  when considering the basis of the full fusion space. This motivates the slight abuse of notation.

This discussion applies to any braiding operator, and we shall often use this abuse of notation, it should always be clear what part of the space that the operator acts on.

**Lemma 4.8.2.** *In a general anyon model, consider the fusion space  $V_{a_1 \dots a_n}^c$  with the standard basis  $(b_1, \dots, b_{n-1})$ , then*

$$\rho(\sigma_j) = B_{b_{j-2} a_j a_{j+1}}^{b_j},$$

*Proof.* Note that  $\rho(\sigma_j)$  is precisely the  $B$ -matrix applied appropriately,

$$\left( \dots \frac{a_j \quad a_{j+1}}{b_{j-2} \quad e \quad b_j} \dots \right) = \sum_{b_{j-1}} \left[ \left( B_{b_{j-2} a_j a_{j+1}}^{b_j} \right)_{e, b_{j-1}} \left( \dots \frac{a_j \quad a_{j+1}}{b_{j-2} \quad b_{j-1} \quad b_j} \dots \right) \right],$$

thus the result follows.  $\square$

With the convention  $b_0 = a_1$  and  $b_{-1} = 1$ , lemma 4.8.2 subsumes lemma 4.8.1, and also the following result, which we state explicitly for convenience. Recall that  $\bar{a}$  denotes the antiparticle of  $a$ .

**Lemma 4.8.3.** *In a general anyon model, consider the fusion space  $V_{a_1 \dots a_n}^1$  with its standard basis  $(b_1, \dots, b_{n-3})$  (note that  $c = 1$  forces  $b_{n-2} = \bar{a}_n$ ), then*

$$\rho(\sigma_{n-1})_{ij} = \delta_{ij} R_{a_{n-1} a_n}^{\bar{a}_n},$$

*acting on the  $b_{n-3}$ -part of the space.*

*Proof.* Since the result of the fusion is assumed to be the trivial particle 1, we must have the indices as follows,

$$\begin{aligned} \rho(\sigma_{n-1}) \left( \dots \frac{a_{n-1} \quad a_n}{b_{n-3} \quad \bar{a}_n \quad 1} \right) &= \left( \dots \frac{a_{n-1} \quad a_n}{b_{n-3} \quad \bar{a}_{n-1} \quad 1} \right) \\ &= \sum_g \left[ \left( B_{b_{n-3} a_{n-1} a_n}^1 \right)_{\bar{a}_{n-1}, g} \left( \dots \frac{a_{n-1} \quad a_n}{b_{n-3} \quad g \quad 1} \right) \right]. \end{aligned}$$

Since the last fusion reads  $g \times a_n = 1$  we must have  $g = \bar{a}_n$  and thus

$$\begin{aligned} \rho(\sigma_{n-1}) &= \left( B_{b_{n-3} a_{n-1} a_n}^1 \right)_{\bar{a}_{n-1}, \bar{a}_n} \\ &= \sum_f \left( (F^{-1})_{b_{n-3} a_n a_{n-1}}^1 \right)_{\bar{a}_{n-1} f} R_{a_{n-1} a_n}^f \left( F_{b_{n-3} a_{n-1} a_n}^1 \right)_{f, \bar{a}_n} \\ &= \overline{R_{a_{n-1} a_n}^{b_{n-3}}} \end{aligned}$$

where the last equality follows from lemma 4.4.1.  $\square$

*Remark 4.8.2.* Note that  $\rho(\sigma_1)$  and  $\rho(\sigma_{n-1})$  are equal in the restricted basis, up to charge conjugation of the basis. Furthermore, note that if time is reversed, the roles of  $\rho(\sigma_1)$  and  $\rho(\sigma_j)$  are interchanged. Reversing the fusion in time means reading the fusion diagram from right/bottom to left/top. This is an example of time reversal symmetry; time reversal corresponds to charge conjugation, see [2].

Time reversal as charge conjugation is also simply manifested in the following lemma, as a result of lemma 4.8.1 and 4.8.3.

**Lemma 4.8.4.** *In the standard basis the  $R$  matrix can be written as*

$$\begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ \text{---} c \end{array} = R_{ab}^c \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \text{---} c \end{array} \iff \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \text{---} 1 \quad b \quad c \end{array} = R_{ab}^c \begin{array}{c} a \quad b \\ \text{---} 1 \quad a \quad c \end{array} \iff \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \text{---} c \quad a \quad 1 \end{array} = R_{ab}^{\bar{c}} \begin{array}{c} a \quad b \\ \text{---} c \quad b \quad 1 \end{array}.$$

## 4.9 Charge sectors

Consider any abstract anyon model with a non-trivial particle label  $t$ . When considering braiding of two intermediate  $t$  anyons in  $V_{t^n}^1$ , label them  $t_1$  and  $t_2$ , it suffices to work with the basis of  $V_{t^n}^1$  restricted to the intermediate charge labels  $a, b$  and  $c$  in

$$\begin{array}{c} t \quad t \\ \text{---} a \quad b \quad c \end{array}.$$

The reason for this is that only these labels are part of the  $B$  matrix describing braiding of  $t_1$  and  $t_1$ . Note that we cannot make any assumptions for the total charge  $c$ , it is not forced to be 1 since we are considering intermediate anyons in the fusion space, only the total charge of the fusion space is 1. The different possible values for  $c$  is referred to as different right charge sector. Similarly, also the  $a$  is unknown, because we are not considering the leftmost particle, there are really more anyons to the left, that fuse to different resulting charges  $a$ . This results in different left charge sectors.

If we fix the both the left and right charge sectors to be trivial, we are really considering braiding of

$$\begin{array}{c} t \quad t \\ \text{---} 1 \quad b \quad 1 \end{array}$$

in  $V_{tt}^1$ , a different fusion space, with much smaller dimension than  $V_{t^n}^1$ . There is just one free intermediate charge label here, compared to three free charge labels when not fixing the charge sectors.

Note that having trivial total charge, as in  $V_{t^n}^1$  represents the fact that we have exactly  $n$  anyons of type  $t$ . If the total charge would be  $t$ , there would really be  $n + 1$  anyons of type  $t$  available.

As an example of the importance of the charge sectors, considering braiding of the last pair of anyons. Then, the charge sector is known, it must be 1, the total charge of the fusion space  $V_{t^n}^1$ . This allows for a straight-forward computation of  $\rho(\sigma_{n-1})$ , as shown in lemma 4.8.3. Similarly, the left charge sector of  $\rho(\sigma_1)$  is 1, c.f. lemma 4.8.1.

Since we seek to characterize general braiding in  $V_{t^n}^1$ , including intermediate anyons where both the left and right charge sector are unfixed, we introduce the following definition for convenience.

**Definition 4.9.1.** The fusion space  $\tilde{V}_{t^n}$  has basis elements on the form

$$\frac{t}{b_1} \frac{t}{b_2} \frac{t}{b_3} \cdots \frac{t}{b_{n-1}} \frac{t}{b_n} \frac{t}{b_{n+1}}$$

where the charge labels  $b_1, b_2, \dots, b_{n+1}$  range over all values allowed by the fusion rules. The space  $\tilde{V}_{t^n}$  should be thought of as the space  $V_{t^n}^1$  but including all possible left and right charge sectors. That is, the space of  $n$  anyons of type  $t$  with unfixed charge sectors.

*Remark 4.9.1.* Recall the notation  $V_{t^n} = \bigoplus_c V_{t^n}^c$ , this is the space of  $n$  anyons of type  $t$  with free right charge sector.

In the space  $\tilde{V}_{t^n}$  with standard basis we naturally have  $n - 1$  braid group generators, we introduce the following notation.

**Definition 4.9.2.** The representation of the  $j$ :th braid group generator  $\sigma_j$  in  $\tilde{V}_{t^n}$  is denoted  $\rho_n(\sigma_j)$ .

## 4.10 Anyonic exchange operator $U_p$

Given any abstract anyon model, single out a non-trivial anyonic charge  $t$  and consider exchange of a pair of anyons around  $p$  enclosed anyons. That is consider the fusion space  $\tilde{V}_{t^{p+2}}$ .

In the standard basis of  $\tilde{V}_{t^n}$ , we can compute the braid group generators  $\rho_n(\sigma_j)$  for  $j = 1, 2, \dots, n - 1$  as shown in section 4.8. The exchange operator  $U_p$  (??) is then

$$U_p = \sigma_1 \sigma_2 \cdots \sigma_p \sigma_{p+1} \sigma_p \cdots \sigma_2 \sigma_1,$$

c.f. fig. 3.2. However, working in the standard basis is rather problematic for computing  $U_p$ , instead we change basis.

With the  $F$  matrix we change basis from the standard basis to a basis where the enclosed  $p$  anyons are fused. If  $p = 2$  we have

$$\frac{t}{a} \frac{t}{b} \frac{t}{d} \frac{t}{e} = \sum_f (F_{btt}^d)_{cf} \frac{t}{a} \frac{t}{b} \frac{t}{f} \frac{t}{d} \frac{t}{e}.$$

This extends to arbitrary number  $p$  of enclosed particles by repeated application of the  $F$  matrix. For any  $p \geq 2$  we can thus change basis from the standard basis to the fused basis

$$\frac{t}{a} \frac{c}{b} \frac{t}{d} \frac{t}{e}$$

where  $c$  is the resulting charge of the fused enclosed  $p$  particles. More explicitly that is

$$\begin{array}{c} t & t & \dots & t & t \\ | & | & & | & | \\ \hline \end{array} \xrightarrow{F} \begin{array}{c} \vdots \\ t \\ \cup^{c_3} \\ t \\ \cup^{c_2} \\ t \\ \cup^{c_1} \\ c \\ | \\ t \end{array} =: \begin{array}{c} t & c & t \\ | & | & | \\ \hline \end{array} \quad (4.5)$$

The intermediate charge  $c$  depends on  $p$  as follows,

$$\begin{aligned} p = 0 &\implies c = 1 \\ p = 1 &\implies c = t \\ p = 2 &\implies c \text{ is a possible result of the fusion } t \times t \\ p = 3 &\implies c \text{ is a possible result of the fusion } t \times t \times t \\ &\vdots \end{aligned}$$

Note that the braid corresponding to  $U_p$  does not depend on the intermediate charges  $c_1, c_2, \dots, c_{p-2}$ . We are now ready to compute  $U_p$  for general  $p$ ,

$$\begin{aligned} U_p \left( \frac{\begin{array}{c} t & c & t \\ | & | & | \\ a & b & d & e \end{array}}{\begin{array}{c} t & c & t \\ | & | & | \\ a & b & d & e \end{array}} \right) &= \begin{array}{c} t & c & t \\ | & | & | \\ \cup & \cup & | \\ \cup & \cup & | \\ \cup & \cup & | \\ a & b & d & e \end{array} \\ &= \sum_f (B_{act}^d)_{bf} \begin{array}{c} t & c & t \\ | & | & | \\ \cup & \cup & | \\ \cup & \cup & | \\ \cup & \cup & | \\ a & f & d & e \end{array} \\ &= \sum_f (B_{act}^d)_{bf} \sum_g (B_{ftt}^e)_{dg} \begin{array}{c} t & c & t \\ | & | & | \\ \cup & \cup & | \\ \cup & \cup & | \\ \cup & \cup & | \\ a & f & g & e \end{array} \\ &= \sum_f (B_{act}^d)_{bf} \sum_g (B_{ftt}^e)_{dg} \sum_h (B_{atc}^g)_{fh} \begin{array}{c} t & c & t \\ | & | & | \\ \cup & \cup & | \\ \cup & \cup & | \\ \cup & \cup & | \\ a & h & g & e \end{array}. \end{aligned} \quad (4.6)$$

Thus, we have the following theorem.

**Theorem 4.10.1.** *In a given abstract anyon model, exchange of a pair of  $t$ -anyons around  $p$  enclosed  $t$ -anyons is described by the exchange operator  $U_p$  given by*

$$U_p \left( \frac{\begin{array}{c} t & c & t \\ | & | & | \\ a & b & d & e \end{array}}{\begin{array}{c} t & c & t \\ | & | & | \\ a & b & d & e \end{array}} \right) = \sum_f (B_{act}^d)_{bf} \sum_g (B_{ftt}^e)_{dg} \sum_h (B_{atc}^g)_{fh} \begin{array}{c} t & c & t \\ | & | & | \\ a & h & g & e \end{array}.$$

### 4.11 Abelian anyons

General Abelian anyons can be modeled in the framework of abstract anyon models by letting the fusion rule be given by

$$[\alpha] \times [\beta] = [\alpha + \beta]$$

where  $[\alpha]$  TODO.

$$R_{ab}^{a+b} = e^{i(a+b)\pi}$$

TODO: Show  $F \equiv 1$ .

**Example 4.11.1.** The set of anyonic charges for the fermion model is given by  $\mathbb{Z}_2 = \{[0], [1]\}$ , the trivial vacuum particle  $[0]$  and one non-trivial particle  $[1]$ , with corresponding fusion rules

$$[1] \times [1] = [0].$$

For simplicity, we drop the  $[\cdot]$  brackets. As we've seen  $F_{abc}^{a+b+c} = 1$  for all  $a, b$  and  $c$ . Thus, the pentagon equation eq. (4.3) is trivial and the hexagon equation eq. (4.4) reduces to

$$\begin{aligned} R_{ac}^g (F_{bac}^d)_e^g R_{ab}^e &= \sum_f (F_{bca}^d)_f^g R_{af}^d (F_{abc}^d)_e^f \iff \\ R_{ac}^{a+c} R_{ab}^{a+b} &= \sum_f R_{af}^{a+b+c}. \end{aligned}$$

We have  $d = a + b + c$  from the  $F$  symbol  $(F_{bac}^d)$ , this gives  $a + f = a + b + c$ , i.e.  $f = b + c$ . The hexagon equation is thus reduced to

$$R_{ac}^{a+c} R_{ab}^{a+b} = R_{a,(b+c)}^{a+b+c}.$$

Since  $R_{ab}^{a+b} = 1$  if  $a$  or  $b$  equals 0. The only non-trivial case is

$$R_{11}^0 R_{11}^0 = 1 \iff R_{11}^0 = \pm 1.$$

$$c = p\alpha$$

$$\frac{\begin{array}{ccc} \alpha & c & \alpha \\ | & | & | \\ 1 & \alpha(p + \cancel{p} + 2)\alpha \end{array}}{\quad}$$

$$U_p = (2p + 1)\alpha$$



## Chapter 5

# Fibonacci anyons

The Fibonacci anyon model is the simplest non-Abelian anyon model, yet containing all the interesting features of non-Abelian anyons. Another common model is the Ising model, however Ising anyons cannot be used for general quantum computation.

This chapter is partly based on [8] and [14]. The literature is rather vague on deriving the properties of Fibonacci anyons, in this chapter we spelled out the details more clearly.

### 5.1 Preliminaries

The Fibonacci anyon model consists of two particle types, 1 (the mandatory trivial particle type) and  $\tau$  (non-trivial) with the corresponding fusion rules

$$1 \times 1 = 1, \quad 1 \times \tau = \tau, \quad \tau \times \tau = 1 + \tau$$

where  $\tau$

The following observation motivates the name of the model. Consider the fusion spaces  $V_{\tau^n}^1$ , where  $\tau^n$  denotes  $n$  repetitions of  $\tau$ . This is the space of possible fusions of  $n$  Fibonacci anyons, having total charge 1. Writing out the canonical basis for these spaces we find

$$\begin{aligned} V_{\tau\tau}^1 &: \left\{ \frac{\tau \ \tau}{1 \ \tau \ 1} \right\} \\ V_{\tau\tau\tau}^1 &: \left\{ \frac{\tau \ \tau \ \tau}{1 \ \tau \ \tau \ 1} \right\} \\ V_{\tau\tau\tau\tau}^1 &: \left\{ \frac{\tau \ \tau \ \tau \ \tau}{1 \ \tau \ 1 \ \tau \ 1}, \frac{\tau \ \tau \ \tau \ \tau}{1 \ \tau \ \tau \ \tau \ 1} \right\} \\ V_{\tau\tau\tau\tau\tau}^1 &: \left\{ \frac{\tau \ \tau \ \tau \ \tau \ \tau}{1 \ \tau \ 1 \ \tau \ \tau \ 1}, \frac{\tau \ \tau \ \tau \ \tau \ \tau}{1 \ \tau \ \tau \ 1 \ \tau \ 1}, \frac{\tau \ \tau \ \tau \ \tau \ \tau}{1 \ \tau \ \tau \ \tau \ \tau \ 1} \right\} \\ V_{\tau\tau\tau\tau\tau\tau}^1 &: \left\{ \frac{\tau \ \tau \ \tau \ \tau \ \tau \ \tau}{1 \ \tau \ 1 \ \tau \ 1 \ \tau \ 1}, \frac{\tau \ \tau \ \tau \ \tau \ \tau \ \tau}{1 \ \tau \ 1 \ \tau \ \tau \ \tau \ 1}, \frac{\tau \ \tau \ \tau \ \tau \ \tau \ \tau}{1 \ \tau \ \tau \ 1 \ \tau \ 1}, \frac{\tau \ \tau \ \tau \ \tau \ \tau \ \tau}{1 \ \tau \ \tau \ 1 \ \tau \ \tau \ 1}, \frac{\tau \ \tau \ \tau \ \tau \ \tau \ \tau}{1 \ \tau \ \tau \ \tau \ \tau \ 1} \right\}. \end{aligned}$$

The reader can continue this list. Note that the bottom line in the fusion diagrams of the fusion states in the bases can be seen as strings of  $\tau$  and 1, having  $1\tau$  at the start and  $\tau 1$  at the end. Furthermore, the string is subject to the condition that 1 may not be followed by 1. Indeed, 1 on the bottom row, fuses with  $\tau$  from the top, to give  $\tau$ . It is thus clear that TODO: Clarify, recursion leads to Fib def.

$$\dim V_{\tau^n}^1 = \text{Fib}(n-1)$$

where  $\text{Fib}(n)$  denotes the  $n$ :th Fibonacci number. That is, the dimension of the fusion space grows with increasing number of  $\tau$ s as the Fibonacci series.

As we have seen in , an anyon model is determined by the corresponding  $F$ - and  $R$ -matrices. We continue by computing these matrices.

## 5.2 Determining the model: Computing the $F$ and $R$ matrices

TODO: Write down computation.

TODO: Discuss the free parameter in  $F$  that is fixed by gauge fixing. Preskill p. 60.

$$R_{\tau\tau} = \begin{pmatrix} R_{\tau\tau}^1 & 0 \\ 0 & R_{\tau\tau}^\tau \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-3\pi i/5} \end{pmatrix},$$

$$F_{\tau\tau\tau} = \begin{pmatrix} F_{\tau\tau\tau}^{11} & F_{\tau\tau\tau}^{1\tau} \\ F_{\tau\tau\tau}^{\tau 1} & F_{\tau\tau\tau}^{\tau\tau} \end{pmatrix} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}.$$

For convenience, when discussing Fibonacci anyons, let

$$F := F_{\tau\tau\tau}^\tau, \quad R := R_{\tau\tau}, \quad B := F^{-1}RF.$$

## 5.3 Braiding of Fibonacci anyons

In this section we shall compute the braid group generators for various number of Fibonacci anyons. Ultimately, we shall compute  $\rho(\sigma_j)$  for  $V_{\tau^n}^1$ . Having trivial total charge represents the fact that we have exactly  $n$  Fibonacci anyons. If the total charge would be  $\tau$ , there would really be  $n+1$  Fibonacci anyons available. As we shall see, there will be some subtleties regarding different charge sectors, i.e. different total charge, when considering braiding of intermediate  $\tau$  anyons in  $V_{\tau^n}^1$ . We begin with some elementary examples.

Recall the decomposition

$$V_{a_1 \cdots a_n} = \bigoplus_c V_{a_1 \cdots a_n}^c,$$

in particular

$$V_{\tau^n} = V_{\tau^n}^1 \oplus V_{\tau^n}^\tau.$$

### 5.3.1 Prototypical examples

**Example 5.3.1** (Braiding in  $V_{\tau^2}$ ). The two charge sectors of  $V_{\tau^2}$  have the standard basis

$$V_{\tau\tau}^1 = \text{span} \left\{ \frac{\tau \quad \tau}{1 \quad \tau \quad 1} \right\}, \quad V_{\tau\tau}^\tau = \text{span} \left\{ \frac{\tau \quad \tau}{1 \quad \tau \quad \tau} \right\}.$$

That is, the fusion space  $V_{\tau^2}$  is two-dimensional and we denote the ordered basis by  $\{1, \tau\}$ . Since there is only two  $\tau$ -anyons, there are only one generator for the braid group,  $\sigma_1$ . We compute  $\rho(\sigma_1)$  by considering its action on the standard fusion states,

$$\begin{aligned} \rho(\sigma_1) \frac{\tau \quad \tau}{1 \quad \tau \quad 1} &= \frac{\tau \quad \tau}{1 \quad \tau \quad 1} = R_{\tau\tau}^1 \frac{\tau \quad \tau}{1 \quad \tau \quad 1} \\ \rho(\sigma_1) \frac{\tau \quad \tau}{1 \quad \tau \quad \tau} &= \frac{\tau \quad \tau}{1 \quad \tau \quad \tau} = R_{\tau\tau}^\tau \frac{\tau \quad \tau}{1 \quad \tau \quad \tau}. \end{aligned}$$

Thus we have

$$\left. \begin{aligned} \rho(\sigma_1)_{11} &= R_{\tau\tau}^1 \\ \rho(\sigma_1)_{1\tau} &= 0 \\ \rho(\sigma_1)_{\tau\tau} &= R_{\tau\tau}^\tau \\ \rho(\sigma_1)_{\tau 1} &= 0 \end{aligned} \right\} \iff \rho(\sigma_1) = \begin{pmatrix} R_{\tau\tau}^1 & 0 \\ 0 & R_{\tau\tau}^\tau \end{pmatrix}.$$

This also follows immediately from lemma 4.8.1.

**Example 5.3.2** (Braiding in  $V_{\tau^3}$ ). The two charge sectors of  $V_{\tau^3}$  have the standard basis

$$V_{\tau\tau\tau}^1 = \text{span} \left\{ \frac{\tau \quad \tau \quad \tau}{1 \quad \tau \quad \tau \quad 1} \right\}, \quad V_{\tau\tau\tau}^\tau = \text{span} \left\{ \frac{\tau \quad \tau \quad \tau}{1 \quad \tau \quad 1 \quad \tau}, \frac{\tau \quad \tau \quad \tau}{1 \quad \tau \quad \tau \quad \tau} \right\}.$$

That is, the fusion space  $V_{\tau^3}$  is three-dimensional, and we denote the ordered basis by  $\{\tau 1, 1\tau, \tau\tau\}$ . Since there are three  $\tau$ -anyons, there are two generators for the braid group,  $\sigma_1$  and  $\sigma_2$ . Lemma 4.8.1 gives

$$\begin{aligned} \rho(\sigma_1)_{(\tau 1), (\tau 1)} &= R_{\tau\tau}^\tau \\ \rho(\sigma_1)_{(1\tau), (1\tau)} &= R_{\tau\tau}^1 \\ \rho(\sigma_1)_{(\tau\tau), (\tau\tau)} &= R_{\tau\tau}^\tau \\ \rho(\sigma_1)_{i,j} &= 0 \text{ for } i \neq j. \end{aligned}$$

In matrix form that is

$$\rho(\sigma_1) = \begin{pmatrix} R_{\tau\tau}^\tau & & \\ & R_{\tau\tau}^1 & \\ & & R_{\tau\tau}^\tau \end{pmatrix} = R_{\tau\tau}^\tau \oplus R.$$

Lemma 4.8.3 gives

$$\rho(\sigma_2)_{(\tau 1), (\tau 1)} = R_{\tau\tau}^\tau.$$

We compute  $\rho(\sigma_2)$  for the  $\tau$ -charge sector by considering its action on the standard fusion states

$$\begin{aligned}\rho(\sigma_2) \frac{\tau \tau \tau}{1 \tau 1 \tau} &= \frac{\tau \tau \tau}{1 \tau 1 \tau} = (B_{\tau\tau\tau}^\tau)_{11} \frac{\tau \tau \tau}{1 \tau 1 \tau} + (B_{\tau\tau\tau}^\tau)_{1\tau} \frac{\tau \tau \tau}{1 \tau \tau \tau}, \\ \rho(\sigma_2) \frac{\tau \tau \tau}{1 \tau \tau \tau} &= \frac{\tau \tau \tau}{1 \tau \tau \tau} = (B_{\tau\tau\tau}^\tau)_{\tau 1} \frac{\tau \tau \tau}{1 \tau 1 \tau} + (B_{\tau\tau\tau}^\tau)_{\tau\tau} \frac{\tau \tau \tau}{1 \tau \tau \tau}.\end{aligned}$$

Thus we have

$$\begin{aligned}\rho(\sigma_2)_{(1\tau), (1\tau)} &= (B_{\tau\tau\tau}^\tau)_{11} \\ \rho(\sigma_2)_{(1\tau), (\tau\tau)} &= (B_{\tau\tau\tau}^\tau)_{1\tau} \\ \rho(\sigma_2)_{(\tau\tau), (1\tau)} &= (B_{\tau\tau\tau}^\tau)_{\tau 1} \\ \rho(\sigma_2)_{(\tau\tau), (\tau\tau)} &= (B_{\tau\tau\tau}^\tau)_{\tau\tau}.\end{aligned}$$

In matrix form that is

$$\rho(\sigma_2) = \begin{pmatrix} R_{\tau\tau}^\tau & & \\ & B_{11} & B_{1\tau} \\ & B_{\tau 1} & B_{\tau\tau} \end{pmatrix} = R_{\tau\tau}^\tau \oplus B.$$

**Example 5.3.3** (Braiding in  $V_{\tau^4}^1$ ). Consider  $V_{\tau^4}^1$ , this is the smallest non-trivial proper fusion space having dimension two. The fusion space is proper in the sense that there is only one charge sector and it is the trivial (vacuum) charge sector. That is, there are really only 4 Fibonacci anyons. We shall compute the corresponding braid group representation determined by exchange of Fibonacci anyons in the standard basis of  $V_{\tau^4}^1$ .

**Proposition 5.3.1.** *The representation of the braid group generators determined by  $V_{\tau^4}^1$ , i.e. four Fibonacci anyons of trivial total charge, is*

$$\rho(\sigma_1) = R, \quad \rho(\sigma_2) = B, \quad \rho(\sigma_3) = R.$$

*Proof.* Lemma 4.8.1 gives

$$\rho(\sigma_1) = B_{1\tau\tau}^\tau = R_{\tau\tau} = \begin{pmatrix} R_{\tau\tau}^1 & 0 \\ 0 & R_{\tau\tau}^\tau \end{pmatrix}.$$

Lemma 4.8.2 gives

$$\rho(\sigma_2)_{ij} = (B_{\tau\tau\tau}^\tau)_{ij} = \sum_f \left( (F^{-1})_{\tau\tau\tau}^\tau \right)_{if} R_{\tau\tau}^f (F_{\tau\tau\tau}^\tau)_{fj}$$

i.e.

$$\rho(\sigma_2) = F^{-1} R F =: B.$$

Finally, lemma 4.8.3 gives

$$\rho(\sigma_3)_{ij} = \delta_{ij} R_{\tau\tau}^{\bar{\tau}} \implies \rho(\sigma_3) = R_{\tau\tau}$$

since  $\tau$  is its own antiparticle,  $\bar{\tau} = \tau$ . □

### 5.3.2 General braiding in $\tilde{V}_{\tau^n}$

Recall the discussion of charge sectors from section 4.9. Since we cannot, in general, restrict the fusion space to a fixed charge sector, we shall compute the braid group generators in all charge sectors for two, three and four  $\tau$  anyons. That is, we shall compute  $\rho_n(\sigma_j)$  in  $V_{\tau^n}$  for  $n = 2, 3, 4$ . (The notation was introduced in definitions 4.9.1 and 4.9.2.) These examples give important insights of how the braid representation grows with the number of anyons, but most importantly these braids shall later be crucial when determining  $U_p$  in theorem 5.4.1.

**Example 5.3.4** (Braiding of 2 anyons in  $V_{\tau^n}^1$ ). Consider  $V_{\tau^n}^1$  with the standard basis. Restrict attention to a pair of neighbouring  $\tau$  anyons. That is, we restrict attention to a part of the space with basis

$$\left\{ \text{valid intermediate charges } abc \text{ in } \frac{\tau \quad \tau}{a \quad b \quad c} \right\} \equiv \left\{ 1\tau 1, 1\tau\tau, \tau\tau 1, \frac{\tau 1\tau}{\tau\tau\tau} \right\}.$$

From lemma 4.8.2 we have

$$\rho_2(\sigma_1) = B_{a\tau\tau}^c \iff \rho_2(\sigma_1)_{ij} = (B_{a\tau\tau}^c)_{ij}$$

where the indices  $i$  and  $j$  run over the given basis element. In the above given order of the basis fusion states the  $B$ -matrix is block diagonal. That is, in the obvious identification of the fusion space with  $\mathbb{C}^5$  we have

$$\begin{aligned} \rho_2(\sigma_1) &= \begin{pmatrix} (B_{1\tau\tau}^1)_{\tau\tau} & & & & \\ & (B_{1\tau\tau}^\tau)_{\tau\tau} & & & \\ & & (B_{\tau\tau\tau}^1)_{\tau\tau} & & \\ & & & (B_{\tau\tau\tau}^\tau)_{11} & (B_{\tau\tau\tau}^\tau)_{1\tau} \\ & & & (B_{\tau\tau\tau}^\tau)_{\tau 1} & (B_{\tau\tau\tau}^\tau)_{\tau\tau} \end{pmatrix} \\ &= \begin{pmatrix} R_{\tau\tau}^1 & & & & \\ & R_{\tau\tau}^\tau & & & \\ & & R_{\tau\tau}^\tau & & \\ & & & B_{11} & B_{1\tau} \\ & & & B_{\tau 1} & B_{\tau\tau} \end{pmatrix} \\ &= R \oplus R_{\tau\tau}^\tau \oplus B. \end{aligned}$$

**Example 5.3.5** (Braiding of 3 anyons in  $V_{\tau^n}^1$ ). Next, taking a third Fibonacci anyon  $\tau$  into account and choosing the basis ordered as follows

$$\left\{ \begin{array}{l} \text{valid intermediate charges} \\ abcd \text{ in } \frac{\tau \quad \tau \quad \tau}{a \quad b \quad c \quad d} \end{array} \right\} \equiv \left\{ 1\tau\tau 1, \frac{1\tau 1\tau}{1\tau\tau\tau}, \frac{\tau 1\tau 1}{\tau\tau\tau 1}, \frac{\tau 1\tau\tau}{\tau\tau 1\tau}, \frac{\tau\tau 1\tau}{\tau\tau\tau\tau} \right\}$$

we have

$$\begin{aligned} \rho_3(\sigma_1) &= R_{\tau\tau\tau}^\tau \oplus R \oplus B \oplus \begin{pmatrix} B_{11} & & B_{12} \\ & R_{\tau\tau}^\tau & \\ B_{21} & & B_{22} \end{pmatrix} \\ \rho_3(\sigma_2) &= R_{\tau\tau\tau}^\tau \oplus B \oplus R \oplus \begin{pmatrix} R_{\tau\tau}^\tau & & \\ & B_{11} & B_{1\tau} \\ & B_{\tau 1} & B_{\tau\tau} \end{pmatrix} \end{aligned}$$

**Example 5.3.6** (Braiding of 4 anyons in  $V_{\tau^n}^1$ ). In the ordered basis

$$\left\{ \begin{array}{c} \text{valid intermediate charges} \\ abcde \text{ in } \frac{\tau \ \tau \ \tau \ \tau}{a \ b \ c \ d \ e} \end{array} \right\} \equiv \left\{ \begin{array}{cccc} \tau 1 \tau 1 \tau, & 1 \tau 1 \tau \tau, & \tau 1 \tau \tau 1, & \tau 1 \tau \tau \tau, \\ 1 \tau 1 \tau 1, & 1 \tau \tau 1 \tau, & \tau \tau 1 \tau 1, & \tau \tau \tau 1 \tau, \\ 1 \tau \tau \tau 1, & 1 \tau \tau \tau \tau, & \tau \tau \tau \tau 1, & \tau \tau 1 \tau \tau, \\ & & & \tau \tau \tau \tau \tau \end{array} \right\}$$

we have

$$\begin{aligned} \rho_4(\sigma_1) &= \begin{pmatrix} R_{\tau\tau}^1 & \\ & R_{\tau\tau}^\tau \end{pmatrix} \oplus \begin{pmatrix} R_{\tau\tau}^1 & & \\ & R_{\tau\tau}^\tau & \\ & & R_{\tau\tau}^\tau \end{pmatrix} \oplus \begin{pmatrix} B_{11} & & B_{1\tau} \\ & R_{\tau\tau}^\tau & \\ B_{\tau 1} & & B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} B_{11} & & & B_{12} \\ & B_{11} & & B_{12} \\ & & R_{\tau\tau}^\tau & \\ B_{21} & & & B_{22} \\ & B_{21} & & B_{22} \end{pmatrix} \\ \rho_4(\sigma_2) &= \begin{pmatrix} B_{11} & B_{1\tau} \\ B_{\tau 1} & B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} B_{11} & & B_{1\tau} \\ & R_{\tau\tau}^\tau & \\ B_{\tau 1} & & B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} R_{\tau\tau}^1 & & & \\ & R_{\tau\tau}^\tau & & \\ & & R_{\tau\tau}^\tau & \\ & & & B_{11} \ B_{12} \\ & & & B_{21} \ B_{22} \end{pmatrix} \\ \rho_4(\sigma_3) &= \begin{pmatrix} R_{\tau\tau}^1 & \\ & R_{\tau\tau}^\tau \end{pmatrix} \oplus \begin{pmatrix} R_{\tau\tau}^\tau & & \\ & B_{11} \ B_{1\tau} \\ & B_{\tau 1} \ B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} R_{\tau\tau}^\tau & & \\ & R_{\tau\tau}^1 & \\ & & R_{\tau\tau}^\tau \end{pmatrix} \oplus \begin{pmatrix} B_{11} & B_{1\tau} & & \\ B_{\tau 1} & B_{\tau\tau} & & \\ & & B_{11} & B_{1\tau} \\ & & & R_{\tau\tau}^\tau \\ & & & & B_{\tau 1} \ B_{\tau\tau} \end{pmatrix} \end{aligned}$$

Note that in this order of the basis, the last block of  $\rho_4(\sigma_2)$  is precisely  $\rho_2(\sigma_1)$ . Indeed, basis elements in the  $\tau\tau$ -sector is ordered by the internal charge sectors.

*Remark 5.3.1.* As is clearly manifested in the three examples, the representation of the braid group generators is always split into four blocks, one block for each two-sided charge sector,  $11, 1\tau, \tau 1$  and  $\tau\tau$ . These blocks will always be disjoint since there is no way of transforming between them. Compare with the decomposition  $V_{ab} = \bigoplus_c V_{ab}^c$ . The dimension of each of these blocks grows as the Fibonacci numbers, simply because the basis states in each block are Fibonacci strings of 1's and  $\tau$ 's. In general we have the dimension of the decomposition for  $n$  Fibonacci anyons given by

$$\dim \rho_n(\sigma_j) = \underbrace{F_{n-2} + F_{n-1}}_{F_n} + \underbrace{F_{n-1} + F_n}_{F_{n+1}} = F_{n+2}.$$

### 5.3.3 Spectrum of $\rho_n(\sigma_j)$

**Theorem 5.3.2.** *The spectrum without multiplicities of the representation of the braid group generator  $\rho_n(\sigma_j)$  is independent of  $n$  and  $j$  and given by*

$$\text{spec}(\rho_n(\sigma_j)) = \{R_{\tau\tau}^1, R_{\tau\tau}^\tau\}.$$

*Proof.* From lemma 2.2.1 we have that the all braid group generators  $\sigma_j$  are conjugate, this translates to that the corresponding representations  $\rho_n(\sigma_j)$  are similar for fixed  $n$ . That is, for fixed  $n$  there exists an invertible matrix  $S$  such

that

$$\rho_n(\sigma_{j+1}) = S\rho_n(\sigma_j)S^{-1}.$$

(In particular we have  $S = \rho_n(\sigma_j)\rho_n(\sigma_{j+1})$  from the proof of lemma 2.2.1.) Thus, the eigenvalues of  $\rho_n(\sigma_j)$  are independent of  $j$ . Fix  $j = 1$ , the representation  $\rho_n(\sigma_1)$  of  $\sigma_1$  acts on  $\tilde{V}_{\tau^n}$  with basis elements

$$\frac{\tau}{b_1} \frac{\tau}{b_2} \frac{\tau}{b_3} \cdots \frac{\tau}{b_{n-1}} \frac{\tau}{b_n} \frac{\tau}{b_{n+1}}.$$

However, only the labels  $b_1, b_2, b_3$  enter in the expression for  $\rho_n(\sigma_1)$ , thus it is really only the space  $\tilde{V}_{\tau^2}$  with basis elements

$$\frac{\tau}{b_1} \frac{\tau}{b_2} \frac{\tau}{b_3}$$

that  $\rho_n(\sigma_1)$  acts on. Thus, as discussed in remark 4.8.1, as  $n$  increases the matrix  $\rho_n(\sigma_1)$  increases and gets repeated block. These repeated blocks give no new eigenvalues (but give increased multiplicity for the existing eigenvalues). Thus, the spectrum of  $\rho_n(\sigma_1)$  is independent of  $n$ , not counting multiplicities.

To sum up, the spectrum of  $\rho_n(\sigma_j)$ , not counting multiplicities, is independent of both  $n$  and  $j$ . Thus, we can compute the spectrum from the special case  $\rho_2(\sigma_1)$ , computed in example 5.3.4.  $\square$

*Remark 5.3.2.* This theorem trivially generalizes to *any* anyon model. However, the specific eigenvalues are of course different.

## 5.4 Exchange operator $U_p$

Consider the fusion space  $V_{\tau^n}^1$  and exchange of a pair of anyons,  $j$  and  $k$ , around  $p$  enclosed anyons,  $j+1, j+2, \dots, k-1$ . We shall use section 4.10, in particular theorem 4.10.1, to compute  $U_p$  for general  $p$  for Fibonacci anyons. Instead of immediately applying theorem 4.10.1, which leads to tedious computation, we take a slightly different route in the proof.

**Theorem 5.4.1.** *Exchange of a pair of anyons around  $p$  enclosed anyons introduces a non-abelian anyonic phase  $U_p$ , given by*

$$\begin{aligned} U_0 &= \rho_2(\sigma_1) \\ U_1 &= \rho_3(\sigma_1)\rho_3(\sigma_2)\rho_3(\sigma_1) \\ U_p &= F^{-1}\rho_4(\sigma_1)\rho_4(\sigma_2)\rho_4(\sigma_3)\rho_4(\sigma_2)\rho_4(\sigma_1)F, \quad p \geq 2 \end{aligned}$$

*expressed in the fused basis eq. (4.5),  $F$  is the corresponding change of basis matrix. The braid generators  $\rho_m(\sigma_j)$  are computed in examples 5.3.4 to 5.3.6.*

*Proof.* The braid corresponding to  $U_0$  is given by theorem 4.10.1 with  $c = 1$ .

That is, (using lemma 4.5.1)

$$\begin{aligned}
 & \begin{array}{c} \tau \quad 1 \quad \tau \\ \text{[Braid diagram with 4 strands labeled } a, b, b, e \text{ at the bottom and } \tau, 1, \tau \text{ at the top]} \\ a \quad b \quad b \quad e \end{array} = \sum_g (B_{a\tau\tau}^e)_{bg} \begin{array}{c} \tau \quad 1 \quad \tau \\ \text{[Braid diagram with 3 strands labeled } a, g, g \text{ at the bottom and } \tau, 1, \tau \text{ at the top]} \\ a \quad g \quad g \quad e \end{array} \\
 \\
 & \Leftrightarrow \begin{array}{c} \tau \quad \tau \\ \text{[Braid diagram with 2 strands labeled } a, b \text{ at the bottom and } \tau, \tau \text{ at the top]} \\ a \quad b \quad e \end{array} = \sum_g (B_{a\tau\tau}^e)_{bg} \begin{array}{c} \tau \quad \tau \\ \text{[Braid diagram with 2 strands labeled } a, g \text{ at the bottom and } \tau, \tau \text{ at the top]} \\ a \quad g \quad e \end{array}
 \end{aligned}$$

This braid is computed in example 5.3.4.

Similarly, the braid corresponding to  $U_1$  is given by theorem 4.10.1 with  $c = \tau$  and is computed in example 5.3.5.

Finally, the braid corresponding to  $U_p$  for  $p \geq 2$  is given by theorem 4.10.1 where the intermediate charge  $c$  can be both 1 and  $\tau$ . The intermediate charges  $c_1, c_2, \dots, c_{p-2}$  in eq. (4.5) do not affect the braid  $U_p$ . Indeed these labels do not enter equation theorem 4.10.1 determining  $U_p$ . Thus, it suffices to consider the subspace of the fusion space containing only the labels  $a, b, c, d, e$ . First, use the  $F$  matrix to change basis as described in eq. (4.5) so that theorem 4.10.1 can be used, finally, change back to the standard basis with  $F^{-1}$ .

To simplify the computation, note that the braid in the proof of theorem 4.10.1 can be modeled by letting the intermediate charge  $c$  be the intermediate charge of fusion of two  $\tau$ -anyons. That is, this braid is same as the braid described in example 5.3.6, up to change of basis by the  $F$ -matrix. Hence the result follows.  $\square$

*Remark 5.4.1.* Note that the case when  $c$  is allowed to be both 1 and  $\tau$  (i.e. when  $p \geq 2$ ) is not the direct sum of the cases  $c = 1$  and  $c = \tau$  because transitions between fusion states with  $c = 1$  and  $c = \tau$  do occur.

*Remark 5.4.2.* As stated in the theorem, the dimension of  $U_p$  is not increasing with  $p \geq 2$ . This is simply due the abuse of notation, as discussed in remark 4.8.1, in which the braid operator is represented in the part of the space containing only labels which enter into the expression for the braid operator. The dimension of  $U_p$  with increasing  $p$  of course grows as the Fibonacci numbers,  $\dim U_p = \text{Fib}(p)$ , cf. remark 5.3.1. Representing  $U_p$  in the full basis results in repeating blocks in the corresponding matrix. Indeed, each intermediate charge  $c_j$  for  $1 \leq j \leq p-2$  gives two additional fusion states (corresponding to  $c_j = 1$  and  $c_j = \tau$ ), thus giving a copy of the block describing the action on the  $abcde$ -part of the fusion state. Note that the order of the fusion state in the bases generally permute the discussed blocks. Compare with section 4.8.

The reason for working in the fused space eq. (4.5) is precisely that this allows for a simpler representation of  $U_p$ . If we would have worked in the standard basis, such repeating blocks would not occur for  $U_p$ , and the representation for  $U_p$  would be much more involved. Indeed, it is in principle straight forward to transform between the two representations, in the fused basis and in the standard basis, by applying the  $F$  matrix to change basis, as described in eq. (4.5).



The fact that  $U_p$  is independent of  $p$  for  $p \geq 2$  is due to the fact that fusion of  $p$  anyons of type  $\tau$  always results in 1 or  $\tau$ . However, if we consider an anyon model with fusion rules

$$t \times t = a, \quad t \times a = t$$

so that

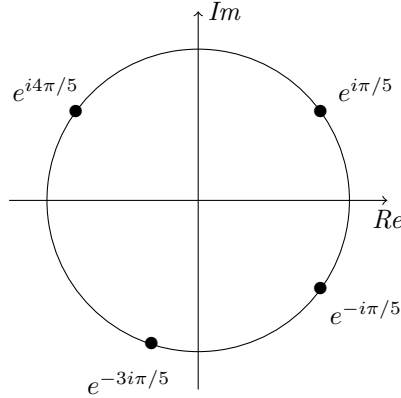
$$t \times t \times t = t$$

then clearly  $U_2$  and  $U_3$  are different, since the intermediate charge  $c$  is different.

We now have an explicit expression for the exchange operator  $U_p$ , straight forward computation gives its eigenvalues.

**Corollary 5.4.2.** *The eigenvalues of  $U_p$  for Fibonacci anyons are*

$$\begin{aligned} \sigma(U_0) &= \{R_{\tau\tau}^1, R_{\tau\tau}^\tau\} \\ &= \{e^{4i\pi/5}, e^{-3i\pi/5}\} \\ \sigma(U_1) &= \{R_{\tau\tau}^1, (R_{\tau\tau}^\tau)^3, (R_{\tau\tau}^\tau)^7\} \\ &= \{e^{4i\pi/5}, e^{i\pi/5}, e^{-i\pi/5}\} \\ \sigma(U_p) &= \{R_{\tau\tau}^1, (R_{\tau\tau}^\tau)^3, (R_{\tau\tau}^\tau)^7, R_{\tau\tau}^\tau\} \quad (p \geq 2) \\ &= \{e^{4i\pi/5}, e^{i\pi/5}, e^{-i\pi/5}, e^{-3i\pi/5}\}. \end{aligned}$$



In chapter 3 we showed the connection between eigenvalues for the exchange operator and bounds for the kinetic energy. The distance of the eigenvalues from 1 along the complex unit circle given the energy bounds. We thus see that simple exchange of two anyons has a higher corresponding kinetic energy than exchange of two anyons around  $p \geq 1$  anyons. In particular  $U_p$  for  $p \geq 1$  has the eigenvalue  $e^{\pm i\pi/5}$  closest to 1.

We see that exchange of Fibonacci described by  $U_p$  always corresponds to a non-zero kinetic energy, it is natural to ask if there are braids that do have zero energy. The following corollary positively answers this.

**Corollary 5.4.3** (Zero energy braid for Fibonacci anyons). *With  $U_p$  as in theorem 5.4.1 we have*

$$\sigma(U_p^5) = \{1, -1\}$$

for all  $p \geq 0$ , by straight-forward computation. This shows that braiding a pair of Fibonacci anyons five times around any number of Fibonacci anyons behaves as both fermions and bosons simultaneously. In particular, the occurrence of the eigenvalue 1 shows that the braid  $U_p^5$  is a zero-energy braid, according to the discussion in chapter 3.

This can be seen as a consequence of the fact that  $(R_{\tau\tau}^1)^5 = 1$  and  $(R_{\tau\tau}^\tau)^5 = -1$ .

## 5.5 Quantum dimension and fusion probabilities

As explained in detail in [8], the quantum dimension  $d_a$  of an anyon of type  $a$  is the rate of growth in dimension of the fusion space  $V_{a^n}^1$  as  $n$  grows. Explicitly that is

$$d_a = \lim_{n \rightarrow \infty} \frac{\dim(V_{a^{n+1}})}{\dim(V_{a^n})}.$$

Recall that  $a^n$  denotes  $\underbrace{\tau \cdots \tau}_n$  if  $a$  is a particle label. The quantum dimension can be immediately computed from the fusion rules. For the Fibonacci anyons we have the fusion rule

$$\tau \times \tau = 1 + \tau$$

which gives the quantum dimension

$$d_\tau^2 = 1 + d_\tau \iff d_\tau = \varphi$$

note that only the positive solution is meaningful. Furthermore, the trivial particle 1 has quantum dimension  $d_1 = 1$ . Indeed, the corresponding fusion space  $V_{1^n}^1$  does not grow in dimension as  $n$  increases.

Using the quantum dimension, [8] shows the following result.

**Proposition 5.5.1.** *The probability  $P(ab \rightarrow c)$  that anyons of type  $a$  and  $b$  fuse to an anyon of type  $c$  is given by*

$$P(ab \rightarrow c) = \frac{N_{ab}^c d_c}{d_a d_b}.$$

Consider fusion of  $n$  Fibonacci  $\tau$  anyons. Proposition 5.5.1 together with  $d_\tau = \varphi$  and  $d_1 = 1$  gives

$$P(\tau\tau \rightarrow \tau) = \varphi^{-1}, \quad P(\tau\tau \rightarrow 1) = \varphi^{-2}.$$

We then have the probability of  $n$  Fibonacci  $\tau$  anyons fusing to charge  $\tau$  given by the recursion relation

$$\begin{aligned} P(\tau^n \rightarrow \tau) &= P(\tau^{n-1} \rightarrow \tau)P(\tau\tau \rightarrow \tau) + P(\tau^{n-1} \rightarrow 1)P(1\tau \rightarrow \tau) \\ &= \varphi^{-1}P(\tau^{n-1} \rightarrow \tau) + P(\tau^{n-1} \rightarrow 1) \\ &= \varphi^{-1}P(\tau^{n-1} \rightarrow \tau) + P(\tau^{n-2} \rightarrow \tau)P(\tau\tau \rightarrow 1) \\ &= \varphi^{-1}P(\tau^{n-1} \rightarrow \tau) + \varphi^{-2}P(\tau^{n-2} \rightarrow \tau). \end{aligned}$$

With initial values  $P(\tau \rightarrow \tau) = 1$  and  $P(\tau\tau \rightarrow \tau) = \varphi^{-1}$  we get

$$P(\tau^n \rightarrow \tau) = \text{Fib}(n)\varphi^{1-n}.$$

Using the closed form expression

$$\text{Fib}(n) = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

we get the limit

$$\lim_{n \rightarrow \infty} P(\tau^n \rightarrow \tau) = \frac{\varphi}{\sqrt{5}} \approx 0.72$$

and thus

$$\lim_{n \rightarrow \infty} P(\tau^n \rightarrow 1) \approx 0.28.$$

TODO: Can this be used to say anything more about the eigenvalues of  $U_p$ ? Some eigenvalues correspond to  $c = 1$ , some correspond to  $c = \tau$ , as we've now seen the latter case is more probable.

## Chapter 6

# Topological Quantum Computation

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