Non-Abelian Anyons: Statistical Repulsion and Topological Quantum Computation

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Chapter 1

Introduction

Chapter 2

How anyons arise

2.1 Particle statistics

2.2 The braid group

The braid group B_n (representing braids on n strands) is the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i-j| \geq 2, \tag{2.1a}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{2.1b}$$

Lemma 2.2.1. *All generators of the braid group are conjugate.*

Proof. First, σ_i is conjugate to σ_{i+1} for all i, as seen by

$$\begin{aligned}(\sigma_i \sigma_{i+1}) \sigma_i (\sigma_i \sigma_{i+1})^{-1} &= (\sigma_i \sigma_{i+1} \sigma_i) \sigma_{i+1}^{-1} \sigma_i^{-1} \\ \text{[eq. (2.1b)]} &= (\sigma_{i+1} \sigma_i \sigma_{i+1}) \sigma_{i+1}^{-1} \sigma_i^{-1} \\ &= \sigma_{i+1} \sigma_i \sigma_i^{-1} \\ &= \sigma_{i+1}.\end{aligned}$$

Finally, conjugation is transitive, thus all braid generators are conjugate. □

Definition 2.2.1. TODO

$$U_p = \sigma_1 \sigma_2 \cdots \sigma_p \sigma_{p+1} \sigma_p \cdots \sigma_2 \sigma_1,$$

2.3 Representation theory

2.4 Physical realizability

2.5 Anyons with or without quantum kinetic energy

Interferometry Bonderson.

Wavefunction density Lundholm.

Chapter 3

Lower bound for the kinetic energy in a gas of anyons

This chapter gives an outline of a method to estimate lower bounds for the kinetic energy of a gas of anyons, more generally known as Lieb-Thirring inequalities. This chapter is base on [6].

3.1 Preliminaries

Consider N anyons in $\mathbb{R}^2$, with coordinate $x_1, \dots, x_N$. This collection of anyons are described by an N-particle wave function $\psi \in L^2(\mathbb{R}^{2N})$, i.e. a square-integrable complex-valued function. The kinetic energy T of the system of N anyons is by definition

$$T \coloneqq \sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} |\Delta_j \psi|^2 dx$$

where Δ_j acts on x_j. In order to give bounds for the kinetic energy we shall determine the kinetic energy for pairs of anyons, by factoring out the dynamics of two anyons from the system of N anyons.

Among the N anyons, let all except two of them be fixed. Let $x_j, x_k \in \mathbb{R}^2$ denote the coordinates for the two free anyons and define

$$x_{\text{cm}} = \frac{1}{2}(x_j + x_k), \quad x_{\text{rel}} = \frac{1}{2}(x_j - x_k).$$

Consider a frame of reference where the center of mass is at the origin, i.e. $x_{\text{cm}} = 0$. Let (r, φ) be the polar coordinates for the relative coordinates x_{rel}. The N-particle wave function can thus be parametrized as

$$\psi = \psi(x', r, \varphi),$$

where x' is the coordinates for the anyons that we considered to be fixed. If we also freeze out the radial dependence by fixing r, we factor out the angle-dependence from the wave function and write

$$\psi(\varphi) \coloneqq \psi(x', r, \varphi).$$

We shall consider the angular dynamics of the two free anyons. As the free anyons circle each other, i.e. as φ increases from 0 to π, a number of fixed anyons may be enclosed. As we shall see, the number of encircled anyons will play a crucial role, therefore we separate the state space $\mathbb{R}^2$ as follows.

Separate the state space into open annuli (regions between two concentric circles) such that none of the fixed $N-2$ anyons are at the interior of an annulus, cf. fig. 3.1. This gives a separation of $\mathbb{R}^2$ into regions with increasing number p of the $N-2$ fixed anyons. More explicitly, each annulus is centered in x_{cm}. Furthermore, let $r_1 \leq r_2 \leq \ldots \leq r_{N-2}$ be the radii of the $N-2$ fixed anyons. The innermost annulus is an open disc (degenerate annulus) with radius r_1, the second innermost annulus is the region between circles of radius r_1 and r_2, etc. Note that the circles with radii $r_1, r_2, \ldots, r_{N-2}$ are not contained in any annuli, since the anyons cannot pass through each other, and we are considering angular motion. If the radii of two fixed anyons coincide, then there is clearly no annulus separating them.

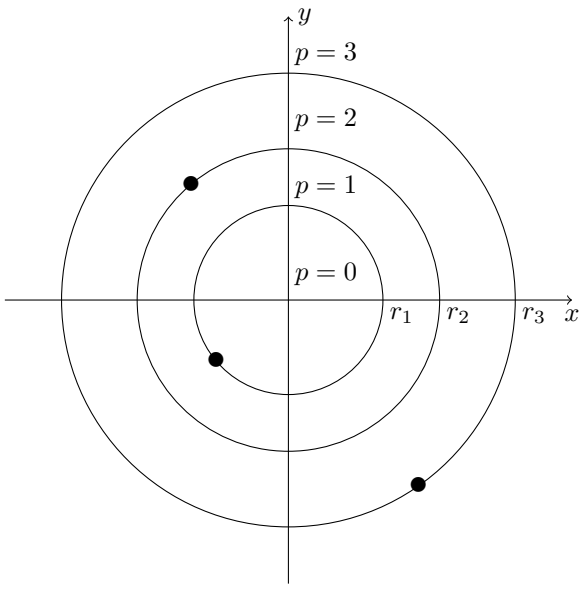


Figure 3.1: Illustration of annuli separating $\mathbb{R}^2$ into regions with increasing number p of contained anyons. A blob ($\bullet$) denotes a fixed anyon. In each annulus, exchange of the two free anyons, i.e. as φ increases from 0 to π, a given number p of fixed anyons will be enclosed.

In the general case, exchange of a pair of anyons introduces an anyonic phase $U_p \in U(n)$ as discussed in chapter 2. The exchange operator U_p may depend on the number p of anyons that get encircled in the exchange loop. With the wave function $\psi = \psi(\varphi)$ parametrized by the relative angular coordinate, we thus get the boundary condition

$$\psi(\pi) = U_p \psi(0). \tag{3.1}$$

This essentially alters the geometry of the space, splitting it in half. It suffices to consider the region $0 \leq \varphi \leq \pi$, i.e. it suffices to consider half annuli.

We have set out to estimate the energy of the system, this amounts to solving

the Schrödinger equation, in particular solving for the lowest energy λ,

$$H\psi = \lambda\psi$$

where $H = -\nabla^2 = -\frac{\partial^2}{\partial\varphi^2}$ is the Hamiltonian of the system. So far we have just one boundary condition, eq. (3.1), while the Schrödinger equation is a second order differential equation, requiring two boundary conditions to give a unique solution. However, since we are primarily interested the energies of the system, it suffices to compute the spectrum of H. With this in mind, we write the Hamiltonian $H = -\nabla^2$ as a square $H = D^2$, where

$$D = -i\partial_\varphi, \quad 0 \leq \varphi \leq \pi, \text{ subject to boundary conditions given by } U_p,$$

and use the spectral theorem to compute the spectrum of H as the squared spectrum of D, i.e.

$$\sigma(H) = \{\lambda^2 : \lambda \in \sigma(D)\}.$$

Similarly, we have that the ground state energy is the infimum of $\sigma(H)$, i.e.

$$\lambda_0 = \inf \sigma(H) = \inf\{\lambda^2 : \lambda \in \sigma(D)\}.$$

The spectrum of D is straight forward to compute and we present the result as a lemma.

Lemma 3.1.1. *The eigenfunctions u and eigenvalues λ of D are such that*

$$Du = \lambda u \quad \iff \quad -iu'(\varphi) = \lambda u(\varphi),$$

having general solution

$$u(\varphi) = Ce^{i\lambda\varphi}$$

for some constant $C \in \mathbb{C}^n$.

We seek an expression for the spectrum of D, the above lemma and the boundary condition eq. (3.1) gives

$$u(\pi) = U_p u(0) \implies Ce^{i\lambda\pi} = U_p C,$$

that is, $e^{i\lambda\pi}$ is an eigenvalues of U_p. We have the following result.

Lemma 3.1.2. *The spectrum of D (and by extension the spectrum of H) is given by*

$$\sigma(D) = \left\{\lambda : e^{i\lambda\pi} \text{ is an eigenvalue of } U_p\right\}.$$

Remark 3.1.1. Recall that U_p is unitary, so all eigenvalues of U_p are on the complex unit circle. Hence, each eigenvalue $e^{i\lambda\pi}$ of U_p with $0 \leq \lambda < 2$, gives a family of possible energy levels $\left\{(\lambda + 2n)^2 \mid n \in \mathbb{Z}\right\}$ to the system.

3.2 Abelian anyons

Although non-Abelian anyons are the main focus of this thesis, we give a quick overview of the the case of Abelian anyons. For more details, see [6]

Abelian anyons are characterized by the fact that the exchange operator is one dimensional, i.e. an element of $U(1)$, hence it is of the form $e^{i\alpha\pi}$. Take this to be U_0, i.e. simple exchange of a pair of Abelian anyons, with the exchange loop enclosing no other anyons. It is then easy to realize that

$$U_p = e^{i\alpha(1+2p)\pi} \tag{3.2}$$

because the left anyon performs an exchange with the p inner anyons, likewise for the right anyon, finally the two anyons undergoing exchange are exchanged with each other, resulting in $1 + 2p$ exchanges of α-type anyons, as illustrated in fig. 3.2.

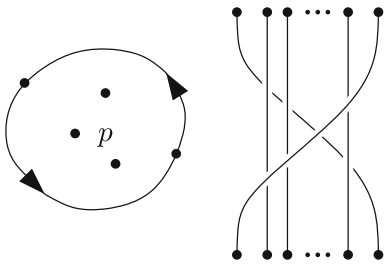


Figure 3.2: Exchange of a pair of Abelian α anyons around p fixed α anyons. Figure taken from [6].

Proposition 3.2.1 (Ground state energy for a pair of Abelian anyons)**.** *Consider a pair of Abelian anyons with anyonic phase α enclosing p fixed anyons (also with anyonic phase α). The ground state energy is bound from below by*

$$\inf_{p,q\in\mathbb{Z}} (\alpha(1+2p)+2q)^2 =$$
$$= \begin{cases} \frac{1}{\nu^2}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ with } \mu \in \mathbb{Z}, \nu \in \mathbb{N}_+ \text{ relatively prime and } \mu \text{ odd,} \\ 0, & \text{otherwise} \end{cases}$$

Proof. By eq. (3.2) we have the anyonic exchange operator as

$$U_p = e^{i\alpha(1+2p)\pi}.$$

By lemma 3.1.2 we have that the boundary condition gives

$$\begin{aligned} & u(\pi) = e^{i\alpha(1+2p)\pi} u(0) \\ \iff \quad & e^{i\lambda\pi} = e^{i\alpha(1+2p)\pi} \\ \iff \quad & \lambda = \alpha(1+2p) + 2q, \quad q \in \mathbb{Z}. \end{aligned}$$

Thus we see that the spectrum for D and $H = D^2$, respectively, is

$$\begin{aligned} \sigma(D) &= \{\alpha(1+2p)+2q : q \in \mathbb{Z}\}, \\ \sigma(H) &= \{(\alpha(1+2p)+2q)^2 : q \in \mathbb{Z}\}. \end{aligned}$$

Hence, the energy λ^2 is minimized for

$$E_0 = \inf_{p,q\in\mathbb{Z}} (\alpha(1+2p)+2q)^2.$$

A number-theoretic result, found as proposition 5 in [6], shows that this can be written as

$$E_0 = \begin{cases} \frac{1}{\nu^2}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ with } \mu \in \mathbb{Z}, \nu \in \mathbb{N}_+ \text{ relatively prime and } \mu \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

□

From this we have an immediate corollary for bosons ($\alpha = 0$) and fermions ($\alpha = 1$).

Corollary 3.2.2. *With $\alpha = 0$, the anyonic phase reads*

$$U_p = e^{i\alpha(1+2p)\pi} = 1$$

for all p, i.e. bosons do not "see" each other, and they have a zero ground state energy.

With $\alpha = 1$, the anyonic phase reads

$$U_p = e^{i\alpha(1+2p)\pi} = e^{i(1+2p)\pi},$$

giving us the boundary condition

$$e^{i\lambda\pi} = e^{i(1+2p)\pi} \iff \lambda = 1+2p+2q, \quad q \in \mathbb{Z}$$

Since p is an integer, this shows that also fermions do not "see" the enclosed p anyons. However, the anyons in the pair do "see" each other, in the sense that the their wave-function changes sign after they have been exchanged, as expected. Finally, this also shows that fermions have a non-zero ground state energy $E_0 = 1$.

3.3 Non-Abelian anyons

Consider the same setting as above, with the distinction that the anyons are now non-Abelian. That is, consider a pair of non-Abelian anyons, with wave-function $\psi \in L^2(\mathbb{R}; \mathbb{C}^n)$ parametrized by the relative angle φ, such that the pair of anyons encloses p fixed anyons as φ increases from 0 to π, giving rise to an anyonic phase $U_p \in U(n)$.

A standard characterization of unitary matrices is as follows.

Lemma 3.3.1. *Let $U \in U(n)$, the n eigenvalues (counted with multiplicity) of U are on the form $e^{i\beta_j\pi}$ for $j = 1, 2, \ldots, n$. Furthermore, $e^{i\alpha\pi}$ where $\alpha = \frac{1}{n}\sum_j \beta_j$ is referred to as the Abelian phase of U_p. It is the Abelian part in the factorization of $U(n) = U(1) \times SU(n)$. Note that if $\det U = e^{i\theta\pi}$ we have $\alpha = \theta/n$.*

Proof. We only show the $U(n) = U(1) \times SU(n)$ decomposition. Rewrite U as $U = e^{i\alpha}\left(e^{-i\alpha}U\right)$ where $e^{i\alpha}$ is the Abelian part and $e^{-i\alpha}U$ is the $SU(n)$ part. It suffices to show that $\det\left(e^{-i\alpha\pi}U\right) = 1$,

$$\begin{aligned}\det\left(e^{-i\alpha\pi}U\right) &= \left(e^{-i\alpha\pi}\right)^n e^{i(\sum_j \beta_j)\pi} \\ &= \left(e^{-i(\sum_j \beta_j)\pi/n}\right)^n e^{i(\sum_j \beta_j)\pi} = 1.\end{aligned}$$

On the other hand, $1 = \det\left(e^{-i\alpha\pi}U\right) = e^{-i\alpha\pi n}e^{i\theta\pi} = e^{i(-\alpha n+\theta)\pi}$, thus $\alpha = \theta/n$ (modulo 2). □

Next, we show that the Abelian phase shifts the eigenvalues uniformly along the complex unit circle.

Proposition 3.3.2. *Consider non-Abelian anyons with arbitrary (unitary) $U_p \in U(n)$ exchange operator. The Abelian part of of the non-Abelian anyonic phase U_p can always be chosen so that the ground state has zero energy.*

Proof. By changing the Abelian phase, each eigenvalue is shifted. This is obvious if we diagonalize U_p. In particular, adding $-\beta_j$ to the Abelian phase, the j:th eigenvalue is shifted to $e^{i(\beta_j-\beta_j)} = 1$, corresponding to zero energy. □

As we shall see in the following chapter, the exchange operator U_p has a rather involved dependence on the particular type of non-Abelian anyons that are considered. In order to characterize the energy bound, via the eigenvalues of U_p, we must first dive into the framework of abstract anyon models in order to characterize U_p.

Chapter 4

Abstract anyon models

In this chapter we shall present the framework of abstract anyon models, culminating in a characterization of anyon braiding. That is, we will show how the braid group representation is computed for a given anyon model.

Anyon models can be modeled by unitary braided tensor categories, see [10, 11]. However, setting up this framework in full generality is redundant for our purposes and we take a more straight forward approach, which is also most common in the literature. The benefit of the categorical approach is that consistency of braiding and fusion is most naturally shown this way. We shall see the ties between the straight forward approach and the categorical model in when discussing the consistency conditions in section 4.6.

This chapter is in part based on [8, 10, 27]. We shall be using the notation that is standard in the literature, and when possible make use of and extend the fusion diagram notation, which greatly clarifies braiding of fusion states.

Starting with section 5.2 we show how a given abstract anyon model gives rise to representations of the braid group.

4.1 Preliminaries

An abstract anyon model consists of a set of labels representing different types of anyons. These labels are known as anyonic charge, topological charge or superselection labels. Anyons can be combined, or fused, to give an anyon of some charge, possibly in different ways. This is modeled by a fusion algebra

$$a \times b = \sum_c N_{ab}^c c$$

representing the possible outcomes from fusion of anyons of type a and b. The fusion multiplicities N_{ab}^c are non-negative integers denoting the number of distinct ways a and b can fuse to c. In each anyon model there is the trivial label 1, representing the vacuum, with the property $N_{a1}^b = N_{1a}^b = \delta_{ab}$, i.e. 1 fuses trivially with every other charge. Furthermore, to each charge a there is a corresponding conjugate charge $\bar{a}$ representing the antiparticle of a, with the property $N_{ab}^1 = \delta_{b\bar{a}}$, i.e. a fuses to the vacuum only with its antiparticle.

The N_{ab}^c distinguishable ways in which a and b can fuse to c can be regarded as an orthonormal basis of a Hilbert space V_{ab}^c. This is the state space, or fusion

space, of anyons of type c resulting from fusion of a and b. The states of V_{ab}^c are called fusion states and we denote the basis for V_{ab}^c by

$$\{|ab;c,\mu\rangle, \quad \mu = 1,2,\ldots,N_{ab}^c\}$$

where $|ab;c,\mu\rangle$ represents the μ:th way in which a and b can fuse to c.

The splitting space V_c^{ab} is the dual space of the fusion space V_{ab}^c, it is the state space of particles with anyonic charge a and b that can be split from an anyon of charge c.

More generally, the fusion space of anyons of type c resulting from fusion of anyons of type $a_1,\ldots,a_n$ is denoted by $V_{a_1a_2\cdots a_n}^c$. This space has a canonical decomposition

$$V_{a_1a_2\cdots a_n}^c \cong \bigoplus_{b_1,b_2,\ldots,b_{n-2}} V_{a_1a_2}^{b_1} \otimes V_{b_1a_3}^{b_2} \otimes V_{b_2a_4}^{b_3} \ldots \otimes V_{b_{n-2}a_n}^c$$

with an associated canonical orthonormal basis with elements being the fusion states

$$|a_1a_2;b_1,\mu_1\rangle \otimes |b_1a_3;b_2,\mu_2\rangle \otimes \cdots \otimes |b_{n-3}a_{n-1};b_{n-2},\mu_{n-2}\rangle \otimes |b_{n-2}a_n;c,\mu_{n-1}\rangle$$

for all possible b_j and μ_j. For convenience we write $b_0 = a_1$ and $b_{-1} = 1$.

Many anyon models of interest have the property that $N_{ab}^c \leq 1$ for all particle types a, b and c. When this is the case, such as for the Fibonacci anyons that we shall consider in chapter 6, the multiplicity label μ can be ignored. This makes the model easier to handle, and we introduce the fusion diagram notation.

4.2 Fusion diagrams

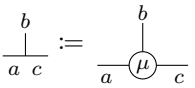
In the previous section we defined

$$|ab;c,\mu\rangle$$

to represent the μ:th way in with a and b fuse to c. It shall be very convenient to use a diagrammatic notation for fusion states when considering braiding of anyons, as we shall see in the following sections. Thus, we introduce the notation

$$\coloneqq |ab;c;\mu\rangle$$

Note that the diagram should be read left/top to right/bottom, i.e. a fuses with b resulting in c. For simplicity we suppress the multiplicity label μ in the notation and write



$$\coloneqq |ab;c;\mu\rangle.$$

This is convenient, since most models we shall consider are such that the fusion multiplicities are trivial, $N_{ab}^c \leq 1$. That is, if fusion of a and b to c is possible, this happens in exactly one way. Then, the multiplicity label μ in $|ab;c,\mu\rangle$ can

be ignored, because the only possibility is $\mu = 1$, if $\mu = 0$ the state is not valid. This is not a major simplification in the notation, it shall always be straight forward to add back the multiplicity labels if needed.

The diagram notation is primarily useful when considering fusion of several anyons and extends in a natural way;

denotes fusion of a, b, c to d such that a fuses with b resulting in the intermediate charge e, finally e fuses with c to give the resulting charge d.

The canonical basis for the fusion space $V^c_{a_1 a_2 \cdots a_n}$ with canonical decomposition

$$V^c_{a_1 \cdots a_n} \cong \bigoplus_{b_1, b_2, \ldots, b_{n-2}} V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes V^{b_3}_{b_2 a_4} \ldots \otimes V^c_{b_{n-2} a_n}$$

can thus be written in terms of fusion diagrams, assuming trivial fusion multiplicities, as

$$\left\{ \quad \begin{array}{ccccccc} & a_2 & a_3 & & & a_{n-1} & a_n \\ & | & | & & & | & | \\ \hline a_1 & b_1 & b_2 & \cdots & b_{n-3} & b_{n-2} & c \end{array} \quad \middle| \quad \begin{array}{c} \text{for all possible intermediate} \\ \text{charges } b_1, b_2, \ldots, b_{n-2} \end{array} \quad \right\}.$$

We can thus represent the basis states as a vector of the possible intermediate charges

$$(b_1, b_2, \ldots, b_{n-2}).$$

TODO: Mention explicitly that we are representing all this with $\mathbb{C}^n$ matrices and how to map back and forth.

The real advantage of writing fusions states with fusion diagrams is that braiding is much easier to represent. This will be extremely useful now as we proceed in developing the abstract model for anyons, and characterize braiding.

4.3 The R operator: Commutativity of fusion

The result of fusing a with b must be the same as fusing b with a. That is, fusion is commutative,

$$a \times b = b \times a.$$

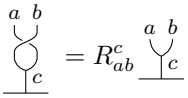
This gives rise to a natural isomorphism

$$R : V^c_{ba} \to V^c_{ab}$$

between the corresponding fusion spaces which can be represented by a unitary matrix R^c_{ab} in the canonical basis

$$|ba, c, \mu\rangle = \sum_{\mu'} (R^c_{ab})^{\mu'}_{\mu} \, |ab, c, \mu'\rangle.$$

As we have seen, if the fusion multiplicities $N_{ab}^c \leq 1$ for all labels a, b, c, we can disregard explicit mention of the fusion multiplicities. From now on we shall do so, and note that it is straight forward to add these indices back if needed. In the diagrammatic notation we then have



(4.1)

In this case it is clear that R-matrix R_{ab} is diagonal in the canonical basis and we have

$$(R_{ab})_{ij} = \delta_{ij} R_{ab}^i.$$

4.4 The F operator: Associativity of fusion

The result of fusing multiple anyons must be independent of which anyons are fused first. That is, fusion is associative,

$$(a \times b) \times c = a \times (b \times c)$$

This gives rise to a natural isomorphism between the two decompositions of the fusion space

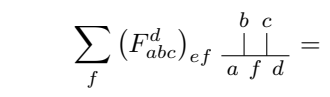
$$V_{abc}^d \cong \bigoplus_f V_{ab}^f \otimes V_{fc}^d \cong \bigoplus_e V_{bc}^e \otimes V_{ae}^d.$$

The first decomposition should be understood as first fusing a with b in all possible ways giving an intermediate charge f, followed by fusing c with f to give the final charge d. The second decomposition should be understood as first fusing b with c in all possible ways giving an intermediate charge e, followed by fusing a with e to give the final charge d.

The first of these two decompositions is referred to as the standard decomposition and the second is the fusion decomposition. We denote this isomorphism by

$$F : \bigoplus_f V_{ab}^f \otimes V_{fc}^d \to \bigoplus_e V_{bc}^e \otimes V_{ae}^d$$

and it can be represented by the matrix F_{abc}^c, satisfying the equation



(4.2)

That is, the F-matrix is the change of basis matrix from the standard basis to the fusion basis of the fusion space V_{abc}^d. Note that this definition looks reversed to how [8] defines the F operator, however it seems [8] defines the inverse. All following results in this thesis are in agreement with [8].

Remark 4.4.1 (Implicit range for the index of summation). In eq. (4.2) the summation index f implicitly ranges over all possible labels in the given anyon model, with the obvious restriction that the corresponding fusion state in the summand is valid. That is, the summation index f must be such that

$$a \times b = f$$
$$f \times c = d.$$

Similarly, the label e is restricted so that

$$b \times c = e$$
$$a \times e = d.$$

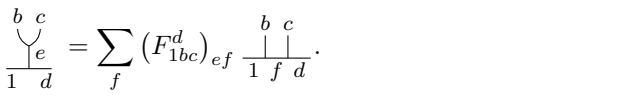
We shall use this convention for all sums with fusion states as summands.

The following lemma will be useful when computing the F-matrix.

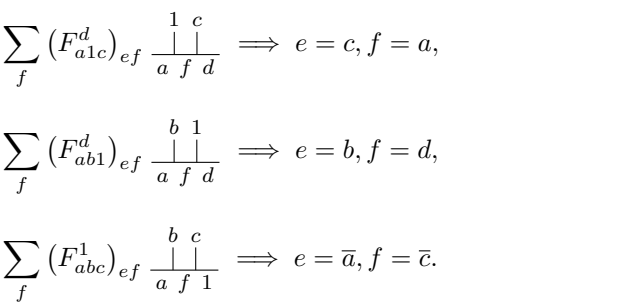
Lemma 4.4.1. *Consider the fusion space V^d_{abc}, when one of the particle types is trivial, i.e. a, b, c or d equals 1, then $\dim V^d_{abc} = 1$ and the corresponding F-matrix F^d_{abc} is trivial. More explicitly we have*

$$F^d_{1bc} = \left(F^d_{1bc}\right)_{db} = 1,$$
$$F^d_{a1c} = \left(F^d_{a1c}\right)_{ca} = 1,$$
$$F^d_{ab1} = \left(F^d_{ab1}\right)_{bd} = 1,$$
$$F^1_{abc} = \left(F^1_{abc}\right)_{\overline{a}\,\overline{c}} = 1.$$

Proof. With $a = 1$ we have, by definition of the F-matrix,



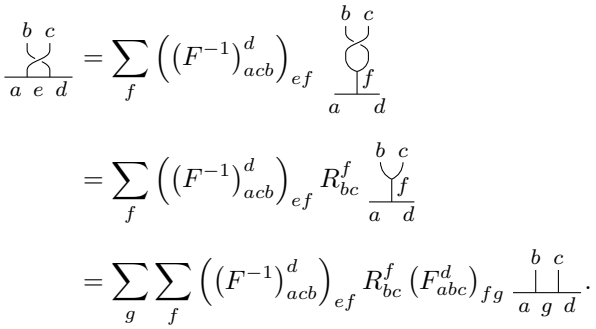
From the fusion diagram on the right hand side we read out $1 \times b = f$ from the first fusion, this is valid only for $b = f$. Similarly, on the left hand side the final fusion reads $1 \times e = d$, implying $e = d$. Since the indices e and f are forced, this implicitly shows that the corresponding fusion spaces is one-dimensional. The other results follow analogously,



The result can also be realized by noting that three anyons of type a, b and c, where one of them is the trivial type 1, is uniquely determined by their total charge. Indeed, these three anyons are really just two, since the trivial particle q fuses trivially, it can be added or removed in the representation without changing anything. Thus, the corresponding F matrix must be trivial in this case. □

4.5 The B operator: Braiding of standard fusion states

We shall now consider braiding on the standard fusion states. This can be realized by applying the F-matrix to put the state in a basis where the R matrix can be applied immediately, followed by reverting back to the standard basis via F^{-1}. That is, using the F and R-matrix we obtain the relation



Recall the implicit summation index convention described in remark 4.4.1. From this we define the B-matrix.

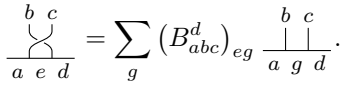
Definition 4.5.1.

$$\left(B^d_{abc}\right)_{eg} = \sum_f \left(\left(F^{-1}\right)^d_{acb}\right)_{ef} R^f_{bc} \left(F^d_{abc}\right)_{fg}.$$

Symbolically we write this as

$$B = F^{-1}RF.$$

To sum up, the B-matrix braids the standard fusion states according to



As a consequence of the definition of the B-matrix and lemma 4.4.1 we have the following lemma, which will be useful when computing the B-matrix.

Lemma 4.5.1. *Consider the fusion space V^d_{abc}, when one of the particle types is trivial, i.e. a, b, c or d equals 1, then $\dim V^d_{abc} = 1$ and the corresponding B-matrix B^d_{abc} is one-dimensional,*

$$\begin{aligned}
B^d_{1bc} &= \left(B^d_{1bc}\right)_{cb} = R^d_{bc}, \\
B^d_{a1c} &= \left(B^d_{a1c}\right)_{da} = R^c_{1c} = 1, \\
B^d_{ab1} &= \left(B^d_{ab1}\right)_{ad} = R^b_{b1} = 1, \\
B^1_{abc} &= \left(B^1_{abc}\right)_{\bar{b}\bar{c}} = R^{\bar{a}}_{bc}.
\end{aligned}$$

4.6 The pentagon and hexagon equations

When considering an anyon model, it is ultimately the B-matrix that determines the properties of interest. The B-matrix gives the phase change introduced when braiding anyons. In the previous section we saw how the B-matrix determine the braid group representation. This is the relevant property both for the study the dynamics of anyons, but also for developing methods for quantum computation with anyons, known as topological quantum computation.

We have seen that the B-matrix is computed from the F and R-matrices. These matrices are in turn determined by what is known as the pentagon equation

$$\left(F^5_{12c}\right)^d_a \left(F^5_{a34}\right)^c_b = \left(F^d_{234}\right)^c_c \left(F^5_{1e4}\right)^d_b \left(F^b_{123}\right)^e_a \tag{4.3}$$

and hexagon equation

$$R^g_{ac} \left(F^d_{bac}\right)^g_e R^e_{ab} = \sum_f \left(F^d_{bca}\right)^g_f R^d_{af} \left(F^d_{abc}\right)^f_e . \tag{4.4}$$

In these equations, all indices are taken as arbitrary particle labels.

These equations are known as coherence conditions for fusion and braiding. The diagrammatic version of these equations, found as commutative diagrams in figs. 4.1 and 4.4 and figs. 4.2 and 4.4 make the point clearer, and shows that these equations are commutativity constraints for fusion and braiding. Indeed, the pentagon equation is the formal constraint for associativity of fusion,

$$(a \times b) \times c = a \times (b \times c).$$

As previously hinted, anyon models can be described by braided tensor categories. In this setting, the pentagon and hexagon equations are precisely Mac Lane's coherence theorem [28], showing that no further conditions are required for consistent fusion and braiding. Further details can be found in [10, 8].

Solving the pentagon and hexagon equations is in general highly non-trivial. The equations are multivariate polynomial equations and require elaborate techniques to be solved. First one must fix the gague freedom that comes from the choice of basis for the fusion space, next an appropriate Gröbner basis can be used to solve the system. See [27] for more details.

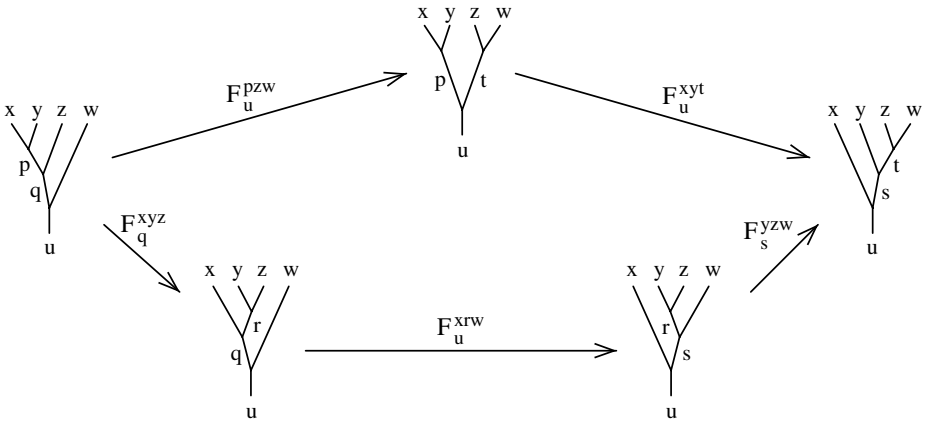


Figure 4.1: The pentagon equation in terms of fusion diagrams. Figure take from [10]. Note that the F-matrix have super- and sub-scripts reversed.

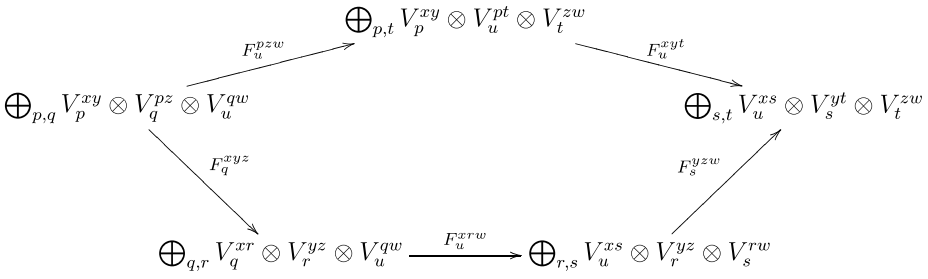


Figure 4.2: The pentagon equation in terms of fusion spaces. Figure take from [10]. Note that the fusion spaces and the F-matrix have super- and sub-scripts reversed.

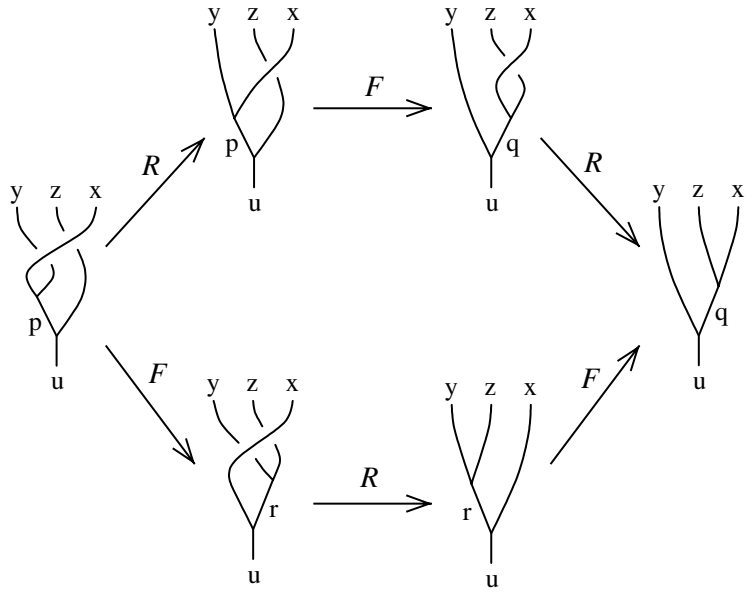


Figure 4.3: The hexagon equation in terms of fusion diagrams. Figure take from [10].

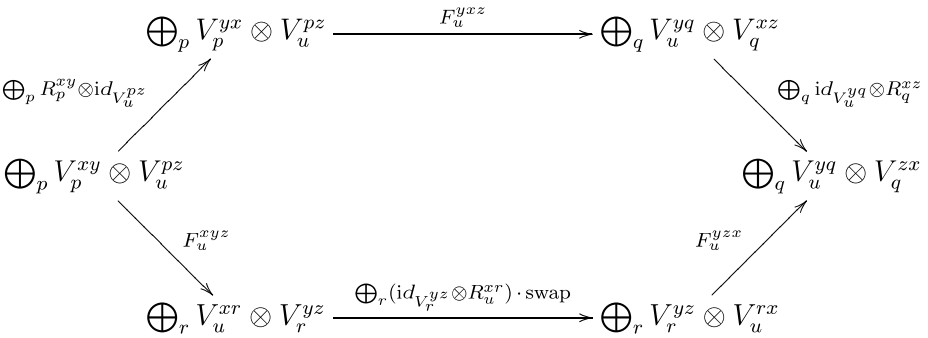


Figure 4.4: The hexagon equation in terms of fusion spaces. Figure take from [10]. Note that the fusion spaces, F-matrix and R-matrix have super- and subscripts reversed.

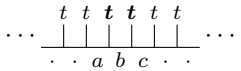
Chapter 5

Anyonic braid group representations

In this chapter we show how a given anyon model gives rise to representations of the braid group. The chapter builds up to a general characterization of the exchange operator U_p of a pair of anyons around p enclosed anyons. To give a detailed account of exactly how the representations of the braid group generators arise, we begin by defining charge sectors of fusion spaces.

5.1 Charge sectors

Consider any abstract anyon model with a non-trivial particle label t. When considering braiding of two intermediate t anyons in $V_{t^n}^1$, it suffices to work with the basis of $V_{t^n}^1$ restricted to the intermediate charge labels a, b and c in



The reason for this is that only these labels are part of the B matrix describing braiding of the two t anyons. Note that we cannot make any assumptions for the charge c, it is not forced to be 1 since we are considering intermediate anyons in the fusion space, only the total charge of the fusion space is specified, in this case 1. The different possible values for c is referred to as different right charge sector. Similarly, also the charge a is unknown, because we are not considering the leftmost particle in the basis, there are really more anyons to the left, that fuse to different resulting charges a. This results in different left charge sectors.

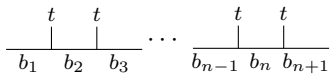
If we fix the both the left and right charge sectors to be trivial, we are really considering braiding of

in V_{tt}^1. This is a different fusion space, with much smaller dimension than $V_{t^n}^1$. There is just one free intermediate charge label here, compared to three free charge labels when not fixing the charge sectors.

Note that having trivial total charge, as in $V_{t^n}^1$ represents the fact that we have exactly n anyons of type t. If the total charge would be t, there would really be $n+1$ anyons of type t available.

Since we seek to characterize general braiding in $V_{t^n}^1$, including intermediate anyons where both the left and right charge sector are unfixed, we introduce the following definition for convenience.

Definition 5.1.1. The fusion space $\widetilde{V}_{t^n}$ has basis elements on the form



where the charge labels $b_1, b_2, \ldots, b_{n+1}$ range over all values allowed by the fusion rules. The space $\widetilde{V}_{t^n}$ should be thought of as the space $V_{t^n}^1$ but including all possible left and right charge sectors. That is, the space of n anyons of type t with unfixed charge sectors.

Remark 5.1.1. We now have three slightly different notations for fusion spaces, to sum up:

$$V_{t^n}^1 := \text{span}\left\{ \begin{array}{c} t\ t \\ \hline 1\ b_1\ b_2 \end{array} \cdots \begin{array}{c} t\ t \\ \hline b_{n-3}\ b_{n-2}\ 1 \end{array} : \text{ for all possible } b_j \right\}$$

$$V_{t^n} := \text{span}\left\{ \begin{array}{c} t\ t \\ \hline 1\ b_1\ b_2 \end{array} \cdots \begin{array}{c} t\ t \\ \hline b_{n-2}\ b_{n-1}\ b_n \end{array} : \text{ for all possible } b_j \right\}$$

$$\widetilde{V}_{t^n} := \text{span}\left\{ \begin{array}{c} t\ t \\ \hline b_1\ b_2\ b_3 \end{array} \cdots \begin{array}{c} t\ t \\ \hline b_{n-1}\ b_n\ b_{n+1} \end{array} : \text{ for all possible } b_j \right\}$$

Recall the notation $V_{t^n} = \bigoplus_c V_{t^n}^c$, this is the space of n anyons of type t with free right charge sector.

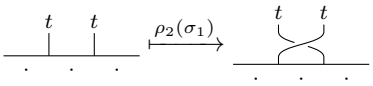
5.2 Braid group representations in $\widetilde{V}_{t^n}$

Each anyon model gives rise to representations of the braid group, the representations depend on the number of considered anyons and the possible charges sectors. In this and the following sections we give a detailed account of how braid group representations comes about in the framework of abstract anyon models. An account of the general theory of braid group representations, not necessarily arising from abstract anyon models, can be found in [22].

Single out a non-trivial anyonic charge and consider the fusion space $\widetilde{V}_{t^n}$ with the standard basis. In this space we naturally have $n-1$ braid group generators by exchanging neighbouring t anyons, we introduce the following notation.

Definition 5.2.1. The representation of the j:th braid group generator σ_j in $\widetilde{V}_{t^n}$ is denoted $\rho_n(\sigma_j)$.

For example, we have



furthermore,

$$\sum_{b_2} \left(B^{b_3}_{b_1 tt}\right)_{eb_2} \quad \begin{array}{ccccc} & t & & t & \\ & | & & | & \\ b_1 & & b_2 & & b_3 \end{array} = \begin{array}{ccccc} & t & & t & \\ & \smile & & \smile & \\ b_1 & & e & & b_3 \end{array}.$$

This shows that $\rho_2(\sigma_1)$ in $\widetilde{V}_{t^2}$ for given b_1 and b_3 is given by

$$\rho_2(\sigma_1) = B^{b_3}_{b_1 tt}.$$

By the nature of the B matrix we see that there is no mixing of the values for b_1 or b_3, this matrix really only acts on the subspace $(b_1, \langle b_2\rangle, b_3)$ generated by the possible values of the label b_2 while keeping the labels b_1 and b_3 fixed. We write this as

$$\rho_2(\sigma_1) \left(\begin{array}{ccccc} & t & & t & \\ & | & & | & \\ b_1 & & \langle b_2\rangle & & b_3 \end{array} \right) = \begin{array}{ccccc} & t & & t & \\ & \smile & & \smile & \\ b_1 & & \langle b_2\rangle & & b_3 \end{array}$$

where

$$\begin{array}{ccccc} & t & & t & \\ & | & & | & \\ b_1 & & \langle b_2\rangle & & b_3 \end{array} \equiv (b_1, \langle b_2\rangle, b_3) \equiv \begin{pmatrix} (b_1, \langle b_2\rangle_1, b_3) \\ (b_1, \langle b_2\rangle_2, b_3) \\ \vdots \end{pmatrix}$$

denotes the vector of fusion states for all possible b_2 with given b_1 and b_3. We denote the j:th possible value for b_2 by $\langle b_2\rangle_j$. Since there is no mixing of the values for the labels b_1 and b_3, we see that the matrix for $\rho_2(\sigma_1)$ splits into blocks given by the left and right charge sectors b_1 and b_3,

$$\rho_2(\sigma_1) = \bigoplus_{b_1,b_3} B^{b_3}_{b_1 tt} = \bigoplus_{b_1,b_3} \begin{pmatrix} \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_1,\langle b_2\rangle_1} & \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_1,\langle b_2\rangle_2} & \cdots \\ \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_2,\langle b_2\rangle_1} & \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_2,\langle b_2\rangle_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Indeed, the action of $\rho_2(\sigma_1)$ on $\langle b_2\rangle_j$ is

$$\rho_2(\sigma_1)\big((b_1, \langle b_2\rangle_j, b_3)\big) = \sum_k \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_j,\langle b_2\rangle_k} (b_1, \langle b_2\rangle_k, b_3).$$

Here we have expressed $\rho_2(\sigma_1)$ in the full basis of the fusion space $\widetilde{V}_{t^2}$ with basis elements

$$(b_1, b_2, b_3) \equiv \begin{array}{ccccc} & t & & t & \\ & | & & | & \\ b_1 & & b_2 & & b_3 \end{array}$$

for all possible values of b_1, b_2 and b_3.

Consider now the larger fusion space $\widetilde{V}_{t^n}$ with standard fusion states

$$(b_1, b_2, \ldots, b_n, b_{n+1}) \equiv \begin{array}{ccccc} & t & & t & \\ & | & & | & \\ b_1 & & b_2 & & b_3 \end{array} \cdots \begin{array}{ccccc} & t & & t & \\ & | & & | & \\ b_{n-1} & & b_n & & b_{n+1} \end{array}.$$

for all possible values of the labels $b_1, b_2, \ldots, b_{n+1}$. The braid generator σ_1 still braids the first two t anyons. However, the basis of the space is now larger, giving us a larger representation $\rho_n(\sigma_1)$. The right charge sector for σ_1 can now be seen to consist of the labels $b_3, b_4, \ldots, b_{n+1}$. With the full basis of $\widetilde{V}_{t^n}$ ordered appropriately we then have

$$\begin{aligned}\rho_n(\sigma_1) &= \bigoplus_{b_1,(b_3,\ldots,b_{n+1})} B^{b_3}_{b_1 tt} \\ &= \bigoplus_{b_1,(b_3,\ldots,b_{n+1})} \begin{pmatrix} \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_1,\langle b_2\rangle_1} & \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_1,\langle b_2\rangle_2} & \cdots \\ \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_2,\langle b_2\rangle_1} & \left(B^{b_3}_{b_1 tt}\right)_{\langle b_2\rangle_2,\langle b_2\rangle_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\end{aligned}$$

In order for this expression to be valid, and for the blocks to not interweave we must have that the basis is ordered such that each for each value of b_2 all basis elements for given b_2 follow immediately after each other, and ordered in the same way for each value of b_2. Such a choice for the order of the basis can of course be made, however, this order will most likely cause $\rho_n(\sigma_2), \rho_n(\sigma_3), \ldots, \rho_n(\sigma_{n-1})$ to have charge sectors appearing in non-contiguous order, giving interweaving blocks. To somewhat remedy this, we shall order the basis by the leftmost and rightmost charge sectors b_1 and b_n. The values for these labels can never be mixed, while all other intermediate charges can be mixed. Thus, all representations of the braid generators $\rho_n(\sigma_j)$ respect these outer charge sector blocks. There is however no ordering of the basis that gives a consistent non-interweaving block structure for $\rho_n(\sigma_2)$, see example 6.3.5 as a concrete example of this. It is clear that for each generator σ_j there is a choice of the order of the basis such that its representation $\rho_n(\sigma_j)$ has this described block form. That is, the basis can be partitioned into subsets where each subset spans an invariant subspace under $\rho_n(\sigma_j)$. We can write this as

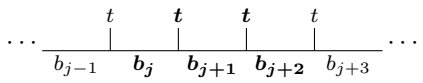
$$\begin{aligned}\rho_n(\sigma_j) &= P_j^{-1}\left[\bigoplus_{(b_1\ldots,b_j),(b_{j+2},\ldots,b_{n+1})} B^{b_{j+2}}_{b_j tt}\right] P_j \\ &= P_j^{-1}\left[\bigoplus_{(b_1\ldots,b_j),(b_{j+2},\ldots,b_{n+1})} \begin{pmatrix} \left(B^{b_{j+2}}_{b_j tt}\right)_{\langle b_{j+1}\rangle_1,\langle b_{j+1}\rangle_1} & \left(B^{b_{j+2}}_{b_j tt}\right)_{\langle b_{j+1}\rangle_1,\langle b_{j+1}\rangle_2} & \cdots \\ \left(B^{b_{j+2}}_{b_j tt}\right)_{\langle b_{j+1}\rangle_2,\langle b_{j+1}\rangle_1} & \left(B^{b_{j+2}}_{b_j tt}\right)_{\langle b_{j+1}\rangle_2,\langle b_{j+1}\rangle_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right] P_j\end{aligned}$$

where P_j is some permutation matrix, permuting the basis of the fusion space as described.

It is now clear that only the immediate left and right charges b_j and b_{j+2} respectively are essential to determine $\rho_n(\sigma_j)$, regardless of n. Writing out $\rho_n(\sigma_j)$ in the full basis of the fusion space $\widetilde{V}_{t^n}$ gives repeated blocks which may be permuted as described. The number of times that a given (permuted) block is repeated is given by the number of corresponding left and right charge sectors. More specifically, consider $\rho_n(\sigma_j)$ and the block associated with b_j and b_{j+2}. The number of times this block is repeated is given by

$$\dim(\langle b_1\rangle, \ldots, \langle b_{j-1}\rangle) + \dim(\langle b_{j+3}\rangle, \ldots, \langle b_{n+1}\rangle).$$

That is, the number of possible combinations for the labels $b_1, \ldots, b_{j-1}$ and $b_{j+3}, \ldots, b_{n+1}$. Indeed, the bold symbols in

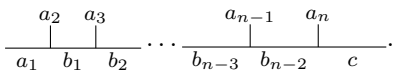


are the ones that correspond to one instance of the corresponding block of $\rho_n(\sigma_j)$. For fixed b_j and b_{j+2} there may be several values for the labels $b_1, \ldots, b_{j-1}$ and $b_{j+3}, \ldots, b_{n+1}$ corresponding the same block.

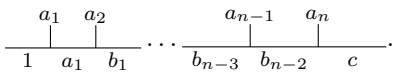
5.3 Computing the braid group generators $\rho(\sigma_j)$

Consider the fusion space $V^c_{a_1 \cdots a_n}$ in a given anyon model, the representation $\rho(\sigma_j)$ gives the anyonic phase introduced to the total wave function when particles a_j and $a_j + 1$ are exchanged (counter)clockwise. Recall from chapter 2 that the braid group is generated by $\sigma_1, \ldots, \sigma_n$, thus it suffices to give expressions for $\rho(\sigma_j)$ in order to be able to compute any braid in a given anyon model.

Before showing how $\rho(\sigma_j)$ is computed, we begin by making our notation slightly more flexible. Consider the fusion space $V^c_{a_1 \ldots a_n}$. As we have seen, the fusion states are on the form



We shall sometimes write such states on the form of the standard basis states of $V^c_{1a_1 \ldots a_n}$, i.e. as



The reason being that it is then simpler to represent braiding of a_1 with a_2. This observation allows us to extend the fusion diagrams with trivial charges when convenient.

Lemma 5.3.1. *In a general anyon model, in the standard basis of the fusion space $V^c_{a_1 \cdots a_n}$ we have*

$$\rho(\sigma_1)_{ij} = \delta_{ij} R^j_{a_1 a_2} \quad \Longleftrightarrow \quad \rho(\sigma_1) = R_{a_1 a_2}.$$

Proof. In the general case, the representation $\rho(\sigma_1)$ for the first generator σ_1 that braids a_1 with a_2, is given by

$$\rho(\sigma_1)\left(\begin{array}{c}\text{\scriptsize } a_1 \; a_2 \\ \text{\scriptsize } 1 \; a_1 \; b_1\end{array}\cdots\right) = \left(\begin{array}{c}\text{\scriptsize } a_1 \; a_2 \\ \text{\scriptsize } 1 \; a_2 \; b_1\end{array}\cdots\right) = \sum_g \left[\left(B^{b_1}_{1a_1a_2}\right)_{a_2 g}\left(\begin{array}{c}\text{\scriptsize } a_1 \; a_2 \\ \text{\scriptsize } 1 \; g \; b_1\end{array}\cdots\right)\right]$$

Since 1 fuses trivially we must have $g = a_1$ and thus $\rho(\sigma_1)$ is one-dimensional with

$$\rho(\sigma_1) = \left(B^{b_1}_{1a_1a_2}\right)_{a_2,a_1} = \sum_f \left(\left(F^{-1}\right)^{b_1}_{1a_2a_1}\right)_{a_2 f} R^f_{a_1 a_2} \left(F^{b_1}_{1a_1a_2}\right)_{f a_1} = R^{b_1}_{a_1 a_2},$$

where the last equality follows from lemma 4.4.1. □

Remark 5.3.1. Above, $\rho(\sigma_1)$ is expressed in the basis for the reduced space $V_{a_1a_2} = \bigoplus_{b_1} V_{a_1a_2}^{b_1}$, having basis determined by the possible anyonic labels b_1. The full space $V_{a_1\cdots a_n}^{c}$ has basis states $(b_1, \ldots, b_{n-2})$ and we implicitly used the operator on the space reduced to fusion states (b_1).

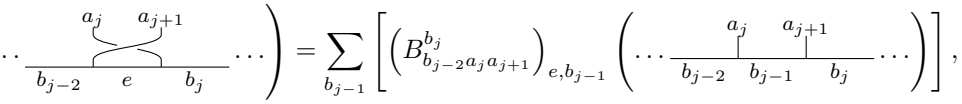
Extending $\rho(\sigma_1)$ to the full space is straight forward: In order to determine the action of $\rho(\sigma_1)$ on a given fusion state in the basis of the full fusion space it suffices to consider the action of $\rho(\sigma_1)$ on the labels a_1 and a_2 in the given fusion state. This information is precisely captured in the representation of $\rho(\sigma_1)$ in the reduced basis. This results in repeating blocks in the matrix representing $\rho(\sigma_1)$ when considering the basis of the full fusion space. This motivates the slight abuse of notation.

This discussion applies to any braiding operator, and we shall often use this abuse of notation, it should always be clear what part of the space that the operator acts on. See section 5.2 for further discussion of the braid group representation.

Lemma 5.3.2. *In a general anyon model, consider the fusion space* $V_{a_1\cdots a_n}^{c}$ *with the standard basis* $(b_1, \ldots, b_{n-1})$*, then*

$$\rho(\sigma_j) = B_{b_{j-2}a_ja_{j+1}}^{b_j},$$

Proof. Note that $\rho(\sigma_j)$ is precisely the B-matrix applied appropriately,



thus the result follows. □

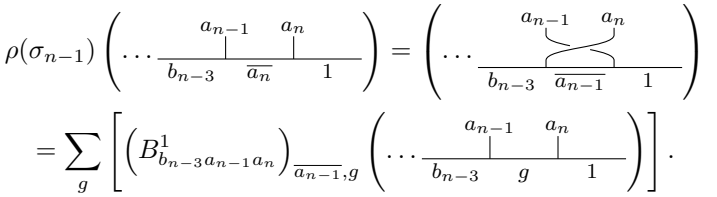
With the convention $b_0 = a_1$ and $b_{-1} = 1$, lemma 5.3.2 subsumes lemma 5.3.1, and also the following result, which we state explicitly for convenience. Recall that $\overline{a}$ denotes the antiparticle of a.

Lemma 5.3.3. *In a general anyon model, consider the fusion space* $V_{a_1\cdots a_n}^{1}$ *with its standard basis* $(b_1, \ldots, b_{n-3})$ *(note that* $c = 1$ *forces* $b_{n-2} = \overline{a_n}$*), then*

$$\rho(\sigma_{n-1})_{ij} = \delta_{ij} R_{a_{n-1}a_n}^{\overline{j}},$$

acting on the b_{n-3}*-part of the space.*

Proof. Since the result of the fusion is assumed to be the trivial particle 1, we must have the indices as follows,



Since the last fusion reads $g \times a_n = 1$ we must have $g = \overline{a_n}$ and thus

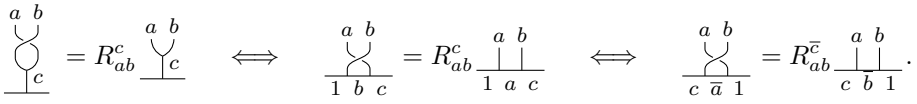
$$\begin{aligned}\rho(\sigma_{n-1}) &= \left(B^1_{b_{n-3}a_{n-1}a_n}\right)_{\overline{a_{n-1}},\overline{a_n}} \\ &= \sum_f \left(\left(F^{-1}\right)^1_{b_{n-3}a_n a_{n-1}}\right)_{\overline{a_{n-1}}f} R^f_{a_{n-1}a_n} \left(F^1_{b_{n-3}a_{n-1}a_n}\right)_{f,\overline{a_n}} \\ &= R^{\overline{b_{n-3}}}_{a_{n-1}a_n}\end{aligned}$$

where the last equality follows from lemma 4.4.1. □

Remark 5.3.2. Note that $\rho(\sigma_1)$ and $\rho(\sigma_{n-1})$ are equal in the restricted basis, up to charge conjugation of the basis. Furthermore, note that if time is reversed, the roles of $\rho(\sigma_1)$ and $\rho(\sigma_j)$ are interchanged. Reversing the fusion in time means reading the fusion diagram from right/bottom to left/top. This is an example of time reversal symmetry; time reversal corresponds to charge conjugation, see [2].

Time reversal as charge conjugation is also simply manifested in the following lemma, as a result of lemma 5.3.1 and 5.3.3.

Lemma 5.3.4. *In the standard basis the R matrix can be written as*



5.4 Anyonic exchange operator U_p

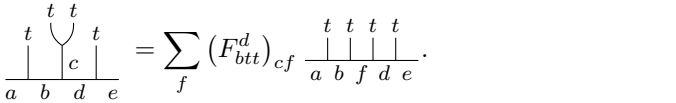
Given any abstract anyon model, single out a non-trivial anyonic charge t and consider exchange of a pair of anyons around p enclosed anyons. That is consider the fusion space $\widetilde{V}_{t^{p+2}}$.

In the standard basis of $\widetilde{V}_{t^n}$, we can compute the braid group generators $\rho_n(\sigma_j)$ for $j = 1, 2, \ldots, n-1$ as shown in section 5.3. The exchange operator U_p (**??**) is then

$$U_p = \sigma_1\sigma_2 \cdots \sigma_p\sigma_{p+1}\sigma_p \cdots \sigma_2\sigma_1,$$

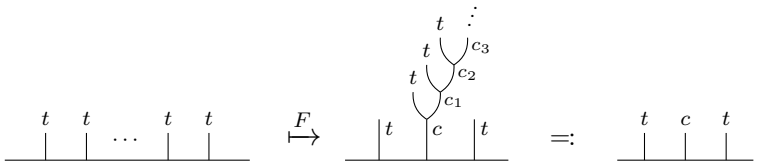
c.f. fig. 3.2. However, working in the standard basis is rather problematic for computing U_p, instead we change basis.

With the F matrix we change basis from the standard basis to a basis where the enclosed p anyons are fused. If $p = 2$ we have



This extends to arbitrary number p of enclosed particles by repeated application of the F matrix. For any $p \geq 2$ we can thus change basis from the standard basis to the fused basis

where c is the resulting charge of the fused enclosed p particles. More explicitly that is



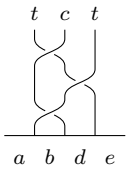
(5.1)

The intermediate charge c depends on p such that c is a possible result of the fusion $\underbrace{t \times t \times \cdots \times t}_{p}$. Explicitly that is

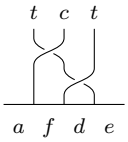
$$
\begin{aligned}
p = 0 &\implies c = 1 \\
p = 1 &\implies c = t \\
p = 2 &\implies c \text{ is a possible result of the fusion } t \times t \\
p = 3 &\implies c \text{ is a possible result of the fusion } t \times t \times t \\
&\vdots
\end{aligned}
$$

Note that the braid corresponding to U_p does not depend on the intermediate charges $c_1, c_2, \ldots, c_{p-2}$. We are now ready to compute U_p for general p,

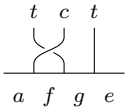
$$U_p \left(\begin{array}{c} t\ c\ t \\ \hline a\ b\ d\ e \end{array} \right) =$$



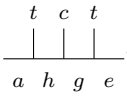
$$= \sum_f \left(B^d_{act}\right)_{bf}$$



$$= \sum_f \left(B^d_{act}\right)_{bf} \sum_g \left(B^e_{ftt}\right)_{dg}$$



$$= \sum_f \left(B^d_{act}\right)_{bf} \sum_g \left(B^e_{ftt}\right)_{dg} \sum_h \left(B^g_{atc}\right)_{fh}$$

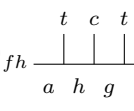


(5.2)

Thus, we have the following theorem.

Theorem 5.4.1. *In a given abstract anyon model, exchange of a pair of t-anyons around p enclosed t-anyons is described by the exchange operator U_p given by*

$$U_p \left(\begin{array}{c} t\ c\ t \\ \hline a\ b\ d\ e \end{array} \right) = \sum_f \left(B^d_{act}\right)_{bf} \sum_g \left(B^e_{ftt}\right)_{dg} \sum_h \left(B^g_{atc}\right)_{fh}$$



5.5 Abelian representations: Abelian anyons

In this section we show how Abelian anyons are modeled in the framework of abstract anyon models. First, note that Abelian anyons are characterized by that each fusion has a unique result, i.e. fusion of a with b results in a unique label c for all a and b,

$$a \times b = c.$$

As a consequence of this we have that any fusion space of Abelian anyons is trivial and one-dimensional. All intermediate charges are uniquely determined by the simple fusion rules.

As a first example we shall see how fermionic statistics fits into the framework of abstract anyon models. In the computations, recall that $R^c_{ab} = R^c_{ba}$. In this section we deviate from our convention and denote the trivial particle by 0 while 1 denotes a non-trivial label.

Example 5.5.1. Let the set of anyonic charges be $\mathbb{Z}_2 = \{0, 1\}$, the trivial vacuum particle 0 and one non-trivial particle 1, with corresponding fusion corresponding to addition in $\mathbb{Z}_2$, in particular

$$1 \times 1 = 0.$$

Since the fusion spaces are trivial, we take $F^{a+b+c}_{abc} = 1$ for all a, b and c. Thus, the pentagon equation eq. (4.3) is trivial and the hexagon equation eq. (4.4) reduces to

$$\begin{aligned} & R^g_{ac} \left(F^d_{bac}\right)^g_e R^e_{ab} = \sum_f \left(F^d_{bca}\right)^g_f R^d_{af} \left(F^d_{abc}\right)^f_e \\ \iff & R^{a+c}_{ac} R^{a+b}_{ab} = \sum_f R^{a+b+c}_{af}. \end{aligned}$$

We have $d = a + b + c$ from the F symbol $\left(F^d_{bac}\right)$, this gives $a + f = a + b + c$, i.e. $f = b + c$. The hexagon equation is thus reduced to

$$R^{a+c}_{ac} R^{a+b}_{ab} = R^{a+b+c}_{a,(b+c)}.$$

Since $R^{a+b}_{ab} = 1$ if a or b equals 0. The only non-trivial case is

$$R^0_{11} R^0_{11} = 1 \quad \iff \quad R^0_{11} = \pm 1.$$

Thus, the chosen model, i.e. charges and fusion modeled by $\mathbb{Z}_2$ gives two distinct Abelian anyon models which we parametrize by $n = 0, 1$ so that we have

$$R^0_{11} = e^{in\pi}.$$

We denote these two models by $\mathbb{Z}_2^{(n)}$ for $n = 0, 1$. In conclusion, we see that fermionic statistics are modeled by $\mathbb{Z}_2^{(1)}$.

In the above example we considered fusion modeled by $\mathbb{Z}_2$ and saw that this gives rise to two distinct models. Before stating the general result, we first extend the example to see more clearly how the hexagon equation determines the R matrices.

Example 5.5.2 ($\mathbb{Z}_3^{(n)}$)**.** Let fusion be modeled on $\mathbb{Z}_3$, so that the the charge label set is $\mathbb{Z}_3 = \{0, 1, 2\}$ and fusion is given by addition modulo 3,

$$a \times b = [a + b]_3.$$

The only non-trivial instances of the hexagon equation are

$$\begin{aligned}
\left(R_{11}^2\right)^2 &= R_{12}^0 \\
\left(R_{12}^2\right)^2 &= R_{11}^2 \\
\left(R_{22}^1\right)^2 &= R_{12}^2 \\
R_{11}^2 R_{12}^0 &= 1 \\
R_{12}^0 R_{22}^1 &= 1.
\end{aligned}$$

The first two equations combined give $\left(R_{11}^2\right)^3 = 1 \iff R_{11}^2 = e^{i2\pi n/3}$. Each of the three solutions for R_{11}^2, corresponding to $n = 0, 1, 2$ give a solution for R_{12}^2 and R_{22}^1. This gives rise to three distinct models:

- $\mathbb{Z}_3^{(0)}$: The trivial solution $R_{ab}^{a+b} \equiv 1$ for all a, b.
- $\mathbb{Z}_3^{(1)}$: Take $R_{11}^2 = e^{i2\pi/3}$, the fourth equation gives $R_{12}^0 = e^{i4\pi/3}$ and the fifth equation gives $R_{22}^1 = e^{i2\pi/3}$.
- $\mathbb{Z}_3^{(2)}$: Take $R_{11}^2 = e^{i4\pi/3}$, the fourth equation gives $R_{12}^0 = e^{i2\pi/3}$ and the fifth equation gives $R_{22}^1 = e^{i2\pi/3}$.

The possible solutions can be compactly written as

$$R_{ab}^{a+b} = e^{i\frac{2\pi n}{3}ab}$$

Indeed, the more general result holds, discussed further in [27], motivating the following definition.

Definition 5.5.1. The Abelian model $\mathbb{Z}_N^{(n)}$ is given by taking the charge label set to be $\mathbb{Z}_N$ and fusion given by addition modulo N,

$$a \times b = [a + b]_N.$$

In this model we have

$$F_{abc}^{a+b+c} \equiv 1 \tag{5.3}$$

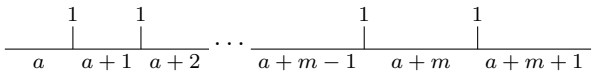
$$R_{ab}^{a+b} = e^{i\frac{2\pi n}{N}ab}. \tag{5.4}$$

By definition 4.5.1 we thus have

$$B_{abc}^{a+b+c} = R_{bc}^{b+c}, \tag{5.5}$$

note that the left charge a does not enter the expression.

Consider $\mathbb{Z}_N^{(n)}$ and the fusion space $\widetilde{V}_{1^m}$ which now is one-dimensional containing one standard fusion state



where a is any charge in $\mathbb{Z}_N$ representing the fact that the m considered anyons may be part of a larger set of anyons. In lemma 5.3.2 we showed

$$\rho(\sigma_j) = B^{b_j}_{b_{j-2}a_j a_{j+1}}. \tag{5.6}$$

Equations (5.4) and (5.5) reduces this to

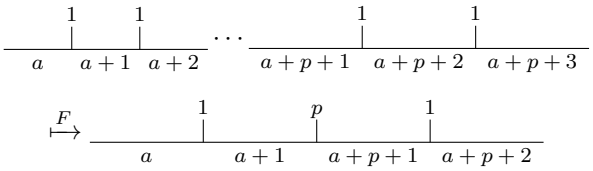
$$\rho(\sigma_j) = R^2_{11} = e^{i\frac{2\pi n}{N}} \tag{5.7}$$

for all j. This precisely models Abelian anyons with anyonic phase $\alpha = \frac{2n}{N}$, i.e. any fractional phase. What we have obtained here precisely corresponds to the discussion of Abelian anyons in section 3.2 which was treated before we had the general framework of abstract anyon models.

Furthermore, in the fusion space $\widetilde{V}_{1^m}$ of the $\mathbb{Z}_N^{(n)}$ model, the exchange operator U_p is

$$\begin{aligned} U_p &= \rho(\sigma_1)\rho(\sigma_2)\cdots\rho(\sigma_p)\rho(\sigma_{p+1})\rho(\sigma_p)\cdots\rho(\sigma_2)\rho(\sigma_1) \\ &= (\rho(\sigma_1))^{2p+1} = e^{i(2p+1)\alpha}. \end{aligned} \tag{5.8}$$

Theorem 5.4.1 is clearly not needed in the Abelian case, we instantly have all information of U_p. However, theorem 5.4.1 still applies, first change basis to the fused basis



and theorem 5.4.1 gives

$$U_p \left(\begin{array}{c} 1\ c\ 1 \\ |\ |\ | \\ \hline a\ b\ d\ e \end{array} \right) = \sum_f \left(B^d_{act}\right)_{bf} \sum_g \left(B^e_{ftt}\right)_{dg} \sum_h \left(B^g_{atc}\right)_{fh} \begin{array}{c} 1\ c\ 1 \\ |\ |\ | \\ \hline a\ h\ g\ e \end{array}. \tag{5.9}$$

As we've seen, the intermediate charges play no role, indeed the fusion space is one dimensional. In the $\mathbb{Z}_N^{(n)}$model this expression for U_p immediately reduces to

$$U_p = R_{c1}R_{11}R_{1c} = e^{ip\alpha\pi}e^{i\alpha\pi}e^{ip\alpha\pi} = e^{i(2p+1)\alpha\pi} \tag{5.10}$$

as expected.

Chapter 6

Fibonacci anyons

The Fibonacci anyon model is the one of the simples non-Abelian anyon model containing all the interesting features of non-Abelian anyons. It is commonly studied as a prototypical example of non-Abelian anyons. Another common model is the Ising model, however Ising anyons cannot be used for general quantum computation.

This chapter is partly based on [8] and [15]. The literature is rather vague on deriving the properties of Fibonacci anyons, in this chapter we spelled out the details more clearly.

6.1 Preliminaries

The Fibonacci anyon model consists of two particle types, 1 (the mandatory trivial particle type) and τ (non-trivial) with the corresponding fusion rules

$$1 \times 1 = 1, \quad 1 \times \tau = \tau, \quad \tau \times \tau = 1 + \tau$$

such that τ is its own anti-particle $\overline{\tau}$. We shall refer to the τ anyon as Fibonacci anyons.

As we shall now see, the dimension of the fusion space of n Fibonacci anyons grows as the Fibonacci numbers as n increases. Thus motivating the name of the model. Consider the fusion spaces $V_{\tau^n}^1$, where τ^n denotes n repetitions of τ. This is the space of possible fusions of n Fibonacci anyons, having total charge

1. Writing out the canonical basis for these spaces we find

$$V^1_{\tau^2} : \left\{ \begin{array}{c} \tau\ \tau \\ |\ | \\ \hline 1\ \tau\ 1 \end{array} \right\}$$

$$V^1_{\tau^3} : \left\{ \begin{array}{c} \tau\ \tau\ \tau \\ |\ |\ | \\ \hline 1\ \tau\ \tau\ 1 \end{array} \right\}$$

$$V^1_{\tau^4} : \left\{ \begin{array}{c} \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ | \\ \hline 1\ \tau\ \mathbf{1}\ \tau\ 1 \end{array}, \begin{array}{c} \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ | \\ \hline 1\ \tau\ \boldsymbol{\tau}\ \tau\ 1 \end{array} \right\}$$

$$V^1_{\tau^5} \simeq \widetilde{V}_\tau : \left\{ \begin{array}{c} \tau\ \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ |\ | \\ \hline 1\ \tau\ \mathbf{1}\ \boldsymbol{\tau}\ \tau\ 1 \end{array}, \begin{array}{c} \tau\ \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ |\ | \\ \hline 1\ \tau\ \boldsymbol{\tau}\ \mathbf{1}\ \tau\ 1 \end{array}, \begin{array}{c} \tau\ \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ |\ | \\ \hline 1\ \tau\ \boldsymbol{\tau}\ \boldsymbol{\tau}\ \tau\ 1 \end{array} \right\}$$

$$V^1_{\tau^6} \simeq \widetilde{V}_{\tau^2} : \left\{ \begin{array}{c} \tau\ \tau\ \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ |\ |\ | \\ \hline 1\ \tau\ \mathbf{1}\ \boldsymbol{\tau}\ \mathbf{1}\ \tau\ 1 \end{array}, \begin{array}{c} \tau\ \tau\ \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ |\ |\ | \\ \hline 1\ \tau\ \mathbf{1}\ \boldsymbol{\tau}\ \boldsymbol{\tau}\ \tau\ 1 \end{array}, \begin{array}{c} \tau\ \tau\ \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ |\ |\ | \\ \hline 1\ \tau\ \boldsymbol{\tau}\ \boldsymbol{\tau}\ \mathbf{1}\ \tau\ 1 \end{array} \begin{array}{c} \tau\ \tau\ \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ |\ |\ | \\ \hline 1\ \tau\ \boldsymbol{\tau}\ \mathbf{1}\ \boldsymbol{\tau}\ \tau\ 1 \end{array}, \begin{array}{c} \tau\ \tau\ \tau\ \tau\ \tau\ \tau \\ |\ |\ |\ |\ |\ | \\ \hline 1\ \tau\ \boldsymbol{\tau}\ \boldsymbol{\tau}\ \boldsymbol{\tau}\ \tau\ 1 \end{array} \right\}.$$

Continuing this list is straight forward. Note that the bottom line in the fusion diagrams of the fusion states in the bases can be seen as strings of τ and 1, having 1τ at the start and $\tau 1$ at the end. Furthermore, the string is subject to the condition that 1 may not be followed by 1. Indeed, 1 on the bottom row fuses with τ from the top, to give τ. We now make this more precise.

Definition 6.1.1. A Fibonacci string of length n on two symbols, say 1 and τ, is a string of length n on the symbols 1 and τ subject to the condition that 1 may not be followed by 1.

There are two Fibonacci string of length one, namely 1 and τ, of length two we have three Fibonacci strings, $1\tau, \tau 1, \tau\tau$, etc. Note that the free charge labels, marked in bold in the above fusion states, are precisely Fibonacci strings. Thus, the dimension of $V^1_{\tau^n}$ is the number of Fibonacci strings of length $n-3$. Alternatively, the dimension of $\widetilde{V}_{\tau^n}$ is the number of Fibonacci strings of length $n+1$.

Lemma 6.1.1. *The number of Fibonacci strings of length n is* $\mathrm{Fib}(n+2)$*, where* $\mathrm{Fib}(n)$ *denotes the n:th Fibonacci number given by*

$$\begin{aligned} \mathrm{Fib}(0) &= 0 \\ \mathrm{Fib}(1) &= 1 \\ \mathrm{Fib}(n+1) &= \mathrm{Fib}(n) + \mathrm{Fib}(n-1). \end{aligned}$$

Proof. Let $\#_n$ denote the length number of Fibonacci strings of length n, and let $\#^a_n$ denote the number of Fibonacci strings of length n ending with a, where $a = 1, \tau$. We then have $\#_n = \#^1_n + \#^\tau_n$. This, together with the definition of Fibonacci strings gives

$$\begin{aligned} \#_{n+1} &= \#^1_{n+1} + \#^\tau_{n+1} \\ &= \#^\tau_n + \left(\#^1_n + \#^\tau_n\right) \\ &= \left(\#^\tau_n + \#^1_n\right) + \left(\#^1_{n-1} + \#^\tau_{n-1}\right) \\ &= \#_n + \#_{n-1}. \end{aligned}$$

This is precisely the recursion relation for the Fibonacci numbers. Finally, initial values $\#_1 = 2, \#_2 = 3$ shows that

$$\#_n = \mathrm{Fib}(n+2).$$

□

Finally, we can use this to characterize the dimension of fusion spaces.

Lemma 6.1.2. *The dimension of the fusion space with n Fibonacci, with different charge sectors, is given by*

$$\begin{aligned}\dim V^1_{\tau^n} &= \mathrm{Fib}(n-1)\\ \dim V^\tau_{\tau^n} &= \mathrm{Fib}(n)\\ \dim \widetilde{V}_{\tau^n} &= \mathrm{Fib}(n+1)\end{aligned}$$

Proof. The first equality follows from

$$\dim V^1_{\tau^n} = \#_{n-3} = \mathrm{Fib}((n-3)+2) = \mathrm{Fib}(n-1).$$

Next,

$$\dim V^\tau_{\tau^n} = \dim V^1_{\tau^{n+1}} = \mathrm{Fib}(n).$$

Finally,

$$\dim \widetilde{V}_{\tau^n} = \#_{n+1} = \mathrm{Fib}((n+1)+1) = \mathrm{Fib}(n+1).$$

□

As we have seen in , an anyon model is determined by the corresponding F- and R-matrices. We continue by computing these matrices.

6.2 Determining the model: Computing the F and R matrices

TODO: Write down computation.

TODO: Discuss the free parameter in F that is fixed by gauge fixing. Preskill p. 60.

$$\begin{aligned}R_{\tau\tau} &= \begin{pmatrix} R^1_{\tau\tau} & 0 \\ 0 & R^\tau_{\tau\tau}\end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-3\pi i/5}\end{pmatrix},\\ F_{\tau\tau\tau} &= \begin{pmatrix} F^{11}_{\tau\tau\tau} & F^{1\tau}_{\tau\tau\tau} \\ F^{\tau 1}_{\tau\tau\tau} & F^{\tau\tau}_{\tau\tau\tau}\end{pmatrix} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1}\end{pmatrix}.\end{aligned}$$

For convenience, when discussing Fibonacci anyons, let

$$F \coloneqq F^\tau_{\tau\tau\tau}, \qquad R \coloneqq R_{\tau\tau}, \qquad B \coloneqq F^{-1}RF.$$

6.3 Braiding of Fibonacci anyons

In this section we shall compute the braid group generators for various number of Fibonacci anyons. Ultimately, we shall compute $\rho(\sigma_j)$ for $V^1_{\tau^n}$. Having trivial total charge represents the fact that we have exactly n Fibonacci anyons. If the total charge would be τ, there would really be $n+1$ Fibonacci anyons available. As we shall see, there will be some subtleties regarding different charge sectors, i.e. different total charge, when considering braiding of intermediate τ anyons in $V^1_{\tau^n}$. We begin with some elementary examples.

Recall the decomposition

$$V_{a_1\cdots a_n} = \bigoplus_c V^c_{a_1\cdots a_n},$$

in particular

$$V_{\tau^n} = V^1_{\tau^n} \oplus V^\tau_{\tau^n}.$$

6.3.1 Prototypical examples

Example 6.3.1 (Braiding in V_{τ^2})**.** The two charge sectors of V_{τ^2} have the standard basis

$$V^1_{\tau\tau} = \operatorname{span}\left\{ \begin{array}{c} \tau\ \tau \\ |\ | \\ 1\ \tau\ \mathbf{1} \end{array} \right\}, \quad V^\tau_{\tau\tau} = \operatorname{span}\left\{ \begin{array}{c} \tau\ \tau \\ |\ | \\ 1\ \tau\ \boldsymbol{\tau} \end{array} \right\}.$$

That is, the fusion space V_{τ^2} is two-dimensional and we denote the ordered basis by $\{1, \tau\}$. Since there is only two τ-anyons, there are only one generator for the braid group, σ_1. We compute $\rho(\sigma_1)$ by considering its action on the standard fusion states,

$$\rho(\sigma_1) \begin{array}{c} \tau\ \tau \\ |\ | \\ 1\ \tau\ 1 \end{array} = \begin{array}{c} \tau\ \tau \\ \smile \\ 1\ \tau\ 1 \end{array} = R^1_{\tau\tau} \begin{array}{c} \tau\ \tau \\ |\ | \\ 1\ \tau\ 1 \end{array}$$

$$\rho(\sigma_1) \begin{array}{c} \tau\ \tau \\ |\ | \\ 1\ \tau\ \tau \end{array} = \begin{array}{c} \tau\ \tau \\ \smile \\ 1\ \tau\ \tau \end{array} = R^\tau_{\tau\tau} \begin{array}{c} \tau\ \tau \\ |\ | \\ 1\ \tau\ \tau \end{array}.$$

Thus we have

$$\left.\begin{aligned} \rho(\sigma_1)_{11} &= R^1_{\tau\tau} \\ \rho(\sigma_1)_{1\tau} &= 0 \\ \rho(\sigma_1)_{\tau\tau} &= R^\tau_{\tau\tau} \\ \rho(\sigma_1)_{\tau 1} &= 0 \end{aligned}\right\} \quad \Longleftrightarrow \quad \rho(\sigma_1) = \begin{pmatrix} R^1_{\tau\tau} & 0 \\ 0 & R^\tau_{\tau\tau} \end{pmatrix}.$$

This also follows immediately from lemma 5.3.1.

Example 6.3.2 (Braiding in V_{τ^3})**.** The two charge sectors of V_{τ^3} have the standard basis

$$V^1_{\tau\tau\tau} = \operatorname{span}\left\{ \begin{array}{c} \tau\ \tau\ \tau \\ |\ |\ | \\ 1\ \tau\ \boldsymbol{\tau}\ \mathbf{1} \end{array} \right\}, \quad V^\tau_{\tau\tau\tau} = \operatorname{span}\left\{ \begin{array}{c} \tau\ \tau\ \tau \\ |\ |\ | \\ 1\ \tau\ \mathbf{1}\ \boldsymbol{\tau} \end{array}, \begin{array}{c} \tau\ \tau\ \tau \\ |\ |\ | \\ 1\ \tau\ \boldsymbol{\tau}\ \boldsymbol{\tau} \end{array} \right\}.$$

That is, the fusion space V_{τ^3} is three-dimensional, and we denote the ordered basis by $\{\tau 1, 1\tau, \tau\tau\}$. Since there are three τ-anyons, there are two generators for the braid group, σ_1 and σ_2. Lemma 5.3.1 gives

$$\begin{aligned}\rho(\sigma_1)_{(\tau 1),(\tau 1)} &= R^\tau_{\tau\tau}\\ \rho(\sigma_1)_{(1\tau),(1\tau)} &= R^1_{\tau\tau}\\ \rho(\sigma_1)_{(\tau\tau),(\tau\tau)} &= R^\tau_{\tau\tau}\\ \rho(\sigma_1)_{i,j} &= 0 \text{ for } i \neq j.\end{aligned}$$

In matrix form that is

$$\rho(\sigma_1) = \begin{pmatrix} R^\tau_{\tau\tau} & & \\ & R^1_{\tau\tau} & \\ & & R^\tau_{\tau\tau} \end{pmatrix} = R^\tau_{\tau\tau} \oplus R.$$

Lemma 5.3.3 gives

$$\rho(\sigma_2)_{(\tau 1),(\tau 1)} = R^\tau_{\tau\tau}.$$

We compute $\rho(\sigma_2)$ for the τ-charge sector by considering its action on the standard fusion states

$$\rho(\sigma_2)\,\underset{1\ \tau\ 1\ \tau}{\overset{\tau\ \tau\ \tau}{|\ |\ |\ |}} = \underset{1\ \tau\ 1\ \tau}{\overset{\tau\ \tau\ \tau}{\smile\ |\ |}} = (B^\tau_{\tau\tau\tau})_{11}\,\underset{1\ \tau\ 1\ \tau}{\overset{\tau\ \tau\ \tau}{|\ |\ |\ |}} + (B^\tau_{\tau\tau\tau})_{1\tau}\,\underset{1\ \tau\ \tau\ \tau}{\overset{\tau\ \tau\ \tau}{|\ |\ |\ |}},$$

$$\rho(\sigma_2)\,\underset{1\ \tau\ \tau\ \tau}{\overset{\tau\ \tau\ \tau}{|\ |\ |\ |}} = \underset{1\ \tau\ \tau\ \tau}{\overset{\tau\ \tau\ \tau}{\smile\ |\ |}} = (B^\tau_{\tau\tau\tau})_{\tau 1}\,\underset{1\ \tau\ 1\ \tau}{\overset{\tau\ \tau\ \tau}{|\ |\ |\ |}} + (B^\tau_{\tau\tau\tau})_{\tau\tau}\,\underset{1\ \tau\ \tau\ \tau}{\overset{\tau\ \tau\ \tau}{|\ |\ |\ |}}.$$

Thus we have

$$\begin{aligned}\rho(\sigma_2)_{(1\tau),(1\tau)} &= (B^\tau_{\tau\tau\tau})_{11}\\ \rho(\sigma_2)_{(1\tau),(\tau\tau)} &= (B^\tau_{\tau\tau\tau})_{1\tau}\\ \rho(\sigma_2)_{(\tau\tau),(1\tau)} &= (B^\tau_{\tau\tau\tau})_{\tau 1}\\ \rho(\sigma_2)_{(\tau\tau),(\tau\tau)} &= (B^\tau_{\tau\tau\tau})_{\tau\tau}\,.\end{aligned}$$

In matrix form that is

$$\rho(\sigma_2) = \begin{pmatrix} R^\tau_{\tau\tau} & & \\ & B_{11} & B_{1\tau} \\ & B_{\tau 1} & B_{\tau\tau} \end{pmatrix} = R^\tau_{\tau\tau} \oplus B.$$

Example 6.3.3 (Braiding in $V^1_{\tau^4}$)**.** Consider $V^1_{\tau^4}$, this is the smallest non-trivial proper fusion space having dimension two. The fusion space is proper in the sense that the there is only one charge sector and it is the trivial (vacuum) charge sector. That is, there are really only 4 Fibonacci anyons. We shall compute the corresponding braid group representation determined by exchange of Fibonacci anyons in the standard basis of $V^1_{\tau^4}$.

Proposition 6.3.1. *The representation of the braid group generators determined by $V^1_{\tau^4}$, i.e. four Fibonacci anyons of trivial total charge, is*

$$\rho(\sigma_1) = R, \quad \rho(\sigma_2) = B, \quad \rho(\sigma_3) = R.$$

Proof. Lemma 5.3.1 gives

$$\rho(\sigma_1) = B^{\tau}_{1\tau\tau} = R_{\tau\tau} = \begin{pmatrix} R^1_{\tau\tau} & 0 \\ 0 & R^{\tau}_{\tau\tau} \end{pmatrix}.$$

Lemma 5.3.2 gives

$$\rho(\sigma_2)_{ij} = (B^{\tau}_{\tau\tau\tau})_{ij} = \sum_f \left(\left(F^{-1}\right)^{\tau}_{\tau\tau\tau} \right)_{if} R^f_{\tau\tau} \left(F^{\tau}_{\tau\tau\tau}\right)_{fj}$$

i.e.

$$\rho(\sigma_2) = F^{-1} R F =: B.$$

Finally, lemma 5.3.3 gives

$$\rho(\sigma_3)_{ij} = \delta_{ij} R^{\bar{j}}_{\tau\tau} \implies \rho(\sigma_3) = R_{\tau\tau}$$

since τ is its own antiparticle, $\overline{\tau} = \tau$. □

6.3.2 General braiding in $\widetilde{V}_{\tau^n}$

Recall the discussion of charge sectors from section 5.1. Since we cannot, in general, restrict the fusion space to a fixed charge sector, we shall compute the braid group generators in all charge sectors for two, three and four τ anyons. That is, we shall compute $\rho_n(\sigma_j)$ in $\widetilde{V}_{\tau^n}$ for $n = 2, 3, 4$. (This notation was introduced in definitions 5.1.1 and 5.2.1.) These examples give important insights of how the braid representation grows with the number of anyons, but most importantly these braids shall later be crucial when determining U_p in theorem 6.4.1.

Example 6.3.4 (Braiding in $\widetilde{V}_{\tau^2}$)**.** The fusion space $\widetilde{V}_{\tau^2}$ has standard

$$\left\{ \text{valid intermediate charges } abc \text{ in } \begin{array}{c} \tau\ \tau \\ |\ \ | \\ a\ b\ c \end{array} \right\} \equiv \left\{ 1\tau 1,\ 1\tau\tau,\ \tau\tau 1,\ \begin{array}{l} \tau 1\tau, \\ \tau\tau\tau \end{array} \right\},$$

grouped into the four charge sector, $11, 1\tau, \tau 1$ and $\tau\tau$, respectively. From lemma 5.3.2 we have

$$\rho_2(\sigma_1) = B^c_{a\tau\tau} \quad \iff \quad \rho_2(\sigma_1)_{ij} = (B^c_{a\tau\tau})_{ij}$$

where the indices i and j run over the given basis element. In the above given order of the basis fusion states the B-matrix is block diagonal. That is, in the obvious identification of the fusion space with $\mathbb{C}^5$ we have

$$\begin{aligned} \rho_2(\sigma_1) &= \begin{pmatrix} (B^1_{1\tau\tau})_{\tau\tau} & & & & \\ & (B^{\tau}_{1\tau\tau})_{\tau\tau} & & & \\ & & (B^1_{\tau\tau\tau})_{\tau\tau} & & \\ & & & (B^{\tau}_{\tau\tau\tau})_{11} & (B^{\tau}_{\tau\tau\tau})_{1\tau} \\ & & & (B^{\tau}_{\tau\tau\tau})_{\tau 1} & (B^{\tau}_{\tau\tau\tau})_{\tau\tau} \end{pmatrix} \\ &= \begin{pmatrix} R^1_{\tau\tau} & & & & \\ & R^{\tau}_{\tau\tau} & & & \\ & & R^{\tau}_{\tau\tau} & & \\ & & & B_{11} & B_{1\tau} \\ & & & B_{\tau 1} & B_{\tau\tau} \end{pmatrix} \\ &= R^1_{\tau\tau} \oplus R^{\tau}_{\tau\tau} \oplus R^{\tau}_{\tau\tau} \oplus B. \end{aligned}$$

Example 6.3.5 (Braiding in $\widetilde{V}_{\tau^3}$)**.** Next, taking a third Fibonacci anyon τ into account gives the fusion space $\widetilde{V}_{\tau^3}$ with standard basis

$$\left\{ \begin{array}{c} \text{valid intermediate charges} \\ abcd \text{ in } \begin{array}{c} \tau\ \tau\ \tau \\ \hline a\ b\ c\ d \end{array} \end{array} \right\} \equiv \left\{ \begin{array}{llll} 1\tau\tau 1, & 1\tau 1\tau, & \tau 1\tau 1, & \tau 1\tau\tau, \\ & 1\tau\tau\tau, & \tau\tau\tau 1, & \tau\tau 1\tau, \\ & & & \tau\tau\tau\tau \end{array} \right\}.$$

In this order of the basis we have

$$\rho_3(\sigma_1) = R^\tau_{\tau\tau} \oplus R \oplus B \oplus \begin{pmatrix} B_{11} & & B_{12} \\ & R^\tau_{\tau\tau} & \\ B_{21} & & B_{22} \end{pmatrix}$$

$$\rho_3(\sigma_2) = R^\tau_{\tau\tau} \oplus B \oplus R \oplus \begin{pmatrix} R^\tau_{\tau\tau} & & \\ & B_{11} & B_{1\tau} \\ & B_{\tau 1} & B_{\tau\tau} \end{pmatrix}$$

Example 6.3.6 (Braiding in $\widetilde{V}_{\tau^4}$)**.** In the ordered basis

$$\left\{ \begin{array}{c} \text{valid intermediate charges} \\ abcde \text{ in } \begin{array}{c} \tau\ \tau\ \tau\ \tau \\ \hline a\ b\ c\ d\ e \end{array} \end{array} \right\} \equiv \left\{ \begin{array}{llll} & & & \tau 1\tau 1\tau, \\ 1\tau 1\tau 1, & 1\tau 1\tau\tau, & \tau 1\tau\tau 1, & \tau 1\tau\tau\tau, \\ 1\tau\tau\tau 1, & 1\tau\tau 1\tau, & \tau\tau 1\tau 1, & \tau\tau\tau 1\tau, \\ & 1\tau\tau\tau\tau, & \tau\tau\tau\tau 1, & \tau\tau 1\tau\tau, \\ & & & \tau\tau\tau\tau\tau \end{array} \right\}$$

we have

$$\rho_4(\sigma_1) = \begin{pmatrix} R^1_{\tau\tau} & \\ & R^\tau_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} R^1_{\tau\tau} & & \\ & R^\tau_{\tau\tau} & \\ & & R^\tau_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} B_{11} & & B_{1\tau} \\ & R^\tau_{\tau\tau} & \\ B_{\tau 1} & & B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} B_{11} & & & B_{12} & \\ & B_{11} & & & B_{12} \\ & & R^\tau_{\tau\tau} & & \\ B_{21} & & & B_{22} & \\ & B_{21} & & & B_{22} \end{pmatrix}$$

$$\rho_4(\sigma_2) = \begin{pmatrix} B_{11} & B_{1\tau} \\ B_{\tau 1} & B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} B_{11} & & B_{1\tau} \\ & R^\tau_{\tau\tau} & \\ B_{\tau 1} & & B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} R^\tau_{\tau\tau} & & \\ & B_{11} & B_{1\tau} \\ & B_{\tau 1} & B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} R^1_{\tau\tau} & & & & \\ & R^\tau_{\tau\tau} & & & \\ & & R^\tau_{\tau\tau} & & \\ & & & B_{11} & B_{12} \\ & & & B_{21} & B_{22} \end{pmatrix}$$

$$\rho_4(\sigma_3) = \begin{pmatrix} R^1_{\tau\tau} & \\ & R^\tau_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} R^\tau_{\tau\tau} & & \\ & B_{11} & B_{1\tau} \\ & B_{\tau 1} & B_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} R^\tau_{\tau\tau} & & \\ & R^1_{\tau\tau} & \\ & & R^\tau_{\tau\tau} \end{pmatrix} \oplus \begin{pmatrix} B_{11} & B_{1\tau} & & & \\ B_{\tau 1} & B_{\tau\tau} & & & \\ & & B_{11} & & B_{1\tau} \\ & & & R^\tau_{\tau\tau} & \\ & & B_{\tau 1} & & B_{\tau\tau} \end{pmatrix}$$

Note that in this order of the basis, the last block of $\rho_4(\sigma_2)$ is precisely $\rho_2(\sigma_1)$. Indeed, basis elements in the $\tau\tau$-sector is ordered by the internal charge sectors.

Remark 6.3.1. As is clearly manifested in the three examples, the representation of the braid group generators is always split into four blocks, one block for each two-sided charge sector, $11, 1\tau, \tau 1$ and $\tau\tau$. These blocks will always be disjoint since there is no way of transforming between them. Compare with the decomposition $V_{ab} = \bigoplus_c V^c_{ab}$. The dimension of each of these blocks grows as the Fibonacci numbers because the basis states in each block are Fibonacci

strings of 1's and τ's, essentially of length $n-3, n-2, n-2, n-1$ respectively (disregarding the fixed labels due to charge sectors). Thus, by lemma 6.1.1 we have that the dimension of $\rho_n(\sigma_j)$ for n Fibonacci anyons is given by

$$\dim \rho_n(\sigma_j) = \underbrace{F_{n-1} + F_n}_{F_{n+1}} + \underbrace{F_n + F_{n+1}}_{F_{n+2}} = F_{n+3}.$$

This can be continued to compute $\rho_n(\sigma_j)$ in $\widetilde{V}_{\tau^n}$ for any n and j. This is done programmatically in appendix A.

6.3.3 Spectrum of $\rho_n(\sigma_j)$

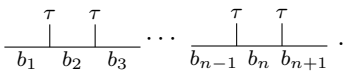
Theorem 6.3.2. *The spectrum without multiplicities of the representation of the braid group generator $\rho_n(\sigma_j)$ is independent of n and j and given by*

$$\operatorname{spec}(\rho_n(\sigma_j)) = \left\{R^1_{\tau\tau}, R^\tau_{\tau\tau}\right\}.$$

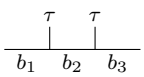
Proof. From lemma 2.2.1 we have that the all braid group generators σ_j are conjugate, this translates to that the corresponding representations $\rho_n(\sigma_j)$ are similar for fixed n. That is, for fixed n there exists an invertible matrix S such that

$$\rho_n(\sigma_{j+1}) = S\rho_n(\sigma_j)S^{-1}.$$

(In particular we have $S = \rho_n(\sigma_j)\rho_n(\sigma_{j+1})$ from the proof of lemma 2.2.1.) Thus, the eigenvalues of $\rho_n(\sigma_j)$ are independent of j. Fix $j = 1$, the representation $\rho_n(\sigma_1)$ of σ_1 acts on $\widetilde{V}_{\tau^n}$ with basis elements



However, only the labels b_1, b_2, b_3 enter in the expression for $\rho_n(\sigma_1)$, thus it is really only the space $\widetilde{V}_{\tau^2}$ with basis elements



that $\rho_n(\sigma_1)$ acts on. Thus, as discussed in remark 5.3.1, as n increases the matrix $\rho_n(\sigma_1)$ increases and gets repeated block. These repeated blocks give no new eigenvalues (but give increased multiplicity for the existing eigenvalues). Thus, the spectrum of $\rho_n(\sigma_1)$ is independent of n, not counting multiplicities.

To sum up, the spectrum of $\rho_n(\sigma_j)$, not counting multiplicities, is independent of both n and j. Thus, we can compute the spectrum from the special case $\rho_2(\sigma_1)$, computed in example 6.3.4. □

Remark 6.3.2. This theorem trivially generalizes to *any* anyon model. However, the specific eigenvalues are of course different.

6.4 Exchange operator U_p

Consider the fusion space $\widetilde{V}_{\tau^n}$ and exchange of a pair of anyons, j and k, around p enclosed anyons, $j+1, j+2, \ldots, k-1$. We shall use section 5.4, in particular theorem 5.4.1, to compute U_p for general p for Fibonacci anyons. As discussed in section 5.2, the dimension of the representation of the braid group depends on the number of considered anyons, in the case of Fibonacci anyons we we saw in remark 6.3.1 that the dimension of $\widetilde{V}_{\tau^n}$ increases as the Fibonacci numbers as n increases. If considering a fusion space with more particles than what enters the represented braid, the braid operator will simply get repeated block. Thus, we shall represent U_p in the fusion space $\widetilde{V}_{p+2}$. This is sufficient to compute the spectrum. Furthermore, representing U_p in a higher-dimensional fusion space is in principle straight forward and described in full generality in section 5.2.

Theorem 6.4.1. *Exchange of a pair of τ anyons around p enclosed τ anyons introduces a non-Abelian anyonic phase U_p, given by*

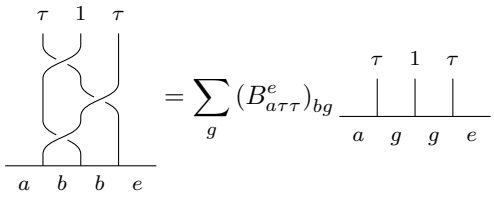
$$\begin{aligned} U_0 &= \rho_2(\sigma_1) \\ U_1 &= \rho_3(\sigma_1)\rho_3(\sigma_2)\rho_3(\sigma_1) \\ U_p &= U_0^{\oplus \mathrm{Fib}(p-1)} \oplus U_1^{\oplus \mathrm{Fib}(p)}, \quad p \geq 2 \end{aligned}$$

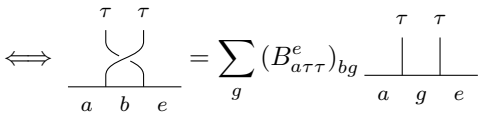
expressed in the fused basis eq. (5.1)*, and the expression for U_p is written in a basis ordered so that all repeating blocks align nicely. The braid generators $\rho_n(\sigma_j)$ are computed in examples 6.3.4 and 6.3.5. Explicitly that is*

$$U_0 = R^1_{\tau\tau} \oplus R^\tau_{\tau\tau} \oplus R^\tau_{\tau\tau} \oplus B$$

$$U_1 = \left(R^\tau_{\tau\tau}\right)^3 \oplus (RBR) \oplus (BRB) \oplus \begin{pmatrix} R^\tau_{\tau\tau}\left(B_{11}\right)^2 + B_{12}B_{21}B_{22} & B_{12}B_{21}R^\tau_{\tau\tau} & B_{12}\left(B_{22}\right)^2 + B_{11}B_{12}R^\tau_{\tau\tau} \\ B_{12}B_{21}R^\tau_{\tau\tau} & B_{11}\left(R^\tau_{\tau\tau}\right)^2 & B_{12}B_{22}R^\tau_{\tau\tau} \\ B_{21}\left(B_{22}\right)^2 + B_{11}B_{21}R^\tau_{\tau\tau} & B_{21}B_{22}R^\tau_{\tau\tau} & \left(B_{22}\right)^3 + B_{12}B_{21}R^\tau_{\tau\tau} \end{pmatrix}$$

Proof. The braid corresponding to U_0 is given by theorem 5.4.1 with $c = 1$. That is, (using lemma 4.5.1)





This braid is computed as $\rho_2(\sigma_1)$ in example 6.3.4.

Similarly, the braid corresponding to U_1 is given by theorem 5.4.1 with $c = \tau$ and is computed as $\rho_3(\sigma_1)\rho_3(\sigma_2)\rho_3(\sigma_1)$ in example 6.3.5.

Finally, for $p \geq 2$ the possible values for c, i.e. the possible results from fusion of p Fibonacci anyons, are 1 and τ. Furthermore, in the fused basis eq. (5.1) the exchange operator U_p does not mix c, only the intermediate charges may

be mixed when braiding. Since the two possible values for c are accounted for in U_0 and U_1, we conclude that $U_2 = U_0 \oplus U_1$. Next, lemma 6.1.2 shows that fusion of p anyons result in 1 in $\mathrm{Fib}(p-1)$ ways and τ in $\mathrm{Fib}(p)$ ways. Thus, we conclude

$$U_p = U_0^{\oplus \mathrm{Fib}(p-1)} \oplus U_1^{\oplus \mathrm{Fib}(p)}.$$

From this we have

$$\begin{aligned}
\dim U_p &= \dim(U_0)\,\mathrm{Fib}(p-1) + \dim(U_1)\,\mathrm{Fib}(p) \\
&= 5\,\mathrm{Fib}(p-1) + 8\,\mathrm{Fib}(p) \\
&= 5\,(\mathrm{Fib}(p-1) + \mathrm{Fib}(p)) + 3\,\mathrm{Fib}(p) \\
&= 5\,\mathrm{Fib}(p+1) + 3\,\mathrm{Fib}(p) \\
&= 3\,(\mathrm{Fib}(p+1) + \mathrm{Fib}(p)) + 2\,\mathrm{Fib}(p+1) \\
&= 3\,\mathrm{Fib}(p+2) + 2\,\mathrm{Fib}(p+1) \\
&= 2\,(\mathrm{Fib}(p+1) + \mathrm{Fib}(p+2)) + \mathrm{Fib}(p+2) \\
&= 2\,\mathrm{Fib}(p+3) + \mathrm{Fib}(p+2) \\
&= (\mathrm{Fib}(p+2) + \mathrm{Fib}(p+3)) + \mathrm{Fib}(p+3) \\
&= \mathrm{Fib}(p+4) + \mathrm{Fib}(p+3) \\
&= \mathrm{Fib}(p+5)
\end{aligned}$$

in agreement with remark 6.3.1, since $n = p + 2$.

□

Remark 6.4.1. The fact that U_p only gets repeated blocks for p for $p \geq 2$ is due to the fact that fusion of p anyons of type τ always results in 1 or τ. However, if we consider an anyon model with fusion rules

$$t \times t = a, \quad t \times a = t$$

so that

$$t \times t \times t = t$$

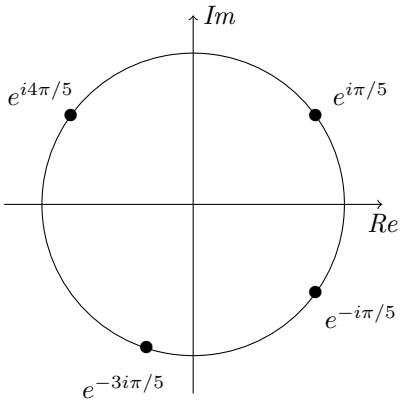
then U_2 and U_3 are different, since the intermediate charge c is different.

We now have an explicit expression for the exchange operator U_p, allowing us to compute the spectrum for U_p.

Corollary 6.4.2. *The eigenvalues of U_p for Fibonacci anyons are*

$$\sigma(U_p) = \left\{ \begin{array}{rcl} e^{i4\pi/5} & \times & \mathrm{Fib}(p+3), \\ e^{i\pi/5} & \times & 2\,\mathrm{Fib}(p), \\ e^{-i\pi/5} & \times & 3\,\mathrm{Fib}(p), \\ e^{-i3\pi/5} & \times & 3\,\mathrm{Fib}(p-1) \end{array} \right\},$$

where $a \times n$ denotes that a occurs with multiplicity n.



Proof. Since U_p consists of repeated block, it suffices to compute the eigenvalues for each block, multiplied by the number of occurrences for each block to get the multiplicities. Note that $B = F^{-1}RF$ is similar to R. The multiplicity for $e^{i4\pi/5}$ is $2\,\mathrm{Fib}(p-1)+3\,\mathrm{Fib}(p) = \mathrm{Fib}(p+3)$, the other multiplicities are straight forward. □

In chapter 3 we showed the connection between eigenvalues for the exchange operator and bounds for the kinetic energy. The distance of the eigenvalues from 1 along the complex unit circle given the energy bounds. We thus see that simple exchange of two anyons has a higher corresponding kinetic energy than exchange of two anyons around $p \geq 1$ anyons. In particular U_p for $p \geq 1$ has the eigenvalue $e^{\pm i\pi/5}$ closest to 1.

We see that exchange of Fibonacci described by U_p always corresponds to a non-zero kinetic energy, it is natural to ask if there are braids that do have zero energy. The following corollary positively answers this.

Corollary 6.4.3 (Zero energy braid for Fibonacci anyons)**.** *With U_p as in theorem 6.4.1 we have*

$$\sigma\left(U_p^5\right) = \{1, -1\}$$

for all $p \geq 0$, by straight-forward computation. This shows that braiding a pair of Fibonacci anyons five times around any number of Fibonacci anyons behaves as both fermions and bosons simultaneously. In particular, the occurrence of the eigenvalue 1 shows that the braid U_p^5 is a zero-energy braid, according to the discussion in chapter 3.

This can be seen as a consequence of the fact that $\left(R^1_{\tau\tau}\right)^5 = 1$ and $\left(R^\tau_{\tau\tau}\right)^5 = -1$.

6.5 Quantum dimension and fusion probabilities

As explained in detail in [8], the quantum dimension d_a of an anyon of type a is the rate of growth in dimension of the fusion space $V^1_{a^n}$ as n grows. Explicitly that is

$$d_a = \lim_{n\to\infty} \frac{\dim\left(V_{a^{n+1}}\right)}{\dim\left(V_{a^n}\right)}.$$

Recall that a^n denotes $\underbrace{\tau \cdots \tau}_{n}$ if a is a particle label. The quantum dimension can be immediately computed from the fusion rules. For the Fibonacci anyons we have the fusion rule

$$\tau \times \tau = 1 + \tau$$

which gives the quantum dimension

$$d_\tau^2 = 1 + d_\tau \iff d_\tau = \varphi$$

note that only the positive solution is meaningful. Furthermore, the trivial particle 1 has quantum dimension $d_1 = 1$. Indeed, the corresponding fusion space $V_{1^n}^1$ does not grow in dimension as n increases.

Using the quantum dimension, [8] shows the following result.

Proposition 6.5.1. *The probability $P(ab \to c)$ that anyons of type a and b fuse to an anyon of type c is given by*

$$P(ab \to c) = \frac{N_{ab}^c d_c}{d_a d_b}.$$

Consider fusion of n Fibonacci τ anyons. Proposition 6.5.1 together with $d_\tau = \varphi$ and $d_1 = 1$ gives

$$P(\tau\tau \to \tau) = \varphi^{-1}, \quad P(\tau\tau \to 1) = \varphi^{-2}.$$

We then have the probability of n Fibonacci τ anyons fusing to charge τ given by the recursion relation

$$\begin{aligned} P(\tau^n \to \tau) &= P(\tau^{n-1} \to \tau)P(\tau\tau \to \tau) + P(\tau^{n-1} \to 1)P(1\tau \to \tau) \\ &= \varphi^{-1}P(\tau^{n-1} \to \tau) + P(\tau^{n-1} \to 1) \\ &= \varphi^{-1}P(\tau^{n-1} \to \tau) + P(\tau^{n-2} \to \tau)P(\tau\tau \to 1) \\ &= \varphi^{-1}P(\tau^{n-1} \to \tau) + \varphi^{-2}P(\tau^{n-2} \to \tau). \end{aligned}$$

With initial values $P(\tau \to \tau) = 1$ and $P(\tau\tau \to \tau) = \varphi^{-1}$ we get

$$P(\tau^n \to \tau) = \mathrm{Fib}(n)\varphi^{1-n}.$$

Using the closed form expression

$$\mathrm{Fib}(n) = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

we get the limit

$$\lim_{n\to\infty} P(\tau^n \to \tau) = \frac{\varphi}{\sqrt{5}} \approx 0.72$$

and thus

$$\lim_{n\to\infty} P(\tau^n \to 1) \approx 0.28.$$

TODO: Can this be used to say anything more abut the eigenvalues of U_p? Some eigenvalues correspond to $c = 1$, some correspond to $c = \tau$, as we've now seen the latter case is more probable.

Chapter 7

Topological Quantum Computation

Appendix A

Computing $\rho_n(\sigma_j)$ for Fibonacci anyons programmatically

In this appendix we compute the representation for the braid group generators $\rho_n(\sigma_j)$ in $\widetilde{V}_{\tau^n}$, for any $2 \geq n$ and $1 \geq j \geq n-1$.

A.1 Symbolic computation

The following Mathematica code symbolically generates $\rho_n(\sigma_j)$ for any n and j.

```
(* FibString[n] returns a list all Fibonacci strings
   of length n, sorted by charge sectors.
   A Fibonacci string is a list of 1's and 2's with no recurring 1's. *)

FibStringRecurse[1] := {{1}, {2}};
FibStringRecurse[n_] := Flatten[
    Map[
        If[
            Last[#] == 2,
            {Flatten[{#, 1}],
            Flatten[{#, 2}]}, {Flatten[{#, 2}]}
        ] &,
        FibString[n - 1]],
    1
];

FibString[n_] := SortBy[FibStringRecurse[n], {First[#], Last[#]} &];

(* FibStringCS[n, csl, csr] returns FibString[n] restricted to left and right
   charge sectors given by csl and csr. *)
FibStringCS[n_, csl_, csr_] := Select[
    FibStringRecurse[n],
    First[#] == csl && Last[#] == csr &
```

```
];

(* [n,j] returns the representation of the j:th braid group generator
   in the fusion space of n Fibonacci anyons, including all possible
   left and right charge sectors. *)
Meta[n_, j_, fusionStates_] := Map[
    Module[{labels, row, idx, idx2, fusionStateCopy},
        fusionStateCopy = #;
        labels = Take[#, {j, j + 2}];
        row = ConstantArray[0, Length[fusionStates]];
        idx = Position[fusionStates, #][[1]][[1]];
        Switch[labels,
            {1, 2, 1}, row[[idx]] = R1,
            {1, 2, 2}, row[[idx]] = R,
            {2, 2, 1}, row[[idx]] = R,
            {2, 1, 2}, (
                row[[idx]] = B11;
                fusionStateCopy[[j + 1]] = 2;
                idx2 = Position[fusionStates, fusionStateCopy][[1]][[1]];
                row[[idx2]] = B12
            ),
            {2, 2, 2}, (
                row[[idx]] = B22;
                fusionStateCopy[[j + 1]] = 1;
                idx2 = Position[fusionStates, fusionStateCopy][[1]][[1]];
                row[[idx2]] = B21;
            )
        ];
        row
    ] &, fusionStates];

[n_, j_] := Meta[n, j, FibString[n + 1]];

[n_, j_, csl_, csr_] := Meta[n, j, FibStringCS[n + 1, csl, csr]];

(* U[p] returns  on p+2 Fibonacci anyons. *)
U[p_] := Dot @@ Flatten[{
    Table[[p + 2, j], {j, 1, p + 1}],
    Table[[p + 2, j], {j, p, 1, -1}]
}, 1];
U[p_, csl_, csr_] := Dot @@ Flatten[{
    Table[[p + 2, j, csl, csr], {j, 1, p + 1}],
    Table[[p + 2, j, csl, csr], {j, p, 1, -1}]
}, 1];

(* Demo *)
Print[MatrixForm[[3, 1]]];
Print[[3, 1].[3, 2].[3, 1] == U[1]];
```

A.2 Numeric computation

The following code builds on the code in the previous section to give numerical values for the braid group generators.

```
(* Plug in values for the parameters. *)
R1 = Exp[4 Pi I/5];
R = Exp[-3 Pi I/5];
R = ({
    {R1, 0},
    {0,  R}
});
 = GoldenRatio;
F = ({
    {^-1,      ^(-1/2)},
    {^(-1/2), -^-1}
});
B = Inverse[F].R.F;
B11 = B[[1, 1]];
B12 = B[[1, 2]];
B21 = B[[2, 1]];
B22 = B[[2, 2]];

(* Compute argument of eigenvalues. Performance boos by computing using N[..],
   then convert back by Rationalize[../Pi]. *)
EigenvaluesArg[data_] := Sort[Rationalize[Arg[Eigenvalues[N[data]]]/Pi]Pi];

(* Demo *)
Print[EigenvaluesArg[[3, 1]]];
```

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